N d'ordre : 14/2020-D/MT.

RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE ET POPULAIRE
MINISTÈRE DE L'ENSEIGNEMENT SUPÉRIEUR ET DE LA RECHERCHE SCIENTIFIQUE
UNIVERSITÉ DES SCIENCES ET DE LA TECHNOLOGIE HOUARI BOUMEDIENNE
FACULTÉ DE MATHÉMATIQUES

THÈSE DE DOCTORAT EN SCIENCES

Présentée pour l'obtention du grâde de Docteur

En : **Mathématiques**

Spécialité : **Analyse: Systèmes Dynamiques**

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Sujet

Etude de certaines classes de problèmes aux limites posés sur des intervalles non bornés

Soutenue publiquement le 16/12/2020, devant le jury composé de :

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Thèse de Doctorat présentée par

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Thème : Etude de certaines classes de problèmes aux limites posés sur des intervalles non bornés

Liste des publications

1. A. Benmezaï, S. Mellal, *Nodal solution for asymptotically linear second-order Bvps on the half line* , Bull. Iran. Math. Soc. 45. 315-335 (2018).

2. A. Benmezaï, S. Mellal, S.K. Ntouyas, *Nodal solution for asymptotically linear second-order Bvps on the real line* , Differ. Equ. and Dyn. Sys. 1-30(2019).

3. A. Benmezaï, S. Mellal , *A class of Sturm-Liouville Bvps with an unintegrable weight*. **Submitted**.

Journal site web

1. Bulletin of the Iranian Mathematical Society
   http://doi.org./ www. Springer.com

2. Differential Equations and Dynamical Systems.
   http://doi.org./ www. Springer.com

Remerciements

Je tiens à exprimer ma reconnaissance et ma gratitude envers tous ceux qui m'ont aidé à mener à bien ce travail.

Tout particulièrement, J'adresse mes vifs remerciements à mon directeur de thèse Pr Abdelhamid Benmezai, professeur à l'U.S.T.H.B, d'avoir accepté de me proposer ce sujet de thèse, je le remercie pour sa confiance et son soutien tout au long de la réalisation de ce travail.

Je tiens aussi à remercier Pr Tarek Aliziane, professeur à l'U.S.T.H.B de me faire l'honneur d'accepter de présider ce jury.

Je tiens aussi à exprimé mes sincères remerciements à

Pr. Toufik Moussaoui professeur à E.N.S Kouba,

Dr. Kamel Bachouche Maître de conference classe A à l'université Alger 1,

Dr. Salima Mechrouk , Maître de conference classe A à l'université de Boumerdes,

De m'avoir honoré en acceptant d'examiner ce travail.

Merci à tous mes amis et proches.

Abstract

Our purpose of this thesis is to study existence of nodal solutions to some classes of Sturm-Liouville boundary value problems having unitegrable weights posed on bounded and unbounded intervals .
The global bifurcation theory of Rabinowitz constitute the principal tool of this thesis.

Resumé

L' objet de cette thèse est de démontrer l'existence de solutions nodales pour certaines classes de problèmes aux limites de Sturm-Liouville avec des poids non integrables et définis sur des intervalles bornés et non bornés. La théorie globale de bifurcation de Rabinowitz constitue l'outil principal de cette thèse.

Contents

Notations

The aim of this chapter is to introduce some basic concepts and elementary results which will be used further.

$C([\xi,\eta])$	Space of continuous functions on $[\xi,\eta]$.
$\|u\|$	$=\sup_{t\in[\xi,\eta]}\|u(t)\|$.
$\|u\|_1$	$=\sup_{t\in[0,1]}\|u'(t)\|$.
$C^k([\xi,\eta])$	Space of k-times continuously differentiable functions on$[\xi,\eta]$.
$C_b^1([\xi,\eta]$	Space of continuously differentiable functions on $[\xi,\eta]$ *with* $\sup_{t\in[\xi,\eta)}\left\|u^{'}(t)\right\|<\infty$.
$L^1([\xi,\eta])$	space of all measurable functions on $[\xi,\eta]$ and satisfying $\|u\|=\int_{\xi}^{\eta}\|u(t)\|\,dt<\infty$.
$\Im^{'}$	The dual space of$\Im^{'}=\{f:\Im\to\mathbb{R}\}$.
$<.,.>$	Duality between $\Im^{'}$and $\Im^{'}$.
$N(A)$	The null space of A.
$B(x_0,r)$	the open ball of radius r and centered at x_0.
i.e	That is.
a.e.	almost everywhere.
$bvp(s)$	Boundary value problem(s).
evp	Eigenvalue problem.

Introduction

This thesis is devoted to study some classes of Sturm-Liouville boundary value problems (SLP) on bounded and unbounded intervals.

C. Sturm and J. Liouville published in the period of 1836 to 1837, a series of papers on second order differential equations including boundary value problems. The impact of these papers went well beyond their subject matter to general linear and nonlinear differential equations and to analysis generally. Prior to this time, the study of differential equations was largely limited to the search for solutions as analytic expressions [60, 61, **?**]. Sturms papers on differential equations are characterized by the general and qualitative nature of the problems. He discussed a general classes of equations not a specific one, and he asked questions about the qualitative properties of the solution, instead to gave the analytic expression of that one. Many authors contribute to the development of the theory of Sturm-Liouville since 1900, for example Herman Weyl (1910) published one of the most widely quoted papers in analysis , just as Sturm and Liouville started the study of regular SLP.
This paper initiated the investigation of singular SLP. Dixon (1912), was the first who replaced the continuity condition of the coefficients by the integrability condition. The proof of general spectral theorem for unbounded self adjoint operators in Hilbert space by Neumann and Stone (1932). The fundamental works of Titchmarsh (1962) provided some results into the spectral theory of Sturm-Liouville opertors.

The main goal of this thesis is to study existence of nodal solutions for some classes of Sturm-Liouville boundary value problems having unintegrable weights. Such a tematic hasn't studied before. This work is organized as follows.

The first chapter is devoted to preliminaries and abstract background, we recall in the concepts of compact operators, positivity as well as Riesz-Schauder Theory. Mainly, a subsection of this chapter is devoted to the global bifurcation of P.H. Rabinowitz [51, 52] which is the tool used to prove the existence and multiplicity results in this work. At the end of this chapter , we present some recent results about element of Sturm-Liouville boundary value problems in integrable case, mainly, the result of Berestyckï [11] concerning the existence of half eigenvalue and the result of Benmezai-Esserhane [7] which extends the result of Berestyckï. We, also give the principal work of Zettl [64] concerning the existence of eigenvalues .

In the second chapter, We investigate the existence of nodal solution of the following boundary value problem

$$\begin{cases} -u''(t) + q(t)u(t) = u(t)f(t, u(t)), \quad t > 0, \\ u(0) = \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{1}$$

where $q \in C(\mathbb{R}^+, \mathbb{R}^+)$ is such that $q(t) > 0$ for all $t \geq T$ and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The first result of this chapter concerns the spectrum of the linear eigenvalue problem associated to the problem (1)

$$\begin{cases} -u''(t) + q(t)u(t) = \mu m(t)u(t), \quad t > 0, \\ u(0) = 0, \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{2}$$

where μ is a real parameter and $m \in C(\mathbb{R}^+, \mathbb{R})$ is such that $\lim_{t\to+\infty} m(t) = 0, m(t) > 0$ a.e. $t \in \mathbb{R}^+$.

We prove that this spectrum consists of an unbounded increasing sequence $(\mu_k(m))_{k\geq 1}$ of eigenvalues and the corresponding eigenfunctions have nodal properties. The main result of this work concerns the existence and multiplicity result for nodal solutions, witch is obtained by means of Rabinowitz global bifurcation theory. It claims that if there is two integers i, j with $1 \leq i \leq j$ such that $\mu_j(f(t,\infty)), \mu_i(f(t,0))$ are oppositely located relatively to 1, then the problem (1) admits a nodal solution.

In the third chapter, we extend the results of second chapter on the real line. We consider the linear eigenvalue problem :

$$\begin{cases} -u''(t) + q(t)u(t) = \mu m(t)u(t), \ t \in \mathbb{R}, \\ \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{3}$$

and the perturbed problem associated to the problem (3)

$$\begin{cases} -u''(t)+q(t)u(t)=\mu u(t)f(t,u(t)),\ t\in\mathbb{R}, \\ \lim_{t\to-\infty}u(t)=\lim_{t\to+\infty}u(t)=0, \end{cases} \tag{4}$$

where μ is real parameter, $q\in C(\mathbb{R},\mathbb{R}^{+})$ is such that $q(t)>0$ for all $|t|\geq T$, $m\in C(\mathbb{R},\mathbb{R})$ is such that $\lim_{t\to-\infty}m(t)=\lim_{t\to+\infty}m(t)=0, m(t)>0$ a.e. $t\in\mathbb{R}$ and $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

The first result concerns the spectrum of problem (3), we prove that this spectrum consists of an unbounded increasing sequence $(\mu_k(m))_{k\geq1}$ of eigenvalues and the corresponding eigenfunctions have nodal properties. The main result of this work concerns the existence and multiplicity result for nodal solutions, witch is obtained by means of Rabinowitz global bifurcation theory. It claims that if there is two integers i,j with $1\leq i\leq j$ such that $\mu_j(f(t,\infty)),\mu_i(f(t,0))$ are oppositely located relatively to μ, then the problem (4) admits a nodal solutions .

In the last chapter, we prove existence of nodal solutions to the following nonlinear boundary value problem

$$\begin{cases} -u''+qu=\rho uf(t,u),\ \text{in}\ (0,1), \\ u(0)=\lim_{t\to1}u(t)=0, \end{cases} \tag{5}$$

where ρ is a positive real parameter, $q\in C([0,1),\mathbb{R})$, is such that $\int_0^1 q=+\infty$ and $f:[0,1]\times(\mathbb{R}\setminus\{0\})\to\mathbb{R}$ is continuous. Nodal solutions appear as eigenfunctions to the half eigenvalue problem

$$\begin{cases} -u''+qu=\sigma mu+\alpha u^{+}-\beta u^{-},\ \text{in}\ (0,1), \\ u(0)=\lim_{t\to1}u(t)=0, \end{cases} \tag{6}$$

where σ is a real parameter, $m,\alpha,\beta\in C([0,1],\mathbb{R})$ such that $m\geq0$ in $(0,1)$, and $m(t_0)>0$ for some $t_0\in[0,1]$.

We prove in the first that the Berestycki's result holds true for the problem (6), that the problem (6) admits two unbounded increasing sequences of simple half-eigenvalues $(\lambda_k^{+}(q,m,\alpha,\beta))_{k\geq1})$ and $(\lambda_k^{-}(q,m,\alpha,\beta))_{k\geq1})$ and the corresponding eigenfunctions have nodal properties..

The main results of this work concerns existence of nodal solutions to the problem (5) in the cases where the nonlinearity $uf(t,u)$ is respectively asymptotically linear, sublinear and superlinear. All are obtained by means of the global bifurcation theory due to P. H. Rabinowitz.

Introduction

On s'intéresse dans cette thèse à l'étude des problèmes aux limites de Sturm-Liouville posés sur des intervalles bornées et non bornées. L'origine des problèmes de Sturm-Liouville remonte à l'époque de 1836 à 1837 quand C. Sturm en collaboration avec J. Liouville publièrent une série d'articles sur les équations différentielles linéaires et non-linéaires du second ordre [60, 61, **?**].

La contribution de Sturm et de Liouville a permis de donner une nouvelle méthodologie concernant les propriétés qualitatives de la solution, non l'expression exacte de cette dernière. Plusieurs auteurs ont contribué au développement de la théorie de Sturm-Liouville de 1900 à 1950, on citera Hermann Weyl (1910) qui a considéré le problème linéaire de Sturm-Liouville dans le cas singulier, Dixon (1912) était le premier à remplacer la continuité des coefficients par une condition d'intégrabilité; M.H. Stone (1932) dans son livre [49] étudia le problème de Sturm-Liouville dans les espaces de Hilbert. Le travail principal de Titchmarch (1962) concerne certains résulats de la théorie spéctral sur les opérateurs de Sturm-Liouville.

L' objet de cette thèse est d' étudier l' existence de solutions nodales pour certaines classes de problèmes aux limites avec des poids non intégrables. Le travail est organisé de la manière suivante:

Le premier chapitre est consacré aux préliminaires. Nous rappelons les notions des opérateurs compacts, la positivité et la théorie de Riesz-Schauder. Nous présentons la théorie de bifurcation globale de Rabinowitz; on terminera ce chapitre par quelques éléments sur les problèmes de Sturm-Liouville dans le cas intégrable.

Dans le deuxième chapitre, on considère le problème au limites suivant

$$\begin{cases} -u''(t) + q(t)u(t) = u(t)f(t, u(t)), \quad t > 0, \\ u(0) = \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{7}$$

où $q \in C(\mathbb{R}^+, \mathbb{R}^+)$ avec $q(t) > 0$ pour tout $t \geq T$ et $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ est continue. Le premier résultat concerne le spèctre du problème linéaire associé au problème (7)

$$\begin{cases} -u''(t) + q(t)u(t) = \mu m(t)u(t), \quad t > 0, \\ u(0) = 0, \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{8}$$

où μ est un paramètre rèel, et $m \in C(\mathbb{R}^+, \mathbb{R})$ tel que $\lim_{t\to+\infty} m(t) = 0, m(t) > 0$ a.e. $t \in \mathbb{R}^+$. On montre que le spèctre consiste en une suite croissante des valeurs propres $(\mu_k(m))_{k\geq 1}$ associés à des vecteurs propres admettant des propriétés nodales. Le résultat principal de ce travail concerne l'existence et la multiplicité de solutions nodales en utilisant la théorie de bifurcation globale de Rabinowitz. On montre que s'il existe deux entiers i, j avec $1 \leq i \leq j$ tels que $\mu_j(f(t, \infty)), \mu_i(f(t, 0))$ sont opposés par rapport a 1, alors le problème (7) admet des solutions nodales.
Dans le troisième chapitre, on étend le résultat de deuxième chapitre sur la droite réelle. On considère le problème linéaire suivant:

$$\begin{cases} -u''(t) + q(t)u(t) = \mu m(t)u(t), \ t \in \mathbb{R}, \\ \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{9}$$

$$\begin{cases} -u''(t) + q(t)u(t) = \mu u(t)f(t, u(t)), \ t \in \mathbb{R}, \\ \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{10}$$

où μ est un paramètre rèel, $q \in C(\mathbb{R}, \mathbb{R}^+)$ avec $q(t) > 0$ pour tout $|t| \geq T$, $m \in C(\mathbb{R}, \mathbb{R})$ avec $\lim_{t\to-\infty} m(t) = \lim_{t\to+\infty} m(t) = 0, m(t) > 0$ a.e. $t \in \mathbb{R}$ et $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ est continue.
On montre que le spèctre consiste en une suite croissante de valeurs propres $(\mu_k(m))_{k\geq 1}$ associées à des vecteurs propres admettant des propriétés nodales. Le résultat principal de ce travail concerne l'existence et la multiplicité de solutions nodales en utilisant la théorie de bifurcation globale de Rabinowitz. On montre que s'il existe deux entiers i, j avec $1 \leq i \leq j$ tels que $\mu_j(f(t, \infty)), \mu_i(f(t, 0))$ sont opposés par rapport a μ, alors le problème (10) admet des solutions nodales.

Dans le dernier chapitre on montre l'existence de solution nodales pour le problème aux limites suivant:

$$\begin{cases} -u'' + qu = \rho u f(t,u), \text{ in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{11}$$

où ρ est un paramètre réel , $q \in C([0,1), \mathbb{R})$, $\int_0^1 q = +\infty$ et $f : [0,1] \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ est continue. Les solutions nodales apparaissent comme des fonctions propres du problème de demi valeurs propres suivant:

$$\begin{cases} -u'' + qu = \sigma m u + \alpha u^+ - \beta u^-, \text{ in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{12}$$

où σ est un paramétre rèel, $q, m, \alpha, \beta \in C([0,1], \mathbb{R})$ tels que $m \geq 0$ dans $[0,1]$ et $m(t_0) > 0$ pour certain $t_0 \in [0,1]$. Au début de ce travail on montre que le résultat de Berestycki reste valide pour le problème (12). Le résultat principal concerne l'existence et la multiplicité de solution nodales du problème (11) dans le cas où la nonlinéarité

$uf(t,u)$ est respectivement asymptotiquement linéaire, sous-linéaire et super-linéaire. Dans tous les cas, on utilise la théorie de bufircation globale de Rabinowitz.

Chapter 1

Preliminaries

1.1 Abstract background

1.1.1 The compactness

We start this section by some definitions about compactness.

Let $\mathcal{E}, \mathcal{F}$ be two Banach spaces

Definition 1.1 ([27]). *A subset M of $\mathcal{E}$ is said to be compact, iff every sequence $(x_n)_{n\in\mathcal{N}} \subset M$ has a convergent subsequence with limit in M.*

Let $\Omega \subset \mathcal{E}$ be an open set.

Definition 1.2. *Let $\mathcal{A} : \Omega \to \mathcal{F}$ be continuous mapping. $\mathcal{A}$ is said to be:*

- *compact, if $\mathcal{A}(\overline{\Omega})$ is compact.*
- *completely continuous mapping if maps bounded sets into relatively compact sets.*

Clearly, all compact mapping are completely continuous mapping, if Ω is a bounded set we have the equivalent.

properties

- A linear combination of compact mappings is compact.
- The product of a compact mapping with a linear bounded mapping is compact.

- If the sequence mappings $\mathcal{A}_n : \Omega \to \mathcal{F}$ are compact and $\mathcal{A} : \Omega \to \mathcal{F}$ such that $\mathcal{A} = \lim_{n\to+\infty} \mathcal{A}_n$ uniformly in any bounded set of Ω, Then $\mathcal{A}$ is a compact mapping.

In this thesis we use the following compactness criteria.

Ascoli-Arzéla compactness criterion

Let (X, d) be a compact metric space and Y a Banach space; then $C(X, Y)$ is a Banach space equipped with the sup norm $\|f\| = \sup_{x\in X} \|f(x)\|_Y$.

Definition 1.3. *Let $H \subset C(X, Y)$ be a family of continuous functions. H is said to be equicontinuous if the set $H(x_0) := \{f(x_0), \quad f \in H\}$ is equicontinuous for all $x_0 \in X$, i.e.*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in X, d(x, x_0) \leq \delta \Longrightarrow \|f(x) - f(x_0)\| \leq \epsilon \ , \forall f \in H. \tag{1.1}$$

Lemma 1.4. *(Ascoli-Arzéla Theorem) [28] Let (X, d) be a compact metric space and Y a Banach space; and let H a subset in $C(X, Y)$. H is said to be relatively compact if and only if*

- *H is equicontinuous,*
- *$\forall x \in X$ the set $\{f(x), f \in H\}$ is relatively compact in Y.*

Corollary 1.5. *$\forall k \in \mathbb{N}$, $C^{k+1}([a, b], \mathbb{R})$ can be embedded compactly in $C^k([a, b], \mathbb{R})$.*

Compactness criteria on noncompact intervals

In this section we present Corduneanu 's compactness criterion, extending the Ascoli-Arzéla lemma.
Let I be a bounded or unbounded interval, and let $C_b =: C_b(I, \mathbb{R})$ denote the vector space of all bounded and continuous function, equipped with the sup norm $\|f\| = \sup_{x\in I} |f(x)|$.

Definition 1.6. *A family $H \subset C_b$ is called equicontinuous on every compact interval I of $\mathbb{R}$ if it satisfies*

$$\forall \epsilon > 0, \exists \delta > 0, \forall t_1, t_2 \in I, |t_1 - t_2| \leq \delta \Longrightarrow |x(t_1) - x(t_2)| \leq \epsilon, forall x \in H. \tag{1.2}$$

Definition 1.7. *A family $H \subset C_b$ is called equiconvergent if it satisfies*

$$\forall \epsilon > 0, \exists T = T(\epsilon) > 0, \forall t_1, t_2 \in I, |t_1| > T, |t_2| > T \Longrightarrow |x(t_1) - x(t_2)| \leq \epsilon, \quad \forall x \in H. \tag{1.3}$$

Theorem 1.8. *(Corduneanau Theorem)[5] A family $H \subset C_b$ is relatively compact if and only if the following conditions are satisfied:*

- *H is uniformly bounded in C_b,*
- *H is equicontinuous on every compact of I,*
- *H is equiconvergent.*

1.1.2 The Riesz Schauder Theory

Let $\mathcal{E}$ be a Banach space and $\mathfrak{L}(\mathcal{E})$ be the Banach space of all bounded linear operators.

Let $\mathcal{A} \in \mathfrak{L}(\mathcal{E})$, we consider the linear operator.

$$\mathcal{A}_\lambda = \lambda \mathcal{I} - \mathcal{A}, \tag{1.4}$$

where $\mathcal{I}$ is the identity operator and $\lambda \in \mathbb{C}$ is a complex number. The distribution of the value λ for which $\mathcal{A}_\lambda$ has an inverse and the properties of this inverse when it exists, are called the spectral theory of the operator $\mathcal{A}_\lambda$.

Definition 1.9 ([13])**.** *Let $\mathcal{A} \in \mathfrak{L}(\mathcal{E})$. The set*

$$\rho(\mathcal{A}) = \{\lambda \in \mathbb{C}, \lambda \mathcal{I} - \mathcal{A} \quad \text{is bijective}\} \tag{1.5}$$

is called the resolvent set of $\mathcal{A}$ and the inverse operator $\mathcal{R}(\lambda; \mathcal{A}) = (\lambda \mathcal{I} - \mathcal{A})^{-1}$ is called the resolvent operator of $\mathcal{A}$ at λ.

Definition 1.10.

- *The spectrum of $\mathcal{A}$, $\sigma(\mathcal{A})$, is the complementary set of $\rho(\mathcal{A})$ in $\mathbb{C}$.*
- *A complex number λ is an eigenvalue of $\mathcal{A}$ if the equation $\lambda x - \mathcal{A} x = 0$ has a solution $x \neq 0$, this solution x is said to be an eigenvector of $\mathcal{A}$ corresponding to λ. The null space $\mathcal{N}(\lambda \mathcal{I} - \mathcal{A})$ is the eigenspace associated with λ, and its dimension is the geometric multiplicity of λ .*

Proposition 1.11. *The spectrum $\sigma(\mathcal{A})$ is a compact set included in the ball $\mathcal{B}(0;\|\mathcal{A}\|)$.*

Definition 1.12. *For every operator $\mathcal{A} \in \mathfrak{L}(\mathcal{E})$, we define*

$$r\left(\mathcal{A}\right) = \sup\{|\lambda|, \quad \lambda \in \sigma(\mathcal{A})\}$$

the spectral radius of $\mathcal{A}$.

Theorem 1.13. *Let $\mathcal{A} \in \mathfrak{L}\left(\mathcal{E}\right)$, the spectral radius of $\mathcal{A}$ is given by*

$$r(\mathcal{A}) = \lim_{n\to+\infty} \|\mathcal{A}^n\|^{\frac{1}{n}} = \inf_{n\in\mathcal{N}} \|\mathcal{A}^n\|^{\frac{1}{n}}$$

and we have

$$r\left(\mathcal{A}\right) \leq \|\mathcal{A}\|$$

.

We denote by $\mathcal{K}(\mathcal{E})$ the space of compact operators from $\mathcal{E}$ to $\mathcal{E}$, which is a closed subspace in $\mathfrak{L}(\mathcal{E})$. In the case of a compact operator, one has a more precise description of the spectrum. This result known as the Riesz-Schauder Theory.

Theorem 1.14. *[13] Let $\mathcal{A} \in \mathcal{K}(\mathcal{E})$, where $\mathcal{E}$ is infinite dimensional space. Then*

1. $0 \in \sigma(\mathcal{A})$.
2. *Each numbers $\lambda \neq 0$ in the spectrum $\sigma(\mathcal{A})$ is an eigenvalue.*
3. *We are in one (and only one) of the following cases*
   - *either $\sigma(\mathcal{A}) = \{0\}$,*
   - *either $\sigma(\mathcal{A})$ is finite,*
   - *or $\sigma(\mathcal{A}) - \{0\}$ may be described as a sequence of distincts points tending to 0.*

Theorem 1.15 ([28, Theorem 11.3.3])**.** *For $\lambda \in \sigma(\mathcal{A})\backslash\{0\}$ there exists $m \in \mathbb{N}$ such that*

$$\mathcal{N}\left((\lambda\mathcal{I} - \mathcal{A})^m\right) = \mathcal{N}\left((\lambda\mathcal{I} - \mathcal{A})^{m+1}\right)$$

and this subspace is finite dimensional.

Since $\mathcal{N}\left(\lambda\mathcal{I} - \mathcal{A}\right) \subset \mathcal{N}\left((\lambda\mathcal{I} - \mathcal{A})^2\right) \subset \mathcal{N}\left((\lambda\mathcal{N})^3\right) \subset \cdots$ then

$$\bigcup_{j\geq 1} \mathcal{N}\left((\lambda\mathcal{I} - \mathcal{A})^j\right) = \mathcal{N}\left((\lambda\mathcal{I} - \mathcal{A})^m\right)$$

and it's finite dimensional. This dimension is called the algebraic multiplicity of λ.

1.1.3 Positivity

Let $\mathcal{E}$ be a real Banach space.

Definition 1.16. *A nonempty closed convex subset $\mathcal{P}$ of $\mathcal{E}$ is called a* cone *if*

i) $tx \in \mathcal{P}$ *for all* $x \in \mathcal{P}$ *and* $t \geq 0$

ii) $x \in \mathcal{P},\ -x \in \mathcal{P}$ *implies* $x = 0$. *(* 0 *denote de zero element of* $\mathcal{E}$*)*

A cone $\mathcal{P}$ is called **solid** *if it contains interior points (i.e. $\dot{\mathcal{P}} \neq \emptyset$). A cone $\mathcal{P}$ is called* **generating** *if $\mathcal{E} = \mathcal{P} - \mathcal{P}$. Moreover if $\overline{\mathcal{P} - \mathcal{P}} = \mathcal{E}$ then the cone $\mathcal{P}$ is said to be* **total***.*

Every cone $\mathcal{P}$ in $\mathcal{E}$ defines a partial order relation $\preceq$ in $\mathcal{E}$ as follows: for $x, y \in \mathcal{E}$,

$$x \preceq y \Leftrightarrow y - x \in \mathcal{P}.$$

We shall write $x \prec y$ to indicate that $x \preceq y$ and $x \neq y$, while $x \prec\prec y$ will always stand for $y - x \in \dot{\mathcal{P}}$ if $\mathcal{P}$ is solid.

Definition 1.17. *A cone $\mathcal{P}$ is said to be* normal *if there exists a positive constant N such that*

$$0 \preceq x \preceq y \Rightarrow \|x\|_{\mathcal{E}} \leq N\|y\|_{\mathcal{E}}. \tag{1.6}$$

Example 1.18. Let $\mathcal{E} = C^1[a, b]$, the space of continuously differentiable functions on $[0, 2\Pi]$ with the norm

$$\| u \| = \max_{t \in [a,b]} |u(t)| + \max_{t \in [a,b]} |u'(t)| \tag{1.7}$$

and let $\mathcal{P}_1 = \{x(t) \in C^1[a, b], x(t) \geq 0\, a \leq t \leq b\}$. Clearly $\mathcal{P}$ is a solid cone in $C^1[a, b]$.

$\mathcal{P}_1$ is not normal. In fact, if $\mathcal{P}_1$ is normal, then there exist an $N > 0$ such that

$$0 \preceq x \preceq y \Rightarrow \|x\|_{\mathcal{E}} \leq N\|y\|_{\mathcal{E}}. \tag{1.8}$$

Let $x_n(t) = 1 - cosnt$, $y_n(t) = 2$. Then we have $0 \preceq x \preceq y$, $\|x_n\| = 2 + n$, and $\|y\| = 2$. Consequently, $2 + n \leq 2N$ $(n = 1, 2, 3, ...)$, which is impossible.

Example 1.19. Let $E = L^p(\Omega)$, where $\Omega \subset \mathcal{R}^n$, $0 < mes\Omega < +\infty$ and $1 \leq p < +\infty$, and $\mathcal{P}_2 = \{x(t) \in L^p(\Omega), x(t) \geq 0, \quad a.e \quad t \in \Omega\}$. It is easy to know that $\mathcal{P}_2$ is a normal cone and its normal constant $N = 1$. Clearly, $int\mathcal{P}_2 = \varnothing$. Thus $\mathcal{P}_2$ is not a solid cone.

The main interest concerning the positive operator, in what follow, is the existence of positive eigenvector. More precisely the condition under which the spectral radius of a positive operator is an eigenvalue.
From now we consider an ordered Banach space $\mathcal{E}$ with respect to a cone $\mathcal{P}$ and denote the partial ordering by $'' \preceq ''$.

Definition 1.20. *Let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be an operator. $\mathcal{A}$ is said to be*

- *Positive if $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$,*
- *Increasing if for $u, v \in \mathcal{P}$*

$$u \preceq v \text{ implies } \mathcal{A}u \preceq \mathcal{A}v.$$

Remark 1.21. The concept of positive operator and increasing operator coincide when the operator $\mathcal{A}$ is a bounded linear operator.

Definition 1.22. *Let $\mathcal{A} \in \mathfrak{L}(\mathcal{E})$ be a positive operator. A real number λ is said to be* positive eigenvalue *of $\mathcal{A}$ if $\lambda > 0$ and there is $x \in \mathcal{P}\backslash\{0\}$ such that*

$$\mathcal{A}x = \lambda x$$

.

The following theorem is known as Krein-Rutman Theorem. This result presents the situation where the spectral radius $r(\mathcal{A})$ of a positive linear compact operator $\mathcal{A}$, is a positive eigenvalue of $\mathcal{A}$.

Theorem 1.23. *[63] Assume that the cone $\mathcal{P}$ is total and $\mathcal{A} \in \mathfrak{L}(\mathcal{E})$ is compact and a positive operator with $r(\mathcal{A}) > 0$. Then $r(\mathcal{A})$ is a positive eigenvalue of $\mathcal{A}$.*

1.1.4 Global bifurcation theory

The Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family, such as the solutions of family of differential equations. A bifurcation occurs when a small smooth change made to the parameter value of a system causes a sudden qualitative or topological change in its behavior. The name "bifurcation" was first introduced by Henri Poincaré in 1885.
Let $\mathcal{E}$ be a real Banach space and $\mathcal{F} : \mathbb{R} \times \mathcal{E} \to \mathcal{E}$ is a continuous mapping. Suppose the equation $\mathcal{F}(\lambda, u) = 0$, possesses a simple curve of solutions given by

$$\mathcal{Z} = \{(\lambda(\tau), u(\tau)), \ \ \tau \in I\}.$$

If for some $\bar{\tau} \in I$. $\mathcal{F}$ possesses zeroes not lying on $\mathcal{Z}$ in every neighborhood of $(\lambda(\bar{\tau}), u(\bar{\tau}))$, then $(\lambda(\bar{\tau}), u(\bar{\tau}))$ is said to be a bifurcation point for $\mathcal{F}$ with respect to the curve $\mathcal{Z}$.

The global theory of bifurcation concerns the equation

$$u = \lambda \mathcal{L} u + \mathcal{H}(\lambda, u), \tag{1.9}$$

where $\mathcal{L} : \mathcal{E} \to \mathcal{E}$ is a compact linear operator and $\mathcal{H} : \mathbb{R} \times \mathcal{E} \to \mathcal{E}$ is completely continuous with: $\mathcal{H} = o(\|u\|_{\mathcal{E}})$ near $u = 0$ uniformly on a bounded interval of λ.
The equation (1.9) possesses a curve of solutions $\{(\mu, 0), \ \ \mu \in \mathbb{R}\}$, which is called curve of trivial solutions.
A bifurcation point with respect to the set of trivial solutions is a point $(\mu, 0)$ such that there is a sequence of nontrivial solutions of (1.9), $(\mu_n, u_n)_n$ which converges to $(\mu, 0)$ in $\mathbb{R} \times E$. It was established that a necessary condition for $(\mu, 0)$ to be a bifurcation point is that μ is a characteristic value of $\mathcal{L}$ (i.e. μ^{-1} is a non zero eigenvalue of $\mathcal{L}$), however this condition is not sufficient as illustrated by the example

$$\mathcal{E} = \mathbb{R}^2, \ u = (x, y)$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -y^3 \\ x^3 \end{pmatrix}.$$

The linear part of equation has $\mu = 1$ as a characteristic value but the equation does not have solutions (λ, u) with $u \neq 0$. The sufficient condition for $(\mu, 0)$ to be a bifurcation point is due to Krasnoselskii [36, 51] as we will see in Theorem 1.24

Theorem 1.24 ([36, 51, 52, Krasnoselskii]). *If μ is a characteristic value of $\mathcal{L}$ with an odd algebraic multiplicity, $(\mu, 0)$ is a bifurcation point.*

The bifurcation phenomenon (which can be local phenomenon or global) gives us informations concerns the structure of the set of nontrivial solutions. If we denote by S the closure of the set of nontrivial solutions of (1.9), Theorem 1.24 implies that the intersection of S with any neighborhood of $(\mu, \ 0)$ is nonempty when μ is of odd multiplicity. For the same hypothesis, Rabinowitz in [51, 52] shows a global phenomenon of bifurcation from a $(\mu, 0)$, more precisely Rabinowitz gives an alternative as we will see in Theorem 1.26.

Definition 1.25. *A component of a topological space is a closed connected subset maximal with respect to inclusion.*

Theorem 1.26 ([51, 52, Rabinowitz]). *If μ is a characteristic value of $\mathcal{L}$ with an odd multiplicity , then S possesses a component of nontrivial solutions C_μ such that $(\mu, 0) \in C_\mu$ and C_μ either :*

- *meets infinity in $\mathbb{R} \times E$ (i.e. C_μ is unbounded); or*
- *meets $(\hat{\mu}, 0)$, where $\hat{\mu}$ is a characteristic value of L with $\hat{\mu} \neq \mu$.*

A stronger result has been obtained by Rabinowitz in [52], when μ is a simple characteristic value of $\mathcal{L}$ (i.e. of multiplicity 1). To describe it, let $v \in E$ denote the eigenvector of $\mathcal{L}$ corresponding to μ normalized so $\|v\|_E = 1$. $l \in \grave{E}$ be the eigenvector of the transpose of $\mathcal{L}$, normalized so that $< l, v >= 1, \tilde{E} = \{u \in E/, \ < l, u >= 0\}$, then $E = \mathbb{R}v \oplus \tilde{\mathcal{E}}$, so for $u \in \mathcal{E}$ we have $u = tv + w$ where $t =< l, u >$ and $w \in \tilde{\mathcal{E}}$. For $\varsigma, \kappa \in \mathbb{R}$ such that $0 < \xi, \ \ 0 < \eta < 1$, we define

$$\mathcal{K}_{\xi,\eta} = \{(\lambda, u) \in \mathbb{R} \times E/|\lambda - \mu| < \xi, < l, u >\geq \eta\|u\|\}.$$

$\mathcal{K}_{\xi,\eta}$ is an open set of $\mathbb{R} \times \mathcal{E}$ and consists of two disjoint convex component $\mathcal{K}^+_{\xi,\eta}$ and $\mathcal{K}^-_{\xi,\eta}$ such that for $\nu = +$ or $-$:

$$\mathcal{K}^\nu_{\xi,\eta} = \{(\lambda, u) \in \mathbb{R} \times \mathcal{E}/|\lambda - \mu| < \xi, < l, \nu u >\geq \eta\|u\|_E\}.$$

For $\zeta > 0$, we denote by $\mathcal{B}_\zeta$ the ball of radius ζ and centred at $(\mu, 0)$.

Lemma 1.27 ([52, Lemma 1.24]). *There exists $\zeta_0 > 0$ such that for all $\zeta < \zeta_0$ $(S - (\mu, 0)) \cap B_\zeta \subset \mathcal{K}_{\xi,\eta}$. If $(\lambda, u) \in (S - (\mu, 0)) \cap B_\zeta)$, then $u = tu + w$ where $t > \eta\|u\|_E$ or $t < -\eta\|u\|_\mathcal{E}$ and $|\lambda - \mu| = 0(1), \|w\|_\mathcal{E} = 0(|t|)$ for t near 0.*

The next Theorem shows that near $(\mu, 0)$, C_μ, consists of two subcontinua which meet only at $(\mu, 0)$.

Theorem 1.28 ([52, Theorem 1.25]). *C_μ possesses a subcontinuum in $\mathcal{K}^+_{\xi,\eta} \cup (\mu, 0)$ and in $\mathcal{K}^-_{\xi,\eta} \cup (\mu, 0)$ each of which meet $(\mu, 0)$ and $\partial\mathcal{B}_\zeta$ for all $\zeta > 0$ sufficiently small.*

Theorem 1.29 ([52, Theorem 1.40]). *$\mathcal{C}_\mu$ can be decomposed into two component $\mathcal{C}_\mu^1$ and $\mathcal{C}_\mu^2$ such that each of them satisfied the alternative :*

- *meets infinity in $\mathbb{R} \times \mathcal{E}$ (i.e. $\mathcal{C}_\mu$ is unbounded); or*
- *meets $(\hat{\mu}, 0)$, where $\hat{\mu}$ is a characteristic value of $\mathcal{L}$ with $\hat{\mu} \neq \mu$.*

For a precise definition of $\mathcal{C}_\mu^1$ and $\mathcal{C}_\mu^2$ see [52].

1.1.5 Elements of Sturm-Liouville boundary value problem

In mathematics and its application a classical Sturm-Liouville theory, named after Jacques Charles Fronçois Sturm (1803-1855) and Joseph Liouville (1809-1882), is the theory of real second order linear differential equation of the form

$$\frac{d}{dx}[p(x)\frac{dy}{dx}] + q(x)y = -\lambda\omega(x)y, \quad \text{in } (a,b), \tag{1.10}$$

where y is a function of the free variable. Here the functions p, q and $\omega > 0$ are specified at the outset. In the simple of cases all coefficients are continuous on the finite closed interval $[a, b]$, and p has continuous derivative. The function $\omega > 0$ is called the weight function, with separated boundary conditions of the form

$$a_1 y(a) - b_1 y'(a) = 0, \quad a_2 u(b) + b_2 y'(b) = 0, \tag{1.11}$$

where a_i, b_i are real numbers such that $|b_i| + |c_i| \neq 0, i = 1, 2$.
In this case the function y is a solution if it is continuously differentiable on (a, b) and satisfies the equation (1.10) at every point in (a, b).
The value of λ is not specified in the equation; finding the value of λ for which there exists a non trivial solution of (1.10) satisfying the boundary conditions is part of the Sturm-Liouville problem.
Such values of λ, when they exist, are called the eigenvalues, and the corresponding solutions (for each such λ) are the eigenfunctions of this problem.

Now we recall some recent results concerning the Sturm-Liouville boundary value problem theory in the integrable case.

The half linear eigenvalue problem

In this section we investigate the half linear eigenvalue problem

$$\begin{cases} -(pu')'(t) + q(t)u(t) = \mu m(t)u(t) + \alpha u^+ - \beta u^-, \quad t \in (\xi, \eta), \\ au(\xi) - bp(\xi)u'(\xi) = 0, \\ cu(\eta) + dp(\eta)u'(\eta) = 0, \end{cases} \tag{1.12}$$

where

- μ is a real parameter.
- $-\infty \leq \xi < \eta \leq +\infty$.
- $p : (\xi, \eta) \to \mathbb{R}^+$ is a measurable function with $p > 0$ a.e on (ξ, η),
- $q, m, \alpha, \beta : (\xi, \eta) \to \mathbb{R}$ are measurable functions.

Noted that $u^+, (u^-)$ the positive part (resp the negative part). The bvp (1.12) is called half-linear since it is linear and positively homogeneous in the cone $u \geq 0$ and $u \leq 0$. Noted that if $\alpha = \beta = 0$ the bvp (1.12) coincide with the linear eigenvalue problem.

In the first, we introduce the concept of the half-eigenvalue.

Definition 1.30. *We say that λ is a half-eigenvalue of (1.12) if there exists a nontrivial solution (λ, u_λ) of (1.12). In this situation, $\{(\lambda, tu_\lambda),\ t > 0\}$ is a half-line of nontrivial solutions of (1.12) and λ is said to be simple if all solutions (λ, v) of (1.12) with v and u having the same sign on a deleted neighborhood of ξ are on this half-line. There may exist another half-line of solutions $\{(\lambda, tv_\lambda),\ t > 0\}$, but then we say that λ is simple if u_λ and v_λ have different signs on a deleted neighborhood of ξ and all solutions (λ, v) of (1.12) lie on these two half lines.*

Let m, α and β be three continuous functions on $[\xi, \eta]$ such that $-\infty < \xi < \eta < +\infty$ with $m > 0$.

Berestycki proved in ([11]) the following theorem

Theorem 1.31. *Assume that $p \in C^1[\xi, \eta]$ and $p > 0$ in $[\xi, \eta]$. Then the set of half eigenvalues of bvp (1.12) consists of two increasing sequences of simple half-eigenvalues $(\lambda_k^+)_{k\geq 1}$ and $(\lambda_k^-)_{k\geq 1}$, such that for all $k \geq 1$ and $\nu = +$ or $-$, the corresponding half-lines of solutions are in $\{\lambda_k^\nu\} \times S_k^\nu$.*

Benmezai and Esserhane [7] proved that the Berestycki result holds for the integrable case in the following lemma .

Lemma 1.32. *[7, lemma 3.7] Assume that $\frac{1}{p}, q, m, \alpha, \beta \in L^1(\xi, \eta)$ with $\frac{1}{p} > 0$ and $m > 0$ a.e. on (ξ, η). Then the set of half eigenvalues of bvp (1.12) consists of two increasing sequences of simple half-eigenvalues $(\lambda_k^+)_{k\geq 1}$ and $(\lambda_k^-)_{k\geq 1}$, such that for all $k \geq 1$ and $\nu = +$ or $-$, the corresponding half-lines of solutions are in $\{\lambda_k^\nu\} \times S_k^\nu$. Furthermore, aside this solutions and the trivial ones, there are no other solution of (1.12).*

In the same paper Benmezai and Esserhane [7], proved the existence of half eigenvalues on Theorem 1.33 when they relaxed the condition $m > 0$ a.e on (ξ, η) to the condition $m \geq 0$ a.e. on (ξ, η) and $m > 0$ on a subset of positive measure.

Theorem 1.33. *Assume that $\frac{1}{p}, q, m, \alpha, \beta \in L^1(\xi, \eta)$ with $\frac{1}{p} > 0$ and $m \geq 0$ a.e. on (ξ, η) with $m > 0$ on a subset of positive measure. Then the bvp (1.12) admits two increasing sequences of simple half-eigenvalues $(\lambda_k^+(q, m, \alpha, \beta))_{k\geq 1}$ and $(\lambda_k^-(q, m, \alpha, \beta))_{k\geq 1}$, such that for all integers $k \geq 1$ and $\nu = +, -$, the corresponding half-line of solutions lies in $\{\lambda_k^\nu(q, m, \alpha, \beta)\} \times S_k^\nu$ and $\lim_{k\to+\infty} \lambda_k^\nu(q, m, \alpha, \beta) = +\infty$. Furthermore, aside from these solutions and the trivial one, there are no other solutions of (1.12). Moreover,*

- *For m fixed in $L^1(\xi, \eta)$ such that $m \geq 0$ a.e. on (ξ, η) and $m > 0$ on a subset of positive measure and $q, \alpha_1, \alpha_2, \beta_1, \beta_2 \in L^1(\xi, \eta)$, the mapping $\lambda_k^\nu(q, m, ., .)$ has the following properties:*

  1. *If $\alpha_1 \leq \alpha_2$ a.e. in (ξ, η) , then $\lambda_k^\nu(q, m, \alpha_1, \beta_1) \geq \lambda_k^\nu(q, m, \alpha_2, \beta_2)$, for all $k \geq 1$ and $\nu = +$ or $-$.*
  2. *If $\beta_1 \leq \beta_2$ a.e. on (ξ, η), then $\lambda_k^\nu(q, m, \alpha_1, \beta_1) \geq \lambda_k^\nu(q, m, \alpha_1, \beta_2)$, for all $k \geq 1$ and $\nu = +$ or $-$.*

- *Let $m, q_1, q_2, \alpha, \beta \in L^1(\xi, \eta)$ such that $m \geq 0$ a.e. on (ξ, η) and $m > 0$ in a subset of positive measure. The mapping $\lambda_k^\nu(m, ., \alpha, \beta)$ has the following properties: If $q_1, q_2 \in L^1(\xi, \eta)$ such that $q_1 \leq q_2$ a.e. on (ξ, η) then $\lambda_k^\nu(q_1, m, \alpha, \beta) \leq \lambda_k^\nu(q_2, m, \alpha, \beta)$, for all $k \geq 1$ and $\nu = +, -$. Moreover, if $q_1 < q_2$ on a subset of positive measure, then $\lambda_k^\nu(q_1, m, \alpha, \beta) < \lambda_k^\nu(q_2, m, \alpha, \beta)$.*

- *Let* $m_n, q_n, m, q \in L^1(\xi,\eta)$ *such that* $m_n, m \geq 0$ *a.e. on* (ξ,η) *and* $m_n, m > 0$ *on a subset of positive measure and if* $q_n \to q$ *and* $m_n \to m$ *in* L^1*. Then for all* $\alpha, \beta \in L^1(\xi,\eta)$*,* $k \geq 1$ *and* $\nu = +, -$*, we have* $\lim_{n\to\infty} \lambda_k^\nu(q_n, m_n, \alpha, \beta) = \lambda_k^\nu(q, m, \alpha, \beta)$*.*

- *For* α, β *fixed in* $L^1(\xi,\eta)$ *and* m_1, m_2 *two functions such that* $m_i \geq 0$ *a.e. on* (ξ,η) *and* $m_i > 0$ *in a subset of positive measure for* $i = 1, 2$*, then the mapping* $\lambda_k^\nu(q, ., \alpha, \beta)$ *has the following properties: If* $m_1 \leq m_2$ *a.e. in* (ξ,η)*,* $m_1 < m_2$ *in a subset of positive measure If either* $\lambda_k^\nu(q, m_1, \alpha, \beta) \geq 0$ *or* $\lambda_k^\nu(q, m_2, \alpha, \beta) \geq 0$ *, then* $\lambda_k^\nu(q, m_1, \alpha, \beta) > \lambda_k^\nu(q, m_2, \alpha, \beta)$*, and if either* $\lambda_k^\nu(q, m_1, \alpha, \beta) \leq 0$ *or* $\lambda_k^\nu(q, m_2, \alpha, \beta) \leq 0$*, then* $\lambda_k^\nu(q, m_1, \alpha, \beta) < \lambda_k^\nu(q, m_2, \alpha, \beta)$*, for all* $k \geq 1$ *and* $\nu = +$ *or* $-$*.*

The linear eigenvalue problem

For the particular case of the problem (1.12) where $\alpha = \beta = 0$, namely for the problem

$$\begin{cases} -(pu')'(t) + q(t)u(t) = \mu m(t)u(t), \quad t \in (\xi,\eta), \\ au(\xi) - bp(\xi)u'(\xi) = 0, \\ cu(\eta) + dp(\eta)u'(\eta) = 0, \end{cases} \tag{1.13}$$

we obtain from Lemma 1.32 the following theorem of Zettl, concerns the basic existence result of eigenvalue for the linear eigenvalue problem (1.13)

Theorem 1.34 ([64, Theorem 4.3.1])**.** *Assume that* $p > 0$ *,* $\frac{1}{p}, q, m \in L^1(\xi,\eta)$ *and* $m > 0$ *a.e. on* (ξ,η)*.*

Then the Sturm-Liouville Problem (1.13) has an infinite but countable number of real eigenvalue and they can be ordred to satisfy

$$-\infty < \mu_1 < \mu_2 < \cdots \text{ and } \lim_{k\to+\infty} \mu_k = \infty$$

. If u_k *is an eigenvalue of* μ_k*, then* u_k *is unique up to constant multiples. Let* n_k *denote the number of zeros of* u_k *in the open interval* (ξ,η)*, then for* $k \geq 1$*,*

$$n_{k+1} = n_k + 1.$$

Zettl proved the existence of eigenvalue on theorem 4.3.2. when they relaxed the condition $m > 0$ a.e. in (ξ,η) to the condition $m \geq 0$ in (ξ,η) and $\int_\xi^\eta m > 0$.

Theorem 1.35 ([64, Theorem 4.3.2]). *Assume that $p > 0$, $\frac{1}{p}, q, m \in L^1(\xi, \eta)$ and $m \geq 0$ on (ξ, η) and $\int_\xi^\eta m > 0$.*

Then the Sturm-Liouville Problem (1.13) has an infinite but countable number of real eigenvalue and they can be ordred to satisfy

$$-\infty < \mu_1 < \mu_2 < \cdots \text{ and } \lim_{k \to +\infty} \mu_k = \infty$$

. If u_k is an eigenvalue of μ_k, then u_k is unique up to constant multiples. Let n_k denote the number of zeros of u_k in the open interval (ξ, η), then for $k \geq 1$

$$n_{k+1} = n_k + 1.$$

Moreover the sufficient but not necessary condition to have $n_1 = 0$ is that $m > 0$ a.e. in (ξ, η).

The integral condition on m eliminate the case when m is identically zero on (ξ, η).

Zettl proved the monotonicity of eigenvalue in theorem 4.9.1.

Theorem 1.36 ([64, Theorem 4.9.1]). *For $\frac{1}{p} > 0, \frac{1}{p}, q, m \in L^1(\xi, \eta)$ and $m > 0$ a.e. in (ξ, η). Then the problem (1.12) admits an unbounded increasing sequence of eigenvalues $(\mu_k(p, q, m),\ k \geq 1)$ such that eigenfunctions associated with $\mu_k(p, q, m)$ belong to S_k. Moreover,*

- *Fix p, m. Suppose $Q \in L^1(\xi, \eta)$ and assume that $Q \geq q$ a.e. on (ξ, η). Then for all $k \geq 1$, $\mu_k(p, Q, m) \geq (\mu_k(p, q, m)$. If $Q > q$ on a subset of positive measure, then for all $k \geq 1$, $\mu_k(p, Q, m) > (\mu_k(p, q, m)$*

- *Fix p, m. Suppose $\frac{1}{P} \in L^1(\xi, \eta)$ and $0 < P \leq p$ a.e. on (ξ, η). Then for all $k \geq 1$, $\mu_k(P, q, m) \geq (\mu_k(p, q, m)$. If $\frac{1}{P} < \frac{1}{p}$ on a subset of positive measure, then for all $k \geq 1$, $\mu_k(P, q, m) < \mu_k(p, q, m)$.*

- *Fix p, q. Suppose $M \in L^1(\xi, \eta)$ and $M \geq m > 0$ a.e. on (ξ, η). Let $k \geq 1$, then $\mu_k(p, q, M) \geq \mu_k(p, q, m)$ if $\mu_k(p, q, M) < 0$ and $\mu_k(p, q, m) < 0$; but $\mu_k(p, q, M) \leq \mu_k(p, q, m)$ if $\mu_k(p, q, M) > 0$ and $\mu_k(p, q, m) > 0$. Furthermore, if strict inequality holds in the hypothesis on a set of positive measure, then strict inequality holds in the conclusion.*

Zettl proved the dependence of eigenvalue on the problem in theorem 4.4.1.

Let $J = (\xi', \eta')$ such that $-\infty \leq \xi' < \xi < \eta < \eta' \leq +\infty$, they study the variation of the eigenvalues with respect to the end point ξ, η as they vary within the interval J.

Theorem 1.37 ([64, Theorem 4.4.1]). *Assume that $\frac{1}{p} > 0$, $\frac{1}{p}, q, m \in L^1_{loc}(\xi', \eta')$. with $m > 0$ a.e. on (ξ', η'). For each $n \in \mathbb{N}$. Let μ_n be the set of the eigenvalue of (1.12) . Then For each $n \in \mathbb{N}$, μ_n is a continuous function of the equation. In particular:*

- *For each $n \in \mathbb{N}$, $\mu_n(\frac{1}{p})$ is a continuous function of $\frac{1}{p} \in L^1(\xi', \eta')$.*
- *For each $n \in \mathbb{N}$, $\mu_n(q)$ is a continuous function of $q \in L^1(\xi', \eta')$.*
- *For each $n \in \mathbb{N}$, $\mu_n(m)$ is a continuous function of $m \in L^1(\xi', \eta')$.*
- *For each $n \in \mathbb{N}$, $\mu_n(\xi)$ is a continuous function of ξ, $\lambda_n(\eta)$ is a continuous function of η.*

Chapter 2

Nodal solutions for asymptotically linear second-order BVPs on the half line

2.1 Introduction

Often motivated by a physical interest, study of existence of solutions to boundary value problems (BVPs for short) associated with second-order ordinary differential equations posed on infinite intervals and their qualitative properties has been the thematic of many articles, see for instance [1, 3, 5, 15, 19, 24, 25, 30, 29, 34, 35, 48] and references therein. Such a study is developed in the papers [24, 25, 30, 29, 48] for the class of BVPs:

$$\begin{cases} -u'' + a(t)u = F(t,u),\ t > T, \\ \mathcal{BC}, \end{cases} \tag{2.1}$$

where $F \in C\left((T,+\infty) \times \mathbb{R}, \mathbb{R}\right)$, $a \in C\left([T,+\infty), \mathbb{R}^{+}\right)$ does not vanish identically and $\mathcal{BC}$ are boundary conditions at T and $+\infty$.

In [24] and [25] is considered the case of BVP (2.1) where the weight a is a positive constant, $\mathcal{BC}$ takes the form $u(T) = \lim_{t\to+\infty} u(t) = 0$ and the nonlinearity F is positive. Notice that for such a weight a, the Green's function associated with BVP (2.1) is given explicitly. This particularity allowed authors to construct a favourable framework to the use of Krasnoselskii's fixed point theorem in a cone, and so to obtain existence and multiplicity results for positive solutions to this particular case of BVP (2.1).

