N° d'ordre : 59/2021-C/MT

République Algérienne Démocratique et Populaire
Ministère de L'Enseignement Supérieur et de la Recherche Scientifique
Université des Sciences et de la Technologie HOUARI BOUMEDIENE

Faculté de Mathématiques

جامعة هواري بومدين
للعلوم والتكنولوجيا
U S T H B

Thèse de doctorat
Présentée pour l'obtention du Grade de **Docteur**
En : MATHEMATIQUES
Spécialité : Contrôle Optimal des EDP

Par : BANOUH Hicham

Sujet

Analyse Harmonique par Ondelettes dans l'Algèbre de Clifford

Soutenue publiquement le 29/05/2021 à la salle Abdelkader Kheladi devant le jury composé de:

Mr. BAHLOUL Djilali	Professeur à l'USTHB	Président
Mr. BEN MABROUK Anouar	Maître de Confèrence/A Directeur de Recherche à l'ISMAI, Kairouan	Encadreur
Mr. KESRI M'hamed	Professeur à l'USTHB	Co-Encadreur
Mr. TOUZALINE Arezki	Professeur à l'USTHB	Examinateur
Mr. MENOUNI Abdelaziz	Professeur à l'Université Mostafa Ben Boulaid, BatnaII	Examinateur
Mr. BENAISSA Abbes	Professeur à l'Université Djilali Liabes, Sidi-Bel-Abbes	Examinateur
Mr LAADJ Toufik	MCA à l'USTHB	Invité

Approved by

First Reader: BEN MABROUK Anouar, Advisor
Associate-Professor, Higher institute of Applied Mathematics and Computer Sciences, University of Kairaouan, Tunisia,

--

Second Reader: KESRI M’hamed, Co-Advisor
Professor, University of Sciences and Technology H. Boumediene,

--

Third Reader: BAHLOUL Djilali, Chairman
Professor, University of Sciences and Technology H. Boumediene,

--

Fourth Reader: TOUZALINE Arezki, Examiner
Professor, University of Sciences and Technology H. Boumediene,

--

Fifth Reader: MENOUNI Abdelaziz, Examiner
Professor, University Mostafa Ben Boulaid, Batna II

--

Sixth Reader: BENAISSA Abbes, Examiner
Professor, University Djilali Liabes, Sidi-Bel-Abbes

--

Seventh Reader: LAADJ Toufik, Invited
Associate-Professor, University of Sciences and Technology H. Boumediene,

--

Acknowledgments

Firstly, I would like to thank my advisor, Dr. Anouar Ben Mabrouk, for his unlimited patience, knowledge, guidance, support, and encouragement. You have inspired me and helped me to grow as a mathematician. I cannot thank you enough.

I thank my co-advisor Professor M'hamed Kesri for his great help during those three last years and for the very usefull conversations we had.

I am very thankful to the chairman and the the members of the Thesis committee for thier reports and thier presence.

To Mom and my late Dad, thank you for instilling in me a love of learning and discovery. It has taken me far and served me well. Thank you for your unconditional love, encouragement, and for putting up with my complaints.

Finally to Mina, Mazigh and Mayas : thank you for your constant love, support and patience. Your unwavering belief in me kept me going when things were toughest. I could not have done this without you.

Analyse Harmonique par Ondelettes dans l'Algèbre de Clifford

(Order No.)

Hicham BANOUH

University of Sciences and Technology Houari Boumediene,

Faculty of Mathematics, 2020

Supervisors: **BEN MABROUK Anouar,**

Associate-Professor, Higher institute of Applied Mathematics and Computer Sciences, University of Kairaouan, Tunisia,

KESRI M'hamed,

Professor, University of Sciences and Technology H. Boumediene,

ABSTRACT

The project falls within the general framework of harmonic analysis on Clifford algebras. We propose more precisely to extend the famous Heisenberg uncertainty principle to the context of Clifford algebras by applying the so-called Clifford wavelets.

In mathematical physics, Clifford analysis has been developed as an extension of the classical harmonic analysis where concepts such as Fourier transforms and wavelets have been extended for the case of Clifford algebras.

In the present work, our aim is to study and establish a new Heisenberg uncertainty principle based on Clifford wavelet transform. We recall that the majority of uncertainty principles in their different forms are based essentially on Fourier transform and the wavelet uncertainty principles already established in the literature did not been extended to general Clifford algebra framework.

ⴰⴳⵣⵓⵍ

ⴰⵙⴻⵏⴼⴰⵔ ⴰⴳⵉ ⵢⴻⵜⵜⵉⵍⵉ ⴷⴻⴳ ⵙⵙⴻⴳⵔⴻⵔ ⴰⵎⴰⵜⵓ ⵏ ⵜⴰⵙⵍⴻⴹⵜ ⵜⴰⵎⵜⴰⵡⴰⵏⵜ ⴷⴻⴳ ⵓⵊⴱⴰⵔⴻⵏ ⵏ Clifford. ⵏⵙⵙⵓⵎⴻⵔ ⴷⴻⴳ ⵓⵥⴷⵉⵇ ⴰⴳⵉ ⴰⵏⵙⵎⴻⵖⵔⴻⴷ ⴰⵣⵡⵉⵔ ⵏ ⵜⴻⴳⵍⴰⴳⵉⵍⵜ ⵏ Heisenberg ⴰⵎⵓⵍⴰⵏ ⴷⴻⴳ ⵓⵙⵙⴻⴳⵔⴻⵔ ⵏ ⵓⵊⴱⴰⵔⴻⵏ ⵏ Clifford ⵙ ⵓⵙⵏⴻⵙⵏ ⵏ ⵜⵉⵎⴷⴻⵙⵡⴰⵍ ⵜⵉⵎⴻⵥⵢⴰⵏⵉⵏ ⵏ Clifford.

ⴷⵉ ⵜⵓⵙⵏⴰⴽⵜ ⵜⴰⵙⴻⵏⴳⴰⵎⴰⵜⵜ, ⵜⴰⵙⵍⴻⴹⵜ ⵏ Clifford ⵜⴻⵜⵜⴻⵡⵙⵏⴻⴼⵍⴻⴷ ⴰⵎ ⵓⵙⴻⵎⵖⴻⵔ ⵏ ⵜⴰⵙⵍⴻⴹⵜ ⵜⴰⵎⵜⴰⵡⴰⵏⵜ ⴰⵏⵉⴷⴰ ⵉⵏⴻⴳⵎⴰⵏ ⴰⵎ ⵜⴱⴻⴷⴷⵉⵍⵜ ⵏ Fourier ⴰⴽⴻⴷ ⵜⵉⵎⴻⵥⵢⴰⵏⵉⵏ ⵜⵜⵡⵙⵎⴻⵖⵔⴻⵏⵜ ⵖⴻⵔ ⵜⴰⵙⵍⴻⴹⵜ ⵏ Clifford.

ⴷⴻⴳ ⵜⵙⴻⵔⵔⵙⵉⵜ ⴰⴳⵉ ⵏⴻⵜⵜⵏⴰⴷⵉ ⴰ ⵏⴻⵍⵎⴻⴷ ⴰⴽⴻⴷ ⴰⴷⵏⴰⴼ ⴰⵣⵡⵉⵔ ⵏ ⵜⴻⴳⵍⴰⴳⵉⵍⵜ ⵏ Heisenberg ⴰⵎⴰⵢⵏⵓⵜ ⵢⴻⵜⵜⵡⴰⴱⵏⴰⵏ ⵖⴻⴼ ⵜⴰⴱⴻⴷⴷⵉⵍⵜ ⵏ ⵜⵉⵎⴻⵥⵢⴰⵏⵉⵏ ⵏ Clifford. ⴰⴷ ⵏⴻⵙⵎⴻⴽⵜⵉ ⴱⴻⵍⵍⵉ ⵜⵉⴳⴷⵉ ⵏ ⵉⵣⵡⵉⵔⴻⵏ ⵏ ⵜⴻⴳⵍⴰⴳⵉⵍⵜ ⴷⵉ ⵜⴰⵍⵖⵉⵡⵉⵏ ⵏⵙⴻⵏ ⵉⵎⴳⴰⵔⴰⴷⴻⵏ ⵜⵜⵡⴰⴱⵏⴰⵏⵜ ⵖⴻⴼ ⵜⴱⴻⴷⵉⵍⵜ ⵏ Fourier ⴻⵔⵏⵓ ⵉⵣⵡⴰⵔⴻⵏ ⵏ ⵜⴻⴳⵍⴰⴳⵉⵍⵜ ⵢⴻⵜⵜⵡⴰⴼⴻⵏ ⴷⵉ ⵜⵙⴻⴽⵍⴰ ⵓⵔ ⵜⵜⵡⵙⵎⴻⵖⵔⴻⵏ ⴰⵔⴰ ⵖⴻⵔ ⵙⴰⵙⵙⴻⴳⵔⴻⵔ ⵏ ⵓⵊⴱⴰⵔⴻⵏ ⵏ Clifford ⴰⵎⴰⵜⵓ.

ملخص

يندرج المشروع في الإطار العام للتحليل التوافقي في جبر كليفورد. نقترح على نحو أكثر دقة توسيع مبدأ عدم اليقين الشهير لهايزنبيرغ في سياق جبر كليفورد من خلال تطبيق ما يسمى .بمويجات كليفورد

في الفيزياء الرياضية، تم تطوير تحليل كليفورد كامتداد للتحليل التوافقي الكلاسيكي حيث تم تمديد مفاهيم مثل تحويلات فورييه و المويجات في حالة جبر كليفورد

في العمل الحالي, هدفنا هو دراسة وإنشاء مبدأ جديد لهايزنبيرغ لعدم اليقين يستند إلى تحويل مويجات كليفورد. ونذكر أن معظم مبادئ عدم اليقين في أشكالها المختلفة تقوم أساسا على تحويل فورييه، وأن مبادئ عدم اليقين التي وضعت بالفعل في المؤلفات لم تمتد إلى إطار الجبر العام لكليفورد.

RESUME

Le projet s'inscrit dans le cadre général de l'analyse harmonique sur les algèbres de Clifford. Nous proposons plus précisément d'étendre le fameux principe d'incertitude d'Heisenberg au contexte des algèbres de Clifford en appliquant les ondelettes de Clifford.

En physique mathématique, l'analyse de Clifford a été développée comme une extension de l'analyse harmonique classique où des concepts tels que les transformations de Fourier et en ondelettes ont été étendus pour le cas de Clifford algèbres.

Dans ce travail, notre objectif est d'étudier et d'établir un nouveau principe d'incertitude d'Heisenberg basé sur la transformation en ondelettes de Clifford. Nous rappelons que la majorité des principes d'incertitude sous leurs différentes formes sont basés essentiellement sur la transformation de Fourier et que les principes d'incertitudes relatifs aux ondelettes n'ont pas été etendus aux algèbres de Clifford générales.

Contents

Chapter 1

Introduction and Main Results

1.1 Introduction

In the present thesis, we seek to establish a new uncertainty Heisenberg principle applied to continuous Clifford wavelet transform in the settings of the non-commutative Clifford algebras (named after the British mathematician and philosopher William Kingdon Clifford (1845-1879). Those Clifford algebras [43, 44] are a generalization of the real and complex numbers and also of Sir William Rowan Hamilton's (1805-1865) quaternions [86, 85] and Hermann Günther Grassmann's (1809-1877) exterior algebra [43, 78]. These algebras incorporate inside one single structure the geometrical and algebraic properties of Euclidean space, that Clifford called them geometrical algebras. They were rediscovered when Paul Adrian Maurice Dirac (1902-1984) used what he called γ-matrices to find a linearisation of the Klein-Gordon equation [63, 62] (those γ-matrices are just generators of a particular Clifford algebra).

The calculus on Clifford algebra treats geometric entities depending on their dimension such as scalars, vectors, bivectors and volume elements, etc. and also that it encompasses all dimensions at once, as opposed to a multi-dimensional tensorial approach with tensor products of one-dimension. The use of Clifford algebras in harmonic analysis uses the fact that holomorphic functions in the complex plane are in the kernel Cauchy-Riemann operator which factorizes the Laplacian. Clifford analysis (study of Clifford algebras valued functions of vector variable) (see for instance [28]) are generalization to higher dimension of the theory of holomorphic functions in the complex plane, their counterpart being monogenic functions (null solutions of a first order differential operator factorizing the multi-dimensional Laplacian). This also means that monogenic functions are harmonic ones. More on functional analysis of Clifford algebra valued functions can be found in [11, 15, 19, 32, 60, 64, 76, 100, 103, 119, 120, 122, 126, 160, 162, 161, 169, 171, 170, 172, 173, 177].

Mathematically and quantitatively speaking wavelet analysis of functions starts by computing a type of transform known as wavelet transform similar to Fourier one and which consists in a convolution product of the function with special copies of one source analysing function called mother wavelet and which plays the role of the exponential in Fourier analysis .

The original work on wavelet analysis has been done by Morlet in [141] to study seismic waves. He also, with Grossman, gave a mathematical study of continuous wavelet transform (see [80]). In [139], Meyer recognized the link between harmonic analysis and Morlet's theory and gave a mathematical foundation to the continuous wavelet theory. The continuous-wavelet analysis of a square integrable function f begins by a convolution with copies of a given mother wavelet ψ translated and dilated respectively by $b \in \mathbb{R}$ and $a > 0$. Such a function ψ has to fulfil an admissibility condition which states that

$$\mathcal{A}_\psi = \int_{\mathbb{R}} \frac{\left|\widehat{\psi}(\xi)\right|^2}{|\xi|} d\xi < +\infty,$$

where $\widehat{\psi}$ is the classical Fourier transform of ψ. More information on real wavelets can be found in [57] and [82].

The connection between the wavelet transform and Clifford analysis was considered in [127, 45, 113, 140]. Clifford analysis/algebra has started to take place especially in signal and image processing. In [159] a wavelet based method has been developed in the quaternion algebra leading to quaternionic representations of face image. In [94] real and complex Fourier transforms are extended to quaternions and Clifford algebras, motivated by applications in nuclear magnetic resonance, electric engineering, colour image and signal processing.

The uncertainty principle also known as Heisenberg's uncertainty principle discovered in 1927 by Heisenberg in [88] is certainly one of the most famous and important concepts of quantum mechanics. It plays an important role in the development and understanding of quantum physics. The physical origin of uncertainty principle is related to quantum systems and states that: the determination of positions by performing measurement on the system disturbs it sufficiently to make the determination of momentum imprecise and vice-versa. It has been described by Heisenberg as [89, Page 30]:

> *One can never know with perfect accuracy both of those two important factors which determine the movement of one of the smallest particles — its position and its velocity. It is impossible to determine accurately both the position and the direction and speed of a particle at the same instant.*

The uncertainty principle has been extended to various transformations in different settings. Using Fourier transform, the authors in [3] established a Stein-Weiss type inequality for the Riesz type potential generated by a Riemann-Liouville operator. Pitt's and Beckner logarithmic uncertainty inequalities have been also proved. The same authors investigated in [4] a Hausdorff-Young inequality for the Fourier transform connected with Riemann-Liouville operator. Such inequality has been applied next to prove an entropy based uncertainty principle and a Heisenberg-Pauli-Weyl inequality (See also [99]). In [145], [144], two types of uncertainty principle such as Heisenberg-Pauli-Weyl and Beurling-Hörmander have been established for the Fourier transform associated with the spherical mean operator in some local framework.

Using real wavelet transform in [155], continuous wavelet transform associated with the spherical mean operator has been introduced yielding a Plancherel formula as well as its inversion. Such findings have been applied next to prove an analogue of Heisenberg's inequality for the introduced wavelet transform (See also [153], [154]).

In [49] the continuous shearlet transform has been investigated to construct mother Shearlet function applied next for an associated general uncertainty principle. Minimizers of such uncertainty have been also developed by means of the new wavelets.

El-Haoui et al in [68] introduced the quaternionic offset linear canonical transform and derived a relationship with the quaternion Fourier transform to establish next Plancherel like rules. These findings have been applied next to prove different uncertainty principles including Heisenberg-Weyl's, Hardy's, Beurling's and logarithmic ones in the case of the new quaternionic offset linear canonical transform. Recently El-Haoui and Fahlaoui established in [67] several uncertainty inequalities in the real Clifford algebra $\mathbb{R}_{p,q}$ such as Hausdorf-Young inequality and qualitative uncertainty principles of Donoho-Stark.

In [70] expansions of signals with respect to Gabor wavelets and short time Fourier transform have been investigated. Using Heisenberg group techniques stable iterative algorithms for signal analysis and synthesis have been developed. These algorithms have been shown to be convergent for a variety of norms and compatibility with the time-frequency localization of signals has been proved.

While the Clifford geometric algebra Fourier Transform is global, the author in [92] introduced the local Clifford geometric algebra wavelet concept using the similitude group $SIM(n)$. As an explicit example, the author introduced Clifford Gabor wavelets. In [93] the same author derived a new directional uncertainty principle for quaternion valued functions by means of quaternion Fourier transformation generalized to the case of Clifford geometric algebras.

In [91] basic concept multivector functions and their vector derivative in geometric algebra have been introduced. Concepts of Fourier transform and Clifford and some useful properties have been also investigated in the same framework of geometric algebras. An uncertainty principle has been next developed for many cases of Clifford wavelets and shown to be useful for signal processing.

In [96] a generalization of the Fourier transform in some Clifford geometric algebras has been extended and adopted for real Clifford geometric algebra Fourier transform. This has been applied next to to define and prove the uncertainty principle for multivector functions in the new Clifford geometric algebras.

In [123] the quaternion ridgelet transform and curvelet transform associated to the quaternion Fourier transform have been investigated and applied to derive an associated reconstruction formulas, reproducing kernels and uncertainty principles.

In [134] an uncertainty principle associated with the quaternion linear canonical transform has been proved by considering the fundamental relationship between such transform and the quaternion Fourier transform. The new principle has been applied to derive an inverse transform and Parseval and Plancherel formulas associated with the quaternion linear canonical transform. (See also [128], [135] for the same authors and similar subject).

Mawardi and Hitzer proposed in [130], [132] and [131] a construction of some Clifford algebra valued wavelets using the similitude groups in a special case. The new framework includes complex Gabor wavelets and extends them to multivectors Clifford Gabor wavelets. A new uncertainty principle for the Clifford Gabor wavelet transform has been proved in the new framework. Generalizations of these results have been conducted by the same authors in [95].

In [133] the quaternionic Fourier transform has been applied to establish an uncertainty principle for its right-sided. Such uncertainty principle has been shown to prescribe a lower bound on the product of the effective widths of quaternion-valued signals in the spatial and frequency domains. Furthermore, Gaussian quaternion signals have been shown to be the only ones minimizing the uncertainty. In the same direction in [114] the linear canonical transform has been revisited and next generalized to quaternion-valued signals.

In [129] the continuous quaternion wavelet transform has been introduced with admissibility condition expressed by means of the right-sided quaternion Fourier transform. An application has been derived to establish a Heisenberg type uncertainty principle for the new extended wavelets.

Recently, Mejjaoli et al considered in [138] a continuous wavelet transform associated with the spherical mean operator relatively to some parameter h. Donoho-Stark and Benedick-type uncertainty principles have been developed.

Yang et al [187] the authors investigated a stronger uncertainty principles in terms of covariance and absolute covariance based on Fourier transform in both directional and the spatial cases for real para-vector-valued signals. Conditions of equality of the studied uncertainty principles have been discussed.

