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# Approximation of the impedance operator for domains coated with thin multilayers

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# Approximation of the impedance operator for domains coated with thin multilayers

## Abstract

We are interested in diffraction problems of an electromagnetic wave by a perfectly conducting planar obstacle coated with thin multilayers of dielectric materials. The aim is to obtain a boundary condition that replaces the effect of dielectric thin layers. This condition is constructed from an approximation of the impedance operator. We first provide the approximations of this operator for planar, circular and arbitrary shaped obstacles in the case of two thin layers for a particular problem. Then we generalize those results in the case of planar obstacles to include all the scattering problems.

**Keywords**:

Electromagnetic scattering, diffraction, thin dielectric layers, impedance operator, abstract Cauchy problem.

# تقريب مُؤثر الـمعَاوقة لـلميَادين الـمطلية بطبقَات متعدِدة وَ رقيقـة

## ملخص

نـهتـم في أطروحة الدكتورَاء هذه بـمـسأَلة إِنكسَار الـموجة الـكهرومغنَاطيسية بوَاسطة حَاجز مسطح موصل تـمَامًا لِلـتّيَار ومغطى بطبقَات رقيقة متعدِدة من الـموَاد العَازلة. الـهدف هو الـحصول علَى شرط حدي يـحل مـحل تأَثير الطبقَات الرقيقة العَازلة. يـتـم إِنشَاء هذَا الشرط من تقريب مُؤثر الـمعَاوقة. نقدم أولاً تقريبَات لهذَا الـمُؤثر في حَالة الـحوَاجز الـمسطحة وَ الـدَائِرية ثـم الـحوَاجز ذَات شكل كيفي في حَالة طبقتين رقيقتين لـمسأَلة خَاصة. ثم نعمّـم تلك النتَائِج في حَالة الـحوَاجز الـمسطحة لتشمل جـميع مسَائِل إِنتشَار الأَموَاج الكهرومغنَاطيسية.

**الكلمَات الـمفتَاحية:**

إِنتشَار الأَموَاج الكهرومغنَاطيسية، إِنكسَار، الطبقَات العَازلة الرقيقة، مُؤثر الـمعَاوقة، مشكلة كوشي المجردة.

# Approximation de l'opÃ©rateur d'impÃ©dance pour les domaines revÃªtus de multicouches minces

## RÃ©sumÃ©

Nous nous intÃ©ressons aux problÃ¨mes de diffraction d'une onde Ã©lectromagnÃ©tique par un obstacle plan parfaitement conducteur revÃªtu de fines multicouches de matÃ©riaux diÃ©lectriques. L'objectif est d'obtenir une condition aux limites qui remplace l'effet des fines couches diÃ©lectriques. Cette condition est construite Ã  partir d'une approximation de l'opÃ©rateur d'impÃ©dance. Nous fournissons d'abord les approximations de cet opÃ©rateur pour des obstacles planaires, circulaires et de forme arbitraire dans le cas de deux couches minces pour un problÃ¨me particulier. Nous gÃ©nÃ©ralisons ensuite ces rÃ©sultats dans le cas d'obstacles plans pour inclure tous les problÃ¨mes de diffraction.

**Mots clÃ©s :**

Diffusion Ã©lectromagnÃ©tique, diffraction, couches minces diÃ©lectriques, opÃ©rateur d'impÃ©dance, problÃ¨me de Cauchy abstrait.

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# Introduction

The diffraction of electromagnetic waves by a perfectly conducting obstacle coated with thin dielectric layers, emerges in many applications in the industrial world such as electromagnetic compatibility problems in embedded systems, antennas, satellites, telecommunications, or also applications involving the detection of objects and radar stealth, see, for example [7], [28], [3], [32], [31], [12], [25], [26], [30] and the references therein.

In this thesis, we are particularly interested in problems of diffraction by perfectly conducting obstacles, covered with homogenous thin dielectric multilayers. These problems are called transmission problems, which consist in solving a system of partial differential equations in an exterior domain with Silver-MÃ¼ller radiation condition at infinity and in an interior domain relating to thin dielectric layers. The governing equations are coupled by connecting conditions set on the common interface between exterior and interior domains and between thin layers as well. Solving numerically these equations is challenging since it requires discretizing on the scale of the layers' thickness. The mesh then contains a very large number of elements, which makes the calculations long and sometimes imprecise [6], [7], [10], [2], [20]. For this reason, we try to replace our problem by another problem that does not bring in any more thin layers.

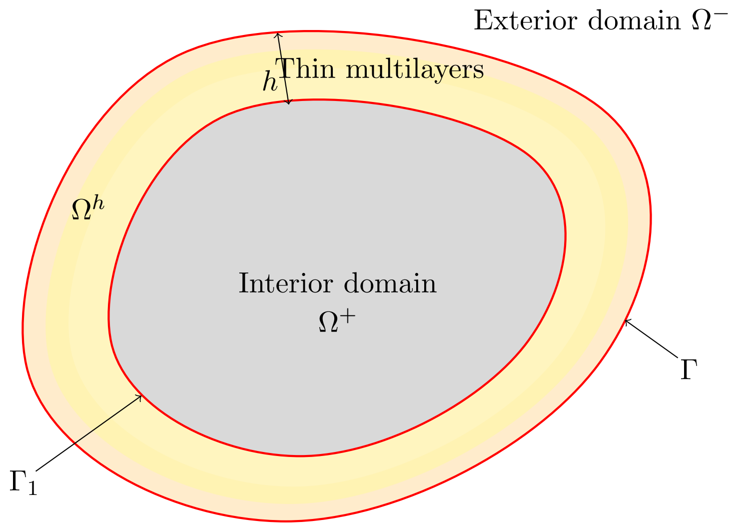


Figure 1: Illustration of the original diffraction domain

The original diffraction problem is given using the transmission problem defined in $\Omega^- \cup \Omega^h$, see figure 1:

$$\begin{cases} \text{Equations in the exterior domain } \Omega^- \\ \text{Equations in thin layers} \\ \text{with transmission conditions on } \Gamma \text{ and interfaces separating thin layers} \\ \text{with perfect conductor condition on } \Gamma_1 \\ \text{with radiation condition at infinity.} \end{cases}$$

The use of so-called Dirichlet-to-Neumann operator, relative to the equations set in thin layers allows to reduce the solving of our original problem to new equivalent or approximate diffraction problem that is posed only in the exterior domain (see figure 2) with an appropriate boundary condition known as Dirichlet-to-Neumann condition [7], [9], [14], [15], [24] and abbreviated DtoN condition, which is also called Steklov-PoincarÃ© condition or impedance condition as well [8], [5]. The new diffraction problem is then expressed as follows:

$$\begin{cases} \text{Equations in the exterior domain } \Omega^- \\ \text{with impedance conditions on } \Gamma \\ \text{with radiation condition at infinity.} \end{cases}$$

The whole difficulty rests on the knowledge of the impedance operator, which is generally

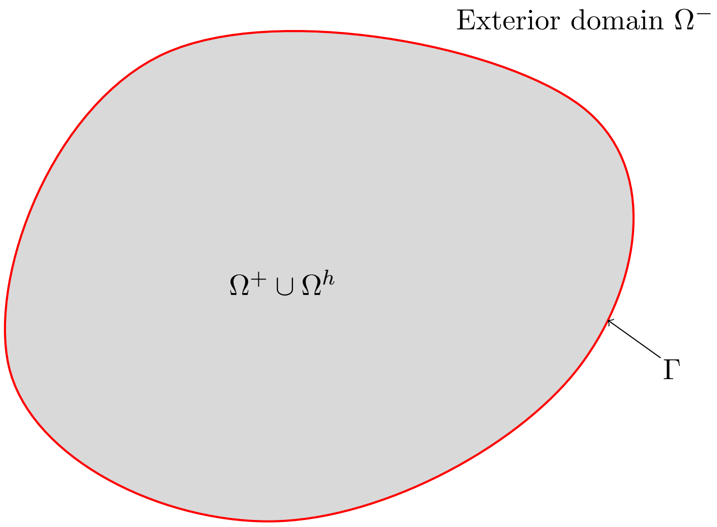


Figure 2: Illustration of the new equivalent or approximate domain

non-explicit [7], [17]. Fortunately, it is possible to explicit and approximate it in many cases as in planar obstacles [18], [22], [1], we are then able to construct the impedance conditions in this case. We will exploit the fact that the thicknesses of thin layers tend to zero to derive the approximations of the impedance operator.

This thesis is organized as follows:

In the first chapter, we begin by presenting briefly the physical problem and the mathematical governing equations.

In the second chapter we are particularly interested in a problem of diffraction of a harmonic wave, by a two-dimensional obstacle coated with thin multilayers of homogeneous dielectric materials. We will provide approximations of the impedance operator for planar, circular and arbitrary shaped obstacles in the case of two thin layers.

In chapter three, we decompose the electromagnetic vector field into its tangential and normal components and then we transform the Maxwell's system from a PDEs system to first order linear abstract Cauchy problem. Then we reformulate our problem using impedance operator, after that, we determine the exact formula of this operator. The third section of this chapter is devoted to constructing approximations of the impedance operator using two approaches: the first one consists of writing Taylor expansions iteratively in the thin layers and the second approach is to use asymptotic expansions. In the last section of this chapter we apply the results obtained in the third section to a particular scattering problem of electromagnetic waves.

# Chapter 1

# Some basic mathematical and physical concepts

The purpose of this preliminary chapter is to provide a brief overview of the physical model that motivates the mathematical work presented in this thesis.

## 1.1 Fundamental equations of electromagnetism

In this paragraph, we delve into exploring Maxwell's equations, which form the foundation of the electromagnetic theory and govern the behavior of electromagnetic waves in theory and application. Details for this section can be found in many references, see for instance [13].

First we mention that, if $\varphi \in L^2(\mathbb{R}^3)$ and $\mathbf{F} \in \left(L^2(\mathbb{R}^3)\right)^3$, then

$$\nabla\varphi = \operatorname{grad}\varphi = \sum_{i=1}^{3} \frac{\partial \varphi}{\partial x_i}\overrightarrow{e_i},$$

$$\nabla \cdot \mathbf{F} = \operatorname{div}\mathbf{F} = \sum_{\mathbf{i=1}}^{\mathbf{3}} \frac{\partial \mathbf{F_i}}{\partial \mathbf{x_i}},$$

and

$$\nabla \times \mathbf{F} = \operatorname{curl} \mathbf{F} = \left(\frac{\partial \mathbf{F}_3}{\partial x_2} - \frac{\partial \mathbf{F}_2}{\partial x_3}\right)\overrightarrow{e_1} + \left(\frac{\partial \mathbf{F}_1}{\partial x_3} - \frac{\partial \mathbf{F}_3}{\partial x_1}\right)\overrightarrow{e_2} + \left(\frac{\partial \mathbf{F}_2}{\partial x_1} - \frac{\partial \mathbf{F}_1}{\partial x_2}\right)\vec{e}_3.$$

The physical phenomena related to electromagnetism in a propagation medium are described using two functions or distributions $E$ (the electric field) and $B$ (the magnetic induction), defined on $\mathbb{R}^3_x \times \mathbb{R}_t \longrightarrow \mathbb{R}^3$ or most often $\mathbb{C}^3_x \times \mathbb{R}_t \longrightarrow \mathbb{C}^3$, defining the electromagnetic field, noted $\{E, B\}$.

The fields $E$ and $B$ are related to four functions (or distributions): the electric charge density $\rho$, the magnetic charge density $\rho_m$, the electric current density $\mathbf{J}$ and the magnetic current density $\mathbf{M}$.

The appearance of induced currents in a fixed conductor, placed in a non-stationary electric field, is translated by the Maxwell-Faraday equation and the conservation law of the flux of the magnetic field:

$$\operatorname{curl} E + \frac{\partial B}{\partial t} = -\mathbf{M},$$

$$\operatorname{div} B = \rho_m.$$

In unsteady state, the volume current $\mathbf{J}$ is no longer at a conservative flow taking into account the conservation law and the variation of the volume charge $\rho$ over time.

The continuity equation (or the conservation of electricity) is then defined by

$$\operatorname{div} \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \tag{1.1}$$

which reflects the fact that, if there is variation over time in the charge contained in a volume then there is a current between the interior and exterior of this volume.

Identically, the magnetic charge conservation equation is defined by

$$\operatorname{div} \mathbf{M} + \frac{\partial \rho_m}{\partial t} = 0. \tag{1.2}$$

The local expression of Gauss's theorem for steady states is expressed by:

$$\operatorname{div} E = \rho.$$

Experiments have shown that the flow of the electric field through any closed surface does not depend on the state of the movement of charges, thus Gauss' theorem can be generalized to unsteady states. However, by applying the generalized Gauss' theorem and using the geometric properties of the electric field, it is easy to establish the Maxwell-AmpÃ¨re equation:

$$\operatorname{curl} B - \frac{\partial E}{\partial t} = \mathbf{J}.$$

Thus in a propagation medium, in the presence of charge, the electromagnetic field $\{E, B\}$ satisfies the following four equations:

$$\operatorname{curl} E + \frac{\partial B}{\partial t} = -\mathbf{M} \quad \text{(Maxwell-Faraday)}, \tag{1.3}$$

$$\operatorname{div} B = \rho_m \quad \text{(Gauss-magnetic)}, \tag{1.4}$$

$$\operatorname{div} E = \rho \quad \text{(Gauss-electric)}, \tag{1.5}$$

$$\operatorname{curl} B - \frac{\partial E}{\partial t} = \mathbf{J} \quad \text{(Maxwell-AmpÃ¨re)}. \tag{1.6}$$

We note that:

- These equations are hyperbolic in $E$ and $B$.
- By derivation with respect to time of the Gauss-electric equation (1.5) and by applying the div operator to the equation (1.6), we find the relation called continuity or conservation of electricity (1.1). An identical demonstration makes it possible to obtain the equation for conservation of the magnetic charge (1.2).

- The evolution of $\{E, B\}$ is given by the Maxwell-AmpÃ¨re and Maxwell-Faraday equations.

- If $\rho, \rho_m, \mathbf{M}$ and $\mathbf{J}$ are zero, then $E$ and $B$ satisfy the wave equation.

## 1.2 Maxwell's equations

In the problems encountered in the electromagnetism, the charge and current densities are not known, or rather are only partially known. Indeed, $\{E, B\}$ creates a charge density $\rho^*$ and a current density $\mathbf{J}^*$ creating in turn an electric field $E^*$ and a magnetic field $B^*$ which are unknown.

Thus, we can decompose $\rho$ and $J$:

$$\rho = \rho^* + \varrho \text{ and } \mathbf{J} = \mathbf{J}^* + J,$$

where $\varrho$ and $J$ respectively represent the given so-called "external" charge and current densities.

The system then becomes:

$$\begin{cases} \operatorname{curl} E + \dfrac{\partial B}{\partial t} = -\mathbf{M}, \\ \operatorname{div} B = \rho_m, \\ \operatorname{div} E - \rho^* = \varrho, \\ \operatorname{curl} B - \dfrac{\partial E}{\partial t} - \mathbf{J}^* = J. \end{cases} \tag{1.7}$$

We now introduce two fields of vectors $P$ and $M$ of $\mathbb{R}^3_x \times \mathbb{R}_t$ related to $\rho^*$ and $\mathbf{J}^*$ by

$$\begin{cases} \operatorname{div}(-P) = \rho^*, \\ \operatorname{curl}(M) - \dfrac{\partial(-P)}{\partial t} = \mathbf{J}^*. \end{cases} \tag{1.8}$$

$P$ called the polarization vector and $M$ the magnetization vector. From (1.8) we can deduce that $\mathbf{J}^*$ and $\rho^*$ also satisfy the continuity equation

$$\frac{\partial \rho^*}{\partial t} + \operatorname{div}(\mathbf{J}^*) = 0.$$

Finally, by setting:

$$\begin{cases} D = E + P, \\ H = B - M, \end{cases} \tag{1.9}$$

where $D$ is called the electric induction or electric displacement and $H$ the magnetic field.

We can verify that the fields $D$ and $H$ satisfy the equations

$$\begin{cases} \operatorname{div} D = \varrho, \\ \operatorname{curl} H - \dfrac{\partial D}{\partial t} = J. \end{cases} \tag{1.10}$$

Finally, we obtain the new system in $\mathbb{R}^3_x \times \mathbb{R}_t$

$$\begin{cases} -\dfrac{\partial D}{\partial t} + \operatorname{curl} H = J, \\ \operatorname{div} D = \varrho, \\ \dfrac{\partial B}{\partial t} + \operatorname{curl} E = -\mathbf{M}, \\ \operatorname{div} B = \rho_m. \end{cases}$$

- $E, B, D$ and $H$ are the unknowns.
- If $\varrho$ and $J$ are assumed to be given, there are therefore twelve scalar unknowns for eight scalar equations (of which only six are independent).

It is therefore necessary to add additional conditions in order to have as many equations as unknowns, to be able to solve the problem. These so-called constitutive conditions are behavioral laws, between $E$ and $D$; $B$ and $H$, they depend on physical considerations and describe the properties of the material considered. Without them, the problem of evolution would be indeterminate.

These conditions are given in the case of a linear medium as:

$D = \varepsilon E + \xi H$ $\quad$ $\varepsilon$ : electrical permittivity of the medium,

$B = \mu H + \zeta E$ $\quad$ $\mu$ : magnetic permeability of the medium.

$\mathbf{J}^*$ and $E$ are related by the equation: $\mathbf{J}^* = \sigma E$, $\sigma$ called electrical conductivity.

These behavioral laws are characteristic of the medium in which the fields propagate. In general, $\varepsilon, \mu, \xi, \zeta, \sigma$ have a tensor character and their value is not necessarily constant (saturation, hysteresis phenomenon, dependence on temperature, etc.).