Inspired by the works in [24] and [25], Ma and Zhu investigate in [48], existence and multiplicity of positive solutions for the case of BVP (2.1) where the weight a is bounded from below and above by positive constants, $\mathcal{BC}$ takes the form $u(T) = \lim_{t\to+\infty} u(t) = 0$

and the nonlinearity F is semipositone. They proved that such a weighted BVP has a Green's function whose properties allowed them to construct an appropriate framework to the use of Krasnoselskii's fixed point theorem in a cone.

In [30] and [29], is considered BVP (2.1) under the conditions that the weight a is bounded from below by a positive constant (a may be unbounded from above) and $\mathcal{BC}$ takes the form $u(T) = u_0$ and u is bounded. Combining the method of upper and lower solutions and sequential arguments, authors obtained existence and multiplicity results.

Many old and recent works, see for instance [6, 7] and references therein, show that under suitable conditions, existence of nodal solutions to BVPs associated with second order ordinary differential equations usually occurs. For this reason, we investigate in this chapter existence of such solutions to BVP (2.1) when $a(t) \geq 0$ for all $t \geq T$ and $\inf_{t \geq T_0} a(t) > 0$ for some $T_0 \geq T$ (a may be unbounded from above) and $\mathcal{BC}$ takes the form $u(T) = \lim_{t \to +\infty} u(t) = 0$. The first main result of this work concerns the spectrum of the linear eigenvalue problem associated with our case of BVP (2.1). It claims that this spectrum consists in an unbounded increasing sequence of eigenvalues and the coresponding eigenfunctions have nodal properties. The second main result of this work is obtained by means of Rabinowitz global bifurcation theory. It claims that if the nonlinearity F has linear approximations at 0 and ∞ satisfying eigenvalue criteria then our version of BVP (2.1) admits nodal solutions.

2.2 Main results

This work deals with existence of nodal solutions to the BVP,

$$\begin{cases} -u''(t) + q(t)u(t) = u(t)f(t, u(t)) & t > 0, \\ u(0) = \lim_{t \to +\infty} u(t) = 0, \end{cases} \tag{2.2}$$

where $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $q \in C(\mathbb{R}^+, \mathbb{R}^+)$.

Statements of the main results of this paper need to introduce some notations. In what follows, we let

$$\Gamma = \{m \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} m(t) = 0\},$$
$$\Gamma^+ = \{m \in \Gamma : m(t) > 0 \text{ a.e. } t \in \mathbb{R}\},$$
$$Q = \{q \in C(\mathbb{R}^+, \mathbb{R}^+) : \exists T \geq 0 \text{ such that } \inf_{t\geq T} q(t) > 0\},$$
$$W = \{u \in C(\mathbb{R}^+, \mathbb{R}) : u(0) = \lim_{t\to+\infty} u(t) = 0\},$$
$$W_k = W \cap C^k(\mathbb{R}^+, \mathbb{R}) \text{ for all integers } k \geq 1.$$

Hereafter, the linear space W is equipped with the norm $\|\cdot\|$, defined for $u \in W$ by $\|u\| = \sup_{t\geq 0} |u(t)|$. Obviously, $(W, \|\cdot\|)$ is a Banach space.

For an integer $k \geq 1$, S_k^+ denotes the set of all the functions $u \in W_1$ having exactly $(k-1)$ zeros in $(0, +\infty)$, all are simple and u is positive in a right neighbourhood of 0, $S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$. For $u \in S_k$, the unique sequence $(z_j)_{j=0}^{j=k}$ such that $0 = z_0 < z_1 < ... < z_k = +\infty$ and $u(z_j) = 0$ for $j = 1, ..., k-1$, is said to be the sequence of zeros of u.

First, we focus our attention on the linear eigenvalue problem associated with BVP (2.2); Namely, we consider for $(q, m) \in Q \times \Gamma^+$ the problem of existence of eigenvalues to the eigenvalue problem (EVP for short):

$$\begin{cases} -u''(t) + q(t)u(t) = \mu m(t)u(t) & t > 0, \\ u(0) = 0, \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{2.3}$$

where μ is a real parameter.

Theorem 2.1. *For all pairs (q, m) in $Q \times \Gamma^+$, the set of eigenvalues of the EVP (2.3) consists in an unbouded increasing sequence of simple eigenvalues $(\mu_k(q, m))_{k\geq 1}$ such that eigenfunctions associated with $\mu_k(q, m)$ belong to S_k. Moreover, for q fixed in Q, the mapping $\mu_k(q, \cdot)$ has the following properties:*

1. *If $m_1, m_2 \in \Gamma^+$ are such that $m_1 \leq m_2$, then $\mu_k(m_1) \geq \mu_k(m_2)$. In addition, $\mu_k(m_1) > \mu_k(m_2)$ whenever $m_1 < m_2$ in a subset of positive measure.*

2. *If $m \in \Gamma^+$ and $(m_n) \subset \Gamma^+$ are such that $\lim m_n = m$ uniformly on $\mathbb{R}^+$, then $\lim_{n\to\infty} \mu_k(q, m_n) = \mu_k(q, m)$.*

Concerning BVP (2.2), we obtain under the assumptions on the nonlinearity f

$$\begin{cases} |f(t, 0)| \in \Gamma^+ \text{ and for all } r > 0, \text{ there exists } \psi_r \in \Gamma^+ \text{ such that} \\ |f(t, u) - f(t, v)| \leq \psi_r(t) |u - v| \text{ for all } t \geq 0 \text{ and } u, v \in [-r, r], \end{cases} \tag{2.4}$$

$$\begin{cases} \text{there exists } \omega \in \Gamma^+ \text{ such that} \\ f(t,u) + \omega(t) \geq 0 \text{ for all } t \geq 0 \text{ and } u \in \mathbb{R}, \end{cases} \tag{2.5}$$

$$\begin{cases} \lim_{u \to 0} f(t,u) = m_0(t) \text{ and} \\ \lim_{|u| \to +\infty} f(t,u) = m_\infty(t) \\ \text{uniformly in } \mathbb{R}^+ \text{ with } m_0, m_\infty \in \Gamma^+, \end{cases} \tag{2.6}$$

the following existence and multiplicity result for nodal solutions:

Theorem 2.2. *Let $q \in Q$ and assume that in addition to Hypotheses (2.4)-(2.6), there exist two integers i, j with $1 \leq i \leq j$ such that one of the following situations holds:*

$$\mu_j(q, m_\infty) < 1 < \mu_i(q, m_0)$$

or

$$\mu_j(q, m_0) < 1 < \mu_i(q, m_\infty).$$

Then BVP (2.2) admits a solution in S_k^ν for all integers $k \in \{i, ..., j\}$ and $\nu = +$ or $-$.

Now, consider the case of the BVP (2.2) where the nonlinearity f is a separable variables function; Namely the case where the BVP (2.2) takes the form

$$\begin{cases} -u'' + q(t)u = m(t)ug(u), \ t > 0, \\ u(0) = \lim_{t \to +\infty} u(t) = 0, \end{cases} \tag{2.7}$$

where $m \in \Gamma^+$ and $g : \mathbb{R} \to \mathbb{R}^+$ is a continuously differentiable function such that

$$\lim_{u \to 0} g(u) = g_0 > 0 \text{ and } \lim_{u \to +\infty} g(u) = g_\infty > 0. \tag{2.8}$$

We deduce, from Theorem 2.2 the following corollary:

Corollary 2.3. *Let $q \in Q$ and assume that in addition to Hypothesis (2.8), there exist two integers i, j with $1 \leq i \leq j$ such that one of the following situations holds:*

$$g_0 < \mu_i(m) < \mu_j(m) < g_\infty, \text{ or}$$

$$g_\infty < \mu_i(m) < \mu_j(m) < g_0.$$

Then BVP (2.7) admits a solution in S_k^ν for all integers $k \in \{i, ..., j\}$ and $\nu = +$ or $-$.

Proof .

Set $f(t,u) = m(t)g(u)$ and notice that such a nonlinearity satisfies Hypotheses (2.5) and (2.6) for

$$m_0(t) = g_0 m(t),\ m_\infty(t) = g_{+\infty} m(t).$$

Since for all integers $k \geq 1$ and $\kappa = 0$ or $+\infty$, $\mu_k(m_\kappa) = \mu_k(m)/g_\kappa$, we have

$$\mu_i(m_0) < 1 < \mu_j(m_\infty) \text{ if } g_\infty < \mu_i(m) < \mu_j(m) < g_0$$

and

$$\mu_j(m_0) < 1 < \mu_i(m_\infty) \text{ if } g_0 < \mu_i(m_0) < \mu_j(m_\infty) < g_\infty.$$

Therefore, Corollary 2.3 is obtained by a simple application of Theorem 2.2. ■

2.3 Preliminaries

2.3.1 The Green's function and fixed point formulation

In all what follows, we let for $q \in Q$, Ψ_q be the unique solution of the initial value problem

$$\begin{cases} -u''(t) + q(t)u(t) = 0, \\ u(0) = 0,\ u'(0) = 1. \end{cases}$$

Lemma 2.4. *For all $q \in Q$, the function Ψ_q has the following properties:*

i) $\Psi_q(t) > 0,\ \Psi'_q(t) > 0$ *and* $\Psi''_q(t) \geq 0$ *for all* $t \in \mathbb{R}^+$.

ii) $\lim_{t\to+\infty} \Psi'_q(t) = +\infty, \lim_{t\to+\infty} \frac{\Psi_q(t)}{1+t} = +\infty,\ \int_t^{+\infty} \dfrac{ds}{\psi_q^2} < \infty$ *for all* $t > 0$.

iii) *The function* Ψ_q/Ψ'_q *is bounded at* $+\infty$.

iv) $\lim_{t\to 0} \Psi_q(t) \int_t^{+\infty} \frac{ds}{\Psi_q^2(s)} = 1$

v) $\lim_{t\to+\infty} \Psi_q(t) \int_t^{+\infty} \frac{ds}{\Psi_q^2(s)} = 0$

Proof .

Let $q \in Q$ and $T > 0$ be such that $\alpha = \inf_{t\geq T} q(t) > 0$.

i)Suppose on the contrary that $\Psi_q'(t_0) = 0$ for some t_0 on $(0, +\infty)$. By the boundary condition $\Psi_q'(0) = 1$, $t_0 > 0$ we may assume that $\Psi_q'(t) > 0$ on $[0, t_0)$. Thus ψ_q is strictly increasing on $[0, t_0)$. On the other hand we have from the equation that $\psi_q''(t_0) = q(t)\psi_q(t_0) \geq 0$, and accordingly t_0 is a minimum value point. This is a contradiction. Then we have for all $t \in \mathbb{R}^+$, $\Psi_q'(t) > 0$, by the boundary condition $\Psi_q(0) = 0$ we obtain that $\psi_q(t) > 0$ for all $t \in \mathbb{R}^+$, and from the equation we have for all $t \in \mathbb{R}^+$, $\psi_q''(t) \geq 0$

ii) Now, we have for all $t \geq T$

$$\Psi_q'(t) = \Psi_q'(T) + \int_T^t \Psi_q'' ds = \Psi_q'(T) + \int_T^t q\Psi_q ds \geq \Psi_q'(T) + \alpha\varepsilon\,(t - T)\,,$$

where $\varepsilon = \inf_{s \geq 0} \Psi_q(s) > 0$. The above inequality shows that $\lim_{t \to +\infty} \Psi_q'(t) = +\infty$. By L'Hopital's rule, we have

$$\lim_{t \to +\infty} \frac{\psi_q(t)}{1+t} = \lim_{t \to +\infty} \psi_q'(t) = +\infty.$$

This shows that $\psi_q^2 \geq (1+t)^2$ for all $t > 0$. By comparaison principle, we have $\int_t^{+\infty} \dfrac{ds}{\psi_q^2} < \infty$ for all $t > 0$.

iii)

$$\begin{aligned}\left(\Psi_q'(t)\right)^2 - \left(\Psi_q'(T)\right)^2 &= 2\int_T^t \Psi_q''(s)\Psi_q'(s)ds = 2\int_T^t q(s)\Psi_q(s)\Psi_q'(s)ds \\ &\geq \alpha\left(\left(\Psi_q(t)\right)^2 - \left(\Psi_q(T)\right)^2\right),\end{aligned}$$

which leads to

$$\left(\Psi_q(t)/\Psi_q'(t)\right)^2 \leq \frac{1}{\alpha} + \left(\Psi_q'(T)/\Psi_q'(t)\right)^2.$$

From this and Property (ii), we deduce existence of $T_\alpha > 0$ such that

$$\Psi_q(t)/\Psi_q'(t) \leq \sqrt{\frac{2}{\alpha}} \text{ for all } t \geq T_\alpha.$$

iv) We have by L'Hopital's rule

$$\lim_{t \to 0} \psi_q(t) \int_t^{+\infty} \frac{ds}{\psi_q^2} = \lim_{t \to 0} \frac{\int_t^{+\infty} \psi_q^{-2} ds}{\left(\psi_q(t)\right)^{-1}} = \lim_{t \to 0} \frac{1}{\psi_q'(t)} = 1.$$

v) Again by L'Hopital's rule we get

$$\lim_{t \to +\infty} \Phi_q(t) \int_t^{+\infty} \frac{ds}{\psi_q^2} = \lim_{t \to +\infty} \frac{1}{\psi_q'(t)} = 0.$$

Proving **v)** and completing the proof of the lemma. ■

Because of Assertions ii), iii), iv) and v) in Lemma 2.4, the function

$$\Phi_q(t) = \begin{cases} \Psi_q(t) \int_t^{+\infty} \frac{ds}{\Psi_q^2(s)}, & \text{if} \quad t > 0, \\ \\ 1, & \text{if} \quad t = 0 \end{cases} \tag{2.9}$$

is well defined and it is the unique solution to the BVP

$$\begin{cases} -u''(t) + q(t)u(t) = 0, \\ u(0) = 1, \ \lim_{t\to+\infty} u(t) = 0. \end{cases}$$

Lemma 2.5. *For all $q \in Q$, the function ϕ_q has the following properties:*

a) $\phi_q(t) > 0$, $\phi_q'(t) < 0$ *and* $\phi_q''(t) \geq 0$ *for all* $t > 0$.

b) $\lim_{t\to+\infty} \phi_q'(t) = 0$

c) *For all* $t > 0$, $\int_t^{+\infty} \phi_q ds < \infty$.

d) *For all* $t > 0$, $\Phi_q(t)\Psi_q'(t) - \Psi_q(t)\Phi_q'(t) = 1$.

e) *The function* ϕ_q/ϕ_q' *is bounded at* $+\infty$.

Proof .

Let $q \in Q$ and $T > 0$ be such that $\alpha = \inf_{t\geq T} q(t) > 0$.

a) Respectively from (2.9) and $\phi_q'' = q\phi_q$, we have $\phi_q(t) > 0$ and $\phi_q''(t) \geq 0$ for all $t > 0$. Since the function ψ_q' is increasing, we obtain from (2.9) that

$$\phi_q'(t) = \psi_q'(t) \int_t^{+\infty} \frac{ds}{\psi_q^2} - \frac{1}{\psi_q(t)} < \int_t^{+\infty} \frac{\psi_q'}{\psi_q^2} ds - \frac{1}{\psi_q(t)} = \frac{-2}{\psi_q} < 0.$$

b) It follows from Assertion **(a)** that the function Φ_q' is nondecreasing and the limit $\lim_{t\to+\infty} \Phi_q'(t)$ exist. Set $l = \lim_{t\to+\infty} \Phi_q'(t)$ and suppose that $l < 0$. We obtain then by the L'Hopital's rule

$$\lim_{t\to+\infty} \frac{\Phi_q(t)}{t} = \lim_{t\to-\infty} \Phi_q'(t) = l < 0,$$

leading to $\lim_{t\to+\infty} \Phi_q(t) = -\infty$. This contradicts $\lim_{t\to+\infty} \Phi_q(t) = 0$ and proves that $\lim_{t\to+\infty} \Phi_q'(t) = 0$.

c) We have for all $s \in (T, +\infty)$

$$\int_T^s \Phi_q dr = \int_T^s \frac{\Phi_q''}{q} dr \leq \frac{1}{\alpha} \int_T^s \Phi_q'' dr = \frac{1}{\alpha}(\Phi_q'(s) - \Phi_q'(T)) \leq -\frac{\Phi_q'(T)}{\alpha}.$$

This proves that $\int_t^{+\infty} \Phi_q(r)dr < \infty$ for all $t > 0$.

d) We have from (2.9) that for all $t > 0$,

$$\Phi_q(t)\Psi_q'(t) - \Psi_q(t)\Phi_q'(t) = \psi_q(t)\left(\psi_q'(t)\int_t^{+\infty}\frac{ds}{\psi_q^2} + \frac{1}{\psi_q(t)}\right) - \psi_q(t)\psi_q'(t)\int_t^t \frac{ds}{\psi_q^2} = 1.$$

e) We have for $t \geq T$:

$$\begin{aligned}\left(-\Phi_q'(t)\right)^2 &= 2\int_t^{+\infty}\Phi_q''(s)\left(-\Phi_q'(s)\right)ds = \int_t^{+\infty} q(s)\Phi_q(s)\left(-\Phi_q'(s)\right)ds \\ &\geq \alpha\left(\Phi_q(t)\right)^2.\end{aligned}$$

leading to

$$\left|\Phi_q(t)/\Phi_q'(t)\right|^2 = \left(\Phi_q(t)/-\Phi_q'(t)\right)^2 \leq \frac{1}{\alpha} \text{ for all } t \geq T,$$

then to,

$$\sup_{t\geq T}\left|\Phi_q(t)/\Phi_q'(t)\right| \leq \frac{1}{\sqrt{\alpha}}.$$

This completes the proof of **(e)** and ends the proof of the lemma. ■

Set for $q \in Q$ and $\theta \geq 0$,

$$\Phi_{q,\theta}(t) = \frac{\Phi_q(t)}{\Phi_q(\theta)},\ \Psi_{q,\theta}(t) = \Phi_q(\theta)\Psi_q(t) - \Psi_q(\theta)\Phi_q(t) \text{ and}$$

$$G_q(\theta,t,s) = \begin{cases} 0 \text{ if } \min(t,s) \leq \theta, \\ \Phi_{q,\theta}(s)\Psi_{q,\theta}(t) \text{ if } \theta \leq t \leq s, \\ \Phi_{q,\theta}(t)\Psi_{q,\theta}(s) \text{ if } \theta \leq s \leq t. \end{cases} \tag{2.10}$$

We have then for $q \in Q$ and $\theta \geq 0$,

$$\Phi_{q,\theta}(t)\Psi_{q,\theta}'(t) - \Psi_{q,\theta}(t)\Phi_{q,\theta}'(t) = 1 \text{ for all } t > 0, \tag{2.11}$$

$$G_q(\theta,t,s) = G_q(t,s) - \frac{\Psi_q(\theta)}{\Phi_q(\theta)}\Phi_q(s)\Phi_q(t), \text{ for } t,s \geq \theta,$$

where

$$G_q(\cdot,\cdot) = G(0,\cdot,\cdot) \tag{2.12}$$

is the Green's function associated with BVP (2.2).

Lemma 2.6. *We have for all functions q in Q :*

i) $\overline{G}_{q,\infty} = \sup_{t,s\in\mathbb{R}^+} G_q(t,s) \leq \sup_{0\leq t<+\infty}\Phi_q(t)\Psi_q(t) < \infty$,

ii) $G_\infty = \sup_{\theta,t,s\in\mathbb{R}^+} G_q(\theta,t,s) < \infty$,

iii) $\tilde{G}_q = \sup_{t\geq 0}\int_0^{+\infty} G_q(t,s)ds < \infty$.

Proof .

Let $q \in Q$ and $T > 0$ be such that $\alpha = \inf_{t \geq T} q(t) > 0$.

i) Taking in consideration that Φ_q is nonincreasing, we obtain from (2.9) that for all $t, s \in \mathbb{R}$,

$$\begin{aligned} G_q(t,s) &\leq \Phi_q(t)\Psi_q(t) = \left(\frac{\Psi_q(t)}{\Psi_q'(t)}\right)\left(\Psi_q(t)\Psi_q'(t)\int_t^{+\infty}\frac{ds}{\Psi_q^2(s)}\right) \\ &\leq \left(\frac{\Psi_q(t)}{\Psi_q'(t)}\right)\left(\Psi_q(t)\int_t^{+\infty}\frac{\Psi_q'(s)ds}{\Psi_q^2(s)}\right) = \left(\frac{\Psi_q(t)}{\Psi_q'(t)}\right). \end{aligned}$$

This together with assertion iii) in the lemma 2.4, leads to

$$\overline{G}_{q,\infty} = \sup_{t,s\in\mathbb{R}^+} G_q(t,s) \leq \sup_{t\in\mathbb{R}} \Phi_q(t)\Psi_q(t) < \infty.$$

ii) Because of Φ_q is decreasing and Ψ_q is increasing, we have for all $s, t \geq \theta$

$$\begin{aligned} 0 &\leq G_q(\theta,t,s) \leq \Phi_q(t)\Psi_q(t) + \frac{\Psi_q(\theta)}{\Phi_q(\theta)}\Phi_q(t)\Phi_q(ts) \\ &\leq \Phi_q(t)\Psi_q(t) + \Psi_q(\theta)\Phi_q(\theta) \\ &\leq 2\sup_{t\geq 0}\Phi_q(t)\Psi_q(t) < \infty, \end{aligned}$$

proving **(ii)**.

iii) We have for all $t \geq T$:

$$\begin{aligned} \int_0^{+\infty} G_q(t,s)ds &= \Phi_q(t)\int_0^t \Psi_q(s)ds + \Psi_q(t)\int_t^{+\infty}\Phi_q(s)ds \\ &= \Phi_q(t)\int_0^t \Psi_q(s)ds + \Psi_q(t)\int_t^{T}\Phi_q(s)ds + \Psi_q(t)\int_T^{+\infty}\Phi_q(s)ds \\ &= \Phi_q(t)\int_0^t \frac{\Psi_q''}{q}ds + \Psi_q(t)\int_t^{T}\frac{\Phi_q''}{q}ds + \Psi_q(t)\int_T^{+\infty}\frac{\Phi_q''}{q}ds \\ &\leq \Phi_q(t)\frac{1}{\alpha}(\Psi_q'(t) - \Psi_q'(0)) - \frac{1}{\alpha}\Psi_q(t)\Phi_q'(t) \\ &\leq \frac{1}{\alpha}(\Phi_q(t)\Psi_q'(t) - \Phi_q'(t)\Psi_q(t) - \Phi_q(t)\Psi_q'(0)). \end{aligned}$$

This together with (d) in lemma 2.5, leads to

$$\begin{aligned} \int_0^{+\infty} G_q(t,s)ds &\leq \frac{1}{\alpha}(1 - \Phi_q(t)) \\ &\leq \frac{1}{\alpha}, \end{aligned}$$

$\sup_{t\geq 0}\int_0^{+\infty} G_q(t,s)ds \leq \frac{1}{\alpha}$.

The proof is complete. ■

The main result of this subsection consists in the following lemma providing a fixed point formulation for BVP (2.2) and EVP (2.3).

Lemma 2.7. *For all functions q in Q, h in Γ and all nonnegative real numbers θ, $u(t) = \int_0^{+\infty} G_q(\theta,t,s)h(s)ds$ is the unique solution in $(\theta,+\infty)$ to the BVP:*

$$\begin{cases} -u''(t)+q(t)u(t)=h(t),\ t>\theta, \\ u(\theta)=\lim_{t\to+\infty} u(t)=0. \end{cases} \tag{2.13}$$

Moreover, for all functions $F \in C(\mathbb{R}^+\times\mathbb{R},\mathbb{R})$ satisfying Hypothesis (2.4), the operator $T_\theta : W \to W$ defined for $u \in W$ by

$$T_\theta u\,(t) = \int_0^{+\infty} G_q(\theta,t,s)u(s)F(s,u(s))ds$$

is completely continuous.

Proof .

Differentiating twice in the relation

$$u(t)=\int_0^{+\infty} G_q(\theta,t,s)h(s)ds = \Phi_{q,\theta}(t)\int_\theta^t \Psi_{q,\theta}(s)\,h(s)ds + \Psi_{q,\theta}(t)\int_t^{+\infty}\Phi_{q,\theta}(s)\,h(s)ds,$$

we get

$$u''(t)=q(t)u(t)+\left(\Phi'_{q,\theta}(t)\,\Psi_{q,\theta}(t)-\Phi_{q,\theta}(t)\,\Psi'_{q,\theta}(t)\right)h(t) \text{ for all } t\geq\theta,$$

then by (d) in lemma 2.5 we obtain

$$-u''(t)+q(t)u(t)=h(t) \text{ for all } t\geq\theta.$$

Because of $G_q(\theta,\theta,s)=0$ for $s>0$, we have $u(\theta)=\int_0^{+\infty} G_q(\theta,\theta,s)h(s)ds=0$.

It remains to show that $\lim_{t\to+\infty} u(t)=0$, we have for all $t>\theta$:

$$u(t)=\Phi_q(t)\int_\theta^t \Psi_q(s)h(s)ds+\Psi_q(t)\int_t^{+\infty}\Phi_q(s)h(s)ds-\frac{\Psi_q(\theta)}{\Phi_q(\theta)}\Phi_q(t)\int_\theta^{+\infty}\Phi_q(s)h(s)ds.$$

Because of (iv) in lemma 2.4, we have $\lim_{t\to+\infty}\Phi_q(t)\int_\theta^{+\infty}\Phi_q(s)h(s)ds=0$ and taking in account (iii) in lemma 2.4, and (i) in Lemma 2.6 and $\lim_{t\to+\infty} h(t)=0$, we obtain by means of the L'Hopital's rule:

$$\begin{aligned} \lim_{t\to+\infty}\Psi_q(t)\int_t^{+\infty}\Phi_q(s)h(s)ds &= \lim_{t\to+\infty}\frac{\int_\theta^t\Phi_q(s)h(s)ds}{\left(\Psi_q(t)\right)^{-1}} \\ &= \lim_{t\to+\infty}-\left(\frac{\Psi_q(t)}{\Psi'_q(t)}\right)\Phi_q(t)\Psi_q(t)h(t)=0. \end{aligned}$$

For the limit of $\Phi_q(t)\int_\theta^t \Psi_q(s)h(s)ds$, if $\int_\theta^{+\infty}\Psi_q(s)h(s)ds < \infty$ then (iv) in lemma 2.4 gives
$\lim_{t\to+\infty}\Phi_q(t)\int_\theta^t \Psi_q(s)h(s)ds = 0$ and if $\int_\theta^{+\infty}\Psi_q(s)h(s)ds = \infty$, then taking in consideration (d) in lemma 2.5 and (i) in Lemma 2.6 and $\lim_{t\to+\infty} h(t) = 0$, we obtain again by means of the L'Hopital's rule:

$$\begin{aligned}\lim_{t\to+\infty}\Phi_q(t)\int_\theta^t \Psi_q(s)h(s)ds &= \lim_{t\to+\infty}\frac{\int_\theta^t \Psi_q(s)h(s)ds}{\left(\Phi_q(t)\right)^{-1}}\\ &= \lim_{t\to+\infty}\left(\frac{\Phi_q(t)}{-\Phi_q'(t)}\right)\left(\Phi_q(t)\Psi_q(t)\right)h(t) = 0.\end{aligned}$$

Uniqueness of u is due to the fact that 0 is the unique solution of BVP (2.13) within $h = 0$. Thus, we have proved that $u(t) = \int_0^{+\infty} G_q(\theta,t,s)h(s)ds$ is the unique solution of BVP (2.13).

Now, we prove that T_θ is a completely continuous operator, let Ω be a subset of W bounded by a constant r.

- Let $\psi_r \in \Gamma^+$ such that $|F(t,x)| \le r\psi_r(t) + |F(t,0)| = \widetilde{\psi}_r(t)$ for all $t \ge 0$ and $x \in [-r,r]$. The following estimates hold for all $u \in \Omega$

$$|T_\theta u(t)| \le r\int_0^{+\infty} G_q(\theta,t,s)\widetilde{\psi}_r(s)ds \le r\,\|U_\theta\| \text{ for all } u \in \Omega.$$

- For any $T > 0$, and $t_1, t_2 \in [0,T]$, we have

$$|T_\theta u(t_2) - T_\theta u(t_1)| \le r\int_0^{+\infty}\left|G_q(\theta,t_2,s) - G_q(\theta,t_1,s)\right|\widetilde{\psi}_r(s)ds, \text{ for all } u \in \Omega \text{ and } t_1,t_2 \ge 0$$

- Since $\lim_{t\to+\infty} T_\theta u(t) = 0$, there exist $T > 0$, for all $t \ge T$, we have

$$|T_\theta u(t)| \le rU_\theta(t)$$

  hold for all $u \in \Omega$, where $U_\theta(t) = \int_0^{+\infty} G_q(\theta,t,s)\widetilde{\psi}_r(s)ds$ satisfies $\lim_{t\to+\infty} U_\theta(t) = 0$.

Together with the Corduneanu criterion of compactness (Lemma 4.1 in [48]) they lead to the compactness of the operator T_θ. The proof is complete. ■

2.3.2 Comparison results

The following lemma will play an important role in the proof of Theorem 2.2.

Lemma 2.8. *Let $(q, m) \in Q \times \Gamma^+$ be such that $\mu_k(q, m) = 1$ for some integer $k \geq 1$. Then there exists $\varepsilon_0 > 0$ such that for all $p \in \Gamma^+$ with $\|p - m\| \leq \varepsilon_0$, $\mu_l(q, p) = 1$ implies $l = k$.*

Proof .

Let $\epsilon_0 > 0$ be such that $\epsilon_0 < \min(\mu_{k+1}(q, m) - \mu_k(q, m), \mu_k(q, m) - \mu_{k-1}(q, m))$, because of Assertion 2 in Theorem 2.1, there exists $\varepsilon_0 > 0$ such that for all $p \in \Gamma^+$, $\|p - m\| \leq \varepsilon_0$ implies

$$\mu_{k-1}(q, m) - \epsilon_0 \leq \mu_{k-1}(q, p) \leq \mu_{k-1}(q, m) + \epsilon_0 \tag{2.14}$$

and

$$\mu_{k+1}(q, m) - \epsilon_0 \leq \mu_{k+1}(q, p) \leq \mu_{k+1}(q, m) + \epsilon_0. \tag{2.15}$$

Let $p \in \Gamma^+$ with $\|p - m\| \leq \varepsilon_0$ and suppose that $\mu_l(q, p) = 1$ for some integer $l \geq 1$. If $l < k$, we have then from (2.14) the contradiction

$$1 = \mu_l(q, p) \leq \mu_{k-1}(q, p) \leq \mu_{k-1}(q, m) + \epsilon_0 < \mu_k(q, m)$$

and if $l > k$, we have then from (2.15) the contradiction

$$1 = \mu_l(q, p) \geq \mu_{k+1}(q, p) \geq \mu_{k+1}(q, m) - \epsilon_0 > \mu_k(q, m) = 1.$$

This shows that $l = k$ and the lemma is proved. ■

We will use extensively the following lemma:

Lemma 2.9 ([11])**.** *Let j and k be two integers such that $j \geq k \geq 2$ and let $(\xi_l)_{l=0}^{l=k}$, $(\eta_l)_{l=0}^{l=j}$ be two families of real numbers such that*

$$\xi_0 = \xi < \xi_1 < \xi_2 < \cdots < \xi_{k-1} < \xi_k = \eta,$$
$$\eta_0 = \xi < \eta_1 < \eta_2 < \cdots < \eta_{j-1} < \eta_j = \eta.$$

If $\xi_1 < \eta_1$, then there exist two integers m and n having the same parity, $1 \leq m \leq k - 1$ and $1 \leq n \leq j - 1$ such that

$$\xi_m < \eta_n \leq \eta_{n+1} \leq \xi_{m+1}.$$

We end this section with the following lemma which is an adapted version of the Sturm's comparison result.

Lemma 2.10. *Let for $i = 1, 2$, $m_i \in \Gamma$ and $w_i \in C^2(\mathbb{R}^+)$ satisfying*

$$-w_i''(t) + q(t)w_i(t) = m_i(t)w_i(t), \; t \in (x_1, x_2)$$

and suppose that w_2 does not vanish identically and $m_1(t) > m_2(t)$ a.e. $t > 0$. If either

1. $x_2 < +\infty$ *and* $w_2(x_1) = w_2(x_2) = 0$*, or*

2. $x_2 = +\infty$ *and* $w_2(x_1) = \lim_{t\to+\infty} w_i(t) = 0$ *for* $i = 1, 2$

then there exists $\tau \in (x_1, x_2)$ *such that* $w_1(\tau) = 0$.

Proof .

1) By the contrary suppose that $w_1 > 0$ in (x_1, x_2) and without loss of generality assume that $w_2 > 0$ in (x_1, x_2), then we have the contradiction:

$$0 \geq w_1(x_2) w_2'(x_2) - w_1(x_1) w_2'(x_1) =$$
$$\int_{x_1}^{x_2} w_2(-w_1'' + qw_1) - w_1(-w_2'' + qw_2) =$$
$$\int_{x_1}^{x_2} (m_1 - m_2) w_1 w_2 > 0.$$

2) By the contrary suppose that $w_1 > 0$ in $(x_1, +\infty)$ and without loss of generality assume that $w_2 > 0$ in $(x_1, +\infty)$, because that $w_i''(t) = (q(t) - m_i(t)) w_i(t)$ and $q(t) - m_i(t) > 0$ for t large, we have that $w_i''(t) > 0$ for t large and $\lim_{t\to+\infty} w_i'(t) = 0$. Therefore, we have for t large

$$(w_1(t) w_2'(t) - w_1(t) w_2'(t)) - w_1(x_1) w_2'(x_1) =$$
$$\int_{x_1}^{t} w_2(-w_1'' + qw_1) - w_1(-w_2'' + qw_2) =$$
$$\int_{x_1}^{t} (m_1 - m_2) w_1 w_2 > 0.$$

Letting $t \to +\infty$, we obtain the contradiction

$$0 \geq -w_1(x_1) w_2'(x_1) = \int_{x_1}^{+\infty} (m_1 - m_2) w_1 w_2 > 0.$$

The proof is complete. ■

2.3.3 On the linear eigenvalue problem

We will present in this subsection two lemmas related to linear eigenvalue problems and needed for the proof of Theorem 2.1. The first one is obtained from Theorem 4.3.2 and Theorem 4.4.1 in [64].

Lemma 2.11. *For all pairs* $(q, m) \in Q \times \Gamma^+$ *and all positive real number* θ*, the EVP*

$$\begin{cases} -u''(t) + q(t)u(t) = \mu m(t)u(t), \ t \in (0, \theta), \\ u(0) = u(\theta) = 0, \end{cases} \tag{2.16}$$

admits an unbounded increasing sequence of simple eigenvalues $\left(\mu_k^-(\theta, q, m)\right)_{k\geq 1}$ *such that:*

1. *if ϕ is an eigenfunction associated with $\mu_k^-(\theta, q, m)$ then ϕ admits $(k-1)$ zeros in $(0, \theta)$ and all are simple.*

2. *Moreover, for (q, m) fixed in $Q \times \Gamma^+$, the function $\theta \to \mu_k^-(\theta) := \mu_k^-(\theta, q, m)$ is continuous and decreasing. We have also $\lim_{\theta\to 0} \mu_k^-(\theta) = +\infty$.*

The next lemma concerns the existence of the positive eigenvalue on the unbounded interval $(\theta, +\infty)$.

Lemma 2.12. *For all pairs $(q, m) \in Q \times \Gamma^+$ and all positive real numbers θ, the EVP*

$$\begin{cases} -u''(t) + q(t)u(t) = \mu m(t)u(t) \quad t > \theta, \\ u(\theta) = 0, \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{2.17}$$

admits a unique positive eigenvalue $\mu_1^+(\theta, q,, m)$. Moreover, for (q, m) fixed in $Q \times \Gamma^+$, the function $\theta \to \mu_1^+(\theta) := \mu_1^+(\theta, q, m)$ is continuous and increasing having $\lim_{\theta\to+\infty} \mu_1^+(\theta) = +\infty$.

Proof .
Let for (q, m) fixed in $Q \times \Gamma^+$, $L_\theta : E \to E$ be the linear compact operator defined by

$$L_\theta u(t) = \int_0^{+\infty} G_q(\theta, t, s)m(s)u(s)ds$$

where the function G_q is that introduced by (2.10), and let $u_\theta \in K$ be the function defined by

$$u_\theta(t) = \begin{cases} 0 & \text{if } t \notin [2\theta, 3\theta], \\ (t-2\theta)(3\theta - t) & \text{if } t \in [2\theta, 3\theta]. \end{cases}$$

We have then $Lu_\theta(t) \geq 0 = u_\theta(t)$ for $t \in [0, 2\theta] \cup [3\theta, +\infty)$ and $Lu_\theta(t), u_\theta(t) > 0$ for $t \in (2\theta, 3\theta)$. This shows that $L_\theta u \geq c_\theta u_\theta$ where $c_\theta = \inf \{Lu_\theta(t)/u_\theta(t) : t \in (2\theta, 3\theta)\} > 0$ and $r(L_\theta) > 0$. Since Lemma 2.7 guarantees that L_θ is compact, we have from the Krein-Rutman theorem, that $r(L_\theta)$ is a positive eigenvalue of L_θ having a eigenvector $\phi_\theta \in K$. By means of Lemma (2.7), we conclude that $\mu_1^+(\theta, q, m) = 1/r(L_\theta)$ is a positive eigenvalue of EVP (2.17).

Now, for λ a positive eigenvalue of EVP (2.17) having an eigenfunction ψ, we have

$$0 = \int_\theta^{+\infty} (-\phi_\theta'' + k^2\phi_\theta)\psi - (-\psi'' + k^2\psi)\phi_\theta = (\mu_1^+(\theta, q, m) - \lambda) \int_\theta^{\xi} m\phi_\theta\psi,$$

leading to $\lambda = \mu_1^+(\theta, q, m)$. Thus, we have proved uniqueness of the positive eigenvalue and that the function $\theta \to \mu_1^+(\theta, q, m)$ is well defined.

Let θ_1, θ_2 be positive real numbers such that $\theta_1 < \theta_2$ and set for $i = 1, 2$, $\mu_i = \mu_1^+(\theta_i, m)$ with the corresponding eigenfunction ψ_i. We have by simple calculations

$$\begin{aligned} 0 > -\psi_2'(\theta_2)\,\psi_1(\theta_2) &= \int_{\theta_2}^{+\infty}((-\psi_1'' + q\psi_1)\psi_2 - (-\psi_2'' + q\psi_2)\psi_1) \\ &= (\mu_1 - \mu_2)\int_{\theta_2}^{+\infty} m\psi_1\psi_2, \end{aligned}$$

leading to $\mu_1 < \mu_2$ and proving that $\theta \to \mu_1(\theta, q, m)$ is an increasing function. The continuity of the function $\mu_1(\cdot, q, m)$ follows from that of the Green's function G and Lemma 2.13 in [10].

Let $[\gamma, \delta]$ be a compact interval and let $\theta_1, \theta_2 \in [\gamma, \delta]$ be such that $\theta_1 < \theta_2$: We have for all $u \in W$ with $\| u \| = 1$

$$\begin{aligned} \left|L_{\theta_2}u(t) - L_{\theta_1}u(t)\right| &= \left|\int_{\theta_2}^{+\infty} G_q(\theta_2, t, s)\, muds - \int_{\theta_1}^{+\infty} G_q(\theta_1, t, s)\, muds\right| \\ &= \begin{cases} 0 & \text{if } t \le \theta_1 < \theta_2, \\ \left|\int_{\theta_1}^{+\infty} G_q(\theta_1, t, s)\, muds\right| & \text{if } \theta_1 < t \le \theta_2, \\ \left|\int_{\theta_2}^{+\infty} G_q(\theta_2, t, s)\, muds - \int_{\theta_1}^{+\infty} G_q(\theta_1, t, s)\, muds\right| & \text{if } \theta_1 < \theta_2 < t. \end{cases} \end{aligned}$$

Set

$$\chi = \|m\| \left[\left(\int_{\gamma}^{+\infty} \phi_q ds\right) \frac{\phi_q(\gamma)}{\phi_q^2(\delta)} + \overline{G}_{q,\infty} + \Phi_q(\gamma)\,\Psi_q(\delta)\right]$$

then we have for $\theta_2 \ge t > \theta_1$

$$\begin{aligned} &\left|\int_{\theta_1}^{+\infty} G_q(\theta_1, t, s)\, muds\right| \le \|m\| \int_{\theta_1}^{+\infty} G_q(\theta_1, t, s)\, ds \\ &= \|m\| \left(\int_{\theta_1}^{+\infty} G_q(t, s)\, ds - \tfrac{\psi_q(\theta_1)}{\phi_q(\theta_1)}\phi_q(t) \int_{\theta_1}^{+\infty} \phi_q ds\right) \\ &= \|m\| \,(\int_{\theta_1}^{t} G_q(t, s)\, ds + \int_{t}^{+\infty} G_q(t, s)\, ds \\ &- \tfrac{\psi_q(\theta_1)}{\phi_q(\theta_1)}\phi_q(t) \int_{\theta_1}^{t} \phi_q ds - \tfrac{\psi_q(\theta_1)}{\phi_q(\theta_1)}\phi_q(t) \int_{t}^{+\infty} \phi_q ds) \\ &= \|m\| \,(\int_{\theta_1}^{t} G_q(t, s)\, ds - \tfrac{\psi_q(\theta_1)}{\phi_q(\theta_1)}\phi_q(t) \int_{\theta_1}^{t} \phi_q ds) + \psi_q(t) \int_{t}^{+\infty} \phi_q ds - \tfrac{\psi_q(\theta_1)}{\phi_q(\theta_1)}\phi_q(t) \int_{t}^{+\infty} \Psi_q ds) \\ &= \|m\| \,(\int_{\theta_1}^{t} G_q(t, s)\, ds - \tfrac{\psi_q(\theta_1)}{\phi_q(\theta_1)}\phi_q(t) \int_{\theta_1}^{t} \phi_q ds) + \int_{t}^{+\infty} \phi_q ds \left(\tfrac{\psi_q(t)}{\phi_q(t)} - \tfrac{\psi_q(\theta_1)}{\phi_q(\theta_1)}\right) \phi_q(t)) \\ &\le \|m\| \left[\left(\int_{\gamma}^{+\infty} \phi_q ds\right) \tfrac{\phi_q(\gamma)}{\phi_q^2(\delta)} + \overline{G}_{q,\infty} + \Phi_q(\gamma)\,\Psi_q(\delta)\right] |\theta_2 - \theta_1| \le \chi\, |\theta_2 - \theta_1| \end{aligned}$$

and for $\theta_1 < \theta_2 < t$,

$$\begin{aligned}
&\left|\int_{\theta_2}^{+\infty} G_q(\theta_2,t,s)\,muds - \int_{\theta_1}^{+\infty} G_q(\theta_1,t,s)\,muds\right| \leq \\
&\left|\int_{\theta_2}^{+\infty} \left(G_q(\theta_2,t,s) - G_q(\theta_1,t,s)\right) muds\right| + \left|\int_{\theta_1}^{\theta_2} G_q(\theta_1,t,s)\,muds\right| \\
&= \left|\left(\int_{\theta_2}^{+\infty} \phi_q muds\right)\left(\frac{\psi_q(\theta_1)}{\phi_q(\theta_1)} - \frac{\psi_q(\theta_2)}{\phi_q(\theta_2)}\right)\phi_q(t)\right| + \left|\int_{\theta_1}^{\theta_2} G_q(\theta_1,t,s)\,muds\right| \\
&\leq \|m\|\left[\left(\int_{\gamma}^{+\infty} \phi_q ds\right)\frac{\phi_q(\gamma)}{\phi_q^2(\delta)} + \overline{G}_{q,\infty}\right]|\theta_2-\theta_1| \leq \chi\,|\theta_2-\theta_1|.
\end{aligned}$$

The above estimates show that the mapping $\theta \to L_\theta$ is locally Lipschitzian and so, it is continuous. Let (θ_n) be a sequence converging to θ_* and let θ_-, θ_+ be such that $(\theta_n) \subset [\theta_-, \theta_+]$. Therefore we have for all $n \geq 1$,

$$0 < \mu_1(\theta_+, q, m) \leq \mu_1(\theta_n, q, m) \leq \mu_1(\theta_-, q, m)$$

and the sequence $(\mu_1(\theta_n, q, m))$ converges (up to a subsequence) to some $\mu_* > 0$. We conclude by Lemma 2.13 in [10] and by uniqueness of that $\mu_* = \mu_1(\theta_*, q, m)$. Thus, the continuity of the mapping $\mu_1(\cdot, q, m)$ is proved.

It remains to prove that $\lim_{\theta\to+\infty} \mu_1^+(\theta, m) = \lim_{\theta\to+\infty}(1/r(L_\theta)) = +\infty$. We have for all $u \in W$ with $\|u\| = 1$

$$\begin{aligned}
|L_\theta u(t)| &\leq \int_\theta^{+\infty} G_q(\theta,t,s)\,m(s)ds \\
&\leq \int_\theta^{+\infty} G_q(t,s)\,m(s)ds + \frac{\Psi_q(\theta)}{\Phi_q(\theta)}\int_\theta^{+\infty} \Phi_q(t)\Phi_q(s)m(s)ds \\
&\leq \int_\theta^{+\infty} G_q(t,s)\,m(s)ds + \Psi_q(\theta)\int_\theta^{+\infty} \Phi_q(s)m(s)ds.
\end{aligned}$$

As in the proof of Lemma 2.7, we have $\lim_{\theta\to+\infty}\Psi_q(\theta)\int_\theta^{+\infty}\Phi_q(s)m(s)ds = 0$ and since $\lim_{t\to+\infty} m(t) = 0$, for $\epsilon > 0$, there exists $\theta_\epsilon > 0$ such that $m(s) \leq \epsilon$ for all $s \geq \theta_\epsilon$. Hence, we have for all $\theta \geq \theta_\epsilon$

$$\int_\theta^{+\infty} G_q(t,s)\,m(s)ds \leq \tilde{G}\epsilon \text{ for all } t \geq 0$$

proving that $\lim_{\theta\to+\infty}\int_\theta^{+\infty} G_q(t,s)\,m(s)ds = 0$ uniformly on $\mathbb{R}^+$. Therefore, we have proved that $\lim_{\theta\to+\infty} r(L_\theta) = \lim_{\theta\to+\infty}\|L_\theta\| = 0$, ending the proof. ■

2.4 Proof of Theorem 2.1

Step 1. Fix (q, m) in $Q \times \Gamma^+$ and let $k \geq 1$ be an integer. Obviously, if $k = 1$ then $\mu_1(q, m) = \mu_1^+(0, q, m)$ is a positive eigenvalue of the EVP (2.3) where $\mu_1^+(0, q, m)$

is that of Lemma 2.12. If $k \geq 2$, then we deduce from Lemmas 2.11 and 2.12 existence of a unique positive real number θ_k^* such that $\mu_1^+(\theta_k^*, q, m) = \mu_{k-1}^-(\theta_k^*, q, m)$. Therefore, if ϕ_{1,θ_k^*} and ψ_{k-1,θ_k^*} are respectively the eigenfunctions associated with $\mu_1^+(\theta_k^*, q, m)$ and $\mu_{k-1}^-(\theta_k^*, q, m)$, then the function

$$\phi_k(t) = \begin{cases} \psi_{k-1,\theta_k^*}(t), & \text{in } \left[0, \theta_k^*\right], \\ \left(\psi'_{k-1,\theta_k^*}\left(\theta_k^*\right) / \phi'_{1,\theta_k^*}\left(\theta_k^*\right)\right) \phi_{1,\theta_k^*}(t), & \text{in } \left[\theta_k^*, +\infty\right), \end{cases}$$

belongs to S_k and is the eigenfunction associated with the eigenvalue $\mu_k(q, m) = \mu_1^+(\theta_k^*, q, m) = \mu_{k-1}^-(\theta_k^*, q, m)$ of the EVP (2.3).

Now, let us prove that $\mu_k(q, m)$ is the unique eigenvalue of the EVP (2.3), having an eigenfunction in S_k. To this aim, let for $i = 1, 2$, $\phi_i \in S_k^+$ be an eigenfunction associated with the eigenvalue μ_i and let $\left(z_j^i\right)_{j=0}^{j=k}$ be the sequence of zeros of ϕ_i. Without loss of generality, suppose that $z_1^1 \leq z_1^2$, we deduce then from Lemma 2.9 existence of two integers $0 \leq n_1, m_1 \leq k-1$ having the same parity such that $z_{n_1}^1 \leq z_{m_1}^2 < z_{m_1+1}^2 \leq z_{n_1+1}^1$. Notice that the fact n_1, m_1 have the same parity means that the functions ϕ_1 and ϕ_2 have the same sign on the interval $\left(z_{m_1}^2, z_{m_1+1}^2\right)$ and after simple calculations, yields

$$0 \leq \int_0^{z_1^1} \phi_2(-\phi_1'' + q\phi_1) - \phi_1(-\phi_2'' + q\phi_2) = (\mu_1 - \mu_2) \int_0^{z_1^1} m\phi_1\phi_2 \text{ and}$$

$$0 \geq \int_{z_{m_1}^2}^{z_{m_1+1}^2} \phi_2(-\phi_1'' + q\phi_1) - \phi_1(-\phi_2'' + q\phi_2) = (\mu_1 - \mu_2) \int_{z_{m_1}^2}^{z_{m_1+1}^2} m\phi_1\phi_2.$$

proving that $\mu_1 = \mu_2$ and $\mu_k(q, m)$ is the unique eigenvalue of the EVP (2.3), having an eigenfunction in S_k.

At this stage, we need to prove that for all integers $k \geq 1$, $\mu_k(q, m)$ has the geometric multiplicity equal to 1. Indeed, if ϕ, ψ are two eigenfunctions associated with the eigenvalue μ and $W = W(\phi, \psi) = \phi\psi' - \phi'\psi$ is their corresponding Wronksian, then we have

$$\begin{aligned} W' &= (\phi\psi' - \phi'\psi)' = \phi\psi'' - \phi''\psi \\ &= \phi(q - \mu m)\psi - (q - \mu m)\phi\psi = 0. \end{aligned}$$

This together with $W(0) = 0$, leads to $W = 0$ and $\psi = c\phi$ for some $c \in \mathbb{R}$ and the geometric simplicity is proved.

Notice that geometric simplicity leads to $\mu_i(q, m) \neq \mu_j(q, m)$ for $i \neq j$ and the sequence $(\mu_k(q, m))$ is infinite. Furtheremore, since for all integers $k \geq 1$, $\mu_k(q, m)$

is a characteristic value of the compact operator $L_m : W \to W$ given by $L_m u(t) = \int_0^{+\infty} G_q(t,s)m(s)u(s)ds$ where G_q is defined in (2.12), we have $\lim_{k\to\infty} \mu_k(q,m) = +\infty$.

In order to prove monotonicity of the sequence $(\mu_k(q,m))$, let for $i = 1,2$, $\phi_i \in S_{k_i}^+$ be an eigenfunction associated with the eigenvalue μ_i of the EVP (2.3), having a sequence of zeros $\left(z_j^i\right)_{j=0}^{j=k_i}$. Suppose that $k_2 > k_1$, we distinguish then the following cases:

Case 1. $z_1^2 \le z_1^1$, in this case we have

$$0 \ge \int_0^{z_1^2} \phi_2(-\phi_1'' + q\phi_1) - \phi_1(-\phi_2'' + q\phi_2) = (\mu_1 - \mu_2) \int_0^{z_1^2} m\phi_1\phi_2,$$

leading to $\mu_1 \le \mu_2$.

Case 2. $z_1^1 \le z_1^2$, in this case, we deduce from Lemma 2.9 existence of two integers n_1, m_1, with $n_1 \le k_1 - 1$, $m_1 \le k_2 - 1$ and such that $z_{n_1}^1 \le z_{m_1}^2 < z_{m_1+1}^2 \le z_{n_1+1}^1$. After simple computations, yields

$$0 \ge \int_{z_{m_1}^2}^{z_{m_1+1}^2} \phi_2(-\phi_1'' + q\phi_1) - \phi_1(-\phi_2'' + q\phi_2) = (\mu_1 - \mu_2) \int_{z_{m_1}^2}^{z_{m_1+1}^2} m\phi_1\phi_2,$$

leading to $\mu_1 \le \mu_2$. This together with $\mu_i(q,m) \ne \mu_j(q,m)$ for $i \ne j$ show that $\mu_1 < \mu_2$.

We end this step by proving that aside the sequence $(\mu_k(q,m))$, the EVP (2.3) has no other eigenvalues. Let μ be an eigenvalue of the EVP (2.3) having an eigenfunction ϕ and by the contrary, suppose that $\mu \ne \mu_k(q,m)$ for all integers $k \ge 1$. Notice that if for some $z_0 \ge 0$, $\phi(z_0) = \phi'(z_0) = 0$, the classical existence and uniqueness result for ODEs leads to the contradiction $\phi = 0$. This shows that all zeros of ϕ are simple and isolated and necessarily, ϕ admits an infinite and increasing sequence of zeros, say (z_n). Therefore, we have $\lim z_n = +\infty$; Indeed, if $\lim z_n = \hat{z} < +\infty$ then we obtain

$$u(\hat{z}) = \lim u(z_n) = 0 \text{ and } u'(\hat{z}) = \lim \frac{u(z_n) - u(\hat{z})}{z_n - \hat{z}} = 0,$$

leading to the contradiction $\phi = 0$.