Finally, Yang and Kou in [188] applied the so-called linear canonical transforms to extend the uncertainty principle for hypercomplex signals in the linear canonical transform domains. Minimizers have been shown to be Gaussian signals, which joins several works mentioned above.

Outline

This thesis is organized as follows :

- In chapter 2 we give a detailed review of the notion of continuous wavelet transform, the definition of an admissible mother wavelet, its proprieties and an inversion formula and a Plancherel-Parseval theorems.

- In the third chapter we give an introduction to Clifford algebras and Analysis including the different definitions of a Clifford algebra, its $\mathbb{Z}_2$-grading, the *Pin* and *Spin* groups, monogenic functions and a Stokes formula and a Cauchy representation formula and some important results and properties of the Clifford-Fourier transform (links with the classical n-dimensional Fourier transform) and the so called Clifford-wavelet are constructed by translation, dilation and *Spin*-rotations. have been investigated.

- In the last chapter, we present a review of the uncertainty principle in different settings as for the Clifford-Fourier transform and we formulate and prove the main results given in this thesis.

1.2 Control Theory

Optimal control theory is a branch of mathematical optimization that deals with finding a control for a dynamical system over a period of time such that an objective function is optimized. It has numerous applications in both science and engineering. For example, the dynamical system might be a spacecraft with controls corresponding to rocket thrusters, and the objective might be to reach the moon with minimum fuel expenditure. Or the dynamical system could be a nation's economy, with the objective to minimize unemployment; the controls in this case could be fiscal and monetary policy. (For a review on the theory of controllability and observability see for instance [110, 109, 107, 108]).

In [179], the authors proposed an alternative feedback control in order to solve affine control system, with quadratic cost functional based on the combination of Haar wavelet and Generalized Hamilton-Jacobi-Bellman equation.

In [111], the authors derived the operational matrices of integration, derivative and production of Hermite wavelets and used a direct numerical method based on Hermite wavelet, for solving optimal control problems.

In [156], the authors gave a numerical method for solving non-linear optimal control problems with inequality constraints by Legendre wavelet approximations.

In his PhD thesis, the author in [79] studied the interaction between problems in control theory for partial differential equations and inequalities of the uncertainty principle type. He also investigated the connection between compactness of localization operators and uncertainty principles from an abstract harmonic analysis perspective and gave results applied to the wavelet transform.

In [112], the authors investigated performance limitations and trade-offs in the control design for linear time-invariant systems and showed that control specifications in time domain and in frequency domain are always mutually exclusive determined by uncertainty relations.

For the moment, a theory of controllability and observability of Clifford algebra valued differential systems is mainly studied and applied for the special case of the algebra of quaternions $\mathbb{H}$. For example in [185, 186] and [184] the authors used quaternion valued equations to determine the best attitude for atmospheric entry (see also [1, 104]).

1.3 Main Results

In this work we obtain anew results stating that the product of variances between Clifford-Fourier transform denoted by $\widehat{f}$ and the Clifford wavelet transform $T_\psi[f]$ (w.r.t an admissible Clifford algebra-valued mother wavelet ψ) of a square integrable function f is lower bounded.

Theorem 1.3.1. Let $\psi \in L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$ be an admissible Clifford mother wavelet. Then for $f \in L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$ the following inequality holds

$$\left(\int_{Spin(n)}\int_{\mathbb{R}+} \|b_k T_\psi[f](a,\bullet,s)\|_2^2 \frac{da}{a^{n+1}} ds\right)^{\frac{1}{2}} \left\|\xi_k \widehat{f}\right\|_2 \geq \frac{(2\pi)^{\frac{n}{2}}}{2}\sqrt{A_\psi}\,\|f\|_2^2, \tag{1.1}$$

where $k = 1, 2, \cdots, n$.

We even found an sharper result as given by the next theorem

Theorem 1.3.2. With the same hypothesis as in Theorem 1.3.1 we have

$$(\int_{Spin(n)}\int_{\mathbb{R}+} \|b_k T_\psi[f](a,\cdot,s)\|_2^2 \frac{da}{a^{n+1}} ds)^{\frac{1}{2}} \left\|\xi_k \widehat{f}\right\|_2 \geq \sqrt{2^{n+1}\pi^n A_\psi}\,\{\|f\|_2^2 + 2\,|\langle f_1, f_2\rangle|\}$$

where

$$\begin{cases} f_1(\underline{x}) &= \frac{1}{A_\psi}\int_{Spin(n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}+} \psi^{a,\underline{b},s}(\underline{x})\partial_{b_k} T_\psi[f](a,\underline{b},s)\frac{da}{a^{n+1}} dV(\underline{b})ds \\ f_2(\underline{x}) &= \frac{1}{A_\psi}\int_{Spin(n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}+} \psi^{a,\underline{b},s}(\underline{x}) b_k T_\psi[f](a,\underline{b},s)\frac{da}{a^{n+1}} dV(\underline{b})ds \end{cases}$$

These results will be detailed in Chapter 4. Moreover, we recall that they have been partially published in the following papers [14, 13]:

- Banouh, H., Ben Mabrouk, A. and Kesri, M. Clifford-Wavelet Transform and the Uncertainty Principle, *Advances in Applied Clifford Algebras*, 2019, Vol. 29, pp. 1-23. DOI:10.1007/s00006-019-1026-4.

- Banouh, H., Ben Mabrouk. A. A Sharp Clifford-Wavelet Heisenberg-type Uncertainty Principle, *J. Math. Phys.*, 2020, Vol. 62, Issue 9. DOI:10.1063/5.0015989.

and presented as a talk [12] in

- Banouh, H. Uncertainty Principle Associated with the Clifford Continuous Wavelet Transform, *12th International Conference on Clifford Algebras and Their Applications in Mathematical Physics*, University of Science and Technology of China, Hefei, China, 3-7 August 2020.

We intend that this result will be extended to stronger inequalities. Some future directions are exposed in the last chapter.

Chapter 2

Wavelet Theory Revisited

2.1 Introduction

Wavelet analysis is a time-scale representation of signals used in physics, mathematics, and engineering in the last few decades [47, 87, 124, 106, 183, 53, 58]. It began with the works of Jean Morlet (a French oil engineer who was analysing geophysical data in the context of oil exploration [77]). Grossmann, Morlet, and Paul [80, 81, 82] proved next that wavelets are simply coherent states associated to the affine group of the real line (action of dilations and translations).

In [124], the discovery of orthonormal bases of regular wavelets has been pointed out, and even with compact supports, as shown in [52], by changing the perspective (of course, the orthonormal basis of the Haar wavelets was known since the beginning of the century, but these are piecewise constant, discontinuous functions). Group theory was replaced by the multiresolution analysis [125].

Most practical signals are non stationary and cover a wide range of frequencies. In addition a direct correlation exists between the frequency of a given segment of the signal and the time duration of that segment. Low-frequency pieces tend to last a long time, whereas high frequencies occur in general for a short moment only. For example, vowels have low frequency and last for a long period of time, whereas consonants short bursts of high frequency. Fourier transform (Fourier 1878) gives us only informations about the frequency domain (symbolized by the variable

ξ) but no information on time localization (the variable x). For this reason, we need a time-frequency representations : transform the signal from a one variable function to a function of two variables : time and frequency as the Short Fourier Transform or Gabor Transform (Denis Gabor in [74]) and the wavelet transform.

We obtain wavelets by starting with a function ψ of the real variable x. This function is called a mother wavelet if it is well localized and oscillating. (It resembles to a wave because it oscillates, and it is a wavelet because it is localized). The localization condition is expressed in the usual way by saying that the function decreases rapidly to zero as $|x| \longrightarrow \infty$. The second condition suggests that ψ vibrates like a wave. Mathematically, we require that the integral of ψ be zero and that preferably the other first N-moments of ψ also vanish.

2.2 Wavelets on $\mathbb{R}$

To deal with and/or to conduct wavelet analysis of functions we usually need and start with one source function which will be called next the mother wavelet and which plays the role of the analysing source. Such function should satisfy several assumptions to be able to analyse functions next.

2.2.1 Admissibility

The analysing function must be square integrable But to have the CWT well defined we add a condition on that analysis wavelet. This condition (admissibility property) assures that the CWT can be inverted and so we can reconstruct the signal again.

Definition 2.1. A function $\psi \in L^2(\mathbb{R}, \mathbb{C}, dx)$ is called an admissible wavelet or mother wavelet if it satisfy the *admissibility condition* ([42]):

$$\mathcal{A}_\psi = 2\pi \int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|} d\xi < \infty, \tag{2.1}$$

where $\widehat{\psi}(\xi) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{\mathbb{R}} e^{-ix\xi} \psi(x) dx$ stands for the classical Fourier transform of ψ.

The admissibility condition on ψ (2.1) can be reformulated as

$$\int_0^\infty \frac{\left|\widehat{\psi}(t\xi)\right|^2}{t} dt < \infty, \ \forall \xi \in \mathbb{R}^*. \tag{2.2}$$

Also, (2.1) means that therefore

$$\widehat{\psi}(0) = \int_{\mathbb{R}} \psi(x)dx = 0.$$

This can be interpreted by the fact that the mother wavelet ψ is an oscillating function. This oscillating behaviour legitimises the use of the denomination wavelet which has been used for decades in digital signal processing and exploration geophysics (one of the first occurrences of the word can be found in [157]). The equivalent French word "ondelette" (little wave) was used by Morlet and Grossmann in the early 1980s in their works on oil prospecting.

In practice, we need more conditions on ψ such as a finite number N of vanishing moments, that is, for $k = 0, 1, \cdots, N \in \mathbb{N}$, we have

$$\int_{\mathbb{R}} x^k \psi(x)dx = 0 \iff \left[\frac{d^k \widehat{\psi}(\xi)}{d\xi^k}\right]_{\xi=0} = 0. \tag{2.3}$$

This will filter the polynomial components (at least of order $\leq N$) which is usually the smoothest part of the signal and will show only the singularities and the sharp parts. For example, if the analysing wavelet has vanishing moment of first order, then the linear aspects of the signal will be ignored.

2.2.2 Rapid examples of wavelets

On $\mathbb{R}$, we fortunately have many examples of explicit wavelets whom construction as well as the proofs of their admissibility is not complicated. The simplest example is known as the Haar wavelets [84, 83] where the mother wavelet is explicitly given by

$$\psi_H(x) = \begin{cases} 1 & , 0 \leq x < \frac{1}{2} \\ -1 & , \frac{1}{2} \leq x < 1 \\ 0 & , \text{otherwise.} \end{cases}$$

A next example is the Mexican Hat wavelet or Marr wavelet [125] obtained as the 2^{nd} derivative of a Gaussian function

$$\psi_{MH}(x) = -\frac{d^2}{dx^2}(e^{-\frac{x^2}{2}}) = (1 - x^2)e^{-\frac{x^2}{2}}.$$

An important example is also due to Morlet [180, 80] and is based on the mother wavelet

$$\psi_M(x) = e^{ic_0 x - \frac{x^2}{2}}.$$

This wavelet is closely related to human perception, both in the processes of audition and vision.

2.2.3 The Continuous Wavelet Transform

To analyse a function with wavelets and relatively to an analysing mother wavelet we pass through the so-called Wavelet Transform of such a function. In wavelet theory and similarly to Fourier one there is two different kinds of wavelet transform : the continuous wavelet transform CWT (see for example [57] and [5]) and the discrete one denoted DWT. In the present section we will introduce the CWT which is the main topic of our work. In fact CWT is more adapted to the continuous or time-wise signals and the DWT is used for discrete cases such as statistical series, images, ... etc. The CWT is used in an analysis and detection and the DWT is used for compression and signal reconstruction. The CWT is based on the projection of a square integrable function (a physicist would say the it is a finite energy signal) on a set of images of a chosen function by a group of symmetries.

The CWT is based on the action of two operators : The translation operator $\mathcal{T}^b$ defined for $b \in \mathbb{R}$ by

$$\mathcal{T}^b\psi(x) = \psi(x-b)$$

and the dilation operator $\mathcal{D}^a$ defined for $a > 0$ by

$$\mathcal{D}^a\psi(x) = \frac{1}{\sqrt{a}}\psi(\frac{x}{a})$$

We create a whole family of admissible wavelets, known sometimes as the daughter wavelets by translating and dilating a mother wavelet ψ

$$\psi^{a,b}(x) = \mathcal{T}^b\mathcal{D}^a\psi(x) = \frac{1}{\sqrt{a}}\psi(\frac{x-b}{a})$$

which may be also seen as the action of the affine group $ax+b$ on ψ.

The parameter a measures the compression or the scale and b is the translation or the position parameter. If $a < 1$, the support of $\psi^{a,b}$ will be smaller than that of the mother wavelet ψ and correspond to high frequencies. By the same way, if $a > 1$, the support of $\psi^{a,b}$ will be wider than that of ψ and will correspond to the lower frequencial part of the signal. The factor $1/\sqrt{a}$ ensures the conservation of the L^2-norm of both ψ and $\psi^{a,b}$ and plays the role of a normalization constant. All these copies $\psi^{a,b}$ have the same L^2-norm as the mother wavelet ψ. Indeed,

$$\left\|\psi^{a,b}\right\|_{L^2(\mathbb{R},\mathbb{C},dx)} = \|\psi\|_{L^2(\mathbb{R},\mathbb{C},dx)}\,.$$

Furthermore, the frequency representation of a daughter wavelet $\psi^{a,b}$ satisfies

$$\widehat{\psi^{a,b}}(\xi) = \sqrt{a}e^{-ib\xi}\widehat{\psi}(a\xi)$$

which yields that all these daughter wavelets are admissible also.

Definition 2.2. Continuous Wavelet Transform (CWT) Let $\psi \in L^2(\mathbb{R}, \mathbb{C}, dx)$ be an admissible wavelet and $f \in L^2(\mathbb{R}, \mathbb{C}, dx)$. The CWT of f at the scale a and the position b relatively to the analysing wavelet ψ is defined by the integral transform

$$T_\psi\,[f]\,(a,b) = \langle f, \psi^{a,b} \rangle = \int_{\mathbb{R}} f(x)\,\left[\psi^{a,b}(x)\right]^c dx.$$

where ^{c} stands for the complex conjugation.

Compared to the Fourier transform we notice that the CWT is a function of two variables: time and frequency or position and scale. The kernel $\psi^{a,b}$ plays the same role as the Fourier mode $e^{ix\xi}$: we project a finite-energy signal f on the space of wavelets and we calculate the correspondence between them. But on the contrary of the Fourier analysis, the analysing function is not a single function but a whole family generated from an admissible function called wavelet. Also, the generated functions in addition to dilation are also translated, so the analysis is performed on all the domain of definition of the signal. In terms of the convolution product we have

$$T_\psi\,[f]\,(a,b) = (f * \widetilde{\psi_a})(b)$$

where $\widetilde{\psi_a}(x) = \frac{1}{\sqrt{a}}\psi(\frac{-x}{a})$. It is interpreted as a filter with a function of zero momentum.

We may remark that if the wavelet ψ has N-vanishing moments, the CWT $T_\psi\,[f]\,(a,b)$ of f will have the same order of magnitude as $a^{N+1+1/2}$ (See [58, Page 102]). Indeed,

$$\begin{aligned} T_\psi[f](a,b) &= \int_{\mathbb{R}} f(x)[\psi^{a,b}(x)]^c dx \\ &= \frac{1}{\sqrt{a}}\{\int_{\mathbb{R}} f(b)[\psi(\frac{x-b}{a})]^c dx + \int_{\mathbb{R}} f'(b)(x-b)[\psi(\frac{x-b}{a})]^c dx \\ &+ \int_{\mathbb{R}} \frac{f''(b)}{2}[\psi(\frac{x-b}{a})]^c dx + \cdots + \int_{\mathbb{R}} \frac{f^{(N)}(b)}{N!}(x-b)^N[\psi(\frac{x-b}{a})]^c dx + \cdots\} \end{aligned}$$

We see that the N first terms are zero. Consequently, putting $x = at + b$ in the $(N+1)$-integral we get an approximation

$$|T_\psi[f]| \sim a^{N+1+1/2}.$$

From the definition of the CWT we may easily notice the following properties. Let ψ and ϕ be two admissible wavelets, $f, g \in L^2(\mathbb{R}, \mathbb{C}, dx)$ and $\alpha, \beta \in \mathbb{C}$.

- Linearity rule

$$T_\psi\,[\alpha f + \beta g]\,(a,b) = \alpha T_\psi\,[f]\,(a,b) + \beta T_\psi\,[g]\,(a,b)$$

- The CWT is translation-invariant in the sense that

$$T_\psi\left[\mathcal{T}_{b_0} f\right](a,b) = T_\psi\left[f\right](a, b - b_0)$$

- The CWT is dilation-invariant in the sense that

$$T_\psi\left[\mathcal{D}_{a_0} f\right](a,b) = \frac{1}{\sqrt{a_0}} T_\psi\left[f\right](\frac{a}{a_0}, \frac{b}{a_0})$$

- Introducing the parity operator $\mathcal{P}f(x) = f(-x)$. We have

$$T_\psi\left[\mathcal{P}f\right](a,b) = T_\psi[f](a,-b)$$

- Anti-linearity

$$T_{\alpha\psi+\beta\phi}[f](a,b) = \alpha^c T_\psi[f](a,b) + \beta^c T_\phi[f](a,b)$$

We may also prove that

- $T_\psi\left[f\right](a,b) = \left[T_\psi\left[f\right](\frac{1}{a}, \frac{-b}{a}\right]^c$.

- $T_{\mathcal{T}_{b_0}\psi}[f](a,b) = T_\psi[f](a, ab_0 + b)$.

- $T_{\mathcal{D}_{a_0}\psi}[f](a,b) = \frac{1}{\sqrt{a_0}} T_\psi[f](a_0 a, b)$.

Besides as in Fourier analysis, we have also the possibility to reconstruct the analysed functions even non periodic one with analogues of Dirichlet, Parseval and Plancherel rules by using the CWT.