**Remark 1.1**

- *If the materials are bi-anisotropic (the most general case), the four tensors are non-zero.*
- *If the materials are anisotropic, the tensors $\xi$ and $\zeta$ are zero, hence $B = \mu H$ and $D = \varepsilon E$. Note that a medium in which these relationships are verified, with $\mu$ and $\varepsilon$ constants, called a ideal or perfect medium.*
- *If the materials are isotropic then the tensors $\varepsilon$ and $\mu$ are diagonal and are written $\varepsilon \equiv \varepsilon I$ and $\mu \equiv \mu I$ where $I$ is the unit diagonal tensor.*
- *In linear media, $\varepsilon$ and $\mu$ are independent of the fields $H$ and $E$.*
- *In conductive materials, the electrical conductivity $\sigma$ is positive, while $\sigma$ is zero in insulators. Note that a perfect insulator will be a medium in which $\sigma = 0$ (non-conductive material).*
- *A perfect conductor corresponds to the limit $\sigma \longrightarrow \infty$. We must have $E = B = 0$ otherwise the power dissipated by the Joule effect, $\sigma E \cdot E$, would be infinite, which is absurd. Maxwell's equations remain valid in a metal.*

### 1.2.1 Transmission conditions

When crossing the interface separating two media, the electromagnetic field undergoes discontinuities. It is, however, possible to consider these discontinuities. Indeed, let $\Omega_1$ and $\Omega_2$ be two continuous media and $\Gamma$ be the interface that separates them, see figure 1.1.

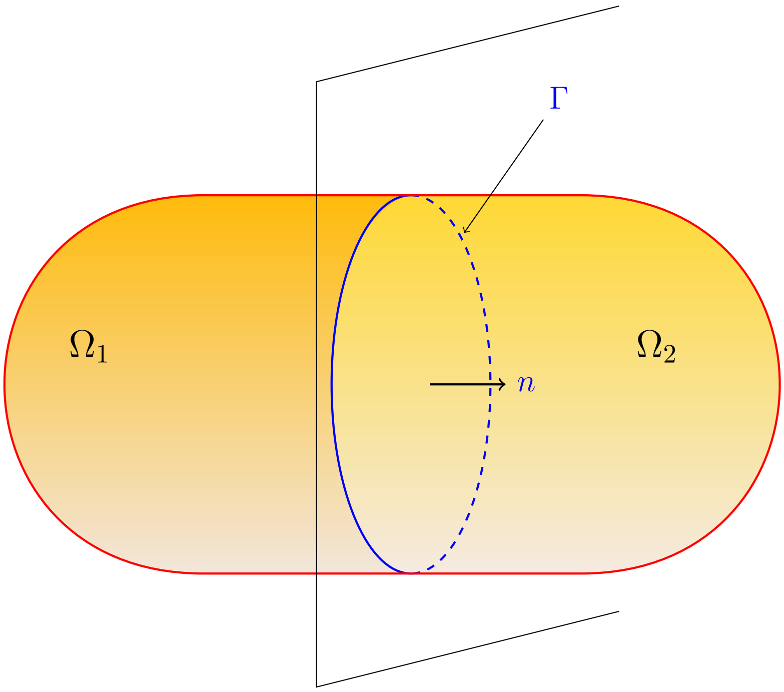


Figure 1.1: Transmission conditions.

Where $n$ is the unit vector normal to $\Gamma$ directed from $\Omega_1$ to $\Omega_2$. We note by $\varrho_\Gamma$ (respectively $\rho_{m\Gamma}$) and $J_\Gamma$ (respectively $M_\Gamma$) the density of charge and electric current (respectively magnetic) created at the interface of the two domains.

The equations (1.3), (1.4), (1.5) and (1.6) integrated on volumes including portions of the surface $\Gamma$, the application of the divergence theorem then gives the transmission conditions [13]:

$$\begin{cases} (D^2 - D^1) \cdot n = \varrho_\Gamma, \\ (H^2 - H^1) \wedge n = -J_\Gamma, \\ (B^2 - B^1) \cdot n = \rho_{m\Gamma}, \\ (E^2 - E^1) \wedge n = -M_\Gamma, \end{cases} \tag{1.11}$$

where `$\cdot$´ designating the scalar product and $\wedge$ the vector product of $\mathbb{R}^3$, and the index $i = 1, 2$ the restriction of the field or the induction to the domain $\Omega_i$.

In the classical case $M_\Gamma = 0$ and $\rho_{m\Gamma} = 0$, the well-known continuity relationships are recovered:

- continuity of the tangential component of the electric field $E$ and of the normal component of the magnetic induction $B$,

- discontinuity of the normal component of the electric induction $D$ measured by the surface charge density $\Gamma$ and of the tangential component of the magnetic field $H$ measured by the surface current density $J_\Gamma$.

Some particular cases are interesting:

1. $\Gamma$ is a perfect electrical conductor, then $M_\Gamma = 0$ and $\rho_{m\Gamma} = 0$,
2. $\Gamma$ is a perfect magnetic conductor, then $J_\Gamma = 0$ and $\varrho_\Gamma = 0$,
3. the media $\Omega_1$ and $\Omega_2$ are perfect dielectrics, then $M_\Gamma = 0$, $\rho_{m\Gamma} = 0$, $J_\Gamma = 0$ and $\varrho_\Gamma = 0$.

### 1.2.2 Boundary conditions

We position ourselves in an external domain $\Omega$ of $\mathbb{R}^3$ with the boundary $\Gamma$. Additionally, assuming that $\mathbb{R}^3 \backslash \Omega$ is a perfect conductor: $D = E = 0$ and $B = H = 0$ in $(\mathbb{R}^3 \backslash \Omega) \times \mathbb{R}_t$. By using this property, along with the transmission conditions and the condition on the interface between two domains from the paragraph 1.2.1, we deduce on $\Gamma$:

$$\begin{cases} D \cdot n = -\varrho_\Gamma, \\ H \wedge n = J_\Gamma, \\ B \cdot n = \rho_{m\Gamma}, \\ E \wedge n = -M_\Gamma. \end{cases} \tag{1.12}$$

In the case of a perfect medium occupying $\Omega$, the previous conditions (1.12) are reduced to the following (refer to [13], page 85):

$$\begin{cases} B \cdot n = 0, \\ E \wedge n = 0. \end{cases} \tag{1.13}$$

## 1.3 Propagation in an anisotropic dielectric medium

Anisotropy can be intrinsic (i.e. caused by the crystal structure of the medium) or extrinsic (i.e. caused by the imposition of an external electric or magnetic field).

In an anisotropic linear dielectric medium, the relationship between the electric induction $D$ and the electric field $E$ is as follows:

$$D = \varepsilon E,$$

where $\varepsilon$ is a second-order tensor called the dielectric tensor. The components of this tensor generally depend on frequency. It can be shown (using Maxwell's equations) that this tensor is symmetric or Hermitian when the tensor is complex (refer to [16]).

As $\varepsilon$ is symmetric, it is possible to diagonalize it through an orthogonal transformation. This means it is possible to choose three mutually perpendicular axes, referred to as principal axes, such that the dielectric tensor is diagonal along these axes. In other words, if we denote the principal axes as $Ox_1, Oy_1, Oz_1$, the dielectric tensor takes the form:

$$\varepsilon = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix},$$

where $\varepsilon_1, \varepsilon_2$ and $\varepsilon_3$ are the eigenvalues of the dielectric tensor matrix.

If we position ourselves in this frame of reference, we can define three different refractive indices, $n_1, n_2$ and $n_3$, along $x_1, y_1$ and $z_1$ respectively:

$$n_1 = \sqrt{\varepsilon_1}, n_2 = \sqrt{\varepsilon_2}, n_3 = \sqrt{\varepsilon_3}.$$

Three different characterizations of crystal systems can be distinguished based on the degree of degeneracy of the dielectric tensor:

1. In an isotropic crystal, all principal axes are arbitrary. The three eigenvalues of the dielectric tensor are equal. This is the case only for the cubic system. Even if a cubic crystal is not isotropic in space at all, its dielectric properties are entirely isotropic, as if the medium were a liquid or glass.

2. In a uniaxial crystal, two of the eigenvalues of the dielectric tensor are equal. There exists an axis of symmetry about which a rotation of the axes does not change the dielectric tensor. Crystal systems such as trigonal, tetragonal, and hexagonal fall into this category.

3. In a biaxial crystal, all three eigenvalues are distinct, and the dielectric tensor does not possess any axis of symmetry. This is observed in crystal systems such as triclinic, monoclinic, and orthorhombic.

Now, let $\mathbf{P}$ be the matrix that transforms the coordinate system $Oxyz$ into the system of principal axes $Ox_1y_1z_1$ :

$$Ox_1 = \mathbf{P}Ox \; ; Oy_1 = \mathbf{P}Oy \; ; \; Oz_1 = \mathbf{P}Oz.$$

If $E = (E_x, E_y, E_z)$ represents the electric field given in the coordinate system $Oxyz$, we can express the new coordinates $E_1 = (E_{x_1}, E_{y_1}, E_{z_1})$ as $E_1 = \mathbf{P}E$.

We can also rewrite the constitutive condition $D = \varepsilon E$ in the new coordinates:

$$D_1 = \mathbf{P}\varepsilon\mathbf{P}^{-1}E_1 = \varepsilon_1 E_1,$$

where $\varepsilon_1 = \mathbf{P}\varepsilon\mathbf{P}^{-1}$ is a diagonal matrix.

## 1.4 Harmonic Maxwell equations

We now consider the harmonic case, where each physical quantity varies periodically in time under the frequency $\omega$, with $\varepsilon$ and $\mu$ being medium-dependent tensors:

$$F(x,t) = \mathrm{Re}\left(F(x)\, e^{i\omega t}\right),$$

where $F$ is one of the physical quantities $B, H, E, D, J, \varrho$.

In the case of perfect mediums, the system is reduced within $\Omega$ to the following equations:

$$\operatorname{curl} H - i\varepsilon\omega E = J, \tag{1.14a}$$

$$\operatorname{div}(\varepsilon E) = \varrho, \tag{1.14b}$$

$$\operatorname{curl} E + i\omega\mu H = 0, \tag{1.14c}$$

$$\operatorname{div}(\mu H) = 0, \tag{1.14d}$$

defining the Maxwell problem in the harmonic regime, with the conditions on $\Gamma$ as those defined in the paragraph 1.2.2.

It can be noted that

***i.*** The equation (1.14d) is redundant. Indeed, by applying the divergence operator to the equation (1.14c), we retrieve the conservation of flux condition (1.14d).

***ii.*** The condition on $\Gamma$ for the electric field combined with equation (1.14a) yields:

$$\operatorname{curl} H \wedge n = J \wedge n \qquad \text{on } \Gamma.$$

Moreover, by applying the curl operator to equation (1.14a) and then combining it with (1.14c), we obtain the following second-order equation:

$$\operatorname{curl}\left(\varepsilon^{-1} \operatorname{curl} H\right) - \omega^2 \mu H = \operatorname{curl}\left(\varepsilon^{-1} J\right), \tag{1.15}$$

which allows us to decouple the problem. We can then focus on the following second-order problem:

$$\begin{cases} \operatorname{curl}(\varepsilon^{-1}\operatorname{curl} H) - \omega^2 \mu H = \operatorname{curl}(\varepsilon^{-1} J), & \text{in } \Omega, \\ \operatorname{curl} H \wedge n = J \wedge n, & \text{on } \Gamma. \end{cases} \tag{1.16}$$

The electric field $E$ is directly derived from $H$ using the equation (1.14a):

$$E = (i\omega\mu)^{-1} (\operatorname{curl} H - J) \qquad \text{in } \Omega. \tag{1.17}$$

Symmetrically, the problem can be reduced to the single unknown $E$:

$$\begin{cases} \operatorname{curl}(\mu^{-1}\operatorname{curl} E) - \omega^2 \varepsilon E = -i\omega J, & \text{in } \Omega, \\ E \wedge n = 0 & \text{on } \Gamma. \end{cases} \tag{1.18}$$

**Remark 1.2** – *In the case where $\varepsilon$ is a constant scalar, using the formula*

$$\operatorname{curl}\operatorname{curl} H = -\Delta H + \operatorname{grad}\operatorname{div} H$$

*and taking into account (1.14d), we see that $H$ satisfies the vector Helmholtz equation:*

$$\Delta H + \omega^2 \varepsilon\mu H = -\operatorname{curl} J \qquad \text{in } \Omega.$$

When the surface is invariant under translation along an axis, it's possible to reduce the 3-dimensional vector system to a 2-dimensional scalar problem. That is the aim of the following paragraph.

### 1.4.1 The case of two-dimensional Maxwell equations

We are interested in a propagation problem of a "cylindrical" nature, that is:

- the propagation domain is of the form:

$$\Omega \times \mathbb{R}, \quad \Omega \subset \mathbb{R}^2,$$

- the coefficients are independent of the coordinate $x_3$:

$$\varepsilon = \varepsilon(x_1, x_2) \quad \mu = \mu(x_1, x_2) \quad \sigma = \sigma(x_1, x_2),$$

- the sources are independent of $x_3$:

$$E_0 = E_0(x_1, x_2) \quad H_0 = H_0(x_1, x_2) \quad J_0 = J_0(x_1, x_2).$$

**The concept of polarization**

The wave vector is defined as $\mathbf{k} = k\nu$, derived from the unit vector $\nu$ indicating the direction of propagation. In the two-dimensional approximation, the axis of translational invariance is $Oz$, and the propagation plane containing the vector $\mathbf{k}$ is defined by $(Ox_1, Ox_2)$. The polarization is said transverse electric $(TE)$ if the electric field $E$ lies within the propagation plane, thus making $H$ collinear with the axis of invariance. When the electric field $E$ is perpendicular to this plane, the polarization is transverse magnetic $(TM)$.

The direction $x_3$ plays a particularly privileged role. It can be shown (refer to [11]) that the solutions $(E, H)$ of (1.14) are, in this context, independent of $x_3$ and that the system (1.14) is decoupled into two subsystems of equations posed within $\Omega$. Namely, $(E_x, E_y, E_z)$ are the coordinates of $E$ in the $(O, x, y, z)$ frame, $(H_x, H_y, H_z)$ are those of $H$, and $(J_x, J_y, J_z)$ are those of $J$:

- **A system in** $(E_x, E_y, H_z)$ **:** $TE$ **Polarization**

$$\begin{cases} i\varepsilon\omega E_x - \partial_y H_z + J_x = 0, \\ i\varepsilon\omega E_y + \partial_x H_z + J_y = 0, \\ i\mu\omega H_z + \partial_x E_y - \partial_y E_x = 0. \end{cases}$$

This is the system of transverse electromagnetic waves. Indeed, the electric field $E$ remains in the "transverse" plane $Oxy$, orthogonal to the invariant direction $Ox_3$.

– **A system in** $(H_x, H_y, E_z)$ **:** $TM$ **Polarization**

$$\begin{cases} i\mu\omega H_x + \partial_y E_z = 0, \\ i\mu\omega H_y - \partial_x E_z = 0, \\ i\varepsilon\omega E_z - (\partial_x H_y - \partial_y H_x) = 0. \end{cases}$$

This is the system of transverse magnetic waves.

All other polarization cases are linear combinations of these two states. There is no depolarization of the incident wave during the diffraction phenomenon (see reference [27]. Therefore, only these two cases should be studied.

**Remark 1.3** – *In $TE$ case, only the components $H_z, E_x$ and $E_y$ are non-zero. Knowing $H_z$ alone is sufficient to determine the components $E_x$ and $E_y$.*

*– In $TM$ case, only the components $H_x, H_y$ and $E_z$ are non-zero. Knowledge of $E_z$ is sufficient to determine the components $H_x$ and $H_y$.*

**Radiation of electromagnetic waves**

Two types of sources can be used to characterize the radiation of objects in the external medium: a plane wave or a dipole.

**Plane waves:** By using the classical method of variable separation or equivalently, using a Fourier transformation, a generic solution to the wave equation is a plane wave in the form:

$$u\,(x, y) = e^{-ik\nu\cdot\mathbf{x}} \tag{1.19}$$

where $\nu$ is the unit vector $\nu = [\cos\theta, \sin\theta]^T$, $\mathbf{x}$ is the position vector $\mathbf{x} = [x, y]^T$, and $\theta$ is the angle of incidence.

**Remark 1.4** *The plane wave defined by (1.19) is a wave with an amplitude of 1. It is obvious that any multiple of this solution also defines a plane wave.*

**Dipole Radiation:** A dipole is modeled by a line of magnetic current in $TE$ polarization or a line of electric current in $TM$ polarization. In both configurations, when this current line is applied at a point $\mathbf{x}_0$, the field created at a point $\mathbf{x}$ is given by

$$u(\mathbf{x}) = G(\mathbf{k}|\mathbf{x}-\mathbf{x}_0|) = \frac{i}{4}H_0^{(1)}(\mathbf{k}|\mathbf{x}-\mathbf{x}_0|).$$

where, $G$ is the Green's function and $H_0^{(1)}$ is the Hankel function of the first kind and order 0.

## 1.5 Theory of Semigroups

### 1.5.1 Semigroups of Linear Operators

Consider $\mathcal{X}$ be a Banach space and let $\mathcal{L}(\mathcal{X})$ be the set of all linear bounded operators from $\mathcal{X}$ into $\mathcal{X}$, norm on $\mathcal{L}(\mathcal{X})$ defined by

$$\|S\|_{\mathcal{L}(\mathcal{X})} = \sup_{\substack{x\in\mathcal{X}\\ x\neq 0}} \frac{\|Sx\|_{\mathcal{X}}}{\|x\|_{\mathcal{X}}}, \tag{1.20}$$

which makes $\mathcal{L}(\mathcal{X})$ a Banach space.

