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par

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Titre

Combinatorial interpretations of classical sequences in graphs

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To my grand-mother ,

my parents and,

my husband

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Interprétation combinatoire des suites classiques dans les graphes

Résumé : Des questions de nature énumérative s'inscrivant dans la combinatoire des suites qui ont un lien directe avec des concepts connus en théorie des graphes, telle que la coloration, ont été posées voire étudiées dans la littérature et ont mené à des recherches fructueuses dans ces domaines. Notre travail s'inscrit dans ce cadre, on se focalise sur les suites de partitions telles que les nombres de Stirling de deuxième espèce, les nombres de Bell et leurs généralisations et / ou restrictions. Des interprétations combinatoires des nombres de Stirling et de Bell dans les graphes ont été proposées voire étudiées pour des classes connues de graphes à travers l'énumération des partitions en stables. Dans une première partie, s'inspirant de ces travaux, on donne une extension de ces interprétations à d'autres classes de graphes en introduisant les nombres Stirling généralisés, cela nous a permis d'établir de nouvelles identités combinatoires et formules explicites concernant les nombres de Stirling, de r-Stirling de seconde espèce, de Bell et de r-Bell. Dans la seconde partie, on donne l'interprétation des nombres de Stirling associés en définissant un nouveau type de partition en utilisant le concept de resolving sets.

Mots clés : Théorie des graphes, Combinatoire énumérative, Nombre de Stirling de deuxième espèce, Nombre de Bell, Partition d'ensemble, Nombre de r-Stirling de deuxième espèce, Nombre de Stirling associé de deuxième espèce, Partition indépendante, Polynome Chromatique.

List of publications and submitted works

- **A. Medjerredine**, H. Belbachir and M. A. Boutiche. Enumerating Some Stable Partitions Involving Stirling and r-Stirling Numbers of the Second Kind Mediterr. J. Math. (2018) 15: 87.
- **A. Medjerredine**, H. Belbachir and M. A. Boutiche. Bell and Stilring numbers for some join graphs, submitted.
- **A.Medjerredine** and H. Belbachir. Enumerating stable partitions for some corona graphs. Submitted.
- **A. Medjerredine** and H. Belbachir. Enumerating resolving set partitions involving the 2-associated Stirling numbers. Submitted

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Notation

$\mathbb{N}$	set of positive integers;
$[n]$	the set $\{1, 2, \ldots, n\}$, for $n \in \mathbb{N}$;
$:=$	equality (affectation);
$\bigcup$	union of sets;
$\delta_{i,j}$	Kronecker delta equal to 1 if $i = j$ and 0 else;
$\lfloor x \rfloor$	lower integer part of x;
$\lceil x \rceil$	upper integer part of x;
$x^{\overline{n}}$	ascending factorial: $x(x+1)\cdots(x+n-1)$ for $n \in \mathbb{N}$ and $x^{\bar{0}} = 1$;
$(x)_n$	falling factorial: $x(x-1)\cdots(x-n+1)$ for $n \in \mathbb{N}$ and $x^{\bar{0}} = 1$;
$\binom{n}{k}$	binomial coefficient: $\frac{n!}{k!(n-k)!}$;
F_n	Fibonacci numbers;
L_n	Lucas numbers;
$S(n, k)$	Stirling numbers of the second kind;
$S^{(s)}(n, k)$	s-associated Stirling numbers of the second kind;
$S_r(n, k)$	r-Stirling numbers of the second kind;
$S(G, k)$	graphical Stirling number of G;
$G + H$	join union of G and H;
$G \cup H$	disjoint union of G and H;
GoH	the corona product of G and H;
G/e	the graph obtained after contracting e;
$G - e$	the graph obtained after deleting e;

(G)	chromatic number of G;
$P(G,\lambda)$	chromatic polynomial of G;
P_n	a path of order n;
E_n	an empty graph of order n;
C_n	a cycle of order n;
S_n	a star of order n;
W_n	a wheel of order n;
T_n	a tree of order n;
T_n^m	a generalized m-tree of order n.

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Introduction

This thesis entitled "Combinatorial interpreting classical sequences in graphs" is related to both graph theory and enumerative combinatorics fields.

Enumerative combinatorics is a branch of combinatorial theory, regarded as the art of counting mathematical objects. Also, graph theory is considered among the branches of combinatorial theory overlapping enumerative combinatorics through the enumeration of graph-theoretic operations.

Many questions in enumerative combinatorics have direct link with graph theoretical concepts, mainly the coloring of graphs. Conversely, several problems in graph theory have combinatorial interpretations involving classical sequences, namely Stirling numbers, Bell numbers, Fibonacci numbers and their restrictions and / or generalizations. Prodinger and Tichy were the first who published a work on interpreting classical sequences in terms of graphs starting with Fibonacci sequence in terms of graphs in [54], where he described the sequence $\{1, \ldots, n\}$ as the vertex set of the path of order n, P_n and $Fib(G)$ the total number of subsets S such that any two vertices of S are not adjacent. Also, important identities related to Lucas numbers have been establihed by Belbachir and Harik [6] and Belbachir et al [7] using elements of graph theory.

Additionally, counting the number of ways to color a simple and finite graph is of interest, besides the fact that it constitutes an important area in graph theory, it gives also attractive questions in enumerative combinatorics.

For $G = (V, E)$ being a simple and finite graph of order n, with V and E are the vertex and the edge set respectively, it is well established by Duncan in 2009 [27] that we can count the number of ways to color some simple and finite graphs with exactly k colors using Stirling numbers of the second kind, while the total number of these colorings is counted by the Bell numbers. The Stirling numbers of the second kind, denoted by $S(n, k)$ count the number of partitions of an n-element set into k non empty subsets and the Bell number is the total number of these partitions, that's why Duncan referred to such colorings as Stirling numbers for graphs or graphical Stirling numbers denoted by $S(G, k)$, $1 \leq k \leq n$.

On the other hand, it is well known that coloring G with k colors amounts to partition their vertex set into k stables (independent vertex sets), thus $S(G, k)$ counts equivalently the number of ways to partition the vertex set of G into k non empty stable sets.

Moreover, the number of such partitions was first introduced and investigated by Duncan and Peel, they have been studied later by [33, 29, 47, 35] although the original idea was due to Yang [69] which was related to Bell numbers for a family of graphs named k-trees.

Furthermore, the polynomial related to these partitions is called the Chromatic polynomial, it was first introduced by Birkhoff in 1912, [10] in an attempt to solve the four color problem, it counts the number of colorings of a graph using at most λ colors. $S(G, k)$ constitutes the coefficients in the chromatic polynomial formula described as follows

$$P(G, \lambda) = \sum_{k=\chi(G)}^{n} S(G, k)(\lambda)_k, \tag{1}$$

where, $(\lambda)_k = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - k + 1)$ for $k \geq 1$ and $(\lambda)_0 = 1$ is the falling factorial of λ.

Our work is presented in two parts: in the first one consisting of four chapters; inspired by the previous works, we extend the interpretation to other families of graphs, in which we show that counting the number of ways to partition some simple graphs depends not only on the Stirling numbers of the second kind but also on the r-Stirling numbers of the second kind, thus an alternative way to provide the

Chromatic polynomial for some classes of graphs, using this fact we establish new identities and explicit formulas related to the Stirling and the r-Stirling numbers of the second kind.

In the second part, we respond to the question: What are the concepts that allow to interpret the associated Stirling numbers of the second kind? Here, we provide a combinatorial interpreting of the 2-associated Stirling numbers [4, 24, 8, 34] in terms of graphs using resolving sets concept.

The document is presented in five chapters described as follows:

In chapter one

Firstly, we give some notations and definitions regarding partition sequences that we need later, in which we propose an overview on Stirling numbers, r-Stirling numbers, associated Stirling numbers of the second kind and Bell numbers, we present some of their properties, their generating functions and examples to illustrate the meaning and we cite by the way some references in this area. As mentioned above, these sequences can be interpreted in terms of graphs, So far as our work is concerned, we provide basic notions and definitions of some elements in graph theory.

In chapter two

We present combinatorial interpretations of Stirling and Bell numbers using elements of graph theory were done by the previously cited authors in different guises. In a far deeper way, this chapter is devoted to show the basic results established for the Stirling and the Bell numbers for special classes of graphs, some technics used in are shown.

In chapter three

We establish that the number of stable partitions into k stable sets for many graph classes such as the thorn graphs, their generalization, n-cyclic graphs and a generalization of cyclic graphs. As consequences, we provide new identities concerning Stirling numbers of the second kind, besides that, explicit formulas related to the generalized Stirling are established, new identities and explicit formulas for Bell and r-Bell numbers are given also.

In chapter four

We apply operations on graphs, namely, join graphs and the corona product, we give explicit formulas to calculate Stirling and Bell numbers for join graphs and some special corona graphs and some known identities related to Lucas and Fibonacci sequences are deduced.

In chapter five

This last chapter is devoted to the 2-associated Stirling numbers of the second kind, in which we interpret these numbers in terms of graphs using resolving set concept and we introduce a resolving set partitions, we show that the 2-associated Stirling numbers are graph theoretically interpreted introducing partition into resolving sets. Conversely, we prove that the number of such partitions for paths and cycles coincide with the 2-associated Stirling numbers of the second kind.

1

Some elements of graph theory and enumerative combinatorics

Outlines

1.1 Introduction

In this Chapter, we shall present at first some notations and definitions in enumerative combinatorics concerning some partition sequences namely the Stirling numbers of the second kind, the Bell numbers and their generalizations. We give their recurrence relations, generating functions and some explicit formulas. Next, we remind the reader of some basic notions in graph theory that we will need in our study and finally we give the graph theoretic encoding of the discussed partition sequences, we propose by the way some examples to illustrate the meaning and we cite some references in this field.

1.2 Partitions and partition sequences

Set partition problems constitute an important area in the field of mathematics specially in enumerative combinatorics, also it is an early concept in number theory, its history began with the Japaneese in the 16^{th} century, considering a game named "genji-ko", described in [43], we cite also the work of Saka 1782 in this context.

Many authors [16, 4, 24, 34, 31, 58, 66] discuss the set partitions and treat the sequences that enumerate these partitions in enumerative combinatorics and others subject to some constraints, namely the Stirling numbers of the second kind, the Bell numbers, the Lah numbers and their generalizations. These sequences have attracted a great attention and interest by many researchers through the years. Intensive investigations were given to the study of their properties, their generating functions, recurrence relations, explicit formulas and many other studies by several researchers, see for instance [Jordan, [24]. all of these partition sequences are intimately linked through the roles they play in enumerating n set partitions. In this area, Mansour in [43] gave an important and a detailed document on enumeration of set partitions.

1.2.1 Stirling numbers of the second kind

As appears in [13], Stirling numbers of the second kind were so named by Nielsen [50] in honor of James Stirling, who computed them in his Methodus Differentialis, [63] in 1730. They arise as coefficients when expressing monomials in terms of falling polynomials. Therefore, they allow to move from the power form to the falling factorial form.

Definition 1. *Let n, k be two non-negative integers. For $k \leq n$, a set $B = \{B_1, \cdots, B_k\}$ is said to be a partition of [n] into k blocks if all B_i's are non empty, their intersection is the empty set and their union gives the whole set B.*

Definition 2. *For n, k two positive integers such that $k \leq n$ the Stirling number of the second kind is the number of set partitions of $[n]$ into k blocks.*

In what follows, we shall use $S(n, k)$ to denote this sequence. This notation was introduced in 1973 by Knuth [38] and used later by Graham, Knuth and Patashnik [31].

We agree that in trivial cases we have, $S(0, 0) = 1$, $S(n, 1) = 1$ for $n \geq 1$, $S(n, n) = 1$, $S(n, n-1) = \binom{n}{2}$ and $S(n, 2) = 2^{n-1} - 1$.

Example 3. *Now, let us use $B_1/B_2/\ldots/B_k$ to denote the blocks of the partition.*

1. $S(5, 2) = 15$,

   the possible situations are: $1, 2/3, 4, 5$; $1, 3/2, 4, 5$; $1, 4/2, 3, 5$; $1, 5/2, 3, 4$; $2, 3/1, 4, 5$; $2, 4/1, 3, 5$; $2, 5/1, 3, 4$; $3, 4/1, 2, 5$; $3, 5/1, 2, 4$; $4, 5/1, 2, 3$; $1/2, 3, 4, 5$; $2/1, 3, 4, 5$; $3/1, 2, 4, 5$; $4/1, 2, 3, 5$; $5/1, 2, 3, 4$.

2. $S(4, 3) = 6$,

   the possible situations are $1/2/3, 4$; $1/3/2, 4$; $1/4/2, 3$; $2/3/1, 4$; $2/4/1, 3$; $3/4/1, 2$.

The Stirling numbers of the second kind satisfy the triangular recurrence relation:

$$S(n, k) = S(n-1, k-1) + kS(n-1, k).$$

This recurrence can be done by direct combinatorial argument; by considering the n'th element, either it forms a block alone, so we have to partition the $n-1$ remaining elements into $k-1$ blocks, or it belongs to a block already formed, in this case we assign this element to the formed blocks.

n k	1	2	3	4	5	6
1	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
6	1	31	90	65	15	1

Table 1.1: Triangle of Stirling numbers of the second kind

We have the following explicit formula:

$$S(n,k) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{(k-j)}\binom{k}{j}j^n,$$

where $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ is the classical binomial coefficient with the convention that $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.

The ordinary generating function of Stirling numbers of the second kind is:

$$\sum_{n\geq k}^{\infty} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

Their exponential generating function is:

$$\sum_{n=k}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x-1)^k.$$

The Stirling numbers constitute the coefficients that allows to move from the falling factorials form to the power form, described as:

$$\sum_{k=0}^{n} S(n,k)x^{\underline{k}} = x^n.$$

Further properties with nice presentations can be found in [24, 56, 16].

1.2.2 Bell numbers

Bell numbers were named after Eric Temple Bell. They were discovered by James Stirling in purely algebraic configuration, [24].

They have many interesting properties and appear in several combinatorial identities [45]. A comprehensive paper is done by Aigner in [1].

Definition 4. *The Bell number counts the number of all set partitions of* $[n]$*, it is denoted by* B_n *and can be expressed as*

$$B_n = \sum_{k=0}^{n} S(n,k).$$

Example 5. $B_5 = \sum_{k=0}^{5} S(5,k) = S(5,1) + S(5,2) + S(5,3) + S(5,4) + S(5,5)$.

We have one possible way to partition $[5]$ *to one block:* $1/2/3/4/5$*, thus* $S(5,1) = 1$*, 15 possible cases for* $S(5,2)$ *:* $1,2/3,4,5$*;* $1,3/2,4,5$*;* $1,4/2,3,5$*;* $1,5/2,3,4$*;* $2,3/1,4,5$*;* $2,4/1,3,5$*;* $2,5/1,3,4$*;* $3,4/1,2,5$*;* $3,5/1,2,4$*;* $4,5/1,2,3$*;* $1/2,3,4,5$*;* $2/1,3,4,5$*;* $3/1,2,4,5$*;* $4/1,2,3,5$*;* $5/1,2,3,4$*,* 25 *cases for* $S(5,3)$*:* $1/2/3,4,5$*;* $1/3/2,4,5$*;* $1/4/2,3,5$*;* $1/5/2,3,4$*;* $2/3/1,4,5$*;* $2/4/1,3,5$*;* $2/5/1,3,4$*;* $3/4/1,2,5$*;* $3/5/1,2,4$*;* $4/5/1,2,3$*;* $1,2/3/4,5$*;* $1,2/4/3,5$*;* $1,2/5/3,4$*;* $1,3/2/4,5$*;* $1,3/4/2,5$*;* $1,3/5/2,4$*;* $1,4/2/3,5$*;* $1,4/3/2,5$*;* $1,4/5/2,3$*;* $1,5/2/3,4$*;* $1,5/3/2,4$*;* $1,5/4/2,3$*;* $2,3/1/4,5$*;*$2,4/1/3,5$*;* $2,5/1/3,4$*,* 10 *possibility for* $S(5,4)$ *and one possible configuration for* $S(5,5)$*, hence,* $B_5 = 1 + 15 + 25 + 10 + 1 = 52$*.*

Bell polynomials are defined as

$$B_n(x) := \sum_{k=1}^{n} S(n,k)x^k.$$

The Bell numbers satisfy the recurrence:

$$B_n = \sum_{i=0}^{n-1} \binom{n-1}{i} B_i.$$

It would be interesting to show the proof of this recurrence in order to familiarize the reader with the combinatorial arguments when it comes to graphs.

Proof. To partition the set $[n]$, consider the element n. If n forms a block alone, which means we have no element to choose from the remaining elements to form that block, then we have $\binom{n-1}{0}$ ways to form that block and we have b_{n-2} ways to partition the $n-1$ remaining elements. If n belongs to a block that contains one other element, in this case, we have $\binom{n-1}{1}$ possibilities to form that block and b_{n-2} possible ways to partition the remaining elements. In general, if n belongs to a block that contains $k-1$ other elements, then there $\binom{n-1}{k-1}$ ways to form the block and b_{n-k} ways to partition the $n-1-(k-1)$ remaining elements. □

The Bell polynomial is an important concept that had a specific focus by many mathematicians mainly in combinatorial analysis [56, 45].

1.2.3 The r-Stirling numbers of the second kind

The so called restricted or generalized r-Stirling numbers of the second kind have been first defined and studied by Broder in 1984, who gave in [13] combinatorial interpretations and many algebraic properties.

Definition 6. *The r-Stirling numbers of the second kind count the restricted set partitions of $[n]$ into k blocks such that the r first elements belong to distinct blocks. We use $S(n,k)_r$, $n,k,r \geq 0$ to denote these numbers.*

Example 7. *As shown in Example 1, we have 15 possible cases for $S(5,2)$, the block partitions are indeed:* $1,2/3,4,5$; $1,3/2,4,5$; $1,4/2,3,5$; $1,5/2,3,4$; $2,3/1,4,5$; $2,4/1,3,5$; $2,5/1,3,4$; $3,4/1,2,5$; $3,5/1,2,4$; $4,5/1,2,3$; $1/2,3,4,5$; $2/1,3,4,5$; $3/1,2,4,5$; $4/1,2,3,5$; $5/1,2,3,4$.
For $r=2$, $S_2(5,2)=8$ with the block partitions: $1,3/2,4,5$; $1,4/2,3,5$; $1,5/2,3,4$; $2,3/1,4,5$; $2,4/1,3,5$; $2,5/1,3,4$; $1/2,3,4,5$; $2/1,3,4,5$. *That is, we have to eliminate those where the elements 1 and 2 share the same block.*

Obviously, some particular values are:

$$S_r(n,k) = 0, \ \texttt{for} \ n < r,$$

$$S_r(n,k) = \delta_{k,r}, \ \texttt{for} \ n = r,$$

$$S_r(n,r) = r^{n-r}, \ \texttt{for} \ n \geq r \ \texttt{and} \ k = r.$$

For $n > r$, they satisfy the recurrence relation:

$$S_r(n,k) = S_r(n-1,k-1) + kS_r(n-1,k)$$

For $r=1$ and $r=0$, these numbers coincide with the classical Stirling numbers of the second kind.

n k	2	3	4	5	6	7
2	1					
3	2	1				
4	4	5	1			
5	8	19	9	1		
6	16	65	55	14	1	
7	32	211	285	125	20	1

Table 1.2: Triangle of 2-Stirling numbers of the second kind

n k	3	4	5	6	7	8
3	1					
4	3	1				
5	9	7	1			
6	27	37	12	1		
7	81	175	97	18	1	
8	243	781	660	205	25	1

Table 1.3: Triangle of 3-Stirling numbers of the second kind

The r-Stirling numbers of the second kind have the explicit formula, for $n, k \geq 0$

$$S(n+k,k)_r = \sum_{1 \leq i_1 \leq \cdots \leq i_n \leq k} i_1 \cdots i_n,$$

The ordinary generating function of the restricted Stirling numbers of the second kind is

$$\sum_k S_r(n,k) z^n = \begin{cases} \frac{z^k}{(1-rz)(1-(r+1)z)\cdots(1-kz)} & \text{for } 0 \leq r \leq k \\ 0 & \text{otherwise.} \end{cases}$$

1.2.4 The r-Bell numbers

The r Bell numbers or in general the r-Bell polynomials appear in different combinatorial identities. Carlitz [14, 15] defined these numbers and proved some identities for them. A specific focus was given by [45] who published a paper which is entirely devoted to these numbers.