Theorem 2.2.1. Let ψ be an admissible wavelet and $f, g \in L^2(\mathbb{R}, \mathbb{C}, dx)$, then

$$\int_{\mathbb{R}} \int_0^{+\infty} T_\psi[f](a,b)[T_\psi[g](a,b)]^c \frac{da}{a^2} db = \mathcal{A}_\psi \left\langle f, g\right\rangle_{L^2(\mathbb{R},\mathbb{C},dx)}. \tag{2.4}$$

Proof. In the frequency domain we have

$$\begin{aligned} T_\psi\left[f\right](a,b) &= \left\langle f, \psi^{a,b}\right\rangle \\ &= \left\langle \widehat{f}, \widehat{\psi^{a,b}}\right\rangle, \text{by Parseval formula for Fourier transform.} \\ &= \int_{\mathbb{R}} \widehat{f}(\xi)[\sqrt{a}e^{-i\xi b}\widehat{\psi}(a\xi)]^c d\xi \\ &= \int_{\mathbb{R}} \sqrt{a}e^{i\xi b}\widehat{f}(\xi)[\widehat{\psi}(a\xi)]^c d\xi \end{aligned}$$

and so

$$[T_\psi[g](a,b)]^c = \int_{\mathbb{R}} \sqrt{a}e^{-i\xi b}[\widehat{g}(\xi)]^c \widehat{\psi}(a\xi) d\xi.$$

This yields that

$$\begin{aligned}\int_{\mathbb{R}}\int_0^{+\infty} T_\psi[f](a,b)[T_\psi[g](a,b)]^c\frac{da}{a^2}db &= \int_{\mathbb{R}}\int_0^{+\infty}\{\int_{\mathbb{R}}\sqrt{a}e^{i\xi b}\widehat{f}(\xi)[\widehat{\psi}(a\xi)]^c d\xi\} \\ &\quad \times\{\int_{\mathbb{R}}\sqrt{a}e^{-i\xi b}[\widehat{g}(\xi)]^c\widehat{\psi}(a\xi)d\xi\}\frac{da}{a^2}db \qquad (2.5)\\ &= \int_{\mathbb{R}}\int_0^{+\infty}\{\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{f}(\xi)[\widehat{g}(\xi)]^c[\widehat{\psi}(a\xi)]^c\widehat{\psi}(a\xi)d\xi d\xi\}\frac{da}{a}db.\end{aligned}$$

By interchanging the order of integration and putting $a\xi = z$ then (2.5) is equal to

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^{+\infty}\{\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{f}(\xi)[\widehat{g}(\xi)]^c\frac{[\widehat{\psi}(a\xi)]^c\widehat{\psi}(a\xi)}{a}d\xi d\xi\}dadb\\ &= \int_{\mathbb{R}}\int_0^{+\infty}\{\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{f}(\xi)[\widehat{g}(\xi)]^c\frac{[\widehat{\psi}(a\xi)]^c\widehat{\psi}(a\xi)}{a}d\xi d\xi\}dadb\\ &= \int_{\mathbb{R}}\widehat{f}(\xi)[\widehat{g}(\xi)]^c\times\int_{\mathbb{R}}\frac{|\widehat{\psi}(z)|^2}{|z|}d\xi\\ &= \mathcal{A}_\psi\langle f,g\rangle.\end{aligned}$$

□

As a consequence of the latter result we have a Plancherel formula

Corollary 2.3. *Let ψ be an admissible wavelet and $f \in L^2(\mathbb{R},\mathbb{C},dx)$, then*

$$\int_{\mathbb{R}}\int_0^{\infty}|T_\psi[f](a,b)|^2\frac{da}{a^2}db = \mathcal{A}_\psi\|f\|^2_{L^2(\mathbb{R},\mathbb{C},dx)}.$$

Also, we have an inversion formula making it possible to retrieve the analysed function from its CWT.

Theorem 2.2.2. Let ψ be an admissible wavelet and $f,g \in L^2(\mathbb{R},\mathbb{C},dx)$, then

$$f(x) = \frac{1}{\mathcal{A}_\psi}\int_{\mathbb{R}}\int_0^{\infty}T_\psi[f](a,b)\psi^{a,b}(x)\frac{da}{a^2}db$$

almost everywhere.

Proof. From above we have

$$\begin{aligned}
\langle f,g\rangle &= \frac{1}{\mathcal{A}_\psi}\int_{\mathbb{R}}\int_0^{+\infty} T_\psi[f](a,b)[T_\psi[g](a,b)]^c\frac{da}{a^2}db\\
&= \frac{1}{\mathcal{A}_\psi}\int_{\mathbb{R}}\int_0^{+\infty} T_\psi[f](a,b)[\int_{\mathbb{R}} g(x)[\psi^{a,b}(x)]^c dx]^c\frac{da}{a^2}db\\
&= \frac{1}{\mathcal{A}_\psi}\int_{\mathbb{R}}\int_0^{+\infty} T_\psi[f](a,b)\int_{\mathbb{R}}[g(x)]^c\psi^{a,b}(x)dx\frac{da}{a^2}db\\
&= \frac{1}{\mathcal{A}_\psi}\int_{\mathbb{R}}(\int_0^{+\infty}\int_{\mathbb{R}} T_\psi[f](a,b)\psi^{a,b}(x)\frac{da}{a^2}db)[g(x)]^c dx\\
&= \frac{1}{\mathcal{A}_\psi}\left\langle \int_0^{+\infty}\int_{\mathbb{R}} T_\psi[f](a,b)\psi^{a,b}(\bullet)\frac{da}{a^2}db, g\right\rangle
\end{aligned}$$

□

Now, we characterize the image of $L^2(\mathbb{R},\mathbb{C},dx)$ by a given CWT and introduce thus some reproducing kernels associated to the CWT. Let $\mathcal{H}_\psi = L^2(\mathbb{R}_*^+\times\mathbb{R},\frac{dadb}{\mathcal{A}_\psi a^2})$. It is a reproducing kernel Hilbert space with kernel

$$K_\psi(a,b;a',b') = \frac{1}{\mathcal{A}_\psi}\left\langle \psi^{a,b},\psi^{a',b'}\right\rangle$$

It is the solution of the integral equation

$$F(a',b') = \int_{\mathbb{R}}\int_0^{\infty} K_\psi(a,b;a',b')F(a,b)\frac{da}{a^2}db.$$

A function $F\in L^2(\mathbb{R}_*^+\times\mathbb{R},C_\psi^{-1}a^{-2}dadb)$ is the CWT of a signal *iff*

$$F(a,b) = \int_{\mathbb{R}}\int_0^{\infty}[K_\psi(a,b;a',b')]^c F(a',b')\frac{da}{a^2}db$$

2.3 Conclusion

In this chapter, we presented the wavelets tool used for analysing functions. We saw some proprieties of those wavelets and proved that under certain conditions, the continuous wavelet transform is a an invertible operator. We will use similar characteristics in the Clifford algebra framework which will be the subject of the next chapter.

Chapter 3

Clifford Algebra/Analysis Toolkit

3.1 Introduction

Clifford analysis may be seen as a generalization of Fourier one in signal processing as it applies real, complex and quaternion numbers. It may be also described with the algebra of Pauli and Dirac matrices for physical space and Minkowski space-time and thus a unifying language for mathematics and physics [25, 163, 23].

In [101] a Clifford algebra based algorithm has been developed for segmentation of blood vessels. In [40] some contributions for Clifford algebra based colour image processing have been reviewed and applied to define colour alterations. Clifford algebras has been proved to be an efficient mathematical tool to investigate the geometry of images. See also [].

In [37] two convolution products for Hypercomplex Fourier transforms are studied for the analysis of higher dimensional signals such as colour images based on Clifford Hermite wavelets. In [75] a sophisticated model based on Marr wavelet kernel has been developed and applied on samples of intensity values for each pixel in an image to estimate the probability density function of the pixel intensity. Marr wavelet has been also applied in [181] to detect and characterize two-dimensional vortex for a synthetic flow and propeller wake.

In [30], the authors studied the historical development of quaternion and Clifford Fourier transforms and wavelets. Basic concepts have been revisited and mathematical formulations has been enlightened. Hypercomplex Fourier transforms and wavelets has been revisited with overviews on quaternion Fourier transforms, Clifford Fourier transforms, quaternion and Clifford wavelets.

In [38], [39], the authors introduced new definition for general geometric Fourier transform covering some Clifford cases. They showed necessary constraints to obtain linearity, scaling and shift theorem. Applications in image/signal processing and mathematical imaging vision in general have been discussed.

In this chapter we propose to state the most relevant results on Clifford algebras and the theory of functions taking value in them. In the first section, we give some preliminary concepts on Clifford algebra as their definitions, the matrix representation of Clifford algebras and the *Spin* group (which will be used to describe rotations in $\mathbb{R}^n$). In the forth section, we give a review on monogenic functions which are counterparts of the holomorphic functions on the complex plan. In the two final sections, we present the generalizations of the classical Fourier transform and the wavelet transform in the Clifford algebras framework.

3.2 Clifford Algebras

In the literature, there are different ways to introduce Clifford algebras. We will present in this part some of them. The readers may be referred to the list of references provided in this document for more details and other constructions.

3.2.1 The original definition made by Clifford

It starts from the Grassmann exterior algebra [78] $\bigwedge \mathbb{R}^n$ of the linear space $\mathbb{R}^n$, which is an associative algebra of dimension 2^n. A basis for such an algebra may be defined from an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ of $\mathbb{R}^n$ by considering the tensor products for $k = 1, 2, \ldots, n$ and $(i_1, i_2, \ldots, i_k) \in \{1; 2; ...; n\}^k$ a multi-index of length $k \geq 1$,

$$e_{i_1 i_2 \ldots i_k} = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}.$$

and $e_\emptyset = 1$ and with the extra assumption

$$e_i \wedge e_j = -e_j \wedge e_i \text{ for } i \neq j \text{ and } e_i \wedge e_i = 0.$$

In [44] the last assumption is replaced by

$$e_i e_i = e_i^2 = 1.$$

The associative 2^n-dimensional algebra, defined above is the Clifford algebra $\mathbb{R}_n$ associated to the euclidean space $\mathbb{R}^n$. We mention also, the case where he put $e_i e_i = -1$ in 1878 which is the Clifford algebra associated with the anti-Euclidean space $\mathbb{R}^{0,n}$ (see for instance [43]).

3.2.2 Riesz's construction

In [158] another construction of Clifford algebras has been provided. Let (V, Q) be a $n-$dimensional quadratic space and $\mathcal{A}$ an associative algebra with the following rules on addition and multiplication :

$$x^2 = Q(x)$$

$$xy + yx = 2B(x, y)$$

where B is the bilinear form associated to Q. If $\{e_1, e_2, \cdots, e_n\}$ is a basis of V then the latter conditions become

$$e_i e_j + e_j e_i = 2B(e_i, e_j).$$

We introduce the Clifford algebra $\mathbb{R}_n$ over $\mathbb{R}^n$ as an associative algebra generated by the basis $\{e_1, e_2, \cdots, e_n\}$ satisfying the rules

$$\begin{cases} e_i e_j + e_j e_i = 0 & \text{if } i \neq j \\ e_i e_i = e_i^2 = 1 & \forall 1 \leq i \leq n. \end{cases}$$

This last construction resembles to the constructions formulated in [61] and [121].

3.2.3 Rapid Examples in low dimensions

The first and simplest example is the Clifford algebra $\mathbb{R}_{0,1}$ where the elements are written on the form $x = x_0 1 + x_1 e_1$ where $x_0, x_1 \in \mathbb{R}$ and $e_1^2 = -1$. We have in fact By making the identity $e_1 = i$, the imaginary unit we have the isomorphism

$$\mathbb{R}_{0,1} \simeq \mathbb{C}.$$

The second example is the well known quaternions algebra [86] denoted $\mathbb{R}_{0,2}$ with its elements $q = q_0 1 + q_1 e_1 + q_2 e_2 + q_{12} e_{12}$ where $q_i \in \mathbb{R}$ for $i = 0, 1, 2$, $q_{12} \in \mathbb{R}$ and where $e_i^2 = -1$, and

$e_ie_j = -e_je_i$ for $i \neq j$. Choosing $e_1 = i$, $e_2 = j$, $e_{12} = k$ the elements of the canonical basis of the Euclidean space $\mathbb{R}^3$ we obtain the Hamilton algebra constructed in [85] known also as the quaternion algebra which satisfies

$$\mathbb{H} \simeq \mathbb{R}_{2,0}.$$

3.3 Gradings of a Clifford Algebra

In the later we mainly consider the Clifford algebra $\mathbb{R}_n$ of the euclidean space $\mathbb{R}^n$.

Any element of the Clifford algebra $\mathbb{R}_n$ can be written as

$$a = \sum_A a_A e_A$$

where A is an arbitrary ordered multi-indices

$$A = e_{i_1 \cdots i_k}, \text{ with } 1 \leq i_1 < i_2 < \cdots < i_k \leq n.$$

For example, the Clifford algebra $\mathbb{R}_3$ is spanned by the family

$$\{1, (e_1, e_2, e_3), (e_{12}, e_{23}, e_{13}), e_{123}\}$$

then an element a in $\mathbb{R}_3$ will be

$$\begin{aligned} a &= \sum a_A e_A \\ &= a_0 + (a_1e_1 + a_2e_2 + a_3e_3) \\ &+ (a_{12}e_{12} + a_{23}e_{23} + a_{13}e_{13}) \\ &+ a_{123}e_{123} \end{aligned}$$

and the a's are real numbers. Let k be the cardinality of A ($|A| = k$), we have

$$a = \sum_A a_A e_A = \sum_{k=0}^{n} \sum_{k=|A|} a_A e_A.$$

The subspace $\mathbb{R}_n^k = Span_{\mathbb{R}} \{e_A | \, |A| = k\}$ will be called subspace of grade k.

For example, the subspace of grade 0 is the field $\mathbb{R}$ whose elements are called scalars, the one of grade 1 is the vector space $\mathbb{R}^n$ composed of vectors, the elements of one of grade 2 are called bivectors and finally, the one dimensional subspace of grade n is called the set of pseudo-scalars. We have a decomposition of $\mathbb{R}_n$

$$\mathbb{R}_n = \mathbb{R}_n^0 \oplus \mathbb{R}_n^1 \oplus .... \oplus \mathbb{R}_n^n$$

So that

$$
\begin{aligned}
a = & \underbrace{a_{\emptyset}1}_{scalars} + \underbrace{(a_1e_1 + ... + a_ne_n)}_{vectors} \\
& + \underbrace{(a_{12}e_{12} + a_{13}e_{13} + ...a_{ij}e_{ij} + ... + a_{n-1n}e_{n-1}e_n)}_{bivectors} \\
& + \cdots + \underbrace{(a_{123...n}e_{123....n})}_{pseudo-scalar}.
\end{aligned}
$$

We may write $a = \sum_{k=0}^{n} [a]_k$ where $[a]_k$ is the projector of a on $\mathbb{R}_n^k$. The operator $[.]_k : \mathbb{R}_n \longrightarrow \mathbb{R}_n^k$ for $k = 0, 1, \cdots, n$ satisfy for $a, b \in \mathbb{R}_n$ and $\lambda \in \mathbb{R}$

$$
\begin{aligned}
[a+b]_k &= [a]_k + [b]_k \\
[\lambda a]_k &= \lambda [a]_k = [a]_k \lambda \\
[[a]_k]_k &= [a]_k
\end{aligned}
$$

We define the complexification of $\mathbb{R}_n$ by $\mathbb{C}_n = \mathbb{C} \otimes \mathbb{R}_n$ which means $a_A \in \mathbb{C}$ in $\sum_A a_A e_A \in \mathbb{C}_n$ or $\lambda = a + ib \in \mathbb{C}_n$ with $a, b \in \mathbb{R}_n$.

The Clifford algebra $\mathbb{R}_n$ is $\mathbb{Z}_2-$grader, which means that it is the direct sum of an even and odd subspaces

$$\mathbb{R}_n = \mathbb{R}_n^+ \oplus \mathbb{R}_n^- = \bigoplus_{k \text{ even}} \mathbb{R}_n^k \oplus \bigoplus_{k \text{ odd}} \mathbb{R}_n^k$$

and any Clifford number a can be written as

$$a = [a]^+ + [a]^-$$

where $[a]^{\pm} \in \mathbb{R}_n^{\pm}$. These two components satisfy to the following inclusions [143]

$$\left.\begin{matrix} \mathbb{R}_n^+\mathbb{R}_n^+ \\ \mathbb{R}_n^-\mathbb{R}_n^- \end{matrix}\right\} \subset \mathbb{R}_n^+ \text{ and } \left.\begin{matrix} \mathbb{R}_n^+\mathbb{R}_n^- \\ \mathbb{R}_n^-\mathbb{R}_n^+ \end{matrix}\right\} \subset \mathbb{R}_n^-.$$

Also, as $\mathbb{R}_n$ is the direct sum of the even and odd subspaces and being of dimension 2^n we have

$$\dim(\mathbb{R}_n^+) = \dim(\mathbb{R}_n^-) = 2^{n-1}.$$

Definition 3.1. The centre of the Clifford algebra $\mathbb{R}_n$ is the set of elements which commute with all the other elements of the algebra. It will be noted by $\mathcal{Z}$

$$\mathcal{Z}(\mathbb{R}_n) := \{a \in \mathbb{R}_n | ab = ba, \forall b \in \mathbb{R}_n\}$$

We have

$$\mathcal{Z}(\mathbb{R}_n) = \begin{cases} \mathbb{R} & \text{for } n \text{ even} \\ \mathbb{R} \oplus \mathbb{R}e_{123\cdots n} & \text{for } n \text{ odd} \end{cases}$$

One remarkable concept in Clifford algebra is its attempt to mix between the Grassmann exterior algebra and the Hamilton quaternion one. Let $\underline{x}$ and $\underline{y}$ be two vectors. We define the *Clifford product* by

$$\begin{aligned}
\underline{x}\underline{y} &= \underline{x}\cdot\underline{y}+\underline{x}\wedge\underline{y}\\
\underline{x}\cdot\underline{y} &= -\sum_{j=1}^{n}x_jy_j = -<\underline{x},\underline{y}>_{\mathbb{R}^n}\\
\underline{x}\wedge\underline{y} &= \sum_{i<j}e_ie_j(x_iy_j-x_jy_i)
\end{aligned}$$

Observing that $\underline{y}\wedge\underline{x}=-\underline{x}\wedge\underline{y}$ we get

$$\underline{y}\underline{x}=\underline{y}\cdot\underline{x}+\underline{y}\wedge\underline{x}=\underline{x}\cdot\underline{y}-\underline{x}\wedge\underline{y}.$$

So that

$$\underline{x}\cdot\underline{y}=\frac{1}{2}(\underline{x}\underline{y}+\underline{y}\underline{x}) \quad \text{and} \quad \underline{x}\wedge\underline{y}=\frac{1}{2}(\underline{x}\underline{y}-\underline{y}\underline{x}).$$

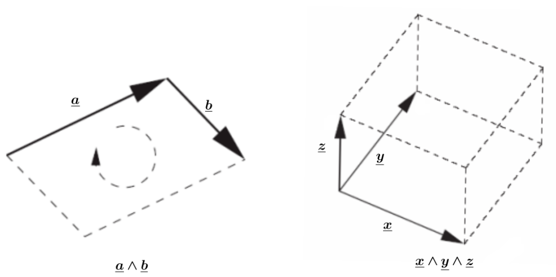

Figure 3.1: wedge product of vectors

All this can be generalized to the product of a vector (grade 1) with a k-grade element A_k which can be decomposed into the sum of an inner product (contraction) and an outer one (extension)

$$\underline{x}\cdot A_k=[\underline{x}A_k]_{k-1}=\frac{1}{2}(\underline{x}A_k-(-1)^kA_k\underline{x}) \tag{3.1}$$

and

$$\underline{x}\wedge A_k=[\underline{x}A_k]_{k+1}=\frac{1}{2}(\underline{x}A_k+(-1)^kA_k\underline{x}). \tag{3.2}$$

In this case the Clifford product will be

$$\underline{x}A_k=\underline{x}\cdot A_k+\underline{x}\wedge A_k.$$

We can even expand to the Clifford product of two $k-$grade,A_k, and $l-$grade, B_l, elements (see

[90, page 6])

$$A_k \cdot B_l = [A_k B_l]_{|k-l|} \text{ with } A_k \cdot B_l = 0 \text{ if } kl = 0$$

and

$$A_k \wedge B_l = [A_k B_l]_{|k+l|}$$

On the real Clifford algebra $\mathbb{R}_n$ we may also define different types of involutions.