**Definition 1.5** *A family $\{T(t); t \geq 0\}$ in $\mathcal{L}(\mathcal{X})$ is a semigroup of bounded linear operators on $\mathcal{X}$ if*

*(i) $T(0) = I$, $I$ is the identity operator on $\mathcal{X}$.*

*(ii) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$.*

*The linear operator $\mathcal{A}$ defined by*

$$\mathcal{D}(\mathcal{A}) = \{x \in \mathcal{X} : \lim_{t\to+\infty} \frac{T(t)x - x}{t} \text{ exists in } \mathcal{X}\} \tag{1.21}$$

*and*

$$\mathcal{A}x = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \quad \text{for } x \in \mathcal{D}(\mathcal{A}) \tag{1.22}$$

*is the infinitesimal generator of the semigroup $T(t)$, $\mathcal{D}(\mathcal{A})$ is the domain of $\mathcal{A}$.*

- *A semigroup of bounded linear operators $T(t)$, is uniformly continuous if*

$$\lim_{t\to 0} \|T(t) - I\| = 0. \tag{1.23}$$

- *A semigroup of linear operators $T(t)$ is called $C_0$ semigroup if*

$$\lim_{t\to 0} T(t)\, x = x, \text{ for each } x \in \mathcal{X}. \tag{1.24}$$

**Theorem 1.6** *A linear operator $\mathcal{A}$ is the infinitesimal generator of a uniformly continuous semigroup if and only if $\mathcal{A}$ in $\mathcal{L}(\mathcal{X})$.*

**Theorem 1.7** *Let $T(t)$ and $S(t)$ be uniformly continuous semigroup of bounded linear operators. If*

$$\lim_{t\to 0} \frac{T(t) - I}{t} = \mathcal{A} = \lim_{t\to 0} \frac{S(t) - I}{t},$$

*then $T(t) = S(t)$ for $t \geq 0$.*

### 1.5.2 Some theorems of $C_0$ semigroups

**The Hille-Yosida Theorem**

**Theorem 1.8** *(Hille-Yosida)*

*A linear (unbounded) operator $A$ is the infinitesimal generator of a $C_0$ semigroup of contractions $T(t), t \geq 0$ if and only if*

*(i) $A$ is closed and $\overline{D(A)} = \mathcal{X}$.*

*(ii) The resolvent set $\rho(A)$ of $A$ contains $\mathbb{R}^+$ and for every $\lambda > 0$*

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}.$$

### The Lumer-Phillips Theorem

**Theorem 1.9** *(Lumer-Phillips) Let* $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ *be a densely defined operator. Then* $A$ *generates a* $C_0$*-semigroup of contractions on* $\mathcal{X}$ *if and only if*

1. $A$ *is dissipative.*

2. *There exists* $\omega > 0$ *such that* $\omega I - A$ *is surjective.*

### The Stone Theorem

The following theorem, credited to Stone , pertains to the scenario of a $C_0$ semigroup of unitary operators on Hilbert spaces. It's worth revisiting that an operator $U \in L(H)$ is termed unitary if $UU^* - U^*U = I$.

**Theorem 1.10** *(Stone Theorem) The necessary and sufficient condition for* $A : D(A) \subseteq H \to H$ *to be the infinitesimal generator of a* $C_0$*-group of unitary operators on* $H$ *is that* $iA$ *be self-adjoint.*

## 1.5.3 The Abstract Cauchy Problem

### Definition

Let $\mathcal{X}$ be a Banach space and let $\mathcal{A}$ be a linear operator from $\mathcal{D}(\mathcal{A}) \subset \mathcal{X}$ into $\mathcal{X}$. The abstract Cauchy problem for $\mathcal{A}$ with initial data $x \in \mathcal{X}$ consists of finding a solution $u(t)$ to the initial value problem

$$\begin{cases} \frac{d}{dt}u\left(t\right) = \mathcal{A}u\left(t\right) \quad ,t>0 \\ u\left(0\right) = x \end{cases} \tag{1.25}$$

**Theorem 1.11** *Let* $\mathcal{A}$ *be a densely defined linear operator with a nonempty resolvent set* $\mathcal{P}(\mathcal{A})$*. The initial value problem (1.25) has a unique solution* $u(t)$*, which is continuously* *differentiable on* $[0, \infty)$*, for every initial value* $x \in \mathcal{D}(\mathcal{A})$ *if and only if* $\mathcal{A}$ *is the infinitesimal generator of a* $C_0$ *semigroup* $T(t)$*.*

### 1.5.4 The Maxwell Operator

The objective of this section is to introduce a partial differential operator that creates a $C_0$ semigroup of unitary operators. This operator undoubtedly ranks among the most pivotal operators in the field of Field Theory.

The development of the intensity of both the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ in the three-dimensional space $\mathbb{R}^3$ devoid of any material presence is expounded by the Maxwell system.

$$\begin{cases} \mathbf{E}_t = -c\nabla \times \mathbf{H} & (t,x) \in \mathbb{R} \times \mathbb{R}^3 \\ \mathbf{H}_t = c\nabla \times \mathbf{E} & (t,x) \in \mathbb{R} \times \mathbb{R}^3 \\ \nabla \cdot \mathbf{E} = 0 \text{ and } \nabla \cdot \mathbf{H} = 0 & (t,x) \in \mathbb{R} \times \mathbb{R}^3 \\ \mathbf{E}(0,x) = \mathbf{E}_0(x) \text{ and } \mathbf{H}(0,x) = \mathbf{H}_0(x) & x \in \mathbb{R}^3, \end{cases}$$

In this context, where $c > 0$ is a positive constant, the system can be reformulated within a carefully selected Hilbert space as follows:

$$\begin{cases} u' = Au \\ u(0) = u_0, \end{cases}$$

where $A$ is the generator of a $C_0$-semigroup of contractions.

**Example 1.1** *(The Maxwell Operator)*

*Let us consider the Hilbert space* $H = \left(L^2\left(\mathbb{R}^3\right)\right)^3 \times \left(L^2\left(\mathbb{R}^3\right)\right)^3$*, and we shall symbolize this by*

$$u = (\mathbf{E}, \mathbf{H}) = (\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}, \mathbf{H_1}, \mathbf{H_2}, \mathbf{H_3})$$

*an arbitrary element in* $H$*. Let* $H_0 = \{u \in H; \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{H} = \mathbf{0}\}$*, where the differential operator* $\nabla$ *is interpreted in the distributional sense, which means that* $\nabla \cdot \mathbf{F} = \mathbf{0}$ *if and only* *if*

$$\int_{\mathbb{R}^3} \nabla g \cdot \mathbf{F d}\omega = \mathbf{0}$$

*for each* $g \in C_0^\infty(\mathbb{R})$. *This means that* $u$ *is in* $H_0$ *if and only if it is orthogonal on each element* $v$ *in* $H$ *of the form* $v = (\nabla\varphi, \nabla\psi)$ *with* $\varphi, \psi \in H^1(\mathbb{R}^3)$. *We shall define the Maxwell operator,* $A : D(A) \subseteq H \to H$, *by*

$$\begin{cases} D(A) = \{(\mathbf{E}, \mathbf{H}) \in H; (-c\nabla \times \mathbf{H}, \mathbf{c}\nabla \times \mathbf{E}) \in H\} \\ A(\mathbf{E}, \mathbf{H}) = (-\mathbf{c}\nabla \times \mathbf{H}, \mathbf{c}\nabla \times \mathbf{E}), \end{cases}$$

*for* $(\mathbf{E}, \mathbf{H}) \in D(A)$. *Let us observe that* $A$ *maps* $D(A)$ *in* $H_0$, *and therefore* $H_0$ *is invariant under* $A$, *because the divergence of a curl is always 0 .This clarifies the reason why, in all subsequent discussions, we will focus on examining the constrained behavior of A limited to* $H_0$, *restriction which, for simplicity's sake, we denote it again as A. We want to stress that the operator* $A$ *is not densely defined in* $H$, *but its restriction to* $H_0$ *does so.*

***Theorem 1.12*** *The operator A, defined as mentioned above, acts as the generator of a* $C_0$*-group of unitary operators.*

***Proof.*** *We show that A satisfies the hypotheses of Stone Theorem 3.9.1. To this aim, let* $C_\sigma^\infty(\mathbb{R}^n) = \{\mathbf{F} \in C_0^\infty(\mathbb{R}^n); \nabla \cdot \mathbf{F} = 0\}$. *Inasmuch as* $C_\sigma^\infty(\mathbb{R}^n) \times C_\sigma^\infty(\mathbb{R}^n)$ *is included in* $D(A)$, *and dense in* $H_0$, *Consequently, it can be deduced that A is densely defined. Subsequently, we proceed to establish that A is skew-adjoint. Initially, let us note that, for every* $\mathbf{E}, \mathbf{H} \in C_\sigma^\infty(\mathbb{R}^n)$, *we have*

$$\langle A(\mathbf{E}, \mathbf{H}), (\mathbf{E}, \mathbf{H})\rangle = 0.$$

*Indeed, considering the integral over* $\mathbb{R}^3$ *of the divergence of a* $C^1$ *function with compact support is 0 , we have*

$$\langle A(\mathbf{E}, \mathbf{H}), (\mathbf{E}, \mathbf{H})\rangle = \int_{\mathbb{R}^3} \operatorname{div}(\mathbf{H} \times \mathbf{E}) d\omega = 0$$

*for each* $(\mathbf{E},\mathbf{H}) \in C_\sigma^\infty(\mathbb{R}^n)$*. Inasmuch as* $C_\sigma^\infty(\mathbb{R}^n) \times C_\sigma^\infty(\mathbb{R}^n)$ *is dense in* $H_0$ *, consequently, it can be deduced that equation (4.3.1) is valid for each* $(\mathbf{E},\mathbf{H}) \in D(A)$*,* $A$ *is skew-symmetric, or equivalently,* $iA$ *is symmetric. To check that* $A$ *is skew-adjoint we prove that* $iA$ *is self-adjoint and, to this aim, we shall prove that* $1 \in \rho(iA)$*. Let us denote by* $\hat{\mathbf{H}}$ *and* $\hat{\mathbf{E}}$ *the Fourier transform of* $\mathbf{H}$ *and respectively of* $\mathbf{E}$*, i.e.*

$$\begin{cases} \hat{E}(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\langle \xi, x\rangle}\mathbf{E}(x)dx, \\ \hat{H}(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i\langle \xi, x\rangle}\mathbf{H}(x)dx. \end{cases}$$

*Then the mapping* $(\mathbf{E},\mathbf{H}) \mapsto (\hat{\mathbf{E}},\hat{\mathbf{H}})$ *is an isomorphism from* $H$ *to a Hilbert space* $\hat{H}$ *analogously defined. More that this, this isomorphism maps* $H_0$ *into a subspace* $\hat{H}_0$ *in* $\hat{H}$*, subspace defined by*

$$\hat{H}_0 = \{(\hat{\mathbf{E}},\hat{\mathbf{H}}) \in \hat{H}; \xi \cdot \hat{\mathbf{E}} = \xi \cdot \hat{\mathbf{H}} = 0\}$$

*and it maps the operator* $A$ *to the operator* $\hat{A} : D(\hat{A}) \subseteq \hat{H}_0 \to \hat{H}_0$*, defined by*

$$\begin{cases} D(\hat{A}) = \left\{(\hat{\mathbf{E}},\hat{\mathbf{H}}) \in \hat{H}_0; (-c\xi \times \hat{\mathbf{H}}, c\xi \times \hat{\mathbf{E}}) \in \hat{H}_0\right\} \\ \hat{A}(\hat{E},\hat{H}) = (-c\xi \times \hat{\mathbf{H}}, c\xi \times \hat{\mathbf{E}}). \end{cases}$$

*Let* $\hat{\mathbf{v}} \in \hat{H}_0, \hat{\mathbf{v}} = (\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2)$*, and now, let's consider the equation*

$$(i\hat{I} - \hat{A})\hat{\mathbf{u}} = \hat{\mathbf{v}},$$

*where* $\hat{I}$ *denotes the identity operator on* $\hat{H}$*. Obviously,* $1 \in \rho(iA)$ *if and only if* $i \in \rho(\hat{A})$*. But this last condition holds if and only if, for each* $\hat{\mathbf{v}} \in \hat{H}_0$*, the equation (4.3.2) has a unique solution* $R_1\hat{\mathbf{v}}$ *and there exists* $K > 0$ *such that*

$$\|R_1\hat{\mathbf{v}}\| \leq K\|\hat{\mathbf{v}}\|$$

*for each* $\hat{\mathbf{v}} \in \hat{H}_0$*.Evidently, within this context, the solution operator* $R_1$ *coincides with* $R(1;\hat{A})$*.Let's note that, when expressed in components, (4.3.2) has the form*

$$\begin{cases} c\xi \times \hat{\mathbf{u}}_2 + i\hat{\mathbf{u}}_1 = \hat{\mathbf{v}}_1 \\ -c\xi \times \hat{\mathbf{u}}_1 + i\hat{\mathbf{u}}_2 = \hat{\mathbf{v}}_2, \end{cases}$$

*system for which the only solution is provided by*

$$\begin{cases} \hat{\mathbf{u}}_1 = \frac{-i\hat{\mathbf{v}}_1 + c\xi \times \hat{\mathbf{v}}_2}{c^2\|\xi\|^2+1} \\ \hat{\mathbf{u}}_2 = \frac{i\hat{\mathbf{v}}_2 - c\xi \times \hat{\mathbf{v}}_1}{c^2\|\xi\|^2+1} \end{cases}$$

*From the equalities above one may observe that* $\hat{\mathbf{u}}$ *is a linear continuous function of* $\hat{\mathbf{v}}$. *Therefore* $i \in \rho(\hat{A})$, *or equivalently* $1 \in \rho(iA)$. *Analogously we deduce that* $-1 \in \rho(iA)$, *shows that* $A$ *is skew-adjoint. The conclusion of this theorem follows from Stone Theorem. The proof is complete.* ■

# Chapter 2

# Study of a model problem

We are particularly interested in a problem of diffraction of a harmonic wave, by a two-dimensional obstacle coated with thin multilayers of homogeneous dielectric materials. The contents of this chapter is inspired from [7] and [21].

## 2.1 Problem statement

We consider the case of a perfectly conducting obstacle made of metal, covered by $p$ thin layers, $p$ being an integer $\geq 1$, of anisotropic dielectric with thickness $h_j, 1 \leq j \leq p$. Inside the obstacle, the fields are considered null. The dielectric with thickness $h_j$ is characterized by a relative permittivity $\varepsilon_j$ and a relative permeability $\mu_j$, $1 \leq j \leq p$. The $\varepsilon_j$ and $\mu_j$ are $3 \times 3$ matrices.

We are interested in the case where $\varepsilon_j$ and $\mu_j$ are diagonal matrices.

$$\varepsilon_j = \begin{bmatrix} \varepsilon_{j,1} & 0 & 0 \\ 0 & \varepsilon_{j,2} & 0 \\ 0 & 0 & \varepsilon_{j,3} \end{bmatrix} \; ; \; \mu_j = \begin{bmatrix} \mu_{j,1} & 0 & 0 \\ 0 & \mu_{j,2} & 0 \\ 0 & 0 & \mu_{j,3} \end{bmatrix}.$$

The metallic obstacle covered with thin layers of dielectric is placed in a propagation medium assumed to be perfect dielectric (i.e., with conductivity $\sigma = 0$, homogeneous, isotropic).

This medium can potentially be a vacuum and is characterized by a permittivity $\varepsilon_0$ and a permeability $\mu_0$.

We illuminate this system with a harmonic incident wave characterized by its frequency $\omega > 0$. This information reflects the sinusoidal dependence of the electromagnetic field through the multiplicative factor $e^{i\omega t}$. When this wave encounters the obstacle, it generates a diffracted wave by the obstacle. In the case of radar stealth problems, the objective is to understand the nature of the wave diffracted by this type of structure.

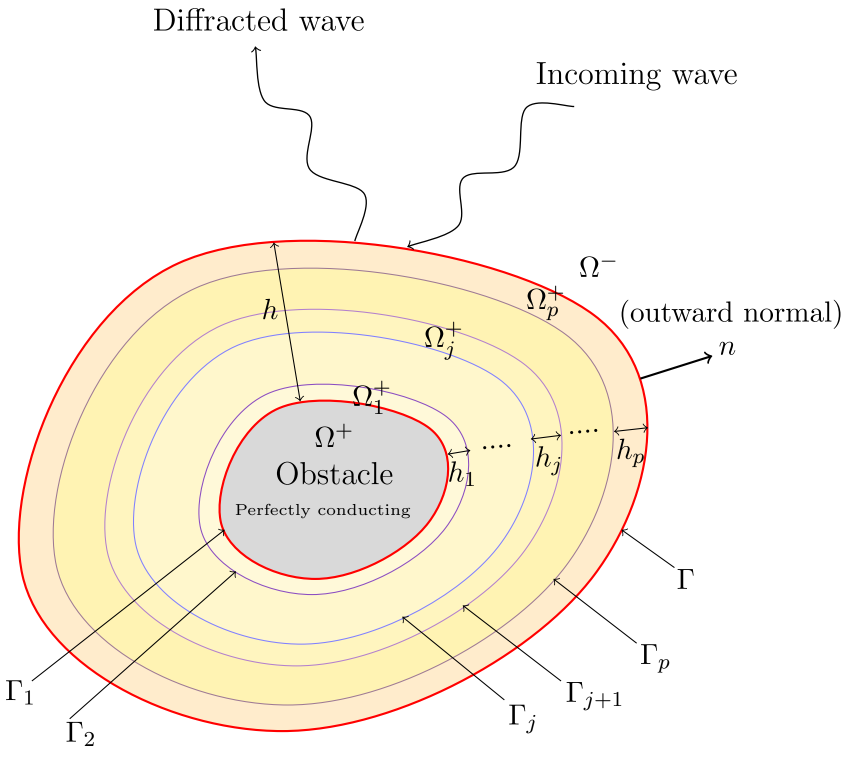


Figure 2.1: Diffraction by a metallic obstacle covered by thin layers of dielectrics.

The metallic obstacle occupies the domain $\Omega^+$ in $\mathbb{R}^d$ in dimension $d = 2$ or 3; the interior of the outer domain $\mathbb{R}^d \backslash \Omega^+$ is denoted as $\Omega_\infty$, the domains

$$\Omega_j^+ = \left\{ x \in \Omega_\infty \; ; \quad \sum_{0 \le l \le j-1} h_l < d\left(x, \partial\Omega^+\right) < \sum_{0 \le l \le j} h_l \right\}, \; 1 \le j \le p, \text{ with } h_0 = 0$$

characterize the thin layers, finally, the external propagation medium is represented by $\Omega^-$. The interface between $\Omega^-$ and $\Omega_p^+$ is called $\Gamma$, the metallic boundary is noted as $\Gamma_1$, the interfaces between the thin layers are denoted $\Gamma_j$, $2 \leq j \leq p$, where

$$\Gamma_j = \left\{ y \in \Omega_\infty \; ; \quad y = x - \sum_{j \leq l \leq p} h_l n\left(x\right), \; x \in \Gamma \right\}. \tag{2.1}$$

$d$ is the distance function from a point $x$ to the boundary $\partial\Omega^+$.