Definition 8. *The r-Bell numbers of the second kind count the set partitions of $[n]$ such that the r first elements belong to distinct blocks. We use $B_{n,r}$ to denote this sequence. They are expressed in terms of the r-Stirling numbers of the second kind as follows:*

$$B_{n,r} = \sum_{k=0}^{n} S(n,k)_r.$$

Example 9. $B_{5,2} = \sum_{k=0}^{5} S_2(5,k) = S_2(5,1) + S_2(5,2) + S_2(5,3) + S_2(5,4) + S_2(5,5)$.

We have one possible way to partition $[5]$ *to one block so that the two first elements belong to distinct blocks:* $1/2/3/4/5$, *thus* $S_2(5,1) = 1$, 8 *possible cases for* $S(5,2)$ *with the possible situations* $1,3/2,4,5$; $1,4/2,3,5$; $1,5/2,3,4$; $2,3/1,4,5$; $2,4/1,3,5$; $2,5/1,3,4$; $1/2,3,4,5$; $2/1,3,4,5$, 19 *cases for* $S_2(5,3)$ *with* $1/2/3,4,5$; $1/3/2,4,5$; $1/4/2,3,5$; $1/5/2,3,4$; $2/3/1,4,5$; $2/4/1,3,5$; $2/5/1,3,4$; $1,3/2/4,5$; $1,3/4/2,5$; $1,3/5/2,4$; $1,4/2/3,5$; $1,4/3/2,5$; $1,4/5/2,3$; $1,5/2/3,4$; $1,5/3/2,4$; $1,5/4/2,3$; $2,3/1/4,5$;$2,4/1/3,5$; $2,5/1/3,4$, 9 *possibility for* $S_2(5,4)$*:* $1,3/2/4/5$; $1,4/2/3/5$; $1,5/2/3/4$; $2,3/1/4/5$; $2,4/1/3/5$;

$2,5/1/3/4$; $3,4/1/2/5$; $3,5/1/2/4$; $4,5/1/2/3$ *and one possible configuration for* $S_2(5,5)$*, hence,* $B_{5,2} = 1 + 8 + 19 + 9 + 1 = 38$.

We deduce by definition that $B_{n,0} = B_n$ as $S(n,k) = S_0(n,k)$

The r-Bell polynomials are defined

$$B_{n,r}(x) = \sum_{k=0}^{n} S(n+r, k+r)_r x^k,$$

and satisfy

$$B_{n,r}(x) = \sum_{k=0}^{n} \sum_{i=0}^{n} \binom{n}{i} S(i,k) r^{n-i} x^k.$$

Also, the r-Bell polynomials are expressed in terms of the Bell polynomial

$$B_{n,r}(x) = \sum_{k=0}^{n} r^k \binom{n}{k} B_{n-k}(x).$$

We must note that $B_{n,r}(1) = B_{n,r}$. Thus, the r-Bell numbers can be deduced.

1.2.5 The s-associated Stirling numbers of the second kind

It is worth defining the so called s-associated Stirling numbers of the second kind in general, although our focus in what follows is limited to the 2-associated Stirling numbers of the second kind.

The s-associated Stirling numbers are defined to be the Stirling numbers of the second kind adding a restriction to the number of elements per blocks. They have been first introduced by Riordan [56]. Comtet [24] defined them later and gave other properties. For other detailed studies see [4, 24, 34].

Definition 10. *The s-associated Stirling numbers of the second kind count the number of partitions of an n-element set into k non empty subsets such that each subset contains at least s elements, we use $S^{(s)}(n,k)$, $n,k,s \geq 0$ to denote these numbers.*

$S^{(s)}(0,0) = 1$, $S^{(s)}(n,0) = 0$, $n \geq 1$ and $S^{(1)}(n,k) = S(n,k)$.

Example 11. *1.* $S^{(3)}(5,2) = 0$*, since* $s.k = 3.2 \geq 5$.

k n	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1
2			3	10	25	56	119
3					15	105	490
4							105

Table 1.4: Triangle of 2-associated Stirling numbers of the second kind

2. *On the other hand,* $S^{(2)}(5,2) = 10$ *with the block partitions* $1,2/3,4,5$*;* $1,3/2,4,5$*;* $1,4/2,3,5$*;* $1,5/2,3,4$*;* $2,3/1,4,5$*;* $2,4/1,3,5$*;* $2,5/1,3,4$*;* $3,4/1,2,5$*;* $3,5/1,2,4$*;* $4,5/1,2,3$.

They obey to the triangular recurrence relation:

$$S(n,k) = kS^{(s)}(n-1,k) + \binom{n-1}{s-1} S^{(s)}(n-s,k-1).$$

They can be done by the following explicit formula

$$S^{(s)}(n,k) = \frac{n!}{k!} \sum_{i_1+i_2+\cdots+i_k=n} \frac{1}{i_1!i_2!\cdots i_k!}, i_j \geq s.$$

For $n = sk$, we get

$$S^{(s)}(sk,k) = \frac{(sk)!}{k!(s!)^k}.$$

Their mixed generating function is

$$\sum_{n,k\geq 0} S^{(s)}(n,k)u^k \frac{t^n}{t!} = exp\{u(\frac{t^s}{s!} + \frac{t^{s+1}}{s+1!} + \cdots)\}.$$

1.3 Some elements of graph theory

1.3.1 Some basic definitions

A graph $G = (V, E)$ consists of a non-empty set $V(G)$ and a (possibly empty) set $E(G)$ of unordered pairs of elements of $V(G)$. The elements of $V(G)$ are called vertices and the elements of $E(G)$ are called edges. If $|V(G)| = n$, then the graph is of order n and if $|E(G)| = m$, then G is said to be of size m. A finite graph is a graph with finite vertex set.

A loop is an edge joining a vertex to itself.

A graph is called a multigraph if it contains multiple edges i.e more than one edge can join some pairs of vertices in G.

A simple graph is a graph without any loops and multiple edges.

All graphs we consider in this document are simple and finite. We may use the notation G_n rather than G to describe a graph of order n.

If there exist an edge joining two vertices u and v in G, then u and v are called adjacents or neighbors, else they are called disjoints. The neighborhood of u in G is the set of all the neighbors of u and is denoted by $N(u)$.

If $v \in V(G)$ has only one neighbor, then it is said to be pendant.

A subgraph of G is a graph having all of its vertices and edges in G. In other words, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H, is a subgraph of G, then G is a supergraph of H.

A spanning subgraph is a subgraph containing all the vertices of G.

An induced subgraph S in G is the subgraph such that their vertices are the vertices of S and the edges are those of G having their ends in S, we use $< S >$ to denote the subgraph induced by the vertex set of S.

For $G = (V, E)$ and $S \subset V$, S is said to be a clique if their vertices are all pairwise adjacent. If they are pairwise disjoint then, S is called a stable or an independent set of G.

A complement of a graph G, denoted $\overline{G}$ is a graph having $V(G)$ as its vertex set and two vertices are adjacent in $\overline{G}$ if and only if they are disjoint in G.

The distance between two vertices u and v is a mapping:

$$\begin{array}{rccl} d: & V \times V & \to & \mathbb{N} \\ & (u, v) & \to & d(u, v) \text{ is the smallest path joining } u \text{ to } v. \end{array}$$

It satisfies, for every $u, v, w \in V(G)$

1. $d(u, v) \geq 0$ and $d(u, v) = 0$ if and only if $u = v$,

2. $d(u, v) = d(v, u)$,

3. $d(u, v) \leq d(u, w) + d(w, v)$.

1.3.2 Paths, cycles and connectivity

The graph with n distinct vertices labeled $v_1, \cdots, v_n$ and edges $v_i v_{i+1}$, $i = 1, \cdots, n-1$ is called a path of order n and length $n-1$, denoted P_n. v_1 and v_n are called the end vertices of P_n.

A cycle of order n, denoted C_n is the graph with n vertices $v_1, \cdots, v_n$ and edges $v_1 v_2, \cdots, v_{n-1} v_n, v_n v_1$. It is called also a closed path.

An odd cycle is a cycle of an odd order while an even cycle is a cycle of an even order.

A graph is connected if there exists a path joining every pair of vertices

1.3.3 Some particular classes of graphs

An empty graph, denoted E_n is a graph without edges while a complete graph K_n is the graph with pairwise adjacent vertices.

$G = (V, E)$ is called bipartite if the vertex set of G can be split into two stables S_1 and S_2 so that each edge of G joins a vertex of S_1 and a vertex of S_2.

A complete bipartite graph is a bipartite graph $G = (S_1 \cup S_2, E)$ in which each vertex of S_1 is adjacent to each vertex in S_2 denoted by K_{n_1}, n_2, where $n_1 = |S_1|$ and $n_2 = |S_2|$. $K_{1,n-1}$ is called a star also denoted by S_n.

A graph $G = (V, E)$ is said to be k-partite if V can be partionned into k stables $S_1, \cdots, S_k$ such that the induced subgraph $< S_i, S_j >$, $i \neq j$ is connected.

A graph is k-complete multipartite if for all $i, j = 1, \cdots, k$ and $i \neq j$ the induced subgraph $< S_i, S_j >$ is complete bipartite.

A tree is a connected graph with no cycle.

A k-tree, $k \in N$ is defined recursively as follows: any complete graph K_k is an k-tree and any k-tree of order $n+1$, T_{n+1}^k is a graph obtained from a k-tree of order n, T_n^k , where $n \geq k$ by adding a new vertex and joining it to each vertex of K_k in T_n^k.

A caterpillar is the graph with the property that the removal of its pendant vertices leaves a path.

A wheel with n spokes, W_n is the graph that consists of a cycle of order n and on additional vertex that is adjacent to all the vertices of the cycle.

1.4 The link among, colorings, Stirling numbers and Bell numbers for graphs

A question arising when considering the problem of graph coloring is: what is the number of ways of coloring a given graph G?

Many interesting problems have been considered and treated by several authors with different interpretations when considering the enumeration of the ways of coloring the vertices of a graph subject to some constraints. A particular interest was given when we disregard the permutations of the colors which means when we consider two colorings as equivalent if they induce the same stable partition. In this context, many sequences had their graph theoretic encoding namely the previously discussed ones. Moreover, many fundamental results were established for them.

1.4.1 Proper λ-Colorings

Definition 12. *For $G = (V, E)$, $k \in N$, a proper λ-coloring of G is a mapping $f : V(G) \to \{1, 2, \cdots, \lambda\}$ with $f(u) \neq f(v)$ whenever u and v are adjacent in G.*

For more simplicity, it would be convenient to use directly the term "colorings" rather than "proper colorings" and agree that by "colorings" of a graph we mean "proper colorings".

The chromatic number of G, denoted by $\chi(G)$ is the smallest λ such that G admits a λ-coloring.

1.4.2 The graph theoretic encoding of Stirling and Bell numbers

Given $G = (V, E)$, the object that encodes the number of λ-colorings of G when the permutations of the colors are disregarded is called the Chromatic polynomial of G, denoted $P(G, \lambda)$, it has been introduced in 1912 by Birkhoff [10]. The study of chromatic polynomials has been expanded by Whitney in 1933 [68].
$P(G, \lambda)$ is defined for all real and complex values of λ, if G is of order n, then $P(G, \lambda)$ is a monic polynomial of degree n. For properties and details on the study of Chromatic polynomials of graphs, see [25, 11].

The chromatic polynomial's formula is

$$P(G, \lambda) = \sum_{k=\chi(G)}^{n} S(G, k)(\lambda)_k. \tag{1.1}$$

Recall that $(\lambda)_k = \lambda(\lambda - 1)(\lambda - 2)\cdots(\lambda - k + 1)$ is the falling factorial of λ.

If G is of order n, then $P(G, \lambda)$ is a monic polynomial of degree n,

The value of this invariant has been determined for several classes of graphs;

Example 13.

- *The chromatic polynomial of a complete graph of order n is* $P(K_n, \lambda) = (\lambda)_n$
- *The chromatic polynomial of a tree of order n is* $P(T_n, \lambda) = \lambda(\lambda - 1)^{n-1}$,
- *The chromatic polynomial of a cycle of order n, $n \geq 3$ is* $P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$,
- *The chromatic polynomial of an m-tree of order n is* $P(T_n^m, \lambda) = \lambda(\lambda - 1)\cdots(\lambda - m + 1)(\lambda - m)^{n-m}$,

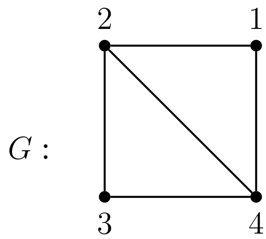
Dong et al in [25] provided a book which is fully devoted to the study of Chromatic polynomials of graphs. According to the authors in this book, there is a bijection between the family of colorings of G using exactly i colors from $\{1, 2, \ldots, \lambda\}$ and the family of partitions of $V(G)$ into i independent sets; and further for any such partition, there are $(\lambda)_i$ λ-colorings of G, thus the number of λ-colorings of G is given by $\sum\limits_{k=\chi(G)}^{n} S(G, k)(\lambda)_k$.

Furthermore, it is not surprising that this coincides with the number of partitions into independent sets, but what was surprising is that the number of partitions of the vertices of a path into independent sets followed the Bell sequence and the number of partitions of a path into exactly k independent sets coincides with the Stirling number of the second kind.

Moreover, the coefficient of the falling factorial in the formula of $P(G, \lambda)$, denoted by $S(G, k)$ is defined to be the graphical Stirling number of G.

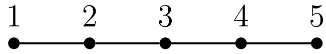
Definition 14. *For $G = (V, E)$, the Stirling number denoted by $S(G, k)$ counts the number of stable partitions (partitions into stable sets called also independent partitions) of $V(G)$ into k non-empty blocks. In other words, this sequence counts the number of ways to color G with exactly k colors as long as we consider each two colorings as similar if one can be obtained from the other applying a permutation on the names of the colors.*

Example 15.



- $S(G, 2) = 0$,
- $S(G, 3) = 1$, *the possible case is* $1, 3/2/4$.
- $S(G, 4) = 1$ *with* $1/2/3/4$.

Example 16.



- $S(P_5, 2) = 1$,
- $S(P_5, 3) = 7$, *the possible situations are indeed* $1/2, 4/3, 5$; $2/1, 4/3, 5$; $3/1, 4/2, 5$; $3/1, 5/2, 4$; $4/2, 5/1, 3$; $5/2, 4/1, 3$; $2/1, 3, 5/4$,
- $S(P_5, 5) = 1$ *with* $1/2/3/4/5$.

Indeed, $S(G, k) = 0$ for $0 \leq k \leq \chi(G) - 1$, where $\chi(G)$ is the chromatic number of G. This follows from the fact that we can not partition the vertex set of G into less than its chromatic number.

A natural consequence that can be deduced is that $S(G,k) = \frac{P(G,k)}{k!}$ when $\chi(G) = k$.

Moreover $S(G_n, n) = 1$ and $S(G_n, n-1) = \binom{n}{2} - m$, where $m = |E(G)|$.

In addition, we should note that $S(G,k)$ represent the number of spanning subgraphs of $\overline{G}$ consisting of k cliques, since an independent set in G corresponds to a clique in $\overline{G}$ and vice versa; a remark mentioned in [28]and [40]. Let $C(\overline{G},k)$ be the number of partitions of G into cliques.

Example 17. *The number of partitions into stables in P_3 corresponds to the number of partitions into cliques in $\overline{P}_3$. Indeed,*

- $C(\overline{P}_3, 1) = 0 = S(P_3, 1)$,
- $C(\overline{P}_3, 2) = 1 = S(P_3, 2)$,
- $C(\overline{P}_3, 3) = 1 = S(P_3, 3)$,

thus its chromatic polynomial,

$$P(P_3, \lambda) = (\lambda)_3 + (\lambda)_2.$$

This graph invariant was first introduced and investigated by [27] and appeared in the literature with different notations under several names, it is considered by Goldman and al [30] and referred to as chromatic vector of G, we find it also under the name chromatic spectrum by Voloshin [65] while Duncan and Peele called it the graphical Stirling number [27]. Also, It should be noted that in 1982, Prodinger and Tichy defined the Fibonacci number of a graph G to be the total number of the stable sets including the empty set, see [54] in which the authors gave the Fibonacci number of paths, trees and cycles.

Maamra and Mihoubi in [41, 42] have used the coefficients of the chromatic polynomial to derive some applications on Stirling numbers of the second kind.

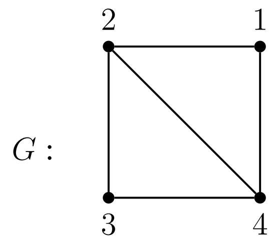
The definition of Stirling number for graphs suggests the definition of Bell numbers for graphs;

Definition 18. *The Bell number for a graph $G = (V, E)$ is defined to be the total number of independent partitions of $V(G)$.*

For a graph G of order n, it is expressed by

$$B_G = \sum_{k=0}^{n} S(G, k). \tag{1.2}$$

Example 19. *Taking the same example as that for the graphical Stirling numbers,*



the total number of independent partitions of $V(G)$ is done by $B_G = S(G, 1) + S(G, 2) + S(G, 3) + S(G, 4) = 0 + 0 + 1 + 1 = 2$.

For an empty graph E_n, there are $S(n, k)$ such colorings, , that's why it was also referred to as the graphical Stirling number or Stirling numbers for graphs [27, 35]. The chromatic polynomial therefore is $\sum_{k=0}^{n} S(n, k)(\lambda)_k$. It has to be noted that, in this case when the falling factorial is replaced by λ^k, the resultant polynomial represents the Bell polynomial discussed previously. Also the Bell number for the empty graph coincide with the familiar Bell number B_n.

For the complete graph denoted by K_n, we have one partitions into stables, when $k = n$ and no possible partitions, otherwise. Thus, the chromatic polynomial $P(K_n, k) = k^n$. The Bell number therefore is equal to 1.

The Stirling numbers for other families of graphs will be further discussed in the next chapter.

2

Basic results on Bell and Stirling numbers for graphs

Outlines

2.1 Introduction

As mentioned in the first Chapter, the number of ways to color a path follows the Bell and Stirling sequences. Many references were made to treat these sequences for other well known graphs [27, 29, 33, 35, 69].

In this chapter, we shall present in full detail the enumeration of graphical Stirling and Bell numbers for some families of graphs such as paths, cycles, trees, wheels, forests $\cdots$ We show also some technics used in, especially those based on combinatorial arguments.