Let for the integer $k \ge 1$, $\phi_k \in S_k$ be the eigenfunction associated with the eigenvalue $\mu_k(q,m)$ and let $(x_j)_{j=1}^{j=k}$ be the sequence of zeros of ϕ_k. We deduce from Lemma 2.9, existence of two integers l, m having the same parity such that $0 \le l \le k-1$ and

$$z_m \le x_l < x_{l+1} \le z_{m+1}.$$

Hence, we have

$$0 \le \int_{x_l}^{x_{l+1}} -\phi_1(\phi_2'' + q\phi_2) - \phi_2(-\phi_1'' + q\phi_1) = (\mu - \mu_k(q,m)) \int_{x_l}^{x_{l+1}} m\phi_1\phi_2$$

leading to $\mu \geq \mu_k(q,,m)$ for all integers $k \geq 1$ then to the contradiction $\mu = \lim_{k\to\infty} \mu_k(q,m) = +\infty$.

Step 2. Monotonicity: Fix q in Q and let m_1, m_2 be two functions in Γ^+ and suppose that $m_1 \leq m_2$ and $m_1 < m_2$ in a subset of positive measure. Set for $i = 1, 2$, $\mu_i = \mu_k(q, m_i)$ and let $\phi_i \in S_k^+$ be the eigenfunction associated with μ_i having a sequence of zeros $\left(z_j^i\right)_{j=0}^{j=k}$. By the contrary suppose that $\mu_1 < \mu_2$, we claim that there exists j_0 such that $z_{j_0}^1 \neq z_{j_0}^2$. Indeed, if $\phi_1\left(z_j^2\right) = 0$ for all $j \in \{1, ..., k-1\}$ then for $j_1 \in \{1, ..., k-1\}$ being such that $meas\left(\{m_2 > m_1\} \cap \left(z_{j_1}^2, z_{j_1+1}^2\right)\right) > 0$ we have since $\phi_1\phi_2 > 0$ in $\left(z_{j_1}^2, z_{j_1+1}^2\right)$, the contradiction

$$\begin{aligned} 0 &= \int_{z_{j_1}^2}^{z_{j_1+1}^2} -\phi_2\phi_1'' + \phi_1\phi_2'' = \int_{z_{j_1}^2}^{z_{j_1+1}^2} (\mu_1 m_1 - \mu_2 m_2)\phi_1\phi_2 \\ &= \int_{z_{j_1}^2}^{z_{j_1+1}^2} (\mu_1 - \mu_2) m_1 \phi_1\phi_2 + \int_{z_{j_1}^2}^{z_{j_1+1}^2} \mu_2\left(m_1 - m_2\right)\phi_1\phi_2 < 0. \end{aligned}$$

Now, let $k_1 = \max\left\{l \leq k : z_j^1 = z_j^2 \text{ for all } j \leq l\right\}$ and $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ be the families defined by $\xi_j = z_{k_1+j}^1$ and $\eta_j = z_{k_1+j}^2$. We distinguish then two cases.

i) $\xi_1 = z_{k_1+1}^1 < \eta_1 = z_{k_1+1}^2$: In this case we have the contradiction

$$\begin{aligned} 0 &< -\phi_2\left(\xi_1\right)\phi_1'\left(\xi_1\right) = \int_{\xi_0}^{\xi_1} -\phi_2\phi_1'' + \phi_1\phi_2'' \\ &= \int_{\xi_0}^{\xi_1} (\mu_1 m_1 - \mu_2 m_2)\phi_1\phi_2 \\ &= \int_{\xi_0}^{\xi_1} (\mu_1 - \mu_2) m_1\phi_1\phi_2 + \int_{\xi_0}^{\xi_1} \mu_2\left(m_1 - m_2\right)\phi_1\phi_2 \leq 0. \end{aligned}$$

ii) $\xi_1 = z_{k_1+1}^1 > \eta_1 = z_{k_1+1}^2$: In this case Lemma 2.9 guarantees existence of two integers m, n having the same parity such that

$$\eta_m = z_{k_1+m}^2 < \xi_n = z_{k_1+n}^1 < \xi_{n+1} = z_{k_1+n+1}^1 \leq \eta_{m+1} = z_{k_1+m+1}^2.$$

As above, we have the contradiction

$$\begin{aligned} 0 &< \int_{\xi_n}^{\xi_{n+1}} -\phi_2\phi_1'' + \phi_1\phi_2'' = \int_{\xi_n}^{\xi_{n+1}} (\mu_1 m_1 - \mu_2 m_2)\phi_1\phi_2 \\ &= \int_{\xi_n}^{\xi_{n+1}} (\mu_1 - \mu_2) m_1\phi_1\phi_2 + \int_{\xi_n}^{\xi_{n+1}} \mu_2\left(m_1 - m_2\right)\phi_1\phi_2 \leq 0. \end{aligned}$$

The monotonicity is proved.

Step 3. Continuity: Fix q in Q, m in Γ^+ and let $(m_n) \subset \Gamma^+$ such that $\lim m_n = m$ uniformly on $\mathbb{R}^+$. Let $L_n, L \in \mathcal{L}(W)$ be defined by

$$L_n u(t) = \int_0^{+\infty} G_q(t,s) m_n(s) u(s) ds \text{ and } Lu(t) = \int_0^{+\infty} G_q(t,s) m(s) u(s) ds.$$

Notice that for all integers $l, n \geq 1$, $\mu_l^n = \mu_l(q, m_n)$ is a characteristic value of L_n, $\mu_l = \mu_l(q,m)$ is a characteristic value of L and because of Assertion iii) in Lemma 2.6, $L_n \to L$ in operator norm.

First, fix $k \geq 1$ and let us prove that if (μ_k^n) admits a subsequence (δ_n) converging to $\delta > 0$, then $\delta = \mu_k$. Indeed, let $\phi_n \in S_k^+$ be the normalized eigenfunction associated with δ_n and let $\psi_n = L\phi_n$. Since L is compact and the sequence (ϕ_n) is bounded, we have up to a subsequence $\psi_n \to \psi$. Thus, we obtain the following estimates,

$$\begin{aligned} \|(\phi_n/\delta_n) - \psi\| &= \|L_n\phi_n - \psi\| \\ &\leq \|L_n\phi_n - L\phi_n\| + \|L\phi_n - \psi\| \\ &\leq \|L_{n_k} - L\| + \|\psi_n - \psi\| \end{aligned}$$

leading to

$$\lim(\phi_n/\delta_n) = \psi \text{ and } \|\psi\| = \lim \|\phi_n\| /\delta_n = 1/\delta > 0.$$

Also, we have

$$\|L_n\phi_n - \delta L\psi\| = \|\delta_n L_{n_k}((\phi_n/\delta_n)) - \delta L\psi\|$$
$$\leq \|\delta_n L_n((\phi_n/\delta_n)) - \delta L_n((\phi_n/\delta_n))\| + \|\delta L_n((\phi_n/\delta_n)) - \delta L((\phi_n/\delta_n))\| + \|\delta L((\phi_n/\delta_n)) - \delta L\psi\|$$
$$\leq |\delta_n - \delta|\, \delta_n \|L_n\| + \tfrac{\delta_n}{\delta}\|L_n - L\| + \tfrac{1}{\delta}\|L\|\,\|(\phi_n/\delta_n) - \psi\|$$

leading to

$$\lim L_n\phi_n = \delta L\psi.$$

Thus, letting $n \to \infty$ in equation $L_n\phi_n = (\phi_n/\delta_n)$ we obtain $L\psi = \psi/\delta$ that is $1/\delta$ is an eigenvalue of L or $\delta = \mu_l(q,m)$ for some integer $l \geq 1$. Then, because of $\lim \delta_n m_n = \delta m$ uniformly on $\mathbb{R}^+$, it follows from Lemma 2.8 that $\delta = \mu_k(q,m)$.

Then, fix $T > 0$ and set for all integers $l, n \geq 1$, $\mu_l^{n,T} = \mu_l^-(T,q,m_n)$ and $\mu_l^T = \mu_l^-(T,q,m)$. We have from Proposition 4.40 in [64] that $\lim_{n\to\infty} \mu_l^{n,T} = \mu_l^T$ for all integers $l \geq 1$ and then there is $c_l > 0$ such that $\mu_l^{n,T} < \mu_l^T + c_l$ for all $n \geq 1$. Fix $k \geq 1$ and denote by $\phi_n \in S_k^+$ the eigenfunction associated with μ_k^n and suppose that ϕ_n admits $(j-1)$ zeros in $(0,T)$. Let $\phi_{n,T}$ be the eigenfunction associated with $\mu_j^{n,T}$ satisfying $\phi'_{n,T}(0) > 0$ and denote by $(x_s)_{s=0}^{s=j}$ the sequence of zeros of $\phi_{n,T}$ and by $(z_s)_{s=0}^{s=j}$ the sequence constituted in zeros of ϕ_n contained in $(0,T)$ and $z_s = T$. We distinguish two cases:

Case 1. $x_1 < z_1$, we have in this case

$$\begin{aligned} 0 > \phi_n(x_1)\,\phi'_{n,T}(x_1) &= \int_0^{x_1} \phi_{n,T}(-\phi''_n + q\phi_n) - \phi_n(-\phi''_{nT} + q\phi_{n,T}) \\ &= \left(\mu_k^n - \mu_j^{n,T}\right) \int_0^{x_1} m_n \phi_{n,T}\phi_n \end{aligned}$$

leading to

$$\mu_k^n \leq \mu_j^{n,T} \leq \max_{1\leq l\leq k}(\mu_l^{n,T}) \leq \max_{1\leq l\leq k}(\mu_l^T + c_l) \leq \mu_k^T + \max_{1\leq l\leq k}(c_l).$$

Case 2. $z_1 \leq x_1$, in this case we deduce from Lemma 4.6 existence of two integers r_T, r having the same parity and such that $z_r \leq x_{r_T} < x_{r_t+1} \leq z_{r+1}$ and $\phi_{n,T}\phi_n > 0$ in (x_{r_T}, x_{r_t+1}). After simple computations yields

$$\begin{aligned} 0 \geq \phi_n(x_{r+1})\,\phi'_{n,T}(x_{r+1}) - \phi_n(x_r)\,\phi'_{n,T}(x_r) &= \int_{x_r}^{x_{r+1}} \phi_{n,T}(-\phi''_n + q\phi_n) - \phi_n(-\phi''_{nT} + q\phi_{n,T}) \\ &= \left(\mu_k^n - \mu_j^{n,T}\right) \int_{x_r}^{x_{r+1}} m_n \phi_{n,T}\phi_n \end{aligned}$$

and we have again

$$\mu_k^n \leq \mu_j^{n,T} \leq \max_{1\leq l\leq k}(\mu_l^{n,T}) \leq \max_{1\leq l\leq k}(\mu_l^T + c_l) \leq \mu_k^T + \max_{1\leq l\leq k}(c_l).$$

At this stage we have proved that the sequence $\left(\mu_k^n\right)$ is bounded, set then $\mu_k^+ = \limsup \mu_k^n$ and $\mu_k^- = \liminf \mu_k^n$. Since $\lim \|L_n\| = \|L\|$, we have $\|L_n\| \geq \|L\|/2$ for n large enough and $\mu_k^n \geq 1/\|L_n\| \geq \|L\|/2$ for n large enough. Therefore, passing to the limit, we obtain $\mu_k^+ \geq \mu_k^- \geq \|L\|/2 > 0$ and taking in account what is showed at the beginning of this proof, we conclude that $\lim \mu_k^n = \mu_k^+ = \mu_k^- = \mu_k$. The continuity is proved.

2.5 Proof of Theorem 2.2

Consider the BVP

$$\begin{cases} -u'' + \widetilde{q}(t)u = \mu u(f(t,u) + 2\omega(t)), \ t > 0, \\ u(0) = \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{2.18}$$

where μ is a real parameter and $\widetilde{q} = q + 2\omega$.

By a solution to BVP (2.18), we mean a pair $(\mu, u) \in \mathbb{R} \times W_2$ satisfying the differential equation in BVP (2.18). Notice that $u \in W_2$ is a solution to BVP (2.2) if and only if $(1, u)$

is a solution to BVP (2.18). For this reason, we will study the bifurcation diagram of the BVP (2.18) and by means of Rabinowitz global bifurcation theory, we will prove that the set of solutions to BVP (2.18) consists in an infinity of unbounded components, each branching from a point on the line $\mathbb{R} \times \{0\}$ (see Lemma 2.13), joining a point on $\mathbb{R} \times \{\infty\}$ (see Lemma 2.14). Obviously, each component having the starting point and the arrival point oppositely located relatively to 1, carries a solution of BVP (2.2) and Theorem 2.2 will be proved once we compute the number of such components. Thus, Theorem 2.2 is the consequence of the following Lemma 2.13, Lemma 2.14 and Lemma 2.15.

Lemma 2.13. *Assume that Hypotheses (2.4)-(2.6) hold, then from each $\mu_l(\widetilde{q}, m_0)$ bifurcate two unbounded components of nontrivial solutions ζ_l^+ and ζ_l^-, such that $\zeta_l^\nu \subset \mathbb{R} \times S_l^\nu$.*

Proof .
It follows from Lemma 2.7 that solutions of BVP (2.18) are those satisfying the fixed point equation

$$u = \mu L_0 u + \mu T_0(u) \tag{2.19}$$

where $L_0, T_0 : W \to W$ are defined as follows

$$L_0 u(t) = \int_0^{+\infty} G_{\widetilde{q}}(t,s)\widetilde{m}_0(s)u(s)ds,$$
$$T_0 u(t) = \int_0^{+\infty} G_{\widetilde{q}}(t,s)u(s)g_0(s,u(s))ds,$$

and $\widetilde{m}_0 = m_0 + 2\omega$, $g_0(s,u) = f(s,u) - m_0(s)$.

Let us prove now, that all characteristic values of L are of algebraic multiplicity one. To this aim, let $u \in N\left((\mu_k(\widetilde{q},\widetilde{m}_0)L_0 - I)^2\right)$ and set $v = (\mu_k(\widetilde{q},\widetilde{m}_0)L_0 u - u$. Then $v \in N(\mu_k(\widetilde{q},\widetilde{m}_0)L_0 - I) = \mathbb{R}\phi_k$ and $\mu_l(\widetilde{q},\widetilde{m}_0)L_0 u - u = \eta\phi_k$ for some $\eta \in \mathbb{R}$. In another way, v satisfies the BVP

$$\begin{cases} -u'' + q(t)u = \mu_k(\widetilde{q},\widetilde{m}_0)\widetilde{m}_0\,(t)\,u - \eta\mu_k(\widetilde{q},\widetilde{m}_0)\widetilde{m}_0(t)\phi_k, \ t > 0, \\ u(0) = \lim_{t\to+\infty} u(t) = 0. \end{cases}$$

Multiplying the differential equation in the above BVP by ϕ_k and integrating on $(0, +\infty)$ we obtain

$$\eta\mu_k(\widetilde{q},\widetilde{m}_0)\int_0^{+\infty} \widetilde{m}_0\phi_k^2 dt = 0,$$

leading to $\eta = 0$ and $u = \mu_k(\widetilde{q},\widetilde{m}_0)Lu \in \mathbb{R}\phi_k$.

Now, we need to prove that $T_0(u) = \circ(\|u\|)$ near 0. Indeed, let $(u_n) \subset W$ with $\lim \|u_n\| = 0$. It follows from Hypothesis (2.6), that for $\epsilon > 0$ there exists $\delta > 0$ such that for all $u \in [-\delta, \delta]$ and $s \geq 0$, $|g_0(s,u)| \leq \epsilon$. Therefore, for n large enough

$$\frac{|T_0 u_n(t)|}{\|u_n\|} \leq \int_0^{+\infty} G_{\tilde{q}}(t,s)\, |g(s,u_n(s))|\, ds \ \leq \epsilon \tilde{G}$$

proving that $T_0(u) = \circ(\|u\|)$ near 0.

Let l_k be the projection of W on $\mathbb{R}\phi_k$, $\widetilde{W} = \{u \in W : l_k u = 0\}$ and let for $\xi > 0$, $\eta \in (0,1)$, $\nu = \pm$

$$K^{\nu}_{\xi,\eta} = \{(\mu, u) \in \mathbb{R} \times W : |\mu - \mu_k(\widetilde{q}, m_0)| < \xi \text{ and } \nu l_k u > \eta \|u\|\}.$$

Since Lemma 2.7 guarantees that the operators L_0 and T_0 are respectively compact and completely continuous, we have from Theorem 1.40 and Theorem 1.25 in [52], that from $(\mu_k(\widetilde{q}, m_0), 0)$ bifurcate two components ζ_k^+ and ζ_k^- of nontrivial solutions to Equation (2.19) such that there is $\zeta_0 > 0$, $\zeta_k^\nu \cap B(0,\zeta) \subset K^\nu_{\xi,\eta}$ for all $\zeta < \zeta_0$ and if $u = \alpha\phi_k + w \in \zeta_k^\nu$ then $|\mu - \mu_k(\widetilde{q}, m_0)| = \circ(1)$, $w = \circ(|\alpha|)$ for α near 0.

We claim that there is $\zeta > 0$ such that $\zeta_k^\nu \cap B(0,\zeta) \subset \mathbb{R} \times S_k^\nu$; Indeed, let $(\mu_n, u_n)_{n\geq 1} \subset \zeta_k^\nu$ be such that $\lim (\mu_n, u_n) = (\mu_k(\widetilde{q}, m_0), 0)$, we have then $\lim \mu_n f(s, u_n(s)) = \mu_k(\widetilde{q}, m_0) m_0(s)$ and Lemma 4.24 guarantees that there is $n_0 \geq 1$ such that $u_n \in S_k$ for all $n \geq n_0$. Moreover, if $u_n = \alpha_n \phi_k + w_n$ then $\lim \frac{u_n}{\alpha_n} = \phi_k$ uniformly in $[0, +\infty)$ proving that $\nu u_n(t) > 0$ for t in a right neighborhood of 0 and $\nu u_n'(0) > 0$ (otherwise, if $u_n'(0) = 0$ then by Cauchy-Lipshitz theorem, $u_n = 0$).

Also, if $(\mu_*, u_*) \in \zeta_k^\nu$ then for all sequence $(\mu_n, u_n)_{n\geq 1} \subset \zeta_k^\nu$ being such that $\lim (\mu_n, u_n) = (\mu_*, u_*)$, we have from Hypothesis (2.4) that $\lim \mu_n f(s, u_n(s)) = \mu_* f(s, u_*(s))$ uniformly in $\mathbb{R}^+$ and Lemma 2.8 guarantees existence of $n_0 \geq 1$ such that $u_n \in S_k$ for all $n \geq n_0$. Moreover, we have

$$\begin{aligned} \left|u_n'(0) - u_*'(0)\right| \ \leq\ & |\mu_* - \mu_n| \int_0^{+\infty} \left|\frac{\partial G_{\widetilde{q}}}{\partial t}(t,s)\right| u_*(s) f(s, u_*(s)) ds \\ & + \mu_n \int_0^{+\infty} \left|\frac{\partial G_{\widetilde{q}}}{\partial t}(t,s)\right| |u_n(s)|\, |[f(s,u_n(s)) - f(s,u_*(s))]|\, ds \\ & + \mu_n \int_0^{+\infty} \left|\frac{\partial G_{\widetilde{q}}}{\partial t}(t,s)\right| |u_n(s) - u_*(s)|\, |f(s,u_*(s))|\, ds. \end{aligned}$$

Hence, we obtain by means of hypothesis (2.4) and Lebesgue dominated convergence theorem that $\lim u_n'(0) = u_*'(0)$ and for n sufficiently large, $u_n'(0) u_*'(0) > 0$. This shows that $\zeta_k^\nu \subset \mathbb{R} \times S_k^\nu$ and ζ_k^ν is unbounded in $\mathbb{R} \times W$, ending the proof. ■

Lemma 2.14. *Assume that Hypotheses (2.4)-(2.6) hold, then for all $k \geq 1$ and $\nu = \pm$, the component ζ_k^ν rejoins the point $(\mu_k(\widetilde{q}, m_\infty), \infty)$.*

Proof .

First, let us prove that for all $k \geq 1$ and $\nu = \pm$, the projection of ζ_k^ν onto the real axis is bounded. Indeed, since 0 is the unique solution to the BVP

$$\begin{cases} -u'' + \widetilde{q}(t)u = 0, \ t > 0, \\ u(0) = \lim_{t\to+\infty} u(t) = 0, \end{cases}$$

the projection of ζ_k^ν onto the real axis is contained in $(0, +\infty)$, namely, if $(\mu, u) \in \zeta_k^\nu$ then $\mu > 0$. Moreover, if $(\mu, u) \in \zeta_k^\nu$ then we read from the BVP (2.18) that $\mu = \mu_k(\widetilde{q}, f(\cdot, u(\cdot) + 2\omega)$, then taking in consideration Hypothesis (2.4), we obtain from Assertion 4 in Theorem 2.1 that $\mu = \mu_k(\widetilde{q}, f(\cdot, u(\cdot)) + 2\omega) \leq \mu_k(\widetilde{q}, \omega)$.

Now, let (μ_n, u_n) be sequence in ζ_k^ν with $\lim_{n\to+\infty} \|u_n\| = +\infty$ then $v_n = \frac{u_n}{\|u_n\|}$ satisfies

$$v_n = \mu_n L_\infty v_n + \mu_n \frac{T_\infty(u_n)}{\|u_n\|} \tag{2.20}$$

where $L, T : E \to E$ are defined as follows

$$L_\infty u(t) = \int_0^{+\infty} G_{\widetilde{q}}(t,s)\widetilde{m}_\infty(s)u(s)ds,$$
$$T_\infty u(t) = \int_0^{+\infty} G_{\widetilde{q}}(t,s)u(s)g_\infty(s, u(s))ds,$$

and $\widetilde{m}_\infty = m_\infty + 2\omega$, $g_\infty(s,u) = f(s,u) - m_\infty(s)$. Note that Hypothesis (2.6) implies that $T_\infty(u) = \circ(\|u\|_\infty)$ at ∞. Combining this with the compactness of L_∞, we obtain from (2.20) existence of $v_+, v_- \in W$ with $\|v_+\| = \|v_-\| = 1$ such that $L_\infty v_+ = \mu_+ v_+$ and $L_\infty v_- = \mu_- v_-$ where $\mu_+ = \limsup \mu_n$ and $\mu_- = \liminf \mu_n$.

Consequently, we have $\mu_+ = \mu_{l_+}(\widetilde{q}, m_\infty)$ and $\mu_- = \mu_{l_-}(\widetilde{q}, m_\infty)$ for some integers l_+, l_- and since each of v_+ and v_- is a limit of a subsequence of $(v_n) \subset S_k^\nu$, we obtain $l_+ = l_- = k$ and $\mu_+ = \mu_- = \mu_k(\widetilde{q}, m_\infty)$. ■

Lemma 2.15. *Assume that there exist two integers i, j with $1 \leq i \leq j$ such that one of the following situations holds*

$$\mu_i(q, m_0) < 1 < \mu_j(q, m_\infty) \text{ or } \mu_j(q, m_0) < 1 < \mu_i(q, m_\infty).$$

Then

$$\mu_i(\widetilde{q}, \widetilde{m}_0) < 1 < \mu_j(\widetilde{q}, \widetilde{m}_\infty) \text{ or } \mu_j(\widetilde{q}, \widetilde{m}_0) < 1 < \mu_i(\widetilde{q}, \widetilde{m}_\infty).$$

Proof .

Let $l \geq 1$ be an integer and $\kappa = 0, \infty$, we have to prove, $\mu_l(q, m_\kappa) < 1$ implies $\mu_l(\widetilde{q}, m_\kappa) < 1$ and $\mu_l(q, m_\kappa) > 1$ implies $\mu_l(\widetilde{q}, m_\kappa) > 1$. We present the proof of the implication: $\mu_l(q, m_\kappa) < 1 \Rightarrow \mu_l(\widetilde{q}, m_\kappa) < 1$, the other is checked similarly. Let $\phi \in S_l$ and $\widetilde{\phi} \in S_l$ be respectively the eigenfunctions associated respectively with $\mu = \mu_l(q, m_\kappa)$ and $\widetilde{\mu} = \mu_l(\widetilde{q}, \widetilde{m}_\kappa)$ and let $(z_j)_{j=0}^{j=l}$ be the sequence of zeros of ϕ. Each of the pairs (μ, ϕ) and $(\widetilde{\mu}, \widetilde{\phi})$ satisfies

$$\begin{cases} -u'' + qu = \mu m_\kappa u \text{ in } (0, +\infty), \\ u(0) = u(+\infty) = 0 \end{cases} \quad \text{and} \quad \begin{cases} -u'' + qu = (\widetilde{\mu} m_\kappa + 2(\widetilde{\mu} - 1)\omega)u \text{ in } (0, +\infty), \\ u(0) = u(+\infty) = 0. \end{cases}$$

By the contrary, suppose that $\widetilde{\mu} \geq 1$, then we have

$$(\widetilde{\mu} m_\kappa + 2(\widetilde{\mu} - 1)\omega) - \mu m_\kappa = (\widetilde{\mu} - \mu)\, m_\kappa + 2(\widetilde{\mu} - 1)\omega > 0 \text{ a.e. } t > 0.$$

Thus, applying Lemma 2.10 we get that in each interval (z_j, z_{j+1}), $j = 0, ..., l-1$, there is a zero of $\widetilde{\phi}$, contradicting $\widetilde{\phi} \in S_l$. This ends the proof. ■

Chapter 3

Nodal solutions for asymptotically linear second-order BVPs on the real line

3.1 Introduction and main results

Because that boundary value problems (bvps for short) associated with second-order ordinary differential equations posed on infinite intervals arise in modeling a variety of physical phenomena, the study of existence of solutions and their qualitative properties to such problems has received a great deal of attention and has been the subject of many old and recent articles, see, for instance [2]-[5], [15]-[40], [59], [62] and references therein. However, to the author's knowledge, there are few papers considering existence of nodal solutions for such type of bvps. The first goal of this chapter is then to fill the gap in this area.

Nodal solutions appear as eigenfunctions to the eigenvalue problem (evp for short)

$$\begin{cases} -(\alpha u')' + \beta u = \sigma \gamma u \text{ in } (\xi, \eta) \text{ a.e.}, \\ a u(\xi) + b \lim_{t\to\xi} p(t) u'(t) = 0, \\ c u(\eta) + d \lim_{t\to\eta} p(t) u'(t) = 0, \end{cases} \tag{3.1}$$

where $-\infty \leq \xi < \eta \leq +\infty$, σ is a real parameter, a, b, c, d are real numbers with $(a^2 + b^2)(c^2 + d^2) \neq 0$ and $\alpha, \beta, \gamma : (\xi, \eta) \to \mathbb{R}$ are three functions.

Theorem 4.9.1 in [64] states that if $\alpha, \gamma > 0$ in (ξ, η) a.e. and $1/\alpha, \beta, \gamma \in L^1(\xi, \eta)$, the evp (3.1) admits an increasing sequence of simple eigenvalues $(\sigma_k)_{k\geq 1}$ such that $\lim_{k\to\infty} \sigma_k = +\infty$ and if ϑ_k is the eigenfunction associated with σ_k, then ϑ_k admits exactly

$(k-1)$ zeros in (ξ, η), all are simple. The condition $\gamma > 0$ in (ξ, η) a.e. has been relaxed in [7] to $\gamma \geq 0$ in (ξ, η) a.e. and $\gamma > 0$ in $[\xi_1, \eta_1] \subset (\xi, \eta)$ a.e..

The second goal of this article is to prove that the existence of nodal solutions holds although the L^1-Carathéodory framework imposed in [64] and [7] is failed. Thus, we consider in this paper the evp:

$$\begin{cases} -u''(t) + q(t)u(t) = \mu m(t)u(t), \ t \in \mathbb{R}, \\ \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{3.2}$$

and the perturbed version of the evp (3.2):

$$\begin{cases} -u''(t) + q(t)u(t) = \mu u(t)f(t, u(t)), \ \ t \in \mathbb{R}, \\ \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{3.3}$$

where μ is a real parameter, the weights q and m belong to $C(\mathbb{R}, \mathbb{R}^+)$, q may be unbounded and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Notice that the evp (3.2) is the version of the evp (3.1) with $(\xi, \eta) = \mathbb{R}$, $a = c = 1$, $b = d = 0$ the and $\alpha = 1$, $\beta = q$. Clearly, with such a weight $\alpha = 1$ and a weight $\beta = q$ being unbounded, the evp (3.2) do not satisfy L^1-Caratheodory framework cited above.

Statements of main results in this paper need to introduce some notations. In what follows, we let

$$\begin{aligned} Q &= \left\{ q \in C(\mathbb{R}, \mathbb{R}^+) : \exists T \geq 0 \text{ such that } \inf_{|t|>T} q(t) > 0 \right\}, \\ W &= \left\{ u \in C(\mathbb{R}, \mathbb{R}) : \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0 \right\}, \\ W_k &= W \cap C^k(\mathbb{R}, \mathbb{R}) \text{ for all integers } k \geq 1, \\ W^+ &= \{ m \in W : m(t) > 0 \text{ a.e. } t \in \mathbb{R} \}. \end{aligned}$$

The linear space W is equipped with the norm $\|\cdot\|$, defined for $u \in W$ by $\|u\| = \sup_{t\in\mathbb{R}} |u(t)|$. Obviously, $(W, \|\cdot\|)$ is a Banach space.

For an integer $k \geq 1$, S_k^+ denotes the set of all the functions $u \in W_1$ having exactly $(k-1)$ zeros in $\mathbb{R}$, all are simple and u is positive in a right neighbourhood of $-\infty$, $S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$. For $u \in S_k$, $(z_j)_{j=0}^{j=k}$ with $-\infty = z_0 < z_1 < \ldots < z_k = +\infty$ and $u(z_j) = 0$ for $j = 1, \ldots, k-1$, is said to be the sequence of zeros of u.

Our first result concerns the evp (3.2), it states that the existence of a sequence of eigenvalues as well as its properties hold for all pairs of functions (q, m) in $Q \times W^+$.

Theorem 3.1. *For all pairs $(q, m) \in Q \times W^+$, the set of eigenvalues of the evp (3.2) consists in an unbounded increasing sequence of simple eigenvalues $(\mu_k(q,m))_{k\geq 1}$ such that eigenfunctions associated with $\mu_k(q,m)$ belong to S_k. Moreover, for q fixed in Q, the mapping $\mu_k(q,\cdot)$ has the following properties:*

1. *If $m_1, m_2 \in W^+$ are such that $m_1 \leq m_2$, then $\mu_k(m_1) \geq \mu_k(m_2)$. In addition, $\mu_k(m_1) > \mu_k(m_2)$ whenever $m_1 < m_2$ in a subset of positive measure.*

2. *If $m \in W^+$ and $(m_n) \subset W^+$ are such that $\lim m_n = m$ in W, then $\lim_{n\to\infty} \mu_k(q, m_n) = \mu_k(q,m)$.*

Concerning the bvp (3.3), we obtain under the assumptions on the nonlinearity f :

$$\begin{cases} |f(t,0)| \in W^+ \text{ and for all } r > 0, \text{ there exists } \psi_r \in W^+ \text{ such that} \\ |f(t,u) - f(t,v)| \leq \psi_r(t)\,|u-v| \text{ for all } t \in \mathbb{R} \text{ and } u,v \in [-r,r]\,, \end{cases} \tag{3.4}$$

$$\begin{cases} \text{there exists } \omega \in W^+ \text{ such that} \\ f(t,u) + \omega\,(t) \geq 0 \text{ for all } t,u \in \mathbb{R}, \end{cases} \tag{3.5}$$

$$\begin{cases} \lim_{u\to 0} f(t,u) = m_0(t) \text{ and} \\ \lim_{|u|\to+\infty} f(t,u) = m_\infty(t) \\ \text{in } W \text{ with } m_0, m_\infty \in W^+, \end{cases} \tag{3.6}$$

the following existence and multiplicity result for nodal solutions:

Theorem 3.2. *Let $q \in Q$ and assume that in addition to Hypotheses (3.4)-(3.6), there exist two integers i, j with $1 \leq i \leq j$ such that one of the following situations holds:*

$$\mu_j(q, m_\infty) < \mu < \mu_i(q, m_0)$$

or

$$\mu_j(q, m_0) < \mu < \mu_i(q, m_\infty).$$

Then for each integer $k \in \{i, \dots, j\}$ and $\nu = \pm$, the bvp (3.3) admits a solution in S_k^ν.

Consider the case of the bvp (3.3) where the nonlinearity f is a separable variables function, namely the case where the bvp (3.3) takes the form

$$\begin{cases} -u'' + q(t)u = \mu m(t) u g(u), \ t \in \mathbb{R}, \\ \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{3.7}$$

where $m \in W^{+}$ and $g : \mathbb{R} \to \mathbb{R}^{+}$ is a continuously differentiable function such that

$$\lim_{u\to 0} g(u) = g_0 > 0 \text{ and } \lim_{u\to +\infty} g(u) = g_\infty > 0. \tag{3.8}$$

We deduce, from Theorem 3.2 the following corollary:

Corollary 3.3. *Let $q \in Q$ and assume that in addition to Hypothesis (3.8), there exist two integers i, j with $1 \le i \le j$ such that one of the following situations holds:*

$$\mu g_0 < \mu_i(q,m) < \mu_j(q,m) < \mu g_\infty,$$

or

$$\mu g_\infty < \mu_i(q,m) < \mu_j(q,m) < \mu g_0.$$

Then for each integer $k \in \{i, \dots, j\}$ and $\nu = \pm$, the bvp (3.7) admits a solution in S_k^ν.

Proof .
Set $f(t,u) = \mu m(t) g(u)$ and note that such a nonlinearity satisfies Hypotheses (3.4) (3.5) and (3.6) with $m_0(t) = g_0 m(t)$, $m_\infty(t) = g_{+\infty} m(t)$. Since for all integers $k \ge 1$ and $\kappa = 0$ or $+\infty$, $\mu_k(m_\kappa) = \mu_k(m)/g_\kappa$, we have $\mu_i(q,m_0) < \mu < \mu_j(q,m_\infty)$ if and only if $\mu g_\infty < \mu_i(q,m) < \mu_j(q,m) < \mu g_0$ and $\mu_j(q,m_0) < \mu < \mu_i(q,m_\infty)$ if and only if $\mu g_0 < \mu_i(q,m_0) < \mu_j(q,m_\infty) < \mu g_\infty$. Therefore, Corollary 3.3 is obtained by a simple application of Theorem 3.2. ■

3.2 Preliminaries

3.2.1 The Green's function and fixed point formulation

Let for $\xi, \eta \in \mathbb{R}$, $\phi_{q,\xi,\eta}$ be the unique solution of the initial value problem

$$\begin{cases} -u''(t) + q(t)u(t) = 0, \\ u(0) = \xi, \\ u'(0) = \eta. \end{cases}$$

It is proved in Section 3.1 in [6] that there is a unique $\eta_0 \in \mathbb{R}$ such that $\phi_{q,1,\eta_0}$ satisfies the bvp

$$\begin{cases} -u''(t) + q(t)u(t) = 0, \\ u(0) = 1, \ \lim_{t\to+\infty} u(t) = 0, \end{cases}$$

with $\phi_{q,1,\eta_0}(t) > 0$ and $\phi'_{q,1,\eta_0}(t) < 0$ for all $t \ge 0$. In all what follows, we let for $q \in Q$, $\Phi_q = \phi_{q,1,\eta_0}$.

Lemma 3.4. *For all $q \in Q$, the function Φ_q has the following properties:*

i) $\Phi_q(t) > 0$ *and* $\Phi_q''(t) \geq 0$ *for all* $t \in \mathbb{R}$.

ii) $\lim_{t\to-\infty} \Phi_q'(t) = -\infty$ *and* $\lim_{t\to+\infty} \Phi_q'(t) = 0$.

iii) $\Phi_q'(t) < 0$ *for all* $t \in \mathbb{R}$.

iv) *For all* $t \in \mathbb{R}$, $\displaystyle\int_t^{+\infty} \Phi_q(s)ds < \infty$.

v) $\displaystyle\lim_{t\to-\infty} \frac{\Phi_q(t)}{1-t} = +\infty$ *and* $\displaystyle\int_{-\infty}^t \frac{ds}{\Phi_q^2} < \infty$ *for all* $t \in \mathbb{R}$.

vi) $\displaystyle\lim_{t\to-\infty} \Phi_q(t) \int_{-\infty}^t \frac{ds}{\Phi_q^2} = 0$.

vii) $\displaystyle\lim_{t\to+\infty} \Phi_q(t) \int_{-\infty}^t \frac{ds}{\Phi_q^2} = +\infty$.

viii) *The function* Φ_q/Φ_q' *is bounded at* $\pm\infty$.

Proof .

Let $q \in Q$ and $T > 0$ be such that $\alpha = \inf_{|t|\geq T} q(t) > 0$.

i) By the way of contradiction, suppose that $\Phi_q(t_0) \leq 0$ for some $t_0 < 0$. In this case, there is an interval $(t_*, 0)$ such that $\Phi_q(t) > 0$ for all $t \in (t_*, 0)$ and $\Phi_q'(t_*) > 0$. Therefore, we have $\Phi_q''(t) \geq 0$ for all $t \in (t_*, 0)$ and Φ_q' is nondecreasing on $(t_*, 0)$. This leads to the contradiction $0 < \Phi_q'(t_*) \leq \Phi_q'(0) < 0$ and proves that $\Phi_q(t) > 0$ for all $t \in \mathbb{R}$. The equation $\Phi_q''(t) = q(t)\Phi_q(t)$ shows that $\Phi_q''(t) \geq 0$ for all $t \in \mathbb{R}$.

ii) It follows from Assertion **i)** that the function Φ_q' is nondecreasing and the limits $\lim_{t\to+\infty} \Phi_q'(t)$ and $\lim_{t\to-\infty} \Phi_q'(t)$ exist. Set $l_+ = \lim_{t\to+\infty} \Phi_q'(t)$ and suppose that $l_+ \neq 0$. We obtain then by the L'Hôpital's rule $\lim_{t\to+\infty} \frac{\Phi_q(t)}{t} = l_+$ and $\lim_{t\to+\infty} \Phi_q(t) = \pm\infty$. This contradicts $\lim_{t\to+\infty} \Phi_q(t) = 0$ and proves that $\lim_{t\to+\infty} \Phi_q'(t) = 0$. Now, we have for all $t \leq -T$,

$$\Phi_q'(t) = \Phi_q'(-T) + \int_{-T}^t \Phi_q'' ds = \Phi_q'(-T) - \int_t^{-T} q\Phi_q ds \leq \Phi_q'(-T) + \alpha\varepsilon(t+T),$$

where $\varepsilon = \inf_{s\leq -T} \Phi_q(s) > 0$.

Clearly, the above inequality proves that $\lim_{t\to-\infty} \Phi_q'(t) = -\infty$.

iii) Since Φ_q' is nondecreasing and $\lim_{t\to+\infty} \Phi_q'(t) = 0$, we have $\Phi_q'(t) < 0$ for all $t \in \mathbb{R}$.

iv) We have for all $s \in (T, +\infty)$,

$$\int_T^s \Phi_q dr = \int_T^s \frac{\Phi_q''}{q} dr \leq \frac{1}{\alpha} \int_T^s \Phi_q'' dr = \frac{1}{\alpha}(\Phi_q'(s) - \Phi_q'(T)) \leq -\frac{\Phi_q'(T)}{\alpha}.$$

This proves that $\int_t^{+\infty} \Phi_q(r) dr < \infty$ for all $t \in \mathbb{R}$.

v) By L'Hôpital's rule, we have

$$\lim_{t \to -\infty} \frac{\Phi_q(t)}{1-t} = \lim_{t \to -\infty} -\Phi_q'(t) = +\infty.$$

This shows that for $\Phi_q^2(s) \geq (1-s)^2$ for all $s \in (-\infty, s_*)$ with s_* near $-\infty$. By the comparaison principle, we have $\int_{-\infty}^t \frac{ds}{\Phi_q^2} < \infty$ for all $t \in \mathbb{R}$.

vi) We have by L'Hôpital's rule

$$\lim_{t \to -\infty} \Phi_q(t) \int_{-\infty}^t \frac{ds}{\Phi_q^2} = \lim_{t \to -\infty} \frac{\int_{-\infty}^t \Phi_q^{-2} ds}{\left(\Phi_q(t)\right)^{-1}} = \lim_{t \to -\infty} -\frac{1}{\Phi_q'(t)} = 0.$$

vii) Again by L'Hôpital's rule we get

$$\lim_{t \to +\infty} \Phi_q(t) \int_{-\infty}^t \frac{ds}{\Phi_q^2} = \lim_{t \to +\infty} -\frac{1}{\Phi_q'(t)} = +\infty.$$

viii) We have for all $t \in \mathbb{R}$, with $t \geq T$,

$$\left(-\Phi_q'(t)\right)^2 = 2 \int_t^{+\infty} \Phi_q'' \left(-\Phi_q'\right) ds = 2 \int_t^{+\infty} q \Phi_q \left(-\Phi_q'\right) ds \geq \alpha \left(\Phi_q(t)\right)^2,$$

leading to

$$\left|\Phi_q(t)/\Phi_q'(t)\right|^2 = \left(\Phi_q(t)/-\Phi_q'(t)\right)^2 \leq \frac{1}{\alpha} \text{ for all } t \in \mathbb{R}, \text{ with } t \geq T,$$

then to,

$$\sup_{t \geq T} \left|\Phi_q(t)/\Phi_q'(t)\right| \leq \frac{1}{\sqrt{\alpha}}.$$

In a similar way, we obtain that

$$\sup_{t \leq -T} \left|\Phi_q(t)/\Phi_q'(t)\right| \leq \frac{1}{\sqrt{\alpha}},$$

proving viii) and completing the proof of the lemma. ■

Because of Assertions v), vi) and vii) in Lemma 3.4, the function

$$\Psi_q(t) = \Phi_q(t) \int_{-\infty}^t \frac{ds}{\Phi_q^2} \tag{3.9}$$

is well defined and it is a solution to the bvp

$$\begin{cases} -u''(t) + q(t)u(t) = 0, \\ \lim_{t \to -\infty} u(t) = 0, \ \lim_{t \to +\infty} u(t) = +\infty. \end{cases}$$

Lemma 3.5. *For all $q \in Q$, the function Ψ_q has the following properties:*

a) $\Psi_q(t) > 0$, $\Psi_q'(t) > 0$ *and* $\Psi_q''(t) \geq 0$ *for all* $t \in \mathbb{R}$.

b) $\lim_{t\to-\infty} \Psi_q'(t) = 0$ *and* $\lim_{t\to+\infty} \Psi_q'(t) = +\infty$.

c) *For all* $t \in \mathbb{R}$, $\int_{-\infty}^{t} \Psi_q ds < \infty$.

d) *For all* $t \in \mathbb{R}$, $\Phi_q(t)\Psi_q'(t) - \Psi_q(t)\Phi_q'(t) = 1$.

e) *The function* Ψ_q/Ψ_q' *is bounded at* $\pm\infty$.

Proof .

Let $q \in Q$ and $T > 0$ be such that $\alpha = \inf_{|t|\geq T} q(t) > 0$.

a) Respectively from (3.9) and $\Psi_q'' = q\Psi_q$, we have $\Psi_q(t) > 0$ and $\Psi_q''(t) \geq 0$ for all $t \in \mathbb{R}$. Since the function Φ_q' is decreasing, we obtain from (3.9) that

$$\Psi_q'(t) = \Phi_q'(t) \int_{-\infty}^{t} \frac{ds}{\Phi_q^2} + \frac{1}{\Phi_q(t)} > \int_{-\infty}^{t} \frac{\Phi_q'}{\Phi_q^2} ds + \frac{1}{\Phi_q(t)} = 0.$$

b) Because that Ψ_q' is a nondecreasing function the limits $\lim_{t\to-\infty} \Psi_q'(t)$ and $\lim_{t\to+\infty} \Psi_q'(t)$ exist. Set $\lim_{t\to-\infty} \Psi_q'(t) = l_+$ and notice that $l_+ \geq 0$. By the way of contradiction, suppose that $l_+ > 0$. We obtain then by means of the L'Hôpital's rule that

$$\lim_{t\to-\infty} \frac{\Psi_q(t)}{t} = \lim_{t\to-\infty} \Psi_q'(t) = l_+ > 0,$$

leading to $\lim_{t\to-\infty} \Psi_q(t) = -\infty$ and contradicting $\lim_{t\to-\infty} \Psi_q(t) = 0$. Therefore, we have proved that $\lim_{t\to-\infty} \Psi_q'(t) = 0$.

Now, we have for all $t \geq T$,

$$\Psi_q'(t) = \Psi_q'(T) + \int_T^t \Psi_q'' ds = \Psi_q'(T) + \int_T^t q\Psi_q ds \geq \Psi_q'(T) + \alpha\varepsilon\,(t - T)\,,$$

where $\varepsilon = \inf_{s\geq s_-} \Psi_q(s) > 0$. The above inequality shows that $\lim_{t\to+\infty} \Psi_q'(t) = +\infty$.

c) We have for all $s \in (-\infty, -T)\,$,

$$\int_s^{-T} \Psi_q(r)dr = \int_s^{-T} \frac{\Psi_q''}{q} dr \leq \frac{1}{\alpha} \int_s^{-T} \Psi_q'' dr = \frac{1}{\alpha}(\Psi_q'(-T) - \Psi_q'(s)) \leq \frac{\Psi_q'(-T)}{\alpha}.$$

This proves that $\int_{-\infty}^{t} \Psi_q dr < \infty$ for all $t \in \mathbb{R}$.

d) We have from (3.9) that for all $t \in \mathbb{R}$,

$$\Phi_q(t)\Psi_q'(t) - \Psi_q(t)\Phi_q'(t) = \Phi_q(t) \left(\Phi_q'(t) \int_{-\infty}^{t} \frac{ds}{\Phi_q^2} + \frac{1}{\Phi_q(t)} \right) - \Phi_q(t)\Phi_q'(t) \int_{-\infty}^{t} \frac{ds}{\Phi_q^2} = 1.$$

e) We have for all $t \geq T$,

$$\begin{aligned}\left(\Psi_q'(t)\right)^2 &= \left(\Psi_q'(T)\right)^2 + 2\int_T^t \Psi_q'\Psi_q'' ds = \left(\Psi_q'(T)\right)^2 + 2\int_T^t q\Psi_q\Psi_q' ds \\ &\geq \left(\Psi_q'(T)\right)^2 + \alpha\left(\left(\Psi_q(t)\right)^2 - \left(\Psi_q(T)\right)^2\right),\end{aligned}$$

from which we obtain that for all $t \geq T$,

$$\frac{\Psi_q(t)}{\Psi_q'(t)} \leq \sqrt{\frac{1}{\alpha} + \frac{\Psi_q(T)}{\Psi_q'(t)}}.$$

This together with Assertion b), we conclude that there is $T_* \geq T$ such that

$$\frac{\Psi_q(t)}{\Psi_q'(t)} \leq \sqrt{\frac{2}{\alpha}}, \quad \text{for all} \quad t \geq T_*.$$

We have for all $t \leq -T$,

$$\left(\Psi_q'(t)\right)^2 = 2\int_{-\infty}^t \Psi_q'\Psi_q'' ds = 2\int_{-\infty}^t q\Psi_q\Psi_q' ds \geq \alpha\left(\Psi_q(t)\right)^2,$$

leading to

$$\frac{\Psi_q(t)}{\Psi_q'(t)} \leq \sqrt{\frac{1}{\alpha}}, \quad \text{for all} \quad t \leq -T.$$

This completes the proof of d) and ends the proof of the lemma. ■

Set for $q \in Q$ and $\theta \in \mathbb{R}$,

$$\Psi_{q,\theta}(t) = \frac{\Psi_q(t)}{\Psi_q(\theta)}, \Phi_{q,\theta}(t) = \Psi_q(\theta)\Phi_q(t) - \Phi_q(\theta)\Psi_q(t),$$

and

$$G_q(\theta,t,s) = \begin{cases} 0, & \text{if } \max(t,s) \geq \theta, \\ \Phi_{q,\theta}(s)\Psi_{q,\theta}(t), & \text{if } t \leq s \leq \theta, \\ \Phi_{q,\theta}(t)\Psi_{q,\theta}(t), & \text{if } s \leq t \leq \theta. \end{cases} \tag{3.10}$$

We have then for all $q \in Q$ and all $\theta \in \mathbb{R}$,

$$\Phi_{q,\theta}(t)\Psi_{q,\theta}'(t) - \Psi_{q,\theta}(t)\Phi_{q,\theta}'(t) = 1, \quad \text{for all} \quad t \in \mathbb{R}, \tag{3.11}$$

and

$$G_q(\theta,t,s) = G_q(t,s) - \frac{\Phi_q(\theta)}{\Psi_q(\theta)}\Psi_q(s)\Psi_q(t), \quad \text{for} \quad t,s \leq \theta,$$

where

$$G_q(t,s) = G_q(+\infty,t,s) = \lim_{\theta\to+\infty} G_q(\theta,t,s) = \begin{cases} \Phi_q(t)\Psi_q(s), & \text{if } t \leq s, \\ \Phi_q(s)\Psi_q(t), & \text{if } s \leq t, \end{cases} \tag{3.12}$$

is the Green's function associated with bvp (3.3).

Lemma 3.6. *We have for all functions q in Q :*

1) $\overline{G}_{q,\infty} = \sup_{t,s\in\mathbb{R}} G_q(t,s) \le \sup_{t\in\mathbb{R}} \Phi_q(t)\Psi_q(t) < \infty$,

2) $G_{q,\infty} = \sup_{\theta,t,s\in\mathbb{R}} G_q(\theta,t,s) < \infty$,

3) $\tilde{G}_{q,\theta} = \sup_{t\in\mathbb{R}} \int_{-\infty}^{+\infty} G_q(\theta,t,s)ds < \infty$ *for all* $\theta \in (-\infty,+\infty]$.

Proof .

Let $q \in Q$ and $T > 0$ be such that $\alpha = \inf_{|t|\ge T} q(t) > 0$.

1) Taking in consideration that Φ_q is nonincreasing, we obtain from (3.9) that for all $t,s \in \mathbb{R}$,

$$\begin{aligned} G_q(t,s) &\le \Phi_q(t)\Psi_q(t) = \left(\frac{\Phi_q(t)}{-\Phi_q'(t)}\right)\left(-\Phi_q'(t)\Phi_q(t)\int_{-\infty}^{t}\frac{ds}{\Phi_q^2}\right) \\ &\le \left(\frac{\Phi_q(t)}{-\Phi_q'(t)}\right)\left(\Phi_q(t)\int_{-\infty}^{t}\frac{-\Phi_q'}{\Phi_q^2}ds\right) = \left(\frac{\Phi_q(t)}{-\Phi_q'(t)}\right). \end{aligned}$$

This together with Assertion viii) in Lemma 3.4, leads to

$$\overline{G}_{q,\infty} = \sup_{t,s\in\mathbb{R}} G_q(t,s) \le \sup_{t\in\mathbb{R}} \Phi_q(t)\Psi_q(t) < \infty.$$

2) Because of Φ_q is decreasing and Ψ_q is increasing, we have for all $s,t \le \theta$,

$$\begin{aligned} 0 \le G_q(\theta,t,s) &\le \Phi_q(t)\Psi_q(t) + \frac{\Phi_q(\theta)}{\Psi_q(\theta)}\Psi_q(t)\Psi_q(s) \\ &\le \Phi_q(t)\Psi_q(t) + \Psi_q(\theta)\Phi_q(\theta) \\ &\le 2\sup_{t\in\mathbb{R}} \Phi_q(t)\Psi_q(t) < \infty, \end{aligned}$$

proving **2)**.

3) Since for all $\theta \in \mathbb{R}$ and $t \in (-\infty,\theta)$,

$$\begin{aligned} \int_{-\infty}^{+\infty} G_q(\theta,t,s)ds &= \int_{-\infty}^{\theta} G_q(t,s)ds - \frac{\Phi_q(\theta)}{\Psi_q(\theta)}\Psi_q(t)\int_{-\infty}^{\theta}\Psi_q ds \\ &\le \int_{-\infty}^{+\infty} G_q(t,s)ds + \Phi_q(\theta)\int_{-\infty}^{\theta}\Psi_q ds, \end{aligned}$$

we have to prove that $\sup_{t\in\mathbb{R}} \int_{-\infty}^{+\infty} G_q(t,s)ds < \infty$. Because of Assertions vi) in Lemma 3.4 and c) in Lemma 3.5, we have for all $t \in \mathbb{R}$,

$$\int_{-\infty}^{+\infty} G_q(t,s)ds = \Phi_q(t)\int_{-\infty}^{t}\Psi_q ds + \Psi_q(t)\int_{t}^{+\infty}\Phi_q ds < \infty,$$

and the function $t \to \int_{-\infty}^{+\infty} G_q(t,s)ds$ belongs $C(\mathbb{R},\mathbb{R})$.