A first type is the **main involution**. It extends the vectorial reflection through the origin to the whole algebra. Sometimes called grade involution, it is denoted $\widetilde{\ }$ and has following proprieties : for $\lambda \in \mathbb{R}, \underline{x} \in \mathbb{R}^n$ we have

$$\widetilde{\lambda} = \lambda, \;\; \widetilde{\underline{x}} = -\underline{x}, \;\; \widetilde{e_i} = -e_i \quad \text{and} \quad \widetilde{e_A} = (-1)^{|A|} e_A.$$

We thus obtain for all $a \in \mathbb{R}_n$

$$\widetilde{\widetilde{a}} = a \quad \text{and} \quad \widetilde{a} = [a]^+ - [a]^-$$

and for all $a, b \in \mathbb{R}_n$,

$$\widetilde{(ab)} = \widetilde{a}\widetilde{b} \quad \text{and} \quad \widetilde{(a+b)} = \widetilde{a} + \widetilde{b}.$$

For a $k-$grade element A_k we have

$$\widetilde{A_k} = (-1)^k A_k.$$

The second inversion type is known as the **reversion** denoted by *. It is defined as follow

$$\lambda^* = \lambda, \;\; \underline{x}^* = \underline{x}, \;\; e_j^* = e_j, \;\; e_A^* = (-1)^{\frac{|A|(|A|-1)}{2}} e_A$$

and for all $a, b \in \mathbb{R}_n$,

$$(ab)^* = b^* a^*, \;\; (a+b)^* = a^* + b^*, \;\; a^* = [a]^+ + [a]^- \;, \;\; a^{*^*} = a.$$

For a product of vectors $\{\underline{v}_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^n$ we have

$$(\prod_{i=1}^{k} \underline{v}_i)^* = \underline{v}_k \underline{v}_{k-1} \cdots \underline{v}_2 \underline{v}_1.$$

For a $k-$grade element A_k we have

$$\widetilde{A_k} = (-1)^{\frac{k(k-1)}{2}} A_k.$$

Next, the **Clifford conjugation** is defined as the composition of the main involution and the reversion as follows,

$$\widetilde{a^*} = \widetilde{a}^*.$$

It corresponds to the complex and quaternion conjugation in the case of $\mathbb{R}_{0,1} \simeq \mathbb{C}$ and $\mathbb{R}_{0,2} \simeq \mathbb{H}$

respectively. The superposition will be denoted by $\bar{\ }$ so we have

$$\overline{a} = \widetilde{a^*} = \widetilde{a}^*.$$

So that

$$\overline{\lambda} = \lambda, \ \ \overline{\underline{x}} = -\underline{x}, \ \ \overline{e_j} = -e_j, \ \ \overline{e_A} = (-1)^{\frac{|A|(|A|+1)}{2}} e_A$$

and for all $a, b \in \mathbb{R}_n$,

$$\overline{(ab)} = \overline{b}\overline{a}, \ \ \overline{(a+b)} = \overline{a} + \overline{b}, \ \ \overline{\overline{a}} = a$$

For a $k-$grade element A_k we have

$$\widetilde{A_k} = (-1)^{\frac{k(k+1)}{2}} A_k.$$

Finally, the **complex Clifford conjugation** is defined on the complexified Clifford algebra $\mathbb{C} \otimes \mathbb{R}_n$. We speak here about another involution by taking the complex conjugation (noted by ^{c}) of the Clifford conjugation. An element $\Lambda \in \mathbb{C} \otimes \mathbb{R}_n$ can be written as $\Lambda = a + ib, \ \ a, b \in \mathbb{R}_n$. The complex Clifford conjugation will be noted by † and defined by

$$\Lambda^\dagger = \overline{a} - i\overline{b}. \tag{3.3}$$

It satisfies the properties

$$(\Lambda\Theta)^\dagger = \Lambda^\dagger\Theta^\dagger \quad \text{and} \quad (\lambda\Lambda + \theta\Theta)^\dagger = \lambda^c\Lambda^\dagger + \theta^c\Theta^\dagger$$

with $\Lambda, \Theta \in \mathbb{C} \otimes \mathbb{R}_n$ and $\lambda, \theta \in \mathbb{C}$. To close with the operations on Clifford algebras we recall the concept of norms on Clifford algebras. We define an inner product on $\mathbb{R}_n$ as

$$\langle a, b \rangle = \left[a\overline{b}\right]_0 = [b\overline{a}]_0$$

and so the Clifford norm $|a|$ of a multivector $a = \sum_A a_A e_A \in \mathbb{R}_n$ satisfy

$$|a|^2 = \overline{a}a = \overline{a}a = \sum_A a_A^2. \tag{3.4}$$

We have for $a, b \in \mathbb{R}_n$, $|a + b| \leq |a| + |b|$ and generally $|ab| \neq |a|\,|b|$. We have instead

$$|ab| \leq 2^n \, |a| \, |b| \, .$$

The equality holds if at least one of them is a vector

$$|a\underline{x}| = |a| \, |\underline{x}| \, , \ \ \forall \underline{x} \in \mathbb{R}^n.$$

3.3.1 *Pin* and *Spin* Groups

Let us now consider the two following groups, the formed by the products of invertible vectors (known as the *versor group*)

$$\Gamma^2_{p,q} := \{\prod_{i=1}^{k} \underline{v_i} : \underline{v_1}, \underline{v_2}, \cdots, \underline{v_k} \in (\mathbb{R}^{p,q})^{\times}\}$$

and

$$\Gamma^1_{p,q} := \{b \in \mathbb{R}^{\times}_{p,q}; \widetilde{b}\underline{x}b^{-1} \text{ for all vector } \underline{x} \in \mathbb{R}^{p,q}\}.$$

We define the *spinor norm*, denoted N, as follows (*cf.* 3.4)

$$N : \mathbb{R}_{p,q} \longrightarrow \mathbb{R}_{p,q}$$

$$a \longmapsto \overline{a}a$$

Definition 3.2. We define the $Pin(p, q)$ group as the subgroup of $\Gamma^1_{p,q}$ of elements b for which $N(b) = \pm 1$. It can be shown that it is also the kernel of the mapping $N : \Gamma^1_{p,q} \longrightarrow \mathbb{R}_*$.

The same way we define $Spin(p, q)$ as the subgroup of $Pin(p, q)$ of product of an even number of elements which the spinor norm equals ± 1 or

$$\begin{aligned} Spin(p, q) &= Pin(p, q) \cap \mathbb{R}^{+}_{p,q} \\ &= \{s \in \mathbb{R}_{p,q}; s = \prod_{j=1}^{2l} \underline{\omega}_j \text{ with } \underline{\omega}_j^2 = \pm 1, 1 \leq j \leq 2l\}. \end{aligned} \tag{3.5}$$

3.4 Clifford Analysis

Clifford analysis, in its most basic form, is a refinement of harmonic analysis in higher dimensional Euclidean spaces. By introducing the so-called Dirac operator, researchers introduced the notion of monogenic functions extending holomorphic ones. In this context, different concepts of real and complex analysis have been extended to the Clifford case such as Fourier transform [19, 28, 60, 54].

3.4.1 Clifford Algebra Valued Functions

We define a multivector function as a mapping f from $\mathbb{R}_n$ to $\mathbb{R}_n$, associating a multivector $f(a)$ to another multivector a. For example we have

Definition 3.3 (Exponential of a multivector)**.** Let $a \in \mathbb{R}_n$, we define the *exponential of a* (noted

e^a or $\exp a$) as the series

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!} \tag{3.6}$$

Proposition 3.4. *For two commuting multivectors a and b we have*

$$e^a e^b = e^{a+b}$$

In the latter, we consider only functions with values on a Clifford algebra but where the variable is a vector in $\mathbb{R}^n$ taken as a part of the Clifford algebra.

We put $\boldsymbol{x} = (x_0, \underline{x}) \in \mathbb{R}^{n+1}$ and where $\underline{x} = \sum_{i=1}^{n} e_i x_i \in \mathbb{R}^n$ so we can write

$$\boldsymbol{x} = e_0 x_0 + \sum_{i=1}^{n} e_i x_i = \sum_{i=0}^{n} e_i x_i$$

The vector space $\mathbb{R}^n$ can be seen as the hyper-plan $\{\boldsymbol{x} = (x_0, \underline{x}) \in \mathbb{R}^{n+1} : x_0 = 0\}$.

Consider functions defined on the $(n+1)$-dimensional vector space $\mathbb{R}^{n+1}$:

$$f : \mathbb{R} \oplus \mathbb{R}^n \simeq \mathbb{R}^{n+1} \longrightarrow \mathbb{R}_n (\text{ or } \mathbb{C}_n).$$

It may be expressed as

$$f(\underline{x}) = \sum_A e_A f_A(\underline{x}), \tag{3.7}$$

where f_A are real-valued functions and $A \subset \{1, 2, \cdots, n\}$. Its conjugate $\overline{f}$ is given by

$$\overline{f}(\underline{x}) = \sum_A \overline{e_A} f_A(\underline{x})$$

for a function with values in the "real" Clifford algebra $\mathbb{R}_n$, and

$$f^\dagger(\underline{x}) = \sum_A e_A^\dagger f_A(\underline{x})$$

for a function with values in the complexified Clifford algebra $\mathbb{C}_n = \mathbb{C} \otimes \mathbb{R}_n$.

The continuity and derivability of f is to be taken component-wise. Denote for Ω an open domain in $\mathbb{R}^n$,

$$\mathcal{C}^{(r)}(\Omega, \mathbb{R}_n) = \{f : \Omega \longrightarrow \mathbb{R}_n; \;\; f = \sum_A f_A e_A \text{ with } f_A \in \mathcal{C}^{(r)}(\Omega, \mathbb{R})\}$$

and

$$\mathcal{C}^{(r)}(\Omega, \mathbb{C}_n) = \{f : \Omega \longrightarrow \mathbb{C}_n; \;\; f = \sum_A f_A e_A \text{ with } f_A \in \mathcal{C}^{(r)}(\Omega, \mathbb{C})\}.$$

The Clifford-valued function f belongs to the *Lebesgue module* $L^p(\Omega, \mathbb{R}_n, dV(\underline{x}))$ if all the components $f_A \in L^p(\Omega, \mathbb{R}, dx), 1 \leq p < \infty$, the norm $2^n \sum_A \left(\int_\Omega |f_A|^p \, dV(\underline{x}) \right)^{1/p}$ being equivalent to the norm

$$\|f\|_p = \left(\int_\Omega |f|^p dV(\underline{x}) \right)^{1/p}$$

where $dV(\underline{x})$ stands for the Lebesgue measure on $\mathbb{R}^n$. We say that $f, g \in L^p(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$ are equal if the set $\{\underline{x} \in \mathbb{R}^n; f(\underline{x}) \neq g(\underline{x})\}$ is negligible.

The inner product on $L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$ is given by

$$\langle f, g \rangle_{L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))} = \int_{\mathbb{R}^n} \overline{f(\underline{x})} g(\underline{x}) dV(\underline{x}) \tag{3.8}$$

and

$$\langle f, g \rangle_{L^2(\mathbb{R}^n, \mathbb{C}_n, dV(\underline{x}))} = \int_{\mathbb{R}^n} [f(\underline{x})]^\dagger g(\underline{x}) dV(\underline{x}). \tag{3.9}$$

More explicitly,

$$\langle f, g \rangle_{L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))} = \sum_{A,B} \int_{\mathbb{R}^n} \overline{f_A(\underline{x})} g_B(\underline{x}) \overline{e_A} e_B.$$

We mainly use the complex Clifford conjugation (see 3.3. The inner product (3.8) satisfies the Cauchy-Schwartz inequality

$$\left| < f, g >_{L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))} \right| \leq \|f\|_{L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))} \|g\|_{L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))} . \tag{3.10}$$

3.4.1.1 Differential and derivative of a Clifford algebra-valued function

Definition 3.5 (Directional Derivative)**.** [46, Page 29] Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}_n$ a multivector-valued function of a vector variable $\underline{x} \in \mathbb{R}^n$ and taking an arbitrary direction $\underline{a} \in \mathbb{R}^n$ we define the vector differential (directional derivative of f in the direction $\underline{a}$)

$$\underline{a} \cdot \nabla f(\underline{x}) := \lim_{\lambda \to 0} \frac{f(\underline{x} + \lambda \underline{a}) - f(\underline{x})}{\lambda} \tag{3.11}$$

provided that the limit exists well defined for all $\underline{a}$, where the limit is taken for scalar λ.

This is similar to the usual definition of a directional derivative but extends it to functions that are not necessarily scalar-valued but also vector valued or multivector valued ones. We have these proprieties

(i) $(\underline{a} + \underline{b}) \cdot \nabla f = \underline{a} \cdot \nabla f + \underline{b} \cdot \nabla f$ for $\underline{a}, \underline{b} \in \mathbb{R}^n$.

(ii) $(t\underline{a}) \cdot \nabla f(\underline{x}) = t(\underline{a} \cdot \nabla f(\underline{x}))$ for $t \in \mathbb{R}$.

(iii) $\underline{a} \cdot \nabla(f + g)(\underline{x}) = \underline{a} \cdot \nabla f(\underline{x}) + \underline{a} \cdot \nabla g(\underline{x})$.

(iv) $\underline{a} \cdot \nabla(fg)(\underline{x}) = (\underline{a} \cdot \nabla f(\underline{x}))g(\underline{x}) + f(\underline{x})(\underline{a} \cdot \nabla g(\underline{x}))$.

(v) $\underline{a} \cdot \nabla \left[f(\underline{x}) \right]_k = \left[\underline{a} \cdot \nabla f(\underline{x}) \right]_k .$

(vi) Let $\tau : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $f = f(\tau(\underline{x}))$ then

$$\underline{a} \cdot \nabla f(\underline{x}) = (\underline{a} \cdot \nabla \tau(\underline{x})) \frac{df}{d\tau}.$$

3.4.1.2 Dirac Operator and Monogenic Functions

Introduce the *Dirac operator* (compare with (3.5))

$$\begin{aligned} \partial_{\underline{x}} : \mathcal{C}^{(r)} &\longrightarrow \mathcal{C}^{(r-1)} \\ f \longmapsto \partial_{\underline{x}} f &= \sum_{k=1}^{n} e_k \frac{\partial f}{\partial_{x_k}} \end{aligned} \tag{3.12}$$

and the generalized *Cauchy-Riemann* operator or *Weyl* operator as

$$D_{\boldsymbol{x}} = \frac{\partial}{\partial x_0} + \partial_{\underline{x}} \tag{3.13}$$

also known as the Fueter-Delanghe operator [72, 59]. For the sake of simplicity we denote from now on

$$\frac{\partial}{\partial x_i} = \partial_i$$

so the operators (3.12) and (3.13) become

$$\partial_{\underline{x}} = \sum_{i=1}^{n} e_i \partial_i \text{ and } D_{\boldsymbol{x}} = \partial_0 + \partial_{\underline{x}}$$

and their conjugates will be

$$\overline{\partial_{\underline{x}}} = \sum_{i=1}^{n} \overline{e_i} \partial_i \text{ and } \overline{D_{\boldsymbol{x}}} = \partial_0 - \overline{\partial_{\underline{x}}}.$$

They have an action from the left

$$\partial_{\underline{x}} f(\underline{x}) = \sum_{i,A} e_i e_A \partial_i f_A(\underline{x})$$

and right

$$f \partial_{\underline{x}}(\underline{x}) = \sum_{i,A} e_A e_i \partial_i f_A(\underline{x}).$$

(see the formulas (3.2)). By the same way

$$\begin{aligned} D_{\boldsymbol{x}} f &= \partial_0 f + \partial_{\underline{x}} f \text{ and} \\ f D_{\boldsymbol{x}} &= f \partial_0 + f \partial_{\underline{x}}. \end{aligned}$$

In the special case where f is vector-valued, then we have $\partial_{\underline{x}} f = \operatorname{div} f + \operatorname{curl} f$ where the divergence and curl of f can be computed as the scalar and bivector parts

$$\operatorname{div} f = \partial_{\underline{x}} \cdot f = \frac{1}{2}(\partial_{\underline{x}} f + f\partial_{\underline{x}})$$

and

$$\operatorname{curl} f = \partial_{\underline{x}} \wedge f = \frac{1}{2}(\partial_{\underline{x}} f - f\partial_{\underline{x}}).$$

We consider the action of the group $Spin(n)$ on a Clifford-valued function f (see [76, Thm 3.6]) given by

$$s \in Spin(n) \to L_s : f(\underline{x}) \to sf(\overline{s}\underline{x}s)\overline{s}.$$

Definition 3.6. A partial differential operator with constant coefficients is called $Spin$-invariant if it commutes with L_s.

Proposition 3.7. *The Dirac operator is Spin-invariant* i.e

$$\partial_{\underline{x}} L_s = L_s \partial_{\underline{x}}.$$

Proof. We have

$$\begin{aligned}
\partial_{\underline{x}} L_s f(\underline{x}) &= \sum_{i=1}^{n} e_i \partial_i \left\{ sf(\overline{s}\underline{x}s)\overline{s} \right\} \\
&= \sum_{i=1}^{n} e_i s \partial_i \left\{ f(\overline{s}\underline{x}s) \right\} \overline{s} \\
&= s(\sum_{i=1}^{n} e_i \partial_i \left\{ f(\overline{s}\underline{x}s) \right\})\overline{s} \\
&= s\partial_{\underline{x}} f(\overline{s}\underline{x}s)\overline{s} \\
&= L_s \partial_{\underline{x}} f(\underline{x}).
\end{aligned}$$

□

Remark 3.8. We remark (see [148, pp. 139]) that the Dirac operator $\partial_{\underline{x}}$ maps even parts to odd parts and odd parts to even parts (see the splitting in (3.3))

$$\partial_{\underline{x}} \left[f(\underline{x}) \right]^{+} = \left[\partial_{\underline{x}} f(\underline{x}) \right]^{-}$$

and

$$\partial_{\underline{x}} \left[f(\underline{x}) \right]^{-} = \left[\partial_{\underline{x}} f(\underline{x}) \right]^{+}$$

As such, f is monogenic if and only if $\partial_{\underline{x}} \left[f(\underline{x}) \right]^{+} = 0$ and $\partial_{\underline{x}} \left[f(\underline{x}) \right]^{-} = 0$. More generally, the equation

$$\partial_{\underline{x}} f = g$$

where f and g are two Clifford algebra-valued functions defined on $\mathbb{R}^n$, can be splitted into a system of two equations

$$\begin{cases} \partial_{\underline{x}}[f]^+ = [g]^- \\ \partial_{\underline{x}}[f]^- = [g]^+ \end{cases}$$

where $[\cdot]^\pm$ is the projection onto even and odd sub-algebras $\mathbb{R}_n^\pm$.