The unit normal to $\Gamma$, oriented outward from $\Omega_p^+$, is designated by $n$.

It is recalled that the speed of electromagnetic waves in a vacuum is

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}.$$

The square root of the ratio of these two constants is the impedance of vacuum

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}.$$

The wavelength is associated with the frequency and speed by

$$\lambda = \frac{2\pi c}{\omega}.$$

The wave number $k$ will be an essential parameter in the problem to characterize the frequency.

$$k = \frac{2\pi}{\lambda}.$$

The previous parameters are related to the propagation domain.

The total wave described by the electromagnetic pair $(E, H)$ satisfies the harmonic Maxwell's equations. In the dielectric medium, these equations are given by

$$\begin{cases} \operatorname{curl} E + ikZ_0\mu_j H = 0, \\ \operatorname{curl} H - ikZ_0^{-1}\varepsilon_j E = 0. \end{cases} \tag{2.2}$$

In $TE$ polarization, simplifications occur in the field components.

$$E = \begin{bmatrix} E_x\left(x, y\right) \\ E_y\left(x, y\right) \\ 0 \end{bmatrix} \qquad H = \begin{bmatrix} 0 \\ 0 \\ u\left(x, y\right) \end{bmatrix}. \tag{2.3}$$

In this $TE$ polarization, the Maxwell-Faraday equation is scalar and is given in each of the $p+1$ mediums by

$$\begin{cases} \operatorname{curl} E + ikZ_0 u = 0 & \text{in } \Omega^-, \\ \operatorname{curl} E + ikZ_0 \mu_j u = 0 & \text{in } \Omega_j^+. \end{cases} \tag{2.4}$$

However, the Maxwell-AmpÃ¨re equation remains vectorial

$$\begin{cases} \operatorname{curl} u - ikZ_0^{-1} E = 0 & \text{in } \Omega^-, \\ \operatorname{curl} u - ikZ_0^{-1} \varepsilon_j E = 0 & \text{in } \Omega_j^+ \end{cases} \tag{2.5}$$

where the expression of the vector curl applied to a function $\varphi(x, y)$ is given by

$$\operatorname{curl} \varphi = \begin{bmatrix} \partial_y \varphi \\ -\partial_x \varphi \end{bmatrix}.$$

To solve the system of equations (2.4)-(2.5), It is necessary to add conditions on the interfaces $\Gamma$ and $\Gamma_j, 1 \leq j \leq p$ which can be classified into three types.

- The transmission conditions when crossing the boundaries $\Gamma$ and $\Gamma_j, 2 \leq j \leq p$. They impose the connection of the electromagnetic field components

$$\begin{aligned} &a)\ \ [E \wedge n]_\Gamma = 0,\ \ [E \wedge n]_{\Gamma_j} = 0, \\ &b)\ \ [\varepsilon E \cdot n]_\Gamma = 0,\ \ [\varepsilon E \cdot n]_{\Gamma_j} = 0, \\ &c)\ \ [H \wedge n]_\Gamma = 0,\ \ [H \wedge n]_{\Gamma j} = 0, \\ &d)\ \ [\mu H \cdot n]_\Gamma = 0,\ \ [\mu H \cdot n]_{\Gamma_j} = 0. \end{aligned} \tag{2.6}$$

  The bracket $[\psi]_\Gamma$ denotes the jump of the trace of the function $\psi$ across the boundary $\Gamma$. This quantity is given by

$$[\psi] = \psi^+_{|\Gamma} - \psi^-_{|\Gamma},$$

  where the function $\psi^-$ (respectively, $\psi^+$) denotes the restriction of the function $\psi$ to the domain $\Omega^-$ (respectively, $\Omega_p^+$). Similarly to $[\psi]_{\Gamma_j}$.

- A perfect conductor condition on the boundary $\Gamma_1$

$$E \wedge n = 0.$$

- A radiation condition at infinity satisfied by each element of the pair

$$\{E - E_{inc}, H - H_{inc}\}$$

which is of the form

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\partial_r \left(u^- - u_{inc}\right) - ik\left(u^- - u_{inc}\right)\right) = 0, \tag{2.7}$$

representing the propagation of energy carried by the diffracted wave from the obstacle to infinity.

It is possible to reduce the vectorial system (2.5) to a scalar problem in two dimensions. Let's recall the two calculation steps. For instance, in the domain $\Omega^-$, it involves multiplying the Maxwell-Faraday equation by $ikZ_0^{-1}$ and applying the vector curl to the Maxwell-AmpÃ¨re relation; this results in the Helmholtz equation for the magnetic field.

$$\Delta u + k^2 u = \mathbf{0} \quad \text{in } \Omega^-. \tag{2.8}$$

In the same way, we have

$$\operatorname{div}\left(\mathcal{A}_j \nabla u\right) + k^2 \mu_{j,3} u = 0 \quad \text{in } \Omega_j^+,$$

where $\mathcal{A}_j$ is a $2 \times 2$ matrix

$$\mathcal{A}_j = \begin{bmatrix} \dfrac{1}{\varepsilon_{j,1}} & 0 \\ 0 & \dfrac{1}{\varepsilon_{j,2}} \end{bmatrix}. \tag{2.9}$$

Noting the magnetic field $u^-$ in $\Omega^-$ and $u_j^+$ in $\Omega_j^+$, the boundary conditions (2.6) in $TE$ mode result in the following boundary conditions:

$$\begin{aligned} &\partial_n u^- = \left(\mathcal{A}_p \nabla u_p^+\right) \cdot n \; ; \; u^- = u_p^+ && \text{on } \Gamma, \\ &\left(\mathcal{A}_j \nabla u_j^+\right) \cdot n = \left(\mathcal{A}_{j-1} \nabla u_{j-1}^+\right) .n \; ; \; u_j^+ = u_{j-1}^+ && \text{on } \Gamma_j, 2 \leq j \leq p, \\ &\left(\mathcal{A}_1 \nabla u_1^+\right) \cdot n = 0 && \text{on } \Gamma_1, \end{aligned} \tag{2.10}$$

indeed, for example, on $\Gamma_p$, the equations (2.6 $a$ and $c$) are written in the form

$$E_p^+ \cdot \tau = E_{p-1}^+ \cdot \tau \ ; \ u_p^+ = u_{p-1}^+,$$

where

$$\tau = \begin{bmatrix} n_y \\ -n_x \end{bmatrix}, E^{p-1} = \begin{bmatrix} \left(E_x^+\right)_{p-1} \\ \left(E_y^+\right)_{p-1} \end{bmatrix} \text{ and } E_p^+ = \begin{bmatrix} \left(E_x^+\right)_p \\ \left(E_y^+\right)_p \end{bmatrix}.$$

Then, using the Maxwell-AmpÃ¨re equation (2.5), we obtain

$$\left(\varepsilon_p^{-1} \operatorname{curl} u_p^+\right) \cdot \tau = \left(\varepsilon_{p-1}^{-1} \operatorname{curl} u_{p-1}^+\right) \cdot \tau \ ; \ u_p^+ = u_{p-1}^+.$$

Since

$$\operatorname{curl} u_{p-1}^+ = \begin{bmatrix} \partial_y u_{p-1}^+ \\ -\partial_x u_{p-1}^+ \end{bmatrix} \ ; \ \operatorname{curl} u_p^+ = \begin{bmatrix} \partial_y u_p^+ \\ -\partial_x u_p^+ \end{bmatrix}$$

we obtain the following conditions on $\Gamma_p$

$$\left(\mathcal{A}_p \nabla u_p^+\right) \cdot n = \left(\mathcal{A}_{p-1} \nabla u_{p-1}^+\right) \cdot n \ ; \ u_p^+ = u_{p-1}^+.$$

**Remark 2.1** *If the unknown $u$ is sought in the Frechet space $H^1_{loc}\left(\overline{\Omega^-}\right)$, its trace on $\Gamma$ is in the space $H^{\frac{1}{2}}(\Gamma)$. Then, the trace of $(\mathcal{A}\nabla u) \cdot n$ on $\Gamma$ is in the space $H^{-\frac{1}{2}}(\Gamma)$, to make sense of the boundary condition.*

Ultimately, it comes to

find the total field $u = \left(u^-, \left(u_j^+\right)_{1\leq j\leq p}\right)$ in $H^1_{loc}\left(\overline{\Omega^-}\right) \underset{1\leq j\leq p}{\cap} H^1\left(\Omega_j^+\right)$ such that

$$\triangle u^- + k^2 u^- = 0 \qquad \text{in } \mathcal{D}'\left(\Omega^-\right), \tag{2.11a}$$

$$\text{div}\left(\mathcal{A}_j \nabla u_j^+\right) + k^2 \mu_{j,3} u_j^+ = 0 \qquad \text{in } \mathcal{D}'\left(\Omega_j^+\right), \tag{2.11b}$$

- with the transmission conditions on $\Gamma$ and $\Gamma_j$ $(2 \leq j \leq p)$

$$\partial_n u^- = \left(\mathcal{A}_p \nabla u_p^+\right)\cdot n \qquad \text{in } H^{-\frac{1}{2}}\left(\Gamma\right), \tag{2.11c}$$

$$u^- = u_p^+ \qquad \text{in } H^{\frac{1}{2}}\left(\Gamma\right), \tag{2.11d}$$

$$\left(\mathcal{A}_j \nabla u_j^+\right)\cdot n = \left(\mathcal{A}_{j-1} \nabla u_{j-1}^+\right)\cdot n \qquad \text{in } H^{-\frac{1}{2}}\left(\Gamma_j\right) \quad (2 \leq j \leq p), \tag{2.11e}$$

$$u_j^+ = u_{j-1}^+ \qquad \text{in } H^{\frac{1}{2}}\left(\Gamma_j\right) \quad (2 \leq j \leq p), \tag{2.11f}$$

- with the perfect conductor condition on $\Gamma_1$

$$\left(\mathcal{A}_1 \nabla u_1^+\right)\cdot n = 0 \qquad \text{in } H^{-\frac{1}{2}}\left(\Gamma_1\right), \tag{2.11g}$$

- with the radiation condition at infinity

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\partial_r \left(u^- - u_{inc}\right) - ik\left(u^- - u_{inc}\right)\right) = 0. \tag{2.11h}$$

If the polarization is chosen to be $TM$, we obtain the same problem with the condition $u_1^+ = 0$ on $\Gamma_1$ instead of $\left(\mathcal{A}_1 \nabla u_1^+\right)\cdot n = 0$ and by substituting $\mu$ for $\varepsilon$.

The data $\varepsilon_j$ and $\mu_j$ verify

$$\text{Re}\left(\varepsilon_j\right)\xi.\overline{\xi} \geq c_1 \left|\xi\right|^2, \;\; \text{Im}\left(\varepsilon_j\right)\xi.\overline{\xi} \leq 0, \;\; \text{Re}\left(\mu_j\right)\xi.\overline{\xi} \geq c_2 \left|\xi\right|^2, \;\; \text{Im}\left(\mu_j\right)\xi.\overline{\xi} \geq 0 \; \forall \xi \in \mathbb{R}^2. \tag{2.12}$$

In the case where one of the inequalities on the imaginary parts are strict, there is dissipation of energy. This results in a property of coercivity, possibly partial, on the variational problem of the system (2.11).

**Theorem 2.2** *The problem (2.11) has one solution and one only. Moreover, when the* *boundary* $\Gamma$ *is* $\mathcal{C}^\infty$*, as assumed here, the solution is in* $\mathcal{C}^\infty\left(\overline{\Omega_1^+}\right) \cap \mathcal{C}^\infty\left(\overline{\Omega_2^+}\right) \cap \mathcal{C}^\infty\left(\Omega^-\right)$.

The problem (2.11) constitutes a simple two-dimensional model for the aforementioned application. We aim to find an approximate problem (2.11) posed solely in $\Omega^-$ with boundary conditions on $\Gamma$ in which the thin layers no longer appear.

### 2.1.1 Reduction to an equation in $\Omega^-$

The numerical resolution of the problem set within the domain with thin layers, as mentioned in introduction, is challenging because it requires discretization at the scale of the layers' thickness. The mesh then contains a very large number of elements, making the calculations lengthy and sometimes imprecise. For this reason, the goal is to replace the initial problem with another problem whose solution is close to the one sought, and which no longer involves thin layers. The use of the impedance operator, concerning the partial differential equation posed within the thin layers, allows the resolution of our initial problem to be reduced to that of a problem posed solely within the propagation medium.

#### Impedance operator

With the aim of reformulating the transmission problem (2.11) as a diffraction problem in the external propagation domain $\Omega^-$, incorporating an appropriate boundary condition on the boundary $\Gamma$. This boundary condition, also known as the impedance condition, relates the tangential components of the electric and magnetic fields. To precisely express this condition, we introduce the Steklov-PoincarÃ© operator. This approach is characteristic of a non-overlapping domain decomposition method.

Let's start by defining this new operator denoted as $S$ [7], also known (up to a multiplicative factor) as the impedance by physicists.

For a sufficiently regular function $\varphi$ defined on $\Gamma$, let $u^+ = \left(u_1^+, u_2^+, \cdots, u_p^+\right)$ be the solution of the boundary problem:

$$\begin{array}{ll}
\operatorname{div}\left(\mathcal{A}_j \nabla u_j^+\right) + k^2 \mu_{j,3} u_j^+ = 0 & \text{in } \mathcal{D}'\left(\Omega_j^+\right), 1 \leq j \leq p, \\
\left(\mathcal{A}_1 \nabla u_1^+\right) \cdot n = 0 & \text{on } \Gamma_1, \text{ in } H^{-\frac{1}{2}}\left(\Gamma_1\right), \text{ for } TE \text{ polarization}, \\
u_1^+ = 0 & \text{on } \Gamma_1, \text{ in } H^{\frac{1}{2}}\left(\Gamma_1\right), \text{ for } TM \text{ polarization}, \\
\left(\mathcal{A}_j \nabla u_j^+\right) \cdot n = \left(\mathcal{A}_{j-1} \nabla u_{j-1}^+\right) \cdot n; & \text{in } H^{-\frac{1}{2}}\left(\Gamma_j\right) \quad (2 \leq j \leq p), \\
u_j^+ = u_{j-1}^+ & \text{in } H^{\frac{1}{2}}\left(\Gamma_j\right) \quad (2 \leq j \leq p), \\
u_p^+ = \varphi & \text{on } \Gamma \text{ with } \varphi \in H^s(\Gamma).
\end{array} \tag{2.13}$$

Regardless of the polarization studied, the operator $S$ on the bounded domain $\bigcup\limits_{1\leq j\leq p} \Omega_j^+$ is defined by the mapping

$$\begin{array}{rcl}
S \ : \ H^{\frac{1}{2}}(\Gamma) & \longrightarrow & H^{-\frac{1}{2}}(\Gamma) \\
\varphi & \longmapsto & S\varphi = \left(\mathcal{A}_p \nabla u_p^+\right) \cdot n|_\Gamma .
\end{array} \tag{2.14}$$

$S^{-1} = T$ is thus the admittance operator.

The exterior problem becomes:

$$\begin{array}{ll}
\boldsymbol{\Delta} u^- + k^2 u^- = 0 & \text{in } \mathcal{D}'\left(\Omega^-\right), \\
\partial_n u^- = S u^- & \text{on } \Gamma, \\
\lim\limits_{|x|\to\infty} |x|^{\frac{1}{2}} \left(\partial_r \left(u - u_{inc}\right) - ik\left(u - u_{inc}\right)\right) = 0. &
\end{array} \tag{2.15}$$

It is well known that $S$ is a pseudo-differential (non-local) operator [7]. This operator accounts for the effect of thin layers and thus leads to an impedance boundary condition. This condition is associated with the Helmholtz equation and the radiation condition to reformulate the initial transmission problem as a boundary value problem, where the primary challenge arises from the boundary condition.

**Theorem 2.3** *The operator $S$ is well-defined, linear, and continuous from $H^{\frac{1}{2}}(\Gamma)$ to $H^{-\frac{1}{2}}(\Gamma)$.*

## 2.2 Impedance operator for planar obstacles

We begin constructing approximations of the impedance operator for thin layers by considering the case of a planar geometry. This model geometry simplifies the description of the thin layers problem, particularly by excluding curvature terms.

We initiate the approximation of the impedance operator by employing a Taylor expansion within the thin layers.

### 2.2.1 Approximation of the impedance operator using a Taylor expansion

We start from the perfect conductor condition at $y = -(h_1 + \cdots + h_p)$ and establish a Taylor expansion at points $(x, y = -(h_2 + \cdots + h_p))$ and $(x, y = -(h_3 + \cdots + h_p))$.

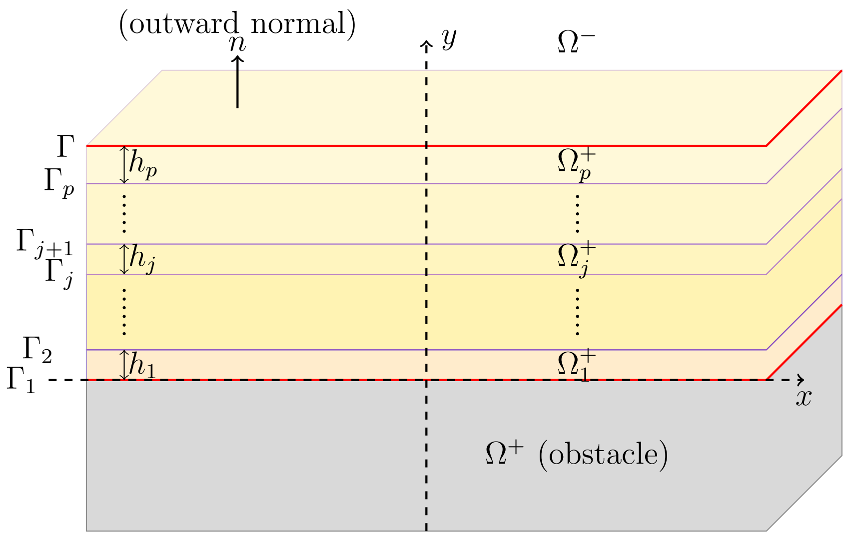


Figure 2.2: Illustration of a planar problem.