2.2 Bell and Stirling numbers for some graph classes

The deletion-contraction principle was a valuable tool and had useful applications in enumerating colorings of a graph. It was firstly used to compute the chromatic polynomial [11], its application remains also valid in the case of counting the number of partitions, this follows from the fact that coloring the vertex set of a graph amounts to partition their vertices into stables.

Theorem 20. *[27] Let G be a simple graph of order n, $e \in E(G)$ and $0 \leq k \leq n-1$, then*

$$S(G,k) = S(G-e,k) - S(G/e,k),$$

where $G-e$ and G/e are the transformation graphs obtained by deleting and contracting edge e from G, respectively.

We must note that the graph G/e may not be a simple graph, but because of the fact that the contraction of distinct vertices will not create any loops, we can ignore multiple edges between vertices as this does not affect the calculation of the number of partitions (as two adjacent vertices remain adjacent regardless of the number of edges between them).

Proof. Partitions of $V(G)$ are those where the ends of e are in different blocks and those where the ends of e share the same block.

On the other hand, $S(G-e,k)$ counts the partitions where the two ends are in the same block and others where the two ends belong to distinct blocks while, $S(G/e,k)$ counts the partitions where the two ends belongs always to the same block. It follows that,

$$S(G,k) = S(G-e,k) - S(G/e,k).$$

□

A proof with similar idea mathematically written appeared in [25]

According to Dong et al [25], the problem of evaluating the chromatic polynomial $P(G,\lambda)$ is at least as hard as that of determining the chromatic number of G, $\chi(G)$. But there are results for evaluating $P(G,\lambda)$ more efficiently for some classes of graphs. They are based on the deletion-contraction principle, a technic that provides a recursive way to compute $P(G,\lambda)$, see [25] for basic results on enumeration of $P(G,\lambda)$.

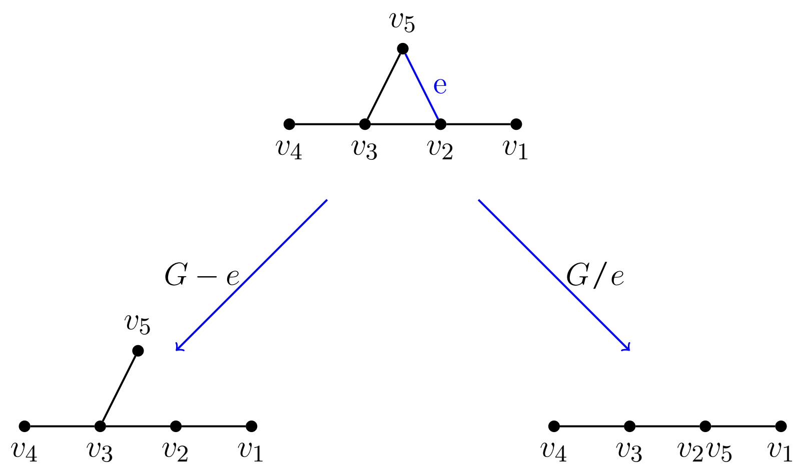


Figure 2.1: Deletion-Contraction procedure.

2.2.1 Paths, Stars, Cycles, Trees, Forests and Wheels

Recall that Bell and Stirling numbers for some graphs are extension of the classical Bell and Stirling numbers of the second kind, since for an empty graph E_n, $S(E_n,k) = S(n,k)$, that's why it was referred as the graphical Stirling number or

Stirling numbers for graphs by Duncan and Peele in [27] and Kereskenyi-Balogh and Nyul in [35].

Before stating the results established for the number of stable partitions of paths, stars and trees, we should give definitions that are equivalent to the stable partitions of these cited classes.

Nonconsecutive partitions are defined to be the partition of $\{x_1, x_2, \cdots, x_n\}$ such that x_i and x_{i+1} don't share the same block for $i = 1, \cdots, n-1$, other denominations are also given to this type of partition namely, Fibonacci, reduced and restricted partitions. Related descriptions can be found in [47, 48, 53, 52, 55, 23].

Counting the number of stable partitions of paths, stars and trees amounts to enumerate non-consecutive partitions. Moreover, it is established that the number of partitions of the vertex set of a path and a star both of order n into k is done by the ordinary Stirling numbers of the second kind, thus the total number of partitions is done by the Bell numbers. This result was first proved by Yang in 1996 [69] using the induction, later Duncan provided a different proof based on the deletion-contraction principle [27] and recently Kereskenyi-Balogh and Nyul [35] gave a more simpler proof using the chromatic polynomial formula, we state here the recent technic used in, those of Kereskenyi.

Theorem 21. *Let $n \geq 1$ and $0 \leq k \leq n$*

$$S(P_n, k) = S(S_n, k) = S(n-1, k-1),$$

and

$$B_{P_n} = B_{S_n} = B_{n-1}.$$

This has been generalized to the m-trees denoted by $T_n^{(m)}$.

Theorem 22. *If $m \geq 1$, $n \geq m+1$ and $0 \leq k \leq n$*

$$S(T_n^m, k) = S(n-m, k-m) \text{ for } m \leq k \leq n \text{ and } 0 \text{ otherwise.} \tag{2.1}$$

Proof. The chromatic polynomial of a generalized m-tree T_n^m with n vertices is

$$P(T_n^m, \lambda) = (\lambda)_m (\lambda - m)^{n-m},$$

we have

$$\sum_{k=0}^{n} = S(n,k)(x)_k = x^n,$$

it follows then that

$$\begin{aligned} P(T_n^m, \lambda) &= (\lambda)_m \textstyle\sum_{k=0}^{n-m} S(n-m,k)(\lambda - m)_k \\ &= \textstyle\sum_{k=m}^{n} S(n-m, k-m)(\lambda)_k, \end{aligned}$$

thus

$$S(T_n^m, k) = \begin{cases} S(n-m, k-m) & \text{for m} \leq k \leq n \\ 0 & \text{for } 0 \leq k \leq m-1. \end{cases}$$

□

According to Berceanu [9], the Bell numbers for graphs can be expressed in terms of ordinary Bell numbers and the coefficients of the chromatic polynomial, his proof was done in 2001 based on some linear operators of the polynomial vector space. Later, Kereskenyi-Balogh and Nyul in 2014 [35] gave a simpler proof by induction using the deletion-contraction principle.

Theorem 23. *If G is a graph, $0 \leq k \leq |V(G)|$, and $P(G, \lambda) = \sum_{j=0}^{|V(G)|} a_j \lambda^j$, then*

$$S(G,k) = \sum_{j=k}^{|V(G)|} a_j S(j,k) \text{ and } B_G = \sum_{j=0}^{|V(G)|} a_j B_j.$$

Also, an explicit formula related to Stirling numbers of graphs was established by Mohr and Porter [47] and later by Kereskenyi [35].

Theorem 24. *If G is a simple graph and $0 \leq k \leq |V(G)|$, then*

$$S(G,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} P(G, k-j).$$

For the complete graph, denoted by K_n, we have one coloring, subject to the above constraints when $k = n$ and no possible colorings, otherwise. Thus, the chromatic polynomial $P(K_n, k) = (k)_n$.

For a cycle of order n, C_n, it is defined by the following alternating sum

Theorem 25. *For $n \geq 3$ and $2 \leq k \leq n$ we have,*

$$S(C_n, k) = \sum_{j=k-1}^{n-1} (-1)^{n-1-j} S(j, k-1) \textit{ and } B_{C_n} = \sum_{j=k-1}^{n-1} (-1)^{n-1-j} B_j. \quad (2.2)$$

For the Wheel consisting of n vertices, denoted by W_n,

Theorem 26. *For $n \geq 3$ and $3 \leq k \leq n$ we have,*

$$S(W_n, k) = \sum_{j=k-2}^{n-2} (-1)^{n-2-j} S(j, k-2),\ B_{W_n} = \sum_{j=k-2}^{n-2} (-1)^{n-2-j} B_j.$$

(2.3)

Also, the Bell and Stirling numbers for the complements of these graph classes were given.

Theorem 27. *For $n \geq 2$ we have,*

$$S(\overline{S}_n, k) = \begin{cases} n-1 & \textit{for k=n-1} \\ \textit{1} & \textit{for k=n} \end{cases} \quad (2.4)$$

and

$$B_{\overline{S}_n} = n. \quad (2.5)$$

Theorem 28. *For $n \geq 2$ and $\frac{n}{2} \leq k \leq n$ we have,*

$$S(\overline{P}_n, k) = \binom{k}{n-k}, \quad (2.6)$$

$$B_{\overline{P}_n} = F_{n+1}.$$

Remark. For $n = 4$, we have $S(P_4, k) = S(\overline{P_4}, k)$ since, $P_4 \simeq \overline{P_4}$.

Theorem 29. *For $n \geq 2$ and $\frac{n}{2} \leq k \leq n$ we have*

$$S(\overline{C}_n, k) = \frac{n}{k}\binom{k}{n-k}, \quad (2.7)$$

$$B_{\overline{C_n}} = L_n,$$

where F_n and L_n are respectively the Fibonacci and the Lucas numbers.
The Fibonacci numbers are defined by the following recurrence relation,

$$F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 2,$$

with $F_0 = 0$, $F_1 = 1$.
The Lucas numbers, named after François Édouard Anatole Lucas (1842 1891), are defined as

$$L_n = L_{n-1} + L_{n-2}, \text{ for } n \geq 2,$$

with $L_0 = 2$, $L_1 = 1$.

For the Complete bipartite graph, The graphical Stirling number is expressed by the convolution sum,

Theorem 30. *For $1 \leq k \leq m+n$ we have*

$$S(K_{m,n}) = \sum_{j=0}^{k} S(m,j)S(n,k-j)$$

and

$$B_{K_{m,n}} = B_m.B_n.$$

For G being a complete l-partite, $K_{n_1,\cdots,n_l}$, a generalization of this result was given by Allagan and Serkan in [2],

Theorem 31. *For $l \geq 1$, $n_i \geq 1, i = 1, \cdots, l$ and $l \leq k \leq \sum_{i=1}^{l} n_i$*

$$S(K_{n_1,\cdots,n_l}, k) = \sum_{j_1+\cdots+j_l=k} S(n_1, j_1) \cdots S(n_l, j_l)$$

and

$$B_{K_{n_1,\cdots,n_l}} = \prod_{i=1}^{l} B_{n_i}.$$

2.2.2 Disjoint union of two graphs

Definition 32. *The disjoint union of two graphs H_1 and H_2 denoted by $H_1 \cup H_2$, is the graph H whose vertex set is $V(H) = V(H_1) \cup V(H_2)$ and whose edge set is $E(H) = E(H_1) \cup E(H_2)$.*

Stirling number for the disjoint union of two graphs has been considered and studied by Duncan in [26]. Also in 2016 Hertz and Melot [33] have given a property of this number in terms of the Stirling numbers of each graph involving the binomial coefficient.

Theorem 33. *Let $H_1 \cup H_2$ be the disjoint union of H_1 and H_2. Then we have,*

$$S(H_1 \cup H_2, k) = \sum_{i=1}^{k} \sum_{j=0}^{i} S(H_1, i) S(H_2, k-j) \binom{i}{i-j} \binom{k-j}{i-j} (i-j)!, \tag{2.8}$$

Proof. The two sums compute $S(H_1 \cup H_2, k)$ as follows. Let $i \leq k$ be the number of colors used for H_1. Let j be an integer such that $i - j$ represents the number of colors that are used both in H_1 and in H_2. The value of j can vary from 0 (that is i colors are shared) to i (that is no color are shared). Observe that in order to use exactly k colors for $H_1 \cup H_2$, H_2 must be colored with exactly $k - j$ colors. Finally, the term $\binom{i}{i-j}$ counts the numbers of ways to choose the $i - j$ shared colors into H_1, the term $\binom{k-j}{i-j}$ does the same for H_2 and $(i - j)!$ counts all the possible permutations for this shared colors. □

As a consequence, they gave the Stirling number for the disjoint union of two completes graphs of orders p and q and $p \leq q$, we have.

Corollary 34. *For $G = K_p \cup K_q$ being the disjoint union of two cliques of orders p and q, $p \leq q$ we have,*

$$B_G = \sum_{k=q}^{p+q} \binom{p}{k-q} \binom{q}{p+q-k} (p+q-k)!$$

We propose a symmetric formula to rewrite the property in Theorem 32.

$$S(H_1 \cup H_2, k) = \sum_{s=k}^{2k} s! \sum_{i+j=s+k} S(H_1, i) S(H_2, j) \binom{i}{s} \binom{j}{s}$$

It can be generalized to the disjoint union of r graphs,

$$S(H_1 \cup H_2 \cup \cdots \cup H_m, k) = \sum_{s=k}^{rk} s! \sum_{i_1+i_2+\cdots+i_m=s_{m-1}+k} S(H_1, i_1) \cdots S(H_m, i_m) \binom{i_1}{s} \cdots \binom{i_2}{s}$$

3

Enumerating stable partitions for some graph classes involving Bell, Stirling, r-Bell and r-Stirling numbers of the second kind

Outlines

3.1 Introduction

Inspired by the previously cited works and exploiting the works carried out in this context, we propose in this chapter to calculate the number of stable partitions for other families of graphs. In what follows, inductive proofs using the deletion-contraction principle, bijective proofs and generating functions are used to demonstrate the obtained results. As consequences, we give new identities concerning the Stirling number of the second kind. Besides that, explicit formulas in terms of the generalized r-Stirling numbers are established.

3.2 Enumerating stable partitions for a generalization of thorn graphs

3.2.1 Notations and Definitions

Given a graph of order n, G_n. For any fixed $p \geq 1$, consider the super-graph obtained by joining a path of order p, P_p to G_n by a bridge as illustrated in Figure 3.1 and let us denote the resulting graph by $G_{n,p}$. If G_n is a cycle of order n, C_n, then the resultant graph is called a tadpole.

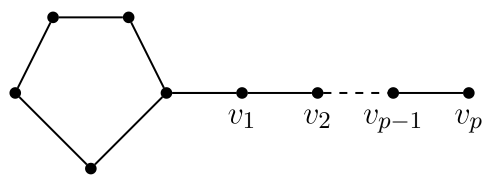


Figure 3.1: Representation of $G_{n,p}$.

More generally, let us consider a graph $G^* = G(t_1, t_2, \ldots, t_n)$ of G_n which is obtained by attaching t_i (≥ 0) new pendant vertices to a vertex v_i of G_n, $i = 1, \ldots, n$. This definition refers to a class of graphs known in the literature as thorn graphs. For G_n being a tree $G^* = T^*$ is called a thorn tree.

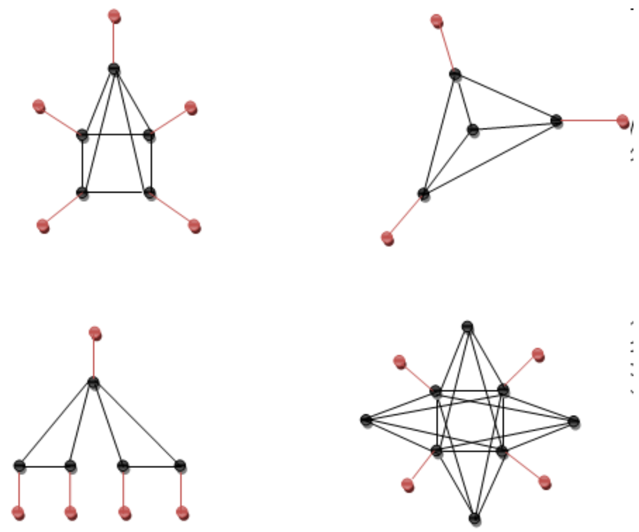


Figure 3.2: Some special thorn graphs.

Moreover, we define a generalized thorn graph $G^{(t)}$ to be a graph obtained from G by attaching new trees T_i of order $t_i \geq 0$ to a given vertex v_i of G_n, $i = 1, \ldots, n$ such that $t = \sum_{i=1}^{n} t_i$. Then, if all T_i's are single vertices then $G^{(t)}$ is a thorn graph. See for instance some generalized thorn graphs in Figure 3.4.

3.2.2 Identities and explicit formulas

The number of stable partitions of $G_{n,p}$ can be expressed in terms of the number of stable partitions of the disjoint unions of paths and the initial graph G_n.

Theorem 35. *For $1 \leq k \leq n$ and $p \geq 1$ we have,*

$$S(G_{n,p}, k) = \sum_{i=0}^{p} (-1)^i S(G_n \cup P_{p-i}, k),$$

where $P_0 = \emptyset$, thus $G_n \cup P_0 = G_n$.

Proof. The proof proceeds by induction on p using the deletion-contraction principle. The recurrence is valid for the trivial case ($p = 1$) with the convention that $G_n \cup P_0 = G_n$ and $P_1 = E_1$ (one isolated vertex) and can be verified using Theorem 20. Now, suppose the identity true for $G_{n,p}$ and let us establish it for $G_{n,p+1}$. From

Theorem 20, we have, (see Figure 3.3 for explanation)

$$S(G_{n,p+1},k) = S(G_n \cup P_{p+1},k) - S(G_{n,p},k),$$

then, using the induction hypothesis, we get,

$$S(G_{n,p+1},k) = S(G_n \cup P_{p+1},k) - \sum_{i=0}^{p}(-1)^i S(G_n \cup P_{p-i},k),$$

we set $j = i+1$ and we obtain,

$$S(G_{n,p+1},k) = \sum_{j=1}^{p+1}(-1)^j S(G_n \cup P_{p-j+1},k) + S(G_n \cup P_{p+1},k), \quad (3.1)$$

which gives the result. □

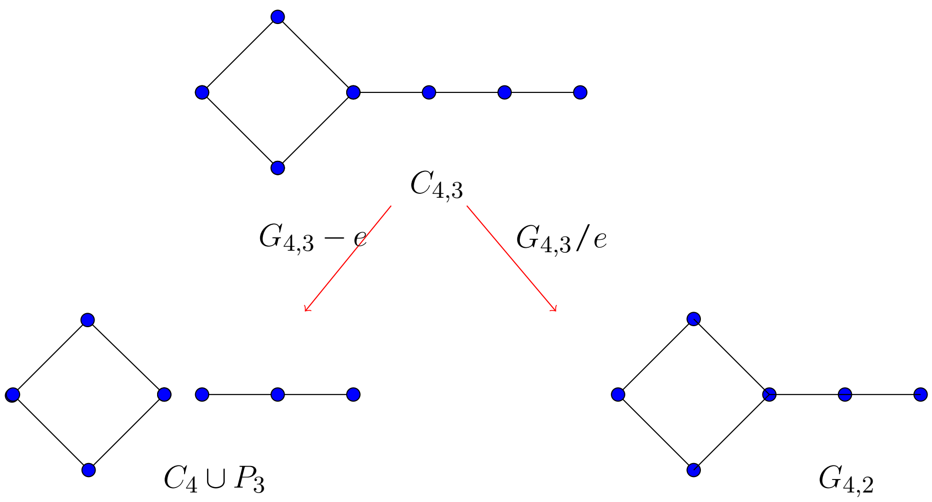


Figure 3.3: Deletion-contraction procedure on the tadpole $G_{4,3}$.

As a corollary, an identity on the classical Stirling numbers of the second kind evaluated with three summations can be derived.