Moreover, we have for all $t \geq T$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} G_q(t,s)ds &= \Phi_q(t)\int_{-\infty}^{t} \Psi_q ds + \Psi_q(t)\int_{t}^{+\infty} \Phi_q ds \\
&= \Phi_q(t)\left(\int_{-\infty}^{-T} \Psi_q ds + \int_{-T}^{T} \Psi_q ds + \int_{T}^{t} \Psi_q ds\right) + \Psi_q(t)\int_{t}^{+\infty} \Phi_q ds \\
&= \Phi_q(t)\left(\int_{-\infty}^{-T} \frac{\Psi_q''}{q} ds + \int_{-T}^{T} \Psi_q ds + \int_{T}^{t} \frac{\Psi_q''}{q} ds\right) + \Psi_q(t)\int_{t}^{+\infty} \frac{\Phi_q''}{q} ds \\
&\leq \Phi_q(t)\left(\frac{1}{\alpha}\int_{-\infty}^{-T} \Psi_q'' ds + 2T\Psi_q(T) + \frac{1}{\alpha}\int_{T}^{t} \Psi_q'' ds\right) + \frac{1}{\alpha}\Psi_q(t)\int_{t}^{+\infty} \Phi_q'' ds \\
&= \Phi_q(t)\left(\frac{1}{\alpha}\Psi_q'(-T) + 2T\Psi_q(T) + \frac{1}{\alpha}(\Psi_q'(t) - \Psi_q'(T)\right) - \frac{1}{\alpha}\Psi_q(t)\Phi_q'(t) \\
&= \Phi_q(t)\frac{1}{\alpha}\Psi_q'(-T) + 2T\Phi_q(t)\Psi_q(T) - \Phi_q(t)\Psi_q'(T) + \frac{1}{\alpha} \\
&\leq \frac{1}{\alpha}\Phi_q(T)\Psi_q'(-T) + 2T\Psi_q(T)\Phi_q(T) + \frac{1}{\alpha},
\end{aligned}$$

and for all $t \leq -T$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} G_q(t,s)ds &= \Phi_q(t)\int_{-\infty}^{t} \Psi_q ds + \Psi_q(t)\int_{t}^{+\infty} \Phi_q ds \\
&= \Phi_q(t)\int_{-\infty}^{t} \frac{\Psi_q''}{q} ds + \Psi_q(t)\left(\int_{t}^{-T} \frac{\Phi_q''}{q} ds + \int_{-T}^{T} \Phi_q ds + \int_{T}^{+\infty} \frac{\Phi_q''}{q} ds\right) \\
&\leq \frac{1}{\alpha}\Phi_q(t)\int_{-\infty}^{t} \Psi_q'' ds + \Psi_q(t)\left(\frac{1}{\alpha}\int_{t}^{-T} \Phi_q'' ds + 2T\Phi_q(-T) + \frac{1}{\alpha}\int_{T}^{+\infty} \Phi_q'' ds\right) \\
&\leq \frac{1}{\alpha}\Phi_q(t)\Psi_q'(t) + \Psi_q(t)\left(\frac{1}{\alpha}\left(\Phi_q'(-T) - \Phi_q'(t)\right) + 2T\Phi_q(-T) - \frac{1}{\alpha}\Phi_q'(T)\right) \\
&= \frac{1}{\alpha} + \frac{1}{\alpha}\Psi_q(t)\Phi_q'(-T) + 2T\Psi_q(t)\Phi_q(-T) - \frac{1}{\alpha}\Psi_q(t)\Phi_q'(T) \\
&\leq \frac{1}{\alpha} + 2T\Phi_q(-T)\Psi_q(-T) - \frac{1}{\alpha}\Psi_q(-T)\Phi_q'(T).
\end{aligned}$$

The above estimates shows that the function $t \to \int_{-\infty}^{+\infty} G_q(t,s)ds$ is bounded at $\pm\infty$ and $\sup_{t\in\mathbb{R}} \int_{-\infty}^{+\infty} G_q(t,s)ds < \infty$. This achieves the proof of the lemma. ■

Lemma 3.7. *For all $q \in Q$ and $\theta \in (-\infty,+\infty]$, the operator $L_\theta : W \to W$ where for $h \in W$ $L_\theta h(t) = \int_{-\infty}^{+\infty} G_q(\theta,t,s)h(s)ds$ is well defined and is continuous.*

Proof .

Let $\theta \in \mathbb{R}$, $h \in W$ and set $u_\theta(t) = L_\theta h(t) = \int_{-\infty}^{+\infty} G_q(\theta,t,s)hds$. We have from the above Assertion 3) that for all $t \in \mathbb{R}$

$$|u_\theta(t)| = \left|\int_{-\infty}^{+\infty} G_q(\theta,t,s)hds\right| \leq \|h\|\int_{-\infty}^{+\infty} G_q(\theta,t,s)ds \leq \|h\|\int_{-\infty}^{+\infty} G_q(\theta,t,s)ds < \infty.$$

Because of $\lim_{t\to\theta} u_\theta(t) = u_\theta(\theta) = 0$, we conclude from the expression

$$u_\theta(t) = \begin{cases} \Phi_q(t)\int_{-\infty}^{t}\Psi_q h ds + \Psi_q(t)\int_{t}^{\theta}\Phi_q h ds - \dfrac{\Phi_q(\theta)}{\Psi_q(\theta)}\Psi_q(t)\int_{-\infty}^{\theta}\Psi_q h ds, & \text{if } t<\theta,\\ 0, & \text{if } t\geq\theta,\end{cases}$$

that the function u_θ belongs to $C(\mathbb{R},\mathbb{R})$.

Clearly, $\lim_{t\to+\infty} u_\theta(t) = u_\theta(\theta) = 0$ and $\lim_{t\to-\infty} \frac{\Phi_q(\theta)}{\Psi_q(\theta)}\Psi_q(t)\int_{-\infty}^{\theta}\Psi_q h ds = 0$. Thus, taking in account Assertions viii) in Lemma 3.4, e) in Lemma 3.5, 1) in Lemma 3.6 and $\lim_{t\to-\infty} h(t) = 0$, we obtain by means of the L'Hôpital's rule

$$\lim_{t\to-\infty}\Phi_q(t)\int_{-\infty}^{t}\Psi_q h ds = \lim_{t\to-\infty}\frac{\int_{-\infty}^{t}\Psi_q h ds}{\left(\Phi_q(t)\right)^{-1}} = \lim_{t\to-\infty} -\left(\frac{\Phi_q(t)}{\Phi_q'(t)}\right)\Phi_q(t)\Psi_q(t)h(t) = 0,$$

$$\lim_{t\to-\infty}\Psi_q(t)\int_{t}^{\theta}\Phi_q h ds = \lim_{t\to-\infty}\frac{\int_{t}^{\theta}\Phi_q h ds}{\left(\Psi_q(t)\right)^{-1}} = \lim_{t\to-\infty}\left(\frac{\Psi_q(t)}{\Psi_q'(t)}\right)\left(\Phi_q(t)\Psi_q(t)\right)h(t) = 0,$$

leading to $\lim_{t\to+\infty} L_\theta u(t) = 0$. All the above show that for $\theta\in\mathbb{R}$, the operator L_θ is well defined. We have also for all $h\in W$,

$$|L_\theta h(t)| = \left|\int_{-\infty}^{+\infty} G_q(\theta,t,s)h ds\right| \leq \int_{-\infty}^{+\infty} G_q(\theta,t,s)ds\,\|h\| \leq \left(\sup_{t\in\mathbb{R}}\int_{-\infty}^{+\infty} G_q(\theta,t,s)ds\right)\|h\|.$$

This leads to $\|L_\theta h\| \leq \tilde{G}_{q,\theta}\|h\|$ for all $h\in W$ and proves that $L_\theta\in\mathcal{L}(W)$.

We have for all $\theta\in\mathbb{R}$ and $h\in W$,

$$\begin{aligned}|L_{+\infty}h(t) - L_\theta h(t)| &= \begin{cases}\left|\Phi_q(t)\int_{-\infty}^{t}\Psi_q h ds + \Psi_q(t)\int_{t}^{+\infty}\Phi_q h ds\right|, & \text{if } t\geq\theta,\\ \left|\Psi_q(t)\int_{\theta}^{+\infty}\Phi_q h ds + \dfrac{\Phi_q(\theta)}{\Psi_q(\theta)}\Psi_q(t)\int_{-\infty}^{\theta}\Psi_q h ds\right|, & \text{if } t<\theta,\end{cases}\\ &\leq \begin{cases}\Phi_q(t)\int_{-\infty}^{t}\Psi_q|h|ds + \Psi_q(t)\int_{t}^{+\infty}\Phi_q|h|ds, & \text{if } t\geq\theta,\\ \Psi_q(\theta)\int_{\theta}^{+\infty}\Phi_q|h|ds + \Phi_q(\theta)\int_{-\infty}^{\theta}\Psi_q|h|ds, & \text{if } t<\theta.\end{cases}\end{aligned}$$

Since

$$\lim_{\theta\to+\infty}\Psi_q(\theta)\int_{\theta}^{+\infty}\Phi_q|h|ds + \Phi_q(\theta)\int_{-\infty}^{\theta}\Psi_q(s)|h|ds = 0,$$

for $\epsilon>0$ there is $\theta_*>0$ such that

$$0 < \Psi_q(t)\int_{t}^{+\infty}\Phi_q|h|ds + \Phi_q(t)\int_{-\infty}^{t}\Psi_q|h|ds < \epsilon, \quad \text{for all}\quad t>\theta_*.$$

Therefore, we have for all $\theta>\theta_*$ $\sup_{t\in\mathbb{R}}|L_{+\infty}h(t) - L_\theta h(t)| \leq \epsilon$ and $L_{+\infty}h = \lim_{\theta\to+\infty} L_\theta h$ in W. This proves that $L_{+\infty}h\in W$ and the operator $L_{+\infty}$ is well defined and because of Assertion 3) in Lemma 3.6, we have for all $h\in W$, $\|L_{+\infty}h\| \leq \tilde{G}_{q,\infty}\|h\|$, showing that $L_{+\infty}\in\mathcal{L}(W)$. The proof of the lemma is complete. ∎

Lemma 3.8. *Let* $\widetilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying Hypothesis (3.4), then for all* $u \in W$ *the function* $\widetilde{F}u$*, with* $\widetilde{F}u(t) = u(t)\widetilde{f}(t, u(t))$ *belongs to* W*. Moreover, the mapping* $\widetilde{F} : W \to W$ *is continuous and bounded.*

Proof .
Let $u \in W$, because of the continuity of the function $\widetilde{f}$, $\widetilde{F}u \in C(\mathbb{R}, \mathbb{R})$. Moreover, we have from Hypothesis (3.4) for all $t \in \mathbb{R}$:

$$\left|u(t)\widetilde{f}(t, u(t))\right| \leq \|u\|^2 \psi_{\|u\|}(t) + \|u\| \left|\widetilde{f}(t, 0)\right| = \widetilde{\psi}(t),$$

with $\widetilde{\psi} \in W$. Therefore $\lim_{|t|\to+\infty} \widetilde{F}u(t) = 0$ and $\widetilde{F}u \in W$.

Now, let $R > 0$ and $u, v \in W$ be such that $\sup(\|u\|, \|v\|) \leq R$. We have from Hypothesis (3.4) that

$$\begin{aligned} \left|\widetilde{F}u(t) - \widetilde{F}v(t)\right| &= \left|u(t)\widetilde{f}(t, u(t)) - v(t)\widetilde{f}(t, v(t))\right| \\ &\leq \left|u(t)\widetilde{f}(t, u(t)) - u(t)\widetilde{f}(t, v(t))\right| + \left|u(t)\widetilde{f}(t, v(t)) - v(t)\widetilde{f}(t, v(t))\right| \\ &\leq R\psi_R(t) |u(t) - v(t)| + R\left(R\psi_R + \left|\widetilde{f}(t, 0)\right|\right) |u(t) - v(t)| \\ &\leq R\left(\|\psi_R\| + R\|\psi_R\| + \left\|\widetilde{f}(t, 0)\right\|\right) \|u - v\|, \end{aligned}$$

leading to

$$\left\|\widetilde{F}u - \widetilde{F}v\right\| \leq R\left(\|\psi_R\| + R\|\psi_R\| + \left\|\widetilde{f}(t, 0)\right\|\right) \|u - v\|,$$

and proving that the mapping $\widetilde{F}$ is locally Lipshitzian, consequently it is continuous and bounded. ■

The main result of this subsection consists in the following lemma providing a fixed point formulation for the evp (3.2) and the bvp (3.3).

Lemma 3.9. *Let* q *be in* Q *and* $\widetilde{f} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying Hypothesis (3.4). Set for* $\theta \in (-\infty, +\infty]$, $T_\theta = L_\theta \circ F$*, then the operator* T_θ *is completely continuous and* $u \in W$ *is a fixed point of* T_θ *if and only if* u *is a solution to the BVP*

$$\begin{cases} -u''(t) + q(t)u(t) = u(t)\widetilde{f}(t, u(t)), \ t \in (-\infty, \theta), \\ \lim_{t\to-\infty} u(t) = \lim_{t\to\theta} u(t) = 0. \end{cases} \tag{3.13}$$

Proof .
First, we prove that fixed points of the operator T_θ are solutions to the bvp (3.13). To this aim, let $u \in W$ be a fixed point of T_θ, from

$$u(t) = \Phi_{q,\theta}(t) \int_{-\infty}^{t} \Psi_{q,\theta} u\widetilde{f}(s, u) ds + \Psi_{q,\theta}(t) \int_{t}^{\theta} \Phi_{q,\theta} u\widetilde{f}(s, u) ds, \quad \text{for } t < \theta,$$

we understand that u belongs to $C^1((-\infty,\theta),\mathbb{R})$ and straightforward computations lead to

$$u'(t) = \Phi'_{q,\theta}(t)\int_{-\infty}^{t}\Psi_{q,\theta}u\widetilde{f}(s,u)ds + \Psi'_{q,\theta}(t)\int_{t}^{\theta}\Phi_{q,\theta}u\widetilde{f}(s,u)ds, \quad \text{for } t<\theta.$$

Again we have $u' \in C^1((-\infty,\theta),\mathbb{R})$ and taking in account (3.11), we obtain for all $t \in (-\infty,\theta)$,

$$\begin{aligned} u''(t) &= \Phi''_{q,\theta}(t)\int_{-\infty}^{t}\Psi_{q,\theta}u\widetilde{f}(s,u)ds + \Psi''_{q,\theta}(t)\int_{t}^{\theta}\Phi_{q,\theta}u\widetilde{f}(s,u)ds \\ &\quad - \left(\Phi_{q,\theta}(t)\Psi'_{q,\theta}(t) - \Phi'_{q,\theta}(t)\Psi_{q,\theta}(t)\right)u(t)\widetilde{f}(t,u(t)) \\ &= q(t)\left(\Phi_{q,\theta}(t)\int_{-\infty}^{t}\Psi_{q,\theta}u\widetilde{f}(s,u)ds + \Psi'_{q,\theta}(t)\int_{t}^{\theta}\Phi_{q,\theta}u\widetilde{f}(s,u)ds\right) - u(t)\widetilde{f}(t,u(t)) \\ &= q(t)u_\theta(t) - u(t)\widetilde{f}(t,u(t)). \end{aligned}$$

Reciprocally, if $u \in W_2$ is a solution to the bvp (3.13) we have then

$$\int_{-\infty}^{+\infty} G_q(t,s)\left(-u''+qu\right)ds = \int_{-\infty}^{+\infty} G_q(t,s)u\widetilde{f}(s,u)ds, \quad \text{for all } t\in\mathbb{R}.$$

Integrating twice by parts the left integral, we obtain that u is a fixed point of T_θ.

Now, we prove that the mapping T_θ is completely continuous for all $\theta \in \mathbb{R}$. To this aim, let Ω be a subset of W bounded by a constant r and $\psi_r \in W^+$ such that

$$\left|\widetilde{f}(t,x) - \widetilde{f}(t,y)\right| \le \psi_r(t)\,|x-y|, \quad \text{for all} \quad t\in\mathbb{R} \quad \text{and} \quad x,y\in[-r,r]. \tag{3.14}$$

Since each of the mapping L_θ and $\widetilde{F}$ is continuous and bounded, the operator T_θ is continuous and bounded. In particular, $T_\theta(\Omega)$ is bounded and we obtain from (3.14):

$$\left|x\widetilde{f}(t,x)\right| \le \widetilde{\psi}_r(t). \quad \text{for all} \quad t\in\mathbb{R} \quad \text{and} \quad x\in[-r,r],$$

where $\widetilde{\psi}_r(t) = r^2\psi_r(t) + r\left|\widetilde{f}(t,0)\right|$.

Therefore, the following estimate hold for all $u \in \Omega$,

$$|T_\theta u(t)| \le U_\theta(t), \quad \text{for all } t\in\mathbb{R}, \quad \text{where } U_\theta = L_\theta\widetilde{\psi}_r \in W,$$

and proves that $T_\theta(\Omega)$ is equiconvergent.

It remains to show that the subset is equicontinuous on compact intervals. Let $[\gamma,\delta]$

be a compact interval and let $t_1, t_2 \in [\gamma, \delta]$ be such that $t_1 < t_2$. We have for all $u \in \Omega$:

$$\begin{aligned}
T_\theta u(t_2) - T_\theta u(t_1) &= 0, \quad \text{if} \quad \theta \le t_1 < t_2, \\
T_\theta u(t_2) - T_\theta u(t_1) &= T_\theta u(\theta) - T_\theta u(t_1) \\
&= \left(\Phi_q(\theta) - \Phi_q(t_1)\right) \int_{-\infty}^{t_1} \Psi_q u \widetilde{f}(s,u) ds + \Phi_q(\theta) \int_{t_1}^{\theta} \Psi_q u \widetilde{f}(s,u) ds \\
&\quad - \left(\Psi_q(\theta) - \Psi_q(t_1)\right) \frac{\Phi_q(\theta)}{\Psi_q(\theta)} \int_{-\infty}^{\theta} \Psi_q u \widetilde{f}(s,u) ds, \quad \text{if} \quad t_1 < \theta \le t_2,
\end{aligned}$$

and

$$\begin{aligned}
T_\theta u(t_2) - T_\theta u(t_1) &= \left(\Phi_q(t_2) - \Phi_q(t_1)\right) \int_{-\infty}^{t_1} \Psi_q u \widetilde{f}(s,u) ds \\
&\quad + \left(\Psi_q(t_2) - \Psi_q(t_1)\right) \int_{t_2}^{\theta} \Psi_q u \widetilde{f}(s,u) ds + \Phi_q(t_2) \int_{t_1}^{t_2} \Psi_q u \widetilde{f}(s,u) ds \\
&\quad - \Psi_q(t_1) \int_{t_1}^{t_2} \Phi_q u \widetilde{f}(s,u) ds \\
&\quad - \left(\Psi_q(t_2) - \Psi_q(t_1)\right) \frac{\Phi_q(\theta)}{\Psi_q(\theta)} \int_{-\infty}^{\theta} \Psi_q u \widetilde{f}(s,u) ds, \quad \text{if} \quad t_1 < t_2 < \theta.
\end{aligned}$$

In all cases, the above estimates lead to:

$$|T_\theta u(t_2) - T_\theta u(t_1)| \le \|\widetilde{\psi}_r\| M \, |t_2 - t_1| \,, \quad \text{for all} \quad u \in \Omega,$$

where

$$M = \left(\left| \Phi_q'(\gamma) \right| + \Psi_q'(\delta) \frac{\Phi_q(\gamma)}{\Psi_q(\gamma)} \right) \int_{-\infty}^{\delta} \Psi_q ds + \Psi_q'(\delta) \int_{\gamma}^{\delta} \Phi_q ds + 2\Phi_q(\gamma)\Psi_q(\delta).$$

This shows that the subset $T_\theta(\Omega)$ is equicontinuous on compact intervals and complete the proof of the compactness of the mapping T_θ for $\theta \in \mathbb{R}$.

We end by proving that $T_{+\infty}$ is completely continuous. Let Λ be a subset in W with $\Lambda \subset B(0_W, R)$ and and $\psi_R \in W^+$ such that

$$\left| \widetilde{f}(t,x) - \widetilde{f}(t,y) \right| \le \psi_R(t) \, |x - y| \,, \quad \text{for all} \quad t \in \mathbb{R} \quad \text{and all} \quad x, y \in [-R, R] \,.$$

Therefore, we have:

$$\left| x \widetilde{f}(t,x) \right| \le \widetilde{\psi}_r(t), \quad \text{for all} \quad t \in \mathbb{R} \quad \text{and all} \quad x \in [-r, r] \,,$$

where $\widetilde{\psi}_r(t) = r^2\psi_r(t) + r\left|\widetilde{f}(t,0)\right|$ and for all $u \in \Lambda$ and $t \in \mathbb{R}$,

$$
\begin{aligned}
&|T_{+\infty}u\,(t) - T_\theta u\,(t)| \\
= &\begin{cases} \left|\Phi_q\,(t)\displaystyle\int_{-\infty}^{t}\Psi_q u\widetilde{f}(s,u)ds + \Psi_q\,(t)\int_{t}^{+\infty}\Phi_q u\widetilde{f}(s,u)\right|, & \text{if} \quad t \geq \theta, \\ \left|\Psi_q\,(t)\displaystyle\int_{\theta}^{+\infty}\Phi_q u\widetilde{f}(s,u) + \frac{\Phi_q(\theta)}{\Psi_q(\theta)}\Psi_q(t)\int_{-\infty}^{\theta}\Psi_q u\widetilde{f}(s,u)ds\right|, & \text{if} \quad t < \theta, \end{cases} \\
\leq &\begin{cases} \Phi_q\,(t)\displaystyle\int_{-\infty}^{t}\Psi_q\widetilde{\psi}_R ds + \Psi_q\,(t)\int_{t}^{+\infty}\Phi_q\widetilde{\psi}_R ds, & \text{if} \quad t \geq \theta, \\ \Psi_q\,(\theta)\displaystyle\int_{\theta}^{+\infty}\Phi_q\widetilde{\psi}_R ds + \Phi_q(\theta)\int_{-\infty}^{\theta}\Psi_q\widetilde{\psi}_R ds, & \text{if} \quad t < \theta. \end{cases}
\end{aligned}
$$

Thus, arguing as in the end of the proof of Lemma 3.7, we obtain that $T_{+\infty} = \lim T_\theta$ in $C_b\left(\overline{\Omega}, W\right)$ and $T_{+\infty}$ is completely continuous. ■

3.2.2 Comparison results

The following lemma will play an important role in the proof of Theorem 3.2.

Lemma 3.10. *Let $(q,m) \in Q \times W^+$ be such that $\mu_k(q,m) = 1$ for some integer $k \geq 1$. Then there exists $\varepsilon_0 > 0$ such that for all $p \in W^+$ with $\|p - m\| \leq \varepsilon_0$, $\mu_l(q,p) = 1$ implies $l = k$.*

Proof .
Let $\epsilon_0 > 0$ be such that $\epsilon_0 < \min(\mu_{k+1}(q,m) - \mu_k(q,m), \mu_k(q,m) - \mu_{k-1}(q,m))$, because of Assertion 2 in Theorem 3.1, there exists $\varepsilon_0 > 0$ such that for all $p \in W^+$, $\|p - m\| \leq \varepsilon_0$ implies

$$\mu_{k-1}(q,m) - \epsilon_0 \leq \mu_{k-1}(q,p) \leq \mu_{k-1}(q,m) + \epsilon_0, \tag{3.15}$$

and

$$\mu_{k+1}(q,m) - \epsilon_0 \leq \mu_{k+1}(q,p) \leq \mu_{k+1}(q,m) + \epsilon_0. \tag{3.16}$$

Let $p \in W^+$ with $\|p - m\| \leq \varepsilon_0$ and suppose that $\mu_l(q,p) = 1$ for some integer $l \geq 1$. If $l < k$, we have then from (3.15) the contradiction

$$1 = \mu_l(q,p) \leq \mu_{k-1}(q,p) \leq \mu_{k-1}(q,m) + \epsilon_0 < \mu_k(q,m),$$

and if $l > k$, we have then from (3.16) the contradiction

$$1 = \mu_l(q,p) \geq \mu_{k+1}(q,p) \geq \mu_{k+1}(q,m) - \epsilon_0 > \mu_k(q,m) = 1.$$

This shows that $l = k$ and the lemma is proved. ■

We will use extensively the following lemma:

Lemma 3.11 ([11]). *Let j and k be two integers such that $j \geq k \geq 2$ and let $(\xi_l)_{l=0}^{l=k}$, $(\eta_l)_{l=0}^{l=j}$ be two families of real numbers such that*

$$\xi_0 = \xi < \xi_1 < \xi_2 < \cdots < \xi_{k-1} < \xi_k = \eta,$$
$$\eta_0 = \xi < \eta_1 < \eta_2 < \cdots < \eta_{j-1} < \eta_j = \eta.$$

If $\xi_1 < \eta_1$, then there exist two integers m and n having the same parity, $1 \leq m \leq k-1$ and $1 \leq n \leq j-1$ such that

$$\xi_m < \eta_n \leq \eta_{n+1} \leq \xi_{m+1}.$$

We end this section with the following lemma which is an adapted version of the Sturmian comparison result.

Lemma 3.12. *Let for $i = 1, 2$, $m_i \in W$ and $w_i \in C^2(\mathbb{R})$ satisfying*

$$-w_i''(t) + q(t)w_i(t) = m_i(t)w_i(t),\ t \in (x_1, x_2),$$

and suppose that w_2 does not vanish identically and $m_1(t) > m_2(t)$ a.e. $t > 0$. If either

1. $x_1 > -\infty$ *and* $w_2(x_1) = w_2(x_2) = 0$, *or*

2. $x_1 = -\infty$, $x_2 < +\infty$ *and* $w_2(x_1) = \lim_{t\to-\infty} w_i(t) = 0$, *for* $i = 1, 2$,

3. $x_1 > -\infty$, $x_2 = +\infty$ *and* $w_2(x_1) = \lim_{t\to+\infty} w_i(t) = 0$, *for* $i = 1, 2$,

4. $x_1 = -\infty$, $x_2 = +\infty$ *and* $\lim_{t\to+\infty} w_i(t) = \lim_{t\to-\infty} w_i(t) = 0$, *for* $i = 1, 2$,

then there exists $\tau \in (x_1, x_2)$ such that $w_1(\tau) = 0$.

Proof .

We present the proofs of Assertions 1) and 4), the other assertions are checked similarly.

1) By the contrary suppose that $w_1 > 0$ in (x_1, x_2) and without loss of generality assume that $w_2 > 0$ in (x_1, x_2), then we have the contradiction

$$\begin{aligned} 0 \ &\geq\ w_1(x_2)\, w_2'(x_2) - w_1(x_1)\, w_2'(x_1) = \int_{x_1}^{x_2} \left(w_2(-w_1'' + qw_1) - w_1(-w_2'' + qw_2)\right) ds \\ &=\ \int_{x_1}^{x_2} (m_1 - m_2) w_1 w_2 ds > 0. \end{aligned}$$

4) By the contrary, suppose that $w_1 > 0$ in $\mathbb{R}$ and without loss of generality assume that $w_2 > 0$ in $\mathbb{R}$. Because that $w_i''(t) = (q(t) - m_i(t))\, w_i(t)$ and $q(t) - m_i(t) > 0$ for $|t|$

large, we have that $w_i''(t) > 0$ for $|t|$ large and $\lim_{|t|\to\infty} w_i'(t) = 0$. Therefore, we have for all $t > 0$,

$$\begin{aligned}
&\left(-w_2(t)\,w_1'(t) + w_1(t)\,w_2'(t)\right) + \left(w_2(-t)\,w_1'(-t) - w_1(-t)\,w_2'(-t)\right) \\
= &\int_{-t}^{+t} \left(w_2(-w_1'' + qw_1) - w_1(-w_2'' + qw_2)\right) ds = \int_{-t}^{t} (m_1 - m_2) w_1 w_2 ds > 0.
\end{aligned}$$

Letting $t \to +\infty$, we obtain the contradiction:

$$0 = \int_{-\infty}^{+\infty} (m_1 - m_2) w_1 w_2 ds > 0.$$

The proof is complete. ■

3.2.3 On the linear eigenvalue problem

We will present in this subsection two lemmas related to linear eigenvalue problems and needed for the proof of Theorem 3.1. The following lemma and its assertions follows from Theorem 2.1 and Lemma 3.7 in [6].

Lemma 3.13. *For all pairs $(q, m) \in Q \times W^+$ and all real numbers θ, the evp*

$$\begin{cases} -u''(t) + q(t)u(t) = \mu m(t)u(t), \ t > \theta, \\ u(\theta) = \lim_{t\to+\infty} u(t) = 0, \end{cases}$$

admits an unbounded increasing sequence of simple eigenvalues $\left(\mu_k^+(\theta, q, m)\right)_{k\geq 1}$ such that:

1. *If ϕ is an eigenfunction associated with $\mu_k^+(\theta, q, m)$ then ϕ admits $(k-1)$ zeros in $(\theta, +\infty)$ and all are simple.*

2. *If $m_1, m_2 \in W^+$ are such that $m_1 \leq m_2$, then $\mu_k(m_1) \geq \mu_k(m_2)$. In addition, $\mu_k(m_1) > \mu_k(m_2)$ whenever $m_1 < m_2$ in a subset of positive measure.*

3. *If $m \in W^+$ and $(m_n) \subset W^+$ are such that $\lim m_n = m$ in W, then $\lim_{n\to\infty} \mu_k(q, m_n) = \mu_k(q, m)$.*

4. *Moreover, for (q, m) fixed in $Q \times W^+$, the function $\theta \to \mu_k^+(\theta) := \mu_k^+(\theta, q, m)$ is continuous and increasing. We have also $\lim_{\theta\to+\infty} \mu_k^+(\theta) = +\infty$.*

The next lemma concerns the existence of the positive eigenvalue on the unbounded interval $(-\infty, \theta)$.

Lemma 3.14. *For all pairs $(q,m) \in Q \times W^{+}$ and all $\theta \in (-\infty,+\infty]$, the evp*

$$\begin{cases} -u''(t) + q(t)u(t) = \mu m(t)u(t), & t < \theta, \\ \lim_{t\to-\infty} u(t) = 0, u(\theta) = 0, \end{cases} \tag{3.17}$$

admits a unique positive eigenvalue $\mu_1^-(\theta,q,,m)$. Moreover, for all $\theta \in \mathbb{R}$, $\mu_1^-(\theta,q,,m)$ is geometrically simple and for (q,m) fixed in $Q \times W^{+}$, the function $\theta \to \mu_1^-(\theta) := \mu_1^-(\theta,q,m)$ is continuous and decreasing having $\lim_{\theta\to-\infty} \mu_1^-(\theta) = +\infty$.

Proof .

Let for (q,m) fixed in $Q \times W^{+}$ and $\theta \in (-\infty,+\infty]$, $L_\theta : W \to W$ be the linear compact operator defined by

$$L_\theta u(t) = \int_{-\infty}^{+\infty} G_q(\theta,t,s) mu ds,$$

where the function G_q is that introduced by (3.10), and let $u_\theta \in K$ be the function defined by

$$u_\theta(t) = \begin{cases} 0, & \text{if } \ t \notin [\sigma_-(\theta), \sigma_+(\theta)], \\ (t - \sigma_-(\theta))(\sigma_+(\theta) - t), & \text{if } \ t \in [\sigma_-(\theta), \sigma_+(\theta)], \end{cases}$$

where

$$\sigma_-(\theta) = \begin{cases} \inf\left(\frac{1}{3}, \frac{\theta}{3}\right), & \text{if } \ \theta > 0, \\ 3\theta - 2, & \text{if } \ \theta \le 0 \end{cases} \quad \text{and } \sigma_+(\theta) = \begin{cases} \inf\left(\frac{1}{2}, \frac{\theta}{2}\right), & \text{if } \ \theta > 0, \\ 2\theta - 1, & \text{if } \ \theta \le 0. \end{cases}$$

We have then $L_\theta u_\theta(t) \ge 0 = u_\theta(t)$ for $t \in (-\infty, \sigma_-(\theta)] \cup [\sigma_+(\theta), \theta)$ and $Lu_\theta(t), u_\theta(t) > 0$ for $t \in (\sigma_-(\theta), \sigma_+(\theta))$. This shows that $L_\theta u \ge c_\theta u_\theta$ where

$$c_\theta = \inf\{Lu_\theta(t)/u_\theta(t) : t \in (\sigma_-(\theta), \sigma_+(\theta))\} > 0,$$

and $r(L_\theta) > 0$.

Since Lemma 3.9 guarantees that L_θ is compact, we have from the Krein-Rutman theorem, that $r(L_\theta)$ is a positive eigenvalue of L_θ having an eigenvector $\phi_\theta \in W^{+}$. By means of Lemma 3.9, we conclude that $\mu_1^-(\theta,q,m) = 1/r(L_\theta)$ is a positive eigenvalue of evp (3.17).

Now, for λ a positive eigenvalue of evp (3.17) having an eigenfunction ψ, we have

$$0 = \int_{-\infty}^{\theta} \left((-\phi_\theta'' + q\phi_\theta)\psi - (-\psi'' + q\psi)\phi_\theta\right) ds = \left(\mu_1^+(\theta,q,m) - \lambda\right) \int_{-\infty}^{\theta} m\phi_\theta\psi ds,$$

leading to $\lambda = \mu_1^-(\theta,q,m)$.

Now we prove that for $\theta \in \mathbb{R}$, $\mu_1^-(\theta, q, m)$ is geometrically simple. Let $\phi \in W^+$ be an eigenfunction associated with $\mu_1^-(\theta, q, m)$ and let $w_\theta = W(\phi_\theta, \phi) = \phi_\theta \phi' - \phi'_\theta \phi$ be the Wronksian of ϕ_θ and ϕ. We have then

$$w'_\theta = W'(\phi_\theta, \phi) = \phi_\theta \phi'' - \phi''_\theta \phi = 0,$$

and

$$w_\theta(\theta) = \phi_\theta(\theta)\phi'(\theta) - \phi'_\theta(\theta)\phi(\theta) = 0,$$

proving that $\mu_1^-(\theta, q, m)$ is geometrically simple.

Let θ_1, θ_2 be real numbers such that $\theta_1 < \theta_2$ and set for $i = 1, 2$, $\mu_i = \mu_1^-(\theta_i, m)$ with the corresponding eigenfunction ψ_i. We have by simple calculations

$$\begin{aligned} 0 < -\psi'_1(\theta_1)\psi_2(\theta_1) &= \int_{-\infty}^{\theta_1}((-\psi''_1 + q\psi_1)\psi_2 - (-\psi''_2 + q\psi_2)\psi_1)ds \\ &= (\mu_1 - \mu_2)\int_{-\infty}^{\theta_1} m\psi_1\psi_2 ds, \end{aligned}$$

leading to $\mu_1 > \mu_2$ and proving that $\theta \to \mu_1(\theta, q, m)$ is an decreasing function.

For the continuity of the function $\mu_1(\cdot, q, m)$, follows from that of the Green's function G and Lemma 2.13 in [10].

Let $[\gamma, \delta]$ be a compact interval and let $\theta_1, \theta_2 \in [\gamma, \delta]$ be such that $\theta_1 < \theta_2$. We have for all $u \in W$ with $\|u\| = 1$,

$$\begin{aligned} |L_{\theta_2}u(t) - L_{\theta_1}u(t)| &= \left| \int_{-\infty}^{\theta_2} G_q(\theta_2, t, s)\, muds - \int_{-\infty}^{\theta_1} G_q(\theta_1, t, s)\, muds \right| \\ &= \begin{cases} 0, & \text{if } t \geq \theta_2 > \theta_1, \\ \left| \int_{-\infty}^{\theta_2} G_q(\theta_2, t, s)\, muds \right|, & \text{if } \theta_2 > t \geq \theta_1, \\ \left| \int_{-\infty}^{\theta_2} G_q(\theta_2, t, s)\, muds - \int_{-\infty}^{\theta_1} G_q(\theta_1, t, s)\, muds \right|, & \text{if } \theta_2 > \theta_1 > t. \end{cases} \end{aligned}$$

Set

$$\chi = \|m\| \left[\left(\int_{-\infty}^{\delta} \Psi_q ds \right) \frac{\Psi_q(\delta)}{\Psi_q^2(\gamma)} + \overline{G}_{q,\infty} + \Phi_q(\gamma)\Psi_q(\delta) \right]$$

then we have for $\theta_2 > t \geq \theta_1$

$$\begin{aligned}
\left| \int_{-\infty}^{\theta_2} G_q(\theta_2, t, s)\, muds \right| &\leq \|m\| \int_{-\infty}^{\theta_2} G_q(\theta_2, t, s)\, ds \\
&= \|m\| \left(\int_{-\infty}^{\theta_2} G_q(t, s)\, ds - \frac{\Phi_q(\theta_2)}{\Psi_q(\theta_2)} \Psi_q(t) \int_{-\infty}^{\theta_2} \Psi_q ds \right) \\
&= \|m\| (\int_{-\infty}^{t} G_q(t, s)\, ds + \int_{t}^{\theta_2} G_q(t, s)\, ds \\
&\quad - \frac{\Phi_q(\theta_2)}{\Psi_q(\theta_2)} \Psi_q(t) \int_{-\infty}^{t} \Psi_q ds - \frac{\Phi_q(\theta_2)}{\Psi_q(\theta_2)} \Psi_q(t) \int_{t}^{\theta_2} \Psi_q ds) \\
&= \|m\| (\Phi_q(t) \int_{-\infty}^{t} \Psi_q ds + \int_{t}^{\theta_2} G_q(t, s)\, ds \\
&\quad - \frac{\Phi_q(\theta_2)}{\Psi_q(\theta_2)} \Psi_q(t) \int_{-\infty}^{t} \Psi_q ds - \frac{\Phi_q(\theta_2)}{\Psi_q(\theta_2)} \Psi_q(t) \int_{t}^{\theta_2} \Psi_q ds) \\
&= \|m\| \left(\left(\int_{-\infty}^{t} \Psi_q ds \right) \left(\frac{\Phi_q(t)}{\Psi_q(t)} - \frac{\Phi_q(\theta_2)}{\Psi_q(\theta_2)} \right) \Psi_q(t) + \int_{t}^{\theta_2} G_q(t, s)\, ds \right. \\
&\quad \left. - \frac{\Phi_q(\theta_2)}{\Psi_q(\theta_2)} \Psi_q(t) \int_{t}^{\theta_2} \Psi_q ds \right) \\
&\leq \|m\| \left[\left(\int_{-\infty}^{\delta} \Psi_q ds \right) \frac{\Psi_q(\delta)}{\Psi_q^2(\gamma)} + \overline{G}_{q,\infty} + \Phi_q(\gamma) \Psi_q(\delta) \right] |\theta_2 - \theta_1| \\
&\leq \chi |\theta_2 - \theta_1|,
\end{aligned}$$

and for $\theta_2 > \theta_1 > t$,

$$\begin{aligned}
&\left| \int_{-\infty}^{\theta_2} G_q(\theta_2, t, s)\, muds - \int_{-\infty}^{\theta_1} G_q(\theta_1, t, s)\, muds \right| \\
\leq\ & \left| \int_{-\infty}^{\theta_1} \left(G_q(\theta_2, t, s) - G_q(\theta_1, t, s) \right) muds \right| + \left| \int_{\theta_1}^{\theta_2} G_q(\theta_2, t, s)\, muds \right| \\
=\ & \left| \left(\int_{-\infty}^{\theta_1} \Psi_q muds \right) \left(\frac{\Phi_q(\theta_1)}{\Psi_q(\theta_1)} - \frac{\Phi_q(\theta_2)}{\Psi_q(\theta_2)} \right) \Psi_q(t) \right| + \left| \int_{\theta_1}^{\theta_2} G_q(\theta_2, t, s)\, muds \right| \\
\leq\ & \|m\| \left[\left(\int_{-\infty}^{\delta} \Psi_q ds \right) \frac{\Psi_q(\delta)}{\Psi_q^2(\gamma)} + \overline{G}_{q,\infty} \right] |\theta_2 - \theta_1| \\
\leq\ & \chi |\theta_2 - \theta_1|.
\end{aligned}$$

The above estimates show that the mapping $\theta \to L_\theta$ is locally Lipschitzian and so, it is continuous. Let (θ_n) be a sequence converging to θ_* and let θ_-, θ_+ be such that $(\theta_n) \subset [\theta_-, \theta_+]$. Therefore we have for all $n \geq 1$,

$$0 < \mu_1(\theta_+, q, m) \leq \mu_1(\theta_n, q, m) \leq \mu_1(\theta_-, q, m),$$

and the sequence $(\mu_1(\theta_n, q, m))$ converges (up to a subsequence) to some $\mu_* > 0$. We conclude by Lemma 2.13 in [10] and by uniqueness of that $\mu_* = \mu_1(\theta_*, q, m)$. Thus, the continuity of the mapping $\mu_1(\cdot, q, m)$ is proved.

It remains to prove that $\lim_{\theta\to-\infty} \mu_1^-(\theta, m) = \lim_{\theta\to-\infty}(1/r(L_\theta)) = +\infty$. We have for all $u \in W$ with $\|u\| = 1$,

$$\begin{aligned} |L_\theta u(t)| &\le \int_{-\infty}^{\theta} G_q(\theta, t, s)\, mds \\ &\le \int_{-\infty}^{\theta} G_q(t, s)\, mds + \frac{\Phi_q(\theta)}{\Psi_q(\theta)} \Psi_q(t) \int_{-\infty}^{\theta} \Psi_q mds \\ &\le \int_{-\infty}^{\theta} G_q(t, s)\, mds + \Phi_q(\theta) \int_{-\infty}^{\theta} \Psi_q mds. \end{aligned}$$

As in the proof of Lemma 3.9, we have $\lim_{\theta\to-\infty} \Phi_q(\theta) \int_{-\infty}^{\theta} \Psi_q mds = 0$ and since $\lim_{t\to-\infty} m(t) = 0$, for $\epsilon > 0$, there exists $\theta_\epsilon > 0$ such that $m(s) \le \epsilon$ for all $s \le \theta_\epsilon$. Hence, we have for all $\theta \le \theta_\epsilon$,

$$\int_{-\infty}^{\theta} G_q(t, s)\, mds \le \tilde{G}_{q,+\infty}\epsilon, \quad \text{for all} \quad t \le \theta,$$

proving that $\lim_{\theta\to-\infty} \left(\sup_{t\le\theta} \int_{-\infty}^{\theta} G_q(t, s)\, mds \right) = 0$. Therefore, we have proved that $\lim_{\theta\to-\infty} r(L_\theta) = \lim_{\theta\to-\infty} \|L_\theta\| = 0$, ending the proof. ■

3.3 Proof of Theorem 3.1

Step 1. Fix (q, m) in $Q \times W^+$ and let $k \ge 1$ be an integer. Existence and uniqueness of $\mu_1(q, m)$ is guaranteed by Lemma 3.14. For $k \ge 2$, we have from Lemmas 3.13 and 3.14 existence of a unique real number θ_k^* such that $\mu_1^-(\theta_k^*, q, m) = \mu_{k-1}^+(\theta_k^*, q, m)$. Therefore, if ϕ_{1,θ_k^*} and ψ_{k-1,θ_k^*} are respectively the eigenfunctions associated with $\mu_1^-(\theta_k^*, q, m)$ and $\mu_{k-1}^+(\theta_k^*, q, m)$, then the function

$$\phi_k(t) = \begin{cases} \psi_{k-1,\theta_k^*}(t), & \text{in } \left[\theta_k^*, +\infty\right), \\ \left(\psi'_{k-1,\theta_k^*}\left(\theta_k^*\right) / \phi'_{1,\theta_k^*}\left(\theta_k^*\right) \right) \phi_{1,\theta_k^*}(t), & \text{in } \left(-\infty, \theta_k^*\right), \end{cases}$$

belongs S_k and is the eigenfunction associated with the eigenvalue $\mu_k(q, m) = \mu_1^-(\theta_k^*, q, m) = \mu_{k-1}^+(\theta_k^*, q, m)$ of the evp (3.2).

Now, let us prove that $\mu_k(q, m)$ is the unique eigenvalue of the evp (3.2), having an eigenfunction in S_k. To this aim, let for $i = 1, 2$, $\phi_i \in S_k^+$ be an eigenfunction associated with the eigenvalue μ_i and let $\left(z_j^i\right)_{j=0}^{j=k}$ be the sequence of zeros of ϕ_i. Without loss of generality, suppose that $z_1^1 \le z_1^2$, we deduce then from Lemma 3.11 existence of two integers $0 \le n_1, m_1 \le k-1$ having the same parity such that $z^1_{n_1} \le z^2_{m_1} < z^2_{m_1+1} \le z^1_{n_1+1}$. Notice that the fact n_1, m_1 have the same parity means that the functions ϕ_1 and ϕ_2 have the same sign on the interval $\left(z^2_{m_1}, z^2_{m_1+1}\right)$ and after simple calculations, yields

$$0 \le \int_{-\infty}^{z_1^1} \left(\phi_2(-\phi_1'' + q\phi_1) - \phi_1(-\phi_2'' + q\phi_2)\right) ds = (\mu_1 - \mu_2) \int_{-\infty}^{z_1^1} m\phi_1\phi_2 ds,$$

and

$$0 \ge \int_{z^2_{m_1}}^{z^2_{m_1+1}} \left(\phi_2(-\phi_1'' + q\phi_1) - \phi_1(-\phi_2'' + q\phi_2)\right) ds = (\mu_1 - \mu_2) \int_{z^2_{m_1}}^{z^2_{m_1+1}} m\phi_1\phi_2 ds,$$

proving that $\mu_1 = \mu_2$ and $\mu_k(q, m)$ is the unique eigenvalue of the evp (3.2), having an eigenfunction in S_k.

At this stage we need to prove that for all positives integers i, j with $i \ne j$, $\mu_i(q, m) \ne \mu_j(q, m)$. By the contrary, suppose that for two positive integers $i \ne j$ we have $\mu_i(q, m) = \mu_j(q, m) = \mu_*$ and let $\phi_i \in S_i^+$ and $\phi_j \in S_j^+$ be their corresponding eigenfunctions. Let $\omega = W(\phi_i, \phi_j) = \phi_i\phi_j' - \phi_j'\phi_j$ be the Wronksian of ϕ_i and ϕ_j, we have then

$$\begin{aligned} \omega' &= (\phi_i\phi_j' - \phi_j'\phi_j)' = \phi_i\phi_j'' - \phi_i''\phi_j \\ &= (q - \mu_* m)\phi_i\phi_j - \phi_i(q - \mu_* m)\phi_j = 0, \end{aligned}$$

leading to $\omega(t) = c$ with $c \in \mathbb{R}$. Moreover, because that ϕ_i, ϕ_j, $q - \mu_* m$ are positive at $-\infty$, $\phi_i'' = (q - \mu_* m)\phi_i$ and $\phi_j'' = (q - m)\phi_j$, we have that $\phi_i'' > 0$, $\phi_j'' > 0$ at $-\infty$, ϕ_i', ϕ_j' are increasing at $-\infty$ and $\lim_{t\to-\infty} \phi_i'(t) = \lim_{t\to-\infty} \phi_j'(t) = 0$. Therefore, $\omega = c = \lim_{t\to-\infty} \left(\phi_i(t)\phi_j'(t) - \phi_i'(t)\phi_j(t)\right) = 0$ and $\phi_j = \alpha\phi_j$ for some $\alpha \in \mathbb{R}$. This contradicts $\phi_i \in S_i^+$ and $\phi_j \in S_j^+$ and proves that for $i \ne j$ we have $\mu_i(q, m) \ne \mu_j(q, m)$.

In order to prove monotonicity of the sequence $(\mu_k(q, m))$, let for $i = 1, 2$, $\phi_i \in S_{k_i}^+$ be an eigenfunction associated with the eigenvalue μ_i of the evp (3.2), having a sequence of zeros $\left(z_j^i\right)_{j=0}^{j=k_i}$. Suppose that $k_2 > k_1$, we distinguish then the following cases:

Case 1. $z_1^2 \le z_1^1$, in this case we have

$$0 \ge \int_{-\infty}^{z_1^2} \left(\phi_2(-\phi_1'' + q\phi_1) - \phi_1(-\phi_2'' + q\phi_2)\right) ds = (\mu_1 - \mu_2) \int_{-\infty}^{z_1^2} m\phi_1\phi_2 ds,$$

leading to $\mu_1 \le \mu_2$.

Case 2. $z_1^1 \le z_1^2$, in this case, we deduce from Lemma 3.11 existence of two integers n_1, m_1, with $n_1 \le k_1 - 1$, $m_1 \le k_2 - 1$ and such that $z^1_{n_1} \le z^2_{m_1} < z^2_{m_1+1} \le z^1_{n_1+1}$. After simple computations, yields

$$0 \ge \int_{z^2_{m_1}}^{z^2_{m_1+1}} \left(\phi_2(-\phi_1'' + q\phi_1) - \phi_1(-\phi_2'' + q\phi_2)\right) ds = (\mu_1 - \mu_2) \int_{z^2_{m_1}}^{z^2_{m_1+1}} m\phi_1\phi_2 ds,$$

leading to $\mu_1 \leq \mu_2$. This together with $\mu_i(q,m) \neq \mu_j(q,m)$ for $i \neq j$ show that $\mu_1 < \mu_2$.

Notice that the sequence $(\mu_k(q,m))$ is infinite and for all integers $k \geq 1$, $\mu_k(q,m)$ is a characteristic value of the compact operator $L_m : W \to W$ given by $L_m u(t) = \int_{-\infty}^{\theta} G_q(t,s)m(s)u(s)ds$ where G_q is defined in (3.12). Therefore, we have $\lim_{k\to\infty} \mu_k(q,m) = +\infty$.

We prove now that aside the sequence $(\mu_k(q,m))$, the evp (3.2) has no other eigenvalues. By the contrary, suppose that the evp (3.2) has an eigenvalue μ having an eigenfunction ϕ and $\mu \neq \mu_k(q,m)$ for all $k \geq 1$. Hence, ϕ has an infinite sequence of simple zeros (z_n) with $\lim z_n = \pm\infty$. Indeed, if for some z_i, $\phi(z_i) = \phi'(z_i) = 0$ then the standard existence and uniqueness result for ODEs leads to the contradiction $\phi = 0$. Also, if $\lim z_n = \hat{z} \in \mathbb{R}$ then

$$u\left(\hat{z}\right) = \lim u\left(z_n\right) = 0 \text{ and } u'\left(\hat{z}\right) = \lim \frac{u\left(z_n\right) - u\left(\hat{z}\right)}{z_n - \hat{z}} = 0,$$

leading again to the contradiction $\phi = 0$. This shows that the limit of (z_n) may be equal to $+\infty$ or $-\infty$.

Let for the integer $k \geq 1$, $\phi_k \in S_k$ be the eigenfunction associated with the eigenvalue $\mu_k(q,m)$ and let $\left(x_j\right)_{j=1}^{j=k}$ be the sequence of zeros of ϕ_k. We deduce from Lemma 3.11, existence of two integers l, m having the same parity such that $0 \leq l \leq k-1$ and $0 \leq m \leq k-1$,

$$x_l \leq z_m < z_{m+1} \leq x_{l+1}.$$

Hence, we have

$$\begin{aligned} 0 &\leq -\phi_k\left(z_{m+1}\right)\phi'\left(z_{m+1}\right) + \phi_k\left(z_m\right)\phi'\left(z_m\right) \\ &= \int_{z_m}^{z_{m+1}} \left(-\phi_k(\phi'' + q\phi) - \phi(-\phi_k'' + q\phi_k)\right) ds \\ &= (\mu - \mu_k(q,m)) \int_{z_m}^{z_{m+1}} m\phi\phi_k ds, \end{aligned}$$

leading to $\mu \geq \mu_k(q,,m)$ for all integers $k \geq 1$, then to the contradiction $\mu = \lim_{k\to\infty} \mu_k(q,m) = +\infty$. Thus, we have proved that aside the sequence $(\mu_k(q,m))$ there are no other eigenvalues.

At the end of this step we have for all integer $k \geq 1$, $\mu_k(q,m)$ is geometrically simple. Indeed, if for some integer $i \geq 1$, $\mu_i(q,m)$ has two eigenfunctions ϕ and ψ with $\phi \in S_i^+$, then necessarily $\psi \in S_i^+$. Otherwise, if ψ has an infinite sequence of zeros then we obtain as above the contradiction $\mu = \lim_{k\to\infty} \mu_k(q,m) = +\infty$, and if $\psi \in S_j^+$ for some integer

$j \neq i$ then because the uniqueness we have the contradiction $\mu_i(q,m) = \mu_j(q,m)$. Set then $\varpi = W(\phi, \psi) = \phi\psi' - \phi'\psi$ be the Wronksian of ϕ and ψ, we have then

$$\begin{aligned} \varpi' &= (\phi\psi' - \phi'\psi)' = \phi\psi'' - \phi''\psi \\ &= (q - \mu_k(q,m)m)\phi\psi - (q - \mu_k(q,m)m)\phi\psi = 0, \end{aligned}$$

leading to $\varpi(t) = c$ with $c \in \mathbb{R}$. Moreover, because that ϕ, ψ, $q - \mu_k(q,m)m$ are positive at $-\infty$, $\phi'' = (q - \mu_k(q,m)m)\,\phi$ and $\psi'' = (q - \mu_k(q,m)m)\,\psi$, we have that $\phi'' > 0$, $\psi'' > 0$ at $-\infty$, ϕ', ψ' are increasing at $-\infty$ and $\lim_{t\to-\infty} \phi'(t) = \lim_{t\to-\infty} \psi'(t) = 0$. Therefore, $\varpi = c = \lim_{t\to-\infty} (\phi(t)\,\psi'(t) - \phi'(t)\,\psi(t)) = 0$ and $\psi = \delta\phi$ for some $\delta \in \mathbb{R}$.