Definition 3.9. A function $f \in \mathcal{C}^1(\Omega, \mathbb{R}_n)$ is called *left-monogenic* (resp. *right-monogenic*) on Ω iff

$$D_{\boldsymbol{x}} f = (\partial_{x_0} + \partial_{\underline{x}}) f(x_0 + \underline{x}) = 0, \tag{3.14}$$

resp.

$$f D_{\boldsymbol{x}} = 0.$$

In the special case where the function f takes values only on the vector space $\mathbb{R}^n$ taken as a component of the real Clifford algebra $\mathbb{R}_n$, so it can be written as

$$f(\underline{x}) = \sum_{i=0}^{n} e_i f_i(\underline{x})$$

where, by analogy with (3.7), the $(n+1)$ functions f_i are real valued. In this case, the monogenicity condition (3.14) becomes

$$\sum_{i,j} e_i e_j \partial_i f_j(\underline{x}) = 0.$$

Or as f is vector-valued $\partial_{\underline{x}} f = \partial_{\underline{x}} \cdot f + \partial_{\underline{x}} \wedge f = 0$ so we have

$$\begin{cases} \partial_{\underline{x}} \cdot f &= -\sum_{j=1}^{n} \partial_{x_j} f_j = -div(f) \\ \partial_{\underline{x}} \wedge f &= \sum_{i<j} e_{ij} \left(\partial_{x_i} f_j - \partial_{x_j} f_i \right) = rot(f) \end{cases}$$

known as the *Riesz system* which describes an irrotational flow [175]. One can show that

$$\partial_{\underline{x}}^2 = -\Delta_n \text{ and } \Delta_{m+1} = D_{\boldsymbol{x}} \overline{D_{\boldsymbol{x}}},$$

where Δ_n and Δ_{n+1} are the Laplacian in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ respectively. This means that a monogenic function is also a harmonic one and hence infinitely differentiable; even more, it is an multivector-valued analytic function in $\Omega \subset \mathbb{R}^n$ and so each of its components f_A is real-analytic in its domain. Now, we will try to write the previous operators in terms of spherical coordinates. For that, we recall that we can write $\overline{x} = r\overline{\underline{\eta}}$ with $\overline{\underline{\eta}} = \sum_{i=0}^{n} \overline{e_i} \eta_i$ and $\eta_i = \frac{x_i}{r}$ for $i = 0, 1, \cdots, n$. More explicitly (see [16, Page 65])

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = r \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \cos\theta_1 \\ \sin\theta_0 \sin\theta_1 \cos\theta_2 \\ \vdots \\ \prod_{j=1}^{i-1} \sin\theta_j \cos\theta_i \\ \vdots \\ \prod_{j=1}^{n-1} \sin\theta_j \cos\theta_n \\ \prod_{j=1}^{n} \sin\theta_j \end{pmatrix}$$

where $0 < \theta_1, \ldots, \theta_{n-1} < \pi$ and $0 < \theta_n < 2\pi$. Using the spherical coordinates, the Dirac operator and its conjugate can be written (see [28, Page 49]) as

$$\partial_{\underline{x}} = \underline{\eta}(\partial_r + \frac{1}{r}\partial_{\underline{\eta}}) \text{ and } \overline{\partial_{\underline{x}}} = \overline{\underline{\eta}}(\partial_r + \frac{1}{r}\partial_{\underline{\eta}})$$

where $\partial_{\underline{\eta}}$ *spherical Dirac operator* acts on the sphere $\mathcal{S}^n$. We have [28, Page 49]

$$\partial_{\underline{\eta}} = \sum_{i=1}^{n} \frac{\frac{\partial \eta}{\partial \theta_i} \cdot \partial_{\theta_i}}{\left|\frac{\partial \eta_i}{\partial \theta_i}\right|^2} \text{ and } \partial_{\underline{\eta}} = \overline{\underline{x}} \wedge \partial_{\underline{x}}.$$

Putting

$$\Gamma_{\underline{\eta}} = \overline{\underline{\eta}}\partial_{\underline{\eta}} \text{ and } \Gamma^{\star}_{\underline{\eta}} = \underline{\eta}\overline{\partial_{\underline{\eta}}} \tag{3.15}$$

called the *spherical Dirac operators*, we also define their adjoints as

$$\widetilde{\Gamma_{\underline{\eta}}} = \overline{\partial_{\underline{\eta}}}\underline{\eta} \text{ and } \widetilde{\Gamma^{\star}_{\underline{\eta}}} = \partial_{\underline{\eta}}\overline{\underline{\eta}}.$$

Consequently,

$$\partial_{\underline{x}} = \underline{\eta}\left(\partial_r + \frac{1}{r}\Gamma_{\underline{\eta}}\right) = (\partial_r + \frac{1}{r}\widetilde{\Gamma^{\star}_{\underline{\eta}}})\underline{\eta}$$

and

$$\overline{\partial_{\underline{x}}} = \left(\partial_r + \frac{1}{r}\widetilde{\Gamma_{\underline{\eta}}}\right)\overline{\underline{\eta}}$$

Observing that

$$\Delta_n = \partial_{\underline{x}}\overline{\partial_{\underline{x}}} = \overline{\partial_{\underline{x}}}\partial_{\underline{x}} \tag{3.16}$$

and that $\underline{\eta}\overline{\underline{\eta}} = \overline{\underline{\eta}}\underline{\eta} = 1$ we obtain

$$\Delta_n = \partial_r^2 + \frac{1}{r}(\Gamma_{\underline{\eta}} + \widetilde{\Gamma_{\underline{\eta}}})\partial_r + \frac{1}{r}(\widetilde{\Gamma_{\underline{\eta}}}\Gamma_{\underline{\eta}} - \Gamma_{\underline{\eta}}).$$

By the same way we get

$$\Delta_n = \partial_r^2 + \frac{1}{r}(\Gamma^{\star}_{\underline{\eta}} + \widetilde{\Gamma^{\star}_{\underline{\eta}}})\partial_r + \frac{1}{r}(\widetilde{\Gamma^{\star}_{\underline{\eta}}}\Gamma^{\star}_{\underline{\eta}} - \Gamma^{\star}_{\underline{\eta}}).$$

This means that

$$\partial_r^2 + \frac{1}{r}(\Gamma_{\underline{\eta}} + \widetilde{\Gamma_{\underline{\eta}}})\partial_r + \frac{1}{r}(\widetilde{\Gamma_{\underline{\eta}}}\Gamma_{\underline{\eta}} - \Gamma_{\underline{\eta}}) = \partial_r^2 + \frac{1}{r}(\Gamma^\star_{\underline{\eta}} + \widetilde{\Gamma^\star_{\underline{\eta}}})\partial_r + \frac{1}{r}(\widetilde{\Gamma^\star_{\underline{\eta}}}\Gamma^\star_{\underline{\eta}} - \Gamma^\star_{\underline{\eta}})$$

Next as we have ([28, Page 50])

$$\Gamma_{\underline{\eta}} + \widetilde{\Gamma_{\underline{\eta}}} = \Gamma^\star_{\underline{\eta}} + \widetilde{\Gamma^\star_{\underline{\eta}}} = n$$

then (3.16) becomes

$$\Delta_n = \partial_r^2 + \frac{n}{r}\partial_r + \Delta_{\underline{\eta}}$$

where $\Delta_{\underline{\eta}}$ is the *Laplace-Beltrami operator*. Denoting next $\boldsymbol{I}$ the identity applications we get

$$\Delta_{\underline{\eta}} = \left((n-1)\,\boldsymbol{I} - \Gamma^\star_{\underline{\eta}}\right)\Gamma^\star_{\underline{\eta}}.$$

Putting $\boldsymbol{E} = \sum_{i=0}^{n} x_i\partial_i$ called the *Euler operator*, we obtain $\boldsymbol{E} = r\partial_r$. As a result, we get the following identities

- $\overline{\boldsymbol{x}}D_{\boldsymbol{x}} = \boldsymbol{E} - \underline{x}\wedge\partial_{\underline{x}} + (x_0\partial_x - \underline{x}\partial_0)\,,$
- $\overline{\boldsymbol{x}}D_{\boldsymbol{x}} = \boldsymbol{E} + \Gamma_{\underline{\eta}},$
- $\underline{x}\partial_{\underline{x}} + \partial_{\underline{x}}\underline{x} = -2\boldsymbol{E} - n,$
- $\Gamma_{\underline{\eta}} = -\sum_{i<j} e_{ij}(x_i\partial_j - x_j\partial_i).$

$\Gamma_{\underline{\eta}}$ is called the *angular Dirac operator* and $L_{ij} = x_i\partial_j - x_j\partial_i$ for $i,j = 1,2,\cdots,n$ are the *angular momentum operators.*

3.4.2 Generating Monogenic Functions

We know that a harmonic function is monogenic so we seek the inverse : given a harmonic function $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, is there a Clifford algebra-valued function g, defined on the same domain Ω such that the restriction of g to its real values is f or

$$f(\underline{x}) = [g(\underline{x})]_0\,,\quad \forall \underline{x} \in \Omega\,?$$

Definition 3.10. [17, Def. 1] An open subset Ω of $\mathbb{R}^n$ is said to be star-shaped with respect to some $\underline{x}_0 \in \Omega$, if for all $\underline{x}$ in the interior of Ω, the subset Ω contains the segment $\{(1-t)\underline{x}_0 + t\underline{x};\;\; 0 \le t < 1\}$.

Proposition 3.11. *[28, Page 48] If Ω is open and star-shaped with respect to the origin and $u:\Omega\to\mathbb{R}$ is harmonic in Ω then the function*

$$g(\underline{x})=f(\underline{x})e_0+\int_0^1 t^{n-1}\overline{\partial_{\underline{x}}}f(t\underline{x})\underline{x}dt-\left[\int_0^1 t^{n-1}\overline{\partial_{\underline{x}}}f(t\underline{x})dt\right]_0$$

is left monogenic in Ω and its scalar part is precisely the function f.

As for the previous result, we seek a monogenic function starting with an analytic one. For this, we use the Cauchy-Kowalevski extension (CK-extension). The idea behind the CK-extension is to characterize solutions of a system of partial differential equations by the restriction of some of their derivatives to a sub-manifold of co-dimension one.

Let f be an analytic function on an open set U of $\mathbb{R}^n$, then we can always monogenically extend it to an open set Ω of $\mathbb{R}^{n+1}$ where $U=\mathbb{R}^n\cap\Omega$ (known as the Cauchy-Kowalevski extension (see for instance the original works [41, 116] and more recently [29, 31, 172, 41][28, Page 111] and [61, Page 151]), then the solution F of the system

$$\begin{cases}\partial_{x_0}F\left(x_0,\underline{x}\right)=-\partial_{\underline{x}}F\left(x_0,\underline{x}\right) \text{ in } \mathbb{R}^{m+1}\\ F(0,\underline{x})=f(\underline{x})\end{cases}$$

is monogenic on $\mathbb{R}^{n+1}$ and its restriction to $\mathbb{R}^n$ is f.

The extension can be realized by the operation

$$F\left(x_0,\underline{x}\right)=e^{-x_0\partial_{\underline{x}}}f(\underline{x})=\sum_{k=0}^{\infty}\frac{\left(-x_0\right)^k}{k!}\partial_{\underline{x}}^k f(\underline{x}). \tag{3.17}$$

understood in the symbolic way. In fact, formally we have

$$\begin{aligned}D_{\boldsymbol{x}}\left(e^{-x_0\partial_{\underline{x}}}f(\underline{x})\right)&=\left(\partial_0+\partial_{\underline{x}}\right)\left(e^{-x_0\partial_{\underline{x}}}f(\underline{x})\right)\\&=\partial_0\left(e^{-x_0\partial_{\underline{x}}}f(\underline{x})\right)+\partial_{\underline{x}}\left(e^{-x_0\partial_{\underline{x}}}f(\underline{x})\right)\\&=-\partial_{\underline{x}}\left(e^{-x_0\partial_{\underline{x}}}f(\underline{x})\right)+\partial_{\underline{x}}\left(e^{-x_0\partial_{\underline{x}}}f(\underline{x})\right)\\&=0\end{aligned}$$

Definition 3.12. [48] A monogenic function f is called *axial monogenic* if it takes the form

$$f\left(x_0,\underline{x}\right)=A\left(x_0,r\right)+\underline{\eta}B\left(x_0,r\right)$$

where $r>0$ and $\underline{\eta}$ are the spherical coordinate given in section (3.4.1.2) and A,B are two scalar valued function. The monogenicity of f leads to the equations

$$\partial_{x_0}A-\partial_r B=\frac{n-1}{r}B \text{ and}$$

$$\partial_{x_0}B+\partial_r A=0$$

called the Vekua system.

Using the following result we can extend a holomorphic complex function into a monogenic one.

Theorem 3.4.1. [164, 72] Let n be odd and let $z = x+iy \longmapsto f(x+iy)$ be holomorphic function. Then the function

$$(\Delta_{n+1})^{\frac{n-1}{2}} f(x_0 + \underline{x})$$

is monogenic.

The authors in [150] proved a similar result for n even. Now, we present another method of extending a function on $\mathbb{R}^n$ into a monogenic one. This is due to [115, 151]

Theorem 3.4.2. Let $f \in L^2(\mathbb{R}^n, \mathbb{C}_n, dV(\underline{x})$. Then f can be monogenically extended to a monogenic function F on $\mathbb{R}^{n+1}$ with the estimate

$$|F(x_0, \underline{x})| \leq ce^{R|(x_0,\underline{x})|}$$

if and only if

$$Supp(\widehat{f}) \subset \overline{B(0, R)}$$

where

$$B(0, R) = \{\underline{x} \in \mathbb{R}^n : |\underline{x}| < R\}$$

In this case we have

$$F(x_0, \underline{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e(x_0, \underline{x}, \underline{\xi})\widehat{f}(\underline{\xi})d\underline{\xi},$$

where

$$e(x_0, \underline{x}, \underline{\xi}) = e^+(x_0, \underline{x}, \underline{\xi}) + e^-(x_0, \underline{x}, \underline{\xi}), \quad e^\pm(x_0, \underline{x}, \underline{\xi}) = e^{i\underline{x}\cdot\underline{\xi}} e^{\mp x_0|\underline{\xi}|} \chi_\pm(\underline{\xi})$$

and

$$\chi_\pm(\underline{\xi}) = \frac{1}{2}\left(1 + \mathrm{i}\frac{\underline{\xi}}{|\underline{\xi}|}\right)$$

Proof. [152, Page 118]. □

Now, we give some results about integration of Clifford-valued function which are similar to those that are well known in the theory of complex-valued functions as a Stokes formula and a Cauchy representation formula.

3.4.3 Cauchy-Clifford Integral Formula

In the sequel, Ω is an open subset of $\mathbb{R}^n$, $U \subset \Omega$ a compact and orientable piecewise differentiable bounded domain in $\mathbb{R}^n$ with lipschitzian boundary ∂U. The surface element on ∂U will be

$$d\sigma_{\underline{x}} = \sum_{j=1}^{n}(-1)^j e_j dx_{[j]}$$

where $dx_{[j]} = dx_1 \wedge dx_2 \wedge \cdots dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n$ (we omit the jth term). If we call $\eta(\underline{x})$ the unit outward pointing normal-vector at $\underline{x} \in \partial U$, then

$$d\sigma_{\underline{x}} = \eta(\underline{x})dS(\underline{x})$$

with $dS(\underline{x})$ being the surface element on ∂U.

Theorem 3.4.3 – Clifford-Stokes Theorem. Let $f, g \in \mathcal{C}^1(\Omega)$ then for $U \subset \Omega$

$$\int_{\partial U} f(\underline{x})d\sigma_{\underline{x}}g(\underline{x}) = \int_U [(f\partial_{\underline{x}})g + f(\partial_{\underline{x}}g)]dV(\underline{x}) \tag{3.18}$$

In particular, if $f \equiv 1$ then

$$\int_{\partial U} d\sigma_{\underline{x}}g(\underline{x}) = \int_U \partial_{\underline{x}}g(\underline{x})dV(\underline{x})$$

which relates the values of a function in a domain to its values on its boundary of the given domain. Let f be left-monogenic and g right-monogenic on an open set $\Omega \subset \mathbb{R}^n$ then

$$\int_{\partial U} g(\underline{x})d\sigma_{\underline{x}}f(\underline{x}) = 0$$

for any $U \subset \Omega$.

As a consequence we have.

Corollary 3.13. *Suppose that g is a right monogenic function on $\Omega \subset \mathbb{R}^n$ then for every subset U of Ω we have*

$$\int_{\partial U} g(\underline{x})d\sigma_{\underline{x}} = 0$$

Theorem 3.4.4. Let f left-monogenic and g right-monogenic on an open $U \subset \Omega$ (as in theorem

3.4.3) and let $\underline{y} \in U$ then

$$f(\underline{y}) = \frac{1}{a_n} \int_{\partial U} E(\underline{x} - \underline{y}) \eta(\underline{x}) f(\underline{x}) d\sigma_{\underline{x}}$$

and

$$g(\underline{y}) = \frac{1}{a_n} \int_{\partial U} g(\underline{x}) \eta(\underline{x}) E(\underline{x} - \underline{y}) d\sigma_{\underline{x}}.$$

Proof. We only prove the result for the left-monogenic function f. We consider the sphere $S^{n-1}(\underline{y}, r)$ for $r > 0$ chosen small enough such that the disc whose boundary is $S^{n-1}(\underline{y}, r)$ is included in U. Applying formula (3.18) on the kernel E and f we get

$$\int_{\partial U} E(\underline{x} - \underline{y}) \eta(\underline{x}) f(\underline{x}) d\sigma_{\underline{x}} = \int_{S^{n-1}(\underline{y}, r)} E(\underline{x} - \underline{y}) \eta(\underline{x}) f(\underline{x}) d\sigma_{\underline{x}}.$$

The sphere $S^{n-1}(\underline{y}, r)$ being of curvature 1, the outward pointing normal-vector η at $\underline{x} \in S^{n-1}(\underline{y}, r)$ is given by (see [161, Page 231])

$$\eta(\underline{x}) = \frac{\underline{x}}{|\underline{x}|} = \frac{\underline{y} - \underline{x}}{|\underline{x} - \underline{y}|} = \frac{\underline{y} - \underline{x}}{r}$$

so on the surface of $S^{n-1}(\underline{y}, r)$

$$E(\underline{x} - \underline{y}) \eta(\underline{x}) = \frac{\underline{x} - \underline{y}}{r^n} \frac{\underline{y} - \underline{x}}{r} = \frac{r^2}{r^{n+1}} = \frac{1}{r^{n-1}}$$

this gives us

$$\begin{aligned}
\int_{S^{n-1}(\underline{y}, r)} E(\underline{x} - \underline{y}) \eta(\underline{x}) f(\underline{x}) d\sigma_{\underline{x}} &= \int_{S^{n-1}(\underline{y}, r)} \frac{1}{r^{n-1}} f(\underline{x}) d\sigma_{\underline{x}} \\
&= \int_{S^{n-1}(\underline{y}, r)} \frac{f(\underline{x}) - f(\underline{y}) + f(\underline{y})}{r^{n-1}} d\sigma_{\underline{x}} \\
&= \int_{S^{n-1}(\underline{y}, r)} \frac{f(\underline{x}) - f(\underline{y})}{r^{n-1}} d\sigma_{\underline{x}} + \int_{S^{n-1}(\underline{y}, r)} f(\underline{y}) d\sigma_{\underline{x}} \\
&= \int_{S^{n-1}(\underline{y}, r)} \frac{f(\underline{x}) - f(\underline{y})}{|\underline{x} - \underline{y}|^{n-1}} d\sigma_{\underline{x}} + f(\underline{y}) \int_{S^{n-1}(\underline{y}, r)} d\sigma_{\underline{x}} \\
&= \frac{1}{a_n} \int_{S^{n-1}(\underline{y}, r)} \frac{f(\underline{x}) - f(\underline{y})}{|\underline{x} - \underline{y}|^{n-1}} d\sigma_{\underline{x}} + \frac{f(\underline{y})}{a_n} \int_{S^{n-1}} d\sigma_{\underline{x}} \\
&= \int_{S^{n-1}(\underline{y}, r)} \frac{f(\underline{x}) - f(\underline{y})}{|\underline{x} - \underline{y}|^{n-1}} d\sigma_{\underline{x}} + a_n f(\underline{y}).
\end{aligned}$$

By continuity we have

$$\lim_{r \to 0} \int_{S^{n-1}(\underline{y}, r)} \frac{f(\underline{x}) - f(\underline{y})}{|\underline{x} - \underline{y}|^{n-1}} d\sigma_{\underline{x}} = 0.$$

Finally, we have $f(\underline{y}) = \frac{1}{a_n} \int_{\partial U} E(\underline{x} - \underline{y}) \eta(\underline{x}) f(\underline{x}) d\sigma_{\underline{x}}$. By a similar procedure we prove the result for g. □

Theorem 3.4.5 – Mean Value Theorem. Let $\underline{y} \in U$ and $R > 0$ and consider $D(\underline{y}, R)$, the disk of centre $\underline{y}$ and radius R such that $\overline{D(\underline{y}, R)} \subset U$ then

$$f(\underline{y}) = \frac{1}{R^n a_n} \int_{D(\underline{y},R)} f(\underline{x}) dV(\underline{x}).$$

Proof. Applying theorem (3.4.4) on $D(\underline{y}, R)$ we obtain

$$\begin{aligned} f(\underline{y}) &= \frac{1}{a_n} \int_{\partial D(\underline{y},R)} E(\underline{x} - \underline{y}) \eta(\underline{x}) f(\underline{x}) d\sigma_{\underline{x}} \\ &= \frac{1}{a_n} \int_{\partial D(\underline{y},R)} E(\underline{x} - \underline{y}) \eta(\underline{x}) f(\underline{x}) d\sigma_{\underline{x}} \\ &= \frac{1}{a_n} \int_{\partial D(\underline{y},R)} \frac{1}{r^{n-1}} f(\underline{x}) d\sigma_{\underline{x}}. \end{aligned}$$

Integrating by respect to r will give us

$$f(\underline{y}) R^n = \frac{1}{a_n} \int_{D(\underline{y},R)} f(\underline{x}) dV(\underline{x}).$$

So that

$$f(\underline{y}) = \frac{1}{a_n R^n} \int_{D(\underline{y},R)} f(\underline{x}) dV(\underline{x}).$$

□

3.5 Clifford-Fourier Transform

In this section we propose to review some basic concepts of the Clifford-Fourier transform. Fore more details we may refer to [71] and [102]. Recall that the classical Fourier transom can be seen as the operator exponential (see [142, 6, 18, 168, 176, 65])

$$\mathcal{F} = \exp\left(-i\frac{\pi}{2}\mathcal{H}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-i\frac{\pi}{2}\right)^k \mathcal{H}^k \tag{3.19}$$

where $\mathcal{H}$ is the scalar-valued operator

$$\mathcal{H} = \frac{-1}{2} \left(\Delta_n + \underline{x}^2 + n\right) \tag{3.20}$$

called *Hermite operator*.