Corollary 36. *For $n \geq 4$, $0 \leq k \leq n$ and $1 \leq l < n$, we have*

$$S(n,k) = \sum_{i,s,t}(-1)^i S(l,s-1)S(n-l-i-1,k-t)\binom{s}{t}\binom{k+1-t}{s-t}(s-t)!,$$

where i,s,t satisfy $0 \leq i < n-l$, $1 \leq s \leq min(k,l)+1$ and $0 \leq t \leq s$.

Proof. Let P_n be a path of order n and P_l a subpath of P_n of order l, $1 \leq l < n$. Therefore, P_n can be written as $G_{l,n-l}$ for $1 \leq l < n$.

Applying Theorem 34 we obtain for $1 \leq k \leq n$, $n \geq 2$ and $1 \leq l < n$,

$$S(P_n, k) = S(G_{l,n-l}, k) = \sum_{i=0}^{n-l} (-1)^i S(P_l \cup P_{n-l-i}, k),$$

we use Theorem 32 (the indices i and j in Relation (2.8) are changed to s and t, respectively), we get, for $0 \leq i \leq n-l$, $1 \leq s \leq k$ and $0 \leq t \leq s$,

$$S(P_n, k) = \sum_{i,s,t} (-1)^i S(P_l, s) S(P_{n-l}, k-t) \binom{s}{s-t} \binom{k-t}{s-t} (s-t)!,$$

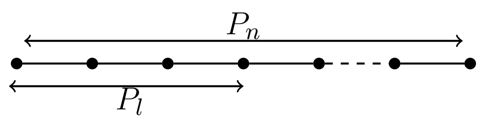
we replace $S(P_n, k)$ by its value given in (2.1), we obtain, for $0 \leq i < n-l$, $1 \leq s \leq k$ and $0 \leq t \leq s$,

$$S(n-1, k-1) = \sum_{i,s,t} (-1)^i S(l-1, s-1) S(n-l-i-1, k-t-1) \binom{s}{s-t} \binom{k-t}{s-t} (s-t)!,$$

by changing of variables k and n, we get, for $n \geq 2$, $1 \leq k \leq n$ and $1 \leq l < n$,

$$S(n, k) = \sum_{i,s,t} (-1)^i S(l-1, s-1) S(n-l-i, k-t) \binom{s}{s-t} \binom{k-t+1}{s-t} (s-t)!,$$

where i, s, t satisfy $0 \leq i < n-l$, $1 \leq s \leq min(k, l) + 1$ and $0 \leq t \leq s$. □



Theorem 37. *For $t \geq 1$ and $1 \leq k \leq n + t$, we have,*

$$S(G^{(t)}, k) = \sum_{i=0}^{k-1} S(G_n, k-i) \sum_{j_1+\cdots+j_{i+1}=t-i} (k-1)^{j_1} \cdots (k-i-1)^{j_{i+1}}.$$

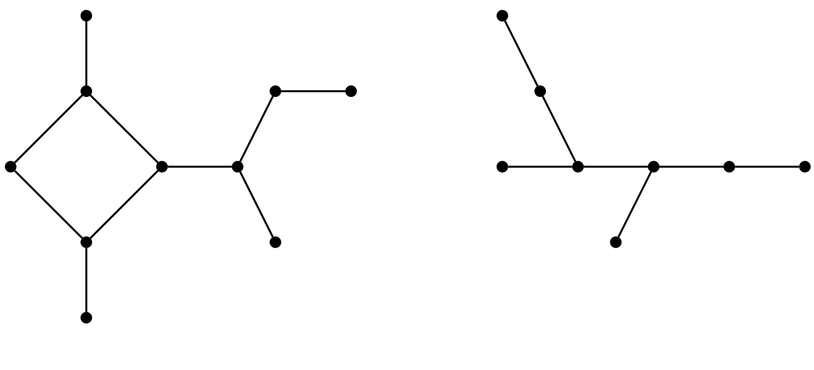


Figure 3.4: Some examples of generalized thorn graphs.

Proof. Recall that $G^{(t)}$ is a generalized thorn graph defined by attaching some trees to some vertices of a simple and finite graph of order n, G_n. Thus $|G^{(t)}| = n + t$. To simplify, we adopt the convention: $\beta_{t,k} = S(G^{(t)}, k)$ and $\beta_{0,k} = S(G, k)$. Firstly, we prove using a bijective combinatorial argument the following recurrence relation

$$\beta_{t+1,k} = (k-1)\beta_{t,k} + \beta_{t,k-1}, \tag{3.2}$$

As was agreed at the beginning of the proof, we have $\beta_{t+1,k} = S(G^{(t+1)}, k)$. Motivated by the connection to colorings, if we take v the vertex of the end of any attached tree T_i in $G^{(t+1)}$ then we have two possible situations. Either the end vertex has one color already used by other vertices or it has its own color, in this case there is $\beta_{t,k-1}$ colorings and it turns out that there is $(k-1)\beta_{t,k}$ possible colorings in the former case, since v has all possible colors except one used by its neighbor. Hence,

$$\beta_{t+1,k} = (k-1)\beta_{t,k} + \beta_{t,k-1}.$$

Now, we use the induction over t to prove the following recurrence

$$\beta_{t,k} = (k-1)^t\beta_{0,k} + \sum_{j_1=0}^{t-1}(k-1)^{j_1}\beta_{t-1-j_1,k-1}. \tag{3.3}$$

It is easy to verify using relation (3.2) for the trivial case ($t = 1$) that the Identity (3.3) is true. By Relation (3.2) and the induction hypothesis we have

$$\beta_{t+1,k} = (k-1)^{t+1}\beta_{0,k} + \sum_{j_1=0}^{t-1}(k-1)^{j_1+1}\beta_{t-1-j_1,k-1} + \beta_{t,k-1}, \tag{3.4}$$

we set $j_1' = j_1 + 1$ and we get,

$$\beta_{t+1,k} = (k-1)^{t+1}\beta_{0,k} + \sum_{j_1'=1}^{t}(k-1)^{j_1'}\beta_{t-j_1',k-1}, \tag{3.5}$$

thus Relation (3.3) is true for $t \geq 1$. Also, we have,

$$\beta_{t-1-j_1,k-1} = (k-2)^{t-1-j_1}\beta_{0,k-1} + \sum_{j_1=0}^{t-1-j_1-1}(k-1)^{j_2}\beta_{t-2-j_1-j_2,k-2}, \tag{3.6}$$

hence, using the same approach as in (3.3), we establish by induction over t that,

$$\begin{aligned}\beta_{t,k} &= (k-1)^t\beta_{0,k} + \beta_{0,k-1}\sum_{j_1+j_2=t-1}(k-1)^{j_1}(k-2)^{j_2} \\ &\quad + \sum_{j_1,j_2/j_1+j_2\leq t-2}(k-1)^{j_1+j_2}\beta_{t-2-j_1-j_2,k-2}.\end{aligned}$$

By developing the sum in the right hand side with the same way and applying the same inductive procedure, we get the result, for $i \geq 1$,

$$\beta_{t,k} = \sum_{i=0}^{k-1} \beta_{0,k-i} \sum_{j_1+j_2+\cdots+j_{i+1}=t-i} (k-1)^{j_1} \cdots (k-i-1)^{j_{i+1}}.$$

□

Paw, tadpole, subdivided star and caterpillar graphs are particular cases of the generalized thorn graphs $G^{(t)}$. See examples of these graph classes in Figure 3.5.

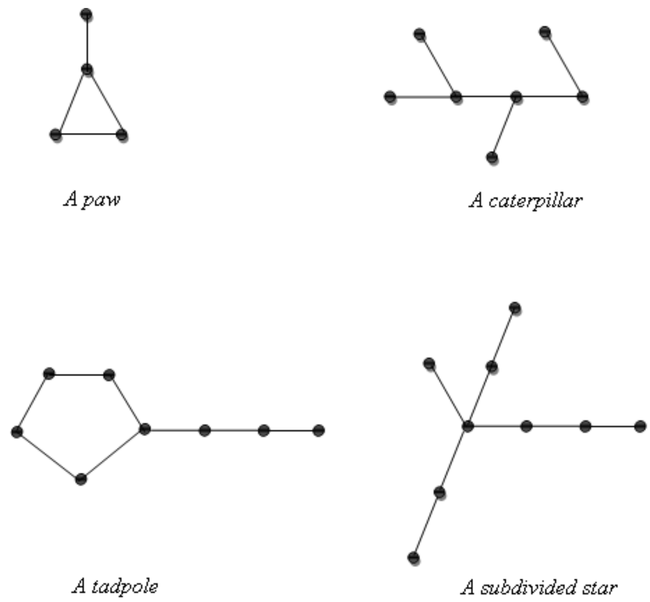


Figure 3.5: Some special classes of graphs.

For $G_n = C_n$ (a cycle), $i = 1$ and T_1 is a path, we obtain a tadpole graph. For $G^{(t)} = G^*$ a thorn graph and G_n a path of length n, we get a caterpillar.

For instance, we propose to count the number of stable partitions into k stable sets for some graphs cited above.
From Theorem 36 and Theorem 25, the number of stable partitions for a tadpole graph $G_{n,p}$ is given by

$$\sum_{i=0}^{k-1} \sum_{j=k-i-1}^{n-1} (-1)^{n-1-j} S(j, k-i-1) \sum_{j_1+\cdots+j_{i+1}=p-i} (k-1)^{j_1} \cdots (k-i-1)^{j_{i+1}}.$$

For a caterpillar, using Theorem 36 and Theorem 21, the number of stable partitions is equal to

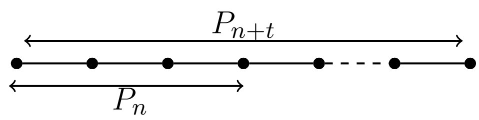
$$\sum_{i=0}^{k-1} S(n-1,k-i-1) \sum_{j_1+\cdots+j_{i+1}=t-i} (k-1)^{j_1}\cdots(k-i-1)^{j_{i+1}}.$$

Observe that the paths, trees and caterpillars are particular cases of the considered class $G^{(t)}$. Moreover, the number of stable partitions of paths have already been done by several authors with several interpretations [33, 26, 27, 35, 47, 29]. Considering this fact, the following identity holds.

Corollary 38. *For $n \geq 1, l \leq n$ and $1 \leq k \leq n$, we have,*

$$S(n+t-1,k-1) = \sum_{i=0}^{k-1} S(n-1,k-i-1) \sum_{j_1+\cdots+j_{i+1}=t-i} (k-1)^{j_1}\cdots(k-i-1)^{j_{i+1}}.$$

Proof. With the same way, as in Corollary 35, we consider a path of length $n+t$ and a path of length n as initial graph. Then, the identity results in by replacing $G^{(t)}$ and evaluating the sequence in Theorem 36. □



Notice that the previous theorem can be extended to the following one, in which $\sum_{j_1+\cdots+j_{i+1}=t-i}(k-1)^{j_1}\cdots(k-i-1)^{j_{i+1}}$ is replaced by the r-Stirling numbers of the second kind.

Theorem 39. *For $t \geq 1$ and $1 \leq k \leq n+t$, we have,*

$$S(G^{(t)},k) = \sum_{i=0}^{k-1} S(G_n,k-i) S_{k-i-1}(t+k-i-1,k-1).$$

Proof. r-Stirling numbers of the second kind have the generating function denoted by $\phi_k(u)$ and described as follows

$$\phi_k(u) = \sum_{n\geq k} S_r(n+r,k+r)u^n = \frac{u^k}{(1-(r+1)u)\cdots(1-(r+k)u)}. \tag{3.7}$$

On other hand, it is well known that

$$\frac{1}{1-ju}=\sum_{n\geq 0}(ju)^n. \tag{3.8}$$

From Relations (3.7) and (3.8) we obtain,

$$\phi_k(u)=u^k\sum_{n_1\geq 0}((r+1)u)^{n_1}\sum_{n_2\geq 0}((r+2)u)^{n_2}\cdots\sum_{n_k\geq 0}((r+k)u)^{n_k}, \tag{3.9}$$

summing by parts, Relation (3.9) can be written as follows

$$\phi_k(u)=u^k\sum_{n_1,n_2,\cdots,n_k\geq 0}(r+1)^{n_1}(r+2)^{n_2}\cdots(r+k)^{n_k}u^{n_1+n_2+\cdots+n_k}, \tag{3.10}$$

also, Relation (3.10) gives

$$\phi_k(u)=u^k\sum_{n\geq 0}(\sum_{n_1+n_2+\cdots+n_k=n}(r+1)^{n_1}(r+2)^{n_2}\cdots(r+k)^{n_k})u^n, \tag{3.11}$$

thus,

$$\phi_k(u)=\sum_{m\geq k}(\sum_{n_1+n_2+\cdots+n_k=m-k}(r+1)^{n_1}(r+2)^{n_2}\cdots(r+k)^{n_k})u^m, \tag{3.12}$$

by identification with the generating function we get,

$$S(n+r,m+r)_r=\sum_{n_1+n_2+\cdots+n_m=n-m}(r+1)^{n_1}(r+2)^{n_2}\cdots(r+m)^{n_m}, \tag{3.13}$$

by changing of variables, we obtain

$$\sum_{j_1+j_2+\cdots+j_k=t-i}(k-1)^{j_1}(k-2)^{j_2}\cdots(k-i-1)^{j_{i+1}}=S_{k-i-1}(t+k-i-1,k-1). \tag{3.14}$$

□

We conclude, for applications that the Stirling number of tadpoles, caterpillars, generalized m-trees is expressed in terms of both Stirling and r-Stirling numbers of the second kind, that is for instance,

for a tadpole graph $G_{n,p}$, $n\geq 3$ $t\geq 2$

$$\sum_{i=0}^{k-1}\sum_{j=k-i-1}^{n-1}(-1)^{n-1-j}S(j,k-i-1)S_{k-i-1}(P+k-i-1,k-1),$$

for a caterpillar constructed with a path of order $n \geq 2$, attached to t pendant vertices,

$$\sum_{i=0}^{k-1} S(n-1, k-i-1) S_{k-i-1}(t+k-i-1, k-1).$$

Consequently, we derive an identity of the Stirling numbers of the second kind in terms of Stirling and the r-Stirling numbers of the second kind.

Corollary 40. *For $t \geq 1$ and $0 \leq k \leq n+t$ we have,*

$$S(n+t, k) = \sum_{i=0}^{k} S(n, k-i) S_{k-i}(t+k-i, k).$$

Proof. From Corollary 37 and Theorem 38, we get the formula. □

This gives rise to an explicit formula related to binomial coefficient which expresses the Stirling numbers of the second kind in terms of the generalized r-Stirling numbers of the second kind, evaluated with two summations.

Corollary 41. *For $0 \leq k \leq n+t$, $n \geq 0$, $t \geq 1$ and $l \leq k$, we have*

$$S(n+t, k) = \sum_{i=0}^{k-l} S(n, k-i) \sum_{j} \binom{t}{j} S_l(t+l-j, i+l)(k-l-i)^j.$$

Proof. This is obtained using Theorem 38 combined with relation (33) in Broder's explicit formulas for the r-Stirling numbers of the second kind [13]. □

Note that for $l = 1$, an identity of the Stirling numbers of the second kind can be deduced.

Corollary 42. *For $0 \leq k \leq n+t$, $n \geq 0$, $t \geq 1$, we have*

$$S(n+t, k) = \sum_{i=0}^{k-1} S(n, k-i) \sum_{j} \binom{t}{j} S(t+1-j, i+1)(k-i-1)^j.$$

3.3 Stirling numbers for a generalization of cyclic graphs

3.3.1 Definitions and Notations

Definition 43. *A bi-cyclic graph is defined to be the graph obtained by attaching two cycles with a bridge edge, we use* $C_{n_1} \star C_{n_2}$ *to denote a bi-cyclic graph of order* $n_1 + n_2$ *constructed with the two cycles* $C_{n_1} and C_{n_2}$ *of order* n_1, n_2 *respectively.*

A more general configuration is to consider any graph instead of one of the two cycles, let consider $G_{n_1} \star C_{n_2}$ to describe this configuration.

3.3.2 Identities and explicit formulas

The number of ways to partition $G_{n_1} \star C_{n_2}$ into k stables is expressed by an alternating sum depending on the Stirling number of the Thorn graph.

Theorem 44. *For* $n_1 \geq 1$, $n_2 \geq 2$ *and* $k \leq n_1 + n_2$ *we have,*

$$S(G_{n_1} \star C_{n_2}, k) = \sum_{i=0}^{n_2-1} (-1)^i S(G_{n_1}^{(n_2-i)}, k).$$

Proof. This can be proved by induction using the deletion-contraction principle. The formula is valid in the trivial case, that is,

$$S(G_{n_1} \star C_1, k) = S(G_{n_1}^{(1)}, k),$$

we suppose the formula true for $G_{n_1} \star C_{n_2}$ and we prove it for $G_{n_1} \star C_{n_2+1}$ hence,

$$S(G_{n_1} \star C_{n_2}, k) = \sum_{i=0}^{n_2-1} (-1)^i S(G_{n_1}^{(n_2-i)}, k)$$

For e being an edge in C_{n_2}, we have from the deletion-contraction theorem, (see an illustration in Figure 3.7)

$$S(G_{n_1} \star C_{n_2+1}, k) = S(G_{n_1} \star C_{n_2+1} - e, k) - S(G_{n_1} \star C_{n_2+1}/e, k),$$

this latter is equivalent to

$$S(G_{n_1} \star C_{n_2+1}, k) = S(G_{n_1}^{(n_2)}, k) - S(G_{n_1} \star C_{n_2}, k),$$

using the induction hypothesis we obtain,

$$S(G_{n_1} \star C_{n_2+1}, k) = S(G_{n_1}^{(n_2)}, k) - \sum_{i=0}^{n_2-1} (-1)^i S(G_{n_1}^{(n_2-i)}, k),$$

thus,

$$S(G_{n_1} \star C_{n_2+1}, k) = \sum_{i=0}^{n_2} (-1)^i S(G_{n_1}^{(n_2-i+1)}, k).$$

□

Definition 45. *We define a generalization of a cyclic graph denoted by* $G_{n_1} \star C_{m,\cdots,1}$ *to be the graph obtained by attaching* m *cycles* $C_{n_1}, C_{n_2}, \cdots, C_{n_m}$ *one to an other by* $m-1$ *bridges and joining the last cycle* C_{n_m} *to* G *with the* m^{th} *bridge. (An example is in Figure* 3.6*)*

Theorem 43 has been extended to the generalized cyclic graph $G_{n_1} \star C_{m,\cdots,1}$,

Theorem 46. *For* $n \geq 1$, $m \geq 3$, $0 \leq k \leq n+M$ *and* $n_1, \cdots, n_m \geq 3$ *we have,*

$$S(G_n * (C_{n_1,\cdots,n_m}), k) = \sum_{i_1,i_2,\cdots,i_n} (-1)^I S(G_n^{(M-I)}, k),$$

$0 \leq i_j \leq n_j - 1$ *for* $1 \leq j \leq m$, *where* $I = i_1 + i_2 + \cdots + i_m$ *and* $M = n_1 + n_2 + \cdots + n_m$