Step 2. Monotonicity: Fix q in Q and let m_1, m_2 be two functions in W^+ and suppose that $m_1 \leq m_2$ and $m_1 < m_2$ in a subset of positive measure. Set for $i = 1,2$, $\mu_i = \mu_k(q, m_i)$ and let $\phi_i \in S_k^+$ be the eigenfunction associated with μ_i having a sequence of zeros $\left(z_j^i\right)_{j=0}^{j=k}$. By the contrary, suppose that $\mu_1 < \mu_2$, then there exists j_0 such that $z_{j_0}^1 \neq z_{j_0}^2$. Indeed, if $\phi_1\left(z_j^2\right) = 0$ for all $j \in \{1, \ldots, k-1\}$ then for $j_1 \in \{1, \ldots, k-1\}$ being such that $meas\left(\{m_2 > m_1\} \cap \left(z_{j_1}^2, z_{j_1+1}^2\right)\right) > 0$, we have since $\phi_1\phi_2 > 0$ in $\left(z_{j_1}^2, z_{j_1+1}^2\right)$, the contradiction

$$\begin{aligned} 0 &= \int_{z_{j_1}^2}^{z_{j_1+1}^2} \left(\phi_2\left(-\phi_1'' + q\phi_1\right) - \phi_1\left(-\phi_2'' + q\phi_2\right)\right) ds \\ &= \int_{z_{j_1}^2}^{z_{j_1+1}^2} (\mu_1 m_1 - \mu_2 m_2)\phi_1\phi_2 ds \\ &= (\mu_1 - \mu_2)\int_{z_{j_1}^2}^{z_{j_1+1}^2} m_1\phi_1\phi_2 ds + \mu_2 \int_{z_{j_1}^2}^{z_{j_1+1}^2} (m_1 - m_2)\,\phi_1\phi_2 ds < 0. \end{aligned}$$

Now, let $k_1 = \max\left\{l \leq k : z_j^1 = z_j^2 \text{ for all } j \leq l\right\}$ and $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ be the families defined by $\xi_j = z_{k_1+j}^1$ and $\eta_j = z_{k_1+j}^2$. We distinguish then two cases.

i) $\xi_1 = z_{k_1+1}^1 < \eta_1 = z_{k_1+1}^2$: In this case we have the contradiction

$$\begin{aligned} 0 &< -\phi_2(\xi_1)\,\phi_1'(\xi_1) = \int_{\xi_0}^{\xi_1} \left(\phi_2\left(-\phi_1'' + q\phi_1\right) - \phi_1\left(-\phi_2'' + q\phi_2\right)\right) ds \\ &= \int_{\xi_0}^{\xi_1} (\mu_1 m_1 - \mu_2 m_2)\phi_1\phi_2 ds \\ &= (\mu_1 - \mu_2)\int_{\xi_0}^{\xi_1} m_1\phi_1\phi_2 ds + \mu_2 \int_{\xi_0}^{\xi_1} (m_1 - m_2)\,\phi_1\phi_2 ds \leq 0. \end{aligned}$$

ii) $\xi_1 = z_{k_1+1}^1 > \eta_1 = z_{k_1+1}^2$: In this case Lemma 3.11 guarantees existence of two integers m, n having the same parity such that

$$\eta_m = z_{k_1+m}^2 < \xi_n = z_{k_1+n}^1 < \xi_{n+1} = z_{k_1+n+1}^1 \leq \eta_{m+1} = z_{k_1+m+1}^2.$$

Again, we have the contradiction

$$\begin{aligned} 0 &< \int_{\xi_n}^{\xi_{n+1}} \left(\phi_2\left(-\phi_1'' + q\phi_1\right) - \phi_1\left(-\phi_2'' + q\phi_2\right)\right) ds \\ &= \int_{\xi_n}^{\xi_{n+1}} (\mu_1 m_1 - \mu_2 m_2)\phi_1\phi_2 ds \\ &= (\mu_1 - \mu_2)\int_{\xi_n}^{\xi_{n+1}} m_1\phi_1\phi_2 ds + \mu_2 \int_{\xi_n}^{\xi_{n+1}} (m_1 - m_2)\,\phi_1\phi_2 ds \leq 0. \end{aligned}$$

The monotonicity is proved.

Step 3. Continuity: Fix q in Q, m in W^+ and let $(m_n) \subset W^+$ such that $\lim m_n = m$ in W. Let $L_n, L \in \mathcal{L}(W)$ be defined by

$$L_n u(t) = \int_{-\infty}^{+\infty} G_q(t,s)m_n(s)u(s)ds \quad \text{and} \quad Lu(t) = \int_{-\infty}^{+\infty} G_q(t,s)m(s)u(s)ds.$$

Notice that for all integers $l, n \geq 1$, $\mu_l^n = \mu_l(q, m_n)$ is a characteristic value of L_n, $\mu_l = \mu_l(q,m)$ is a characteristic value of L and $L_n \to L$ in operator norm.

First, fix $k \geq 1$ and let us prove that if $\left(\mu_k^n\right)$ admits a subsequence (δ_n) converging to $\delta > 0$, then $\delta = \mu_k$. Indeed, let $\phi_n \in S_k^+$ be the normalized eigenfunction associated with δ_n and let $\psi_n = L\phi_n$. Since L is compact and the sequence (ϕ_n) is bounded, we have up to a subsequence $\psi_n \to \psi$ in W. Thus, we obtain the following estimates,

$$\begin{aligned} \|(\phi_n/\delta_n) - \psi\| &= \|L_n\phi_n - \psi\| \\ &\leq \|L_n\phi_n - L\phi_n\| + \|L\phi_n - \psi\| \\ &\leq \|L_{n_k} - L\| + \|\psi_n - \psi\|, \end{aligned}$$

leading to

$$\lim(\phi_n/\delta_n) = \psi \text{ in } W \text{ and } \|\psi\| = \lim \|\phi_n\|/\delta_n = 1/\delta > 0.$$

Also, we have

$$\begin{aligned} \|L_n\phi_n - \delta L\psi\| &= \|\delta_n L_{n_k}\left((\phi_n/\delta_n)\right) - \delta L\psi\| \\ &\leq \|\delta_n L_n\left((\phi_n/\delta_n)\right) - \delta L_n\left((\phi_n/\delta_n)\right)\| + \|\delta L_n\left((\phi_n/\delta_n)\right) - \delta L\left((\phi_n/\delta_n)\right)\| \\ &\quad + \|\delta L\left((\phi_n/\delta_n)\right) - \delta L\psi\| \\ &\leq |\delta_n - \delta|\,\delta_n\,\|L_n\| + \frac{\delta_n}{\delta}\|L_n - L\| + \frac{1}{\delta}\|L\|\,\|(\phi_n/\delta_n) - \psi\|, \end{aligned}$$

leading to $\lim L_n\phi_n = \delta L\psi$ in W.

Thus, letting $n \to \infty$ in equation $L_n\phi_n = (\phi_n/\delta_n)$ we obtain $L\psi = \psi/\delta$ that is $1/\delta$ is an eigenvalue of L or $\delta = \mu_l(q,m)$ for some integer $l \geq 1$. Then, because of $\lim \delta_n m_n = \delta m$ in W, it follows from Lemma 3.10 that $\delta = \mu_k(q,m)$.

Then, fix T in $\mathbb{R}$, and set for all integers $l, n \geq 1$, $\mu_l^{n,T} = \mu_l^+(T,q,m_n)$ and $\mu_l^T = \mu_l^+(T,q,m)$. We have from Assertion 3) in Lemma 3.13 that $\lim_{n\to\infty} \mu_l^{n,T} = \mu_l^T$ for all integers $l \geq 1$ and then there is $c_l > 0$ such that $\mu_l^{n,T} < \mu_l^T + c_l$ for all $n \geq 1$. Fix $k \geq 1$ and denote by $\phi_n \in S_k^+$ the eigenfunction associated with μ_k^n and suppose that ϕ_n admits $(j-1)$ zeros in $(T,+\infty)$ and $\phi_n > 0$ in a left neightborhood of T. Let $\phi_{n,T}$ be the eigenfunction associated with $\mu_j^{n,T}$ satisfying $\phi'_{n,T}(T) > 0$ and denote by $(x_s)_{s=0}^{s=j}$ the sequence of zeros of $\phi_{n,T}$ and by $(z_s)_{s=0}^{s=j}$ the sequence constituted in zeros of ϕ_n contained in $(T,+\infty)$ with $z_0 = T$ and $z_j = +\infty$. We distinguish two cases:

Case 1. $x_1 < z_1$, we have in this case

$$\begin{aligned} 0 > \phi_n(x_1)\phi'_{n,T}(x_1) - \phi_n(T)\phi'_{n,T}(T) &= \int_T^{x_1} \left(\phi_{n,T}(-\phi''_n + q\phi_n) - \phi_n(-\phi''_{nT} + q\phi_{n,T})\right) ds \\ &= \left(\mu_k^n - \mu_j^{n,T}\right) \int_T^{x_1} m_n \phi_{n,T}\phi_n ds, \end{aligned}$$

leading to

$$\mu_k^n \leq \mu_j^{n,T} \leq \max_{1\leq l\leq k}(\mu_l^{n,T}) \leq \max_{1\leq l\leq k}(\mu_l^T + c_l) \leq \mu_k^T + \max_{1\leq l\leq k}(c_l).$$

Case 2. $z_1 \leq x_1$, in this case we deduce from Lemma 4.6 existence of two integers r_T, r having the same parity and such that $z_r \leq x_{r_T} < x_{r_T+1} \leq z_{r+1}$ and $\phi_{n,T}\phi_n > 0$ in (x_{r_T}, x_{r_T+1}). After simple computations yields

$$\begin{aligned} 0 &\geq \phi_n(x_{r_T+1})\phi'_{n,T}(x_{r_T+1}) - \phi_n(x_{r_T})\phi'_{n,T}(x_{r_T}) \\ &= \int_{x_{r_T}}^{x_{r_T+1}} \left(\phi_{n,T}(-\phi''_n + q\phi_n) - \phi_n(-\phi''_{nT} + q\phi_{n,T})\right) ds \\ &= \left(\mu_k^n - \mu_j^{n,T}\right) \int_{x_{r_T}}^{x_{r_T+1}} m_n \phi_{n,T}\phi_n ds, \end{aligned}$$

and we have again

$$\mu_k^n \leq \mu_j^{n,T} \leq \max_{1\leq l\leq k}(\mu_l^{n,T}) \leq \max_{1\leq l\leq k}(\mu_l^T + c_l) \leq \mu_k^T + \max_{1\leq l\leq k}(c_l).$$

At this stage we have proved that the sequence (μ_k^n) is bounded, set then $\mu_k^+ = \limsup \mu_k^n$ and $\mu_k^- = \liminf \mu_k^n$. Since $\lim \|L_n\| = \|L\|$, we have $\|L_n\| \geq \|L\|/2$ for n large enough and $\mu_k^n \geq 1/\|L_n\| \geq \|L\|/2$ for n large enough. Therefore, passing to the limit, we obtain $\mu_k^+ \geq \mu_k^- \geq \|L\|/2 > 0$ and taking in account what is showed at the beginning of this proof, we conclude that $\lim \mu_k^n = \mu_k^+ = \mu_k^- = \mu_k$. The continuity is proved.

3.4 Proof of Theorem 3.2

Consider the bvp

$$\begin{cases} -u'' + \widetilde{q}(t)u = \lambda\mu u(f(t,u) + 2\omega(t)), \ t \in \mathbb{R}, \\ \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0, \end{cases} \tag{3.18}$$

where λ is a real parameter and $\widetilde{q} = q + 2\mu\omega$.

By a solution to the bvp (3.18), we mean a pair $(\lambda, u) \in \mathbb{R} \times W_2$ satisfying the differential equation in the bvp (3.18). Notice that $u \in W_2$ is a solution to the bvp (3.3) if and only if $(1, u)$ is a solution to the bvp (3.18). For this reason, we will study the bifurcation diagram of the bvp (3.18) and by means of Rabinowitz's global bifurcation theory, we will prove that the set of solutions to the bvp (3.18) consists in an infinity of unbounded components, each branching from a point on the line $\mathbb{R} \times \{0\}$ (see Lemma 3.15), joining a point on $\mathbb{R} \times \{\infty\}$ (see Lemma 3.16). Obviously, each component having the starting point and the arrival point oppositely located relatively to 1, carries a solution of the bvp (3.3) and Theorem 3.2 will be proved once we compute the number of such components. Thus, Theorem 3.2 is the consequence of the following Lemma 3.15, Lemma 3.16 and Lemma 3.17. First, let us introduce some notations. In all this section, we let

$$\widetilde{m}_0 = \mu\left(m_0 + 2\omega\right), \qquad \widetilde{m}_\infty = \mu\left(m_\infty + 2\omega\right),$$
$$g_0(s,u) = \mu\left(f(s,u) - m_0(s)\right), \qquad g_\infty(s,u) = \mu\left(f(s,u) - m_\infty(s)\right),$$

and for $\nu = 0$ or ∞, $L_\nu, T_\nu : W \to W$ are defined as follows:

$$\begin{aligned} L_\nu u(t) &= \mu \int_{-\infty}^{+\infty} G_{\widetilde{q}}(t,s)\widetilde{m}_\nu u ds, \\ T_\nu u(t) &= \mu \int_{-\infty}^{+\infty} G_{\widetilde{q}}(t,s) u g_\nu(s,u) ds. \end{aligned}$$

Lemma 3.15. *Assume that Hypotheses (3.4)-(3.6) hold, then from each $\mu_l(\widetilde{q}, \widetilde{m}_0)$ bifurcate two unbounded components of nontrivial solutions ζ_l^+ and ζ_l^-, such that $\zeta_l^\nu \subset \mathbb{R} \times S_l^\nu$.*

Proof .

It follows from Lemma 3.9 that solutions to the bvp (3.18) are those satisfying the fixed point equation

$$u = \lambda L_0 u + \lambda T_0(u). \tag{3.19}$$

Let us prove now, that all characteristic values of L_0 are of algebraic multiplicity one. To this aim, let $u \in N\left((\mu_k(\widetilde{q}, \widetilde{m}_0)L_0 - I)^2\right)$ and set $v = (\mu_k(\widetilde{q}, \widetilde{m}_0)L_0 u - u$. Then

$v \in N(\mu_k(\widetilde{q}, \widetilde{m}_0)L_0 - I) = \mathbb{R}\phi_k$ and $\mu_l(\widetilde{q}, \widetilde{m}_0)L_0 u - u = \eta\phi_k$ for some $\eta \in \mathbb{R}$. In another way, v satisfies the bvp

$$\begin{cases} -u'' + q(t)u = \mu_k(\widetilde{q}, \widetilde{m}_0)\widetilde{m}_0\,(t)\,u - \eta\mu_k(\widetilde{q}, \widetilde{m}_0)\widetilde{m}_0(t)\phi_k, \; t \in \mathbb{R}, \\ \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0. \end{cases}$$

Multiplying the differential equation in the above bvp by ϕ_k and integrating on $\mathbb{R}$, we obtain

$$\eta\mu_k(\widetilde{q}, \widetilde{m}_0) \int_{-\infty}^{+\infty} \widetilde{m}_0\phi_k^2 dt = 0,$$

leading to $\eta = 0$ and $u = \mu_k(\widetilde{q}, \widetilde{m}_0)L_0 u \in \mathbb{R}\phi_k$.

Now, we need to prove that $T_0(u) = \circ(\|u\|)$ near 0. Indeed, let $(u_n) \subset W$ with $\lim \|u_n\| = 0$. It follows from Hypothesis (3.6), that for $\epsilon > 0$ there exists $\delta > 0$ such that for all $u \in [-\delta, \delta]$ and $s \geq 0$, $|g_0\,(s, u)| \leq \epsilon$. Therefore, for n large enough

$$\frac{|T_0 u_n(t)|}{\|u_n\|} \leq \int_{-\infty}^{+\infty} G_{\widetilde{q}}(t,s)\,|g_0(s, u_n)|\,ds \leq \tilde{G}_{\widetilde{q},+\infty}\epsilon,$$

proving that $T_0(u) = \circ(\|u\|)$ near 0.

Let l_k be the projection of W on $\mathbb{R}\phi_k$, $\widetilde{W} = \{u \in W : l_k u = 0\}$ and let for $\xi > 0$, $\eta \in (0,1)$, $\nu = \pm$

$$K^{\nu}_{\xi,\eta} = \{(\lambda, u) \in \mathbb{R} \times W : |\lambda - \mu_k(\widetilde{q}, \widetilde{m}_0)| < \xi \text{ and } \nu l_k u > \eta\,\|u\|\}\,.$$

Since Lemma 3.9 guarantees that the operators L_0 and T_0 are respectively compact and completely continuous, we have from Theorem 1.40 and Theorem 1.25 in [52], that from $(\mu_k(\widetilde{q}, \widetilde{m}_0), 0)$ bifurcate two components ζ_k^+ and ζ_k^- of nontrivial solutions to Equation (3.19) such that there is $\zeta_0 > 0$, $\zeta_k^\nu \cap B(0,\zeta) \subset K^\nu_{\xi,\eta}$ for all $\zeta < \zeta_0$ and if $u = \alpha\phi_k + w \in \zeta_k^\nu$ then $|\lambda - \mu_k(\widetilde{q}, \widetilde{m}_0)| = \circ\,(1)$, $w = \circ\,(|\alpha|)$ for α near 0.

We claim that there is $\delta > 0$ such that $\zeta_k^\nu \cap B(0,\zeta) \subset \mathbb{R} \times S_k^\nu$; Indeed, let $(\lambda_n, u_n)_{n\geq 1} \subset \zeta_k^\nu$ be such that $\lim\,(\lambda_n, u_n) = (\mu_k(\widetilde{q}, \widetilde{m}_0), 0)$, we have then $\lim \lambda_n f(s, u_n(s)) = \mu_k(\widetilde{q}, \widetilde{m}_0)\widetilde{m}_0(s)$ and Lemma 3.10 guarantees that there is $n_0 \geq 1$ such that $u_n \in S_k$ for all $n \geq n_0$. Moreover, if $u_n = \alpha_n\phi_k + w_n$ then $\lim \frac{u_n}{\alpha_n} = \phi_k$ in W, proving that $\nu u_n(t) > 0$ for t in a right neighborhood of $-\infty$.

Also, if $(\lambda_*, u_*) \in \zeta_k^\nu$ then for all sequence $(\lambda_n, u_n)_{n\geq 1} \subset \zeta_k^\nu$ being such that $\lim\,(\lambda_n, u_n) = (\lambda_*, u_*)$, we have from Hypothesis (3.4) that $\lim \lambda_n f(s, u_n(s)) = \lambda_* f(s, u_*(s))$ in W and Lemma 3.10 guarantees existence of $n_0 \geq 1$ such that $u_n \in S_k$ for all $n \geq n_0$. This shows that $\zeta_k^\nu \subset \mathbb{R} \times S_k^\nu$ and ζ_k^ν is unbounded in $\mathbb{R} \times W$, ending the proof. ∎

Lemma 3.16. *Assume that Hypotheses (3.4)-(3.6) hold, then for all $k \geq 1$ and $\nu = \pm$, the component ζ_k^ν rejoins the point $(\mu_k(\widetilde{q}, \widetilde{m}_\infty), \infty)$.*

Proof .
First, let us prove that for all $k \geq 1$ and $\nu = \pm$, the projection of ζ_k^ν onto the real axis is bounded. Indeed, since 0 is the unique solution to the bvp

$$\begin{cases} -u'' + \widetilde{q}(t)u = 0, \ t \in \mathbb{R}, \\ \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0, \end{cases}$$

the projection of ζ_k^ν onto the real axis is contained in $(0, +\infty)$, namely, if $(\mu, u) \in \zeta_k^\nu$ then $\mu > 0$. Moreover, if $(\mu, u) \in \zeta_k^\nu$ then we read from the bvp (3.18) that $\mu = \mu_k(\widetilde{q}, f(\cdot, u(\cdot) + 2\omega)$, then taking in consideration Hypothesis (3.4), we obtain from Assertion 4 in Theorem 3.1 that $\mu = \mu_k(\widetilde{q}, f(\cdot, u(\cdot)) + 2\omega) \leq \mu_k(\widetilde{q}, \omega)$.

Now, let (μ_n, u_n) be sequence in ζ_k^ν with $\lim_{n\to+\infty} \|u_n\| = +\infty$ then $v_n = \frac{u_n}{\|u_n\|}$ satisfies

$$v_n = \lambda_n L_\infty v_n + \lambda_n \frac{T_\infty(u_n)}{\|u_n\|}. \tag{3.20}$$

Notice that Hypothesis (3.6) implies that $T_\infty(u) = \circ(\|u\|_\infty)$ at ∞. Combining this with the compactness of L_∞, we obtain from (3.20) existence of $v_+, v_- \in W$ with $\|v_+\| = \|v_-\| = 1$ such that $L_\infty v_+ = \mu_+ v_+$ and $L_\infty v_- = \mu_- v_-$ where $\mu_+ = \limsup \mu_n$ and $\mu_- = \liminf \mu_n$.

Consequently, we have $\mu_+ = \mu_{l_+}(\widetilde{q}, \widetilde{m}_\infty)$ and $\mu_- = \mu_{l_-}(\widetilde{q}, \widetilde{m}_\infty)$ for some integers l_+, l_- and since each of v_+ and v_- is a limit of a subsequence of $(v_n) \subset S_k^\nu$, we obtain $l_+ = l_- = k$ and $\mu_+ = \mu_- = \mu_k(\widetilde{q}, \widetilde{m}_\infty)$. ■

Lemma 3.17. *Assume that there exist two integers i, j with $1 \leq i \leq j$ such that one of the following situations holds*

$$\mu_i(q, m_0) < \mu < \mu_j(q, m_\infty) \text{ or } \mu_j(q, m_0) < \mu < \mu_i(q, m_\infty).$$

Then

$$\mu_i(\widetilde{q}, \widetilde{m}_0) < 1 < \mu_j(\widetilde{q}, \widetilde{m}_\infty) \text{ or } \mu_j(\widetilde{q}, \widetilde{m}_0) < 1 < \mu_i(\widetilde{q}, \widetilde{m}_\infty).$$

Proof .
Let $l \geq 1$ be an integer and $\kappa = 0, \infty$, we have to prove, $\mu_l(q, m_\kappa) < \mu$ implies $\mu_l(\widetilde{q}, \widetilde{m}_\kappa) < 1$ and $\mu_l(q, m_\kappa) > \mu$ implies $\mu_l(\widetilde{q}, \widetilde{m}_\kappa) > 1$. We present the proof of the implication: $\mu_l(q, m_\kappa) < \mu \Rightarrow \mu_l(\widetilde{q}, \widetilde{m}_\kappa) < 1$, the other is checked similarly. Let $\phi \in S_l$ and $\widetilde{\phi} \in S_l$

be respectively the eigenfunctions associated respectively with $\mu_l = \mu_l(q, m_\kappa)$ and $\widetilde{\mu}_l = \mu_l(\widetilde{q}, \widetilde{m}_\kappa)$ and let $(z_j)_{j=0}^{j=l}$ be the sequence of zeros of ϕ. Each of the pairs (μ, ϕ) and $(\widetilde{\mu}, \widetilde{\phi})$ satisfies

$$\begin{cases} -u'' + qu = \mu_l m_\kappa u, \ t \in \mathbb{R}, \\ \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0, \end{cases} \quad \text{and} \quad \begin{cases} -u'' + qu = (\widetilde{\mu}_l \mu m_\kappa + 2(\widetilde{\mu}_l - 1)\mu\omega)u, \ t \in \mathbb{R}, \\ \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0. \end{cases}$$

By the contrary, suppose that $\widetilde{\mu} \geq 1$, then we have

$$(\widetilde{\mu}_l m_\kappa + 2(\widetilde{\mu}_l - 1)\omega) - \mu_l m_\kappa = (\widetilde{\mu}_l \mu - \mu_l)\, m_\kappa + 2(\widetilde{\mu} - 1)\mu\omega > 0 \text{ a.e. } t \in \mathbb{R}.$$

Thus, applying Lemma 3.12 we get that in each interval (z_j, z_{j+1}), $j = 0, \ldots, l-1$, there is a zero of $\widetilde{\phi}$, contradicting $\widetilde{\phi} \in S_l$. This ends the proof. ■

Remark 3.18. Let q and c be two positive constants and notice that the solution of the ordinary differential equation $-u'' + qu = c$ is $\phi(t) = \frac{c}{q} + \alpha e^{-\sqrt{q}t} + \beta e^{\sqrt{q}t}$ where α, β are real numbers. Since $\lim_{t\to-\infty} \phi(t) = \lim_{t\to+\infty} \phi(t) = \frac{c}{q}$, the bvp

$$\begin{cases} -u'' + qu = c \text{ in } \mathbb{R}, \\ \lim_{t\to-\infty} u(t) = \lim_{t\to+\infty} u(t) = 0, \end{cases}$$

admits no solution. This shows that Hypothesis (3.4) is indispensable for existence of solutions.

Chapter 4

A class of Sturm-Liouville BVPs with an unintegrable weight

4.1 Introduction

Sturm-Liouville boundary value problems (bvp for short) have been the subject of hundreds of articles during the previous five decades, where existence and multiplicity of solutions have been investigated. Many of these articles concern existence of nodal solutions for second order differential equations subject to various boundary conditions; see, for example, [11], [12], [20], [31], [32], [33] [42], [41], [43], [44], [45], [46], [49], [50], [52], [53] [54], [55], [56], [57], [58] and references therein.

Nodal solutions appear as eigenfunctions to the half eigenvalue problem

$$\begin{cases} -u'' + qu = \sigma m u + \alpha u^+ - \beta u^- \text{ in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{4.1}$$

where σ is a real parameter, $q, m, \alpha, \beta \in C([0,1], \mathbb{R})$ and $m > 0$ in $[0,1]$.

To the author's knowledge, such a bvp has been studied for the first time in [11], where H. Berestycki introduced the concept of half-eigenvalue. He proved that the bvp (4.1) admits two increasing sequences of half-eigenvalues $(\sigma_k^+)_{k\geq 1}$ and $(\sigma_k^-)_{k\geq 1}$ such that $\vartheta_{k,\nu}$, the eigenfunction associated with σ_k^ν, admits exactly $(k-1)$ zeros in $(0,1)$, all are simple and $\nu\vartheta'_{k,\nu}(0) > 0$. The conditions $q, m, \alpha, \beta \in C([0,1], \mathbb{R})$ and $m > 0$ in $[0,1]$ have been relaxed in [8] to $q, m, \alpha, \beta \in L^1([0,1], \mathbb{R})$, $m \geq 0$ a.e. in $(0,1)$ $m > 0$ a.e. in a subinterval (ξ, η) of $[0,1]$. Notice that the concept of half-eigenvalue generalizes that of eigenvalue and for the role played by this notion, we refer the reader to [11], [14], [32], [55], [56], and [57].

In this chapter, we consider the case of the bvp (4.1) where $m, \alpha, \beta \in C([0,1], \mathbb{R})$, $m \geq 0$ in $(0,1)$, $m(t_0) > 0$ for some $t_0 \in [0,1]$, and $q \in C([0,1), \mathbb{R})$ with $\int_0^1 q(t)dt = +\infty$. Notice that the results obtained in [11] and in [8] do not cover such a situation. However, we prove in Section 3 that the Berysticki's result holds true for such a version of the bvp (4.1).

In Section 4, we investigate existence and multiplicity of nodal solutions to the bvp

$$\begin{cases} -u'' + qu = g(t,u) \text{ in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{4.2}$$

where $q \in C([0,1), \mathbb{R})$ with $\int_0^1 q(t)dt = +\infty$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. The nonlinearity g is supposed to be sublinear, assymptotically linear and superlinear. This interest is mainly motivated by that in [49], [46] , [45] and [44] where is considerd the version of the bvp (4.1) with $q = 0$ and the nonlinearity g is separable variable; Namely

$$\begin{cases} -u''(t) = a(t) g(u(t)), \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{4.3}$$

with $a : [0,1] \to [0,+\infty)$ is continuous and does not vanish identically and $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Let $g_0 = \lim_{s\to 0} g(s)/s$, $g_\infty = \lim_{|s|\to\infty} g(s)/s$ and $(\mu_k)_{k\geq 1}$ be the sequence of eigenvalues of the bvp

$$\begin{cases} -u''(t) = \mu a(t) u(t), \quad t \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

Authors of the paper [49] under the assumptions that

(A) $a > 0$ in $[0,1]$,

(B) a is continuously differentiable,

(C) $g(-s) = -g(s)$ for all $s \in \mathbb{R}$,

(D) $g(s)s > 0$ for all $s \neq 0$,

(E) g is locally Lipschitzian,

(F) in the case where $g_0 = \infty$, g is nondecreasing and $g(s)/s$ is nonincreasing on $(0, s_0]$ for some $s_0 > 0$,

proved by means of a shooting method, that if for some integer k, $\lambda_k < g(s)/s < \lambda_{k+1}$ for all $s \neq 0$, then except the trivial function the bvp (4.3) has no solution and if $g_0 < \lambda_k < g_\infty$ or $g_\infty < \lambda_k < g_0$, then the bvp (4.3) has a solution having exactly $k-1$ zeros in $(0,1)$, all are simple.

In [45], R. Ma and B. Thompson improved the existence result in [49]. Just under Hypotheses **(A)** and **(D),** they proved that if $0 < g_0 < \lambda_k < g_\infty < \infty$ or $0 < g_\infty < \lambda_k < g_0 < \infty$, then the bvp (4.3) has two solution u_+ and u_-, each having exactly $k-1$ zeros in $(0,1)$, all are simple and for $\nu = +$ or $-$, $\nu u'_\nu(0) > 0$. In [46], where Hypothesis **(A)** relaxed to:

(A') $a \geq 0$ in $[0,1]$ and does not vanish identically on any subinterval of $[0,1]$,

they obtained the same result.

As it is mentioned in [45], we conclude from the above result that if Hypotheses **(A')** and **(D)** hold and if there are integers k, i such that $0 < g_0 < \lambda_k \leq \lambda_{k+i} < g_\infty < \infty$ or $0 < g_\infty < \lambda_k \leq \lambda_{k+i} < g_0 < \infty$, then for each $j \in \{0, 1, ..i\}$ the bvp (4.3) has two solutions $u_{+,j}$ and u_-, j, each having exactly $k+j-1$ zeros in $(0,1)$, all are simple and for $\nu = +$ or $-$, $\nu u'_{\nu,j}(0) > 0$.

In [44], authors consider the cases where the nonlinearity f is superlinear and sublinear. They proved that if Hypotheses **(A)**, **(D)** hold and $g_0 = 0$, $g_\infty = \infty$ or Hypotheses **(A)**, **(D)**, **(F)** hold and $g_\infty = 0$, then for each $j \in \mathbb{N} = \{1, ...\}$ the bvp (4.3) has two solution $u_{j,+}$ and $u_{j,-}$, each having exactly $j-1$ zeros in $(0,1)$, all are simple and for $\nu = +$ or $-$, $\nu u'_{j,\nu}(0) > 0$.

Main results of Section 4 concern nodal solutions to the bvp (4.2) in the cases where the nonlinearity g is respectively asymptotically linear, superlinear and sublinear. All are obtained by means of the global bifurcation theory due to P. H. Rabinowitz and they provide existence and multiplicity of nodal solutions with less conditions relatively to that obtained in the above cited papers.

4.2 Preliminaries

4.2.1 General setting

Statements of main results in this chapter need to introduce some notations: in what follows, we let

$$
\begin{aligned}
&E = C\left([0,1],\mathbb{R}\right), \quad E^+ = \{m \in E : m \geq 0 \text{ in } [0,1]\},\\
&\Gamma^+ = \{m \in E^+ : m > 0 \text{ in a subinterval of } [0,1]\},\\
&\Gamma^{++} = \{m \in \Gamma^+ : m > 0 \text{ in } [0,1]\},\\
&Q = \left\{q \in C([0,1),\mathbb{R}) : \textstyle\int_0^1 q(s)ds = +\infty\right\},\\
&Q^+ = \{q \in Q : q(t) \geq 0 \text{ for all } t \in (0,1)\},\\
&Q_\# = \left\{q \in Q : \textstyle\int_0^1 (1-s)q(s)ds < \infty\right\},\\
&W = \{u \in C([0,1),\mathbb{R}) : u(0) = \lim_{t\to 1} u(t) = 0\},\\
&C_b^1\left([0,1),\mathbb{R}\right) = \left\{u \in C^1\left([0,1),\mathbb{R}\right) : \sup_{t\in[0,1)} \left|u'\left(t\right)\right| < \infty\right\}\\
&W^1 = W \cap C_b^1\left([0,1),\mathbb{R}\right), \quad W^2 = W^1 \cap C^2\left([0,1),\mathbb{R}\right).
\end{aligned}
$$

The linear spaces W and W_1 are respectively equipped with the norms $\|\cdot\|$ and $\|\cdot\|_1$ defined by $\|u\| = \sup_{t\in[0,1]} |u(t)|$ and $\|u\|_1 = \sup_{t\in[0,1]} |u'(t)|$. Obviously, $(W,\|\cdot\|)$ and $(W^1,\|\cdot\|_1)$ are Banach spaces.

For an integer $k \geq 1$, S_k^+ denotes the set of all the functions u in W^1 having exactly $(k-1)$ zeros in $(0,1)$, all are simple and u is positive in a right neighbourhood of 0, $S_k^- = -\,S_k^+$ and $S_k = S_k^+ \cup S_k^-$. For $u \in S_k$, $\left(z_j\right)_{j=0}^{j=k}$ with $0 = z_0 < z_1 < ... < z_k = 1$ and $u\left(z_j\right) = 0$ for $j = 1,...,k-1$, is said to be the sequence of zeros of u.

Throughout this paper, for $q \in Q$ the operator $\mathcal{L}_q : C^2\left([0,1),\mathbb{R}\right) \to C\left([0,1),\mathbb{R}\right)$ is defined by $\mathcal{L}_q u = -u'' + qu$.

For $\nu = +$ or $-$, let $I^\nu : W \to W$ be the operator defined for $u \in W$ by $I^\nu u(x) = \max(\nu u(x), 0) = u^\nu(x)$. We have for all $u \in W$

$$u = I^+u - I^-u \quad \text{and} \quad |u| = I^+u + I^-u.$$

This implies that, for all $u, v \in W$,

$$
\begin{aligned}
|I^+u - I^+v| &\leq \left(\tfrac{|u-v|}{2} + \tfrac{||u|-|v||}{2}\right) \leq |u-v|\,,\\
|I^-u - I^-v| &\leq \left(\tfrac{|u-v|}{2} + \tfrac{||u|-|v||}{2}\right) \leq |u-v|,
\end{aligned}
\tag{4.4}
$$

and the operators I^+, I^- are continuous.

4.2.2 The Green's function and fixed point formulation

In all what follows, we let for $q \in Q^+$, Ψ_q be the unique solution of the initial value problem

$$\begin{cases} \mathcal{L}_q u = 0, \\ u(0) = 0,\ u'(0) = 1. \end{cases}$$

Lemma 4.1. *For all $q \in Q^+$, the function ψ_q has the following properties:*

i) $\Psi_q(t) > 0$, $\Psi_q'(t) > 0$ *and* $\Psi_q''(t) \geq 0$ *for all* $t \in (0,1]$.

ii) $\lim_{t\to 1} \Psi_q'(t) = +\infty$.

iii) *The function* Ψ_q/Ψ_q' *is bounded at* $t = 1$.

iv) $\lim_{t\to 0} \Psi_q(t) \int_t^1 \frac{ds}{\Psi_q^2(s)} = 1$.

v) $\lim_{t\to 1} \Psi_q(t) \int_t^1 \frac{ds}{\Psi_q^2(s)} = 0$.

vi) *If $q \in Q_\#$ then* $\Psi_q(1) = \lim_{t\to 1} \Psi_q(t) < \infty$.

Proof .

Let $q \in Q^+$ and let $a \in (0,1)$ be such that $\varrho = \inf_{s\in(a,1)} \Psi_q(s) > 0$.

i) Suppose on the contrary that $\Psi_q'(t_0) = 0$ for some t_0 on $(0,1)$. By the boundary condition $\Psi_q'(0) = 1$, $t_0 > 0$ we assume that $\Psi_q'(t) > 0$ on $[0, t_0)$. Thus ψ_q is strictly increasing on $[0, t_0)$. On the other hand we have from the equation $-u''(t) + q(t)u(t) = 0$ that $\Psi_q''(t_0) = q(t)\Psi_q(t_0) > 0$, and accordingly t_0 is a minimum value point. This is a contradiction, then $\Psi_q > 0$ thus $\Psi_q''(t) \geq 0$ and since $\Psi_q'(0) = 1$, we obtain $\Psi_q' > 0$.

ii) We have for all $t \in (a,1)$

$$\begin{aligned} \Psi_q'(t) &= \left(\Psi_q'(a) + \int_a^t \Psi_q''(s)\,ds\right) \\ &= \left(\Psi_q'(a) + \int_a^t q(s)\Psi_q(s)\,ds\right) \\ &\geq \left(\Psi_q'(a) + \varrho\int_a^t q(s)\,ds\right) \end{aligned}$$

leading to $\lim_{t\to 1} \Psi_q'(t) = +\infty$.

iii) We have for all $t \geq a$

$$\begin{aligned}\left(\Psi_q'(t)\right)^2-\left(\Psi_q'(a)\right)^2 &= 2\int_a^t \Psi_q''(s)\Psi_q'(s)ds = 2\int_a^t q(s)\Psi_q(s)\Psi_q'(s)ds \\ &\geq \varrho\left(\left(\Psi_q(t)\right)^2-\left(\Psi_q(a)\right)^2\right),\end{aligned}$$

leading to

$$\left(\Psi_q(t)/\Psi_q'(t)\right)^2 \leq \frac{1}{\varrho}+\left(\Psi_q'(a)/\Psi_q'(t)\right)^2 \text{ for all } t \geq a.$$

Hence, we deduce from Assertion **ii)**, existence of $a_* \in (a,1)$ such that

$$\Psi_q(t)/\Psi_q'(t) \leq \sqrt{\frac{2}{\varrho}} \text{ for all } t \geq a_*.$$

iv) By means of L'Hopital's rule we obtain

$$\lim_{t\to 0}\Psi_q(t)\int_t^1 \frac{ds}{\Psi_q^2(s)} = \lim_{t\to 0}\frac{\int_t^1 \Psi_q^{-2}ds}{\left(\Psi_q(t)\right)^{-1}} = \lim_{t\to 0}\frac{1}{\Psi_q'(t)} = 1.$$

v) Again by means of L'Hopital's rule we obtain

$$\lim_{t\to 1}\Psi_q(t)\int_t^1 \frac{ds}{\Psi_q^2(s)} = \lim_{t\to 1}\frac{1}{\Psi_q'(t)} = 0.$$

vi) First, notice that if $q \in Q_\#$ then for all $t \in (a,1)$

$$\begin{aligned}\int_a^t\int_a^s q(\tau)\,d\tau ds &\leq \int_0^t\int_0^s q(\tau)\,d\tau ds \\ &= -(1-t)\int_0^t q(s)\,ds+\int_0^t (1-s)q(s)\,ds \\ &\leq 2\int_0^1 (1-s)q(s)\,ds.\end{aligned}$$

Then, for all $s \in (a,1)$

$$\begin{aligned}\Psi_q'(s) &= \left(\Psi_q'(a)+\int_a^t \Psi_q''(s)\,ds\right) = \left(\Psi_q'(a)+\int_a^s q(s)\Psi_q(s)\,ds\right) \\ &\leq \left(\Psi_q'(a)+\Psi_q(s)\int_a^s q(\tau)\,d\tau\right)\end{aligned}$$

leading to

$$\frac{\Psi_q'(s)}{\Psi_q(s)} \leq \frac{\Psi_q'(a)}{\Psi_q(s)}+\int_a^s q(\tau)\,d\tau \leq \frac{\Psi_q'(a)}{\Psi_q(a)}+\int_a^s q(\tau)\,d\tau.$$

Integrating on (a,t), we obtain

$$\ln\left(\frac{\Psi_q(t)}{\Psi_q(a)}\right) \leq \frac{\Psi_q'(a)}{\Psi_q(a)}+\int_a^t\int_a^s q(\tau)\,d\tau ds \leq \frac{\Psi_q'(a)}{\Psi_q(a)}+2\int_0^1 (1-s)q(s)\,ds,$$

leading to

$$\Psi_q(t) \leq \Psi_q(a) \exp\left(\frac{\Psi_q'(a)}{\Psi_q(a)} + 2\int_0^1 (1-s)q(s)\, ds\right).$$

As Ψ_q is increasing, we have $\Psi_q(1) = \lim_{t\to 1} \Psi_q(t) < +\infty$.

The proof of Lemma 4.1 is complete. ■

Because of Properties (ii), (iii), (iv) and (v) in Lemma 4.1, the function

$$\Phi_q(t) = \begin{cases} 1, & \text{if} \quad t = 0, \\ \Psi_q(t) \int_t^1 \frac{ds}{\Psi_q^2(s)}, & \text{if} \quad t \in (0,1), \\ 0, & \text{if} \quad t = 1, \end{cases} \tag{4.5}$$

is well defined and it is the unique solution of the bvp

$$\begin{cases} \mathcal{L}_q u = 0, \text{ in } (0,1), \\ u(0) = 1, \lim_{t\to 1} u(t) = 0. \end{cases}$$

Lemma 4.2. *For all $q \in Q^+$, the function Φ_q has the following properties:*

a) $\Phi_q(t) > 0$, $\Phi_q'(t) < 0$ *and* $\Phi_q''(t) \geq 0$ *for all* $t \in (0,1)$,

b) *For all* $t \in [0,1]$, $\Phi_q(t)\Psi_q'(t) - \Psi_q(t)\Phi_q'(t) = 1$,

c) *The function* Φ_q/Φ_q' *is bounded at* 1.

Proof .

Let $q \in Q^+$ and $a \in (0,1)$ be such that $\alpha = \inf_{t\geq a} q(t) > 0$.

a) We have respectively from (4.5) and $\Phi_q'' = q\Phi_q$, that $\Phi_q(t) > 0$ and $\Phi_q''(t) \geq 0$ for all $t \in (0,1)$. Since the function Ψ_q' is increasing, we obtain from (4.5) that for all $t \in (0,1)$,

$$\Phi_q'(t) = \Psi_q'(t)\int_t^1 \frac{ds}{\Psi_q^2} - \frac{1}{\Psi_q(t)} < \int_t^1 \frac{\Psi_q'}{\Psi_q^2} ds - \frac{1}{\Psi_q(t)} < -\frac{1}{\lim_{t\to 1}\Psi_q(t)} \leq 0.$$

b) We have from (4.5) that for all $t \in [0,1]$

$$\Phi_q(t)\Psi_q'(t) - \Psi_q(t)\Phi_q'(t) = \Psi_q(t)\Psi_q'(t)\int_t^1 \frac{ds}{\Psi_q^2} - \Psi_q(t)\left(\Psi_q'(t)\int_t^1 \frac{ds}{\Psi_q^2} - \frac{1}{\Psi_q(t)}\right) = 1.$$

c) We have for $t \geq a$:

$$\begin{aligned} \left(-\Phi_q'(t)\right)^2 &= 2\int_t^1 \Phi_q''(s)\left(-\Phi_q'(s)\right) ds = \int_t^1 q(s)\Phi_q(s)\left(-\Phi_q'(s)\right) ds \\ &\geq \alpha\left(\Phi_q(t)\right)^2. \end{aligned}$$

This leads to

$$\left|\Phi_q(t)/\Phi_q'(t)\right|^2 = \left(\Phi_q(t)/-\Phi_q'(t)\right)^2 \leq \frac{1}{\alpha} \text{ for all } t \geq T,$$

and so,

$$\sup_{t\geq T}\left|\Phi_q(t)/\Phi_q'(t)\right| \leq \frac{1}{\sqrt{\alpha}}.$$

The proof of Lemma 4.2 is complete. ■

Set for $q \in Q^+$ and $0 \leq \theta < \eta < 1$

$$\begin{aligned}
\Psi_{q,\theta}(t) &= \Phi_q(\theta)\Psi_q(t) - \Psi_q(\theta)\Phi_q(t),\\
\Phi_{q,\theta,\eta}(t) &= \frac{\Psi_q(\eta)\Phi_q(t) - \Phi_q(\eta)\Psi_q(t)}{\Psi_{q,\theta}(\eta)},\\
\Phi_{q,\theta}(t) &= \lim_{\eta\to 1}\Phi_{q,\theta,\eta}(t) = \frac{\Phi_q(t)}{\Phi_q(\theta)},\\
G_q(\theta,\eta,t,s) &= \begin{cases} 0, \text{ if } \min(t,s) \leq \theta,\\ \Phi_{q,\theta,\eta}(s)\Psi_{q,\theta}(t), \text{ if } \theta \leq t \leq s \leq \eta,\\ \Phi_{q,\theta}(t)\Psi_{q,\theta}(s), \text{ if } \theta \leq s \leq t \leq \eta\\ 0 \text{ if } \min(t,s) \geq \eta,\end{cases}\\
G_q(\theta,t,s) &= \lim_{\eta\to 1} G_q(\theta,\eta,t,s) = \begin{cases} 0, \text{ if } \min(t,s) \leq \theta,\\ \Phi_{q,\theta}(s)\Psi_{q,\theta}(t), \text{ if } \theta \leq t \leq s,\\ \Phi_{q,\theta}(t)\Psi_{q,\theta}(s), \text{ if } \theta \leq s \leq t.\end{cases}
\end{aligned}$$

$$\begin{aligned}
\Phi_{q,\theta}(t) &= \frac{\Phi_q(t)}{\Phi_q(\theta)}, \Psi_{q,\theta}(t) = \Phi_q(\theta)\Psi_q(t) - \Psi_q(\theta)\Phi_q(t) \text{ and}\\
G_q(\theta,t,s) &= \begin{cases} 0, \text{ if } \min(t,s) \leq \theta,\\ \Phi_{q,\theta}(s)\Psi_{q,\theta}(t), \text{ if } \theta \leq t \leq s,\\ \Phi_{q,\theta}(t)\Psi_{q,\theta}(s), \text{ if } \theta \leq s \leq t.\end{cases}
\end{aligned}$$

We have then for all $q \in Q^+$ and all $\theta, \eta \in [0,1]$

$$\Phi_{q,\theta,\eta}\Psi_{q,\theta}' - \Phi_{q,\theta,\eta}'\Psi_{q,\theta} = \Phi_{q,\theta}\Psi_{q,\theta}' - \Phi_{q,\theta}'\Psi_{q,\theta} = 1 \text{ in } [0,1] \tag{4.6}$$

and

$$G_q(\theta,t,s) = G_q(t,s) - \frac{\Psi_q(\theta)}{\Phi_q(\theta)}\Phi_q(s)\Phi_q(t) \text{ for } t,s \geq \theta.$$

where

$$G_q(t,s) = G_q(0,t,s) = \begin{cases} \Phi_q(t)\Psi_q(s), \text{ if } 0 \leq t \leq s < 1,\\ \Phi_q(s)\Psi_q(t), \text{ if } 0 \leq s \leq t < 1.\end{cases} \tag{4.7}$$

Lemma 4.3. *We have for all $q \in Q^+$ and $\theta, \eta \in [0,1)$ with $\theta < \eta$:*

1. $G_q(\theta, \eta, t, s) \leq G_q(\theta, \eta, s, s)$ *for all* $t, s \in [\theta, \eta]$,
2. $G_q(\theta, t, s) \leq G_q(\theta, s, s)$ *for all* $t, s \in [\theta, 1]$,
3. $G_q(\theta, \eta, t, s) \geq \rho_{\theta,\eta}(t)\, G_q(\theta, \eta, s, s)$ *for all* $t, s \in [\theta, \eta]$ *where* $\rho_{\theta,\eta}(t) = \min(t-\theta, \eta - t) / \Psi_{q,\theta}(\eta)$. *Moreover, if* $q \in Q_\#$ *then* $\Psi_q(1) = \lim_{t\to 1} \Psi_q(t) < \infty$ *and* $G_q(\theta, \eta, t, s) \geq \rho^*_{\theta,\eta}(t)\, G_q(\theta, \eta, s, s)$ *for all* $t, s \in [\theta, \eta]$ *where* $\rho^*_{\theta,\eta}(t) = \min(t-\theta, \eta - t) / \Psi_{q,\theta}(1)$.

Proof .

Assertions 1 and 2 are obtained from the monotonicity of the functions $\Phi_{q,\theta,\eta}$, $\Phi_{q,\theta}$ and $\Psi_{q,\theta}$. We have

$$\begin{aligned} \frac{G_q(\theta,\eta,t,s)}{G_q(\theta,\eta,s,s)} &= \begin{cases} \frac{\Psi_{q,\theta}(t)}{\Psi_{q,\theta}(s)}, & \text{if } \theta \leq t \leq s \leq \eta, \\ \frac{\Phi_{q,\theta,\eta}(t)}{\Phi_{q,\theta,\eta}(s)}, & \text{if } \theta \leq s \leq t \leq \eta, \end{cases} \\ &\geq \begin{cases} \frac{\Psi_{q,\theta}(t)}{\Psi_{q,\theta}(\eta)}, & \text{if } \theta \leq t \leq s \leq \eta, \\ \Phi_{q,\theta,\eta}(t), & \text{if } \theta \leq s \leq t \leq \eta. \end{cases} \end{aligned} \tag{4.8}$$

Since

$$\Psi_{q,\theta}(t) = \int_\theta^t \Psi'_{q,\theta}(s)\, ds \geq \int_\theta^t \Psi'_{q,\theta}(\theta)\, ds = t - \theta$$

and

$$\Phi_{q,\theta,\eta}(t) = \int_t^\eta \left(-\Phi'_{q,\theta,\eta}(s)\right) ds \geq \int_t^\eta \left(-\Phi'_{q,\theta,\eta}(\eta)\right) ds = \frac{\eta - t}{\Psi_{q,\theta}(\eta)},$$

we obtain from (4.8),

$$\frac{G_q(\theta,\eta,t,s)}{G_q(\theta,\eta,s,s)} \geq \begin{cases} \frac{t-\theta}{\Psi_{q,\theta}(\eta)}, & \text{if } \theta \leq t \leq s \leq \eta, \\ \frac{\eta-t}{\Psi_{q,\theta}(\eta)}, & \text{if } \theta \leq s \leq t \leq \eta. \end{cases} \geq \rho_{\theta,\eta}(t).$$

This ends the proof. ∎

Lemma 4.4. *We have for all $q \in Q^+$*

i) $\overline{G}_q = \sup_{s,t\in[0,1]} G_q(t,s) = \sup_{0\leq t\leq 1} \Phi_q(t)\, \Psi_q(t) < \infty,$

ii) $\widetilde{G}_q = \sup_{\theta,t,s\in[0,1]} G_q(\theta,t,s) < \infty.$

Proof .

Let $q \in Q^+$ and $T \in (0,1)$ be such that $\alpha = \inf_{t \geq T} q(t) > 0$.

i) Taking in consideration that Ψ_q is increasing, we obtain from (4.5), that for all $t, s \in [0,1]$

$$\begin{aligned} G_q(t,s) \leq \Phi_q(t)\Psi_q(t) &= \left(\frac{\Psi_q(t)}{\Psi_q'(t)}\right)\left(\Psi_q(t)\Psi_q'(t)\int_t^1 \frac{ds}{\Psi_q^2(s)}\right) \\ &\leq \left(\frac{\Psi_q(t)}{\Psi_q'(t)}\right)\left(\Psi_q(t)\int_t^1 \frac{\Psi_q'(s)ds}{\Psi_q^2(s)}\right) \leq \left(\frac{\Psi_q(t)}{\Psi_q'(t)}\right). \end{aligned}$$

This together with **iii)** in lemma 4.1 leads to

$$\overline{G}_q = \sup_{t,s\in[0,1]} G_q(t,s) \leq \sup_{t\in[0,1]} \Phi_q(t)\Psi_q(t) < \infty.$$

ii) Because of Φ_q is decreasing and Ψ_q is increasing we have for all $s, t \geq \theta$

$$\begin{aligned} 0 &\leq G_q(\theta,t,s) \leq \Phi_q(t)\Psi_q(t) + \frac{\Psi_q(\theta)}{\Phi_q(\theta)}\Phi_q(t)\Phi_q(s) \\ &\leq \Phi_q(t)\Psi_q(t) + \Psi_q(\theta)\Phi_q(\theta) \\ &\leq 2\sup_{t\in[0,1]} \Phi_q(t)\Psi_q(t) < \infty. \end{aligned}$$

The proof of Lemma 4.4 is complete. ∎

Lemma 4.5. *For all $q \in Q^+$, $\theta \in [0,1)$ and $h \in W$, $L_{q,\theta}h(t) = \int_0^1 G_q(\theta,t,s)h(s)ds$ is the unique solution in $(\theta,1)$ to the bvp:*

$$\begin{cases} \mathcal{L}_q u = h(t), \quad \theta < t < 1, \\ u(\theta) = \lim_{t\to 1} u(t) = 0 \end{cases}$$

and the operator $L_{q,\theta} : W \to W^1$ is continuous. Moreover, if $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $F(0,0) = F(1,0) = 0$, then the operator $T_{q,\theta} : W \to W$ defined for $v \in W$ by

$$T_{q,\theta}u\,(t) = \int_0^1 G_q(\theta,t,s)F(s,v(s))ds$$

is completely continuous and $u \in W$ is a fixed point of $T_{q,\theta}$ if and only if u is a solution to the bvp

$$\begin{cases} \mathcal{L}_q v = F(t,v(t)), \quad \theta < t < 1, \\ v(\theta) = v(1) = 0. \end{cases}$$

Proof .