By utilizing the fact that $u_1^+$ *(respectively* $u_2^+$*)* satisfies the Helmholtz equation in $\Omega_1^+$ *(respectively* $\Omega_2^+$*)* and considering the transmission conditions, we derive a formula connecting $\partial_y u^+$ and $u^+$ at the point $(x, -(h_3 + \cdots + h_p))$. By applying recursively, the obtained formula and taking into account the continuity through the interfaces between the thin layers, we obtain a formula connecting $\partial_y u^+(x,0)$ and $u^+(x,0)$. Subsequently, we can easily deduce an approximation of the impedance operator; the order of the expansion will determine the order of the approximation.

Here we present the approximation of order 2 for planar obstacles covered with two thin layers, for details see [21]. The general formulas for the case of $p$ thin layers at any order will be studied in the following chapter.

#### Case of $TE$ polarization

We introduce the operators

$$L(\varepsilon_i) = \varepsilon_{2,2}\left(\partial_x \frac{1}{\varepsilon_{i,1}}\partial_x + k^2\mu_{i,3}\right) \;;\; M(\varepsilon_i) = \varepsilon_{i,2}\left[\partial_x\left(\partial_y \frac{1}{\varepsilon_{i,1}}\right)\partial_x + k^2\left(\partial_y \mu_{i,3}\right)\right] \; i = 1,2,$$
$$\Lambda_1 = 1 - \frac{h_1^2}{2}L(\varepsilon_1) \quad ; \quad \Lambda_2 = 1 - \frac{h_2^2}{2}L(\varepsilon_2),$$

$$P = h_2\Lambda_1\left(L(\varepsilon_2) - \frac{h_2}{2}M(\varepsilon_2)\right) + \frac{\varepsilon_{2,2}}{\varepsilon_{1,2}}h_1\left(L(\varepsilon_1) - \frac{h_1}{2}M(\varepsilon_1)\right)\Lambda_2,$$
$$Q = \Lambda_1\Lambda_2 - \frac{\varepsilon_{2,2}}{\varepsilon_{1,2}}h_1h_2\left(L(\varepsilon_1) - \frac{h_1}{2}M(\varepsilon_1)\right)\left(1 + \frac{h_2}{2}\varepsilon_{2,2}\left(\partial_y \frac{1}{\varepsilon_{2,1}}\right)\right).$$

A second-order $(2,2)$ approximation of the impedance operator $S$ by a Taylor expansion is given by

$$QS\varphi = -P\varphi.$$

#### Case of $TM$ polarization

In this polarization, the operators $P$ and $Q$ are defined as

$$P = \frac{\mu_{2,2}}{\mu_{1,2}}\Lambda_1\Lambda_2 - h_1h_2\left(1 + \frac{h_1}{2}\mu_{1,2}\left(\partial_y \frac{1}{\mu_{1,1}}\right)\right)\left(L(\mu_2) - \frac{h_2}{2}M(\mu_2)\right),$$
$$Q = -\frac{\mu_{2,2}}{\mu_{1,2}}h_2\Lambda_1\left(1 + \frac{h_2}{2}\mu_{2,2}\left(\partial_y \frac{1}{\mu_{2,1}}\right)\right) - h_1\left(1 + \frac{h_1}{2}\mu_{1,2}\left(\partial_y \frac{1}{\mu_{1,2}}\right)\right)\Lambda_2.$$

At order $(2,2)$, we obtain a result similar to the $TE$ polarization case

$$QS\varphi = -P\varphi.$$

The associated impedance conditions are referred to as "quasi-local impedance conditions."

### 2.2.2 Plane wave analysis

We are going to introduce another approach used by physicists to describe the effects of thin dielectric layers covering a perfect conductor. In the case where the incident field is a plane wave decomposed into Fourier-Hankel modes, the impedance operator is explicitly determined through its symbol. Approximating this symbol by a polynomial or a rational fraction allows us to construct the different impedance conditions.

Here, we limit ourselves to the case where the thin dielectric layers are isotropic and homogeneous (*i.e.*, $\varepsilon_1, \varepsilon_2, \mu_1$ and $\mu_2$ are constant scalar values).

#### Exact impedance operator

The total symbol $\sigma_S$ of the impedance operator $S$ is given by

$$\sigma_S(\xi) = -\frac{\alpha_2(\xi)\tan(h_2\alpha_2(\xi)) + \alpha_1(\xi)\dfrac{\varepsilon_2}{\varepsilon_1}\tan(h_1\alpha_1(\xi))}{1 - \dfrac{\alpha_1(\xi)\varepsilon_2}{\alpha_2(\xi)\varepsilon_1}\tan(h_2\alpha_2(\xi))\tan(h_1\alpha_1(\xi))} \quad \text{for } TE \text{ polarization,} \tag{2.16a}$$

$$\sigma_S(\xi) = \frac{\dfrac{\varepsilon_2}{\varepsilon_1} - \alpha_2(\xi)\tan(h_2\alpha_2(\xi))\dfrac{\tan(h_1\alpha_1(\xi))}{\alpha_1(\xi)}}{\varepsilon_2\dfrac{\tan(h_2\alpha_2(\xi))}{\alpha_2(\xi)\varepsilon_1} + \dfrac{\tan(h_1\alpha_1(\xi))}{\alpha_1(\xi)}} \quad \text{for } TM \text{ polarization,} \tag{2.16b}$$

where

$$\alpha_i(\xi) = \sqrt{k^2 n_i^2 - \xi^2}\ ;\ n_i = \sqrt{\varepsilon_i\mu_i} \quad i = 1,2.$$

The total symbol $\sigma_S$ determines the impedance operator $S$ by the formula

$$S\varphi(x) = \frac{1}{2\pi}\int e^{ix\xi}\sigma_S(\xi)\,\widehat{\varphi}(\xi)\,d\xi. \tag{2.17}$$

### Construction of the impedance operator

The symbol $\sigma_S$ of the impedance operator (pseudo-differential) $S$ is given using a trigonometric fraction by the formulas (2.16). As it is not a rational fraction in $\xi$, this symbol does not correspond to a differential operator. Our goal now is to approximate the symbol $\sigma_S$ with a symbol corresponding to a local or quasi-local operator, in practice, a rational fraction with respect to $\xi$.

**Case of $TE$ polarization** To provide an approximation of the impedance operator, the most straightforward approach is to approximate the symbol (2.16) using a Taylor series expansion. Upon expanding the complete symbol (2.16) up to the $(3,3)$ order, we arrive at the approximation

$$S = -\frac{\varepsilon_2}{\varepsilon_1} h_1 \left(1 + h_2^2 L_2 + \left(\frac{\varepsilon_2}{\varepsilon_1} h_1 h_2 + \frac{1}{3} h_1^2\right) L_1\right) L_1 - h_2 \left(1 + \frac{1}{3} h_2^2 L_2\right) L_2. \tag{2.18}$$

Where we set

$$L_i = \partial_x^2 + k^2 n_i^2,\ \ i = 1, 2.$$

To derive lower-order conditions, one simply needs to successively eliminate the terms containing $h_i^3$, $h_i^2$ $(i = 1, 2)$,....

By letting $h_1$ tend towards 0, we obtain the following approximation of the operator $S$.

$$S = -h_2 \left(\partial_x^2 + k^2 n_2^2\right) - \frac{h_2^3}{3} \left(\partial_x^2 + k^2 n_2^2\right)^2. \tag{2.19}$$

The associated impedance condition is identical to that obtained in [9] and [7], for more details see [21]. This condition involves a fourth-order operator that is difficult to discretize.

One idea to avoid these high-order operators [7] is to rewrite the total symbol $\sigma_S$ in the form

$$\sigma_S = -\frac{\alpha_2 \sin(\alpha_2 h_2)\cos(\alpha_1 h_1) + \alpha_1 \dfrac{\varepsilon_2}{\varepsilon_1}\sin(\alpha_1 h_1)\cos(\alpha_2 h_2)}{\cos(\alpha_1 h_1)\cos(\alpha_2 h_2) - \dfrac{\varepsilon_2 \alpha_1}{\varepsilon_1 \alpha_2}\sin(\alpha_1 h_1)\sin(\alpha_2 h_2)} = -Q^{-1}P,$$

A Taylor expansion of $P$ and $Q$ up to the $(3,3)$ order yields

$$P = \frac{\varepsilon_2}{\varepsilon_1}h_1 L_1 + h_2 L_2 - \frac{1}{2}\left(h_2 h_1^2 + \frac{\varepsilon_2}{\varepsilon_1}h_1 h_2^2\right)L_1 L_2,$$
$$Q = 1 - \frac{1}{2}h_2^2 L_2 - \left(\frac{1}{2}h_1^2 + \frac{\varepsilon_2}{\varepsilon_1}h_1 h_2\right)L_1 + \frac{1}{4}h_2^2 h_1^2 L_1 L_2.$$

By setting $h_1 = 0$, we find the condition in [7]

$$\partial_y u^-(x,0) + \frac{h}{\varepsilon_2}\left(\partial_x Q^{-1}\partial_x + k^2 n_2^2 Q^{-1}\right) = 0. \tag{2.20}$$

**Case of $TM$ polarization** In this polarization, we are concerned with the approximation of the admittance operator $T$. Its total symbol is given by:

$$\sigma_T = \frac{1}{\sigma_S} = \frac{\dfrac{\mu_2}{\mu_1}\dfrac{\tan(h_2\alpha_2)}{\alpha_2} + \dfrac{\tan(h_1\alpha_1)}{\alpha_1}}{\dfrac{\mu_2}{\mu_1} - \alpha_2\tan(h_2\alpha_2)\dfrac{\tan(h_1\alpha_1)}{\alpha_1}}.$$

An approximation up to order $(3,3)$ provided by

$$\sigma_T = \frac{\mu_1}{\mu_2}h_1\left(1 + \frac{1}{3}h_1^2\alpha_1^2\right) + h_2\left(1 + \left(\frac{\mu_1}{\mu_2}h_1\left(\frac{\mu_1}{\mu_2}h_1 + h_2\right) + \frac{1}{3}h_2^2\right)\alpha_2^2\right).$$

Another approach to construct approximations of the impedance operator based on asymptotic expansion relative to the small parameters of the problem helps to recover the above approximations.

## 2.3 Curved shapes

### 2.3.1 Impedance operator for a circular boundary

As seen in the previous paragraph, planar geometry enabled us to determine the exact impedance operator using its symbol. Similarly, circular geometry allows the explicit expression of this operator through a Fourier series decomposition. The advantage of this approach, compared to the previous one, is its consideration of curvature effects through the radius $R$. Once the exact impedance operator is calculated, one can establish an approximation process at different orders.

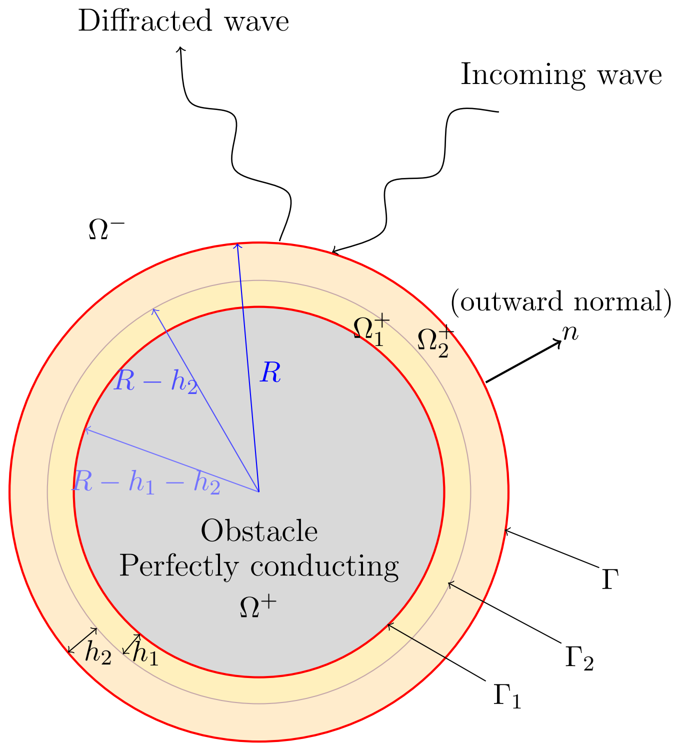


Figure 2.3: Illustration of a circular shaped obstacle.

#### Case of $TE$ polarization

An approximation of order $(3,3)$ for the operator $S$ is given by

$$\begin{aligned} S\left(h_1,h_2\right) = \; & -\frac{\varepsilon_2}{\varepsilon_1}h_1L_1 - h_2L_2 + \frac{\varepsilon_2}{\varepsilon_1}\frac{h_1^2}{2R}\left(L_1 - 2\frac{\partial_\theta^2}{R^2}\right) + \frac{\varepsilon_2}{\varepsilon_1}\frac{h_1h_2}{R}\left(L_1 - 2\frac{\partial_\theta^2}{R^2}\right) \\ & + \frac{h_2^2}{2R}\left(L_2 - 2\frac{\partial_\theta^2}{R^2}\right) - \frac{\varepsilon_2}{\varepsilon_1}\frac{h_1^3}{3}\left(L_1^2 + \frac{\partial_\theta^2}{R^4}\right) - \frac{\varepsilon_2}{\varepsilon_1}h_1^2h_2\left(\frac{\varepsilon_2}{\varepsilon_1}L_1^2 + \frac{\partial_\theta^2}{R^4}\right) \\ & - \frac{\varepsilon_2}{\varepsilon_1}h_1h_2^2\left(L_1L_2 + \frac{\partial_\theta^2}{R^4}\right) - \frac{h_2^3}{3}\left(L_2^2 + \frac{\partial_\theta^2}{R^4}\right), \end{aligned} \tag{2.21}$$

where we set

$$L_i = k^2n_i^2 + \frac{\partial_\theta^2}{R^2} \quad i = 1,2. \tag{2.22}$$

**Case of $TM$ polarization**

An approximation of order $(3,3)$ for the admittance operator $T$ is expressed as

$$T(h_1,h_2) = \frac{\mu_1}{\mu_2}h_1 + h_2 + \frac{\mu_1}{\mu_2}\frac{h_1^2}{2R} + \frac{\mu_1}{\mu_2}\frac{h_1 h_2}{R} + \frac{h_2^2}{2R} + \frac{\mu_1}{\mu_2}\frac{h_1^3}{3}\left(L_1 + \frac{1}{R^2}\right) + \frac{\mu_1}{\mu_2}h_1^2 h_2\left(\frac{\mu_1}{\mu_2}L_2 + \frac{1}{R^2}\right) + \left(\frac{\mu_1}{\mu_2}h_1 h_2^2 + \frac{h_2^3}{3}\right)\left(L_2 + \frac{1}{R^2}\right).$$

### 2.3.2 Extension to arbitrary shaped obstacles

Starting from the circular canonical case, it is easy to extend the approximations of the impedance operator to a more general framework. This allows on the one hand to be able to deal with diffraction problems by obstacles of arbitrary geometries and, on the other hand, the validation of new conditions.

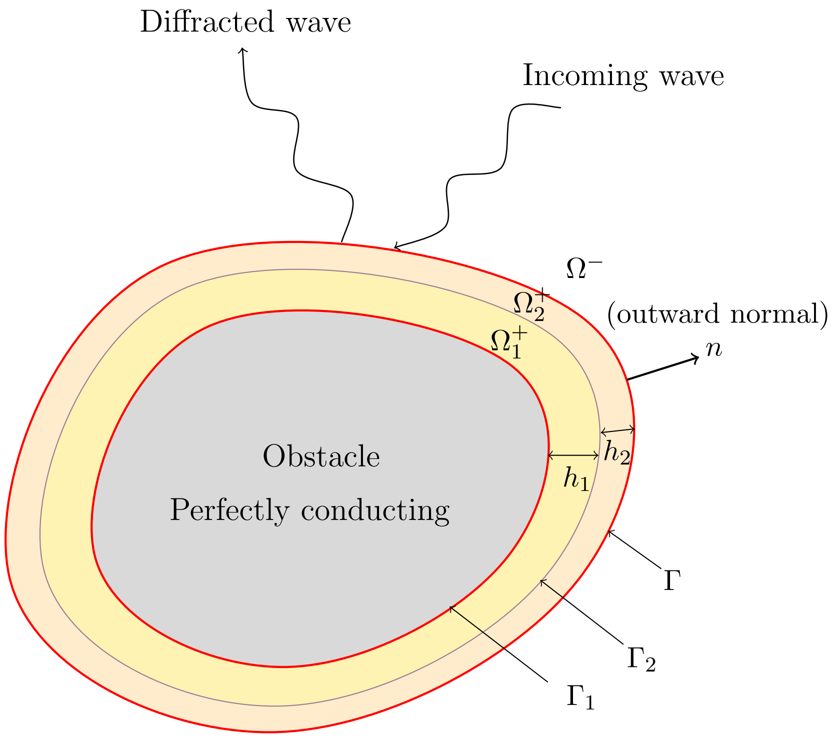


Figure 2.4: Illustration of an arbitrary shaped obstacle.

We use the following formal substitutions:

$$\begin{cases} \partial_r \longleftrightarrow \partial_n \\ \dfrac{1}{R} \longleftrightarrow \kappa(s) \\ \dfrac{\partial_\theta}{R} \longleftrightarrow \partial_s = \nabla_\Gamma, \end{cases} \tag{2.23}$$

to derive impedance conditions on a boundary of an arbitrary domain. Here, $\kappa(s)$ represents the curvature of the boundary (oriented in the positive sense) at the curvilinear abscissa point $s$.