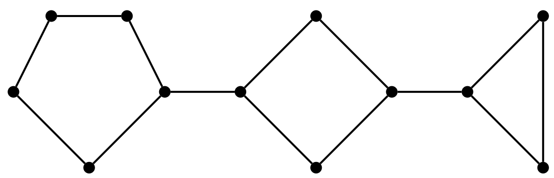


Figure 3.6: Example of generalized cyclic graph

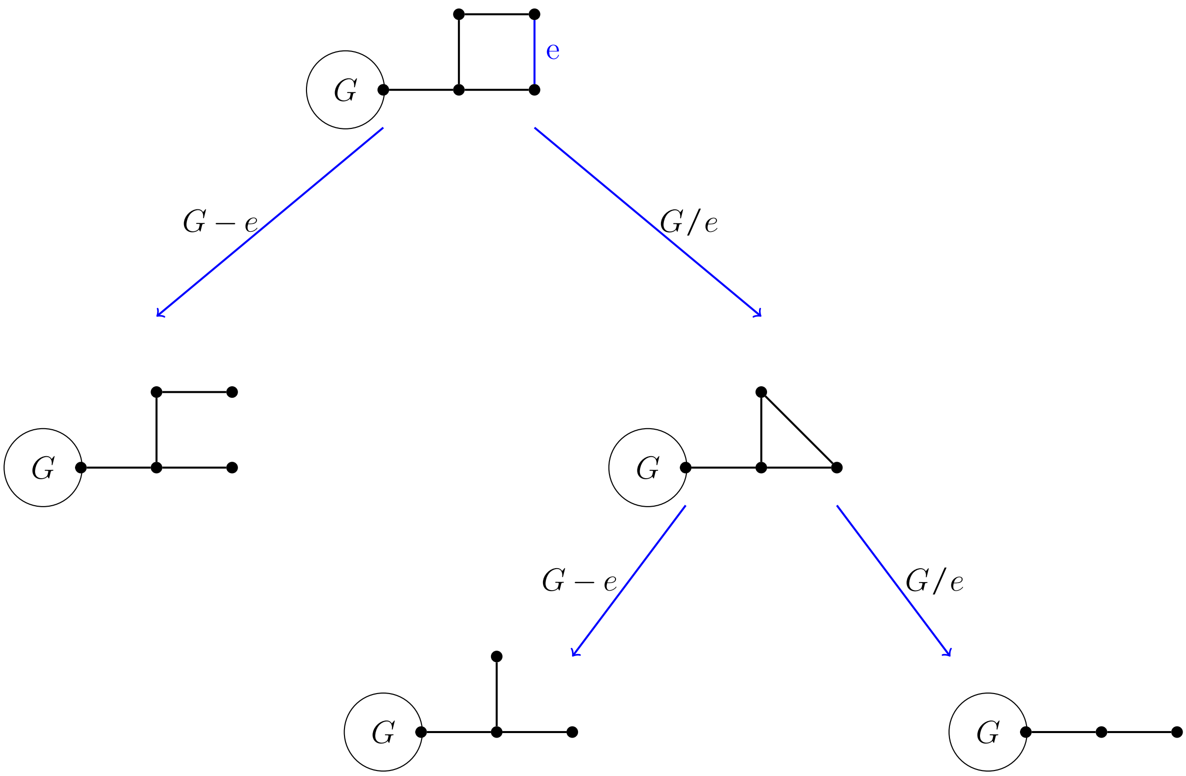


Figure 3.7: Deletion contraction procedure on a special case of generalized cyclic graph

3.3.3 Applications

The result can be applied to bi-cyclic graphs. For $n_1 \geq 1, n_2 \geq 3$ we have,

$$\begin{aligned} S(C_{n_1} \star C_{n_2}) &= \sum_{i=0}^{n_2}(-1)^i \sum_{j=0}^{k-1} S(C_{n_1}, k-j) S_{k-j-1}(n_2 - i + k - j - 1, k-1) \\ &= \sum_{i=0}^{n_2}(-1)^i \sum_{j=0}^{k-1} \sum_{s=k-j-1}^{n_1-1} (-1)^{n_1-1-s} S(s, k-j-1) S_{k-j-1}(n_2 - i + k - j - 1, k-1 \end{aligned}$$

Let T_{n_1} be a tree of order n_1. For $n_1 \geq 1, n_2 \geq 3$ we have,

$$S(T_{n_1} \star C_{n_2}) = \sum_{i=0}^{n_2-1} (-1)^i \sum_{j=0}^{k-1} S(n_1 - 1, k-j-1) S_{k-j-1}(n_2 - i + k - j - 1, k-1).$$

Let G be a generalized m-tree, $T_{n_1}^m$. For $n_1 \geq m+1$ and $n_2 \geq 3$ we have

$$S(T_{n_1^m} \star C_{n_2}) = \sum_{i=0}^{n_2-1} (-1)^i \sum_{j=0}^{k-1} S(n_1 - m, k-j-m) S_{k-j-1}(n_2 - i + k - j - 1, k-1).$$

For G being a complement of path, $\overline{P_{n_1}}$, $n_1 \geq 1$, $n_2 \geq 3$ then,

$$S(\overline{P}_{n_1} \star C_{n_2}) = \sum_{i=0}^{n_2-1} (-1)^i \sum_{j=0}^{k-1} \binom{k-j}{n-k+j} S_{k-j-1}(n_2 - i + k - j - 1, k-1).$$

4

Stirling and Bell numbers for join graphs and some special corona product graphs

Outlines

4.1 Introduction

In the current chapter, we apply two operations on graphs, namely, the join and the corona product. Based on bijective proofs, we establish explicit formulas to calculate Stirling and Bell numbers for join graphs, consequently we derive well-known identities related to Fibonacci and Lucas sequences. On the other hand, other families of graphs whose Stirling numbers depend on the generalized Stirling numbers of the second kind are found out.

4.2 Bell and Stirling numbers for join graphs

4.2.1 Notations and Definition

Definition 47. *Let G and H be any graphs. The join graph $G+H$ is the graph obtained by adding a new edge from every vertex of G to every vertex of H.*

In this section, we give a general formula for the Stirling numbers of the join of G and H and some consequences on the join of some special graphs. We can see for instance some join graphs in the figure 4.1.

4.2.2 Identities and explicit formulas

Theorem 48. *Let G and H be graphs of order $n_1 \geq 1$ and $n_2 \geq 1$ respectively, with $\chi(G) + \chi(H) \leq k \leq n_1 + n_2$. Then,*

$$S(G+H,k) = \sum_{j=0}^{k} S(G,j)S(H,k-j) \tag{4.1}$$

and

$$B_{G+H} = B_G.B_H \tag{4.2}$$

Proof. This can be done by direct combinatorial argument. We partition the vertex set of the join graph $G+H$ into k non-empty independent subsets, which means we have to form k blocks in such a way that the vertex set of G can not be in the same block as the elements of $V(H)$. So, a partition of $V(G+H)$ can be done

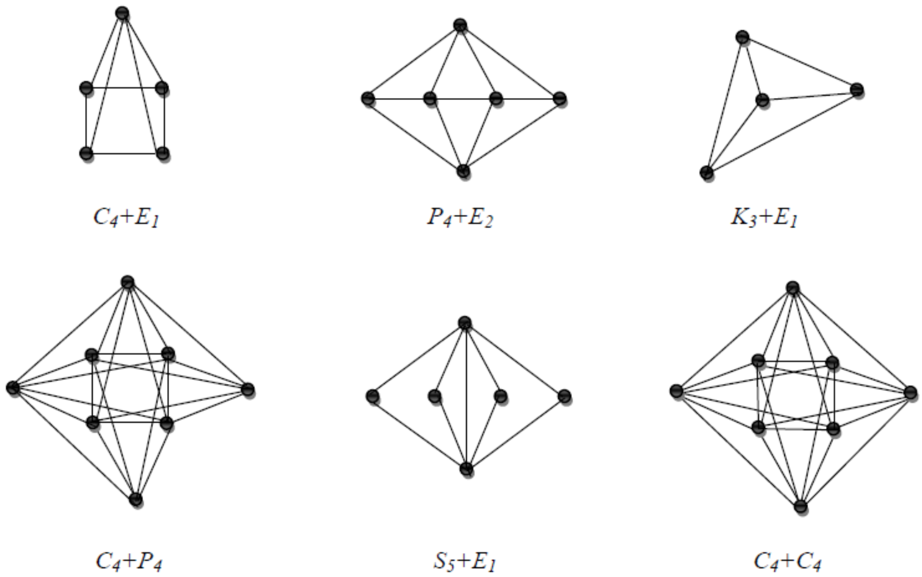


Figure 4.1: Join of some well-known graphs.

by constructing j blocks $(0 \leq j \leq k)$ from partitions of $V(G)$, for each formed block we form the $k - j$ remaining blocks from partitions of $V(H)$ and we sum over all possibilities of j. Summing over all possible values of k we get the Bell number. □ □

A natural generalization is to consider the join of $r(\geq 2)$ graphs.

Theorem 49. *Let $G_1, \dots, G_r$ be r graphs and denote by $G_1 + \cdots + G_r$ their join, with $|G_i| = n_i$, $\chi(G_i) = \chi_i$ and $k \in \left[\sum_{i=1}^{r} \chi_i, \sum_{i=1}^{r} n_i\right]$. Then*

$S(G_1 + \cdots + G_r, k) = \sum_{k_1+\cdots+k_r=k} S(G_1, k_1)S(G_2, k_2) \cdots S(G_r, k_r)$.(4.3)*and*

$$B_{G_1+\cdots+G_r} = \prod_{i=1}^{r} B_{G_i}. \tag{4.4}$$

Proof. The proof is done by induction on r. The result in trivial case $(r = 2)$ verified by Theorem 46. We set the induction hypothesis.

$$S(G_1 + \cdots + G_{r-1}, k) = \sum_{k_1+\cdots+k_{r-1}=k} S(G_1, k_1) \cdots S(G_{r-1}, k_{r-1}) \tag{4.5}$$

and

$$B_{G_1+\cdots+G_{r-1}} = B_{G_1}\cdots B_{G_{r-1}} \tag{4.6}$$

From Theorem 46 we have

$$S(G_1+\cdots+G_{r-1}+G_r,k) = \sum_{j=0}^{k}\sum_{j_1+\cdots+j_{r-1}=j} S(G_1,j_1)\cdots S(G_{r-1},j_{r-1})S(G_r,k-j) \tag{4.7}$$

Using the induction hypothesis Relation 4.7 can be rewritten as

$$S(G_1+\cdots+G_{r-1}+G_r,k) = \sum_{j_1+\cdots+j_r=k} S(G_1,j_1)\cdots S(G_r,j_r) \tag{4.8}$$

which completes the proof □

Applying this generalization to some special graphs we get these two well known identities,

Corollary 50. *Let $n_1, n_2, \cdots, n_r$ be positive integers and $n = n_1+n_2+\cdots+n_r$ with $n_1, n_2, \cdots, n_r$ not equal zero.*

$$\sum_{k=0}^{n}\sum_{j_1+\cdots+j_r=n-k} \frac{n_1}{n_1-j_1}\binom{n_1-j_1}{j_1}\cdots\frac{n_r}{n_r-j_r}\binom{n_r-j_r}{j_r} = l_{n_1}\cdots l_{n_r}, \tag{4.9}$$

Proof. From the Theorem 28 we have for $n \geq 2$, $\lceil \frac{n}{2} \rceil \leq k \leq n$

$$S(\overline{P_n},k) = \binom{k}{n-k}$$

and

$$B_{\overline{P_n}=f_{n+1}}$$

We compute using Theorem 47 the Bell number for the join of r complements of paths. By identification with the definition of Bell numbers we get the result. □

Corollary 51. *Let $n_1, n_2, \cdots, n_r$ be positive integers and $n = n_1+n_2+\cdots+n_r$ with $n_1, n_2, \cdots, n_r$ not equal zero.*

$$\sum_{k=0}^{n}\sum_{j_1+\cdots+j_r=n-k} \binom{n_1-j_1}{j_1}\cdots\binom{n_r-j_r}{j_r} = f_{n_1+1}\cdots f_{n_r+1}, \tag{4.10}$$

Proof. With the same way as in the previous corollary, we compute the Bell number of the join of r complements of cycles. □

4.2.3 Applications

Knowing the Stirling and the Bell numbers of some fewer graphs (section 3 in [35]); E_n, K_n, S_n, $\overline{S}_n$, P_n, $\overline{P}_n$, C_n, $\overline{C}_n$, $K_{m,n}$ that denote the empty graph, the complete graph, the star graph, the complementary of the star graph, the path graph, the complementary of the path graph, the cycle graph, the complementary of the cycle graph and the complete bipartite graph, respectively, we compute the Stirling and the Bell numbers of some join of two special graphs listed above. The table below summarizes these values.

While we set

$$C_{n,j} := \sum_{i=j}^{n} (-1)^{n-i} S(i,j) \text{ and } B'_n := \sum_{i=1}^{n} (-1)^{n-i} B_i.$$

as the Stirling numbers and the Bell numbers of the Cycle graph of order n respectively, for $\chi(C_n) \leq j \leq n$. We should mention that graphical Stirling numbers for join graphs in the following table with non-listed parameters k are equal to 0 and Stirling numbers of the second kind $S(n,k)$ are equal to 0 for $k < 0$ or $k > n$. 4pt

	G	H	$S(G+H,k)$	B_{G+H}	Conditions
1	E_{n_1}	E_{n_2}	$\sum_{j=0}^{k} S(n_1,j)S(n_2,k-j)$	$B_{n_1}\cdot B_{n_2}$	$2 \le k \le n_1+n_2$
2	K_{n_1}	K_{n_2}	1	1	$k = n_1+n_2$
3	S_{n_1+1}, P_{n_1+1}	S_{n_2+1}, P_{n_2+1}	$\sum_{j=0}^{k} S(n_1,j)S(n_2,k-j)$	$B_{n_1}\cdot B_{n_2}$	$2 \le k \le n_1+n_2+2$
4	$C_{n_1+1}(n_1\ge 2)$	$C_{n_2+1}(n_2\ge 2)$	$\sum_{j=0}^{k} (C_{n_1,j}\cdot C_{n_2,k-j})$	$B'_{n_1}\cdot B'_{n_2}$	$4 \le k \le n_1+n_2+2$
5	K_{m_1,n_1}	K_{m_2,n_2}	$\sum_{i_1+i_2+i_3+i_4=k} S(m_1,i_1)S(n_1,i_2)S(m_2,i_3)S(n_2,i_4)$	$B_{m_1}\cdot B_{n_1}\cdot B_{m_2}\cdot B_{n_2}$	$2 \le k \le n_1+n_2+m_1+m_2$
6	$\overline{S}_{n_1+1}$	$\overline{S}_{n_2+1}$	$\begin{cases} n_1\cdot n_2 & if \quad k=n_1+n_2 \\ 1 & if \quad k=n_1+n_2+2 \end{cases}$	$n_1\cdot n_2$	
7	$\overline{P}_{n_1}(n_1\ge 2)$	$\overline{P}_{n_2}(n_2\ge 2)$	$\sum_{j=0}^{k} \binom{j}{n_1-j}\binom{k-j}{n_2-k+j}$	$F_{n_1+1}\cdot F_{n_2+1}$	$\lceil (n_1+n_2)/2\rceil \le k \le n_1+n_2$
8	K_{n_1}	E_{n_2}	$S(n_2,k-n_1)$	B_{n_2}	$n_1+1 \le k \le n_1+n_2$
9	K_{n_1}	$C_{n_2+1}(n_2\ge 2)$	$C_{n_2,k-n_1}$	B'_{n_2}	$n_1+2 \le k \le n_1+n_2+1$
10	E_{n_1}	$\overline{S}_{n_2}(n_2\ge 2)$	$(n_2-1)S(n_1,k-n_2+1)+S(n_1,k-n_2)$	$n_2\cdot B_{n_1}$	$n_2-1 \le k \le n_1+n_2$
11	P_{n_1+1}	$C_{n_2+1}(n_2\ge 2)$	$\sum_{j=0}^{k} S(n_1,j)\cdot C_{n_2,k-j}$	$B_{n_1}\cdot B'_{n_2}$	$4 \le k \le n_1+n_2+2$
12	$\overline{S}_{n_1}(n_1\ge 2)$	$\overline{P}_{n_2}(n_2\ge 2)$	$(n_1-1)\binom{k-n_1+1}{n_2-k+n_1-1}+\binom{k-n_1}{n_2-k+n_1}$	$n_1\cdot F_{n_2+1}$	$\lceil (n_2+1)/2\rceil \le k \le n_1+n_2$
13	$\overline{P}_{n_1}(n_1\ge 2)$	$\overline{C}_{n2}(n_2\ge 4)$	$\sum_{j=1}^{k} \frac{n_1}{j}\binom{j}{n_1-j}\binom{k-j}{n_2-(k-j)}$	$F_{n_1+1}\cdot L_{n_2}$	$\lceil (n_1+n_2)/2\rceil \le k \le n_1+n_2$

Applying the generalization to the join of r graphs of orders $n_1, \cdots, n_r$ respectively, we give also;

For the join of r empty graphs, $n_1 \geq 2, \cdots, n_r \geq 2$ we have,

$S(E_{n_1} + \cdots + E_{n_r}, k) = \sum_{j_1+\cdots+j_r=k} S(n_1, j_1) \dots S(n_r, j_r)$.

For the join of r complete graphs, $n_1 \geq 2, \cdots, n_r \geq 2$, we have,

$S(K_{n_1} + \cdots + K_{n_r}, k) = 1$ for $k = n_1 + \cdots + n_r$ and 0 otherwise.

For the join of r trees, $n_1 \geq 2, \cdots, n_r \geq 2$, we have

$S(T_{n_1} + \cdots + T_{n_r}, k) = \sum_{j_1+\cdots+j_r=k} S(n_1 - 1, j_1 - 1) \cdots S(n_r - 1, j_r - 1)$.

For the join of r generalized m-trees, $n_1 \geq m+1, \cdots, n_r \geq m+1$

$S(T_{n_1^m} + \cdots + T_{n_r^m}, k) = \sum_{j_1+\cdots+j_r=k} S(n_1 - m, j_1 - m) \cdots S(n_r - m, j_r - m)$.

4.3 Bell and Stirling numbers for some corona graphs

4.3.1 Definitions

Definition 52. *The corona of two graphs G and H is the graph GoH formed from one copy of G and $|V(G)|$ copies of H, where the i'th vertex of G is adjacent to every vertex in the i'th copy of H. See in Figure 4.2 some special corona graphs.*

Let G be a graph of order n, P_m a path of order m and let $G \odot P_m$ be the super graph of order $n + m$ obtained by joining P_m to one vertex of G with m edges. An example is in Figure 4.3.