An eigenfunction for the n-dimensional Fourier transform is given by the Gaussian function $\underline{x} \longmapsto G(\underline{x}) = e^{-|\underline{x}|^2/2}$ which satisfies

$$\widehat{G}(\underline{\xi}) = G(\underline{\xi}).$$

Proposition 3.14. *The two operators $\mathcal{H}$ and $\exp(-i\frac{\pi}{2}\mathcal{H})$ are Fourier invariant in the sense* *that*

$$\widehat{\mathcal{H}(f)} = \mathcal{H}(\widehat{f})$$

and that

$$\widehat{\exp(-i\frac{\pi}{2}\mathcal{H})(f)} = \exp(-i\frac{\pi}{2}\mathcal{H})(\widehat{f}).$$

Proof. We have

$$\begin{aligned}
\widehat{\mathcal{H}(f)}(\underline{\xi}) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{-1}{2}(\Delta_n f(\underline{x}) + \underline{x}^2 f(\underline{x}) + nf(\underline{x}))e^{-i\underline{x}\cdot\underline{\xi}}dV(\underline{x}) \\
&= \frac{-1}{2}[\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\partial_{\underline{x}}^2 f(\underline{x})e^{-i\underline{x}\cdot\underline{\xi}}dV(\underline{x}) + \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\underline{x}^2 f(\underline{x})e^{-i\underline{x}\cdot\underline{\xi}}dV(\underline{x}) \\
&+ n\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(\underline{x})e^{-i\underline{x}\cdot\underline{\xi}}dV(\underline{x})] \\
&= \frac{-1}{2}\left[\underline{\xi}^2\widehat{f}(\underline{\xi}) + \partial_{\underline{\xi}}^2\widehat{f}(\underline{\xi}) + n\widehat{f}(\underline{\xi})\right] \\
&= \mathcal{H}(\widehat{f})(\underline{\xi}).
\end{aligned}$$

For the second assertion we have

$$\begin{aligned}
\widehat{\exp(-i\frac{\pi}{2}\mathcal{H})(f)}(\underline{\xi}) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\exp(-i\frac{\pi}{2}\mathcal{H})(f)(\underline{x})e^{-i\underline{x}\cdot\underline{\xi}}dV(\underline{x}) \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\left\{\sum_{k=0}^{\infty}\frac{\left(-i\frac{\pi}{2}\right)^k}{k!}\mathcal{H}^k(f)(\underline{x})\right\}e^{-i\underline{x}\cdot\underline{\xi}}dV(\underline{x}) \\
&= \sum_{k=0}^{\infty}\frac{\left(-i\frac{\pi}{2}\right)^k}{k!}\left\{\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\mathcal{H}^k(f)(\underline{x})e^{-i\underline{x}\cdot\underline{\xi}}dV(\underline{x})\right\} \\
&= \sum_{k=0}^{\infty}\frac{\left(-i\frac{\pi}{2}\right)^k}{k!}\widehat{\mathcal{H}^k(f)}(\underline{\xi}).
\end{aligned}$$

Lets calculate $\widehat{\mathcal{H}^k(f)}$. We have from the first assertion :

$$\begin{aligned}
\widehat{\mathcal{H}^k(f)} &= \mathcal{H}\left\{\widehat{\mathcal{H}^{k-1}}(f)\right\} \\
&= \mathcal{H}\widehat{\mathcal{H}^{k-1}(f)} \\
&= \mathcal{H}\mathcal{H}\left\{\widehat{\mathcal{H}^{k-2}}(f)\right\} \\
&= \mathcal{H}^2\widehat{\mathcal{H}^{k-2}(f)} \\
&\vdots \\
&= \mathcal{H}^k\widehat{\mathcal{H}^0(f)} \\
&= \mathcal{H}^k\widehat{f}.
\end{aligned}$$

This gives us

$$\begin{aligned}\widehat{\exp(-i\frac{\pi}{2}\mathcal{H})(f)}(\underline{\xi}) &= \sum_{k=0}^{\infty}\frac{\left(-i\frac{\pi}{2}\right)^k}{k!}\mathcal{H}^k\widehat{f}(\underline{\xi})\\ &= \exp(-i\frac{\pi}{2}\mathcal{H}\widehat{f}).\end{aligned}$$

□

The idea behind the extension of the classical Fourier transform to Clifford algebra-valued one resides in the generalization of the Hermite operator into a multivector-valued one. We use the factorization of the Laplace operator by the Dirac operator to obtain two new Clifford-valued operators that factorize the Hermite operator. This method has been developed in [26, 56, 22, Ch. 12]. For that purpose we introduce the following operators (see [24])

$$\begin{aligned}O_1 &= \frac{1}{2}\left(\partial_{\underline{x}} - \underline{x}\right)\left(\partial_{\underline{x}} + \underline{x}\right)\\ O_2 &= \frac{1}{2}\left(\partial_{\underline{x}} + \underline{x}\right)\left(\partial_{\underline{x}} - \underline{x}\right)\end{aligned}$$

This operators has the following properties

Proposition 3.15.

$$\begin{aligned}O_1 &= \mathcal{H} + \Gamma_{\underline{x}}\\ O_2 &= \mathcal{H} - \Gamma_{\underline{x}} + n\\ O_1 + O_2 &= 2\left(\mathcal{H} + \tfrac{n}{2}\right)\\ O_1 - O_2 &= 2\left(\Gamma_{\underline{x}} - \tfrac{n}{2}\right)\end{aligned}$$

Proof. We have that $\Gamma_{\underline{x}} = \frac{-1}{2}(\underline{x}\partial_{\underline{x}} - \partial_{\underline{x}}\underline{x} - n)$. So

$$\begin{aligned}O_1 &= \frac{1}{2}\left(\partial_{\underline{x}} - \underline{x}\right)\left(\partial_{\underline{x}} + \underline{x}\right)\\ &= \frac{1}{2}\left(\partial_{\underline{x}}^2 + \partial_{\underline{x}}\underline{x} - \underline{x}\partial_{\underline{x}} - \underline{x}^2\right)\\ &= \frac{1}{2}\left(\partial_{\underline{x}}^2 - \underline{x}^2 - n\right) - \frac{1}{2}\left(\underline{x}\partial_{\underline{x}} - \partial_{\underline{x}}\underline{x} - n\right)\\ &= \mathcal{H} + \Gamma_{\underline{x}}.\end{aligned}$$

And

$$\begin{aligned} O_2 &= \frac{1}{2}\left(\partial_{\underline{x}} + \underline{x}\right)\left(\partial_{\underline{x}} - \underline{x}\right) \\ &= \frac{1}{2}\left(\partial_{\underline{x}}^2 - \underline{x}^2 - n + \underline{x}\partial_{\underline{x}} - \partial_{\underline{x}}\underline{x} + n\right) \\ &= \mathcal{H} + \frac{1}{2}\left(\underline{x}\partial_{\underline{x}} - \partial_{\underline{x}}\underline{x} + 2n - n\right) \\ &= \mathcal{H} - \Gamma_{\underline{x}} + n. \end{aligned}$$

We have also

$$\begin{aligned} O_1 + O_2 &= \mathcal{H} + \Gamma_{\underline{x}} + \mathcal{H} - \Gamma_{\underline{x}} + n \\ &= 2\left(\mathcal{H} + \frac{n}{2}\right) \\ O_1 - O_2 &= \mathcal{H} + \Gamma_{\underline{x}} - \mathcal{H} + \Gamma_{\underline{x}} - n \\ &= 2\left(\Gamma_{\underline{x}} - \frac{n}{2}\right). \end{aligned}$$

□

For the two operators O_1 and O_2 to be used in the definition of the new Fourier transform in the Clifford algebra's setting, they have to be Fourier-invariant. We have from proposition (3.14)

$$\widehat{\mathcal{H}(f)} = \mathcal{H}(\widehat{f}).$$

So we have just to prove that

$$\widehat{\Gamma_{\underline{x}}(f)} = \Gamma_{\underline{\xi}}(\widehat{f}).$$

Indeed, formally we can write

$$\begin{aligned} \widehat{\Gamma_{\underline{x}}(f)} &= \frac{-1}{2}\widehat{(\underline{x}\partial_{\underline{x}} - \partial_{\underline{x}}\underline{x} - n)} \\ &= \frac{-1}{2}\left(\widehat{\underline{x}\partial_{\underline{x}}f} - \widehat{\partial_{\underline{x}}\underline{x}f} - n\widehat{f}\right) \\ &= \frac{-1}{2}\left(\partial_{\underline{\xi}}\widehat{\partial_{\underline{x}}f} - \underline{\xi}\widehat{\underline{x}f} - n\widehat{f}\right) \\ &= \frac{-1}{2}\left(\partial_{\underline{\xi}}\underline{\xi} - \underline{\xi}\partial_{\underline{\xi}} - n\right)\widehat{f} \\ &= \Gamma_{\underline{\xi}}(\widehat{f}). \end{aligned}$$

As

$$\begin{cases} O_1 &= \mathcal{H} + \Gamma_{\underline{x}} \\ O_2 &= \mathcal{H} - \Gamma_{\underline{x}} + n \end{cases}$$

then

$$\begin{cases} \widehat{O_1 f} & = \widehat{\mathcal{H}f} + \widehat{\Gamma_{\underline{x}} f} = O_1 \widehat{f} \\ \widehat{O_2} & = \widehat{\mathcal{H}\{} - \widehat{\Gamma_{\underline{x}} f} + n\widehat{f} = O_2 \widehat{f}. \end{cases}$$

Now, we are able to define the Clifford-Fourier transform

Definition 3.16. The Clifford-Fourier transform is the pair of exponential operators

$$\mathcal{F}_+ = \exp(-i\frac{\pi}{2}\mathcal{H}_+) \text{ and } \mathcal{F}_- = \exp(-i\frac{\pi}{2}\mathcal{H}_-).$$

If we want that the classical Fourier transform $\mathcal{F}$ to be the harmonic average of the couple $\mathcal{F}_+$ and $\mathcal{F}_-$

$$\mathcal{F}^2 = \mathcal{F}_+ \mathcal{F}_-$$

and as $\mathcal{F} = \exp(-i\frac{\pi}{2}\mathcal{H})$ we have to choose $\mathcal{H}_+$ and $\mathcal{H}_-$ to satisfy

$$\mathcal{H} = \frac{1}{2}(\mathcal{H}_+ + \mathcal{H}_-).$$

But having $O_1 + O_2 = 2\mathcal{H} + n$ we get

$$\mathcal{H}_+ + \mathcal{H}_- = O_1 + O_2 - n.$$

So we set

$$\begin{cases} \mathcal{H}_+ & = O_1 - \frac{n}{2} \\ \mathcal{H}_- & = O_2 - \frac{n}{2} \end{cases}$$

hence

$$\begin{cases} \mathcal{H}_+ & = \mathcal{H} + \left[\Gamma_{\underline{x}} - \frac{n}{2}\right] \\ \mathcal{H}_- & = \mathcal{H} - \left[\Gamma_{\underline{x}} - \frac{n}{2}\right]. \end{cases} \tag{3.21}$$

One alternative for the operators $\mathcal{H}_+$ and $\mathcal{H}_-$ is given in [56, Sec. 12.3.2] and in [26, Def. 4.2]

$$\begin{cases} \mathcal{H}_+ & = O_1 \\ \mathcal{H}_- & = O_2 - n \end{cases}$$

and in this case we have

$$\begin{cases} \mathcal{H}_+ & = \mathcal{H} + \Gamma_{\underline{x}} \\ \mathcal{H}_- & = \mathcal{H} - \Gamma_{\underline{x}}. \end{cases}$$

Keeping our operators as in (3.21) we obtain in terms of the exponential operator

$$\begin{aligned}\mathcal{F}_+ &= \exp(-i\frac{\pi}{2}\mathcal{H}_+)\\ &= \exp\left(-i\frac{\pi}{2}\left(\mathcal{H}+\left[\Gamma_{\underline{x}}-\frac{n}{2}\right]\right)\right)\\ &= \exp\left(-i\frac{\pi}{2}\left(\Gamma_{\underline{x}}-\frac{n}{2}\right)\right)\exp\left(-i\frac{\pi}{2}\mathcal{H}\right)\\ &= \exp\left(-i\frac{\pi}{2}\left(\Gamma_{\underline{x}}-\frac{n}{2}\right)\right)\mathcal{F}\end{aligned} \tag{3.22}$$

and in the same way

$$\mathcal{F}_- = \exp\left(i\frac{\pi}{2}\left(\Gamma_{\underline{x}}-\frac{n}{2}\right)\right)\mathcal{F}. \tag{3.23}$$

We obtain an integral representation for the new Clifford-Fourier transform:

$$\mathcal{F}_+[f](\underline{\xi}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\exp\left(-i\frac{\pi}{2}\left(\Gamma_{\underline{\xi}}-\frac{n}{2}\right)\right)e^{-i\underline{x}\cdot\underline{\xi}}f(\underline{x})dV(\underline{x})$$

and

$$\mathcal{F}_-[f](\underline{\xi}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\exp\left(i\frac{\pi}{2}\left(\Gamma_{\underline{\xi}}-\frac{n}{2}\right)\right)e^{-i\underline{x}\cdot\underline{\xi}}f(\underline{x})dV(\underline{x}). \tag{3.24}$$

We have defined the Clifford-Fourier transform as a pair of operators satisfying

$$\mathcal{F}^2 = \mathcal{F}_+\mathcal{F}_-$$

so we can write

$$\mathcal{F} = \mathcal{F}_+^{\frac{1}{2}}\mathcal{F}_-^{\frac{1}{2}}$$

where the square root is the Fractional Fourier Transform (see [149] and [56, Ch.11]) given by

$$\begin{aligned}\mathcal{F}_+^{\frac{1}{2}} &= \exp\left(-i\frac{\pi}{4}\mathcal{H}_+\right)\\ \mathcal{F}_-^{\frac{1}{2}} &= \exp\left(-i\frac{\pi}{4}\mathcal{H}_-\right).\end{aligned}$$

We can factorize the classical Fourier transform as the products

$$\begin{aligned}\mathcal{F} &= \exp\left(-i\frac{\pi}{4}\mathcal{H}_+\right)\exp\left(-i\frac{\pi}{4}\mathcal{H}_-\right)\\ &= \exp\left(-i\frac{\pi}{4}\mathcal{H}_-\right)\exp\left(-i\frac{\pi}{4}\mathcal{H}_+\right)\end{aligned}$$

This transform can be inverted. Using (3.22) and (3.23) we have

$$\mathcal{F}_+^{-1} = \exp\left(i\frac{\pi}{2}\mathcal{H}_+\right)$$

and

$$\mathcal{F}_-^{-1} = \exp\left(i\frac{\pi}{2}\mathcal{H}_-\right)$$

which gives

$$\mathcal{F}_+^{-1} = \exp\left(i\frac{\pi}{2}\left(\Gamma_{\underline{x}} - \frac{n}{2}\right)\right)\mathcal{F}^{-1}$$

and

$$\mathcal{F}_-^{-1} = \exp\left(-i\frac{\pi}{2}\left(\Gamma_{\underline{x}} - \frac{n}{2}\right)\right)\mathcal{F}^{-1}$$

The Clifford-Fourier transform has properties similar to those of the classical Fourier transform.