Let $h \in W$ and set $H(t) = L_{q,\theta}h(t)$. We have

$$H(\theta) = \int_0^1 G(\theta, \theta, s)h(s)ds = 0$$

and differentiating twice in the relation

$$H(t) = \int_0^1 G(\theta, t, s)h(s)ds = \Phi_{q,\theta}(t) \int_\theta^t \Psi_{q,\theta}(s) h(s)ds + \Psi_{q,\theta}(t) \int_t^1 \Phi_{q,\theta}(s) h(s)ds$$

we obtain

$$H''(t) = q(t)H(t) + \left(\Phi'_{q,\theta}(t) \Psi_{q,\theta}(t) - \Phi_{q,\theta}(t) \Psi'_{q,\theta}(t)\right) h(t) \text{ for all } t \geq \theta.$$

This together with (4.6) lead to

$$\mathcal{L}_q H(t) = h(t) \text{ for all } t \geq \theta.$$

We have for all $t > \theta$:

$$H(t) = \Phi_q(t) \int_\theta^t \Psi_q(s)h(s)ds + \Psi_q(t) \int_t^1 \Phi_q(s)h(s)ds - \frac{\Psi_q(\theta)}{\Phi_q(\theta)} \Phi_q(t) \int_\theta^1 \Phi_q(s)h(s)ds.$$

Let us prove that $\lim_{t\to 1} H(t) = 0$. Clearly, if $\int_\theta^1 \Psi_q(s)h(s)ds < \infty$ then $\lim_{t\to 1} \Phi_q(t) \int_\theta^t \Psi_q(s)h(s)ds = 0$ and if $\int_\theta^1 \Psi_q(s)h(s)ds = \infty$ then taking in consideration Assertions **d)** in lemma 4.2, **i)** of Lemma (4.4) and $\lim_{t\to 1} h(t) = 0$, we obtain by means of the L'Hopital's rule

$$\begin{aligned} \lim_{t\to 1} \Phi_q(t) \int_\theta^t \Psi_q(s)h(s)ds &= \lim_{t\to 1} \frac{\int_\theta^t \Psi_q(s)h(s)ds}{\left(\Phi_q(t)\right)^{-1}} \\ &= \lim_{t\to 1} \left(\frac{\Phi_q(t)}{-\Phi'_q(t)}\right) \left(\Phi_q(t)\Psi_q(t)\right) h(t) = 0. \end{aligned}$$

Similarly, if $\lim_{t\to 1} \Psi_q(t) < \infty$ then $\lim_{t\to 1} \Psi_q(t) \int_t^1 \Phi_q(s)h(s)ds = 0$ and if $\lim_{t\to 1} \Psi_q(t) = +\infty$ then taking in consideration **iii)** in lemma 4.1, **i)** of Lemma 4.4 and $\lim_{t\to 1} h(t) = 0$, we obtain by means of the L'Hopital's rule

$$\begin{aligned} \lim_{t\to 1} \Psi_q(t) \int_t^1 \Phi_q(s)h(s)ds &= \lim_{t\to 1} \frac{\int_t^1 \Phi_q(s)h(s)ds}{\left(\Psi_q(t)\right)^{-1}} \\ &= \lim_{t\to 1} \left(\frac{\Psi_q(t)}{\Psi'_q(t)}\right) \Phi_q(t)\Psi_q(t)h(t) = 0. \end{aligned}$$

Now, for any $h \in W$, we have

$$\left\|L_{q,\theta}h\right\| = \sup_{t\in[0,1]} \left|L_{q,\theta}h(t)\right| = \sup_{t\in[0,1]} \left|\int_0^1 G(\theta, t, s)h(s)ds\right| \leq G_q \|h\|$$

and taking in consideration (4.6) we obtain

$$\begin{aligned}\left\|\left(L_{q,\theta}h\right)'\right\| &= \sup_{t\in(0,1)}\left|\left(L_{q,\theta}h\right)'(t)\right| = \sup_{t\in(0,1)}\left|\Phi'_{q,\theta}(t)\int_\theta^t \Psi_{q,\theta}(s)\,h(s)ds + \Psi'_{q,\theta}(t)\int_t^1 \Phi_{q,\theta}(s)\,h(s)ds\right| \\ &\leq \sup_{t\in(0,1)}\left(-\Phi'_{q,\theta}(t)\int_\theta^t \Psi_{q,\theta}(s)\,|h(s)|\,ds + \Psi'_{q,\theta}(t)\int_t^1 \Phi_{q,\theta}(s)\,|h(s)|\,ds\right) \\ &\leq \sup_{t\in(0,1)}\left(-\Phi'_{q,\theta}(t)\,\Psi_{q,\theta}(t)\int_\theta^t ds + \Psi'_{q,\theta}(t)\,\Phi_{q,\theta}(t)\int_t^1 ds\right)\|h\| \\ &\leq \|h\|.\end{aligned}$$

The above estimates prove that the operator $L_{q,\theta}$ is well defined and is continuous.

Now, We proof that $T_{q,\theta}$ is completely continuous. Notice that $T_{q,\theta} = \mathbb{I}\circ L_{q,\theta}\circ\mathbb{F}$ where $\mathbb{F} : W \to W$ is defined by $\mathbb{F}u(t) = F(t, u(t))$ and $\mathbb{I}$ is the compact embeding of W^1 in W. Because that the mapping $\mathbb{F}$ is continuous and bounded, the operator $T_{q,\theta}$ is completely continuous.

At the end, if u is a fixed point of $T_{q,\theta}$ and $h = \mathbb{F}u$, then $u = L_{q,\theta}h$ and

$$\begin{cases} \mathcal{L}_q u(t) = h(t) = F(t, v(t)),\ \theta < t < 1, \\ u(\theta) = \lim_{t\to 1} u(t) = 0. \end{cases}$$

■

In the reminder of this chapter, for $q \in Q^+$ and $m \in E$, we let $L_{q,m}, L^+_{q,m}, L^-_{q,m} : W \to W$ the operators defined by

$$\begin{aligned} L_{q,m}u(t) &= \textstyle\int_0^1 G_q(t,s)m(s)u(s)ds, \\ L^+_{q,m}u(t) &= \left(L_{q,m}\circ I^+\right)u(t) = L_{q,m}u^+(t), \\ L^-_{q,m}u(t) &= \left(L_{q,m}\circ I^-\right)u(t) = L_{q,m}u^-(t). \end{aligned}$$

It follows from Lemma 4.5 that $L_{q,m}$ is compact and for $\nu = +$ or $-$, $L^\nu_{q,m}$ is completely continuous.

4.2.3 Comparison results

The following three lemmas will play important roles in this work.

Lemma 4.6 ([8]). *Let j and k be two integers such that $j \geq k \geq 2$ and let $(\xi_l)_{l=0}^{l=k}$, $(\eta_l)_{l=0}^{l=j}$ be two families of real numbers such that*

$$\xi_0 = \xi < \xi_1 < \xi_2 < \cdots < \xi_{k-1} < \xi_k = \eta,$$

$$\eta_0 = \xi < \eta_1 < \eta_2 < \cdots < \eta_{j-1} < \eta_j = \eta.$$

If $\xi_1 < \eta_1$, then there exist two integers m and n having the same parity, $1 \le m \le k-1$ and $1 \le n \le j-1$ such that

$$\xi_m < \eta_n \le \eta_{n+1} \le \xi_{m+1}.$$

Lemma 4.7. *For $i = 1, 2$ let $\phi_i \in S_\rho^{k_i,\nu} \cap W_2$ having a sequence of zeros $\left(z_j^i\right)_{j=0}^{j=k_i}$. If for some integers m, n with $m \le k_1 - 1$ we have $n \le k_2 - 1$ $z_m^1 \le z_n^2 < z_{n+1}^2 \le z_{m+1}^1$ and $\phi_1\phi_2 > 0$, then*

$$\int_{z_n^2}^{z_{n+1}^2} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 \begin{cases} > 0, \text{ if } z_m^1 < z_n^2 \text{ or } z_{n+1}^2 < z_{m+1}^1, \\ = 0, \text{ if } z_m^1 = z_n^2 < z_{n+1}^2 = z_{m+1}^1. \end{cases}$$

Proof .

Let $Wr = \phi_1\phi_2' - \phi_2\phi_1'$ be the Wronksian of ϕ_1 and ϕ_2 and without loss of generality, suppose that $\phi_1, \phi_2 > 0$ in $\left(z_n^2, z_{n+1}^2\right)$. We have then $Wr(0) = \lim_{t\to 1} Wr(t) = 0$ and

$$\int_{z_n^2}^{z_{n+1}^2} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 = Wr\left(z_n^2\right) - \lim_{t \to z_{n+1}^2} Wr\left(t\right).$$

Therefore, we distinguish the following cases:

i) $z_m^1 \le z_n^2 < z_{n+1}^2 = z_{m+1}^1$: In this case we have

$$\phi_1\left(z_n^2\right) = \phi_2\left(z_m^1\right) = \phi_1\left(z_{n+1}^2\right) = \phi_1\left(z_{m+1}^2\right) = 0,$$

leading to

$$\begin{aligned} \int_{z_n^2}^{z_{n+1}^2} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 &= Wr\left(z_n^2\right) - \lim_{t \to z_{n+1}^2} Wr\left(t\right) \\ &= Wr\left(z_m^1\right) - \lim_{t \to z_{m+1}^2} Wr\left(t\right) = 0. \end{aligned}$$

ii) $z_m^1 \le z_n^2 < z_{n+1}^2 < z_{m+1}^1$: In this case we have

$$z_{n+1}^2 < 1,\ \phi_1\left(z_{n+1}^2\right) > 0,\ \phi_2\left(z_n^2\right) = \phi_2\left(z_{n+1}^2\right) = 0,\ \phi_1\left(z_{n+1}^2\right) > 0 \text{ and } \phi_2'\left(z_{n+1}^2\right) < 0,$$

leading to

$$\begin{aligned} \int_{z_n^2}^{z_{n+1}^2} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 &= Wr\left(z_n^2\right) - Wr\left(z_{n+1}^2\right) \\ &\ge -Wr\left(z_{n+1}^2\right) = -\phi_1\left(z_{n+1}^2\right)\phi_2'\left(z_{n+1}^2\right) > 0. \end{aligned}$$

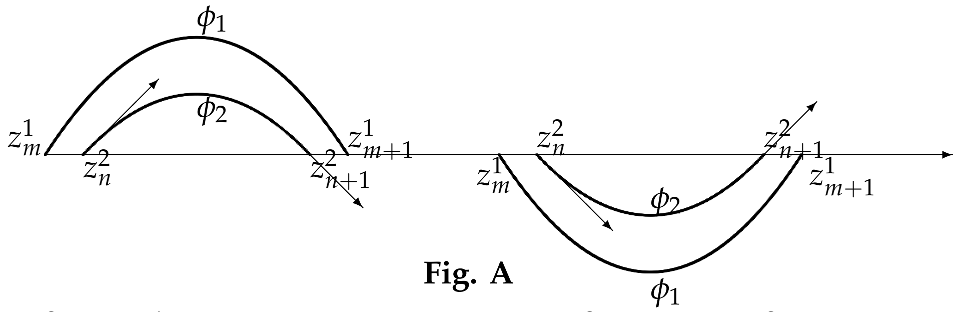

Fig. A

iii) $z_m^1 < z_n^2 < z_{n+1}^2 \leq z_{m+1}^1$: In this case we have $\phi_1\left(z_n^2\right) > 0$, $\phi_2'\left(z_n^2\right) >$ and

$$\lim_{t\to z_{n+1}^2} Wr\left(t\right) = \begin{cases} 0, & \text{if } z_{n+1}^2 = 1, \\ \phi_1\left(z_{n+1}^2\right)\phi_2'\left(z_{n+1}^2\right), & \text{if } z_{n+1}^2 < 1. \end{cases}$$

Thus, we obtain

$$\begin{aligned} \int_{z_n^2}^{z_{n+1}^2} \phi_1\mathcal{L}_q\phi_2 - \phi_2\mathcal{L}_q\phi_1 &= Wr\left(z_n^2\right) - \lim_{t\to z_{n+1}^2} Wr\left(t\right) \\ &\geq Wr\left(z_n^2\right) = \phi_1\left(z_n^2\right)\phi_2'\left(z_n^2\right) > 0. \end{aligned}$$

This ends the proof.

■

We end this section with the following lemma which is an adapted version of the Sturmian comparison result.

Lemma 4.8. *Let $q \in Q$ and for $i = 1, 2$, $m_i \in \Gamma^+$ and $w_i \in C^2\left([0,1), \mathbb{R}\right)$ satisfying*

$$\mathcal{L}_q w_i = m_i w_i \text{ in } \left(x_1, x_2\right)$$

and suppose that w_2 does not vanish identically, $m_1 \geq m_2$ and $m_1 \geq m_2$ in a subset of positive measure. If either

i) *$x_2 < 1$ and $w_2(x_1) = w_2(x_2) = 0$, or*

ii) *$x_2 = 1$ and $w_2(x_1) = \lim_{t\to 1} w_i(t) = 0$ for $i = 1, 2$*

then there exists $\tau \in (x_1, x_2)$ such that $W^1(\tau) = 0$.

Proof .

i) By the contrary suppose that $w_1 > 0$ in (x_1, x_2) and without loss of generality assume that $w_2 > 0$ in (x_1, x_2), then we have the contradiction:

$$0 \geq w_1\left(x_2\right)w_2'\left(x_2\right) - w_1\left(x_1\right)w_2'\left(x_1\right) =$$
$$\int_{x_1}^{x_2} w_2\mathcal{L}_q w_1 - w_1\mathcal{L}_q w_2 = \int_{x_1}^{x_2}(m_1 - m_2)w_1 w_2 > 0.$$

ii) By the contrary suppose that $w_1 > 0$ in $(x_1, 1)$ and without loss of generality assume that $w_2 > 0$ in $(x_1, 1)$, we have for $t > x_1$ that

$$\begin{gathered}(w_1(t)\,w_2'(t) - w_1(t)\,w_2'(t)) - w_1(x_1)\,w_2'(x_1) = \\ \int_{x_1}^t w_2\mathcal{L}_q w_1 - w_1\mathcal{L}_q w_2 = \int_{x_1}^t (m_1 - m_2)w_1w_2 > 0.\end{gathered}$$

Since from lemma 4.5, $w_1, w_2 \in W^1$, we have

$$\lim_{t\to 1}\left(w_1(t)\,w_2'(t) - w_1'(t)\,w_2(t)\right) = 0 \tag{4.9}$$

and so, the contradiction

$$0 \geq -w_1(x_1)\,w_2'(x_1) = \int_{x_1}^1 (m_1 - m_2)w_1w_2 > 0.$$

The proof is complete. ∎

4.2.4 The positive eigenvalue

The main result of this subsection concerns the existence of positive eigenvalue on the bounded interval $[\theta, 1]$.

Theorem 4.9. *For all $q \in Q$, $m \in \Gamma^{++}$ and $\theta \in [0, 1)$, the eigenvalue problem*

$$\begin{cases} \mathcal{L}_q = \mu m u, \ \ in\ \ (\theta, 1), \\ u(\theta) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{4.10}$$

admits a unique positive eigenvalue $\mu_1^+(q, m, \theta)$. Moreover for q, m fixed, the function $\theta \to \mu_1(\theta) := \mu_1(q, m, \theta)$ is continuous increasing and we have $\lim_{\theta\to 1}\mu_1(\theta) = +\infty$.

Proof .

Let $q \in Q$, $m \in \Gamma^{++}$, $\theta \in [0, 1)$ and let ϖ be a positive constant such that $\widehat{q} = q + \varpi m \geq 0$ in $[0, 1]$. Consider the eigenvalue problem

$$\begin{cases} \mathcal{L}_{\widehat{q}} u = \mu m u, \ \ \text{in}\ \ (\theta, 1) \\ u(\theta) = \lim_{t\to 1} u(t) = 0 \end{cases} \tag{4.11}$$

and notice that μ_0 is a positive eigenvalue of the bvp (4.11) if and only if $\mu_0 - \varpi$ is a positive eigenvalue of the bvp (4.10).

We have from Lemma 4.5 that μ is a positive eigenvalue of (4.11) if and only if μ^{-1} is a positive eigenvalue of the linear compact operator $L_{\widehat{q},\theta} : W \to W$ where

$$L_{\widehat{q},m,\theta}u(t) = \int_0^1 G_{\widehat{q}}(\theta, t, s)m(s)u(s)ds.$$

Let $u_\theta \in W$ be the function defined by

$$u_\theta(t) = \begin{cases} 0, & \text{if } t \notin \left[\frac{2\theta+1}{3}, \frac{2+\theta}{3}\right], \\ (t - \frac{2\theta+1}{3})(\frac{2+\theta}{3} - t), & \text{if } t \in \left[\frac{2\theta+1}{3}, \frac{2+\theta}{3}\right], \end{cases}$$

we have then $L_{\widehat{q},m,\theta}u_\theta(t) \geq 0 = u_\theta(t)$ for $t \in \left[0, \frac{2\theta+1}{3}\right] \cup \left[\frac{2+\theta}{3}, 1\right]$ and $L_{\widehat{q},m,\theta}u_\theta(t), u_\theta(t) > 0$ for $t \in \left(\frac{2\theta+1}{3}, \frac{2+\theta}{3}\right)$.
This shows that $L_{\widehat{q},m,\theta}u_\theta \geq c_\theta u_\theta$ where $c_\theta = \inf\left\{L_{\widehat{q},\theta}u_\theta(t)/u_\theta(t) : t \in \left(\frac{2\theta+1}{3}, \frac{2+\theta}{3}\right)\right\} > 0$ and $r(L_{\widehat{q},m,\theta}) > 0$. We have from the Krein-Rutman theorem, that $r(L_{\widehat{q},\theta})$ is a positive eigenvalue of L_θ having a positive eigenvector ϕ_θ. Obviously, $\widehat{\mu}_1(\theta, \widehat{q}, m) = 1/r(L_{\widehat{q},m,\theta})$ is a positive eigenvalue of the eigenvalue problem (4.11) and $\mu_1(\theta, q, m) = \widehat{\mu}_1(\theta, \widehat{q}, m) - \varpi$ is a positive eigenvalue of the eigenvalue problem (4.10).

Now, let us prove uniqueness of the positive eigenvalue. Suppose that λ is a positive eigenvalue of the eigenvalue problem (4.10) having an eigenfunction ψ, we have then

$$0 = \int_\theta^1 \psi \mathcal{L}_{\widehat{q}}\phi_\theta + \phi_\theta \mathcal{L}_{\widehat{q}}\psi = (\mu_1(\theta, q, m) - \lambda) \int_\theta^1 m\phi_\theta \psi$$

leading to $\lambda = \mu_1(\theta, q, m)$.

Let now $\theta_1, \theta_2 \in (0,1)$ be such that $\theta_1 < \theta_2$ and set for $i = 1, 2$, $\mu_i = \mu_1(\theta_i, q, m)$ with the corresponding eigenfunction ψ_i. We have

$$\begin{aligned} 0 > -\psi_2'(\theta_2)\psi_1(\theta_2) &= \textstyle\int_{\theta_2}^1 \psi_2 \mathcal{L}_{\widehat{q}}\psi_1 - \psi_1 \mathcal{L}_{\widehat{q}}\psi_2'' \\ &= (\mu_1 - \mu_2) \textstyle\int_{\theta_2}^1 m\psi_1\psi_2 \end{aligned}$$

leading to $\mu_1 < \mu_2$, proving that the function $\theta \to \mu_1(\cdot)$ is increasing.

At this stage let us prove the continuity of the function $\theta \to \mu_1(\cdot)$. Let $[\gamma, \delta] \subset [0,1]$ and $\theta_1, \theta_2 \in [\gamma, \delta]$ be such that $\theta_1 < \theta_2$. We have for all $u \in W$ with $\| u \| = 1$

$$\begin{aligned} \left|L_{\widehat{q},m,\theta_2}u(t) - L_{\widehat{q},m,\theta_1}u(t)\right| &= \left|\int_{\theta_2}^1 G_{\widehat{q}}(\theta_2, t, s)\, muds - \int_{\theta_1}^1 G_{\widehat{q}}(\theta_1, t, s)\, muds\right| \\ &= \begin{cases} 0, & \text{if } t \leq \theta_1 < \theta_2, \\ \left|\int_{\theta_1}^1 G_{\widehat{q}}(\theta_1, t, s)\, muds\right|, & \text{if } \theta_1 < t \leq \theta_2, \\ \left|\int_{\theta_2}^1 G_{\widehat{q}}(\theta_2, t, s)\, muds - \int_{\theta_1}^1 G_{\widehat{q}}(\theta_1, t, s)\, muds\right|, & \text{if } \theta_1 < \theta_2 < t. \end{cases} \end{aligned}$$

Set

$$\chi = \|m\| \left[\left(\int_\gamma^1 \phi_{\widehat{q}} ds\right) \frac{\phi_{\widehat{q}}(\gamma)}{\phi_{\widehat{q}}^2(\delta)} + \overline{G}_{\widehat{q}} + \Phi_{\widehat{q}}(\gamma)\, \Psi_{\widehat{q}}(\delta)\right]$$

then we have for $\theta_2 \geq t > \theta_1$,

$$\left|\int_{\theta_1}^1 G_{\widehat{q}}(\theta_1,t,s)\,muds\right| \leq \|m\| \int_{\theta_1}^1 G_{\widehat{q}}(\theta_1,t,s)\,ds$$
$$= \|m\| \left(\int_{\theta_1}^1 G_{\widehat{q}}(t,s)\,ds - \frac{\psi_q(\theta_1)}{\phi_q(\theta_1)}\phi_{\widehat{q}}(t)\int_{\theta_1}^1 \phi_{\widehat{q}}ds\right)$$
$$= \|m\| \,(\int_{\theta_1}^t G_{\widehat{q}}(t,s)\,ds + \int_t^1 G_{\widehat{q}}(t,s)\,ds$$
$$-\frac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)}\phi_{\widehat{q}}(t)\int_{\theta_1}^t \phi_{\widehat{q}}ds - \frac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)}\phi_{\widehat{q}}(t)\int_t^1 \phi_{\widehat{q}}ds)$$
$$= \|m\| \,(\int_{\theta_1}^t G_{\widehat{q}}(t,s)\,ds - \frac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)}\phi_{\widehat{q}}(t)\int_{\theta_1}^t \phi_{\widehat{q}}ds) + \psi_{\widehat{q}}(t)\int_t^1 \phi_{\widehat{q}}ds - \frac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)}\phi_{\widehat{q}}(t)\int_t^1 \Psi_{\widehat{q}}ds)$$
$$= \|m\| \,(\int_{\theta_1}^t G_{\widehat{q}}(t,s)\,ds - \frac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)}\phi_{\widehat{q}}(t)\int_{\theta_1}^t \phi_{\widehat{q}}ds) + \int_t^1 \phi_{\widehat{q}}ds\left(\frac{\psi_{\widehat{q}}(t)}{\phi_{\widehat{q}}(t)} - \frac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)}\right)\phi_{\widehat{q}}(t))$$
$$\leq \|m\| \left[\left(\int_\gamma^1 \phi_{\widehat{q}}ds\right)\frac{\phi_{\widehat{q}}(\gamma)}{\phi_{\widehat{q}}^2(\delta)} + \overline{G}_{\widehat{q}} + \Phi_{\widehat{q}}(\gamma)\,\Psi_q(\delta)\right]|\theta_2-\theta_1| \leq \chi\,|\theta_2-\theta_1|$$

and for $\theta_1 < \theta_2 < t$,

$$\left|\int_{\theta_2}^1 G_{\widehat{q}}(\theta_2,t,s)\,muds - \int_{\theta_1}^1 G_{\widehat{q}}(\theta_1,t,s)\,muds\right| \leq$$
$$\left|\int_{\theta_2}^1 \left(G_{\widehat{q}}(\theta_2,t,s) - G_{\widehat{q}}(\theta_1,t,s)\right)muds\right| + \left|\int_{\theta_1}^{\theta_2} G_{\widehat{q}}(\theta_1,t,s)\,muds\right|$$
$$= \left|\left(\int_{\theta_2}^1 \phi_{\widehat{q}}muds\right)\left(\frac{\psi_{\widehat{q}}(\theta_1)}{\phi_{\widehat{q}}(\theta_1)} - \frac{\psi_{\widehat{q}}(\theta_2)}{\phi_{\widehat{q}}(\theta_2)}\right)\phi_{\widehat{q}}(t)\right| + \left|\int_{\theta_1}^{\theta_2} G_{\widehat{q}}(\theta_1,t,s)\,muds\right|$$
$$\leq \|m\| \left[\left(\int_\gamma^1 \phi_{\widehat{q}}ds\right)\frac{\phi_{\widehat{q}}(\gamma)}{\phi_{\widehat{q}}^2(\delta)} + \overline{G}_{\widehat{q}}\right]|\theta_2-\theta_1| \leq \chi\,|\theta_2-\theta_1|\,.$$

The above estimates show that the mapping $\theta \to L_{\widehat{q},m,\theta}$ is locally Lipschitzian and so, it is continuous. Let (θ_n) be a sequence converging to θ_* and let θ_-,θ_+ be such that $(\theta_n) \subset [\theta_-,\theta_+]$. Therefore we have for all $n \geq 1$,

$$0 < \mu_1(\theta_+) \leq \mu_1(\theta_n) \leq \mu_1(\theta_-)$$

and the sequence $(\mu_1(\theta_n,\widehat{q},m))$ converges (up to a subsequence) to some $\mu_* > 0$. We conclude by Lemma 2.13 in [10] and by uniqueness of the positive eigenvalue that $\mu_* = \mu_1(\theta_*)$. Thus, the continuity of the mapping $\mu_1(\cdot)$ is proved.

It remains to prove that

$$\lim_{\theta\to 1}\mu_1^+(\theta) = \lim_{\theta\to 1}\frac{1}{r(L_{\widehat{q},m,\theta})} = +\infty.$$

We have for all $u \in W$ with $\|u\| = 1$,

$$\begin{aligned}\left|L_{\widehat{q},m,\theta}u(t)\right| &\leq \int_\theta^1 G_{\widehat{q}}(\theta,t,s)\,m(s)ds\\ &\leq \int_\theta^1 G_{\widehat{q}}(t,s)\,m(s)ds + \frac{\Psi_{\widehat{q}}(\theta)}{\Phi_{\widehat{q}}(\theta)}\int_\theta^1 \Phi_{\widehat{q}}(t)\Phi_{\widehat{q}}(s)m(s)ds\\ &\leq \int_\theta^1 G_{\widehat{q}}(t,s)\,m(s)ds + \Psi_{\widehat{q}}(\theta)\int_\theta^1 \Phi_{\widehat{q}}(s)m(s)ds.\end{aligned}$$

Arguing as in the proof of Lemma 4.5, we obtain $\lim_{\theta\to 1}\Psi_{\widehat{q}}(\theta)\int_\theta^1\Phi_{\widehat{q}}(s)m(s)ds = 0$ and because of $\int_\theta^1 G_{\widehat{q}}(t,s)\,m(s)ds \le \overline{G}_{\widehat{q}}\int_\theta^1 m(s)ds$, we have $\lim_{\theta\to 1}\int_\theta^1 G_{\widehat{q}}(t,s)\,m(s)ds = 0$ uniformely on $[0,1]$. Therefore, we have proved that $\lim_{\theta\to 1} r(L_{\widehat{q},m,\theta}) = \lim_{\theta\to+\infty}\left\|L_{\widehat{q},m,\theta}\right\| = 0$ and this ends the proof. ∎

4.3 The half-eigenvalue problem

Consider for $q\in Q,\ m\in\Gamma^+$ and $\alpha,\beta\in E$ the bvp:

$$\begin{cases}\mathcal{L}_q = \lambda mu + \alpha u^+ - \beta u^-, & \text{in } (0,1)\\ u(0) = \lim_{t\to 1} u(t) = 0,\end{cases}\tag{4.12}$$

where λ is a real parameter.

Because that the function $u \to \lambda mu + \alpha u^+ - \beta u^-$ is linear on the cones $\{u\in E : u\ge 0$ in $[0,1]\}$ and $\{u\in E : u\le 0$ in $[0,1]\}$, the bvp (4.12) is said to be half-linear.

Definition 4.10. *We say that λ_0 is a half-eigenvalue of (4.12) if there exists a nontrivial solution (λ_0,u_0) of (4.12). In this situation, $\{(\lambda_0,tu_0),\ t>0\}$ is a half-line of nontrivial solutions of (4.12) and μ_0 is said to be simple if all solutions (λ_0,u) of (4.12), with $uu_0>0$ in a right neighborhood of 0, are on this half-line. There may exist another half-line of solutions $\{(\lambda_0,tv_0),\ t>0\}$, but then we say that λ_0 is simple, if $u_0v_0<0$ in a right neighborhood of 0 and all solutions (λ_0,v) of (4.12) lie on these two half lines.*

The case of the bvp (4.12) where $q\in E$ has been considered by Berestycki in [11]. He has proved that the bvp (4.12) admits two increasing sequences of half-eigenvalues. So, the main goal of this section is to prove that the Berestycki's result holds true for the case $q\in Q$. We begin with the following list of lemmas.

Proposition 4.11. *Let $q\in Q$, $m\in\Gamma^+$ and $\alpha,\beta\in E$. If (λ,ϕ) is a nontrivial solution to the bvp (4.12), then $\phi\in S_k^\nu$ for some integer $k\ge 1$ and $\nu = +$ or $-$.*

Proof .
Let $\varepsilon>0$ be small enough and let $A>0$ be such that $\mu_1(q-\alpha,m+\varepsilon)>-A$. Consider the bvp

$$\begin{cases}\mathcal{L}_{q+Am}u = \lambda mu + \alpha u^+ - \beta u^- \text{ in } (0,1),\\ u\,(0) = \lim_{t\to 1} u\,(t) = 0,\end{cases}\tag{4.13}$$

and notice that λ is a half-eigenvalue of the bvp (4.13) if and only if $(\lambda - A)$ is a half-eigenvalue to the bvp (4.12). Thus, we have to prove that if (λ, ϕ) is a nontrivial solution to the bvp (4.13), then $\phi \in S_k^\nu$ for some integer $k \geq 1$ and $\nu = +$ or $-$. To this aim, let (λ, ϕ) is a nontrivial solution to the bvp (4.13), we claim first that all zeros of ϕ in $[0, 1)$ are simple. Indeed, noticing that the right hand-side in (4.13) is lipschitzian, if $\phi(x_*) = \phi'(x_*) = 0$ for some $x_* \in [0, 1)$ then the standard existence and uniqueness result of a solution to initial value problem leads to $\phi = 0$. This contradicts the fact that (λ, ϕ) is a nontrivial solution to the bvp (4.13).

Now, we claim that ϕ has a finite number of zeros. By the contrary, assume that ϕ has an infinite sequence of zeros, say (z_n) such that $\lim z_n = z_*$, we distinguish then the following two cases:

i. $z_* \in [0, 1)$, in this situation we have

$$\phi(z_*) = \lim \phi(z_n) = 0 \text{ and } \phi'(z_*) = \lim \frac{\phi(z_n) - \phi(z_*)}{z_n - z_*} = 0.$$

This contradicts the simplicity of zeros of ϕ in $[0, 1)$.

ii. $z_* = 1$, in this case ϕ satisfies for all $n \geq 1$

$$\begin{cases} \mathcal{L}_{q+Am} u = \lambda m u + \alpha u^+ - \beta u^- \text{ in } (0, 1), \\ u(z_n) = \lim_{t \to 1} u(t) = 0. \end{cases}$$

Let for all $n \geq 1$ $\mu_n = \mu_1(q + Am - \alpha, m + \varepsilon, z_n)$ the positive eigenvalue given by Theorem 4.9 and let ψ_n the normalized positive eigenfunction associated with μ_n. Notice that

$$\mu_n = \mu_1(q + Am - \alpha, m + \varepsilon, z_n) \geq \mu_1(q + Am - \alpha, m + \varepsilon) = \mu_1(q - \alpha, m + \varepsilon) + A > 0.$$

We claim now that for all integers $n \geq 1$, $\lambda > \mu_n$. Indeed, let $l \geq n$ be such that $\phi > 0$ in (z_l, z_{l+1}), we obtain Lemma 4.7 that

$$0 < \int_{z_l}^{z_{l+1}} -\psi_n \mathcal{L}_q \phi + \phi \mathcal{L}_q \psi_n + \mu_n \varepsilon \int_{z_l}^{z_{l+1}} \phi \psi_n = (\lambda - \mu_n) \int_{z_l}^{z_{l+1}} m \phi \psi_n$$

leading to $\lambda > \mu_n$.

Therefore, we obtain from Theorm 4.9 the contradiction

$$\lambda \geq \lim \mu_n = \lim \mu_1(q + Am - \alpha, m + \varepsilon, z_n) = +\infty.$$

This completes the proof of the lemma. ∎

Proposition 4.12. *For $q \in Q$, $m \in \Gamma^+$, $\alpha, \beta \in E$, $k \geq 1$ and $\nu = +$ or $-$ the bvp (4.12) admits at most one half eigenvalue having an eigenfunction in S_k^ν.*

Proof .

Let (λ_1, ϕ_1) and (λ_2, ϕ_2) be two nontrivial solutions to the bvp (4.12) such that $\lambda_1 \neq \lambda_2$ and $\phi_1, \phi_2 \in S_k^\nu$ for some integer $k \geq 1$ and $\nu = +, -$, and denote for $i = 1, 2$ $\left(z_j^i\right)_{j=0}^{j=k}$ the sequence of zeros of ϕ_i. First, we claim that there exists j_0 such that $z_{j_0}^1 \neq z_{j_0}^2$; indeed, assume that $\phi_1\left(z_j^2\right) = 0$ for all $j \in \{1, ..., k-1\}$ and $\lambda_1 < \lambda_2$ and note that there exists $j_1 \in \{1, ..., k-1\}$ such that $meas\left(\{m > 0\} \cap \left(z_{j_1}^2, z_{j_1+1}^2\right)\right) > 0$ and $\phi_1\phi_2 > 0$ in $\left(z_{j_1}^2, z_{j_1+1}^2\right)$. Applying Lemma 4.8, we conclude that there is $\tau \in \left(z_{j_1}^2, z_{j_1+1}^2\right)$ such that $\phi_1(\tau) = 0$ and this contradicts $\phi_1 \in S_k^\nu$.

Now, let $k_1 = \max\left\{l \leq k : z_j^1 = z_j^2 \text{ for all } j \leq l\right\}$ and $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ be the families defined by $\xi_j = z_{k_1+j}^1$ and $\eta_j = z_{k_1+j}^2$ and without loss of generality, assume that $\xi_1 = z_{k_1+1}^1 < \eta_1 = z_{k_1+1}^2$. We obtain from Lemma 4.6 that there exist two integers $m, n \geq 1$ having the same parity such that

$$\xi_m = z_{k_1+m}^1 < \eta_n = z_{k_1+n}^2 < \eta_{n+1} = z_{k_1+n+1}^2 \leq \xi_{m+1} = z_{k_1+m+1}^1$$

and we have from Lemma 4.7 that

$$0 < \int_{\xi_0}^{\xi_1} \phi_2\mathcal{L}_q\phi_1 - \phi_1\mathcal{L}_q\phi_2 = (\lambda_1 - \lambda_2)\int_{\xi_0}^{\xi_1} m\phi_1\phi_2 \tag{4.14}$$

$$0 < \int_{\eta_n}^{\eta_{n+1}} \phi_1\mathcal{L}_q\phi_2 - \phi_2\mathcal{L}_q\phi_1 = (\lambda_2 - \lambda_1)\int_{\eta_n}^{\eta_{n+1}} m\phi_1\phi_2. \tag{4.15}$$

Therefore, we obtain from (4.14) that $\lambda_1 > \lambda_2$, and from (4.15) the contradiction $\lambda_1 < \lambda_2$. This ends the proof. ■

Proposition 4.13. *Let $q \in Q$, $m \in \Gamma^+$, $\alpha, \beta \in E$ and assume that $(\lambda_1, \phi_1), (\lambda_2, \phi_2)$ are two solutions of the bvp (4.12) such that $\phi_i \in S_\rho^{k_i,\nu}$ for $i = 1, 2$. If $k_2 > k_1$ then $\lambda_2 > \lambda_1$.*

Proof .

By the way of contradiction assume that $\lambda_2 \leq \lambda_1$ and let for $i = 1, 2$, $\left(z_j^i\right)_{j=0}^{j=k}$ be the sequence of zeros of ϕ_i. Set $k_* = \max\left\{l \leq k : z_j^1 = z_j^2 \text{ for all } j \leq l\right\}$ and consider $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ the families defined by $\xi_j = z_{k_*+j}^1$ and $\eta_j = z_{k_*+j}^2$. We distinguish then two cases.

i) $\xi_1 = z^1_{k_*+1} > \eta_1 = z^2_{k_*+1}$. In this case we have from Lemma 4.7

$$0 < \int_{\eta_0}^{\eta_1} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 = (\lambda_2 - \lambda_1) \int_{\eta_0}^{\eta_1} m\phi_1\phi_2$$

leading to the contradiction $\lambda_1 < \lambda_2$.

ii) $\xi_1 = z^1_{k_*+1} < \eta_1 = z^2_{k_*+1}$. In this case, Lemma 4.6 guarantees existence of two integers m, n having the same parity such that

$$\xi_m = z^1_{k_1+m} < \eta_n = z^2_{k_1+n} < \eta_{n+1} = z^2_{k_1+n+1} \leq \xi_{m+1} = z^1_{k_1+m+1},$$

and we have from Lemma 4.7

$$0 < \int_{\eta_n}^{\eta_{n+1}} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 = (\lambda_2 - \lambda_1) \int_{\eta_n}^{\eta_{n+1}} m\phi_1\phi_2,$$

leading also to the contradiction $\lambda_1 < \lambda_2$.

This ends the proof. ■

Proposition 4.14. *Let $q \in Q$, $m \in \Gamma^+$ and $\alpha, \beta \in E$. If λ is a half-eigenvalue of the bvp (4.12), then λ is simple.*

Proof .
Let λ be a half-eigenvalue of the bvp (4.12) having two eigenfunctions ϕ_1, ϕ_2 and without loss of generality, assume that $\phi_1, \phi_2 > 0$ in a right neighborhood of 0. Because of Proposition 4.13 we have that $\phi_1, \phi_2 \in S^+_k$ for some integer $k \geq 1$. For $i = 1, 2$, let $(z^i_j)_{j=0}^{j=k-1}$ be the sequence of zeros of ϕ_i. We have that $z^1_j = z^2_j$ for all $j = 0, \ldots, k$. By induction, clearly $z^1_0 = z^2_0 = 0$ and if $z^1_j = z^2_j$ then $z^1_{j+1} = z^2_{j+1}$. Indeed, if for example $z^1_{j+1} < z^2_{j+1}$, then Lemma 4.7 leads to the contradiction

$$0 < \int_{z^1_j}^{z^1_{j+1}} \phi_2 \mathcal{L}_q \phi_1 - \phi_1 \mathcal{L}_q \phi_2 = 0.$$

Because of the positive homogeneity of (4.12) and $\phi_1, \phi_2 \in S^+_k$, $\phi_1'(0) > 0$, $\phi_2'(0) > 0$ and $\psi_1 = (\phi_1'(0))^{-1} \phi_1$, $\psi_2 = (\phi_2'(0))^{-1} \phi_2$ are eigenfunctions associated with λ satisfying

$$\psi_1(0) = \psi_2(0) = 0 \text{ and } \psi_1'(0) = \psi_2'(0) = 1.$$

Therefore, $\psi = \psi_1 - \psi_2$ satisfies

$$\begin{cases} \mathcal{L}_q \psi = \mu m \psi + \alpha \psi^+ - \beta \psi^- \text{ in } (0, z^1_j), \\ \psi(0) = \psi'(0) = 0, \end{cases}$$

proving that $\psi_1 = \psi_2$ in $[0,1]$. This completes the proof. ■

In what follows and when for functions $q \in Q$, $m \in \Gamma^+$ and $\alpha, \beta \in E$ the half-eigenvalue of the bvp (4.12) associated with an eigenfunction in S_k^ν exists, this will be denoted by $\lambda_k^\nu(q, m, \alpha, \beta)$.

Proposition 4.15. *Let $q_1, q_2 \in Q$, $m \in \Gamma^+$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in E$ and assume that for some $k \geq 1$ and $\nu = \pm$, $\lambda_k^\nu(q_1, m, \alpha_1, \beta_1)$, $\lambda_k^\nu(q_2, m, \alpha_1, \beta_1)$, $\lambda_k^\nu(q_1, m, \alpha_2, \beta_1)$ and $\lambda_k^\nu(q_1, m, \alpha_1, \beta_2)$ exist.*

1. *If $\alpha_1 \leq \alpha_2$ a.e. in $(0,1)$, then $\lambda_k^\nu(q_1, m, \alpha_1, \beta_1) \geq \lambda_k^\nu(q_1, m, \alpha_2, \beta_1)$.*

2. *If $\beta_1 \leq \beta_2$ a.e. in $(0,1)$, then $\lambda_k^\nu(q_1, m, \alpha_1, \beta_1) \geq \lambda_k^\nu(q_1, m, \alpha_1, \beta_2)$.*

3. *If $q_1 \leq q_2$ a.e. in $(0,1)$, then $\lambda_k^\nu(q_1, m, \alpha_1, \beta_1) \leq \lambda_k^\nu(q_2, m, \alpha_1, \beta_1)$.*

Proof .
We present the proof of Assertion 1, Assertion is checked similarly and Assertion 3 is a consequence of Assertions 2 and 3. Suppose that $\alpha_1 \leq \alpha_2$ and for $i = 1, 2$, set $\lambda_i = \lambda_k^\nu(m, \alpha_i, \beta_1)$. Let ϕ_i be the eigenfunction associated with λ_i having a sequence of zeros $(z_j^i)_{j=0}^{j=k}$. We distinguish two cases:

i). $z_j^1 = z_j^2$ for all $j \in \{1, \ldots, k-1\}$. Let $j_1 \in \{1, \ldots, k-1\}$ be such that $\operatorname{meas}(\{m > 0\} \cap (z_{j_1}^2, z_{j_1+1}^2)) > 0$, we have

$$\begin{aligned} 0 &= \int_{z_{j_1}^2}^{z_{j_1+1}^2} \phi_2 \mathcal{L}_q \phi_1 - \phi_1 \mathcal{L}_q \phi_2 = (\lambda_1 - \lambda_2) \int_{z_{j_1}^2}^{z_{j_1+1}^2} m\phi_1\phi_2 \\ &+ \int_{z_{j_1}^2}^{z_{j_1+1}^2} \left(\alpha_1 \phi_1^+ \phi_2 - \alpha_2 \phi_2^+ \phi_1\right) + \int_{z_{j_1}^2}^{z_{j_1+1}^2} \left(\beta_1 \phi_1^- \phi_2 - \beta_1 \phi_2^- \phi_1\right) \\ &= (\lambda_1 - \lambda_2) \int_{z_{j_1}^2}^{z_{j_1+1}^2} m\phi_1\phi_2 + \int_{z_{j_1}^2}^{z_{j_1+1}^2} \left(\alpha_1 \phi_1^+ \phi_2 - \alpha_2 \phi_2^+ \phi_1\right). \end{aligned} \tag{4.16}$$

Thus, from (4.16) in both the case $\phi_1, \phi_2 > 0$ in $(z_{j_1}^2, z_{j_1+1}^2)$ and the case $\phi_1, \phi_2 < 0$ in $(z_{j_1}^2, z_{j_1+1}^2)$, we obtain $\lambda_1 \geq \lambda_2$.

ii) $z_{j_0}^1 \neq z_{j_0}^2$ for some j_0: In this case set $k_1 = \max\{l \leq k : z_j^1 = z_j^2 \text{ for all } j \leq l\}$. If $z_{k_1+1}^1 < z_{k_1+1}^2$, then

$$0 < \int_{z_{k_1}^1}^{z_{k_1+1}^1} \phi_2 \mathcal{L}_q \phi_1 - \phi_1 \mathcal{L}_q \phi_2 = (\lambda_1 - \lambda_2) \int_{z_{k_1}^1}^{z_{k_1+1}^1} m\phi_1\phi_2 + \int_{z_{k_1}^1}^{z_{k_1+1}^1} (\alpha_1 - \alpha_2)\phi_1\phi_2$$

proving that $\mu_1 > \mu_2$ and if $z_{k_1+1}^2 < z_{k_1+1}^1$ then considering the families $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ with $\xi_j = z_{k_1+j}^1$ and $\eta_j = z_{k_1+j}^2$, we obtain from Lemma 4.6 that there exist two integers $m, n \geq 1$ having the same parity such that

$$\xi_m = z^2_{k_1+m} < \eta_n = z^1_{k_1+n} < \eta_{n+1} = z^1_{k_1+n+1} \leq \xi_{m+1} = z^2_{k_1+m+1}.$$

Therefore, we obtain from Lemma 4.7

$$0 < \int_{\eta_n}^{\eta_{n+1}} \phi_2 \mathcal{L}_q \phi_1 - \phi_1 \mathcal{L}_q \phi_2 = (\lambda_1 - \lambda_2) \int_{\eta_n}^{\eta_{n+1}} m\phi_1\phi_2 + \int_{\eta_n}^{\eta_{n+1}} (\alpha_1 - \alpha_2)\phi_1\phi_2,$$

leading to $\lambda_1 > \lambda_2$.

This completes the proof. ■

Proposition 4.16. *Let $q \in Q$, $m_1, m_2 \in \Gamma^+$ and $\alpha, \beta \in E$. Assume that $m_1 \leq m_2$ in $(0,1)$, $m_1 < m_2$ in a subset of positive measure and $\lambda_k^\nu(q, m_1, \alpha, \beta)$, $\lambda_k^\nu(q, m_2, \alpha, \beta)$ exist for some integer $k \geq 1$ and $\nu = +$ or $-$. If either $\lambda_k^\nu(q, m_1, \alpha, \beta) \geq 0$ or $\lambda_k^\nu(q, m_2, \alpha, \beta) \geq 0$, then $\lambda_k^\nu(q, m_1, \alpha, \beta) > \lambda_k^\nu(q, m_2, \alpha, \beta)$ and if either $\lambda_k^\nu(q, m_1, \alpha, \beta) \leq 0$ or $\lambda_k^\nu(q, m_2, \alpha, \beta) \leq 0$, then $\lambda_k^\nu(q, m_1, \alpha, \beta) < \lambda_k^\nu(q, m_2, \alpha, \beta)$.*

Proof .
Assume that for $i = 1, 2$ $\lambda_i = \lambda_k^\nu(m_1, \alpha, \beta)$ exists and has an eigenfunction ϕ_i having a sequence of zeros $(z^i_j)_{j=0}^{j=k}$. First, we claim that there exists j_0 such that $z^1_{j_0} \neq z^2_{j_0}$. Indeed, if $\phi_1(z^2_j) = 0$ for all $j \in \{1, \ldots, k-1\}$ and $j_1 \in \{1, \ldots, k-1\}$ is such that $\text{meas}(\{m_2 > m_1\} \cap (z^2_{j_1}, z^2_{j_1+1})) > 0$, then taking in account that $\phi_1\phi_2 > 0$ in $(z^2_{j_1}, z^2_{j_1+1})$, we obtain by means of Lemma 4.8 in the case $\lambda_1 \leq \lambda_2$ (the other caes is checked similarly) that there exists $\tau \in (z^2_{j_1}, z^2_{j_1+1})$ such that $\phi_1(\tau) = 0$. Obviously, this contradicts $\phi_1 \in S_k^\nu$.

Now, let $k_1 = \max\{l \leq k : z^1_j = z^2_j \text{ for all } j \leq l\}$, and $(\xi_j)_{j=0}^{j=k-k_1}$ and $(\eta_j)_{j=0}^{j=k-k_1}$ be the families defined by $\xi_j = z^1_{k_1+j}$ and $\eta_j = z^2_{k_1+j}$. Assume that $\lambda_1 \geq 0$ or $\lambda_2 \geq 0$, we distinguish then two cases.

i. $\xi_1 = z^1_{k_1+1} < \eta_1 = z^2_{k_1+1}$: In this case we have from Lemma 4.7

$$\begin{aligned} 0 < \int_{\xi_0}^{\xi_1} \phi_2 \mathcal{L}_q \phi_1 - \phi_1 \mathcal{L}_q \phi_2 &= \int_{\xi_0}^{\xi_1} (\lambda_1 m_1 - \lambda_2 m_2)\phi_1\phi_2 \\ &= (\lambda_1 - \lambda_2) \int_{\xi_0}^{\xi_1} m_1\phi_1\phi_2 + \lambda_2 \int_{\xi_0}^{\xi_1} (m_1 - m_2)\phi_1\phi_2 \\ &= (\lambda_1 - \lambda_2) \int_{\xi_0}^{\xi_1} m_2\phi_1\phi_2 + \lambda_1 \int_{\xi_0}^{\xi_1} (m_1 - m_2)\phi_1\phi_2 \end{aligned}$$

and this proves that in both the cases $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, we have $\lambda_1 > \lambda_2$.

ii. $\xi_1 = z^1_{k_1+1} > \eta_1 = z^2_{k_1+1}$: In this case Lemma 4.6 guarantees existence of two integers m, n having the same parity such that

$$\eta_n = z^2_{k_1+n} < \xi_m = z^1_{k_1+m} < \xi_{m+1} = z^1_{k_1+m+1} \leq \eta_{n+1} = z^2_{k_1+n+1}.$$

As above, we have from Lemma 4.7

$$\begin{aligned}0 &< \int_{\xi_m}^{\xi_{m+1}} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 = \int_{\xi_m}^{\xi_{m+1}} (\lambda_2 m_2 - \lambda_1 m_1)\phi_1\phi_2 \\ &= (\lambda_2 - \lambda_1) \int_{\xi_m}^{\xi_{m+1}} m_2 \phi_1 \phi_2 + \lambda_1 \int_{\xi_m}^{\xi_{m+1}} (m_2 - m_1)\phi_1\phi_2 \\ &= (\lambda_1 - \lambda_2) \int_{\xi_m}^{\xi_{m+1}} m_2 \phi_1 \phi_2 + \lambda_1 \int_{\xi_m}^{\xi_{m+1}} (m_1 - m_2)\phi_1\phi_2\end{aligned}$$

and this proves that in both the cases $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, we have $\lambda_1 > \lambda_2$.

Assume that $\lambda_1 \leq 0$ or $\lambda_2 \leq 0$, we distinguish then two cases.

iii. $\xi_1 = z^1_{k_1+1} > \eta_1 = z^2_{k_1+1}$: In this case we have from Lemma 4.7

$$\begin{aligned}0 &> \int_{\eta_0}^{\eta_1} \phi_2 \mathcal{L}_q \phi_1 - \phi_1 \mathcal{L}_q \phi_2 = \int_{\eta_0}^{\eta_1} (\lambda_1 m_1 - \lambda_2 m_2)\phi_1\phi_2 \\ &= (\lambda_1 - \lambda_2) \int_{\eta_0}^{\eta_1} m_1 \phi_1 \phi_2 + \lambda_2 \int_{\eta_0}^{\eta_1} (m_1 - m_2)\phi_1\phi_2 \\ &= (\lambda_1 - \lambda_2) \int_{\eta_0}^{\eta_1} m_2 \phi_1 \phi_2 + \lambda_1 \int_{\eta_0}^{\eta_1} (m_1 - m_2)\phi_1\phi_2\end{aligned}$$

and this proves that in both the cases $\lambda_1 \leq 0$ and $\lambda_2 \leq 0$, we have $\lambda_1 < \lambda_2$.

iv. $\xi_1 = z^1_{k_1+1} < \eta_1 = z^2_{k_1+1}$: In this case Lemma 4.6 guarantees existence of two integers m, n having the same parity such that

$$\xi_m = z^1_{k_1+m} < \eta_n = z^2_{k_1+n} < \eta_{n+1} = z^2_{k_1+n+1} \leq \xi_{m+1} = z^1_{k_1+m+1}.$$

As above, we have from Lemma 4.7

$$\begin{aligned}0 &> \int_{\eta_n}^{\eta_{n+1}} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 = \int_{\eta_n}^{\eta_{n+1}} (\lambda_2 m_2 - \lambda_1 m_1)\phi_1\phi_2 \\ &= (\lambda_2 - \lambda_1) \int_{\eta_n}^{\eta_{n+1}} m_2 \phi_1 \phi_2 + \lambda_1 \int_{\eta_n}^{\eta_{n+1}} (m_2 - m_1)\phi_1\phi_2 \\ &= (\lambda_1 - \lambda_2) \int_{\eta_n}^{\eta_{n+1}} m_2 \phi_1 \phi_2 + \lambda_1 \int_{\eta_n}^{\eta_{n+1}} (m_1 - m_2)\phi_1\phi_2\end{aligned}$$

and this proves that in both the cases $\lambda_1 \leq 0$ and $\lambda_2 \leq 0$, we have $\lambda_1 < \lambda_2$. The proof is complete. ■

Lemma 4.17. *Let (ϕ_n) be a sequence in S^{ν}_k converging in W^1 to some $\phi \in S^{\kappa}_l$, then $l \leq k$ and $\kappa = \nu$.*

Proof .

On the contrary suppose that $\phi \in S^{\nu}_l$ for some $l > k$ and let $(z_j)_{j=0}^{j=l}$ be the sequence of zeros of ϕ. Let $\delta > 0$ small enough, there exists an integer $n_* \geq 1$ such that $\phi\phi_n > 0$ in the intervals $[\delta, z_1 - \delta]$ and $\left[z_j + \delta, z_{j+1} - \delta\right]$ for $j = 1, ..., l-2$.

Also, for each integer $j \in \{1, ..., l-1\}$ there exists $n_j \geq n_*$ such that the function ϕ_n has exactly one zero in $\left[z_j + \delta, z_{j+1} - \delta\right]$. Otherwise if there is a subsequence (ϕ_{n_i}) such that for all $i \geq 1$, ϕ_{n_i} has at least two zeros, then we can choose $x^1_{n_i}$ and $x^2_{n_i}$ in $\left[z_j + \delta, z_{j+1} - \delta\right]$ such that

$$\phi'_{n_i}\left(x^1_{n_i}\right) \leq 0 \leq \phi'_{n_i}\left(x^2_{n_i}\right).$$

Let

$$x^1_{\inf} = \liminf x^1_{n_i}, \quad x^1_{\sup} = \limsup x^1_{n_i},$$
$$x^2_{\inf} = \liminf x^2_{n_i}, \quad x^2_{\sup} = \liminf x^2_{n_i}.$$

Hence, we have since $\phi = \lim \phi_n$ in W^1,

$$\phi\left(x^1_{\inf}\right) = \phi\left(x^2_{\inf}\right) = \phi\left(x^1_{\sup}\right) = \phi\left(x^2_{\sup}\right) = 0$$

leading to $\lim x^1_{n_i} = \lim x^2_{n_i} = z_j$ then to

$$\phi'(z_j) = \lim \phi'_{n_l}\left(x^1_{n_i}\right) = \lim \phi'_{n_l}\left(x^2_{n_i}\right) = 0.$$

Contradicting the simplicity of z_j.