**Case of $TE$ polarization**

An asymptotic expansion up to order $(3,3)$ provides an approximation of order $(3,3)$ for the impedance operator $S$, which is given by

$$\begin{aligned} S(h_1,h_2) = &-\frac{\varepsilon_2}{\varepsilon_1}\mathcal{L}_1 h_1 - \mathcal{L}_2 h_2 + \frac{1}{2}\frac{\varepsilon_2}{\varepsilon_1}\left(\kappa k^2 n_1^2 - \partial_s(\kappa\partial_s)\right)h_1^2 + \frac{\varepsilon_2}{\varepsilon_1}\left(\kappa k^2 n_1^2 - \partial_s(\kappa\partial_s)\right)h_1 h_2 \\ &+\frac{1}{2}\left(\kappa k^2 n_2^2 - \partial_s(\kappa\partial_s)\right)h_2^2 - \frac{1}{3}\frac{\varepsilon_2}{\varepsilon_1}\left(\mathcal{L}_1^2 + \partial_s(\kappa^2\partial_s)\right)h_1^3 - \frac{\varepsilon_2}{\varepsilon_1}\left(\frac{\varepsilon_2}{\varepsilon_1}\mathcal{L}_1^2 + \partial_s(\kappa^2\partial_s)\right)h_1^2 h_2 \\ &-\frac{\varepsilon_2}{\varepsilon_1}\left(\mathcal{L}_1\mathcal{L}_2 + \partial_s(\kappa^2\partial_s)\right)h_1 h_2^2 - \frac{1}{3}\left(\mathcal{L}_2^2 + \partial_s(\kappa^2\partial_s)\right)h_2^3, \end{aligned} \tag{2.24}$$

where we set

$$\mathcal{L}_1 = \left(\partial_s^2 + k^2 n_1^2\right)\ ;\ \mathcal{L}_2 = \left(\partial_s^2 + k^2 n_2^2\right). \tag{2.25}$$

**Case of $TM$ polarization**

An approximation of order $(3,3)$ for the admittance operator $T$ is given by

$$\begin{aligned} T(h_1,h_2) = &\frac{\mu_1}{\mu_2}h_1 + h_2 + \frac{1}{2}\frac{\mu_1}{\mu_2}\kappa h_1^2 + \frac{\mu_1}{\mu_2}\kappa h_1 h_2 + \frac{1}{2}\kappa h_2^2 + \frac{1}{3}\frac{\mu_1}{\mu_2}\left(\mathcal{L}_1 + \kappa^2\right)h_1^3 \\ &+\frac{\mu_1}{\mu_2}\left(\frac{\mu_1}{\mu_2}\mathcal{L}_2 + \kappa^2\right)h_1^2 h_2 + \frac{\mu_1}{\mu_2}\left(\mathcal{L}_2 + \kappa^2\right)h_1 h_2^2 + \frac{1}{3}\left(\mathcal{L}_2 + \kappa^2\right)h_2^3. \end{aligned}$$

# Chapter 3

# Approximation of impedance operator for planar obstacles

In this chapter, we generalize the results presented in the previous chapter in the case of planar obstacles to include all the scattering problems of electromagnetic waves by a perfectly conducting obstacle coated with thin dielectric multilayers. The content of this chapter has been published in [4].

## 3.1 Problem statement

We consider the case of a perfectly conducting obstacle (made of metal) coated with $p$ parallel thin dielectric layers of thicknesses $h_j$, $j = 1, \ldots, p$. The dielectric of thickness $h_j$ is characterized by a relative permittivity $\varepsilon_j$ and a relative permeability $\mu_j, j = 1, \ldots, p$. The metallic obstacle coated with thin dielectric layers is placed in a dielectric medium (propagation medium). This medium can be the vacuum and it is characterized by a permittivity $\varepsilon_0$ and permeability $\mu_0$. This system is illuminated by an incident wave characterized by its number $k > 0$. When this wave encounters the obstacle, it generates a wave diffracted by this latter.

The metallic obstacle occupies a three-dimensional planar domain $\Omega$; the thin layers are denoted by $\Omega_j$ with interior boundary $\partial_{int}\Omega_j, j = 1, \ldots, p.$. The domain $\Omega$ adding to it the $p$ thin layers is denoted by $\Omega^+$ with boundary $\Gamma$ and unit outward normal vector $n$. The exterior domain of $\Omega^+$ is designated $\Omega^-$. The thickness of the layers from the first till the $j^{th}$ is $\tilde{h}_j = h_1 + \ldots + h_j$. We set $\tilde{h}_p = h$ and $h_j = \beta_j h$ with $\sum_{j=1}^{p} \beta_j = 1$, by convention $\tilde{h}_0 = h_0 = \beta_0 = 0$.

We introduce the family $\Gamma(s)$ of parallel surfaces

$$\Gamma(s) = \{y; y = x - sn(x), x \in \Gamma\}, s \in (-\infty, h].$$

We notice that $\Gamma(0) = \Gamma$, $\partial_{int}\Omega_j = \Gamma\left(\tilde{h}_j\right)$ and $\Omega_j$ is the domain limited by $\Gamma\left(\tilde{h}_j\right)$ and $\Gamma\left(\tilde{h}_{j-1}\right)$. We set $\Gamma_j = \Gamma\left(\tilde{h}_j\right), j = 1, \ldots, p$. See Figure 3.1.

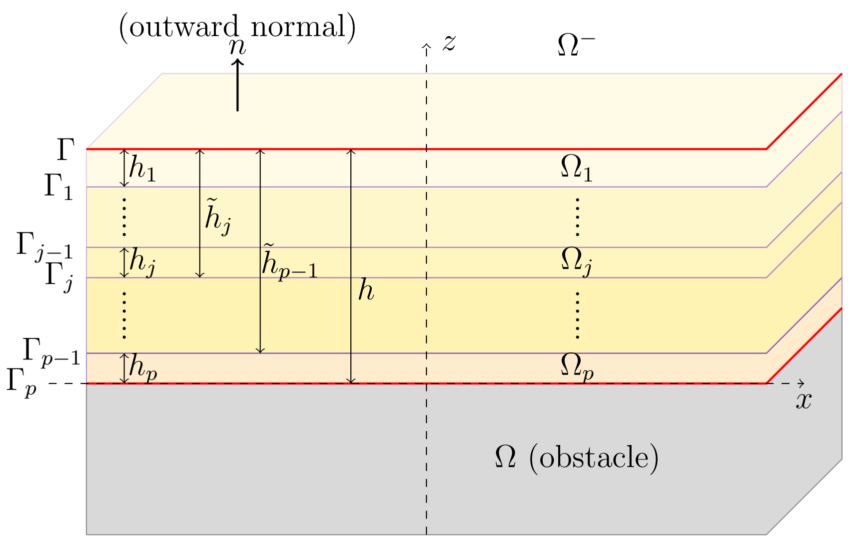


Figure 3.1: Illustration of a planar obstacle covered with $p$ thin layers of different dielectric materials.

Decomposing the electromagnetic vector field into its tangential and normal components [28], [17], the Maxwell's system can be transformed from a PDEs system to first order linear abstract Cauchy problem [17]. Therefore, scattering problems of electromagnetic waves by a perfectly conducting obstacle coated with thin dielectric layers can be represented in curvilinear coordinates [28] by the equations

$$
\left.\begin{aligned}
&\tfrac{\partial}{\partial s}\mathcal{Y}_p(s) = \mathcal{M}_p\mathcal{Y}_p(s) && \text{in } \mathcal{C}\left(\left(\tilde{h}_{p-1},\tilde{h}_p\right);X\right), \\
&\mathcal{Y}_p\left(\tilde{h}_p\right) = [\varphi_1,\varphi_2]^{\mathsf{T}}, && \text{with } \sigma_1\varphi_1+\sigma_2\varphi_2=\varphi, \\
&\tfrac{\partial}{\partial s}\mathcal{Y}_j(s) = \mathcal{M}_j\mathcal{Y}_j(s) && \text{in } \mathcal{C}\left(\left(\tilde{h}_{j-1},\tilde{h}_j\right);X\right), \quad j=1,\dots,p-1, \\
&\mathcal{Y}_j\left(\tilde{h}_j\right) = \mathcal{Y}_{j+1}\left(\tilde{h}_j\right) && \qquad\qquad\qquad\qquad\quad j=1,\dots,p-1, \\
&\tfrac{\partial}{\partial s}\mathcal{Y}_0(s) = \mathcal{M}_0\mathcal{Y}_0(s) && \qquad\qquad\qquad\qquad\quad \text{in } \mathcal{C}\left((-\infty,0);X\right), \\
&\mathcal{Y}_0(0) = \mathcal{Y}_1(0), && \\
&+ \text{ Silver-MÃ¼ller radiation condition for } s\to-\infty. &&
\end{aligned}\right\} \tag{3.1}
$$

Where $\mathcal{Y}_j = [U_j, V_j]^{\mathsf{T}}$ is in $\mathcal{C}^1\left(\left(\tilde{h}_{j-1},\tilde{h}_j\right);X\right)$, see Figure 3.2. The matrices $\mathcal{M}_j = \begin{bmatrix} A_j & G_j \\ F_j & B_j \end{bmatrix}$ are linear differential operators at most of second-order with values in a Sobolev space $X$ on $\mathbb{R}^n, n = 1,2,3$. Note that $\varphi_1$ represents Dirichlet's condition and $\varphi_2$ represents Neumann's condition which are linearly combined as $\sigma_1\varphi_1 + \sigma_2\varphi_2 = \varphi$ with $\sigma_1, \sigma_2$ and $\varphi$ being given constants.

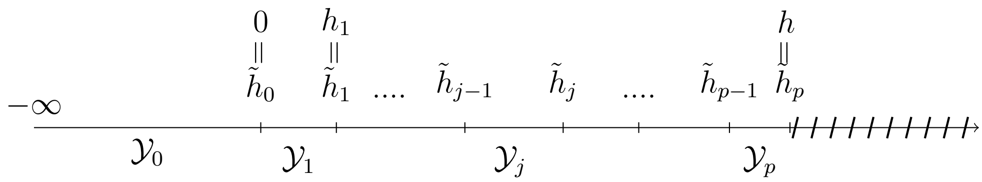


Figure 3.2: Domain representation of the planar obstacle and the $p$ thin layers.

For arbitrary shaped obstacles, the operators $\mathcal{M}_j$ are depending on $s$, however for planar obstacles, they are independent of $s$ and in this case the well-posedness of the problem (3.1) follows immediately from the theory of linear abstract Cauchy problems, see [19, page 83] and [29, page 104].

### 3.1.1 Dirichlet-to-Neumann operator

As we mentioned in the introduction, solving numerically the problem (3.1) is challenging since it requires discretizing on the scale of the layers' thickness. The mesh then contains a very large number of elements, which makes the calculations long and sometimes imprecise. For this reason, we reformulate our problem (3.1) and replace it by another problem that does not bring in any more thin layers. The use of Dirichlet-to-Neumann operator, relative to the equations set in thin layers allows to reduce the solving of our original problem to a problem that is posed only in the exterior domain $\Omega^-$ corresponding to $s \in (-\infty, 0)$.

Our goal, therefore, is to rewrite the problem (3.1) as a problem in the exterior domain with an appropriate boundary condition on $\Gamma$ corresponding to $s = 0$, which is known as Dirichlet-to-Neumann condition. To express this condition accurately, we introduce the Dirichlet-to-Neumann operator, abbreviated DtoN, which is also called Steklov-PoincarÃ© operator and is known as impedance operator as well [8], [5].

We begin by defining this new operator. For $\phi = [\phi_1, \phi_2]^{\mathrm{T}}$ sufficiently smooth defined on $s = 0$, we consider $\mathcal{Y}^+ = (\mathcal{Y}_1, \ldots, \mathcal{Y}_p)$ the solution of the following problem:

$$\left.\begin{array}{lll}
\frac{\partial}{\partial s}\mathcal{Y}_1(s) = \mathcal{M}_1\mathcal{Y}_1(s) & \text{in } \mathcal{C}\left(\left(0, \tilde{h}_1\right); X\right), & \\
\mathcal{Y}_1(0) = [\phi_1, \phi_2]^{\mathrm{T}}, & & \\
& & \\
\frac{\partial}{\partial s}\mathcal{Y}_j(s) = \mathcal{M}_j\mathcal{Y}_j(s) & \text{in } \mathcal{C}\left(\left(\tilde{h}_{j-1}, \tilde{h}_j\right); X\right), & j = 2, \ldots, p, \\
\mathcal{Y}_j\left(\tilde{h}_{j-1}\right) = \mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right) & & j = 2, \ldots, p, \\
& & \\
\mathcal{Y}_p\left(\tilde{h}_p\right) = [\varphi_1, \varphi_2]^{\mathrm{T}} & \text{with } \sigma_1\varphi_1 + \sigma_2\varphi_2 = \varphi. &
\end{array}\right\} \tag{3.2}$$

**Definition 3.1** *We define the DtoN operator by the mapping*

$$S \;: \phi_1 \longmapsto S\phi_1 = \phi_2. \tag{3.3}$$

The problem for $s \in (-\infty, 0)$, becomes then

$$\left.\begin{array}{ll} \frac{\partial}{\partial s}\mathcal{Y}_0(s) = \mathcal{M}_0\mathcal{Y}_0(s) & \text{in } \mathcal{C}((-\infty,0);X), \\ (\mathcal{Y}_0(0))_2 = S(\mathcal{Y}_0(0))_1, & \\ + \text{ Silver-MÃ¼ller radiation condition for } s \to -\infty. & \end{array}\right\} \quad (3.4)$$

The inverse operator $S^{-1} : \phi_2 \longmapsto S^{-1}\phi_2 = \phi_1$ is called Neumann-to-Dirichlet operator [23].

**Remark 3.2** *If we are interested in the values inside the thin layers, we define in a similar manner, the DtoN operator $S$ posed in the exterior domain.*

### 3.1.2 Determination of the exact Dirichlet-to-Neumann operator

Note that the calculation of the DtoN operator returns to express $\mathcal{Y}_p\left(\tilde{h}_p\right) = [\varphi_1, \varphi_2]^{\mathrm{T}}$ in terms of $\mathcal{Y}_0(0) = [\phi_1, \phi_2]^{\mathrm{T}}$.

**Theorem 3.3** *The exact Dirichlet-to-Neumann operator is given by*

$$S \;: \phi_1 \longmapsto S\phi_1 = (\sigma_1 Q_1 + \sigma_2 Q_2)^{-1}(\varphi - (\sigma_1 P_1 + \sigma_2 P_2)\phi_1). \quad (3.5)$$

*where*

$$\begin{bmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{bmatrix} = \exp(h_p\mathcal{M}_p)\ldots\exp(h_j\mathcal{M}_j)\ldots\exp(h_1\mathcal{M}_1).$$

**Proof.** Existence and uniqueness of the DtoN operator $S$ comes from solving successively linear abstract Cauchy problems [19], [29].

The unique solution of the Cauchy problem

$$\left.\begin{array}{ll} \frac{\partial}{\partial s}\mathcal{Y}_j(s) = \mathcal{M}_j\mathcal{Y}_j(s) & \text{in } \mathcal{C}\left(\left(\tilde{h}_{j-1}, \tilde{h}_j\right); X\right), \\ \mathcal{Y}_j\left(\tilde{h}_{j-1}\right) = \mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right), & \end{array}\right\} \quad (3.6)$$

is given by

$$\mathcal{Y}_j(s) = \exp\left(\left(s - \tilde{h}_{j-1}\right)\mathcal{M}_j\right)\mathcal{Y}_j\left(\tilde{h}_{j-1}\right),\; s \in \left(\tilde{h}_{j-1}, \tilde{h}_j\right)$$

therefore for $s = \tilde{h}_j$ and replacing $\mathcal{Y}_j\left(\tilde{h}_{j-1}\right)$ by $\mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right)$ yields

$$\mathcal{Y}_j\left(\tilde{h}_j\right) = \exp\left(\left(\tilde{h}_j - \tilde{h}_{j-1}\right)\mathcal{M}_j\right)\mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right) = \exp\left(h_j\mathcal{M}_j\right)\mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right).$$

By induction it follows that

$$[\varphi_1, \varphi_2]^{\mathrm{T}} = \mathcal{Y}_p\left(\tilde{h}_p\right) = \tilde{\mathcal{M}}_p\,[\phi_1, \phi_2]^{\mathrm{T}}, \tag{3.7}$$

where

$$\tilde{\mathcal{M}}_p \equiv \begin{bmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{bmatrix} = \exp\left(h_p\mathcal{M}_p\right) \ldots \exp\left(h_j\mathcal{M}_j\right) \ldots \exp\left(h_1\mathcal{M}_1\right). \tag{3.8}$$

The equation (3.7) is equivalent to

$$\left.\begin{aligned} P_1\phi_1 + Q_1\phi_2 &= \varphi_1, \\ P_2\phi_1 + Q_2\phi_2 &= \varphi_2. \end{aligned}\right\}$$

Since $\sigma_1\varphi_1 + \sigma_2\varphi_2 = \varphi$ then we get

$$\left(\sigma_1 P_1 + \sigma_2 P_2\right)\phi_1 + \left(\sigma_1 Q_1 + \sigma_2 Q_2\right)\phi_2 = \varphi.$$

Consequently

$$S\ : \phi_1 \longmapsto S\phi_1 = \phi_2 = \left(\sigma_1 Q_1 + \sigma_2 Q_2\right)^{-1}\left(\varphi - \left(\sigma_1 P_1 + \sigma_2 P_2\right)\phi_1\right).$$

■

In most cases either $\sigma_1$ or $\sigma_2$ is equal to zero.

In the case where $\sigma_2 = 0$ the DtoN operator $S_1$ is

$$S_1\ : \phi_1 \longmapsto S_1\phi_1 = Q_1^{-1}\left(\varphi_1 - P_1\phi_1\right). \tag{3.9}$$

Similarly if $\sigma_1 = 0$ the DtoN operator $S_2$ is

$$S_2\ : \phi_1 \longmapsto S_2\phi_1 = Q_2^{-1}\left(\varphi_2 - P_2\phi_1\right). \tag{3.10}$$

Unfortunately, the formula of the exact DtoN operator is not practical for computation and it will be useful and interesting to approximate it. Our goal here is to approximate this operator by an operator that is a rational fraction with respect to the thickness of thin layers.

## 3.2 Approximation of Dirichlet-to-Neumann operator

We present two different approaches to approximate the DtoN operator. A first approach consists in using a Taylor expansions. A second approach concerns the asymptotic analysis of the problem with respect to the thickness of thin layers.

In order to simplify the formulas of the approximate DtoN operator, we introduce the multi-index notation.