Proposition 3.17. *For two Clifford algebra-valued functions f and g and $a, b \in \mathbb{C}_n$ we have*

$$\mathcal{F}_+\left[fa + gb\right] = \mathcal{F}_+\left[f\right]a + \mathcal{F}_+\left[g\right]b$$

and

$$\mathcal{F}_-\left[fa + gb\right] = \mathcal{F}_-\left[f\right]a + \mathcal{F}_-\left[g\right]b.$$

Proof. This results from (3.24). □

Proposition 3.18. *For $\lambda > 0$ we have*

$$\mathcal{F}_+\left[f(\lambda\bullet)\right](\underline{\xi}) = \frac{1}{\lambda^n}\mathcal{F}_+\left[f\right](\frac{\underline{\xi}}{\lambda})$$

and

$$\mathcal{F}_-\left[f(\lambda\bullet)\right](\underline{\xi}) = \frac{1}{\lambda^n}\mathcal{F}_-\left[f\right](\frac{\underline{\xi}}{\lambda}).$$

Proof. First, let us prove that

$$\mathcal{F}\left[f(\lambda\bullet)\right](\underline{\xi}) = \frac{1}{\lambda^n}\mathcal{F}\left[f\right](\frac{\underline{\xi}}{\lambda}).$$

We have

$$\mathcal{F}\left[f(\lambda\bullet)\right](\underline{\xi}) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(\lambda\underline{x})e^{-i\underline{x}\cdot\underline{\xi}}dV(\underline{x}).$$

If we put $\underline{y} = \lambda\underline{x}$ then $dV(\underline{x}) = \frac{1}{\lambda^n}dV(\underline{y})$, so

$$\begin{aligned}\mathcal{F}\left[f(\lambda\bullet)\right](\underline{\xi}) &= \frac{1}{\lambda^n}\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(\underline{y})e^{-i\underline{x}\cdot\frac{\underline{\xi}}{\lambda}}dV(\underline{y}) \\ &= \frac{1}{\lambda^n}\widehat{f}(\frac{\underline{\xi}}{\lambda}).\end{aligned}$$

Then from (3.22) and (3.23)

$$\mathcal{F}_+\left[f(\lambda\bullet)\right](\underline{\xi}) = \exp\left(-i\frac{\pi}{2}\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)\right)\frac{1}{\lambda^n}\widehat{f}(\frac{\underline{\xi}}{\lambda})$$

and

$$\mathcal{F}_-\left[f(\lambda\bullet)\right](\underline{\xi}) = \exp\left(i\frac{\pi}{2}\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)\right)\frac{1}{\lambda^n}\widehat{f}(\frac{\underline{\xi}}{\lambda}).$$

Next, we prove that $\Gamma_{\underline{x}} = \Gamma_{\frac{\underline{x}}{\lambda}}$. We have

$$\Gamma_{\frac{\underline{x}}{\lambda}} = -\frac{\underline{x}}{\lambda} \wedge \partial_{\frac{\underline{x}}{\lambda}},$$

if we put $\underline{y} = \frac{\underline{x}}{\lambda} \Longleftrightarrow x_i = \lambda y_i$ and so

$$\partial_{\underline{y}} = \sum_{i=1}^{n} e_i \frac{\partial}{\partial y_i} = \sum_{i=1}^{n} e_i \lambda \frac{\partial}{\partial x_i} = \lambda \partial_{\underline{x}}.$$

So finally,

$$\Gamma_{\frac{\underline{x}}{\lambda}} = -\frac{\underline{x}}{\lambda} \wedge \partial_{\frac{\underline{x}}{\lambda}} = -\frac{\underline{x}}{\lambda} \wedge \lambda \partial_{\underline{x}} = \Gamma_{\underline{x}}.$$

We obtain the formulas

$$\mathcal{F}_{+}\left[f(\lambda \bullet)\right](\underline{\xi}) = \exp\left(-i\frac{\pi}{2}\left(\Gamma_{\frac{\underline{\xi}}{\lambda}} - \frac{n}{2}\right)\right)\frac{1}{\lambda^n}\widehat{f}(\frac{\underline{\xi}}{\lambda}) = \frac{1}{\lambda^n}\mathcal{F}_{+}\left[f\right](\frac{\underline{\xi}}{\lambda})$$

and

$$\mathcal{F}_{-}\left[f(\lambda \bullet)\right](\underline{\xi}) = \exp\left(i\frac{\pi}{2}\left(\Gamma_{\frac{\underline{\xi}}{\lambda}} - \frac{n}{2}\right)\right)\frac{1}{\lambda^n}\widehat{f}(\frac{\underline{\xi}}{\lambda}) = \frac{1}{\lambda^n}\mathcal{F}_{-}\left[f\right](\frac{\underline{\xi}}{\lambda}).$$

□

Proposition 3.19. *For all Clifford-valued function f we have*

$$\mathcal{F}_{+}\left[\bullet f(\bullet)\right](\underline{\xi}) = -(-i)^n \partial_{\underline{\xi}} \mathcal{F}_{+}\left[f\right](\underline{\xi})$$

and

$$\mathcal{F}_{-}\left[\bullet f(\bullet)\right](\underline{\xi}) = (i)^n \partial_{\underline{\xi}} \mathcal{F}_{-}\left[f\right](\underline{\xi})$$

Proof. We know from the proprieties of the classical Fourier transform that

$$\mathcal{F}\left[\bullet f(\bullet)\right](\underline{\xi}) = i\partial_{\underline{\xi}} \mathcal{F}\left[f\right](\underline{\xi}).$$

Then

$$\begin{aligned}
\mathcal{F}_{+}\left[\bullet f(\bullet)\right](\underline{\xi}) &= \exp\left(-i\frac{\pi}{2}\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)\right)\mathcal{F}\left[\bullet f(\bullet)\right](\underline{\xi}) \\
&= i\exp\left(-i\frac{\pi}{2}\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)\right)\partial_{\underline{\xi}}\mathcal{F}\left[f\right](\underline{\xi}) \\
&= i\left\{\sum_{k=0}^{\infty}\frac{(-i\frac{\pi}{2})^k}{k!}\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)^k\right\}\partial_{\underline{\xi}}\mathcal{F}\left[f\right](\underline{\xi}). \qquad (3.25)
\end{aligned}$$

We know from [142, Thm 2.4] that

$$\partial_{\underline{\xi}}\Gamma_{\underline{\xi}} + \Gamma_{\underline{\xi}}\partial_{\underline{\xi}} = (n-1)\partial_{\underline{\xi}}.$$

Then

$$\Gamma_{\underline{\xi}}\partial_{\underline{\xi}} = \partial_{\underline{\xi}}\left[n - 1 - \Gamma_{\underline{\xi}}\right],$$

and

$$\frac{n}{2}\partial_{\underline{\xi}} = \partial_{\underline{\xi}}\frac{n}{2}.$$

So

$$\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)\partial_{\underline{\xi}} = \partial_{\underline{\xi}}\left(\frac{n-2}{2} - \Gamma_{\underline{\xi}}\right).$$

Applying $\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)$ gives us

$$\begin{aligned}\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)\partial_{\underline{\xi}} &= \left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)\partial_{\underline{\xi}}\left(\frac{n-2}{2} - \Gamma_{\underline{\xi}}\right)\\ &= \partial_{\underline{\xi}}\left(\frac{n-2}{2} - \Gamma_{\underline{\xi}}\right)\left(\frac{n-2}{2} - \Gamma_{\underline{\xi}}\right).\end{aligned}$$

Repeating this k times we obtain

$$\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)^k\partial_{\underline{\xi}} = \partial_{\underline{\xi}}\left(\frac{n-2}{2} - \Gamma_{\underline{\xi}}\right)^k.$$

Then (3.25) becomes

$$\begin{aligned}\mathcal{F}_+\left[\bullet f(\bullet)\right](\underline{\xi}) &= i\left\{\sum_{k=0}^{\infty}\frac{(-i\frac{\pi}{2})^k}{k!}\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)^k\right\}\partial_{\underline{\xi}}\mathcal{F}\left[f\right](\underline{\xi})\\ &= i\left\{\sum_{k=0}^{\infty}\frac{(-i\frac{\pi}{2})^k}{k!}\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)^k\partial_{\underline{\xi}}\right\}\mathcal{F}\left[f\right](\underline{\xi})\\ &= i\left\{\sum_{k=0}^{\infty}\frac{(-i\frac{\pi}{2})^k}{k!}\partial_{\underline{\xi}}\left(\frac{n-2}{2} - \Gamma_{\underline{\xi}}\right)^k\right\}\mathcal{F}\left[f\right](\underline{\xi})\\ &= i\partial_{\underline{\xi}}\exp\left(i\frac{\pi}{2}\left(\Gamma_{\underline{\xi}} - \frac{n}{2} + 1\right)\right)\mathcal{F}\left[f\right](\underline{\xi})\\ &= i\exp\left(i\frac{\pi}{2}\right)\partial_{\underline{\xi}}\exp\left(i\frac{\pi}{2}\left(\Gamma_{\underline{\xi}} - \frac{n}{2}\right)\right)\mathcal{F}\left[f\right](\underline{\xi})\\ &= -\partial_{\underline{\xi}}\mathcal{F}_-\left[f\right](\underline{\xi}).\end{aligned}$$

The same yields for $\mathcal{F}_-$ and we have

$$\mathcal{F}_-\left[\bullet f(\bullet)\right](\underline{\xi}) = \partial_{\underline{\xi}}\mathcal{F}_+\left[f\right](\underline{\xi}).$$

□

These results may be generalized to

$$\mathcal{F}_+\left[\underline{x}^{2k}f\right](\underline{\xi}) = (-1)^k\partial_{\underline{\xi}}^{2k}\mathcal{F}_+\left[f\right](\underline{\xi})$$

and

$$\mathcal{F}_+\left[\underline{x}^{2k+1}f\right](\underline{\xi}) = -(-1)^k\partial_{\underline{\xi}}^{2k+1}\mathcal{F}_-\left[f\right](\underline{\xi})$$

By applying the recurrence rule.

The explicit form of the kernel of (3.24) is a difficult problem. In the case where $n = 2$ it has been given in [25, Sec. 4]

$$\mathcal{F}_+[f](\underline{\xi}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \exp(\underline{\xi}\wedge\underline{x})f(\underline{x})dV(\underline{x})$$

and

$$\mathcal{F}_-[f](\underline{\xi}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \exp(\underline{x}\wedge\underline{\xi})f(\underline{x})dV(\underline{x}).$$

For even dimensions, a first attempt was done in [27], for example for $n = 4$ we have in the term of the Bessel function [6, Sec. 2.2.2]

$$K_+(\underline{x},\underline{\xi}) = \sqrt{\frac{\pi}{2}}|\underline{x}\wedge\underline{\xi}|^{-1/2}\left((1+\underline{x}\cdot\underline{\xi})J_{1/2}(|\underline{x}\wedge\underline{\xi}|) + \frac{(\underline{\xi}\wedge\underline{x})}{|\underline{x}\wedge\underline{\xi}|}J_{3/2}(|\underline{x}\wedge\underline{\xi}|)(\underline{x}\cdot\underline{\xi})\right)$$

and

$$K_-(\underline{x},\underline{\xi}) = \sqrt{\frac{\pi}{2}}|\underline{x}\wedge\underline{\xi}|^{-1/2}\left((1-\underline{x}\cdot\underline{\xi})J_{1/2}(|\underline{x}\wedge\underline{\xi}|) + \frac{(\underline{\xi}\wedge\underline{x})}{|\underline{x}\wedge\underline{\xi}|}J_{3/2}(|\underline{x}\wedge\underline{\xi}|)(\underline{x}\cdot\underline{\xi})\right)$$

and for $n = 6$ we have

$$K_+(\underline{x},\underline{\xi}) = \sqrt{\tfrac{\pi}{2}}\{|\underline{x}\wedge\underline{\xi}|^{-1/2}\left(J_{1/2}(|\underline{x}\wedge\underline{\xi}|) + \tfrac{(\underline{x}\cdot\underline{\xi})^2}{|\underline{x}\wedge\underline{\xi}|}J_{3/2}(|\underline{x}\wedge\underline{\xi}|)\right)$$
$$+|\underline{x}\wedge\underline{\xi}|^{-3/2}\left(2(\underline{x}\cdot\underline{\xi})J_{3/2}(|\underline{x}\wedge\underline{\xi}|) + (\underline{\xi}\wedge\underline{x})J_{3/2}(|\underline{x}\wedge\underline{\xi}|) + \tfrac{(\underline{x}\cdot\underline{\xi})^2}{|\underline{x}\wedge\underline{\xi}|}(\underline{\xi}\wedge\underline{x})J_{5/2}(|\underline{x}\wedge\underline{\xi}|)\right)\}$$

and

$$K_-(\underline{x},\underline{\xi}) = \sqrt{\tfrac{\pi}{2}}\{|\underline{x}\wedge\underline{\xi}|^{-1/2}\left(J_{1/2}(|\underline{x}\wedge\underline{\xi}|) + \tfrac{(\underline{x}\cdot\underline{\xi})^2}{|\underline{x}\wedge\underline{\xi}|}J_{3/2}(|\underline{x}\wedge\underline{\xi}|)\right)$$
$$-|\underline{x}\wedge\underline{\xi}|^{-3/2}\left(2(\underline{x}\cdot\underline{\xi})J_{3/2}(|\underline{x}\wedge\underline{\xi}|) + (\underline{\xi}\wedge\underline{x})J_{3/2}(|\underline{x}\wedge\underline{\xi}|) + \tfrac{(\underline{x}\cdot\underline{\xi})^2}{|\underline{x}\wedge\underline{\xi}|}(\underline{\xi}\wedge\underline{x})J_{5/2}(|\underline{x}\wedge\underline{\xi}|)\right)\}.$$

In [55], the authors found a general expression for the kernels for all even dimensions.

3.6 Clifford Wavelet Transform

We introduce the concept of the Clifford-wavelet transform and some of its important properties to be used later. Besides of the translation and dilation, we will use rotation using the action of the *spin* group which is a double-cover of the special orthogonal group in $\mathbb{R}^n$. In this context, a function $\psi \in L^1 \cap L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$ will be considered as a Clifford mother wavelet. To join the admissibility assumptions in the case of wavelets on $\mathbb{R}$, here-also we define

Definition 3.20 (Clifford Wavelet)**.** Let $\psi \in L^1 \cap L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$ such that

- $\widehat{\psi}(\underline{\xi})\left[\widehat{\psi}(\underline{\xi})\right]^\dagger$ is scalar.
- The admissibility condition : $\mathcal{A}_\psi = (2\pi)^n \displaystyle\int_{\mathbb{R}^n} \frac{\widehat{\psi}(\underline{\xi})\left[\widehat{\psi}(\underline{\xi})\right]^\dagger}{|\underline{\xi}|^n} dV(\underline{\xi}) < \infty.$

The function ψ is called an *admissible Clifford mother wavelet* and $\mathcal{A}_\psi$ is its admissibility constant. We can see that this condition implies that

$$\widehat{\psi}(\underline{0}) = 0 \Longleftrightarrow \int_{\mathbb{R}^n} \psi(\underline{x})dV(\underline{x}) = 0.$$

Starting with an admissible wavelet, we create a whole set of *daughter wavelets* by translating, dilating and *Spin*-rotating the mother wavelet.

Example 3.21. In [34], the authors defined a generalization of the n-dimensional Mexican Hat wavelet [125] as the CK-extension of

$$\begin{aligned}\psi(\underline{x}) &= \exp(\frac{1}{2}\underline{x}^2)H_n(\underline{x}) \\ &= (-1)^n\partial_{\underline{x}}\exp(\frac{\underline{x}^2}{2})\end{aligned}$$

where H_n are the radial Hermite polynomials given in [33, 34, 9, 10]. Its Fourier transform is

$$\widehat{\psi}(\underline{\xi}) = (2\pi)^{\frac{n}{2}}(-i)^n\underline{\xi}^n\exp(\frac{\underline{\xi}^2}{2})$$

and so it is an admissible Clifford algebra-valued mother wavelet since

$$\begin{aligned}\mathcal{A}_\psi &= (2\pi)^n\int_{\mathbb{R}^n}\frac{\left|\widehat{\psi}(\underline{\xi})\right|^2}{\left|\underline{\xi}\right|^n}dV(\underline{\xi}) \\ &= (2\pi)^{2n}\int_{\mathbb{R}^n}\left|\exp(\frac{\underline{\xi}^2}{2})\right|^2 dV(\underline{\xi}) \\ &< \infty.\end{aligned}$$

Example 3.22. From [6] we know that for $0 < t < \frac{1-n-2\alpha}{2}$ we have

$$\int_{\mathbb{R}^n}\underline{x}^k G_{n,t,\alpha+t}(\underline{x})\left(1+|\underline{x}|^2\right)^\alpha dV(\underline{x}) = 0$$

where $G_{n,t,\alpha+t}$ are the Clifford-Gegenbauer polynomials and specially

$$\begin{aligned}\int_{\mathbb{R}^n} G_{n,t,\alpha+t}(\underline{x})\left(1+|\underline{x}|^2\right)^\alpha dV(\underline{x}) &= \int_{\mathbb{R}^n}(-1)^t\partial_{\underline{x}}^t\left(1+|\underline{x}|^2\right)^{\alpha+t}dV(\underline{x}) \\ &= 0\end{aligned}$$

and this is just the admissibility condition 3.6. So the functions $\underline{x} \longmapsto \psi_{n,t,\alpha}(\underline{x}) = (-1)^t\partial_{\underline{x}}^t\left(1+|\underline{x}|^2\right)^{\alpha+t}$ can be taken as mother wavelets. We call them *Clifford-Gegenbauer Wavelets.*

Example 3.23. In [20] the authors defined the so called *Clifford-Laguerre Wavelets* as

$$(-1)^k\partial_{\underline{x}}^k\left(\exp(-|\underline{x}|)|\underline{x}|^{\alpha+2k}P^+\right)$$

for $\alpha > -n$, $l > 0$ and $P^+(\underline{x}) = \frac{1}{2}(1 + i\frac{\underline{x}}{|\underline{x}|})$ is the Clifford-Heaviside function.

Definition 3.24. For $(a, \underline{b}, s) \in \mathbb{R}^+ \times \mathbb{R}^n \times Spin(n)$, we denote

$$\psi^{a,\underline{b},s}(\underline{x}) = \frac{1}{a^{\frac{n}{2}}} s\psi(\frac{\overline{s}(\underline{x}-\underline{b})s}{a})\overline{s}.$$

It holds in fact that these copies are also admissible and that

$$\mathcal{A}_{\psi^{a,\underline{b},s}} = \frac{a^{n/2}}{(2\pi)^n}\mathcal{A}_\psi < \infty.$$

Now we will see that indeed the family of wavelets $\psi^{a,\underline{b},s}$ can be used to analyse or decompose square integrable Clifford-valued functions, for that we have

Proposition 3.25. *The set* $\left\{\psi^{a,\underline{b},s} : a > 0, \underline{b} \in \mathbb{R}^n,\ s \in Spin(n)\right\}$ *is dense in* $L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$.