4.3.2 Identities and explicit formulas

The Stirling number of $G \odot P_m$ can be calculated using the following recurrence,

Lemma 53. *For $1 \leq k \leq m + n$ we have,*

$$S(G \odot P_m, k) = S(G \odot P_{m-1}, k - 1) + (k - 2)S(G \odot P_{m-1}, k).$$

Proof. The recurrence can be proved with a bijective proof, choosing a vertex v_1 from P_m there is two possible ways of partition of $G \odot P_m$; either v_1 forms a block alone, in this case we partition $G \odot P_{m-1}$ into $k - 1$ blocks and we have $S(G \odot P_{m-1}, k -$

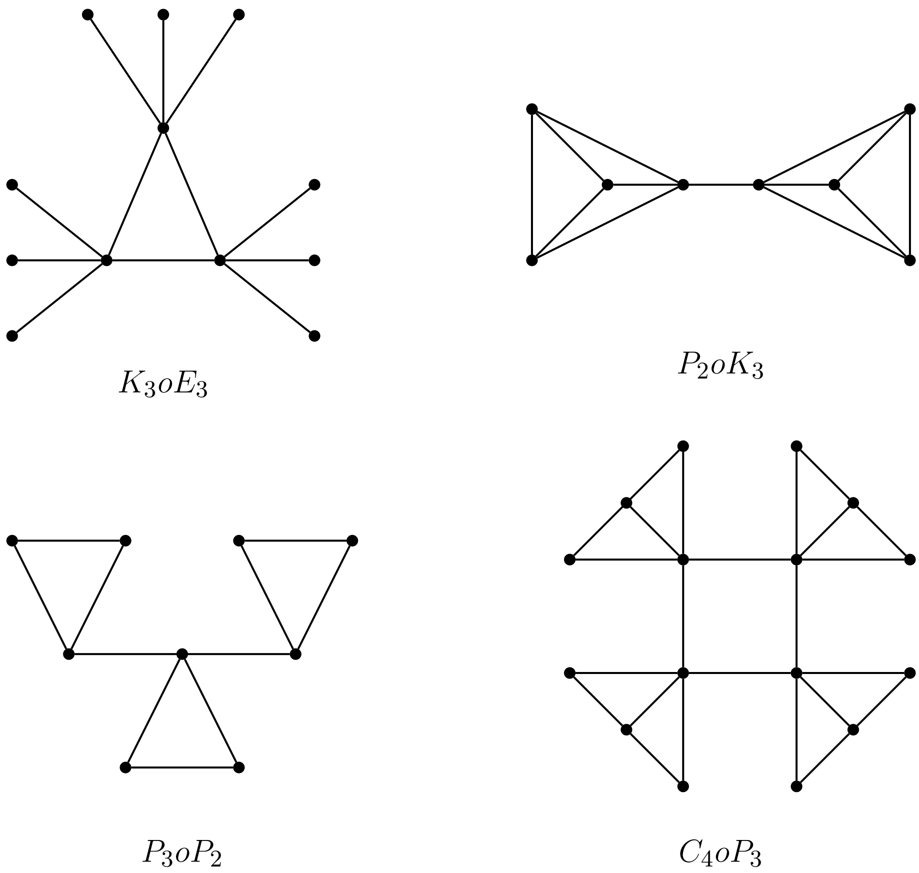


Figure 4.2: Some special Corona graphs.

1) ways to do it, or v_1 belongs to the blocks of the partition, hence we have to partition $G \odot P_{m-1}$ into k blocks and assign v_1 to blocks with $(k-2)S(G \odot P_{m-1}, k)$ possibility. □

Theorem 54. *For $m \geq 1$, $k \leq m+n$ we have*

$$S(G \odot P_{m+1}, k+1) = \sum_{j=0}^{k-1} S(G^{(1)}, k-j+1) \sum_{l=0}^{m-j} (k-j-1)^{m-l-j} \sum_{i_1+\cdots+i_j=l} (k-1)^{i_1} \cdots (k-j)^{i_j}.$$

Proof. For simplicity, we adopt the notation $\omega_{m,k} = S(G \odot P_m, k), m \geq 1$. From the Lemma 51 we have

$$\omega_{m+1,k+1} = \omega_{m,k} + (k-1)\omega_{m,k+1}.$$

We prove by induction over m the following recurrence

$$\omega_{m+1,k+1} = \sum_{i_1=0}^{m-1} (k-1)^{i_1} \omega_{m-i_1,k} + (k-1)^m \omega_{1,k+1}, \tag{4.11}$$

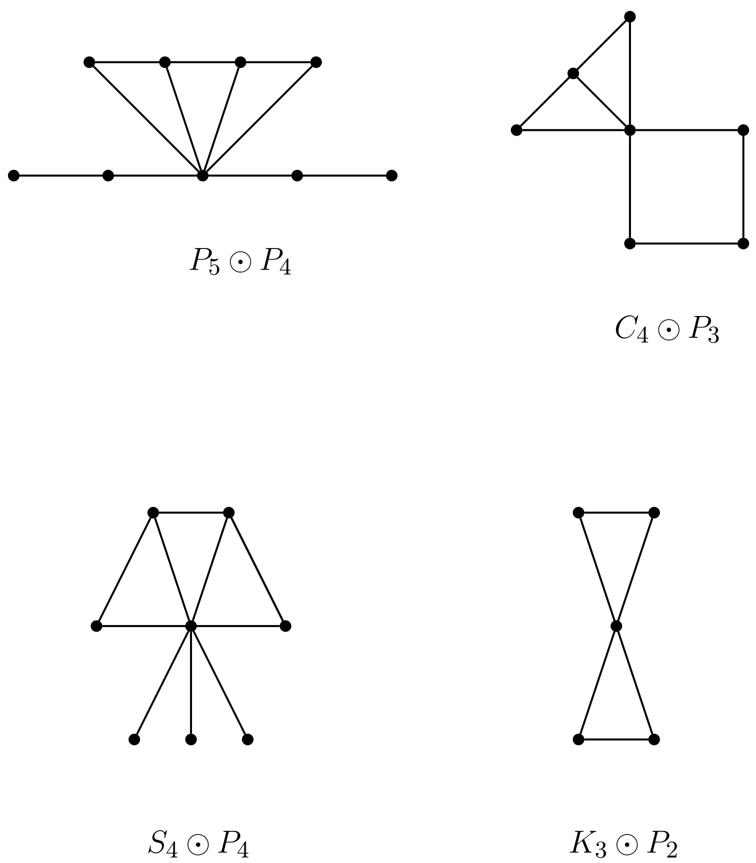


Figure 4.3: Examples of $G \odot H$

we can verify by the previous Lemma that Relation (1) is true for the trivial case $(m = 1)$, hence

$$\omega_{2,k+1} = \omega_{1,k} + (k-1)\omega_{1,k},$$

now, assume that Relation (4.11) is verified for $m+1$ and prove it for $m+2$. By the previous Lemma and the induction hypothesis we have,

$$\omega_{m+2,k+1} = \sum_{i_1=0}^{m-1} (k-1)^{i_1+1} \omega_{m-i_1,k} + (k-1)^{m+1}\omega_{1,k+1} + \omega_{m+1,k}, \tag{4.12}$$

we set $i_1\prime = i_1 + 1$ and we obtain

$$\omega_{m+2,k+1} = \sum_{i_1\prime=1}^{m} (k-1)^{i_1\prime}\omega_{m-i_1\prime+1,k} + \omega_{m+1,k} + (k-1)^{m+1}\omega_{1,k+1},$$

Relation (4.12) is equivalent to,

$$\omega_{m+2,k+1} = \sum_{i_1=0}^{m} (k-1)^{i_1}\omega_{m-i_1+1,k} + (k-1)^{m+1}\omega_{1,k+1},$$

thus the Relation (4.11) is true for $m \geq 1$. Also, we have,

$$\omega_{m-i_1,k} = \sum_{i_2=0}^{m-i_1-2} (k-2)^{i_2}\omega_{m-i_1-i_2-1,k-1} + (k-2)^{m-i_1-1}\omega_{1,k},$$

hence, using the same approach as for Relation (4.11), we establish by induction over m,

$$\begin{aligned}\omega_{m+1,k+1} = &\sum_{i_1=0}^{m-1}\sum_{i_2=0}^{m-i_1-2} (k-1)^{i_1}(k-2)^{i_2}\omega_{m-i_1-i_2-1,k-1} \\ &+ \sum_{i_1=0}^{m-1}(k-2)^{m-i_1-1}\omega_{1,k} + (k-1)^m\omega_{1,k+1}.\end{aligned}$$

By developing the sum in the right hand side with the same way and applying the same inductive procedure, we get,

$$\begin{aligned}\omega_{m+1,k+1} = &(k-1)^m\omega_{1,k+1} + \\ &\sum_{i_1=0}^{m-1}(k-1)^{i_1}(k-2)^{m-i_1-1}\omega_{1,k} \\ &\sum_{i_1=0}^{m-1}\sum_{i_2=0}^{m-i_1-2}(k-1)^{i_1}(k-2)^{i_2}(k-3)^{m-i_1-i_2-2}\omega_{1,k-1} + \\ &\vdots \\ &\sum_{i_1=0}^{m-1}\sum_{i_2=0}^{m-i_1-2}\cdots\sum_{i_{k-1}=0}^{m-i_1-\cdots-i_{k-2}-(k-1)}(k-1)^{i_1}(k-2)^{i_2}\cdots 1^{i_{k-1}}0^{m-i_1-\cdots-i_{k-1}-(k-1)}\omega_{1,2} + \\ &\sum_{i_1=0}^{m-1}\sum_{i_2=0}^{m-i_1-2}\cdots\sum_{i_{k-1}=0}^{m-i_1-\cdots-i_{k-2}-(k-1)}\sum_{i_k=0}^{m-i_1-\cdots-i_{k-1}-(k)}(k-1)^{i_1}\cdots 1^{i_{k-1}}0^{i_k}\omega_{m-i_1-\cdots-i_k-(k-1),1}.\end{aligned}$$

The last term is equal to 0, since $\omega_{m-i_1-\cdots-i_k-(k-1),1} = 0$ we have $m \geq 1$ and the only case for which $\omega_{m-i_1-\cdots-i_k-(k-1),1}$ take a value is when $(G_n, P_m) = (E_n, P_0)$, otherwise we can not partition $G_n \odot P_m$ into only one block. Thus the relation (3) is equivalent to,

$$\omega_{m+1,k+1} = \sum_{j=0}^{k-1} \omega_{1,k-j+1} (\sum_{i_1=0}^{m-1} \sum_{i_2=0}^{m-i_1-2} \cdots$$

$$\sum_{i_j=0}^{m-i_1-\cdots-i_{j-1}-j} (k-1)^{i_1}(k-2)^{i_2} \cdots (k-j)^{i_j}(k-j-1)^{m-\cdots-i_j-j}),$$

this can be written in a simplified form as,

$$\omega_{m+1,k+1} = \sum_{j=0}^{k-1} \omega_{1,k-j+1} \sum_{l=0}^{m-j} (k-j-1)^{m-l-j} \sum_{i_1+\cdots+i_j=l} (k-1)^{i_1} \cdots (k-j)^{i_j}.$$

□

We have to note that the previous theorem can be extended to the following one, as the case for the generalization of thorn graphs, that is, an other class whose Stirling number depends on the generalized Stirling number of the second kind is $G_n \odot P_m$;

Theorem 55. *For $m \geq 1$, $k \leq m+n$ we have,*

$$S(G_n \odot P_m, k) = \sum_{j=0}^{k-1} S(G^{(1)}, k-j+1) \sum_{l=0}^{m-j} (k-j-1)^{m-l-j} S_{k-j}(l+k-1, k-1).$$

Proof. As proved in Theorem 38

$$\sum_{i_1+\cdots+i_{j+1}=t-j} (k-1)^{i_1} \cdots (k-j-1)^{i_{j+1}} = S_{k-j-1}(t+k-j-1, k-1).$$

We set $j\prime = j+1$ we get,

$$\sum_{i_1+\cdots+i_{j\prime}=t-j\prime+1} (k-1)^{i_1} \cdots (k-j\prime)^{i_{j\prime}} = S_{k-j\prime}(t+k-j\prime, k-1),$$

we fix $t = m$ and $l = m-j+1$ we obtain,

$$\sum_{i_1+\cdots+i_{j\prime}=l+1} (k-1)^{i_1} \cdots (k-j\prime)^{i_{j\prime}} = S_{k-j\prime}(l+k-1, k-1),$$

we replace in Theorem 52, we get the result. □

This formula can be generalized to compute the Stirling number of the corona product GoP_m.

Theorem 56. *For $m \geq 1$, $k \leq m+n$ we have*

$$S(G_n o P_{m+1}, k+1) = \sum_{j_1=0}^{k-1} \cdots \sum_{j_n=0}^{k-j_1-\cdots-j_{n-1}-1} S(G_n^{(n)}, k-j_1-\cdots-j_s+1)$$
$$\sum_{s=1}^{n} \sum_{l=0}^{m-j_s} (k-j_s-1)^{m-l-j_s} S_{k-j_s}(l+k-1, k-1).$$

Proof. By definition we have, $GoP_m = (((\underbrace{G \odot P_m) \odot P_m) \cdots \odot P_m}_{ntimes})$, doing the operation $\odot$ repeatedly to every vertex of G ($|G| = n$ times) and applying Theorem 53 to the resultant graph for each step we get the result. □

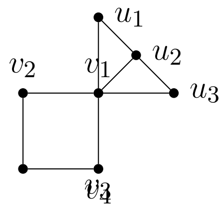


Figure 4.4: Representation of $C_4 \odot P_3$

Now, let us take a cycle of order m, C_m, we set $G \odot C_m$ a graph obtained by joining C_m to a vertex of G with m edges. See an example in Figure 4.5.

Theorem 57. *For $m \geq 3$, $k \leq m+n$ we have,*

$$S(G \odot C_m, k) = \sum_{i=0}^{m-1} (-1)^i S(G \odot P_{m-i}, k).$$

Proof. We prove the result by induction using the deletion-contraction principle. It is easy to verify the formula for the trivial case, thus we have the equality $S(G_n \odot C_1, k) = S(G_n \odot P_1, k)$ (graph in Figure in the right side), now suppose

$$S(G \odot C_m, k) = \sum_{i=0}^{m-1} (-1)^i S(G \odot P_{m-i}, k),$$

and prove the following

$$S(G \odot C_{m+1}, k) = \sum_{i=0}^{m} (-1)^i S(G \odot P_{m-i+1}, k),$$

let e be an edge in C_m, hence from the deletion contraction theorem we have,

$$S(G \odot C_{m+1}, k) = S(G \odot C_{m+1} - e, k) - S(G \odot C_{m+1}/e, k),$$

the latter is equivalent to

$$S(G \odot C_{m+1}, k) = S(G \odot P_{m+1}, k) - S(G \odot C_m, k),$$

using the induction hypothesis we obtain,

$$S(G \odot C_{m+1}, k) = S(G \odot P_{m+1}, k) - \sum_{i=0}^{m-1} (-1)^i S(G \odot P_{m-i}, k),$$

thus,

$$S(G \odot C_{m+1}, k) = \sum_{i=0}^{m} (-1)^i S(G \odot P_{m-i+1}, k).$$

□

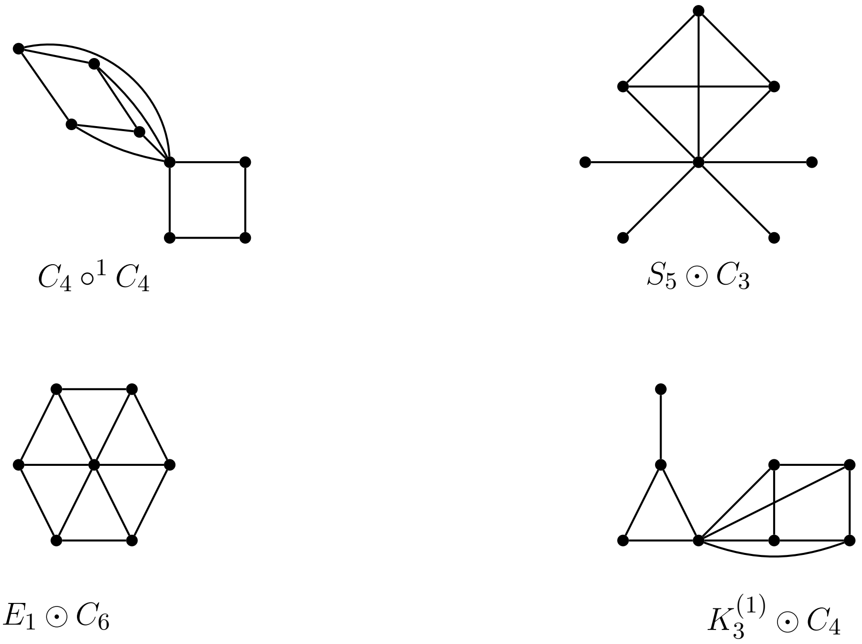


Figure 4.5: Examples of $G \odot C_n$.

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5

Enumerating restricted resolving partitions involving the 2-associated Stirling numbers of the second kind

Outlines

5.1 Introduction

The concept of resolving sets was first introduced by Slater [60, 59] under the name locating set and he referred to the minimum cardinality among all resolving sets as location number $loc(G)$, studied later under the name metric dimension. Later, under the same denominations used by Slater, Harary and Melter defined independently the resolving sets and the metric dimension.

A huge attention was given to the study of these concepts [32, 60, 59, 20, 19], since they have several applications in various real world problems mainly in chemistry for representing chemical compounds [18, 19], network discovery, robot navigation [20] and pharmaceutical chemistry.

According to Chartrand et al [19], a fundamental mathematical problem in the study of chemical structures is that of providing a mathematical classification of chemical compounds which can be studied using mathematical modeling objects that allow to understand the chemical structures, we refer to the thesis [57] for more details of the applications of resolving sets and metric dimension.

Additionally, Slater in [60, 59] gave the utility of these concepts when working for an electronic positioning system such as U. S sonar and Coast Guard Loran stations.

We return to the same example that appears in [57] to show a direct application of resolving sets.

Suppose that a certain facility consists of five rooms R_1, R_2, R_3, R_4, R_5 (shown in Figure 5.1). The distance between rooms R_1 and R_3 is 2 and the distance between R_2 and R_4 is also 2. The distance between all other pairs of distinct rooms is 1. The distance between a room and itself is 0. Suppose that a certain (red) sensor is placed in one of the rooms. If a fire should take place in one of the rooms, then the sensor is able to detect the distance from the room with the red sensor to the room containing the fire. Suppose, for example, that the sensor is placed in R_1. If a fire occurs in R_3 , then the sensor alerts us that a fire has occurred in a room at distance 2 from R_1, that is, the fire is in R_3 since R_3 is the only room at distance 2 from R_1. If the fire is in R_1, then the sensor indicates that the fire has occurred in a room at distance 0 from R_1, that is, the fire is in R_1. However, if the fire is in any of the other three rooms, then the sensor tells us that there is a fire in a room at distance 1 from R_1. But with this information, we cannot tell exactly in which room the fire has occurred. In fact, there is no room in which the (red) sensor can be placed to identify the exact location of a fire in every instance.

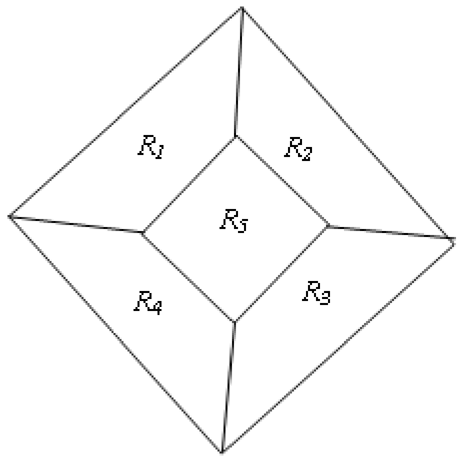


Figure 5.1: A facility consisting of five rooms.

On the other hand, if we place the red sensor in R_1 and a blue sensor in R_2, and a fire occurs in R_4 , say, then the red sensor in R_1 tells us that there is a fire in a room at distance 1 from R_1, while the blue sensor tells us that the fire is in a room at distance 2 from R_2 , that is, R_4 has the code $(1, 2)$. Since the codes are distinct for all rooms, the minimum number of sensors required to detect the exact location of any fire is two. Even though 2 is the answer, care must be taken as to where the two sensors are placed. For example, we cannot place sensors in R_1 and R_3 since, in this case, the codes of R_2, R_4 , and R_5 are all $(1, 1)$, and we cannot distinguish the precise location of the fire.