Now, we claim that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\phi\phi_n > 0$ in $(0, \delta)$. Indeed, if there a subsequence (ϕ_{n_i}) such that for all $i \geq 1$, ϕ_{n_i} has at least a zero x_{n_i} with $\nu\phi'_{n_i}(x_{n_i}) < 0$, then we obtain as above for $x_- = \liminf x_{n_i}$ and $x_+ = \limsup x_{n_i}$ $\phi(x_-) = \phi(x_+) = 0$ and $x_- = x_+ = 0$. Therefore, we have

$$0 < \nu\phi'(0) = \lim \nu\phi'_{n_i}(x_{n_i}) \leq 0,$$

contradicting the simplicity of the zero $z_0 = 0$. The proof of the lemma is complete. ■

Proposition 4.18. *Let $q \in Q$, $m \in \Gamma^+$, $\alpha, \beta \in E$ and let (m_n) be a sequence of functions in Γ^+ such that $\lim m_n = m$ in E. If for some integer $k \geq 1$ and $\nu = +$ or $-$, $\lambda^\nu_k(q, m_n, \alpha, \beta)$ exits for all $n \geq 1$ with $\lim_{n\to+\infty} \lambda^\nu_k(q, m_n, \alpha, \beta) = \lambda \in \mathbb{R}$, then $\lambda = \lambda^\nu_k(q, m, \alpha, \beta)$.*

Proof .

Let for all integers $n \geq 1$ $\phi_n \in S^\nu_k$ be the normalized eigenfunction associated with $\lambda^\nu_{k,n} = \lambda^\nu_k(q, m_n, \alpha, \beta) = \lambda^\nu_k(q^+, m_n, \alpha + q^-, \beta + q^-)$. Therefore, we have for all integers $n \geq 1$

$$\phi_n(t) = \lambda^\nu_{k,n} L_{q^+, m_n}\phi_n(t) + L^+_{q^+, \alpha+q^-}\phi_n(t) - L^-_{q^+, \beta+q^-}\phi_n(t).$$

Since all the operators in the above equation are compact and (ϕ_n) is bounded, up to a subsequence, (ϕ_n) converges to some ϕ with $\|\phi\| = 1$ and

$$\phi(t) = \lambda_k^\nu L_{q^+,m}\phi(t) + L^+_{q^+,\alpha+q^-}\phi(t) - L^-_{q^+,\beta+q^-}\phi(t).$$

This proves that λ_k^ν is a half-eigenvalue of the bvp (4.12).

We have from Lemma 4.17 that $\phi \in S_l^\nu$ with $l \le k$. Let us prove that $l = k$. We claim that there is an integer $n_+ \ge 1$ such that $\phi\phi_n > 0$ in $(z_{l-1} + \delta, 1)$. Indeed, if there a subsequence (ϕ_{n_i}) such that for all $i \ge 1$, ϕ_{n_i} has a zero $x_{n_i} \in (z_{l-1} + \delta, 1)$ and ϕ_{n_i} does not vanish in $(x_{n_i}, 1)$ then

$$\lambda_{k,n}^\nu = \mu_1(q + Am - \omega, m + \varepsilon, x_{n_i}) \ge \mu_1(q + Am - \omega, m + \varepsilon, x_{n_i})$$

where

$$\omega = \begin{cases} \alpha, \text{ if } \phi_{n_i} > 0 \text{ in } (x_{n_i}, 1), \\ \beta, \text{ if } \phi_{n_i} < 0 \text{ in } (x_{n_i}, 1) \end{cases}$$

and $\omega = \max(|\alpha|, |\beta|)$.

Passing to the limit, we obtain the contradiction

$$+\infty > \lambda_k^\nu \ge \lim \mu_1(q + Am - \omega, m + \varepsilon, x_{n_i}) = +\infty.$$

From all the above, we obtain for all $n \ge \max\{n^*, n_+, n_1, ...n_{l-1}\}$ ϕ_{n_i} belongs to S_l^ν, and $l = k$. The proof is complete. ■

Lemma 4.19. *([7]) Let $q \in Q$, $m \in \Gamma^{++}$ and $\alpha, \beta \in E$. For all $\theta \in (0,1)$ the bvp*

$$\begin{cases} \mathcal{L}_q u = \lambda m u + \alpha u^+ - \beta u^-, \text{ in } (0,\theta), \\ u(0) = u(\theta) = 0, \end{cases}$$

admits two increasing sequence of simple half eigenvalues $\left(\lambda_k^+(q, m, \alpha, \beta, \theta)\right)_{k\ge1}$ and $\left(\lambda_k^-(q, m, \alpha, \beta, \theta)\right)_{k\ge1}$ such that for all integers $k \ge 1$ and $\nu = +$ or $-$, the corresponding half-line of solutions lies on $\{\lambda_k^\nu(q, m, \alpha, \beta, \theta)\} \times S_k^\nu$,. Moreover, for all integers $k \ge 1$ and $\nu = +$ or $-$, the function $\theta \to \lambda_k^\nu(\theta) := \lambda_k^\nu(\theta, q, m, \alpha, \beta, \theta)$ is continuous decreasing and $\lim_{\theta\to0} \lambda_k^\nu(\theta) = +\infty$.

Lemma 4.20. *For all functions $q \in Q$, $m \in \Gamma^{++}$ and $\alpha, \beta \in E$, the bvp (4.12) admits two increasing sequences of half-eigenvalues $\left(\lambda_k^+(q, m, \alpha, \beta)\right)_{k\ge1}$ and $\left(\lambda_k^-(q, m, \alpha, \beta)\right)_{k\ge1}$ such that for all integers $k \ge 1$ and $\nu = +$ or $-$, the corresponding half-line of solutions lies on $\{\mu_k^\nu(m, \alpha, \beta)\} \times S_k^\nu$.*

Proof .

Let $q \in, Q$, $m \in \Gamma^{++}$ and $\alpha, \beta \in E$. Clearly for $k = 1$, we have $\lambda_k^+(q, m, \alpha, \beta) = \mu_1^+(q - \alpha, m, 0)$ and $\lambda_k^-(q, m, \alpha, \beta) = \mu_1^+(q - \beta, m, 0)$ that existence is guaranteed by Theorem 4.9. Fix $k \geq 2$, $\nu = +$ or $-$ and set $\omega_1 = \alpha$ and $\omega_2 = \beta$. Let for $\theta \in (0,1)$, $\lambda_{k-1}^\nu(\theta) = \lambda_{k-1}^\nu(q,, m, \alpha, \beta, \theta)$ and for $i = 1,2$, $\mu_i(\theta) = \mu_l^\nu(q - \omega_i, m, \theta)$ given respectively by Lemma 4.19 and Theorem 4.9. Because that the function $\lambda_{k-1}^\nu(\cdot)$ is decreasing, the functions $\mu_i(\cdot)$ are increasing and

$$\lim_{\theta \to 0} \lambda_{k-1}^\nu(\theta) = \lim_{\theta \to 1} \mu_i(\theta) = +\infty,$$

the equation $\lambda_{k-1}^\nu(\theta) = \mu_i(\theta)$ admits a unique solution $\theta_{k,i} \in (0,1)$.

Let for $\theta \in (0,1)$, ψ_θ be the eigenfunction associated with $\lambda_{k-1}^\nu(\theta)$ and for $i = 1,2$ $\phi_{\theta,i}$ be the eigenfunction associated with $\mu_i(\theta)$. We distinguish the following cases:

a) $\psi_\theta'(\theta) > 0$ for all $\theta \in (0,1)$. In this case $\lambda_k^\nu = \lambda_{k-1}^\nu(\theta_{k,1}) = \mu_i(\theta_{k,1})$ is the half-eigenvalue having as an eigenfunction the function $\psi_k \in S_k^\nu$ defined by

$$\psi_k(t) = \begin{cases} \psi_{\theta_{k,1}}(t), & \text{for } t \in [0, \theta_{k,1}], \\ \phi_{\theta_{k,1},1}(t) \left(\psi_{\theta_{k,1}}(\theta_{k,1}) / \phi'_{\theta_{k,1},1}(\theta_{k,1}) \right), & \text{for } t \in [\theta_{k,1}, 1]. \end{cases}$$

b) $\psi_\theta'(\theta) < 0$ for all $\theta \in (0,1)$. In this case $\lambda_k^\nu = \lambda_{k-1}^\nu(\theta_{k,2}) = \mu_i(\theta_{k,2})$ is the half-eigenvalue having as an eigenfunction the function $\psi_k \in S_k^\nu$ defined by

$$\psi_k(t) = \begin{cases} \psi_{\theta_{k,2}}(t), & \text{for } t \in [0, \theta_{k,2}], \\ \phi_{\theta_{k,2}}(t) \left(\psi_{\theta_{k,2}}(\theta_{k,2}) / \phi'_{\theta_{k,2}}(\theta_{k,2}) \right), & \text{for } t \in [\theta_{k,2}, 1]. \end{cases}$$

This ends the proof. ■

Lemma 4.21. *Let $q \in Q$, $m \in \Gamma^{++}$ and set for all $k \geq 1$*

$$\mu_k(q, m) = \lambda_k^+(q, m, 0, 0) = \lambda_k^-(q, m, 0, 0).$$

Then for any interval $[\gamma, \delta] \subset (0,1)$, $\mu_k(q, m) < \mu_k(q, m, [\gamma, \delta])$ where $(\mu_k(q, m, [\gamma, \delta]))$ is the sequence of eigenvalues of the bvp

$$\begin{cases} \mathcal{L}_q u = \mu m u, \ in \ (\gamma, \delta), \\ u(\gamma) = u(\delta) = 0. \end{cases}$$

Proof .

Fix $k \geq 1$ and set $\mu_1 = \mu_k(q, m)$ and $\mu_2 = \mu_k(q, m, [\gamma, \delta])$. Let for $i = 1,2$, ϕ_i be an eigenfunction associated with μ_i, having a sequence of zeros $\left(z_j^i\right)_{j=0}^{j=k}$, and without loss of generality, suppose that $\phi_1\phi_2 > 0$ in a right neighborhood of γ. We distinguish two cases.

i) $\phi_2 > 0$ in (γ, δ) (i.e. $k = 1$): In this case we obtain by Lemma 4.7

$$0 < \int_\gamma^\delta \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 = (\mu_2 - \mu_1) \int_\gamma^\delta m\phi_1\phi_2$$

leading to $\mu_2 > \mu_1$.

ii) $\phi_2(t_0) = 0$ for some $t_0 \in (\gamma, \delta)$: In this case consider the family $(\xi_j)_{j=0}^{j=k_0}$ defined by $\xi_0 = \gamma$, $\xi_{k_0} = \delta$ and $\phi_1(\xi_j) = 0$ for $j \in \{1, ..., k_0 - 1\}$ and note that $k_0 \leq k$. Thus, we have from Lemma 4.6 that there exist two integers m, n having the same parity, such that $\xi_m < z_n^2 < z_{n+1}^2 \leq \xi_{m+1}$. Therefore, we have $\phi_1, \phi_2 > 0$ in (z_n^2, z_{n+1}^2) and we obtain by Lemma 4.7

$$\begin{aligned} 0 &< \int_{z_n^2}^{z_{n+1}^2} \phi_1 \mathcal{L}_q \phi_2 - \phi_2 \mathcal{L}_q \phi_1 \\ &= (\mu_2 - \mu_1) \int_{z_n^2}^{z_{n+1}^2} m\phi_1\phi_2 \end{aligned}$$

leading to $\mu_2 > \mu_1$.

This ends the proof. ■

Theorem 4.22. *For all $q \in Q$, $m \in \Gamma^+$ and $\alpha, \beta \in E$ the bvp (4.12) admits two increasing sequences of simple half-eigenvalues $\left(\lambda_k^+(q, m, \alpha, \beta)\right)_{k\geq 1}$ and $\left(\lambda_k^-(q, m, \alpha, \beta)\right)_{k\geq 1}$ such that for all integers $k \geq 1$, the corresponding half-line of solutions lies on $\{\mu_k^\nu(m, \alpha, \beta)\} \times S_k^\nu$, $\nu = +, -$ with $\lim_{k\to\infty} \mu_k^\nu(q, m, \alpha, \beta) = +\infty$, aside from these solutions and the trivial one, there are no other solutions of (4.12). Furthermore, for $k \geq 1$ and $\nu = +$ or $-$, the half-eigenvalue $\lambda_k^\nu(\cdot, \cdot, \cdot, \cdot)$ has the following properties:*

1. *Let $q \in Q$, $m \in \Gamma^+$ and $\alpha_1, \alpha_2, \beta \in E$. If $\alpha_1 \leq \alpha_2$ in $(0,1)$, then $\lambda_k^\nu(q, m, \alpha_1, \beta) \geq \lambda_k^\nu(q, m, \alpha_2, \beta)$.*

2. *Let $q \in Q$, $m \in \Gamma^+$ and $\alpha, \beta_1, \beta_2 \in E$. If $\beta_1 \leq \beta_2$ in $(0,1)$, then $\lambda_k^\nu(q, m, \alpha, \beta_1) \geq \lambda_k^\nu(q, m, \alpha, \beta_2)$.*

3. *Let $q_1, q_2 \in Q$, $m \in \Gamma^+$ and $\alpha, \beta \in E$. If $q_1 \leq q_2$ in $(0,1)$, then $\lambda_k^\nu(q_1, m, \alpha, \beta) \leq \lambda_k^\nu(q_2, m, \alpha, \beta)$.*

4. *Let $m_1, m_2 \in \Gamma^+$, $\alpha, \beta \in E$, with $m_1 \leq m_2$ in $(0,1)$ and $m_1 < m_2$ in a subset of positive measure. If $\lambda_k^\nu(m_1, \alpha, \beta) \geq 0$ or $\lambda_k^\nu(m_2, \alpha, \beta) \geq 0$, then $\lambda_k^\nu(q, m_1, \alpha, \beta) > \lambda_k^\nu(q, m_2, \alpha, \beta)$ and if $\lambda_k^\nu(q, m_1, \alpha, \beta) \leq 0$ or $\lambda_k^\nu(q, m_2, \alpha, \beta) \leq 0$, then $\lambda_k^\nu(q, m_1, \alpha, \beta) < \lambda_k^\nu(q, m_2, \alpha, \beta)$.*

5. *If $m \in \Gamma^+$ and $(m_n) \subset \Gamma^+$ are such that $\lim m_n = m$ in E, then $\lim\limits_{n\to\infty} \lambda_k^\nu(q, m_n, \alpha, \beta) = \lambda_k^\nu(q, m, \alpha, \beta)$ for all $\alpha, \beta \in E$.*

Proof .

Let $q \in Q$, $m \in \Gamma^+$, $\alpha, \beta \in E$ and (ϵ_n) be a decreasing sequence of real numbers converging to 0 and let $A > 0$ be such that $\min(\mu_1(q-\alpha, m+\epsilon_1), \mu_1(q-\beta, m+\epsilon_1)) > -A$. Consider the BVP

$$\begin{cases} \mathcal{L}_{q+Am} u = \lambda m u + \alpha u^+ - \beta u^- \text{ in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{4.17}$$

and notice that λ is a half-eigenvalue of the (4.17) if and only if $(\lambda - A)$ is a half-eigenvalue of the bvp (4.12). Let for k and ν fixed, $\lambda_{k,n}^\nu = \lambda_k^\nu(q + Am, m + \epsilon_n, \alpha, \beta)$ and let $[\gamma, \delta] \subset (\xi, \eta)$ be such that $m > 0$ a.e. in (γ, δ).

First, because of

$$\begin{aligned} \lambda_{k,1}^\nu = \lambda_k^\nu(q + Am, m + \epsilon_1, \alpha, \beta) &\geq \lambda_1^\nu(q, m + \epsilon_1, \alpha, \beta) + A \\ &\geq \min(\mu_1(q-\alpha, m+\epsilon_1), \mu_1(q-\beta, m+\epsilon_1)) + A > 0, \end{aligned}$$

we have by Proposition 4.16 that for all $n \in \mathbb{N}$, $\lambda_{k,n+1}^\nu \geq \lambda_{k,n}^\nu \geq \lambda_{k,1}^\nu > 0$.

Set $\widetilde{q} = q + Am + (|\alpha| + |\beta|)$, Proposition 4.15, Lemma 4.21 and Proposition 4.16 lead to

$$0 < \lambda_{k,n}^\nu \leq \mu_k(\widetilde{q}, m + \epsilon_n) \leq \mu_k(\widetilde{q}, m + \epsilon_n, [\gamma, \delta]) \leq \mu_k(\widetilde{q}, m, [\gamma, \delta])$$

proving that $\lim \lambda_{k,n}^\nu = \lambda_k^\nu \in \mathbb{R}$. Thus, we conclude from Proposition 4.19 that $\lambda_k^\nu = \lambda_k^\nu(q + Am, m, \alpha, \beta)$.

Now, we need to prove that $\lim_{k\to\infty} \lambda_k^\nu(q + Am, m, \alpha, \beta) = +\infty$. To this aim set $\omega = |\alpha| + |\beta|$ and let $B > 0$ be such that $\overline{q} = q + Am - \omega + B(m + \epsilon_1) > 0$ in $[0,1)$. We have then from Propositions 4.15 and 4.16:

$$\begin{aligned} \lambda_k^\nu(q + Am, m, \alpha, \beta) &\geq \lambda_k^\nu(q + Am, m + \epsilon_1, \alpha, \beta) \\ &\geq \lambda_k^\nu(q + Am, m + \epsilon_1, \omega, \omega) \\ &\geq \lambda_k^\nu(q + Am, m + \epsilon_1, \omega, \omega) \\ &= \mu_k(q + Am - \omega, m + \epsilon_1) \\ &= \mu_k(\overline{q}, m + \epsilon_1) - B. \end{aligned}$$

Because that $(\mu_k(\overline{q}, m+\epsilon_1))$ is the sequence of characteristic-values of the positive compact operator $L_{\overline{q},m+\epsilon_1} : W \to W$ defined for $u \in W$ by

$$L_{\overline{q},m+\epsilon_1}u(t) = \int_0^1 G_{\overline{q}}(t,s)\,(m\,(s)+\epsilon_1)\,u(s)ds,$$

we have that $\lim_k \mu_k(\overline{q}, m+\epsilon_1) = +\infty$, proving that $\lim_k \lambda_k^\nu(q+Am, m, \alpha, \beta) = +\infty$.

At the end, Assertions 1, 2, 3, 4 and 5 follow from Propositions 4.11-4.18. ■

For the particular case of the bvp (4.12) where $\alpha = \beta = 0$, namely for the bvp

$$\begin{cases} \mathcal{L}_q = \mu m u, \ \text{ in } \ (0,1) \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{4.18}$$

we obtain from Theorem the following corollary.

Corollary 4.23. *For all pairs (q,m) in $Q \times \Gamma^+$, the set of eigenvalues of the bvp (4.18) consists in an unbounded increasing sequence of simple eigenvalues $(\mu_k(q,m))_{k\geq 1}$ such that eigenfunctions associated with $\mu_k(q,m)$ belong to S_k. Moreover, the mapping $\mu_k(\cdot,\cdot)$ has the following properties:*

1. *Let $q \in Q$, $m_1, m_2 \in \Gamma^+$ with $m_1 \leq m_2$ in $(0,1)$ and $m_1 < m_2$ in a subset of positive measure. If $\mu_k(q,m_1) \geq 0$ or $\mu_k(q,m_2) \geq 0$, then $\mu_k(q,m_1) > \mu_k(q,m_2)$ and if $\mu_k(q,m_1) \leq 0$ or $\mu_k(q,m_2) \leq 0$, then $\mu_k(q,m_1) < \mu_k(q,m_2)$.*

2. *If $m \in \Gamma^+$ and $(m_n) \subset \Gamma^+$ are such that $\lim m_n = m$ in E, then $\lim_{n\to\infty} \mu_k(q,m_n) = \mu_k(q,m)$.*

3. *Let $q_1, q_2 \in Q$ and $m \in \Gamma^+$. If $q_1 \leq q_2$ then , $\mu_k(q_1,m) \leq \mu_k(q_2,m)$ for all $k \geq 1$.*

The following proposition is a consequence of Assertion 2 in Corollary 4.23 and it will be used in the following section.

Proposition 4.24. *Let $q \in Q$ and $m \in \Gamma^+$ be such that $\mu_k(q,m) = 1$ for some integer $k \geq 1$. Then there exists $\varepsilon_0 > 0$ such that for all $p \in \Gamma^+$ with $\|p-m\| \leq \varepsilon_0$, $\mu_l(q,p) = 1$ implies $l = k$.*

Proof .

Let $\epsilon_0 > 0$ be such that $\epsilon_0 < \min(\mu_{k+1}(q,m) - \mu_k(q,m), \mu_k(q,m) - \mu_{k-1}(q,m))$, because of the continuity of the functions $\mu_{k-1}(q,m),\ \mu_{k+1}(q,m)$, there exists $\varepsilon_0 > 0$ such that for all $p \in \Gamma^+,\ \|p-m\| \leq \varepsilon_0$ implies

$$\mu_{k-1}(q,m) - \epsilon_0 \leq \mu_{k-1}(q,p) \leq \mu_{k-1}(q,m) + \epsilon_0 \tag{4.19}$$

and

$$\mu_{k+1}(q,m) - \epsilon_0 \leq \mu_{k+1}(q,p) \leq \mu_{k+1}(q,m) + \epsilon_0. \tag{4.20}$$

Let $p \in \Gamma^+$ with $\|p - m\| \leq \varepsilon_0$ and suppose that $\mu_l(q,p) = 1$ for some integer $l \geq 1$. If $l < k$, we have then from (4.19) the contradiction

$$1 = \mu_l(q,p) \leq \mu_{k-1}(q,p) \leq \mu_{k-1}(q,m) + \epsilon_0 < \mu_k(q,m)$$

and If $l > k$, we have then from (4.20) the contradiction

$$1 = \mu_l(q,p) \geq \mu_{k+1}(q,p) \geq \mu_{k+1}(q,m) - \epsilon_0 > \mu_k(q,m) = 1.$$

This shows that $l = k$ and the lemma is proved. ■

4.4 Nodal solutions to the nonlinear bvp

4.4.1 Main results

In all this section, ρ is a positive real parameter, q is a function in Q, m, α and β are functions in E and $f : [0,1] \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ is a continuous function. Main results of this section concern existence of nodal solutions to the bvp

$$\begin{cases} \mathcal{L}_q u = \rho u f(t,u) \text{ in } (0,1) \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{4.21}$$

where the function f is assumed to satisfy one of the following Hypotheses (4.22), (4.23) and (4.24).

$$\begin{cases} \lim_{u\to 0} f(t,u) = m(t), \\ \lim_{u\to -\infty} f(t,u) = \beta(t) \text{ and} \\ \lim_{u\to +\infty} f(t,u) = \alpha(t) \text{ in } E. \end{cases} \tag{4.22}$$

$$\begin{cases} \lim_{u\to 0} f(t,u) = m(t) \text{ in } E \text{ and} \\ \lim_{|u|\to +\infty} \left(\inf_{t\in[0,1]} f(t,u)\right) = +\infty. \end{cases} \tag{4.23}$$

$$\begin{cases} \lim_{u\to 0} u f(t,u) = 0, \\ \lim_{u\to 0} \left(\inf_{t\in[0,1]} f(t,u)\right) = +\infty, \\ \lim_{u\to -\infty} f(t,u) = \beta(t) \text{ and} \\ \lim_{u\to +\infty} f(t,u) = \alpha(t) \text{ in } E. \end{cases} \tag{4.24}$$

Remark 4.25. Notice that if the nonlinearity f satisfies one of the Hypotheses (4.22), (4.24) and (4.23), then there is $\omega_0 \in \Gamma^{++}$ such that $f(t,u) + \omega_0(t) > 0$ for all $t \in [0,1]$ and $u \in \mathbb{R}$.

The statement of the main results of this section and there proofs need to introduce some notations. In all this section we let:

$$\widetilde{q} = q^+ + \rho\left(m^- + 2\omega_0\right), \quad \widetilde{m} = \rho\left(m^+ + 2\omega_0\right) + q^-, \quad \widetilde{f}(t,u) = \rho\left(f(t,u) - m\right),$$

$$\widetilde{\alpha} = \rho\left(\alpha - m\right), \quad \widetilde{\beta} = \rho\left(\beta - m\right), \quad \varphi = \inf\left(\alpha, \beta\right) \quad \text{and} \quad \psi = \sup\left(\alpha^+, \beta^+\right),$$

where ω_0 is that in Remark 4.25.

Since in all this section the weight q is fixed in Q, we let for all $\chi \in \Gamma^+$ and all $k \geq 1$, $\mu_k(\chi) = \mu_k(q,\chi)$. In particular we let for all $k \geq 1$ and $\nu = +$ or $-$,

$$\widetilde{\mu}_k = \mu_k\left(\widetilde{q}, \widetilde{m}\right), \quad \widetilde{\lambda}_k^\nu = \lambda_k^\nu\left(\widetilde{q}, \widetilde{m}, \widetilde{\alpha}, \widetilde{\beta}\right).$$

The operators $T_0,\ T_\infty : W \to W$ are defined as follows

$$\begin{aligned} T_0u(t) &= \textstyle\int_0^1 G_{\widetilde{q}}(t,s)u(s)\widetilde{f}(s,u(s))ds, \\ T_\infty u(t) &= T_0u(t) - L^+_{\widetilde{q},\widetilde{\alpha}}u(t) + L^-_{\widetilde{q},\widetilde{\beta}}u(t) \\ &= \textstyle\int_0^1 G_{\widetilde{q}}(t,s)u(s)f^*(s,u(s))ds, \end{aligned}$$

where $f^*(s,u) = u\widetilde{f}(s,u) - \widetilde{\alpha}u^+ + \widetilde{\beta}u^-$. We have from Lemma 4.5 that $T_0,\ T_\infty$ are completely continuous.

The following Theorems 4.26, 4.28 and 4.27 are the main results of this section. They provide respectively existence and multiplicity results for the cases where the nonlinearity f is asymptoticaly linear, sublinear and superlinear.

Theorem 4.26. *Assume that Hypothesis (4.22) holds true.*

1. *Let i, j be two integers such that $i \geq j \geq 1$. The bvp (4.21) admits in each of $S_j^+, \ldots, S_i^+, S_j^-,, \ldots, S_i^-$ a solution if one of the following Hypothesis (4.25), (4.26), (4.27) and (4.28) holds true.*

$$\varphi, m^+ \in \Gamma^+ \text{ and } \mu_i(\varphi) < \rho < \mu_j(m^+), \tag{4.25}$$

$$\begin{cases} \varphi \in \Gamma^+, m^+ = 0,\ \mu_i(\varphi) < \rho \text{ and} \\ \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+ \end{cases} \tag{4.26}$$

$$\psi, m \in \Gamma^+ \text{ and } \mu_i < \rho < \mu_j(\psi), \tag{4.27}$$

$$\begin{cases} m \in \Gamma^+, \psi = 0,\ \mu_i(m) < \rho \text{ and} \\ \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{4.28}$$

2. *Let i, j be two integers such that $i \geq j \geq 1$ and $i \geq 2(j-1)$. The bvp (4.21) admits in each of $S^+_{2j}, \ldots, S^+_i, S^-_{2j-1}, \ldots, S^-_i$ a solution if one of the following Hypothesis (4.29) and (4.30) holds true.*

$$m, \beta^+ \in \Gamma^+ \text{ and } \mu_i(m) < \rho < \mu_j(\beta^+), \tag{4.29}$$

$$\begin{cases} m \in \Gamma^+, \beta^+ = 0,\ \mu_i(m) < \rho \text{ and} \\ \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{4.30}$$

3. *Let i, j be two integers such that $i \geq j \geq 1$ and $i \geq 2(j-1)$. The bvp (4.21) admits in each of $S^+_{2j-1}, \ldots, S^+_i, S^-_{2j}, \ldots, S^-_i$ a solution if one of the following Hypothesis (4.31) and (4.32) holds true.*

$$m, \alpha^+ \in \Gamma^+ \text{ and } \mu_i(m) < \rho < \mu_j(\alpha^+), \tag{4.31}$$

$$\begin{cases} m \in \Gamma^+, \alpha^+ = 0,\ \mu_i(m) < \rho \text{ and} \\ \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{4.32}$$

Theorem 4.27. *Assume that Hypothesis (4.23) holds true and let $j \geq 1$. The bvp (4.21) admits for all $k \geq j$ a solution in S^+_k and in S^-_k if one of the following Hypotheses (4.33) and (4.34) holds true.*

$$m^+ \in \Gamma^+ \text{ and } \mu_j(m^+) > \rho, \tag{4.33}$$

$$m^+ = 0 \text{ and } \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \tag{4.34}$$

Theorem 4.28. *Assume that Hypothesis (4.24) holds true, $q \in Q_\#$ and let $j \geq 1$.*

1. *The bvp (4.21) admits for all $k \geq j$ a solution in S^+_k and in S^-_k if one of the following Hypotheses (4.35) and (4.36) holds true.*

$$\psi \in \Gamma^+ \text{ and } \mu_j(\psi) > \rho, \tag{4.35}$$

$$\psi = 0 \text{ and } \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \tag{4.36}$$

2. *The bvp (4.21) admits a solution in S_k^+ for all $k \geq 2j$ and a solution in S_k^- for all $k \geq 2j-1$ if one of the following Hypotheses (4.37) and (4.38) holds true.*

$$\beta^+ \in \Gamma^+ \text{ and } \mu_j(\beta^+) > \rho, \tag{4.37}$$

$$\beta^+ = 0 \text{ and } \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \tag{4.38}$$

3. *The bvp (4.21) admits a solution in S_k^+ for all $k \geq 2j-1$ and a solution in S_k^- for all $k \geq 2j$ if one of the following Hypotheses (4.39) and (4.40) holds true.*

$$\alpha^+ \in \Gamma^+ \text{ and } \mu_j(\alpha^+) > \rho, \tag{4.39}$$

$$\alpha^+ = 0 \text{ and } \mu_j(\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \tag{4.40}$$

4.4.2 Related Lemmas

In this subsection we prove some intermediate results.

Lemma 4.29.

1. *If $m \in \Gamma^+$ and $\mu_l(m) < \rho$ for some $l \geq 1$, then $\widetilde{\mu}_k < 1$ for all $k \leq l$.*

2. *If $m^+ \in \Gamma^+$ and $\mu_l\left(m^+\right) > \rho$ for some $l \geq 1$, then $\widetilde{\mu}_k > 1$ for all $k \geq l$.*

3. *If $m = -m^-$ and $\mu_l\left(\chi_0\right) > 0$ for some $l \geq 1$ and $\chi_0 \in \Gamma^+$, then $\widetilde{\mu}_k > 1$ for all $k \geq l$.*

Proof .
If $m^+ \in \Gamma^+$, we have then

$$\begin{aligned}
\widetilde{\mu}_k &= \mu_k\left(q^+ + \rho\left(m^- + 2\omega_0\right), \rho\left(m^+ + 2\rho\omega_0\right) + q^-\right) \\
&= \mu_k\left(q^+ + 2\rho\omega_0 + \rho m^- - \widetilde{\mu}_k\left(2\rho\omega_0 + q^-\right), \rho m^+\right) \\
&= \mu_k\left(q + \left(1 - \widetilde{\mu}_l\right)\left(2\rho\omega_0 + q^-\right), \rho m^+\right) \\
&= \left(\mu_k\left(q + \left(1 - \widetilde{\mu}_l\right)\left(2\rho\omega_0 + q^-\right) + \rho m^-, m^+\right)/\rho\right).
\end{aligned} \tag{4.41}$$

Suppose that $m = m^+ \in \Gamma^+$, $\mu_l\left(m\right) < \rho$ for some $l \geq 1$ and $\widetilde{\mu}_k \geq 1$ for some $k \leq l$. We obtain from (4.41) and Assertion 3 in Proposition 4.15 the contradiction

$$1 \leq \widetilde{\mu}_k = \left(\mu_k\left(q + \left(1 - \widetilde{\mu}_k\right)\left(2\rho\omega_0 + q^-\right), m\right)/\rho\right) \leq \left(\mu_k\left(m\right)/\rho\right) \leq \left(\mu_l(m)/\rho\right) < 1.$$

This proves Assertion 1.

Similarly, suppose that $m^+ \in \Gamma^+$, $\mu_l(m) < \rho$ for some $l \geq 1$ and $\widetilde{\mu}_k \leq 1$ for some $k \geq l$. We obtain from (4.41) and Assertion 3 in Proposition 4.15 the contradiction

$$1 \geq \widetilde{\mu}_k = \left(\mu_k\left(q + (1-\widetilde{\mu}_k)\left(2\rho\omega_0 + q^-\right) + \rho m^-, m\right)/\rho\right) \leq \left(\mu_k(m)/\rho\right) \geq \left(\mu_l(m)/\rho\right) > 1.$$

This proves Assertion 2.

Suppose that $m = -m^-$ (i.e. $m^+ = 0$), $\mu_l(q, \chi_0) > 0$ for some $l \geq 1$ and $\chi_0 \in \Gamma^+$ and $\widetilde{\mu}_k \leq 1$ for some $k \geq l$. We read from

$$\begin{aligned} \widetilde{\mu}_k &= \mu_k\left(q^+ + \rho\left(m^- + 2\omega_0\right), \rho\left(m^+ + 2\omega_0\right) + q^-\right) \\ &= \mu_k\left(q^+ + \rho\left(m + 2\omega_0\right), 2\rho\omega_0 + q^-\right) \end{aligned}$$

that

$$\mu_k\left(q + (1-\widetilde{\mu}_k)\left(2\rho\omega_0 + q^-\right), \chi\right) = 0 \text{ for all } \chi \in \Gamma^+.$$

Therefore, Assertion 3 in Proposition 4.15 leads to the contradiction

$$0 = \mu_k\left(q + (1-\widetilde{\mu}_k)\left(2\rho\omega_0 + q^-\right), \chi_0\right) \geq \mu_k(\chi_0) \geq \mu_l(\chi_0) > 0.$$

This Proves Assertion 3 and ends the proof. ■

Lemma 4.30. *For all integers $l \geq 1$ and $\nu = +$ or $-$:*

1. *If $\varphi \in \Gamma^+$ and $\mu_l(\varphi) < \rho$ for some $l \geq 1$, then $\widetilde{\lambda}_k^\nu < 1$ for all $k \leq l$.*
2. *If $\psi \in \Gamma^+$ and $\mu_l(\psi) > \rho$ for some $l \geq 1$, then $\widetilde{\lambda}_l^\nu > 1$ for all $k \geq l$.*
3. *If $\psi = 0$ and $\mu_l(\chi_0) > 0$ for some $l \geq 1$ and $\chi_0 \in \Gamma^+$, then $\widetilde{\mu}_k > 1$ for all $k \geq l$.*

Proof .

To prove Assertion 1, we have to show that $\widetilde{\lambda}_l^\nu > 1$. By the way of contradiction, suppose that $\mu_l(\varphi) < \rho$ and $\widetilde{\lambda}_l^\nu \geq 1$ and let $u,v \in S_l^\nu$ be the eigenfunctions associated respectively with $\mu_l(\rho\varphi) = (\mu_l(\varphi)/\rho)$ and $\widetilde{\lambda}_l^\nu$. Notice that

$$\begin{cases} \mathcal{L}_q u = \mu_l(\rho\varphi)\rho\varphi u, & \text{in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases}$$

$$\begin{cases} \mathcal{L}_q v = \left(\widetilde{\lambda}_l^+ - 1\right)(\rho m + 2\rho\omega_0 + q^-)v + \rho\alpha v^+ - \rho\beta v^- \text{ in } (0,1), \\ v(0) = \lim_{t\to 1} v(t) = 0, \end{cases}$$

Let $(x_j)_{j=0}^{j=l}$ and $(y_j)_{j=0}^{j=l}$ be respectively the sequences of zeros of u and v. We distinguish then the following two cases:

i) $x_1 \le y_1$: in this case we have the contradiction:

$$\begin{aligned} 0 &\le \int_{x_0}^{x_1} v\mathcal{L}_q u - u\mathcal{L}_q v \\ &\le \int_{x_0}^{x_1} \mu_l(\rho\varphi)\rho\varphi uv - \left(\rho\alpha v^+ - \rho\beta v^-\right) u \\ &= \int_{x_0}^{x_1} \left(\mu_l(\rho\varphi)\varphi - \alpha\right)\rho u^+v^+ + \left(\mu_l(\rho\varphi)\varphi - \beta\right)\rho u^-v^- < 0. \end{aligned}$$

ii) $y_1 < x_1$: in this case Lemma 4.6 guarantees existence of two integers m, n having the same parity such that $y_m < x_n < x_{n+1} \le y_{m+1}$ and Lemma 4.7 leads to the contradiction:

$$\begin{aligned} 0 &< \int_{x_n}^{x_{n+1}} v\mathcal{L}_q u - u\mathcal{L}_q v \\ &\le \int_{x_n}^{x_{n+1}} \mu_l(\rho\varphi)\rho\varphi uv - \left(\rho\alpha v^+ - \rho\beta v^-\right) u \\ &= \int_{x_n}^{x_{n+1}} \left(\mu_l(\rho\varphi)\varphi - \alpha\right)\rho u^+v^+ + \left(\mu_l(\rho\varphi)\varphi - \beta\right)\rho u^-v^- < 0. \end{aligned}$$

We prove Assertion 2 by the same way. Suppose that $\mu_l(\psi) > \rho$ and $\widetilde{\lambda}_l^\nu \le 1$ and let $u,v \in S_l^\nu$ be the eigenfunctions associated respectively with $\mu_l(\rho\psi) = \mu_l(\psi)/\rho$ and $\widetilde{\lambda}_l^\nu$. We have that

$$\begin{cases} \mathcal{L}_q u = \mu_l(\rho\psi)\rho\psi u, \ \text{ in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases}$$

$$\begin{cases} \mathcal{L}_q v = \left(\widetilde{\lambda}_l^\nu - 1\right)(\rho m + 2\rho\omega_0 + q^-)v + \rho\alpha v^+ - \rho\beta v^- \text{ in } (0,1), \\ v(0) = \lim_{t\to 1} v(t) = 0. \end{cases}$$

Let $(x_j)_{j=0}^{j=l}$ and $(y_j)_{j=0}^{j=l}$ be respectively the sequences of zeros of u and v. We distinguish then the following two cases:

a) $x_1 \le y_1$: in this case we have the contradiction:

$$\begin{aligned} 0 &\le \int_{x_0}^{x_1} v\mathcal{L}_q u - u\mathcal{L}_q v \\ &\le \int_{x_0}^{x_1} \mu_l(\rho\psi)\rho\psi uv - \left(\rho\alpha v^+ - \rho\beta v^-\right) u \\ &= \int_{x_0}^{x_1} \left(\mu_l(\rho\psi)\psi - \alpha\right)\rho u^+v^+ + \left(\mu_l(\rho\varphi)\varphi - \beta\right)\rho u^-v^- < 0. \end{aligned}$$

b) $y_1 < x_1$: in this case Lemma 4.6 guarantees existence of two integers m, n having the same parity such that $y_m < x_n < x_{n+1} \leq y_{m+1}$ and Lemma 4.7 leads to the contradiction:

$$\begin{aligned} 0 &< \int_{x_n}^{x_{n+1}} v\mathcal{L}_q u - u\mathcal{L}_q v \\ &\leq \int_{x_n}^{x_{n+1}} \mu_l(\rho\psi)\rho\psi uv - \left(\rho\alpha v^+ - \rho\beta v^-\right) u \\ &= \int_{x_n}^{x_{n+1}} (\mu_l(\rho\psi)\psi - \alpha)\,\rho u^+ v^+ + (\mu_l(\rho\varphi)\varphi - \beta)\,\rho u^- v^- < 0. \end{aligned}$$

We have for all $k \geq 1$ and $\nu = +$ or $-$,

$$\begin{aligned} \widetilde{\lambda}_k^\nu &= \lambda_k^\nu \left(q^+ + \rho(m^- + 2\omega_0) + q^-, \rho(m^+ + 2\omega_0) + q^-, \rho\left(\alpha - m\right), \rho\left(\beta - m\right)\right) \\ &= \lambda_k^\nu \left(q^+ + \rho(m^+ + 2\omega_0) + q^-, \rho(m^+ + 2\omega_0) + q^-, \rho\alpha, \rho\beta\right). \end{aligned}$$

This can be read that for all $\chi \in \Gamma^+$

$$0 = \lambda_k^\nu \left(q + \left(1 - \widetilde{\lambda}_k^\nu\right)\left(\rho(m^+ + 2\omega_0) + q^-\right), \chi, \rho\alpha, \rho\beta\right).$$

Therefore, if $\psi = 0$, $\mu_l\left(\chi_0\right) > 0$ for some $l \geq 1$ and $\chi_0 \in \Gamma^+$ and $\widetilde{\lambda}_k^\nu \leq 1$ for some $k \geq l$, Proposition 4.15 leads to the contradiction

$$\begin{aligned} 0 &= \lambda_k^\nu \left(q + \left(1 - \widetilde{\lambda}_k^\nu\right)\left(\rho(m^+ + 2\omega_0) + q^-\right), \chi_0, \rho\alpha, \rho\beta\right) \\ &\geq \lambda_k^\nu \left(q, \chi_0, 0, 0\right) = \mu_k\left(\chi_0\right) \geq \mu_l\left(\chi_0\right) > 0. \end{aligned}$$

The proof is complete. ∎

Lemma 4.31.

1. *If $\alpha^+ \in \Gamma^+$ and $\mu_l(\alpha^+) > \rho$ for some $l \geq 1$, then $\widetilde{\lambda}_k^+ > 1$ for all $k \geq 2l - 1$ and $\widetilde{\lambda}_k^- > 1$ for all $k \geq 2l$.*

2. *If $\alpha^+ = 0$ and $\mu_l(\chi_0) > \rho$ for some $l \geq 1$ and $\chi_0 \in \Gamma^+$, then $\widetilde{\lambda}_k^+ > 1$ for all $k \geq 2l - 1$ and $\widetilde{\lambda}_k^- > 1$ for all $k \geq 2l$.*

3. *If $\beta^+ \in \Gamma^+$ and $\mu_l(\beta^+) > \rho$ for some $l \geq 1$, then $\widetilde{\lambda}_k^+ > 1$ for all $k \geq 2l$ and $\widetilde{\lambda}_k^- > 1$ for all $k \geq 2l - 1$.*

4. *If $\beta^+ = 0$ and $\mu_l(\chi_0) > \rho$ for some $l \geq 1$ and $\chi_0 \in \Gamma^+$, then $\widetilde{\lambda}_k^+ > 1$ for all $k \geq 2l - 1$ and $\widetilde{\lambda}_k^- > 1$ for all $k \geq 2l$.*

Proof .

To be brief, we present the proof of Assertions 1 and 2, the other assertions are obtained similarly. Suppose that $\alpha^+ \in \Gamma^+$ and $\mu_l(\alpha^+) > \rho$ and let ϕ, ϑ, ψ be respectively the eigenfunctions associated respectively with $\mu_l(\alpha)$, $\widetilde{\lambda}^+_{2l-1}$ and $\widetilde{\lambda}^-_{2l}$. Thus ϕ, ϑ, ψ satisfy

$$\begin{cases} \mathcal{L}_q\phi = \mu_l(\rho\alpha)\rho\alpha\phi, \ \text{ in } (0,1), \\ \phi\,(0) = \lim_{t\to 1}\phi(t) = 0, \end{cases}$$

$$\begin{cases} \mathcal{L}_q\vartheta = \left(\widetilde{\lambda}^+_{2l-1} - 1\right)(\rho m + 2\rho\omega_0 + q^-)\vartheta + \rho\alpha\vartheta^+ - \rho\beta\vartheta^-, \text{ in } (0,1), \\ \vartheta\,(0) = \lim_{t\to 1}\vartheta(t) = 0, \end{cases}$$

$$\begin{cases} \mathcal{L}_q\psi = \left(\widetilde{\lambda}^-_{2l} - 1\right)(\rho m + 2\rho\omega_0 + q^-)\psi \ + \rho\alpha\psi^+ - \rho\beta\psi^-, \text{ in } (0,1), \\ \psi\,(0) = \lim_{t\to 1}\psi(t) = 0. \end{cases}$$

Let $(x_j)_{j=0}^{j=l}$, $(y_j)_{j=0}^{j=2l-1}$ and $(y_j)_{j=0}^{j=2l}$ be respectively the sequences of zeros of ϕ, ϑ and ψ. Thus, if $\widetilde{\lambda}^+_{2l-1} \le 1$, then

$$\left(\widetilde{\lambda}^+_{2l-1} - 1\right)(\rho m + 2\rho\omega_0 + q^-) + \rho\alpha < \rho\alpha < \frac{\mu_l\,(\alpha)}{\rho}\rho\alpha = \mu_l\,(\rho\alpha)\,\rho\alpha$$

and we obtain from Lemma 4.8 that in each interval (y_{2j}, y_{2j+1}), $j = 0, ..., l-1$, ϕ admits a zero. This contradicts $\phi \in S_l$.

Similarly, if $\widetilde{\lambda}^-_{2l} \le 1$ then

$$\left(\widetilde{\lambda}^-_{2l} - 1\right)(\rho m + 2\rho\omega_0 + q^-) + \rho\alpha < \mu_l\,(\rho\alpha)\,\rho\alpha$$

and we obtain from Lemma 4.8 that in each interval (y_{2j+1}, y_{2j+2}), $j = 0, ..., l-1$, ϕ admits a zero. This contradicts $\phi \in S_l$.

Suppose that $\alpha^+ = 0$, $\mu_l(\chi_0) > 0$ for some $l \ge 1$ and $\chi_0 \in \Gamma^+$ and let ϕ, ϑ, ψ be respectively the eigenfunctions associated respectively with $\mu_l(\chi_0)$, $\widetilde{\lambda}^+_{2l-1}$ and $\widetilde{\lambda}^-_{2l}$. Thus ϕ, ϑ, ψ satisfy

$$\begin{cases} \mathcal{L}_q\phi = \mu_l(\chi_0)\chi_0\phi, \ \text{ in } (0,1), \\ \phi\,(0) = \lim_{t\to 1}\phi(t) = 0, \end{cases}$$

$$\begin{cases} \mathcal{L}_q\vartheta = \left(\widetilde{\lambda}^+_{2l-1} - 1\right)(\rho m^+ + 2\rho\omega_0 + q^-)\vartheta + \rho\alpha\vartheta^+ - \rho\beta\vartheta^-, \text{ in } (0,1), \\ \vartheta\,(0) = \lim_{t\to 1}\vartheta(t) = 0, \end{cases}$$

$$\begin{cases} \mathcal{L}_q \psi = \left(\widetilde{\lambda}_{2l}^{-} - 1\right)(\rho m + 2\rho\omega_0 + q^-)\psi \; + \rho\alpha\psi^+ - \rho\beta\psi^-, \text{ in } (0,1), \\ \psi(0) = \lim_{t\to 1}\psi(t) = 0. \end{cases}$$

Let $(x_j)_{j=0}^{j=l}$, $(y_j)_{j=0}^{j=2l-1}$ and $(y_j)_{j=0}^{j=2l}$ be respectively the sequences of zeros of ϕ, ϑ and ψ. Thus, if $\widetilde{\lambda}_{2l-1}^{+} \leq 1$ then

$$\left(\widetilde{\lambda}_{2l-1}^{+} - 1\right)(\rho m + 2\rho\omega_0 + q^-) + \rho\alpha < \rho\alpha < \frac{\mu_l(\alpha)}{\rho}\rho\alpha = \mu_l(\rho\alpha)\,\rho\alpha$$

and we obtain from Lemma 4.8 that in each interval (y_{2j}, y_{2j+1}), $j = 0, ..., l-1$, ϕ admits a zero. This contradicts $\phi \in S_l$.

Similarly, if $\widetilde{\lambda}_{2l}^{-} \leq 1$ then

$$\left(\widetilde{\lambda}_{2l}^{-} - 1\right)(\rho m + 2\rho\omega_0 + q^-) + \rho\alpha < \mu_l(\rho\alpha)\,\rho\alpha$$

and we obtain from Lemma 4.8 that in each interval (y_{2j+1}, y_{2j+2}), $j = 0, ..., l-1$, ϕ admits a zero. This contradicts $\phi \in S_l$. ■

Lemma 4.32. *Let (m_n) be a sequence in Γ^+ such that $\lim_{n\to+\infty}(\inf_{t\in[0,1]} m_n(t)) = +\infty$. Then for all $q \in Q$ and $k \geq 1$, $\lim_{n\to+\infty}\mu_k(m_n) = 0$.*

Proof .

For arbitrary $A > 0$, there is $n_A \geq 1$ such that $m_n \geq A$ for all $n \geq n_A$. Thus, we obtain by means of Assertion 1 in Corollary 4.23 that for all $k \geq 1$ and $n \geq n_A$,

$$|\mu_k(m_n)| \leq |\mu_k(A)| = (|\mu_k(1)| / A),$$

proving that $\lim_{n\to+\infty}\mu_k(m_n) = 0$. ■

Lemma 4.33. *Assume that $q \in Q_\#$ and let u be a nontrivial solution to the bvp (4.21), then either $u \in S_k^\nu$ for some $k \geq 1$ and $\nu = +, -$ or u has an infinite monotone sequence of simple zeros.*

Proof .

We distinguish two cases:

i) u has a finite number of zeros $(z_j)_{j=0}^{j=l}$, in this case we have for all j, $0 \leq j \leq l-1$,

$$|u(t)| \geq \rho^*_{z_j, z_{j+1}}(t) \sup_{t\in[z_j, z_{j+1}]} |u(t)|, \text{ in } [z_j, z_{j+1}]$$

leading to

$$\begin{aligned} \left| \frac{u(t)}{t - z_j} \right| &\geq \sup_{t \in [z_j, z_{j+1}]} |u(t)| / \Psi_q(1) \text{ for } t \text{ near } z_j \text{ and} \\ \left| \frac{u(t)}{t - z_{j+1}} \right| &\geq \sup_{t \in [z_j, z_{j+1}]} |u(t)| / \Psi_q(1). \end{aligned}$$

Passing to the limits we obtain that $|u'(z_j)| > 0$ and $|u'(z_{j+1})| > 0$. This proves that all zeros of u are simple and $u \in S_l^\nu$ for some $\nu = +$ or $-$.

ii) u has an infinite number of zeros, in this case there is $z_* \in [0,1]$ such that $u(z_*) = u'(z_*) = 0$. We claim that there is a monotone sequence of simple zeros (t_n) such that $\lim t_n = z_*$. Indeed, if this fails then there is an interval $[a,b] \subsetneq [0,1]$ such that $u = 0$ in $[a,b]$ and $z_* \in [a,b]$. Set then

$$\begin{aligned} t_+ &= \sup \{ t \geq b : u(s) = 0 \text{ for all } s \in [b,t] \}, \\ t_- &= \inf \{ t \leq a : u(s) = 0 \text{ for all } s \in [t,a] \}. \end{aligned}$$

Since u is a nontrivial solution, we have $t_- > 0$ or $t_+ < 1$. Without loss of generality, suppose that $t_+ < 1$ and $u > 0$ in (t_+, t_*) where $t_* = \sup \{ t > t_+ : u(t) > 0 \}$. In one hand, we have

$$u'(t_+) = \lim_{t \overset{\leq}{\to} t_+} \frac{u(t)}{t - t_+} = 0.$$

In the other, we obtain from Lemma 4.3 the contradiction

$$u'(t_+) = \lim_{t \overset{\geq}{\to} t_+} \frac{u(t)}{t - t_+} \geq \left(\sup_{t \in [t_+, t_*]} u(t) / \Psi_q(1) \right) > 0.$$

This proves that there is a monotone sequence of zeros (t_n) of u and the simplicity of t_n is obtained again by means of Lemma 4.3. This achieves the proof. ∎

The following lemma is a adapted version of Corduneanu compacteness criterion:

Lemma 4.34. *A nonempty bounded subset Ω is relatively compact in W if*

(a) Ω is locally equicontinuous on $[0,1)$, that is, equicontinuous on every compact interval of $[0,1)$ and

(b) Ω is equiconvergent at 1, that is, given $\epsilon > 0$, there corresponds $T(\epsilon) \in (0,1)$ such that $|x(t)| < \epsilon$ for any $t \geq T(\epsilon)$ and $x \in \Omega$.

4.4.3 Proofs of Theorems 4.26 and 4.27

An associated bifurcation bvp

Consider the bvp

$$\begin{cases} \mathcal{L}_{\widetilde{q}}u = \mu\widetilde{m}u + u\widetilde{f}(t,u) \text{ in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{4.42}$$

where μ is real parameter.

By a solution to the bvp (4.42), we mean a pair $(\mu, u) \in \mathbb{R} \times W^2$ satisfying the differential equation in the bvp (4.42). Notice that $u \in W^2$ is a solution to the bvp (4.21) if and only if $(1, u)$ is a solution to the bvp (4.42). For this reason, we will study the bifurcation diagram of the bvp (4.42) and by means of Rabinowitz's global bifurcation theory, we will prove that the set of solutions to the bvp (4.42) consists in an infinity of unbounded components, each branching from a point on the line $\mathbb{R} \times \{0\}$ joining a point on $\overline{\mathbb{R}} \times \{\infty\}$. Obviously, each component having the starting point and the arrival point oppositely located relatively to 1, carries a solution of the bvp (4.21) and main results of this section will be proved once we compute the number of such components.

Lemma 4.35. *From each $\widetilde{\mu}_l$ bifurcate two unbounded components of nontrivial solutions to the bvp (4.42) ζ_l^+ and ζ_l^-, such that $\zeta_l^\nu \subset \mathbb{R} \times S_l^\nu$.*

Proof .
It follows from Lemma 4.5 that solutions to the bvp (4.42) are those satisfying to fixed point equation

$$u = \mu L_{\widetilde{q},\widetilde{m}}u + T_0(u). \tag{4.43}$$

In order to use the global bifurcation theory, let us prove that all characteristic value of $L_{\widetilde{q},\widetilde{m}}$ are of algebraic multiplicity one. To this aim let $u \in N\left((I - \widetilde{\mu}_k L_{\widetilde{q},\widetilde{m}})^2\right)$ and set $v = u - \widetilde{\mu}_k L_{\widetilde{q},\widetilde{m}}u$, then $v \in N(I - \widetilde{\mu}_k L_{\widetilde{q},\widetilde{m}}) = \mathbb{R}\phi_k$ and $u - \widetilde{\mu}_k L_{\widetilde{q},\widetilde{m}}u = \eta\phi_k$ for some $\eta \in \mathbb{R}$. In another way, v satisfies the bvp

$$\begin{cases} -v'' + \widetilde{q}v = \widetilde{\mu}_k\widetilde{m}v - \eta\widetilde{m}\phi_k, \text{ in } (0,1) \\ u(0) = \lim_{t\to 1} u(t) = 0. \end{cases}$$

Multiplying the differential equation in the above bvp by ϕ_k and integrating on $(0,1)$ we obtain

$$\eta\widetilde{\mu}_k \int_0^1 \widetilde{m}\phi_k^2 dt = 0.$$

leading to $\eta = 0$ and $u = \widetilde{\mu}_k L_{\widetilde{m}} u \in \mathbb{R}\phi_k$.

