A novel decomposition scheme to solve parametric nonconvex programs as they arise in Nonlinear Model Predictive Control (NMPC) is presented. It consists of a fixed number of alternating proximal gradient steps and a dual update per time step. Hence, the proposed approach is attractive in a real-time distributed context. Assuming that the Nonlinear Program (NLP) is semi-algebraic and that its critical points are strongly regular, contraction of the sequence of primal-dual iterates is proven, implying stability of the sub-optimality error, under some mild assumptions. Moreover, it is shown that the performance of the optimality-tracking scheme can be enhanced via a continuation technique. The efficacy of the proposed decomposition method is demonstrated by solving a centralized NMPC problem to control a dc motor and a distributed NMPC program for collaborative tracking of unicycles, both within a real-time framework. Furthermore, an analysis of the sub-optimality error as a function of the sampling period is proposed given a fixed computational power.