The facility that we have just described can be represented by a graph, whose vertices are the rooms (see Figure 5.2). (Notice that this is actually the same graph as that in Figure , except for the way in which the vertices are labeled.) For the graph G of Figure then, the dimension of G is 2 and R_1, R_2 is a minimum resolving set for G.

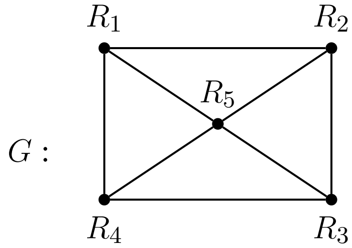


Figure 5.2: A graph representing a facility of five rooms.

5.2 Basic results on resolving sets and resolving partitions

5.2.1 Resolving sets

A vertex $w \in V(G)$ resolves a pair of vertices $u, v \in V(G)$ if $d(u, w) \neq d(v, w)$ in G.

For an ordered set $W = \{w_1, \cdots, w_k\} \subseteq V(G)$ and a vertex v of G, we refer to the k-vector

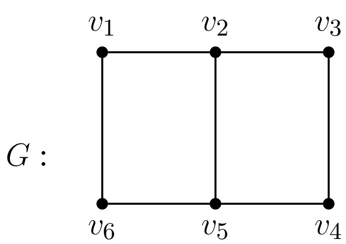
$$c_W(v) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$$

as the code of v with respect to W .

Definition 58. *A set $S \in V(G)$ is called a resolving set of G if, for each pair of vertices u, v there exist at least one vertex $w \in S$, such that w resolves u, v.*

In other words, the set S is called a resolving set for G if distinct vertices in G have distinct codes.

The metric dimension of G, denoted by $dim(G)$ is the minimum cardinality among all the resolving sets.



Example 59.

The ordered set $S_1 = \{v_2, v_5\}$ *is not a resolving set for* G *since* $c_{S_1}(v_3) = (1, 2) = c_{S_1}(v_1)$*, that is,* G *contains two vertices with the same code.*

On the other hand, $S_1 = \{v_1, v_6\}$ *is a resolving sets, since the codes of all the vertices are pairwise distinct, that is,*

$c_{S_1}(v_2) = (1, 2)$*,* $c_{S_1}(v_3) = (2, 3)$*,* $c_{S_1}(v_4) = (3, 2)$ *and* $c_{S_1}(v_5) = (2, 1)$*.*

On the other hand, $W_2 = \{v_1, v_2, v_3\}$ is a resolving set for G since the codes for the vertices of G with respect to W_2 are

$$c_{W_2}(v_1) = (0, 1, 1), c_{W_2}(v_2) = (1, 0, 1), c_{W_2}(v_3) = (1, 1, 0),$$

$$c_{W_2}(v_4) = (1, 2, 1), c_{W_2}(v_5) = (2, 1, 1).$$

However,W_2 is not a minimum resolving set for G since $W_3 = \{v_1, v_2\}$ is also a resolving set.

The dimension of some well-known classes of graphs have been established in [32, 60, 59, 19].

Theorem 60. *Let* G *be a connected graph of order* $n \geq 2$*. We have,*

- $dim(G) = 1$ *if and only if* $G = P_n$*, the path of oredr* n*.*
- $dim(G) = n - 1$ *if and only if* $G = K_n$*, the complete graph of order* n*.*
- *For* $n \geq 3$*,* $dim(C_n) = 2$*, where* C_n *is the cycle of order* n*.*
- *For* $n \geq 4$*,* $dim(G) = n - 2$*, if and only if* $G = K_{r,s}$ $(r, s \geq 1)$*,* $G = K_r + \overline{K_s}$ $(r \geq 1, s \geq 2)$*, or* $G = K_r + (K_1 \cup K_s)$ $(r, s \geq 1)$*.*

Before introducing the concept of k-resolving set partition, we should give the definitions of resolving partitions appearing in the literature, we state by the way some results associated to this type of partitions.

5.2.2 Resolving partitions

Let S be a set of vertices of G.

Definition 61. *The distance between a vertex $v \in G$ and S is defined to be the minimum distance between v and elements of S, that is*

$$d(v, S) = min\{d(v, x) : x \in S\}.$$

Definition 62. *For $\pi = \{S_1, S_2, \cdots, S_k\}$ being an ordered k-partition of $V(G)$, the code of a vertex $v \in G$ with respect to π is defined as the k-vector*

$$c_\pi(v) = (d(v, S_1), d(v, S_2), \cdots, d(v, S_k)).$$

The partition is called a resolving partition of G if the distinct vertices of G have distinct codes with respect to π.

The minimum k for which there is a resolving k-partition of $V(G)$ is the partition dimension $pd(G)$ of G.

A resolving partition of $V(G)$ containing $pd(G)$ elements is called a minimum resolving partition.

The resolving partitions and the partition dimension of a some graph classes were studied by Chartrand et al [21, 22] who established the partition dimension for some particular classes of trees. Also, they gave the partition dimensions of double stars and certain caterpillars. They also characterized all graphs with partition dimension 2, and all graphs on n vertices with partition dimension $n-1$ or n. Lately, Tomescu characterized all graph of order n having partition dimension n - 2.

Theorem 63. *If G is nontrivial connected graph, then $pd(G) \leq dim(G) + 1$.*

Theorem 64. *Let G be a connected graph of order $n \geq 2$. Then*

- $pd(G) = 2$ *if and only if* $G = P_n$.
- $pd(G) = n$ *if and only if* $G = K_n$.
- $pd(G) = n - 1$ *if and only if* $G \in \{\}$

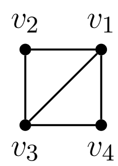
5.3 Enumerating k-resolving set partitions involving the 2-associated Stirling numbers of the second kind

5.3.1 k-resolving set partitions

Definition 65. *We define the number of k- resolving set partitions of a graph G to be the number of partitions into k resolving sets, denoted by $RS(G, k)$, in other words it corresponds to the number resolving partitions with all their k parts are resolving sets.*

We use a combinatorial concept to count the number of restricted resolving partitions for some special graphs, thus some lower bounds for the number of resolving partitions for these classes.

G :



Example 66.

It is easy to verify that each singleton doesn't constitute a resolving set.

The number of 2-resolving set partitions of G is done by $RS(G, 2) = 2$, with the possible partitions: $\{\{1, 2, 3, 4\}\}$ *and* $\{\{1, 4\}, \{2, 3\}\}$.

5.3.2 Enumerating restricted resolving partitions for paths, cycles and Wheels involving associated Stirling numbers

The resolving sets in a complete graph K_n and a star S_n should take at least all the vertices except one.

$$RS(K_n, k) = RS(S_n, k) = \begin{cases} 1 & \text{for k=1} \\ 0 & \text{otherwise} \end{cases}$$

Lemma 67. *For a cycle of an odd order, every pair of vertices is a resolving set.*

Proof. Assume that there exist a pair $x, y \in C_n$ such that $S = \{x, y\}$ is not a resolving set which means, $\exists\ u_i, u_j \in V(C_n)$ such that

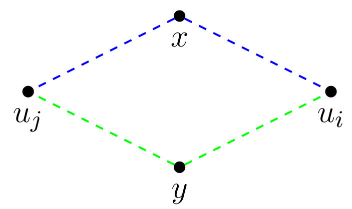
$$d(x, u_i) = d(x, u_j), , \ x \text{ is in the middle of the vertices } u_i \text{ and } u_j$$

and

$$d(y, u_i) = d(y, u_j), \ y \text{ is in the middle of the vertices } u_i \text{ and } u_j.$$

Hence,

$$d(x, u_i) + d(y, u_i) = d(x, u_j) + d(y, u_j).$$



Thus, there exist two distinct paths of the same length that link x to y. Contradiction with the definition of an odd cycle. □

From this lemma and the first Theorem 60, we can conclude the following.

Theorem 68. *Let C_n be a cycle of an odd order n, for $0 \le k \le n$ we have,*

$$RS(C_n, k) = S^{(2)}(n, k).$$

Lemma 69. *Let P_n be a path of order n,*

- *every pair of distinct vertices in P_n forms a resolving set,*
- *if S is a resolving set of P_n then $|S| = 1$ if and only if S consists of one of the two ends of P_n.*

Proof. For the first point, the proof proceeds with the same way as in the previous lemma so that we obtain a contradiction with the definition of a path (between every two vertices there exist a unique path).

Now let S be a resolving set in P_n and u_1, u_n be the two end vertices of P_n. Assume that S is neither u_1 nor u_n but $S = \{u_i\}, i \neq 1, n$, then $\exists\ u_{i-1}, u_{i+1}$ (the predecessor and the successor respectively of u_i) such that $d(u_i, u_{i-1}) = d(u_i, u_{i+1}) = 1$, contradiction with the definition of a resolving set. □

Theorem 70. *For $0 \leq k \leq n$ we have,*

$$RS(P_n, k) = S^{(2)}(n, k) + S^{(2)}(n-1, k-1) + S^{(2)}(n-2, k-2).$$

Proof. To recapitulate from Theorem 58 and the previous lemma, the blocks of the partitions would be either blocks that contain at least two elements or singletons consisting on the end vertices. In the first situation, this amounts to partition an n-element set into k subsets so that each subset contains at least two elements, this coincides with the 2-associated Stirling numbers of the second kind $S^{(2)}(n, k)$. In the latter situation, either each end vertex forms a block alone, we have $S^{(2)}(n-2, k-2)$ possible ways to do it, or only one of them forms a singleton, hence we have $2S^{(2)}(n-1, k-1)$ possible ways. $\square$

Conclusion

Our aim was to combinatorially interpret classical sequences using graph theoretical concepts.

As mentioned in [25], the problem of evaluating the chromatic polynomial is at least as hard as the problem of finding the chromatic number which is in its general form NP-complete. We provided an alternative way to compute the chromatic polynomial by recurrence formulas which are more efficiently for some classes of graphs, although the complexity of these methods is not studied in the present document, this can be improved in future studies.

We showed how combinatorial interpretations can allow to construct identities and explicit formulas beginning with simple configurations.

Along this thesis, we explored the Stirling, the r-Stirling, the 2-associated Stirling numbers of the second kind and we conclude for the Bell numbers, for different classes of graphs, namely, Thorn and generalized Thorn graphs, generalized cyclic graphs, some join graphs and special corona product graphs. For some families, using recursions based on combinatorial interpretations, we established explicit formulas to count the number of independent partitions, in which we proved that the number of independent partitions for that classes depends on the Stirling and the r-Stirling numbers of the second kind. For the others, inspired by the idea of deletion-contraction, we gave an extension to the study of Bell and Stirling numbers. As consequences, this gave rise to new properties and identities related to Stirling and r-Stirling numbers of the second kind, besides that, this study allowed us to further derive well-known identities related to Fibonacci and Lucas sequences.

In the last chapter, we introduced a new concept, that is, k-resolving set partitions, we opened a direction to explore the 2-associated Stiling numbers of the second kind and it was surprising that the number of k-resolving set partitions coincides with the 2-associated Stirling numbers of the second kind for cycles, also we established a recurrence to evaluate the number of that partitions in terms of the 2-associated Stirling numbers of the second kind.

Open ends, lingering questions and ideas will be discussed in the perspectives

Perspectives

Problem 1

The Lah numbers denoted by $L(n,k)$ count the number of partitions of $[n]$ into k ordered lists. See [4, 51, 58, 66]

They satisfy

$$x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} L(n,k)x(x-1)\cdots(x+k-1).$$

They obey to the recurrence relation

$$L(n,k) = L(n-1,k-1) + (n+k-1)L(n-1,k).$$

They have the following explicit formula for n, k nonnegative integers

$$L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}.$$

We define the Lah number for a graph G to be the number of independent partitions of $V(G)$ into k ordered stables.

For $G = E_n$ an empty graph, $L(E_n,k)) = L(n,k)$.

For $G = K_n$ a complete graph, $L(K_n,k) = 1$ if $k = n$ and 0 otherwise.

It is of interest to find the recurrence relation to evaluate $L(G,k)$ for well-known graphs.

We take for instance a path of order n, P_n and we seek a recurrence relation to count the number of independent partitions of P_n into k ordered stables.

If we choose an end vertex n, then we have two possible situations; either n forms a block alone, in this case **no problem** we have to partition the remaining vertices into $k-1$ independent lists with $L(P_{n-1}, k-1)$ ways; or n belongs to any block in the partition except the one who contains its neighbor: the vertex $n-1$ since they are linked, therefore, the assignment of the vertex n and its position depends on the list containing its neighbor and the cardinal of that list!

If we have the partitions of $V(P_n)$ into independent blocks, it is clear that we can deduce the partitions into independent lists associated to this partition whenever we know the cardinal of each block.

Example 71. *let* $P = \{\{1\}, \{2,4\}, \{3,5\}\}$ *be an independent partition of* P_5 *into* 3 *blocks. The partitions into independent lists associated to this partition* P *are:* $\{(1), (2,4), (3,5)\}; \{(1), (4,2), (3,5)\}; \{(1), (2,4), (5,3)\}; \{(1), (4,2), (5,3)\}$

Furthermore, if we have the number of independent partitions of P_n into k blocks, $S(P_n, k)$ with the size of each block, then we can compute the number of partitions into k independent lists $L(P_n, k)$.

In order to describe explicitly the situation, we need parameters containing the sizes of the blocks for each partition, this pushed us to think for introducing **the graphical partial Bell polynomial** for a simple graph G, denoted by $B_{G_n,k}(a_1, a_2, \cdots)$, where the a_i's interprets the presence of a block of size i in the partition. The sum of the coefficients of this polynomial corresponds to the Stirling number of G, the sum of its indices in each term is n and the sum of the powers is k. we refer to [3] to see ordinary partial bell polynomial

Example 72. *Given a path of order* 5*,* P_5*.*

There is one possibility to partition P_5 *into* 2 *blocks which are of sizes* 2 *and* 3*, that is, 2,4/1,3,5, thus* $B_{P_5,2} = 1.a_2a_3$*.*

It follows that, $S(P_5, 2) = 1$ *and* $L(P_5, 2) = 1.2!.3! = 12$*.*

While, $B_{P_5,3} = 1.a_1a_1a_3 + 6.a_1a_2a_2$*, since, we have one way to partition* P_5 *into* 3 *blocks of sizes* $1,1$ *and* 3*, that is,* $1,3,5/2/4$ *and six ways for blocks of sizes* $1,2$ *and* 2*, the possible situations are,* $1,3/2,5/4$*,* $1,3/2,4/5$*,* $1,4/2,5/3$*,* $1,4/3,5/2$*,* $1,5/2,4/3$*,* $1/2,4/3,5$*.*

It follows that $S(P_5, 3) = 1 + 6 = 7$ *and* $L(P_5, 3) = 1.1!.3! + 6.1!.2!.2! = 30$*.*

We propose in the following table to evaluate $B_{P_n,k}$ for some values of k and n:

n/k	1	2	3	4	5	6	7
1	a_1	0	0	0	0	0	0
2	0	a_1^2	a_1a_2	a_2^2	a_3a_2	a_3^2	a_3a_4
3	0	0	a_13	$3a_1^2a_2$	$a_1^2a_3+6a_1a_2^2$	$10a_1a_2a_3+5a_2^3$	$3a_1a_2a_4+8a_1a_3^2+19a_2^2a_3$
4	0	0	0	a_1^4	$6a_1^3a_2$	$21a_1^2a_2^2+4a_1^3a_3$	
5	0	0	0	0	a_1^5	$10a_1^4a_2$	
6	0	0	0	0	0	a_1^6	$15a_1^5a_2$
7	0	0	0	0	0	0	a_1^7

The problem consists of mathematically characterizing the polynomial $B_{P_n,k}$ (and $B_{G_n,k}$ in general), which means, defining a recurrence relation to determine this polynomial for different classes of graphs.

Problem 2

We define the s-associated Stirling numbers of the second kind for a graph G denoted by $S^{(s)}(G_n,k)$ to be the number of partitions of $V(G)$ into k independent subsets so that each subset contains at least s elements.

$S^{(s)}(E_n,k) = S^{(s)}(n,k)$ and $S^{(s)}(K_n,k) = 1$ if $n = k$ and $s = 1$ and 0 otherwise.

In terms of coloring, it can be thought of as the number of ways to color a given graph using exactly k colors in such a way that each color is used at least s times.

Equivalently, it represents the number of partitions of $\overline{G}$ into k cliques such that each clique is of order at least s.

In particular, when $s = 2$ and $n = 2k$, $S^{(2)}(2k,k)$ corresponds to the number of perfect matchings in $\overline{G}$, where, a matching is a set of pairwise disjoint edges and a perfect matching is a matching taking all the vertices of the graph.

It follows that the problem in its general form is at least as hard as the problem of finding the number of perfect matching in a graph.

A lingering question is to determine that number for graphs (beginning with well-known graphs), its properties, its relation with the graphical Stirling number and the associated Stirling number if exist and the polynomial encoding that colorings in order to get new identities and explicit formulas in this context.

Problem 3

As described in the last chapter, the number of k-resolving set partitions for cycles coincides with the 2-associated Stirling number of the second kind, in the same direction, it would be interesting to treat other classes of graphs and generalize the idea to the s-associated Stirling numbers of the second kind, also, determine the link among that type of partitions, the s-associated Stirling numbers of the second kind and the concept of resolving partitions studied in the literature.

Furthermore, a resolving partition into k blocks that satisfies: each block is an independent vertex set is called a resolving independent partition, it has been studied in [18, 17] and referred as locating independent partition, the number of such partitions is equal to the number of colorings using k colors that distinguishes all vertices of G in terms of their distances from the color classes, it is called a resolving coloring. A minimum number that uses a resolving coloring is called a resolving chromatic number which has been established for some well-known graphs by [18, 17].

Inspired by these works, we add the constraint of independence in blocks to the k-resolving set partition, this creates a new problem that may have connections with the graphical Stirling number and the number of colorings encoded by the associated Stirling numbers of the second kind (defined in Problem 2).