Now, we need to prove that $T_0(u) = \circ(\|u\|)$ near 0. Indeed, let $(u_n) \subset W$ with $\lim \|u_n\| = 0$, we have

$$\frac{|T_0 u_n(t)|}{\|u_n\|} \leq \int_0^1 G_{\widetilde{q}}(t,s) \left| \widetilde{f}(s, u_n(s)) \right| ds \ \leq \overline{G}_{\widetilde{q}} \int_0^1 \left| \widetilde{f}(s, u_n(s)) \right| ds$$

We have from Hypothesis (4.22) that $\widetilde{f}(s, u_n(s)) \to 0$ as $n \to +\infty$ for all $s \in (0,1)$. Thus, we conclude by the Dominated convergence Theorem that $T_0(u) = \circ(\|u\|)$ near 0.

Let l_k be the projection of W on $\mathbb{R}\phi_k$, $\widetilde{W} = \{u \in W : l_k u = 0\}$ and for $\xi > 0$, $\eta \in (0,1)$ and $\nu = +$ or $-$,

$$K^{\nu}_{\xi,\eta} = \{(\mu, u) \in \mathbb{R} \times W : |\mu - \widetilde{\mu}_k| < \xi \text{ and } \nu l_k u > \eta \|u\|\} .$$

Since Lemma 4.5 guarantees that the operators $L_{\widetilde{m}}$ and T_0 are respectively compact and completely continuous, we have from Theorem 1.40 and Theorem 1.25 in [52], that from $(\widetilde{\mu}_k, 0)$ bifurcate two components ζ_k^+ and ζ_k^- of nontrivial solutions to Equation (4.43) such that there is $\varsigma_0 > 0$, $\zeta_k^{\nu} \cap B(0,\varsigma) \subset K^{\nu}_{\xi,\eta}$ for all $\varsigma < \varsigma_0$ and if $u = \alpha\phi_k + w \in \zeta_k^{\nu}$ then $|\mu - \widetilde{\mu}_k| = \circ(1)$, $w = \circ(|\alpha|)$ for α near 0.

We claim that there is $\delta > 0$ such that $\zeta_k^{\nu} \cap B(0,\zeta) \subset \mathbb{R} \times S_k^{\nu}$; for all $\varsigma < \delta$. Indeed, let $(\mu_n, u_n)_{n\geq 1} \subset \zeta_k^{\nu}$ be such that $\lim (\mu_n, u_n) = (\widetilde{\mu}_k, 0)$, we have from Hypothesis (4.22) that $f(s, u_n(s)) \to m$, that is $\lim \mu_n f(s, u_n(s)) = \mu_k m(s)$ and Lemma 4.24 guarantees that there is $n_0 \geq 1$ such that $u_n \in S_k$ for all $n \geq n_0$. Moreover, if $u_n = \alpha_n\phi_k + w_n$ then $\lim \frac{u_n}{\alpha_n} = \phi_k$ in E proving that $\nu u_n(t) > 0$ for t in a right neighborhood of 0 and $\nu u_n'(0) > 0$ (otherwise, if $u_n'(0) = 0$ then the existence and uniqueness result for ODEs leads to $u_n = 0$).

Also, if $(\mu_*, u_*) \in \zeta_k^{\nu}$ then for all sequence $(\mu_n, u_n)_{n\geq 1} \subset \zeta_k^{\nu}$ be such that $\lim (\mu_n, u_n) = (\mu_*, u_*)$, we have $\lim \mu_n f(s, u_n(s)) = \mu_* f(s, u_*(s))$ in E and Lemma 4.24 guarantees existence of $n_0 \geq 1$ such that $u_n \in S_k$ for all $n \geq n_0$. This shows that $\zeta_k^{\nu} \subset \mathbb{R} \times S_k^{\nu}$ and ζ_k^{ν} is unbounded in $\mathbb{R} \times W$. The lemma is proved. ■

Proof of Theorem 4.26

Step 1. In this step we prove that for all $l \geq 1$ and $\nu = +$ or $-$, the projection of the component ζ_l^{ν} on the real axis is bounded. Since the nonlinearity f satisfies Hypothesis (4.22), there is $\gamma \in \Gamma^{++}$ be such that

$$-\gamma(t) \leq f(t,u) \leq \gamma(t) \text{ for all } t \in [0,1] \text{ and } u \in \mathbb{R}.$$

Let for $\kappa = +$ or $-$, $\psi_{k,\kappa} \in S_k^\nu$ be the eigenfunction associated with $\mu_{k,\kappa} = \mu_k(\widetilde{q} - \rho(m + \kappa\gamma), \widetilde{m})$ and $(\mu, u) \in \zeta_k^\kappa$. It follows from Lemma 4.6 and Lemma 4.7 that there exist two intervals (ξ_1, η_1) and (ξ_2, η_2) where $u\psi_{k,\kappa} \geq 0$ and such that

$$\begin{aligned} 0 &\geq \int_{\xi_1}^{\eta_1} \psi_{k,+}\mathcal{L}_{\widetilde{q}}u - u\mathcal{L}_{\widetilde{q}}\psi_{k,+} = \int_{\xi_1}^{\eta_1} (\mu - \mu_{k,+})\, \widetilde{m}\psi_{k,+}u + (f(s,u) + \gamma)\, u\psi_{k,+} \\ &\geq (\mu - \mu_{k,+}) \int_{\xi_1}^{\eta_1} \widetilde{m}\psi_{k,+}u ds, \end{aligned}$$

$$\begin{aligned} 0 &\leq \int_{\xi_2}^{\eta_2} \psi_{k,-}\mathcal{L}_{\widetilde{q}}u - u\mathcal{L}_{\widetilde{q}}\psi_{k,-} = \int_{\xi_2}^{\eta_2} ((\mu - \mu_{k,-})\, \widetilde{m}u\psi_{k,-} + (f(s,u) - \gamma)\, u\psi_{k,-})\, ds \\ &\leq (\mu - \mu_{k,-}) \int_{\xi_2}^{\eta_2} \widetilde{m}u\psi_{k,-} ds. \end{aligned}$$

The above inequalities lead to $\mu_{k,+} \leq \mu \leq \mu_{k,-}$.

Step 2. In this step we prove that for all $l \geq 1$ and $\nu = +$ or $-$, the component ζ_l^ν rejoins the point $(\widetilde{\lambda}_l^\nu, \infty)$. Notice that (4.43) is equivalent to

$$u = \mu L_{\widetilde{m}}u + L_{\widetilde{\alpha}-\widetilde{m}}I^+u - L_{\widetilde{\beta}-\widetilde{m}}I^-u + T_\infty u. \tag{4.44}$$

We prove that $K(u_n) = \circ(\|u_n\|)$ near ∞. Indeed; from lemma (4.4) in (**i**) we have

$$(|T_\infty u_n(t)| / \|u_n\|) \leq \int_0^1 P_n(s)ds,$$

where

$$P_n(s) = \overline{G}_{\widetilde{q}} \left| f(s, u_n(s)) \frac{u_n(s)}{\|u_n\|} - \widetilde{\alpha}(s) \frac{u_n^+(s)}{\|u_n\|} + \widetilde{\beta}(s) \frac{u_n^-(s)}{\|u_n\|} \right|.$$

Therefore, we have to prove that $\int_0^1 P_n(s)ds \to 0$ as $n \to \infty$.

We distinguish the following three cases:

i) $\lim u_n(s) = +\infty$: In this case, from (4.22) we obtain

$$P_n(s) \leq \overline{G}_{\widetilde{q}}|\widetilde{f}(s, u_n(s)) - \widetilde{\alpha}(s)| \to 0 \quad \text{as } n \to +\infty.$$

ii) $\lim u_n(s) = -\infty$: in this case, from (4.22) we obtain

$$P_n(s) \leq \overline{G}_{\widetilde{q}}|\widetilde{f}(s, u_n(s)) - \widetilde{\beta}(s)| \to 0 \quad \text{as } n \to +\infty.$$

iii) $\lim u_n(s) \neq \pm\infty$: in this case there may exist subsequences $(u_{n_k^1}(s))$ and $(u_{n_k^2}(s))$ such that $(u_{n_k^1}(s))$ is bounded and $\lim u_{n_k^2}(s) = \pm\infty$. Arguing as in the above two cases we obtain that $\lim P_{n_k^2}(s) = 0$ and we have from (4.22)

$$P_{n_k^1}(s) \leq \overline{G}_{\widetilde{q}} \left(\widetilde{f}(t, u(t)) + \widetilde{\alpha}(s) + \widetilde{\beta}(s) \right) \left(|u_{n_k^1}(s)| / \|u_{n_k^1}\| \right) \to 0 \quad \text{as } k \to +\infty,$$

proving that $T_\infty(u_n) = \circ(\|u_n\|)$ at ∞.

Now, let (μ_n, u_n) be sequence in ζ_k^ν with $\lim_{n\to+\infty} \|u_n\| = +\infty$ then $v_n = (u_n/\|u_n\|)$ satisfies

$$v_n = \mu_n L_{\widetilde{q},\widetilde{m}} v_n + L^+_{\widetilde{q},\widetilde{\alpha}} v_n - L^-_{\widetilde{q},\widetilde{\beta}} v_n + (T_\infty(u_n)/\|u_n\|) \tag{4.45}$$

with $T_\infty(u_n) = o(\|u_n\|)$ at ∞. By the compactness of the operators $L_{\widetilde{m}}, L_{\widetilde{\alpha}-\widetilde{m}}, L_{\widetilde{\beta}-\widetilde{m}}$, we obtain from (4.45) existence of $v_+, v_- \in W$ such that for $\kappa = +$ or $-$, $\|v_\kappa\| = 1$ and

$$v_\kappa = \mu_\kappa L_{\widetilde{q},\widetilde{m}} v_\kappa + L^+_{\widetilde{q},\widetilde{\alpha}} v_\kappa - L^-_{\widetilde{q},\widetilde{\beta}} v_\kappa,$$

where $\mu_+ = \limsup \mu_n$ and $\mu_- = \liminf \mu_n$. We have from Lemma 4.17 that for $\kappa = +$ or $-$, $v_\kappa \in S_l^\nu$ with $l \le k$. We claim that there is an integer $n_+ \ge 1$ such that $v_\kappa v_n > 0$ in $(z_{l-1} + \delta, 1)$. Indeed, if there a subsequence (v_{n_i}) such that for all $i \ge 1$, v_{n_i} has at a zero $x_{n_i} \in (z_{l-1} + \delta, 1)$ and v_{n_i} does not vanish in $(x_{n_i}, 1)$ then

$$\mu_n = \mu_1(\widetilde{q} - \widetilde{f}(s, u_n), \widetilde{m}, x_{n_i}) \ge \mu_1(q - \widetilde{\gamma}, \widetilde{m}, x_{n_i}).$$

Passing to the limit, we obtain from Theorem 4.9 the contradiction

$$+\infty > \mu_\kappa \ge \lim \mu_1(q - \widetilde{\gamma}, \widetilde{m}, x_{n_i}) = +\infty.$$

From all the above, we obtain that for all $n \ge n_+$, v_{n_i} belongs to S_l^ν and $l = k$.

Step 3. Notice that $u \in W^1 \cap C^2([0,1), \mathbb{R})$ is a solution to the bvp (4.21) if and only if $(1, u)$ is a solution to the bvp (4.42). This means that any component ζ_k^ν having the starting point $(\widetilde{\mu}_k, 0)$ and the arrival point $(\widetilde{\lambda}_k^\nu, \infty)$, oppositely located relatively to 1, carries a solution of the bvp (4.21). Therefore, we have to compute in each of the cases stated in Theorem 4.26 the number of such components. To be brief, we present only the proofs of Assertions 1 and 3.

Suppose that there is two integers i and j such that $i \ge j \ge 1$ and $\max(\mu_i(\alpha), \mu_i(\beta)) < \rho < \mu_j(m)$. We have then from Assertion 1 in Lemma 4.29 and Assertion 1 in Lemma 4.31 that $\widetilde{\mu}_j > 1$ and $\widetilde{\lambda}_i^\nu < 1$. Therefore, for all integers $l \in \{j, ..i\}$ and $\nu = +$ or $-$, the component ζ_l^ν crosses the hyperplan $\{1\} \times W$.

Now, Suppose that there is two integers i and j such that $i \ge j \ge 1$, with $i \ge 2(j-1)$ and $\mu_i(m) < \rho < \mu_j(\beta)$. We have then from Assertion 1 in Lemma 4.29 and Assertion 2 in Lemma 4.30 that $\widetilde{\mu}_i < 1$, $\widetilde{\lambda}^-_{2j-1} > 1$ and $\widetilde{\lambda}^+_{2j} > 1$. Therefore, for all integers $l \in \{2j-1, ..i\}$, the component ζ_l^- crosses the hyperplan $\{1\} \times W$ and for all integers $l \in \{2j, ..i\}$, the component ζ_l^+ crosses the hyperplan $\{1\} \times W$.

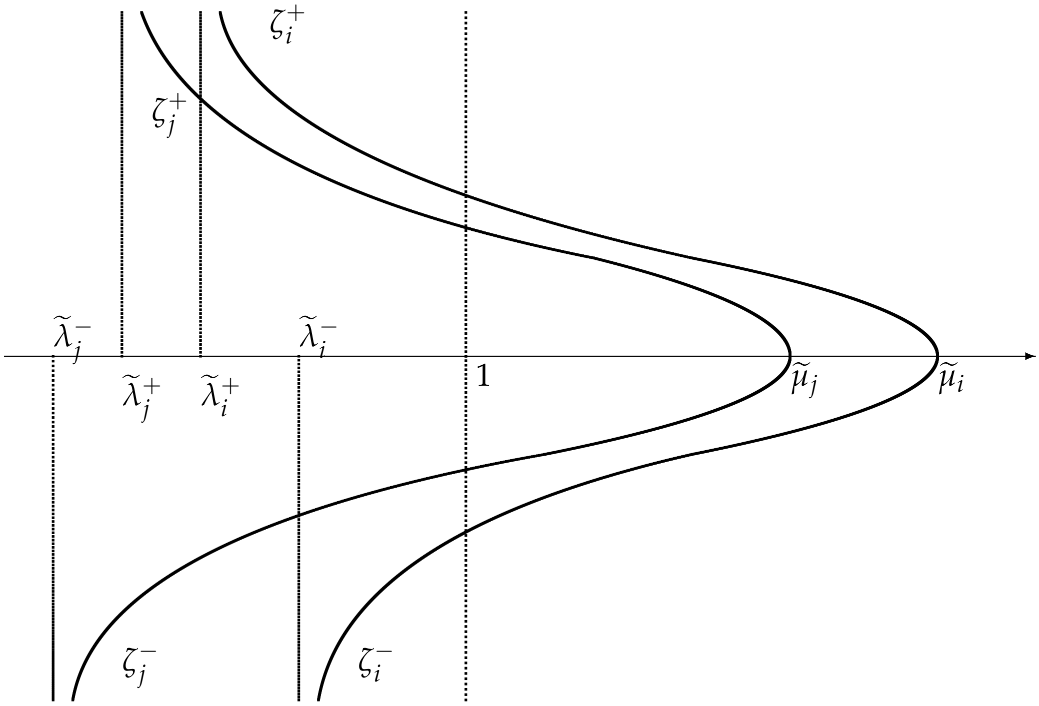

Fig. B: $\mu_i(\varphi) < \rho < \mu_j(m^+)$

Proof of Theorem 4.27

Step 1. In this step we prove that for all $l \geq 1$ and $\nu = +$ or $-$, the projection of the component ζ_l^ν on the real axis is upper bounded . Since the nonlinearity f satisfies Hypothesis (4.23), there is $\gamma \in \Gamma^{++}$ be such that

$$f(t,u) \geq -\gamma(t) \text{ for all } t \in [0,1] \text{ and } u \in \mathbb{R}.$$

Because the nonlinearity f satisfies Hypothesis (4.23) there is $\gamma \in \Gamma^{++}$ such that

$$f(t,u) \geq -\gamma(t) \text{ for all } t \in [0,1] \text{ and } u \in \mathbb{R}.$$

Fix k and ν and let us prove first that if $(\mu, u) \in \zeta_k^\nu$ then $\mu \leq \mu_{k,-} = \mu_k(\widetilde{q} - \rho(m-\gamma), \widetilde{m})$. To this aim, let $\psi_k \in S_k^\nu$ be the eigenfunction associated with $\mu_{k,-}$, it follows from Lemma 4.6 and Lemma 4.7 that there exists an interval (ξ, η) where $u\psi_k \geq 0$ and we have

$$\begin{aligned} 0 &\leq \int_\xi^\eta \psi_k \mathcal{L}_{\widetilde{q}} u - u \mathcal{L}_{\widetilde{q}} \psi_k = \int_\xi^\eta \left((\mu - \mu_{k,-})\, \widetilde{m} u \psi_k + (f(s,u) - \gamma)\, u\psi_k \right) ds \\ &\leq (\mu - \mu_{k,-}) \int_\xi^\eta \widetilde{m} u \psi_k ds \end{aligned}$$

leading to $\mu \leq \mu_{k,-}$.

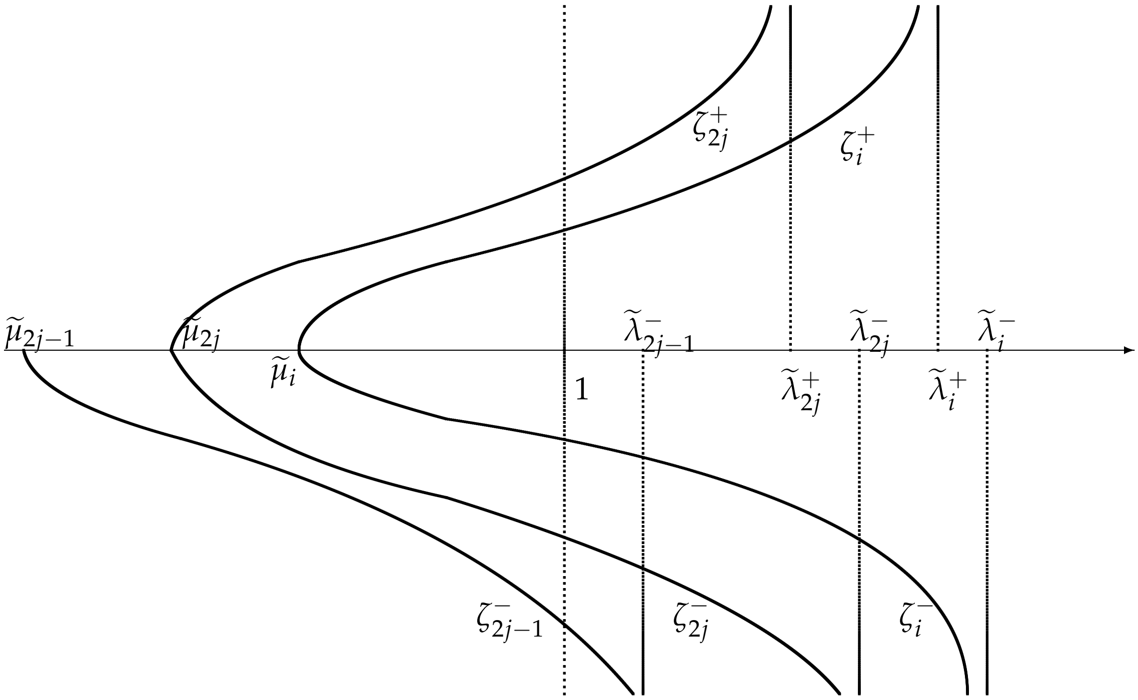

Fig. C: $\mu_i(m) < \rho < \mu_j(\alpha^+)$

Step 2. In this step we prove that for all $l \geq 1$ and $\nu = +$ or $-$, the component ζ_l^ν rejoins the point $(-\infty, \infty)$. Thus, we have to prove that for all $\mu < \mu_{k,-}$, there is a positive constant M_k^ν such that

$$\zeta_k^\nu \cap ([\mu, \mu_{k,-}] \times W) \subset [\mu, \mu_{k,-}] \times \overline{B}(0, M_k^\nu).$$

On the contrary, suppose that this fails and there is a sequence $(\mu_n, u_n)_{n\geq 1}$ in $\zeta_k^\nu \cap ([\mu, \mu_{k,-}] \times W)$ such that $\lim_{l\to\infty} \|u_n\| = +\infty$. That is for all $n \geq 1$

$$\begin{cases} \mathcal{L}_{\widetilde{q}} u_n = u_n \left(\mu_n + \widetilde{f}(t, u_n)\right), \text{ in } (0,1) \\ u_n(0) = \lim_{t\to 1} u_n(t) = 0, \end{cases}$$

from which we read that for all $n \geq 1$

$$\mu_k(\widetilde{q}, w_n) = 1, \tag{4.46}$$

where $w_n(t) = \mu_n + \widetilde{f}(t, u_n(t))$.

Let $\left(z_j^n\right)_{j=0}^{j=k}$ be the sequence of zeros of u_n, $I_j^n = \left[z_{j-1}^n, z_j^n\right]$, $\rho_j^n = \sup_{t\in I_j^n} |u_n(t)| = \left|u_n\left(y_j^n\right)\right|$ with $y_j^n \in I_j^n$. Because $\lim_{n\to\infty} \|u_n\| = +\infty$, there is j_n such that $\lim \rho_{j_n}^n = +\infty$. We claim that there is $a_* \in (0,1)$ such that if (n_s) is a sequence of integers such that $\lim_{s\to\infty} \rho_{j_{n_s}}^{n_s} = +\infty$ then $y_{j_{n,l_s}}^{n,l_s} \in (0, a_*)$. Indeed, if for any sequence (l_s) of integers such that $\lim_{s\to\infty}\rho^{n_s}_{j_{n_s}} = +\infty$ we have $\lim_{s\to\infty} y^{n_s}_{j_{n_s}} = 1$, then (u_{n_s}) is bounded on any interval $[0,a] \subset [0,1)$. Therefore, from the equation

$$u_n(t) = \int_0^1 G_{\widetilde{q}}(t,s)\, u_n(s) \left(\mu_n + \widetilde{f}(s, u_n(s))\right) ds$$

we conclude that (u_{n_s}) converges uniformly to $u \in W$ in all intervals $[0,a] \subset [0,1)$ and

$$u(t) = \int_0^1 G_{\widetilde{q}}(t,s)\, u(s) \widetilde{f}(s, u_n(s))\, ds.$$

Since for all $t \in [0,1)$

$$|u_n(t) - u(t)| \le \int_0^1 G_{\widetilde{q}}(s,s) \left| u_n(s)\widetilde{f}(s,u_n(s)) - u(s)\widetilde{f}(s,u(s)) \right| ds,$$

we obtain by means of the Lebesgue dominated convergence theorem that $u_n \to u$ in W, leading to the contradiction $\|u\| = \lim_{n\to\infty} \|u_n\| = +\infty$.

Set $q_* = \sup_{t\in[0,a_*]} q(t)$ and let $A_* > 0$ be such that $f(t,u) > q_*$ for all $t \in [0,a_*]$ and $|u| > A_*$. We prove now, that if $I^n_j \subset [0,a_*]$ then $\lim \rho^n_j = +\infty$. On the contrary suppose that $\lim \rho^n_{j_n-1} \neq +\infty$ and $\lim \rho^n_{j_n+1} \neq +\infty$, that is (u_n) is bounded in $I^n_{j_n-1} \cup I^n_{j_n+1}$ and let ϖ be such that $\max\left(\rho^n_{j_n-1}, \rho^n_{j_n+1}\right) \le \varpi$. Let $\alpha^n_{j_n} \in \left(z^n_{j_n-1}, y^n_{j_n}\right)$ and $\beta^n_{j_n} \in \left(y^n_{j_n}, z^n_{j_n}\right)$ be such that $\left|u_n\left(\alpha^n_{j_n}\right)\right| = \left|u_n\left(\beta^n_{j_n}\right)\right| = A_*$. Thus, we have

$$-u''_n(t)\, u_n(t) = u_n^2(t)\left(f(t,u_n(t)) - q(t)\right) \ge u_n^2(t)\left(q_* - q(t)\right) \ge 0, \text{ in } \left(\alpha^n_{j_n}, \beta^n_{j_n}\right)$$

leading to $\left|u'_n\left(\alpha^n_{j_n}\right)\right| = \sup_{t\in\left(\alpha^n_{j_n}, y^n_{j_n}\right)} |u'_n(t)|$ and $\left|u'_n\left(\beta^n_{j_n}\right)\right| = \sup_{t\in\left(y^n_{j_n}, \beta^n_{j_n}\right)} |u'_n(t)|$.

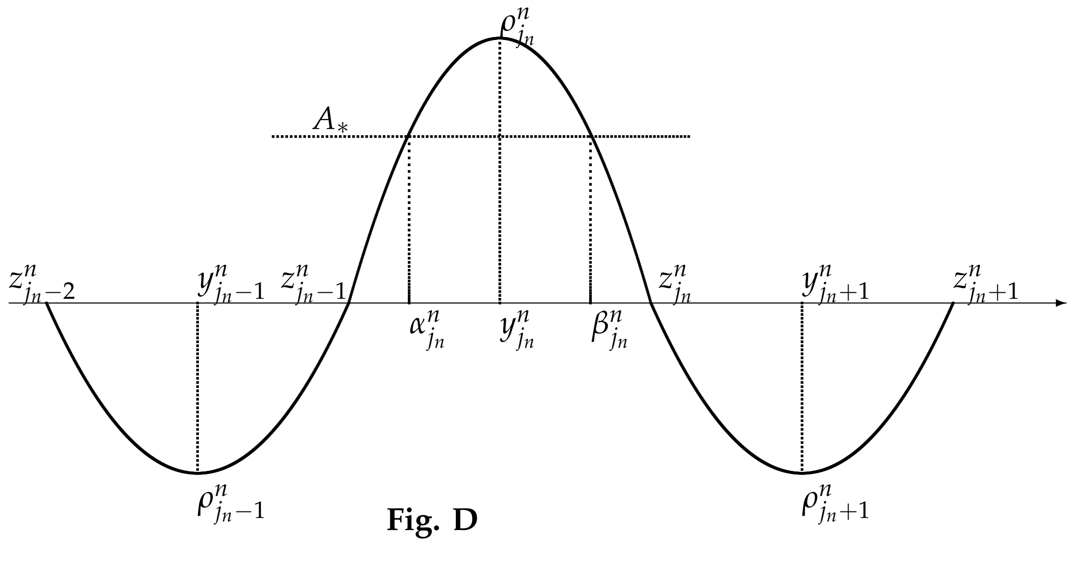

Fig. D

In one hand, we have

$$\begin{aligned}\lim\left|u_n'\left(\alpha_{j_n}^n\right)\right| &= \lim\left(\sup_{t\in\left(\alpha_{j_n}^n,y_{j_n}^n\right)}\left|u_n'(t)\right|\right) = \lim\left|u_n'\left(\beta_{j_n}^n\right)\right| \\ &= \lim\left(\sup_{t\in\left(y_{j_n}^n,\beta_{j_n}^n\right)}\left|u_n'(t)\right|\right) = +\infty.\end{aligned}$$

Indeed, if for instance u_n' is bounded by a constant A in $\left(\alpha_{j_n}^n, y_{j_n}^n\right)$ then

$$\rho_{j_n}^n \leq A_* + \int_{\alpha_{j_n}^n}^{y_{j_n}^n}\left|u_n'(s)\right| ds \leq A_* + A,$$

contradicting $\lim \rho_{j_n}^n = +\infty$.

In the other hand, we have the contradiction

$$\left|u_n'\left(\alpha_{j_n}^n\right)\right| = \left|\int_{y_{j_n-1}^n}^{\alpha_{j_n}^n} u_n(s)\left(f\left(s,u_n(s)\right)-q(s)\right)ds\right| \leq \max\left(\varpi, A_T\right)\left(q_*+\theta\right) < \infty,$$

$$\left|u_n'\left(\beta_{j_n}^n\right)\right| = \left|\int_{\beta_{j_n}^n}^{y_{j_n+1}^n} u_n(s)\left(f\left(s,u_n(s)\right)-q(s)\right)ds\right| \leq \max\left(\varpi, A_T\right)\left(q_*+\theta\right) < \infty,$$

where $\theta = \sup\left\{\left|f\left(s,u\right)\right| : s\in[0,1] \text{ and } u \in \left[-\max\left(\varpi,A_T\right), \max\left(\varpi,A_T\right)\right]\right\}$. This shows that all bumps of u_n contained in $[0,a_*]$ are unbounded.

At this stage, for all $n \geq 1$, there is an interval $I_{j_n}^n = \left[z_{j_n-1}^n, z_{j_n}^n\right] \subset [0,a_*]$ such that $z_{j_n}^n - z_{j_n-1}^n \geq \frac{a_*}{k}$ and Lemma 4.3 leads to $|u_n(t)| \geq \frac{\rho_{j_n}^n}{4}$ for all $t \in \left[\gamma_{j_n}^n, \delta_{j_n}^n\right]$, where

$$\gamma_{j_n}^n = z_{j_n-1}^n + \frac{z_{j_n}^n - z_{j_n-1}^n}{4} \text{ and } \delta_{j_n}^n = z_{j_n}^n - \frac{z_{j_n}^n - z_{j_n-1}^n}{4}.$$

Set $\gamma_0 = \sup \gamma_{j_n}^n$ and $\delta_0 = \inf \gamma_{j_n}^n$ and notice that $\delta_0 - \gamma_0 = \inf\left(\delta_{j_n}^n - \gamma_{j_n}^n\right) \geq \frac{T}{2k}$. Because of

$$u_n(t) = \int_{z_{j_n-1}^n}^{z_{j_n}^n} G\left(z_{j_n-1}^n, z_{j_n}^n, t, s\right) u(s)_n \widetilde{f}\left(s, u_n(s)\right) ds,$$

we obtain from Lemma 4.3 that

$$|u_n(t)| \geq \frac{\min\left(t - z_{j_n-1}^n, z_{j_n}^n - t\right)}{\Psi_{q,\theta}\left(z_{j_n}^n\right)}\rho_{j_n}^n \geq \frac{\min\left(t - z_{j_n-1}^n, z_{j_n}^n - t\right)}{\Psi_{q,\theta}\left(T\right)}\rho_{j_n}^n \to +\infty$$

for all $t \in [\gamma_0, \delta_0]$. Thus, we obtain from Lemma 4.21 and (4.46) that

$$\mu_k(\widetilde{q}, w_n, [\gamma_0, \delta_0]) > \mu_k(\widetilde{q}, w_n) = 1. \tag{4.47}$$

Let $A > \mu_k(\widetilde{q}, 1, [\gamma_0, \delta_0])$, there is $n_A \geq 1$ such that $w_n(t) = \mu_n + \widetilde{f}(t, u_n(t)) \geq A$, for all $n \geq n_A$ and $t \in [\gamma_0, \delta_0]$. Hence, we obtain by Assertion 1 in Corollary 4.23 the contradiction

$$1 < \mu_k(\widetilde{q}, w_n, [\gamma_0, \delta_0]) \leq \mu_k(\widetilde{q}, A, [\gamma_0, \delta_0]) = \frac{\mu_k(\widetilde{q}, 1, [\gamma_0, \delta_0])}{A} < 1.$$

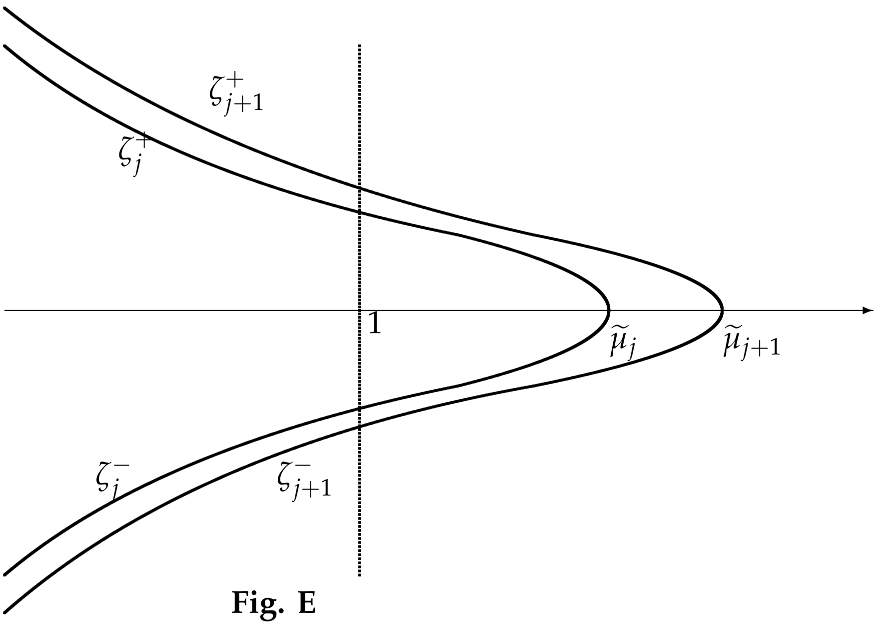

Fig. E

Step 3. At this stage, we have only to compute components that crossing the hyperplane $\mu = 1$. Assume that Hypothesis (4.33) holds, then we have from Assertion 2 in Lemma 4.29 that $\widetilde{\mu}_k > 1$ for all $k \geq j$. Since for all $k \geq 1$ and $\nu = \pm$ the component ζ_k^ν reachs $(-\infty, \infty)$, ζ_k^ν crosses the hyperplane $\mu = 1$ for all $k \geq j$. Thus, the bvp (4.21) admits for all $k \geq j$ a solution in S_k^+ and in S_k^-. The case where $m^+ = 0$ and $\mu_j(\chi_0) > 0$ for some $\chi_0 \in \Gamma^+$ is obtained by means of Assertion 3 in Lemma 4.29.

The proof of Theorem 4.27 is complete.

4.4.4 Proof of Theorem 4.28

Set for $n \geq 1$

$$f_n(t,u) = \begin{cases} f(t,u), \text{ if } |u| \geq \frac{1}{n}, \\ f(t,n), \text{ if } |u| < \frac{1}{n}, \end{cases}$$

and consider the bvp

$$\begin{cases} -u'' + qu = \rho u f_n(t,u), \text{ in } (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0. \end{cases} \tag{4.48}$$

We have then

$$\lim_{u\to+\infty} f_n(t,u) = \alpha(t), \lim_{u\to-\infty} f_n(t,u) = \beta(t) \text{ and } \lim_{u\to 0} f_n(t,u) = f(t, \frac{1}{n}) \text{ in } E.$$

To be brief, we present the proof of Assertion 1, the other Assertions are checked similarly. Because of $\lim_{n\to\infty}\left(\inf_{t\in[0,1]} \widetilde{f}(t, \frac{1}{n})\right) = +\infty$, for all $l \geq 1$ there exists $n_l \geq 1$ such that for all $n \geq n_l$, $\mu_l(\widetilde{q}, \widetilde{f}(t, \frac{1}{n})) < \rho$.

Fix $k \geq j$ and $\nu = +$ or $-$. For all $n \geq n_k$ Assertion 3 in Theorem 4.26 guarantees existence of $u_n \in S_k^\nu$ solution to the bvp (4.48).

Let ω_0 be that in Remark 4.25,

$$\overline{q} = q^+ + 2\rho\omega_0 \quad , \quad \overline{f}_n(t,u) = \rho\left(f_n(t,u) + 2\omega_0\right) + q^-$$

and observe that v is a solution to (4.48) if and only if v is a solution to the bvp

$$\begin{cases} -u'' + \overline{q}u = \rho u \overline{f}_n(t,u) \text{ in } (0,1) \\ u(0) = \lim_{t\to 1} u(t) = 0. \end{cases} \tag{4.49}$$

We claim that there is a positive constant m_k^ν such that $\|u_n\| \geq m_k^\nu$. By the contrary, suppose that (u_n) admits a subsequence (u_s) such that $\lim u_s = 0$ in E and let $A >$

$\mu_k(\overline{q},1)$. There is $\gamma_A > 0$ such that for all $u \in \mathbb{R}$, $|u| < \gamma_A$ implies $\inf_{t\in[0,1]} \overline{f}_n(t,u_s) > A$ and there is s_A such that $\|u_s\| < \gamma_A$ for all $s > s_A$. Thus, for all $s \geq \sup(1/\gamma_A, s_A)$, $\inf_{t\in[0,1]} \overline{f}_n(t,u_n(t)) > A$ and this leads to the contradiction

$$1 = \mu_k(\overline{q}, \overline{f}_n(t,u_n(t))) < \mu_k(\overline{q},A) = \frac{\mu_k(\overline{q},1)}{A} < 1.$$

We prove now that there is positive constant M_k^ν such that $\|u_n\| \leq M_k^\nu$. By the contrary suppose that there is a subsequence (u_r) of (u_n) such that $\lim \|u_r\| = \infty$. Arguing as in Step 2 in the proof of Theorem 4.27, we obtain that $(v_r) = (u_r/\|u_r\|)$ converges, up to a subsequence, to $v \in S_k^\nu$ satisfying

$$\begin{cases} \mathcal{L}_q v = \rho\alpha v^+ - \rho\beta v^-, \text{ in } (0,1), \\ v(0) = \lim_{t\to 1} v(t) = 0. \end{cases}$$

Let $\phi \in S_k^\nu$ be the eigenfunction associated with $\mu_k(\rho\psi)$, that is ϕ satisfies

$$\begin{cases} \mathcal{L}_q \phi = \mu_k(\rho\psi)\rho\psi\phi, \text{ in } (0,1), \\ \phi(0) = \lim_{t\to 1} \phi(t) = 0. \end{cases}$$

Let $(x_j)_{j=0}^{j=l}$ and $(y_j)_{j=0}^{j=l}$ be respectively the sequences of zeros of v and ϕ. We distinguish then the following two cases:

i) $x_1 \leq y_1$: in this case we have the contradiction:

$$\begin{aligned} 0 &\leq \int_{x_0}^{x_1} v\mathcal{L}_q\phi - \phi\mathcal{L}_q v \\ &\leq \int_{x_0}^{x_1} \mu_k(\rho\psi)\rho\psi\phi v - \left(\rho\alpha v^+ - \rho\beta v^-\right)\phi \\ &= \int_{x_0}^{x_1} (\mu_k(\rho\psi)\rho\psi - \alpha)\rho\phi^+ v^+ + (\mu_k(\rho\psi)\rho\psi - \beta)\rho\phi^- v^- < 0. \end{aligned}$$

ii) $y_1 < x_1$: in this case Lemma 4.6 guarantees existence of two integers m, n having the same parity such that $y_m < x_n < x_{n+1} \leq y_{m+1}$ and Lemma 4.7 leads to the contradiction:

$$\begin{aligned} 0 &< \int_{x_n}^{x_{n+1}} v\mathcal{L}_q\phi - \phi\mathcal{L}_q v \\ &\leq \int_{x_n}^{x_{n+1}} \mu_k(\rho\psi)\rho\psi\phi v - \left(\rho\alpha v^+ - \rho\beta v^-\right)\phi \\ &= \int_{x_n}^{x_{n+1}} (\mu_k(\rho\psi)\rho\psi - \alpha)\rho\phi^+ v^+ + (\mu_k(\rho\psi)\rho\psi - \beta)\rho\phi^- v^- < 0. \end{aligned}$$

At this stage by means of Theorem 4.34 we prove that the sequence (u_n) is relatively compact. Let $[0,a] \subset [0,1)$, $t_1, t_2 \in [0,a]$ be such that $t_1 < t_2$ and

$C_k^\nu = \sup\left\{ \left| u\overline{f}(t,u) \right| : t \in [0,1] \text{ and } u \in \left[-M_k^\nu, M_k^\nu\right] \right\}$. We have

$$\begin{aligned} &|u_n(t_2) - u_n(t_1)| \le C_k^\nu \int_0^1 \left|G_{\overline{q}}(t_2,s) - G_{\overline{q}}(t_1,s)\right| ds \le C_k^\nu (\left|\Phi_{\overline{q}}(t_2) - \Phi_{\overline{q}}(t_2)\right| \int_0^{t_1} \Psi_{\overline{q}}(s)\, ds \\ &+ \int_{t_1}^{t_2} \left|\Phi_{\overline{q}}(t_2)\Psi_{\overline{q}}(s) - \Phi_{\overline{q}}(s)\Psi_{\overline{q}}(t_1)\right| ds + \left|\Psi_{\overline{q}}(t_2) - \Psi_{\overline{q}}(t_1)\right| \int_{t_2}^1 \Phi_{\overline{q}}(s)\, ds) \\ &\le C_k^\nu \left(\left|\Phi'_{\widetilde{q}}(0)\right| \int_0^a \Psi_{\overline{q}}(s)\, ds + 2\Psi_{\overline{q}}(a) + \Psi'_{\overline{q}}(a) \int_0^1 \Phi_{\overline{q}}(s)\, ds + \right) |t_2 - t_1| . \end{aligned}$$

This proves that (u_n) is equicontinuous on any interval $[0,a]$ contained in $[0,1)$.

By the mean value theorem, for all $n \ge 1$ and all $t \in [0,1)$ there is $t_n \in (t,1)$ such that

$$\left|\frac{u_n(t)}{1-t}\right| = \left|u_n'(t_n)\right| = \left|\int_0^1 \frac{\partial G_{\overline{q}}}{\partial t}(t_n,s) u_n(s) \overline{f}_n(s, u_n(s))\, ds\right| \le C_k^\nu.$$

This proves that the sequence (u_n) is equiconvergente at $t_0 = 1$.

Therefore, $\lim u_n = u$ (up to a subsequence) and $u(t) = \int_0^1 G_{\widetilde{q}}(t,s)\overline{f}(s,u(s))\, ds$ proving that u is a solution to the bvp (4.21). Furthermore, combining Lemma 4.33 with Lemma 4.17 we see that $u \in S_k^\nu$. This ends the proof.

4.4.5 Separable variable case

Consider the case of the bvp (4.21) where the nonlinearity f is a separable variables function, namely the case where the bvp (4.21) takes the form

$$\begin{cases} \mathcal{L}_q u = \rho\varkappa u h(u),\ t \in (0,1), \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{4.50}$$

where $\varkappa \in \Gamma^+$ and $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a continuous function satisfying

$$\lim_{u\to 0} h(u) = h_0, \quad \lim_{u\to+\infty} h(u) = h_+, \quad \lim_{u\to-\infty} h(u) = h_-. \tag{4.51}$$

We obtain from Theorems 4.26, 4.27 and 4.28 the following corollary:

Corollary 4.36. *Assume that (4.51) holds.*

1. *Let i, j be two integers such that $i \ge j \ge 1$. The bvp (4.50) admits in each of $S_j^+, \ldots, S_i^+, S_j^-, , \ldots, S_i^-$ a solution if one of the following Hypotheses (4.52), (4.53), (4.54) and (4.55) holds true.*

$$\begin{cases} h_0, h_+, h_- \in (0,+\infty) \text{ and} \\ (\mu_j(q,\varkappa)/\min(h_+,h_-)) < \rho < (\mu_i(q,\varkappa)/h_0), \end{cases} \tag{4.52}$$

$$\begin{cases} h_0 \le 0,\ h_+, h_- \in (0,+\infty),\ (\mu_i(q,\varkappa)/\min(h_+,h_-)) < \rho \\ \text{and } \mu_j(q,\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+, \end{cases} \tag{4.53}$$

$$\begin{cases} h_0, h_+, h_- \in (0, +\infty) \text{ and} \\ (\mu_i(q, \varkappa)/h_0) < \rho < (\mu_j(q, \varkappa)/\max(h_+, h_-)), \end{cases} \tag{4.54}$$

$$\begin{cases} h_0 \leq 0,\ h_+, h_- \in (0, +\infty),\ (\mu_j(q, \varkappa)/\max(h_+, h_-)) > \rho \\ \text{and } \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{4.55}$$

2. *Let i, j be two integers such that $i \geq j \geq 1$ and $i \geq 2(j-1)$. The bvp (4.50) admits in each of $S^+_{2j}, \ldots, S^+_i, S^-_{2j-1}, \ldots, S^-_i$ a solution if one of the following Hypotheses (4.56), (4.57) holds true.*

$$\begin{cases} h_0, h_- \in (0, +\infty) \text{ and} \\ (\mu_i(q, \varkappa)/h_0) < \rho < (\mu_j(q, \varkappa)/h_-), \end{cases} \tag{4.56}$$

$$\begin{cases} h_0 > 0,\ h_- \leq 0,\ (\mu_i(q, \varkappa)/h_0) < \rho \\ \text{and } \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{4.57}$$

3. *Let i, j be two integers such that $i \geq j \geq 1$ and $i \geq 2(j-1)$. The bvp (4.50) admits in each of $S^+_{2j-1}, \ldots, S^+_i, S^-_{2j}, \ldots, S^-_i$ a solution if one of the following Hypotheses (4.58), (4.59) holds true.*

$$\begin{cases} h_0, h_+ \in (0, +\infty) \text{ and} \\ (\mu_i(q, \varkappa)/h_0) < \rho < (\mu_j(q, \varkappa)/h_+), \end{cases} \tag{4.58}$$

$$\begin{cases} h_0 > 0,\ h_+ \leq 0,\ (\mu_i(q, \varkappa)/h_0) < \rho \\ \text{and } \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{4.59}$$

4. *The bvp (4.50) admits for all $k \geq j$ a solution in each of S^+_k and S^-_k if one of the following Hypotheses (4.60), (4.61), (4.62) and (4.63) holds true.*

$$\begin{cases} h_0 > 0,\ h_- = h_+ = +\infty \text{ and} \\ (\mu_j(q, \varkappa)/h_0) > \rho, \end{cases} \tag{4.60}$$

$$\begin{cases} h_0 \leq 0,\ h_- = h_+ = +\infty \text{ and} \\ \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+, \end{cases} \tag{4.61}$$

$$\begin{cases} h_-, h_+ \in (0, +\infty),\ h_0 = +\infty \text{ and} \\ (\mu_j(q, \varkappa)/\max(h_-, h_+)) > \rho, \end{cases} \tag{4.62}$$

$$\begin{cases} h_-, h_+ \leq 0,\ h_0 = +\infty \text{ and} \\ \mu_j(q, \chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+, \end{cases} \tag{4.63}$$

5. *The bvp (4.50) admits a solution in S_k^+ for all $k \geq 2j$ and a solution in S_k^- for all $k \geq 2j-1$, if one of the following Hypotheses (4.64), (4.65) holds true.*

$$\begin{cases} h_- > 0,\ h_0 = +\infty \text{ and} \\ \left(\mu_j(q,\varkappa)/h_-\right) > \rho, \end{cases} \tag{4.64}$$

$$\begin{cases} h_- \leq 0,\ h_0 = +\infty \text{ and} \\ \mu_j(q,\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+, \end{cases} \tag{4.65}$$

6. *The bvp (4.50) admits a solution in S_k^+ for all $k \geq 2j-1$ and a solution in S_k^- for all $k \geq 2j$, if one of the following Hypotheses (4.66), (4.67) holds true.*

$$\begin{cases} h_+ > 0,\ h_0 = +\infty \text{ and} \\ \left(\mu_j(q,\varkappa)/h_+\right) > \rho, \end{cases} \tag{4.66}$$

$$\begin{cases} h_+ \leq 0,\ h_0 = +\infty \text{ and} \\ \mu_j(q,\chi_0) > 0 \text{ for some } \chi_0 \in \Gamma^+. \end{cases} \tag{4.67}$$

4.4.6 Comments

1. Under one of the Hypotheses (4.22), (4.23) and (4.24), the set of solutions to the bvp (4.21) is contained in $\cup_{k\geq 1,\nu=\pm} S_k^\nu$. Indeed, we have seen above that u is a solution to the bvp (4.21) if and only if u satisfies

$$\begin{cases} \mathcal{L}_{\widetilde{q}} u = u\widetilde{f}(t,u), \text{ in } (0,1) \\ u(0) = \lim_{t\to 1} u(t) = 0, \end{cases} \tag{4.68}$$

where $\widetilde{q} = q + \omega_1$, $\widetilde{f}(t,u) = f(t,u) + \omega_1$ and $\omega_1 \in \Gamma^{++}$ is that in Remark 4.25. We read from (4.68) that u is a solution to bvp

$$\begin{cases} \mathcal{L}_{\widetilde{q}} v = v\widetilde{f}(t,u), \text{ in } (0,1)\,, \\ v(0) = \lim_{t\to 1} v(t) = 0, \end{cases}$$

that is $\mu_l\left(\widetilde{q},\widetilde{f}(t,u)\right) = 1$ for some $l \geq 1$ and the associated eigenfunction $u \in S_l^\nu$.

2. Let u be a solution to the bvp (4.21); according to the above comment, there is $k \geq 1$ such that $u \in S_k$. Let $(z_j)_{j=0}^{j=k}$ be the sequence and $t_q \in (0,1)$ be such that $q(t) > 0$

for all $t \geq t_q$. Set $t^* = \max(t_q, z_{k-1})$ and let $y_j \in (z_{k-1}, 1)$ be such that $u'(y_j) = 0$. We have then for all $t \geq t^*$

$$-u'(t) + \int_{y_j}^{t} q(s)u(s)ds = \int_{y_j}^{t} u(s)f(s,u(s))ds \tag{4.69}$$

leading to

$$\left| \int_{y_j}^{t} q(s)u(s)ds \right| = \left| u'(t) \right| + \int_{y_j}^{t} \left| u(s)f(s,u(s)) \right| ds < \infty.$$

We deduce from the above inequality for both the cases $u > 0$ in $(z_{k-1}, 1)$ and $u < 0$ in $(z_{k-1}, 1)$ that

$$\int_{y_j}^{1} q(s)u(s)ds = \lim_{t \to \to 1} \int_{y_j}^{t} q(s)u(s)ds < \infty.$$

This proves that if u is a solution to the bvp (4.21) then $\int_0^1 q(s)u(s)ds$ converges.

Therefore, we obtain from (4.69) that

$$\begin{aligned} \lim_{t \to 1} u'(t) &= \lim_{t \to 1} \left(\int_{y_j}^{t} q(s)u(s)ds - \int_{y_j}^{t} u(s)f(s,u(s))ds \right) \\ &= \int_{y_j}^{1} q(s)u(s)ds - \int_{y_j}^{1} u(s)f(s,u(s))ds. \end{aligned}$$

3. Let $q \in Q$, notice that if for some $m \in \Gamma^+$ and $l \geq 1$ $\mu_l(q,m) = 0$, then $\mu_l(q,\chi) = 0$ for all $\chi \in \Gamma^+$. Therefore, if $\mu_l(q,m) > 0$ (resp. < 0) for some $m \in \Gamma^+$ and $l \geq 1$ then $\mu_l(q,\chi) > 0$ (resp. < 0) for all $\chi \in \Gamma^+$. Indeed, if $\mu_l(q,\chi_0) > 0$ and $\mu_l(q,\chi_1) < 0$ for some $\chi_0, \chi_1 \in \Gamma^+$ and $l \geq 1$, then the continuity of the mapping

$$\mu_l(q,\cdot) : \{(1-r)\chi_0 + r\chi_1 : r \in [0,1]\} \to \mathbb{R}$$

leads to the existence of $r_0 \in (0,1)$ such that $\mu_l(q,(1-r_0)\chi_0 + r_0\chi_1) = 0$, then to the contradiction $\mu_l(q,\chi) = 0$, for all $\chi \in \Gamma^+$.

4. Let $q \in Q^+$ and $\chi_0 \in \Gamma^+$. The operator L_{q,χ_0} is then positive and we have for all $l \geq 1$

$$\mu_l(q,\chi_0) \geq \mu_1(q,\chi_0) = \frac{1}{r(L_{q,\chi_0})} > 0.$$

Therefore, $q \in Q^+$ is a particular situation where Assertion 3 in Lemmas 4.29 and 4.30 and Assertions 2 and 4 in Lemma 4.31 are satisfied.

Conclusion

This thesis was devoted to the investigation of some classes of nonlinear boundary value problems having unintegrable weights posed on bounded and unbounded interval.

In chapter 2 and chapter 3, we have obtained a new results concerns the existence of eigenvalue associated to the linear eigenvalue problems. The main results of these chapters concerns the existence and multiplicity of nodal solutions to the nonlinear boundary value problems by means of Rabinowitz global bifurcation theory where the nonlinearity is asymtotically linear. In chapter 4, we have obtained a new results concerns the existence of half eigenvalue associated to the half linear eigenvalue problem. The main results of this chapter concerns the existence and multiplicity for nodal solutions to the nonlinear boundary value problems by means of Rabinowitz global bifurcation theory where the nonlinearity is asymtotically linear, sublinear and superlinear.

Conclusion

On s'intéresse dans cette thèse à l'étude de certaines classes de problèmes aux limites avec des poids non intégrable posés sur des intervalles bornés et non bornés.

Dans le deuxième et troisième chapitre, nous avons obtenus des nouveaux résultats concernant l'existence des valeurs propres pour les problèmes linéaires associés. Nous avons fait recours à la théorie de bifurcation global de Rabinowitz pour obtenir les résultats d'existence et de multiplicité de solutions nodales des problémes nons linéaires où la non linéairité est asymptotiquement linéaire. Dans le quatrième chapitre, Nous avons obtenus des nouveaux résultats concernant l'existence des demi valeurs propres pour le problème linéaire associé. Nous avons fait recours à la théorie de bifurcation global de Rabinowitz pour obtenir les résultats d'existence et de multiplicité de solutions nodales de problème nons linéaire où la non linéairité est asymptotiquement linéaire, souslinéaire et superlinéaire.

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