### 3.2.1 Multi-index notation

A $p$-dimensional multi-index is an $p$-tuple $\alpha = (\alpha_1, \alpha_2, ..., \alpha_p)$ of non-negative integers, which is in the set $p$-dimensional natural numbers, denoted $\mathbb{N}_0^p$.
For multi-indices $\alpha$ and $\eta$ in $\mathbb{N}_0^p$ we define:

Componentwise sum and difference as $\alpha \pm \eta = (\alpha_1 \pm \eta_1, \alpha_2 \pm \eta_2, \ldots, \alpha_p \pm \eta_p)$.

Sum of components or absolute value as $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_p$.

Factorial as $\alpha! = \alpha_1!\alpha_2!\ldots\alpha_p!$.

A vector $\mathbf{V} = (V_1, V_2, \ldots, V_p)$ to the power of multi-index $\alpha$ as $\mathbf{V}^\alpha = V_1^{\alpha_1} V_2^{\alpha_2} \ldots V_p^{\alpha_p}$.

### 3.2.2 Approximation of the DtoN operator by Taylor expansions

Recall that the calculation of the DtoN operator returns to express $\mathcal{Y}_p\left(\tilde{h}_p\right)$ in terms of $\mathcal{Y}_0\left(0\right)$.

We start from the condition of electrical conductor at $s = \tilde{h}_p$ which is $\mathcal{Y}_p\left(\tilde{h}_p\right) = [\varphi_1, \varphi_2]^{\mathrm{T}}$ and we write a Taylor expansion at the points $s = \tilde{h}_{j-1}, j = p, \ldots, 1$. Using the fact that $\mathcal{Y}_j(s)$ satisfies the equation $\frac{\partial}{\partial s}\mathcal{Y}_j(s) = \mathcal{M}_j\mathcal{Y}_j(s)$ in $\left(\tilde{h}_{j-1}, \tilde{h}_j\right)$ and taking into account of the transmission conditions $\mathcal{Y}_j\left(\tilde{h}_{j-1}\right) = \mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right)$, we obtain a formula that connects $\mathcal{Y}_p\left(\tilde{h}_p\right)$ and $\mathcal{Y}_0(0)$. Then we can easily derive an approximation of the DtoN operator; the order of Taylor expansion will give the order of the approximation.

**Theorem 3.4** *An approximation of order n for DtoN operator (3.5) is given by*

$$S\ : \phi_1 \longmapsto S\phi_1 = \left(\sigma_1 Q_{1,n} + \sigma_2 Q_{2,n}\right)^{-1}\left(\varphi - \left(\sigma_1 P_{1,n} + \sigma_2 P_{2,n}\right)\phi_1\right), \tag{3.11}$$

*where*

$$P_{1,n} = \sum_{l=0}^{n}\mathbf{A}_l h^l, \quad Q_{1,n} = \sum_{l=0}^{n}\mathbf{G}_l h^l, \quad P_{2,n} = \sum_{l=0}^{n}\mathbf{F}_l h^l, \quad Q_{2,n} = \sum_{l=0}^{n}\mathbf{B}_l h^l, \tag{3.12}$$

*with*

$$N_l \equiv \begin{bmatrix} \mathbf{A}_l & \mathbf{G}_l \\ \mathbf{F}_l & \mathbf{B}_l \end{bmatrix} = \sum_{|\alpha|=l}\frac{\mathbf{M}^\alpha}{\alpha!}, \quad \mathbf{M} = \left(\beta_p\mathcal{M}_p, \ldots, \beta_1\mathcal{M}_1\right).$$

**Proof.** By Taylor expansions

$$\mathcal{Y}_j\left(\tilde{h}_j\right) = \sum_{l=0}^{n}\mathcal{Y}_j^{(l)}\left(\tilde{h}_{j-1}\right)\frac{\left(h_j\right)^l}{l!}, \tag{3.13}$$

where $\mathcal{Y}_j^{(l)}$ is the derivative of order $l$ of $\mathcal{Y}_j$ with respect to $s$ with the convention $\mathcal{Y}_j^{(0)} \equiv \mathcal{Y}_j$. For simplicity in writing we omitted the term $o\left(\left(h_j\right)^n\right)$.

Since the matrix operator $\mathcal{M}_j$ is independent of $s$, we can easily see that

$$\mathcal{Y}_j^{(l)}(s) = \left(\mathcal{M}_j\right)^l\mathcal{Y}_j(s). \tag{3.14}$$

Replacing $\mathcal{Y}_j^{(l)}\left(\tilde{h}_{j-1}\right)$ by its value of (3.14) in (3.13), then substituting $\mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right)$ for $\mathcal{Y}_j\left(\tilde{h}_{j-1}\right)$, we obtain

$$\mathcal{Y}_j\left(\tilde{h}_j\right) = M_{j,n}\mathcal{Y}_{j-1}\left(\tilde{h}_{j-1}\right), \tag{3.15}$$

where

$$M_{j,n} = \sum_{l=0}^{n} \left(\mathcal{M}_j\right)^l \frac{\left(h_j\right)^l}{l!} = \sum_{l=0}^{n} \left(\beta_j \mathcal{M}_j\right)^l \frac{h^l}{l!}, \tag{3.16}$$

with $\left(\mathcal{M}_j\right)^0$ is the $2 \times 2$ identity matrix. By induction we obtain

$$\mathcal{Y}_p\left(\tilde{h}_p\right) = \tilde{M}_{p,n}\mathcal{Y}_0\left(0\right), \tag{3.17}$$

with

$$\tilde{M}_{p,n} = M_{p,n}M_{p-1,n}...M_{2,n}M_{1,n}. \tag{3.18}$$

According to the formula of exact DtoN operator (3.5), its approximation of order $n$ can be expressed as

$$S \ : \phi_1 \longmapsto S\phi_1 = \left(\sigma_1 Q_{1,n} + \sigma_2 Q_{2,n}\right)^{-1}\left(\varphi - \left(\sigma_1 P_{1,n} + \sigma_2 P_{2,n}\right)\phi_1\right), \tag{3.19}$$

where

$$\begin{bmatrix} P_{1,n} & Q_{1,n} \\ P_{2,n} & Q_{2,n} \end{bmatrix} = \tilde{M}_{p,n}. \tag{3.20}$$

To get approximation of any order with respect to $h$, we need to express $\tilde{M}_{p,n}$ as a polynomial ordered by increasing powers of $h$.

The matrix $\tilde{M}_{p,n}$ can be written as

$$\tilde{M}_{p,n} = M_{p,n} \dots M_{2,n}M_{1,n} = \left(\sum_{l_p=0}^{n} \left(\beta_p \mathcal{M}_p\right)^{l_p} \frac{h^{l_p}}{l_p!}\right) \dots \left(\sum_{l_1=0}^{n} \left(\beta_1 \mathcal{M}_1\right)^{l_1} \frac{h^{l_1}}{l_1!}\right), \tag{3.21}$$

which can be rearranged to the conventional form

$$\tilde{M}_{p,n} = \sum_{l=0}^{n} N_l h^l, \tag{3.22}$$

where

$$N_l \equiv \begin{bmatrix} \mathbf{A}_l & \mathbf{G}_l \\ \mathbf{F}_l & \mathbf{B}_l \end{bmatrix} = \sum_{l_1+l_2+\ldots+l_p=l} \left(\frac{\left(\beta_p \mathcal{M}_p\right)^{l_p}}{l_p!} \dots \frac{\left(\beta_2 \mathcal{M}_2\right)^{l_2}}{l_2!}\frac{\left(\beta_1 \mathcal{M}_1\right)^{l_1}}{l_1!}\right), \tag{3.23}$$

or alternatively it can be written as

$$N_l = \sum_{0 \le l_1 \le l_2 \le \ldots \le l_{p-1} \le l} \left(\frac{\left(\beta_p \mathcal{M}_p\right)^{l-l_{p-1}}}{\left(l-l_{p-1}\right)!} \dots \frac{\left(\beta_2 \mathcal{M}_2\right)^{l_2-l_1}}{\left(l_2-l_1\right)!}\frac{\left(\beta_1 \mathcal{M}_1\right)^{l_1}}{l_1!}\right).$$

With the multi-indices notations introduced above, the term $N_l$ can simply be written as

$$N_l = \sum_{|\alpha|=l} \frac{\mathbf{M}^{\alpha}}{\alpha!}, \tag{3.24}$$

where $\mathbf{M} = (\beta_p \mathcal{M}_p, \ldots, \beta_1 \mathcal{M}_1)$ . The calculation of $N_l, l = 1, \ldots, n$ determines the approximation of order $n$ of DtoN operator, which is given by the formula (3.19) with

$$P_{1,n} = \sum_{l=0}^{n} \mathbf{A}_l h^l, \quad Q_{1,n} = \sum_{l=0}^{n} \mathbf{G}_l h^l, \quad P_{2,n} = \sum_{l=0}^{n} \mathbf{F}_l h^l, \quad Q_{2,n} = \sum_{l=0}^{n} \mathbf{B}_l h^l.$$

■

**Approximation of order 0**

We begin the calculations with something that is more simple, i.e. an approximation of order 0. In this case the corresponding matrix $N_0$ is a $2 \times 2$ identity matrix. The approximation of order 0 is therefore given by

$$S\phi_1 = \frac{1}{\sigma_2}(\varphi - \sigma_1 \phi_1) \quad \text{if} \quad \sigma_2 \neq 0 \quad \text{and} \quad S^{-1}\phi_2 = \varphi_1 \quad \text{if} \quad \sigma_2 = 0.$$

The associated DtoN conditions are

$$(\mathcal{Y}_0(0))_2 = \frac{1}{\sigma_2}(\varphi - \sigma_1 (\mathcal{Y}_0(0))_1) \quad \text{if} \quad \sigma_2 \neq 0 \quad \text{and} \quad (\mathcal{Y}_0(0))_1 = \varphi_1 \quad \text{if} \quad \sigma_2 = 0.$$

These conditions are in fact quite reasonable, they simply consist of completely removing the thin layers. However, they are uninteresting because they do not take into account the effect of thin layers. They are not satisfactory only when the thicknesses of the layers become almost zero. We should therefore go further in our Taylor expansion to lead to conditions of higher order that are more useful. These conditions of order 0 must be recovered in all higher order approximations by letting the thickness $h$ tend to zero.

### Approximation of order 1

Let us now examine the approximation of order 1. The matrix $N_1$ is given by

$$N_1 \equiv \begin{bmatrix} \mathbf{A}_1 & \mathbf{G}_1 \\ \mathbf{F}_1 & \mathbf{B}_1 \end{bmatrix} = \sum_{j=1}^{p} \beta_j \mathcal{M}_j = \sum_{j=1}^{p} \beta_j \begin{bmatrix} A_j & G_j \\ F_j & B_j \end{bmatrix}.$$

Therefore, the approximate DtoN operator of order 1 is

$$S\phi_1 = \left(\sigma_2 + \left(\sigma_1 \mathbf{G}_1 + \sigma_2 \mathbf{B}_1\right) h\right)^{-1} \left(\varphi - \left(\sigma_1 + \left(\sigma_1 \mathbf{A}_1 + \sigma_2 \mathbf{F}_1\right) h\right) \phi_1\right),$$

where

$$\mathbf{A}_1 = \sum_{j=1}^{p} \beta_j A_j, \quad \mathbf{G}_1 = \sum_{j=1}^{p} \beta_j G_j, \quad \mathbf{F}_1 = \sum_{j=1}^{p} \beta_j F_j, \quad \mathbf{B}_1 = \sum_{j=1}^{p} \beta_j B_j.$$

### Approximation of order 2

The matrix $N_2$ can be written as

$$N_2 = \sum_{i,j,i>j}^{p} \beta_i \beta_j \mathcal{M}_i \mathcal{M}_j + \frac{1}{2} \sum_{j=1}^{p} \beta_j^2 \mathcal{M}_j^2 \equiv \begin{bmatrix} \mathbf{A}_2 & \mathbf{G}_2 \\ \mathbf{F}_2 & \mathbf{B}_2 \end{bmatrix}, \tag{3.25}$$

and thus the approximation of DtoN operator in this case is

$$S\phi_1 = Q^{-1} \left(\varphi - \left(\sigma_1 + \left(\sigma_1 \mathbf{A}_1 + \sigma_2 \mathbf{F}_1\right) h + \left(\sigma_1 \mathbf{A}_2 + \sigma_2 \mathbf{F}_2\right) h^2\right) \phi_1\right), \tag{3.26}$$

$$Q = \sigma_2 + \left(\sigma_1 \mathbf{G}_1 + \sigma_2 \mathbf{B}_1\right) h + \left(\sigma_1 \mathbf{G}_2 + \sigma_2 \mathbf{B}_2\right) h^2, \tag{3.27}$$

with

$$\mathbf{A}_2 = \sum_{i,j,i>j}^{p} \beta_i \beta_j \left(A_i A_j + G_i F_j\right) + \frac{1}{2} \sum_{j=1}^{p} \beta_j^2 \left(A_j^2 + G_j F_j\right), \tag{3.28}$$

$$\mathbf{G}_2 = \sum_{i,j,i>j}^{p} \beta_i \beta_j \left(A_i G_j + G_i B_j\right) + \frac{1}{2} \sum_{j=1}^{p} \beta_j^2 \left(A_j G_j + G_j B_j\right), \tag{3.29}$$

$$\mathbf{F}_2 = \sum_{i,j,i>j}^{p} \beta_i \beta_j \left(F_i A_j + B_i F_j\right) + \frac{1}{2} \sum_{j=1}^{p} \beta_j^2 \left(F_j A_j + B_j F_j\right), \tag{3.30}$$

$$\mathbf{B}_2 = \sum_{i,j,i>j}^{p} \beta_i \beta_j \left(F_i G_j + B_i B_j\right) + \frac{1}{2} \sum_{j=1}^{p} \beta_j^2 \left(F_j G_j + B_j^2\right). \tag{3.31}$$

Obtaining the expression of $S$ in terms of $A_j, G_j, F_j$ and $B_j$ for the approximations of higher order is not such a simple task. The computations are cumbersome, the formulas are too long and we would rather not give them here.

### 3.2.3 Asymptotic analysis

We will now present another approach of constructing approximations of the DtoN operator, based on the construction of an asymptotic expansion with respect to the thickness of thin layers.

#### Problem reformulation

The determination of the approximated DtoN operator by asymptotic expansions, based primarily on reformulating the problem (3.2), which helps eliminate the dependence of the problem geometry on the small parameter $h$. This can be done by the following change of variable:

$$t = \frac{s - \tilde{h}_j}{h_j} + j, \quad \tilde{h}_{j-1} \leq s \leq \tilde{h}_j, \quad j = 1, ..., p. \tag{3.32}$$

We then set

$$\mathcal{E}_j(t) = \mathcal{Y}_j(s), \quad j - 1 \leq t \leq j, \quad j = 1, ..., p. \tag{3.33}$$

We are now able to write the equations of the problem (3.2) verified by the new unknowns $\mathcal{E}_j(t), \ j = 1, ..., p.$

The derivative of $\mathcal{E}$ with respect to the new variable $t$ is written as

$$\tfrac{\partial}{\partial t}\mathcal{E}_j(t) = h_j \tfrac{\partial}{\partial s}\mathcal{Y}_j(s), \quad j - 1 \leq t \leq j, \quad j = 1, ..., p. \tag{3.34}$$

By inserting these formulas in the problem (3.2), we obtain

$$\left.\begin{array}{lll}
\frac{\partial}{\partial t}\mathcal{E}_1(t) = h_1\mathcal{M}_1\mathcal{E}_1(t) & \text{in } \mathcal{C}((0,1);X), & \\
\mathcal{E}_1(0) = [\phi_1,\phi_2]^{\mathrm{T}}, & & \\
 & & \\
\frac{\partial}{\partial t}\mathcal{E}_j(t) = h_j\mathcal{M}_j\mathcal{E}_j(t) & \text{in } \mathcal{C}((j-1,j);X), & j=2,\dots,p, \\
\mathcal{E}_j(j-1) = \mathcal{E}_{j-1}(j-1) & & j=2,\dots,p, \\
 & & \\
\mathcal{E}_p(p) = [\varphi_1,\varphi_2]^{\mathrm{T}}. & &
\end{array}\right\} \tag{3.35}$$

**Asymptotic expansion**

The thickness $h$ of the thin layers is assumed to be small enough. This allows us to postulate the existence of an asymptotic expansion for the solution of the problem (3.35) in the following form:

$$\mathcal{E}_j(t) = \sum_{l=0}^{\infty}\mathcal{E}_{j,l}(t)\,h^l, \quad j-1\le t\le j, \quad j=1,\dots,p, \tag{3.36}$$

where the functions $\mathcal{E}_{j,l}$ are independent of $h$.

By inserting these expressions in our problem (3.35) and formally identifying the same powers in $h^l$, it will lead to systems of equations that are independent of $h$. They allow to determine iteratively the terms of our asymptotic expansion.

We will start by writing the auxiliary problems arising from this formal identification in the equations of the problem (3.35).

$$\left.\begin{array}{lll}
\frac{\partial}{\partial t}\mathcal{E}_{1,0} = 0; \quad \frac{\partial}{\partial t}\mathcal{E}_{1,l} = \beta_1\mathcal{M}_1\mathcal{E}_{1,l-1}, l\ge 1 & \text{in } \mathcal{C}((0,1);X), & \\
\mathcal{E}_{1,0}(0) = [\phi_1,\phi_2]^{\mathrm{T}}; \quad \mathcal{E}_{1,l}(0) = 0, l\ge 1 & & \\
 & & \\
\frac{\partial}{\partial t}\mathcal{E}_{j,0} = 0; \quad \frac{\partial}{\partial t}\mathcal{E}_{j,l} = \beta_j\mathcal{M}_j\mathcal{E}_{j,l-1}, l\ge 1 & \text{in } \mathcal{C}((j-1,j);X), & j=2,\dots,p, \\
\mathcal{E}_{j,l}(j-1) = \mathcal{E}_{j-1,l}(j-1), l\ge 0, & & j=2,\dots,p.
\end{array}\right\} \tag{3.37}$$

Solving these equations allows us to proceed to the determination of DtoN operator approximations. We immediately observe that

$$\mathcal{E}_{1,l}(t) = \frac{1}{l!}(t\beta_1\mathcal{M}_1)^l [\phi_1, \phi_2]^{\mathsf{T}}, \ l \geq 0 \ \text{ in } \mathcal{C}((0,1);X).$$

Then solving iteratively for $\mathcal{E}_{j,l}$ we obtain

$$\mathcal{E}_{p,l}(t) = \left(\sum_{|\alpha|=l} \frac{(\mathbf{M}(t))^\alpha}{\alpha!}\right)[\phi_1, \phi_2]^{\mathsf{T}}, \ l \geq 0, \ \text{in } \mathcal{C}((p-1,p);X). \tag{3.38}$$

where $\mathbf{M}(t) = ((t-p+1)\beta_p\mathcal{M}_p, \beta_{p-1}\mathcal{M}_{p-1}, ..., \beta_2\mathcal{M}_2, \beta_1\mathcal{M}_1)$.