A

Some special graphs derived from the treated graph classes

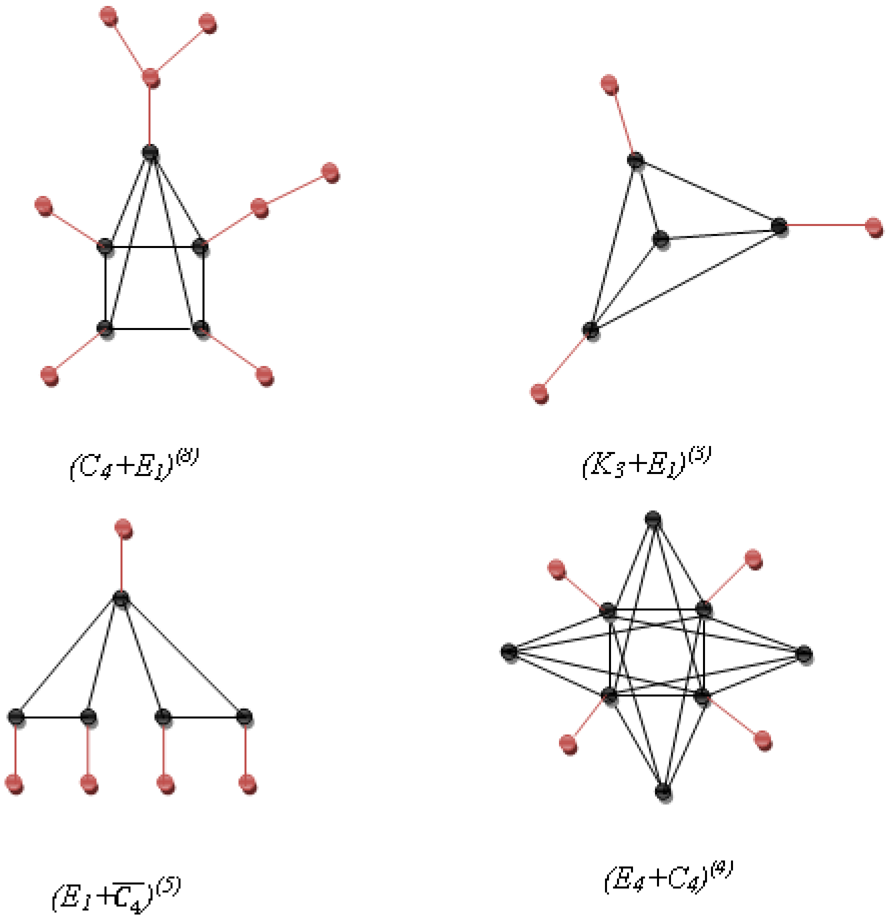


Figure A.1: Graphs of families $(G + H)^{(t)}$.

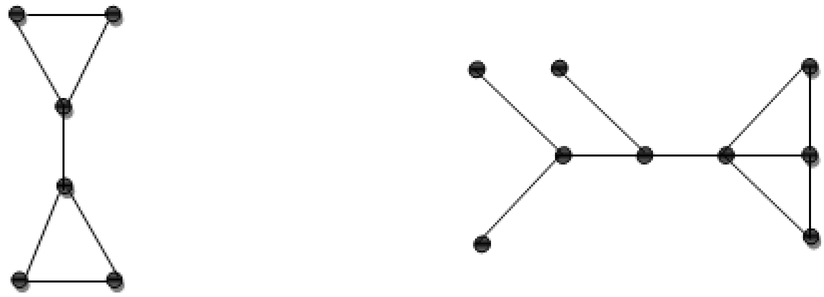


Figure A.2: Graphs of family $G^{(t)} \odot P_m$.

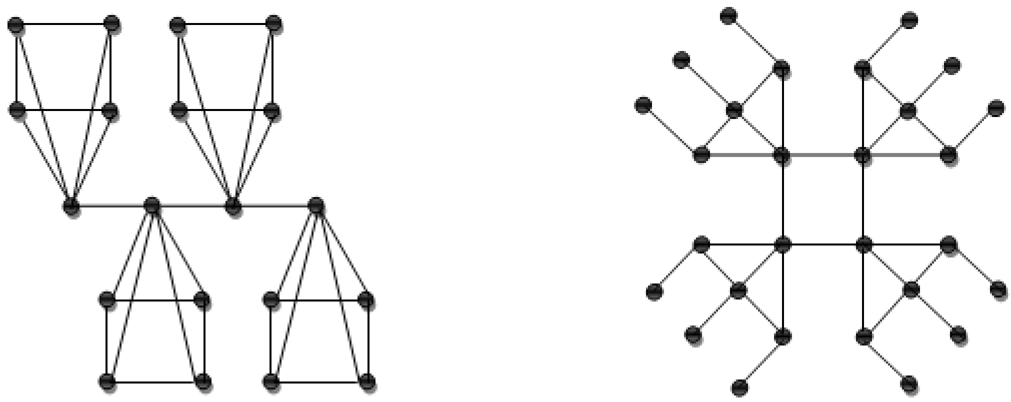


Figure A.3: Graphs of families GoC_m and $(GoP_m)^{(t)}$.

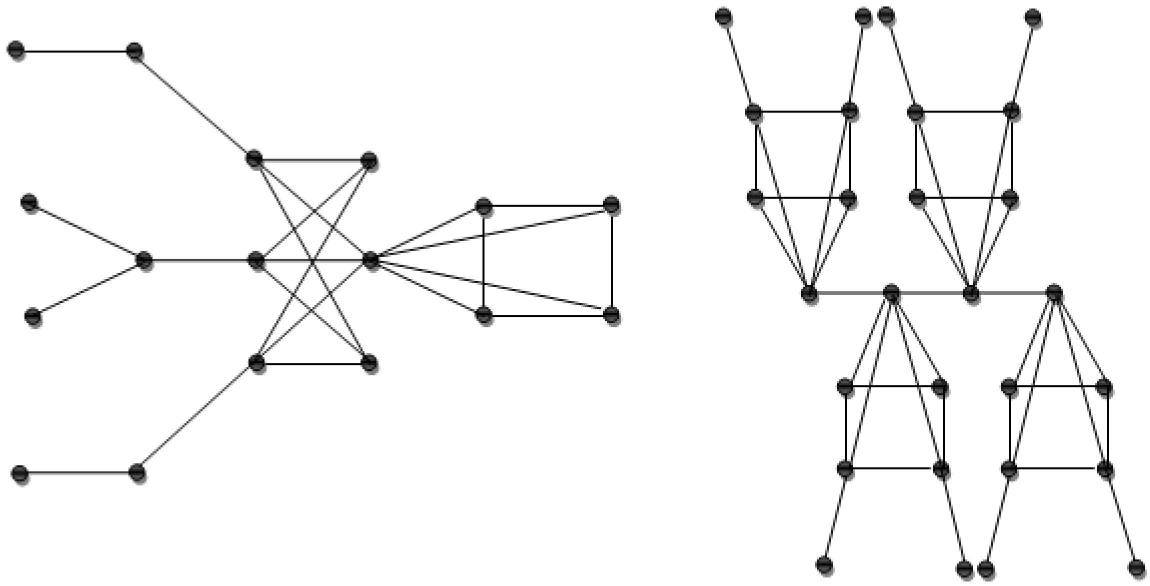


Figure A.4: Graphs of families $G \odot C_m$ and $(GoC_m)^{(t)}$.

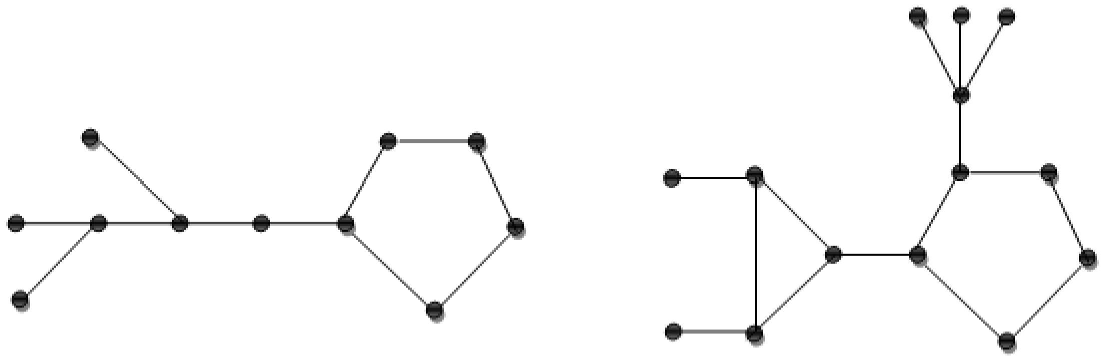


Figure A.5: Graphs of families $G^{(t)\star C_n}$ and $G^{(t)\star C_{1,2}}$.

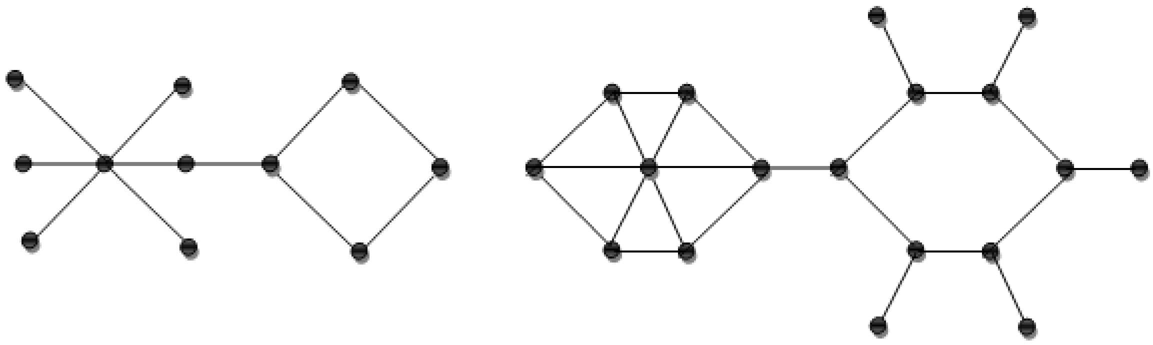


Figure A.6: Graphs of classes $G \star C_m$ and $G \star C_m^{(t)}$.

Bibliography

[1] M. Aigner, A characterization of the Bell numbers, Discrete Math. 205 (1999), 207210. 10

[2] J. Allagan and C. Serkan, Bell Numbers of Complete Multipartite Graphs, Computer Science Journal of Moldova, **24**, no.2(71), 2016. 30

[3] H. Belbachir, S. Bouroubi and A. Khelladi, Connection between ordinary multinomials, generalized Fibonacci numbers, partial Bell partition polynomials and convolution powers of discrete uniform distribution, Ann. Math. Inform. 35 (2008), 21–30. 75

[4] H. Belbachir and I. E. Bousbaa, *Associated Lah numbers and r-Stirling numbers*, arXiv:1404.5573v1, (2014). 3, 7, 14, 74

[5] H. Belbachir, M.A. Boutiche and A. Medjerredine, *Enumerating Some Stable Partitions Involving Stirling and r-Stirling Numbers of the Second Kind*, Mediterr. J. Math. (2018) 15: 87.7.

[6] H. Belbachir and H. Harik, *Link between hosoya index and Fibonacci numbers.* to appear in Miskolc. Math. Notes, MMN-1415. 1

[7] H. Belbachir, H. Harik and S. Pirzada, *Determining Lucas identities by using Hosoya index*, CREAT. MATH, **26** (2) (2017), 145–151. 1

[8] H. Belbachir and A. F. Tebtoub. Les nombres de Stirling associés avec succession dordre 2, nombres de Fibonacci Stirling et unimodalité. Comptes Rendus Mathematique, 353(9) :767-771, 2015. 3

[9] C. Berceanu, Chromatic polynomials and k-trees, Demonstratio Math. 34 (2001), 743-748. 28

[10] G. D. Birkhoff, *A determinant formula for the number of ways of coloring a map.* Ann. Math. Harvard Coll. **14** (1912), 42–46. 2, 19

[11] G. D. Birkhoff and D.C. Lewis, Chromatic polynomials. Trans. Amer. Math. Soc. 60:355451, 1946. 19, 25

[12] A. Brandstädt, V. B. Le and J. P. Spinard, *Graph classes A survey.* Society for Industrial and Applied Mathematics, Pheladelphia USA 1999.

[13] A. Z. Broder, *The r-Stirling numbers.* Disc. Math. **49** (1984), 241–259. 7, 11, 43

[14] L. Carlitz, Weighted Stirling numbers of the first and second kind–I, Thd Fibonacci Quarterly 18 (1980) 147-162. 13

[15] L. Carlitz, Weighted Stirling numbers of the first and second kind— , The Fibonacci Quarterly 18 (1980) 242-257. 13

[16] C. A. Charalambides, *Enumerative combinatorics*, CRC Press, 2002. 7, 9

[17] G. Chartrand, D. Erwin, M. A. Henning, P. J. Slater, and P. Zhang, On the locating-chromatic number of a graph. Preprint. 77

[18] G. Chartrand, D. Erwin, M. A. Henning, P. J. Slater, and P. Zhang, The locating-chromatic number of a graph. Bull. Inst. Combin. Appl. To appear. 64, 77

[19] G. Chartrand, L. Eroh, M. A. Johnson and O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph. Discrete Appl. Math. 105 (2000) 99-113. 64, 67

[20] G. Chartrand, C. Poisson, and P. Zhang, Resolvability and the upper dimension of graphs. J. Comput. Math. Appl. 39 (2000) 19-28. 64

[21] G. Chartrand, E. Salehi, and P. Zhang, On the partition dimension of a graph. Congress. Numer. 131 (1998) 55-66. 68

[22] G. Chartrand, E. Salehi, and P. Zhang, *The partition dimension of a graph.* Aequationes Math. 59 (2000) 45-54. 68

[23] D. I.A. Cohen, *Basic Techniques of Combinatorial Theory*, John Wiley Sons, 1978. 27

[24] L. Comtet. *Advanced Combinatorics: The art of finite and infinite expansions*, Springer Science and Business Media, 1974. 3, 7, 9, 14

[25] F. M. Dong, K. M. Koh and K. L. Teo , *Chromatic Polynomials and Chromaticity of Graphs.* World Scientic, British library, 2005. 19, 26, 72

[26] B. Duncan, *Bell and Stirling numbers for disjoint unions of graphs*, Congr. Numer. 206 (2010), 215217. 31, 41

[27] B. Duncan and R. Peele, *Bell and Stirling numbers for graphs.* J. Integer Seq. **12** (2009), 09.7.1. 2, 21, 22, 25, 27, 41

[28] R. W. Frucht and R. E. Giudici, *A note on the matching numbers of tiangle-free graphs*, J. Graph Theory 9, 1985, 455–458. 21

[29] D. Galvin and D. T. Thanh, *Stirling numbers of forests and cycles.* Electron. J. Combin., **20** (1) (2013), P73. 2, 25, 41

[30] J. Goldman, J. Joichi and D. White, *Rook Theory III. Rook polynomials and the Chromatic structure of graphs.* J. Combin. Th. Ser. **25** (1978), 135–142. 21

[31] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics.* Massachusetts : Addison-Wesley, 1989. 7, 8

[32] F. Harary and R. A. Melter, *On the metric dimension of a graph. Ars Combin.* 2 (1976) 191-195. 64, 67

[33] A. Hertz and H. Melot . *Counting the number of non-equivalent vertex colorings of a graph.* Disc. App. Math. **203** (2016), 62–71. 2, 25, 31, 41

[34] F. Howard. *Associated Stirling numbers.* Fibonacci Quart, 18(4) :303–315, 1980. 3, 7, 14

[35] Z. Kereskényi-Balogh and G. Nyul, *Stirling numbers of the second kind and Bell numbers for graphs.* Australasian J. Combinatorics. **58** (2) (2014), 264–274. 2, 22, 25, 27, 28, 41, 53

[36] D.E. Knuth, *The Art of Computer Programming, Vol. 1.* (Addison-Wesley, Reading, MA, 1973). 8

[37] D.E. Knuth, *The Art of Computer Programming, Vol. 2.* (Addison-Wesley, Reading, MA, 1981).

[38] D.E. Knuth. *The Art of Computer Programming, Vol. 3.* (Addison-Wesley, Reading, MA, 1973). 8

[39] R. Korfhage, *σ-polynomials and graph coloring.* J. Comb. Theory Ser. B 24 (1978), 137-153.

[40] R. Y. Liu, *A new method to find chromatic polynomial of graph and its applications*, Kexue Tongbao 32, 1508–1509.

[41] M. S. Maamra and M. Mihoubi, *Some applications of the chromatic polynomials.* arXiv, 1402.0731V1, (2015) .

[42] M. S. Maamra and M. Mihoubi, *Note on some restricted Stirling numbers of the second kind.* Comptes Rendus Mathematique. **354** (3) (2016), 231–234. 21

21

21

[43] T. Mansour, *Combinatorics of set partitions*, CRC Press Taylor Francis Group, 2013. 7

[44] R. A. Melter, I. Tomescu, *Metric bases in digital geometry*, Computer Vision, Graphics, and Image Processing 25(1984), 113-121.

[45] I. Mezo, *The r-Bell numbers*, J. Integer Seq. 14 (2011), Article 11.1.1. 10, 11, 13

[46] M. Mihoubi and M. S. Maamra. *The* $(r1, ..., rp)$*-Stirling numbers of the second kind.* Intgers, 12(5) :1047-1059, 2012.

[47] A. Mohr and T.D. Porter, *Applications of chromatic polynomials involving Stirling numbers*, J. Combin. Math. Combin. Comput. 70 (2009), 5764. 2, 27, 28, 41

[48] A.O. Munagi, *k-complementing subsets of nonnegative integers*, Int. J. Math. Math. Sci. (2005), 215–224. 27

[49] A. O. Munagi, *Set partitions with successions and separations*, Int. J. Math. Math. Sci. (2005), 451-463.

[50] N. Nielsen, *Handbuch der Theorie der Gammafunktion* (B.G. Teubner, Leipzig, 1906); reprinted under the title: Die Gammafunktion (Chelsea, New York, 1965). 7

[51] G. Nyul and G. Rácz. *The r-Lah numbers.* Discrete Mathematics, 338(10) :1660–1666, 2015. 74

[52] H. Prodinger, *Analysis of an algorithm to construct Fibonacci partitions*, RAIRO Inform. Théor. 18 (1984), 387–394. 27

[53] H. Prodinger, *On the number of Fibonacci partitions of a set*, Fibonacci Quart. 19 (1981), 463–465. 27

[54] H. Prodinger and R. Tichy. *Fibonacci numbers of graphs.* Fibonacci Quarterly. **20** (1) (1982), 16–21. 1, 21

[55] R.C. Read, *An introduction to chromatic polynomials*, J. Combin. Theory 4 (1968), 52–71. 27

[56] J. Riordan, *An Introduction to Combinatorial Analysis.* New York, NY, USA: Wiley, 1958. 9, 11, 14

[57] V. Saephotphat, *Resolvability in Graphs*, Western Michigan University Kalamazoo, Michigan, 2002. http://scholarworks.wmich.edu/dissertations/1301

[58] M. Shattuck, *Generalized r- Lah numbers*, arXiv14128721v1, 2014. 64

[59] P. J. Slater, *Dominating and reference sets in graphs.* J. Math. Phys. Sci. 22 (1988) 445-455. 7, 74

[60] P. J. Slater, *Leaves of trees.* Congress. Numer. 14 (1975) 549-559. 64, 67

[61] R. P. Stanley. *Enumerative Combinatorics.* Springer, 1986. 64, 67

[62] R.P. Stanley, *Enumerative Combinatorics, Volume 1*, Cambridge University Press, 1997.

[63] J. Stirling, *Methodus Differentialis*, Sire Tractatus De Summatione et Interpolazione Serierum Infinitorum (London, 1730).

[64] C. Tweedie, James Stirling, *A Sketch of his Life and Works* (Clarendon Press, Oxford, 1922). 7

[65] V. I. Voloshin. *Coloring Mixed Hypergraphs.* Theory, Algorithms and Applications (AMS and Fields Institute Monographs), Provi-dence, RI, USA: AMS, 2002.

21

[66] C. G. Wagner, *Generalized Stirling and Lah numbers, Discrete. Math*, 160, 199–218, 1996. 7, 74

[67] D. Wagner. *The partition polynomial of a finite set system.* J. Comb. Theory Ser. A 56 (1991), 138–159.

[68] H. Whitney. *On the colouring of graphs.* Ann. Math, (12) 33, (1933), 688–718.

[69] W. Yang. *Bell numbers and k-trees.* Disc. Math. **156** (1996), 247–252. 19

2, 25, 27