The need of fast distributed solvers for optimization problems in networked systems has motivated the recent development of the Fast-Lipschitz optimization framework. In such an optimization, problems satisfying certain qualifying conditions, such as monotonicity of the objective function and contractivity of the constraints, have a unique optimal solution obtained via fast distributed algorithms that compute the fixed point of the constraints. This paper extends the set of problems for which the Fast-Lipschitz framework applies. Existing assumptions on the problem form are relaxed and new and generalized qualifying conditions are established by novel results based on Lagrangian duality. It is shown for which cases of more constraints than decision variables, and less constraints than decision variables Fast-Lipschitz optimization applies. New results are obtained by imposing non strict monotonicity of the objective functions. The extended Fast-Lipschitz framework is illustrated by a number of examples, including network optimization and optimal control problems.