Recall that the asymptotic expansion of the solution $\mathcal{E}_p$ is given by

$$\mathcal{E}_p(t) = \sum_{l=0}^{\infty} \mathcal{E}_{p,l}(t) h^l.$$

Substituting $p$ for $t$ in $\mathcal{E}_p(t)$, we obtain

$$\mathcal{E}_p(p) = \sum_{l=0}^{\infty} \mathcal{E}_{p,l}(p) h^l = [\varphi_1, \varphi_2]^{\mathsf{T}}.$$

with

$$\mathcal{E}_{p,l}(p) = \left(\sum_{|\alpha|=l} \frac{\mathbf{M}^\alpha}{\alpha!}\right)[\phi_1, \phi_2]^{\mathsf{T}}, \ l \geq 0,$$

where $\mathbf{M} = \mathbf{M}(p) = (\beta_p\mathcal{M}_p, \beta_{p-1}\mathcal{M}_{p-1}, \ldots, \beta_2\mathcal{M}_2, \beta_1\mathcal{M}_1)$.

Finally we obtain the following formula:

$$\left(\sum_{l=0}^{\infty}\left(\sum_{|\alpha|=l} \frac{\mathbf{M}^\alpha}{\alpha!}\right) h^l\right)[\phi_1, \phi_2]^{\mathsf{T}} = [\varphi_1, \varphi_2]^{\mathsf{T}},$$

which allows to determine the asymptotic expansion of DtoN operator that is given by

$$S\phi_1 = (\sigma_1 Q_1 + \sigma_2 Q_2)^{-1}(\varphi - (\sigma_1 P_1 + \sigma_2 P_2)\phi_1),$$

where

$$\begin{bmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{bmatrix} = \sum_{l=0}^{\infty}\left(\sum_{|\alpha|=l} \frac{\mathbf{M}^\alpha}{\alpha!}\right) h^l.$$

We restrict the sum over $l$ from 0 to $n$ to get an approximation of order $n$. We observe that is the same formula as obtained in Taylor expansions.

We comment that if $\boldsymbol{\mathcal{Y}}_h$ is the solution obtained by solving the problem (3.1) with approximate boundary condition and $\boldsymbol{\mathcal{Y}}$ its exact solution, then we postulated the convergence in the following meaning:

$$\|\boldsymbol{\mathcal{Y}}_h - \boldsymbol{\mathcal{Y}}\|_{\mathbf{X}} \leq ch^r, \qquad c, r > 0.$$

## 3.3 Applications

In this section we apply the results obtained in approximating the DtoN operator, to a problem of scattering of a transverse electric ($TE$) electromagnetic wave by perfectly conducting planar obstacles, covered with thin homogenous dielectric multilayers.

In $TE$ electromagnetic waves, there will be simplifications in the components of electric and magnetic fields

$$E = \begin{bmatrix} E_x(x,y) \\ E_y(x,y) \\ 0 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 0 \\ H_z(x,y) \end{bmatrix}.$$

The total electromagnetic wave $(E, H)$ can be represented only by its non zero magnetic component, which is a scalar two variables $x$ and $y$ function denoted $u(x, y)$. In this case the Maxwell equations are reduced in the domains $\Omega^-, \Omega_j, j = 1, ..., p$ to Helmholtz equations

$$\triangle u_j + \kappa_j u_j = 0, \;\; j = 0, \ldots, p,$$

where $\kappa_j = k^2 \varepsilon_j \mu_j, j = 0, ..., p$.

The transmission conditions $[E \wedge n]_{\Gamma_j} = 0$ and $[H \wedge n]_{\Gamma_j} = 0$ that impose the connection of the components of the electromagnetic field can be reduced to

$$\frac{1}{\varepsilon_j}\partial_n u_j = \frac{1}{\varepsilon_{j+1}}\partial_n u_{j+1}, \quad u_j = u_{j+1} \text{ on } \Gamma_j, j = 0, \ldots, p-1,$$

where [.] denotes the jump across the boundary $\Gamma_j$.

The above conditions have to be complemented by the Silver-MÃ¼ller radiation condition

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r}\left(u_0-u_{\text{inc}}\right)-ik\left(u_0-u_{\text{inc}}\right)\right)=0,$$

and perfect conductor condition

$$\partial_n u_p=0 \text{ on } \Gamma_p,$$

with $r=\sqrt{x^2+y^2}$ and $u_{\text{inc}}$ is the incident wave.

Since the normal derivative $\partial_n$ in planar domains is the derivative with respect to $y$, then the scattering problem can be represented by the following scalar problem in dimension two.

$$\left.\begin{array}{ll}
\triangle u_p+\kappa_p u_p=0 & \text{in } \Omega_p,\\
\dfrac{\partial u_p}{\partial y}\left(x,-\tilde{h}_p\right)=0, & x\in\mathbb{R},\\
 & \\
\triangle u_j+\kappa_j u_j=0 & \text{in } \Omega_j,\quad j=1,\dots,p-1,\\
\dfrac{1}{\varepsilon_j}\dfrac{\partial u_j}{\partial y}\left(x,-\tilde{h}_j\right)=\dfrac{1}{\varepsilon_{j+1}}\dfrac{\partial u_{j+1}}{\partial y}\left(x,-\tilde{h}_j\right); & x\in\mathbb{R},\quad j=1,\dots,p-1,\\
u_j\left(x,-\tilde{h}_j\right)=u_{j+1}\left(x,-\tilde{h}_j\right) & x\in\mathbb{R},\quad j=1,\dots,p-1,\\
 & \\
\triangle u_0+\kappa_0 u_0=0 & \text{in } \Omega^-,\\
\dfrac{1}{\varepsilon_0}\dfrac{\partial u_0}{\partial y}(x,0)=\dfrac{1}{\varepsilon_1}\dfrac{\partial u_1}{\partial y}(x,0);\ u_0(x,0)=u_1(x,0), & x\in\mathbb{R},\\
\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial}{\partial r}\left(u_0-u_{\text{inc}}\right)-ik\left(u_0-u_{\text{inc}}\right)\right)=0, &
\end{array}\right\}$$

Note that this problem has been handled by a similar approach in [17] and [18]. For one thin layer, is also treated in [6] and they demonstrated the efficiency of using approximations of DtoN operator by numerical experiments.

Rewriting the Helmholtz equation $\triangle u_j+\kappa_j u_j=0$ in the form $\dfrac{-\partial}{\partial y}\left(\frac{1}{\varepsilon_j}\dfrac{\partial u_j}{\partial y}\right)=\frac{1}{\varepsilon_j}\left(\kappa_j+\dfrac{\partial^2}{\partial x^2}\right)u_j$ provides us an idea to set

$$\mathcal{Y}_j(s)=\left[U_j(s),V_j(s)\right]^{\mathrm{T}}=\left[u_j(x,-s),\frac{1}{\varepsilon_j}\frac{\partial u_j}{\partial y}(x,-s)\right]^{\mathrm{T}}.$$

Then the matrix $\mathcal{M}_j$ in the corresponding problem (3.2) will be

$$\mathcal{M}_j = \begin{bmatrix} A_j & G_j \\ F_j & B_j \end{bmatrix} = \begin{bmatrix} 0 & -\varepsilon_j \\ \dfrac{1}{\varepsilon_j}\left(\kappa_j + \dfrac{\partial^2}{\partial x^2}\right) & 0 \end{bmatrix}.$$

Since the perfect conductor condition $\dfrac{\partial u_p}{\partial y}\left(x, -\tilde{h}_p\right) = 0$ is a Neumann condition, then $\sigma_1 = \varphi = 0$ and $\sigma_2 \neq 0$.

In this case, the approximated DtoN operator of order $n$ is

$$S : \phi \longmapsto S\phi = -\left(Q_{2,n}\right)^{-1} P_{2,n}\phi, \tag{3.39}$$

where $[P_{2,n}, Q_{2,n}]$ is the second row of the $2 \times 2$ matrix $\tilde{M}_{p,n} = \sum_{l=0}^{n} N_l h^l$, $N_l = \sum_{|\alpha|=l} \dfrac{\mathbf{M}^\alpha}{\alpha!}$ with $\mathbf{M} = \left(\beta_p \mathcal{M}_p, \ldots, \beta_1 \mathcal{M}_1\right)$.

We can prove easily that $N_l$ is a diagonal matrix if $l$ is even and it is with zeros in its diagonal if $l$ is odd number. Consequently, $P_{2,n}$ has only odd powers of $h$ and $Q_{2,n}$ has only even ones.

**Remark 3.5** *If the scattered wave is transverse magnetic (TM), it will be reduced to the same problem with the condition* $u_p\left(x, -\tilde{h}_p\right) = 0$ *instead of* $\dfrac{\partial u_p}{\partial y}\left(x, -\tilde{h}_p\right) = 0$ *and substituting* $\mu$ *for* $\varepsilon$.

## 3.3.1 Approximation of order 0

Since $\sigma_1 = \varphi = 0$, the approximate DtoN operator of order 0 is simply $S\phi = 0$. The associated DtoN condition is $\dfrac{\partial u_0}{\partial y}(x, 0) = 0$, which corresponds to the case where the thin layers are completely neglected.

### 3.3.2 Approximation of order 1

The matrix $\tilde{M}_{p,1}$ for the approximation of order 1 is $\tilde{M}_{p,1} = N_0 + N_1 h$. As we have seen before, the matrices $N_0$ and $N_1$ are

$$N_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \; N_1 = \sum_{j=1}^{p} \beta_j \begin{bmatrix} A_j & G_j \\ F_j & B_j \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{G}_1 \\ \mathbf{F}_1 & 0 \end{bmatrix}, \tag{3.40}$$

where

$$\mathbf{G}_1 = -\sum_{j=1}^{p} \beta_j \varepsilon_j, \quad \mathbf{F}_1 = \sum_{j=1}^{p} \frac{\beta_j}{\varepsilon_j} L_j \text{ with } L_j = \kappa_j + \frac{\partial^2}{\partial x^2}. \tag{3.41}$$

Since $\sigma_1 = \varphi = 0$ and $\mathbf{A}_1 = \mathbf{B}_1 = 0$, then the approximate DtoN operator of order 1 is

$$S : \phi \longmapsto S\phi = -\mathbf{F}_1 h \phi.$$

### Approximation of order 2

The matrix $\tilde{M}_{p,2}$ for the approximation of order 2 is $\tilde{M}_{p,2} = N_0 + N_1 h + N_2 h^2$. Using the results obtained in the formulas (3.25)-(3.31) we see that the matrix $N_2$ can be written as

$$N_2 = \begin{bmatrix} \mathbf{A}_2 & 0 \\ 0 & \mathbf{B}_2 \end{bmatrix},$$

where

$$\mathbf{A}_2 = -\sum_{j=1}^{p} \left( \tfrac{1}{2} \beta_j \varepsilon_j + \sum_{k=j+1}^{p} \beta_k \varepsilon_k \right) \frac{\beta_j}{\varepsilon_j} L_j, \quad \mathbf{B}_2 = -\sum_{j=1}^{p} \left( \tfrac{1}{2} \beta_j \varepsilon_j + \sum_{k=1}^{j-1} \beta_k \varepsilon_k \right) \frac{\beta_j}{\varepsilon_j} L_j.$$

Thus, the approximate DtoN operator of order 2 is

$$S : \phi \longmapsto S\phi = -\left(1 + \mathbf{B}_2 h^2\right)^{-1} \mathbf{F}_1 h \phi. \tag{3.42}$$

Let's take a look at the higher order. We will provide the approximation of order 4. That of order 3 can be recovered by replacing $h^4$ by 0.

## Approximation of order 4

As we mentioned above the matrices $N_3$ and $N_4$ are in the form

$$N_3 = \begin{bmatrix} 0 & \mathbf{G}_3 \\ \mathbf{F}_3 & 0 \end{bmatrix}, \; N_4 = \begin{bmatrix} \mathbf{A}_4 & 0 \\ 0 & \mathbf{B}_4 \end{bmatrix}.$$

Since the approximated DtoN operator of order 4 is

$$S : \phi \longmapsto S\phi = -\left(1 + \mathbf{B}_2 h^2 + \mathbf{B}_4 h^4\right)^{-1} \left(\mathbf{F}_1 h + \mathbf{F}_3 h^3\right) \phi, \tag{3.43}$$

we need to calculate only $\mathbf{F}_3$ and $\mathbf{B}_4$.

Using the general formula (3.24) for calculating $N_l$ we see that

$$\mathbf{F}_3 = -\sum_{i,j,i>j}^{p} \omega_{ij} \tfrac{\beta_i \beta_j}{\varepsilon_i \varepsilon_j} L_j L_i - \tfrac{1}{6} \sum_{j=1}^{p} \tfrac{\beta_j^3}{\varepsilon_j} L_j^2,$$
$$\mathbf{B}_4 = \sum_{i,j,i>j}^{p} \gamma_{ij} \tfrac{\beta_i \beta_j}{\varepsilon_i \varepsilon_j} L_j L_i + \tfrac{1}{6} \sum_{j=1}^{p} b_j \tfrac{\beta_j^3}{\varepsilon_j} L_j^2,$$

where

$$\omega_{ij} = \tfrac{1}{2}\left(\beta_i \varepsilon_i + \beta_j \varepsilon_j\right) + \sum_{k=j+1}^{i-1} \beta_k \varepsilon_k, \quad \gamma_{ij} = \omega_{ij} a_j - \tfrac{1}{12} \beta_j^2 \varepsilon_j^2,$$
$$a_j = \tfrac{1}{2} \beta_j \varepsilon_j + \sum_{k=1}^{j-1} \beta_k \varepsilon_k, \quad b_j = \tfrac{1}{4} \beta_j \varepsilon_j + \sum_{k=1}^{j-1} \beta_k \varepsilon_k.$$

An approximation of order 4 that is a polynomial with respect to the thickness of thin layers is

$$S : \phi \longmapsto S\phi = \left(-\mathbf{F}_1 h - \left(\mathbf{F}_3 - \mathbf{F}_1 \mathbf{B}_2\right) h^3\right) \phi,$$

where

$$\mathbf{F}_3 - \mathbf{F}_1 \mathbf{B}_2 = 2 \sum_{i,j,i>j}^{p} a_j \tfrac{\beta_i \beta_j}{\varepsilon_i \varepsilon_j} L_j L_i + \sum_{j=1}^{p} c_j \tfrac{\beta_j^2}{\varepsilon_j^2} L_j^2,$$

with

$$a_j = \tfrac{1}{2} \beta_j \varepsilon_j + \sum_{k=1}^{j-1} \beta_k \varepsilon_k, \quad c_j = \tfrac{1}{3} \beta_j \varepsilon_j + \sum_{k=1}^{j-1} \beta_k \varepsilon_k.$$

In the case where the scattered wave is transverse magnetic ($TM$), the approximated DtoN operator of order 4 is

$$S : \phi \longmapsto S\phi = -\left(\mathbf{G}_1 h + \mathbf{G}_3 h^3\right)^{-1}\left(1 + \mathbf{A}_2 h^2 + \mathbf{A}_4 h^4\right)\phi, \tag{3.44}$$

where

$$\begin{aligned} &\mathbf{G}_1 = -\sum_{i=1}^{p} \beta_i \mu_i, \;\; \mathbf{A}_2 = -\sum_{i=1}^{p} a_i \tfrac{\beta_i}{\mu_i} L_i, \;\; \mathbf{G}_3 = \sum_{i=1}^{p} c_i \tfrac{\beta_i}{\mu_i} L_i, \\ &\mathbf{A}_4 = \sum_{i,j,i>j}^{p} \gamma_{ij} \tfrac{\beta_i \beta_j}{\mu_i \mu_j} L_j L_i + \tfrac{1}{6} \sum_{i=1}^{p} b_i \tfrac{\beta_i^3}{\mu_i} L_i^2, \end{aligned}$$

with

$$\begin{aligned} &\omega_{ij} = \tfrac{1}{2}\left(\beta_i \mu_i + \beta_j \mu_j\right) + \sum_{k=j+1}^{i-1} \beta_k \mu_k, \quad \gamma_{ij} = \omega_{ij} a_i - \tfrac{1}{12} \beta_i^2 \mu_i^2, \\ &a_i = \tfrac{1}{2} \beta_i \mu_i + \sum_{k=i+1}^{p} \beta_k \mu_k, \;\; b_i = \tfrac{1}{4} \beta_i \mu_i + \sum_{k=i+1}^{p} \beta_k \mu_k, \\ &c_i = \left(\tfrac{1}{2} \beta_i \mu_i + \sum_{k=1}^{i-1} \beta_k \mu_k\right) a_i - \tfrac{1}{12} \beta_i^2 \mu_i^2. \end{aligned}$$

# Conclusion and perspectives

The scattering problems of electromagnetic waves by a perfectly conducting obstacle coated with thin multilayered dielectric materials can be transformed, with the help of curvilinear coordinates, into evolution problems and in planar obstacles into abstract Cauchy problems. The use of impedance operator replaces the effect of the thin layers by a boundary condition. The main result in this thesis is analyzing the construction and the approximation of this operator in the case of planar obstacles using two approaches: Taylor and asymptotic expansions.

Obtaining the expression of approximated impedance operator for higher order is not such a simple task. The computations are cumbersome and the formulas are too long.

The study presented here can be extended to arbitrarily shaped obstacles, but simplifying the expressions remains a challenge.

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