Proof. Let f be an analysed function such that

$$< \psi^{a,\underline{b},s}, f >_{L^2(\mathbb{R}^n,\mathbb{R}_n,dV(\underline{x}))} = 0,\ \forall a > 0,\ \underline{b} \in \mathbb{R}^n \text{ and } s \in Spin(n).$$

We shall prove that $f = 0$. Using the Parseval identity of the Clifford-Fourier transform : for $f, g \in L^1 \cap L^2(\mathbb{R}^n, \mathbb{C}, dV(\underline{x})$

$$\langle f, g\rangle_{L^2(\mathbb{R}^n,\mathbb{C},dV(\underline{x})} = \left\langle \widehat{f}, \widehat{g}\right\rangle_{L^2(\mathbb{R}^n,\mathbb{C},dV(\underline{\xi})}. \tag{3.26}$$

we obtain

$$< \psi^{a,\underline{b},s}, f >_{L^2(\mathbb{R}^n,\mathbb{R}_n,dV(\underline{x}))} = < \widehat{\psi^{a,\underline{b},s}}, \widehat{f} >_{L^2(\mathbb{R}^n,\mathbb{R}_n,dV(\underline{x}))} = 0.$$

Since

$$< \widehat{\psi^{a,\underline{b},s}}, \widehat{f} >_{L^2(\mathbb{R}^n,\mathbb{R}_n,dV(\underline{x}))} = a^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\underline{b}\cdot\underline{\xi}} s\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger \overline{s}\widehat{f}(\underline{\xi})dV(\underline{\xi}) = 0,$$

then necessarily

$$s\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger \overline{s}\widehat{f}(\underline{\xi}) = 0, \forall \underline{\xi} \in \mathbb{R}^n.$$

Recall now that for a fixed $\underline{\xi} \neq 0$ in $\mathbb{R}^n$ (see [56], pages 48 and 49)

$$\left\{a\overline{s}\underline{\xi}s, a > 0 \text{ and } s \in Spin(n)\right\} = \mathbb{R}^n.$$

It results that

$$\widehat{f} = 0 \text{ and so } f = 0.$$

□

As for the real case, we define the Clifford Wavelet Transform as the projection of the signal f on the set of admissible Clifford wavelets

Definition 3.26. The Clifford-wavelet transform of a function $f \in L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$ with respect to an admissible mother wavelet ψ is [1]

$$T_\psi\left[f\right](a,\underline{b},s) = < \psi^{a,\underline{b},s}, f >_{L^2(\mathbb{R}^n,\mathbb{R}_n,dV(\underline{x}))}$$

$$= \int_{\mathbb{R}^n} \left[\psi^{a,\underline{b},s}(\underline{x})\right]^\dagger f(\underline{x})dV(\underline{x}) \tag{3.27}$$

$$= \frac{1}{a^{\frac{n}{2}}} \int_{\mathbb{R}^n} s\left[\psi\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\right]^\dagger \overline{s}f(\underline{x})dV(\underline{x}). \tag{3.28}$$

3.6.1 Proprieties of the Clifford Wavelet Transform

It has the following covariance proprieties

- Covariance by translation

$$T_\psi\left[f(\bullet - \underline{c}\right](a,\underline{b},s) = T_\psi\left[f\right](a,\underline{b}-\underline{c},s).$$

- Covariance by Dilation

$$T_\psi\left[\frac{1}{\lambda^{\frac{n}{2}}}f(\frac{\bullet}{\lambda})\right](a,\underline{b},s) = T_\psi\left[f\right](\frac{a}{\lambda},\frac{b}{\lambda},s).$$

- Covariance by *Spin* rotation

$$T_\psi\left[L_t f\right](a,\underline{b},s) = tT_\psi\left[f\right](a,\overline{t}\underline{b}t,\overline{t}s)\overline{t}.$$

Proof. We have

$$T_\psi\left[f(\bullet - \underline{c}\right](a,\underline{b},s) = \frac{1}{a^{\frac{n}{2}}} \int_{\mathbb{R}^n} s\left[\psi\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\right]^\dagger \overline{s}f(\underline{x}-\underline{c})dV(\underline{x}).$$

Put $\underline{y} = \underline{x} - \underline{c} \Longrightarrow \underline{x} - \underline{b} = \underline{y} - (\underline{b} - \underline{c})$ and $dV(\underline{x}) = dV(\underline{y})$ so

$$T_\psi\left[f(\bullet - \underline{c}\right](a,\underline{b},s) = \frac{1}{a^{\frac{n}{2}}} \int_{\mathbb{R}^n} s\left[\psi\left(\frac{\overline{s}(\underline{y}-(\underline{b}-\underline{c}))s}{a}\right)\right]^\dagger \overline{s}f(\underline{y})dV(\underline{y})$$

$$= T_\psi\left[f\right](a,\underline{b}-\underline{c},s).$$

For the covariance by dilation

$$T_\psi\left[\frac{1}{\lambda^{\frac{n}{2}}}f(\frac{\bullet}{\lambda})\right](a,\underline{b},s) = \frac{1}{a^{\frac{n}{2}}} \int_{\mathbb{R}^n} s\left[\psi\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\right]^\dagger \overline{s}\frac{1}{\lambda^{\frac{n}{2}}}f(\frac{\underline{x}}{\lambda})dV(\underline{x}),$$

put $\underline{y} = \frac{\underline{y}}{\lambda} \Longrightarrow \underline{x} = \lambda\underline{y}, \underline{x} - \underline{b} = \lambda\underline{y} - \underline{b}$ and $dV(\underline{x}) = \lambda^n dV(\underline{y})$ so

$$T_\psi\left[\frac{1}{\lambda^{\frac{n}{2}}}f(\frac{\bullet}{\lambda})\right](a,\underline{b},s) = (\frac{\lambda}{a})^{\frac{n}{2}} \int_{\mathbb{R}^n} s\left[\psi\left(\frac{\overline{s}(\lambda\underline{y}-\underline{b})s}{a}\right)\right]^\dagger \overline{s}f(\underline{y})dV(\underline{y})$$

$$= \frac{1}{(\frac{a}{\lambda})^{\frac{n}{2}}} \int_{\mathbb{R}^n} s\left[\psi\left(\frac{\overline{s}(\underline{y}-\frac{\underline{b}}{\lambda})s}{\frac{a}{\lambda}}\right)\right]^\dagger \overline{s}f(\underline{y})dV(\underline{y})$$

$$= T_\psi\left[f\right](\frac{a}{\lambda},\frac{b}{\lambda},s).$$

[1]We used the complex Clifford conjugation † (see (3.3)) but we could have used (3.3) too.

Finally, for the case of the *Spin* rotation first recall the action of the *Spin* group (3.4.1.2)

$$L_s : f(\underline{x}) \to sf(\overline{s}\underline{x}s)\overline{s}$$

Then we have

$$T_\psi\left[L_t f\right](a,\underline{b},s) = \frac{1}{a^{\frac{n}{2}}}\int_{\mathbb{R}^n} s\left[\psi\left(\frac{\overline{s}(\underline{x}-\underline{b})s}{a}\right)\right]^\dagger \overline{s}tf(\overline{t}\underline{x}t)\overline{t}dV(\underline{x})$$

put $\underline{y} = \overline{t}\underline{x}t \Longrightarrow \underline{x} = t\underline{y}\overline{t}$ and $dV(\underline{x}) = \overline{t}dV(\underline{y})t$. So

$$T_\psi\left[L_t f\right](a,\underline{b},s) = \frac{1}{a^{\frac{n}{2}}}\int_{\mathbb{R}^n} s\left[\psi\left(\frac{\overline{s}(t\underline{y}\overline{t}-\underline{b})s}{a}\right)\right]^\dagger \overline{s}tf(\overline{t}\underline{y}t)\overline{t}dV(\underline{y}),$$

having $\overline{t}t = t\overline{t} = 1$, then

$$\begin{aligned} T_\psi\left[L_t f\right](a,\underline{b},s) &= \frac{1}{a^{\frac{n}{2}}}\int_{\mathbb{R}^n} s\left[\psi\left(\frac{\overline{s}(t\underline{y}\overline{t}-\underline{b})s}{a}\right)\right]^\dagger \overline{s}tf(\overline{t}\underline{y}t)\overline{t}dV(\underline{y}) \\ &= t\frac{1}{a^{\frac{n}{2}}}\int_{\mathbb{R}^n} \{\overline{t}s\}\left[\psi\left(\frac{\overline{s}t(\underline{y}-\overline{t}\underline{b}t)\overline{t}s}{a}\right)\right]^\dagger \{\overline{s}t\}\, f(\overline{t}\underline{y}t)\overline{t}dV(\underline{y})\overline{t} \\ &= tT_\psi\left[f\right](a,\overline{t}\underline{b}t,\overline{t}s)\overline{t} \end{aligned}$$

□

Definition 3.27. (Inner product relation) Let $\mathcal{H}_\psi = \left\{T_\psi\left[f\right],\ f \in L^2(\mathbb{R}^n,\mathbb{R}_n,dV(\underline{x}))\right\}$ be the image of $L^2(\mathbb{R}^n,\mathbb{R}_n,dV(\underline{x}))$ relatively to the operator T_ψ. We define the inner product for two square integrable functions f and g by

$$\left[T_\psi\left[f\right], T_\psi\left[g\right]\right] = \frac{1}{\mathcal{A}_\psi}\int\limits_{Spin(n)}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^+}(T_\psi\left[f\right](a,\underline{b},s))^\dagger T_\psi\left[g\right](a,\underline{b},s)\frac{da}{a^{n+1}}dV(\underline{b})ds,$$

where ds stands for the Haar measure on $Spin(n)$.

Proposition 3.28. *The range of an isometry $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}'$ is a closed subspace of $\mathcal{H}'$.*

Knowing all that, we can now introduce a result that permits us to invert the Clifford wavelet transform

Proposition 3.29. *$T_\psi : L^2(\mathbb{R}^n,\mathbb{R}_n,dV(\underline{x})) \longrightarrow \mathcal{H}_\psi$ is an isometry.*

Proof. We have to show that

$$\left[T_\psi\left[f\right], T_\psi\left[g\right]\right] = < f,g >_{L^2(\mathbb{R}^n,\mathbb{R}_n,dV(\underline{x}))}\,. \tag{3.29}$$

Put

$$\Phi_\psi(a,s,\underline{\xi})\left[f\right](-\underline{b}) = \left[\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger \overline{s}\widehat{f}(\underline{\xi})\right](-\underline{b})$$

and similarly

$$\Phi_\psi(a,s,\underline{\xi})\,[g]\,(-\underline{b}) = \left[\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger \overline{s}\widehat{g}(\underline{\xi})\right](-\underline{b}).$$

We obtain

$$T_\psi\,[f]\,(a,\underline{b},s) = a^{\frac{n}{2}} s(2\pi)^{\frac{n}{2}}\, \Phi_\psi\widehat{(a,\underline{\xi},s)}\,[f]\,(-\underline{b})$$

and

$$T_\psi\,[g]\,(a,\underline{b},s) = a^{\frac{n}{2}} s(2\pi)^{\frac{n}{2}}\, \Phi_\psi\widehat{(a,\underline{\xi},s)}\,[g]\,(-\underline{b}).$$

Applying Parseval formula we get

$$\left\langle \Phi_\psi\widehat{(a,\bullet,s)}\,[f], \Phi_\psi\widehat{(a,\bullet,s)}\,[g]\right\rangle = \left\langle \Phi_\psi(a,\bullet,s)\,[f]\,, \Phi_\psi(a,\bullet,s)\,[g]\right\rangle.$$

And so

$$\begin{aligned}
[T_\psi\,[f]\,, T_\psi\,[g]] &= \frac{1}{(2\pi)^n\mathcal{A}_\psi}\int\limits_{Spin(n)}\int\limits_{\mathbb{R}^+}\left\{\int_{\mathbb{R}^n}(\Phi_\psi(a,\underline{\xi},s)\,[f]\,(\underline{\xi}))^\dagger\Phi_\psi(a,\underline{\xi},s)\,[g]\,(\underline{\xi})dV(\underline{b})\right\}\frac{da}{a}ds\\
&= \frac{1}{(2\pi)^n\mathcal{A}_\psi}\int\limits_{Spin(n)}\int\limits_{\mathbb{R}^+}\left\{\int_{\mathbb{R}^n}\left[(\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger\overline{s}\widehat{f}(\underline{\xi})\right]^\dagger\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger\overline{s}\widehat{g}(\underline{\xi})dV(\underline{\xi})\right\}\frac{da}{a}ds\\
&= \frac{1}{(2\pi)^n\mathcal{A}_\psi}\int_{Spin(n)}\int\limits_{\mathbb{R}^+}\left\{\int_{\mathbb{R}^n}\left[\widehat{f}(\underline{\xi})\right]^\dagger s\widehat{\psi}(a\overline{s}\underline{\xi}s)\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger\overline{s}\widehat{g}(\underline{\xi})dV(\underline{\xi})\right\}\frac{da}{a}ds\\
&= \frac{1}{(2\pi)^n\mathcal{A}_\psi}\int_{\mathbb{R}^n}\left[\widehat{f}(\underline{\xi})\right]^\dagger\left\{\int_{Spin(n)}\int\limits_{\mathbb{R}^+}s\widehat{\psi}(a\overline{s}\underline{\xi}s)\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger\overline{s}\frac{da}{a}ds\right\}\widehat{g}(\underline{\xi})dV(\underline{\xi}).
\end{aligned}$$

Observing now that

$$\int_{Spin(n)}\int\limits_{\mathbb{R}^+}s\widehat{\psi}(a\overline{s}\underline{\xi}s)\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger\overline{s}\frac{da}{a}ds = \frac{\mathcal{A}_\psi}{(2\pi)^n}, \tag{3.30}$$

we get immediately

$$\begin{aligned}
[T_\psi\,[f]\,, T_\psi\,[g]] &= \int_{\mathbb{R}^n}\left[\widehat{f}(\underline{\xi})\right]^\dagger\widehat{g}(\underline{\xi})dV(\underline{\xi})\\
&=< \widehat{f},\widehat{g} >\\
&=< f,g >\,.
\end{aligned} \tag{3.31}$$

□

We can also say that it is an isometry between the two spaces of integrable functions $L^2(\mathbb{R}^n,\mathbb{R}_n,dV(\underline{x}))$ and $L^2(\mathbb{R}_+\times\mathbb{R}^n\times Spin(n),\mathcal{A}_\psi^{-1}a^{-(n+1)}dadV(\underline{b})ds)$. This is an analogue to Parseval's formula and as a result we have also a Plancherel's formula

$$\int\limits_{Spin(n)}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^+}|T_\psi\,[f]\,(a,\underline{b},s)|^2\,\frac{da}{a^{n+1}}dV(\underline{b})ds = \mathcal{A}_\psi\,\|f\|_2^2\,. \tag{3.32}$$

As a result of the last Proposition and as in the real case, we have here a Clifford-wavelet reconstruction formula.

Proposition 3.30. *For all $f \in L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$ we have*

$$f(\underline{x}) = \frac{1}{A_\psi} \int\limits_{Spin(n)} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^+} \psi^{a,\underline{b},s}(\underline{x}) T_\psi [f] (a, \underline{b}, s) \frac{da}{a^{n+1}} dV(\underline{b}) ds$$

which holds weakly in $L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$.

In other words, the Clifford wavelet transform decomposes the signal f in terms of the analysing wavelets $\psi^{a,b,s}$ with coefficients $T_\psi [f]$.

Proof. Let f and g two square integrable Clifford-valued functions with Clifford wavelet transforms (with respect to a mother wavelet ψ) $T_\psi [f]$ and $T_\psi [g]$ respectively. We have from proposition (3.29) and using (3.28)

$$\begin{aligned}
< f, g >_{L^2} &= \frac{1}{\mathcal{A}_\psi} \int\limits_{Spin(n)} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^+} (T_\psi [f] (a, \underline{b}, s))^\dagger T_\psi [g] (a, \underline{b}, s) \frac{da}{a^{n+1}} dV(\underline{b}) ds \\
&= \frac{1}{\mathcal{A}_\psi} \int\limits_{Spin(n)} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^+} [T_\psi [f] (a, \underline{b}, s)]^\dagger T_\psi [g] (a, \underline{b}, s) \frac{da}{a^{n+1}} dV(\underline{b}) ds \\
&= \frac{1}{\mathcal{A}_\psi} \int_{Spin(n)} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^+} < \psi^{a,\underline{b},s}, f >_{L^2}^\dagger T_\psi[g](a, \underline{b}, s) \frac{da}{a^{n+1}} dV(\underline{b}) ds \\
&= \frac{1}{\mathcal{A}_\psi} \int_{Spin(n)} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^+} < f, \psi^{a,\underline{b},s} >_{L^2} T_\psi[g](a, \underline{b}, s) \frac{da}{a^{n+1}} dV(\underline{b}) ds \\
&=< f, \frac{1}{\mathcal{A}_\psi} \int_{Spin(n)} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^+} \psi^{a,\underline{b},s}(\underline{x}) T_\psi[g](a, \underline{b}, s) \frac{da}{a^{n+1}} dV(\underline{b}) ds >_{L^2}
\end{aligned}$$

Then

$$g(\underline{x}) = \frac{1}{\mathcal{A}_\psi} \int_{Spin(n)} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^+} \psi^{a,\underline{b},s}(\underline{x}) T_\psi[g](a, \underline{b}, s) \frac{da}{a^{n+1}} dV(\underline{b}) ds$$

where the equality is understood in the L^2-sense. □

Now, we present a result similar to the reproducing kernel given by (2.2.3).

Theorem 3.6.1. A function $F \in L^2\left(\mathbb{R}_+ \times \mathbb{R}^n \times Spin(n), \mathcal{A}_\psi^{-1} a^{-(n+1)} da dV(\underline{b}) ds\right)$ is the Clifford wavelet transform of a square integrable function f iff

$$F(a,\underline{b},s)=\frac{1}{C_\psi}\int_{Spin(n)}\int_{\mathbb{R}^n}\int_0^{+\infty}\left(K_\psi(a,\underline{b},s;\tilde{a},\tilde{\underline{b}},\tilde{s})\right)^\dagger F(\tilde{a},\tilde{\underline{b}},\tilde{s})\frac{d\tilde{a}}{\tilde{a}^{n+1}}dV(\tilde{\underline{b}})d\tilde{s}$$

where $K_\psi(a,\underline{b},s;\tilde{a},\tilde{\underline{b}},\tilde{s})=T_\psi\left[\psi^{a,b,s}\right](\tilde{a},\tilde{\underline{b}},\tilde{s})=<\psi^{\tilde{a},\tilde{b},\tilde{s}},\psi^{a,b,s}>$ is the reproducing kernel.

3.7 Conclusion

In this part, we gave an introduction to the theory of Clifford algebras and we saw that modulo a condition similar to the Cauchy-Riemann equations in the complex plan, we generalized the notion of holomorphic and harmonic functions to n-dimensions. Also, we presented the extension of the well known Fourier transform to Clifford algebra settings and presented the main topic of this thesis namely the Clifford wavelet transform.

Chapter 4

Clifford Wavelet Uncertainty Principle

4.1 Introduction

The uncertainty principle is central for information processing and quantum physics. In the Clifford algebras framework, the uncertainty principle provides us data about the way a multivector valued function and its Clifford-Fourier transform are related. We will see that the uncertainty principle for the Clifford wavelet transform establishes a lower bound of the product of the variances of Clifford wavelet transform of a square integrable multivector-valued function and its Clifford-Fourier transform.

4.2 Old uncertainty principle revisited

First, we recall some results concerning the classical Heisenberg uncertainty principle. For more backgrounds on the uncertainty principle, its variants, Fourier and wavelet transforms on the Euclidean space $\mathbb{R}^n$ the readers may be referred also to [105], [146], [166] and [174]. Mathematically, the uncertainty principle states that a non-zero function and its Fourier transform cannot both be sharply localized. The next theorem formally summarizes the Heisenberg's Uncertainty Principle.

Theorem 4.2.1. Uncertainty Principle [182]

Let A and B be two self-adjoint operators on a Hilbert space $\mathcal{X}$ with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively and $[A, B] = AB - BA$ their commutator. Then

$$\|Af\|_2 \|Bf\|_2 \geq \frac{1}{2} |< [A, B] f, f >| , \forall f \in \mathcal{D}([A, B]). \tag{4.1}$$

Proof. Let $f \in \mathcal{D}([A, B])$. Then $f \in \mathcal{D}(A) \cap \mathcal{D}(B)$. Hence ,

$$| \langle [A, B] f, f \rangle | = | \langle ABf, f \rangle - \langle BAf, f \rangle |.$$

As A and B are two self-adjoints operators, we get

$$| \langle [A, B] f, f \rangle | = | \langle Af, Bf \rangle - \langle Bf, Af \rangle | = 2|\mathfrak{Im} \{\langle Af, Bf \rangle\} |.$$

Applying next the Cauchy-Schwartz inequality, we obtain

$$| \langle [A, B] f, f \rangle | \leq 2\|Af\|_2\|Bf\|_2.$$

Which reads as

$$\|Af\|_2 \|Bf\|_2 \geq \frac{1}{2} |\langle [A, B] f, f \rangle| . \tag{4.2}$$

which completes the proof. □

Lets apply theorem (4.2.1) to the case of the Fourier transform. We mainly review the results of [71] and [102]. Let $f \in L^1 \cap L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$ We define for $k \in \{1, 2, \cdots, n\}$ the two families of operators

$$A_k f(\underline{x}) = x_k f(\underline{x})$$

$$B_k f(\underline{x}) = \partial_{x_k} f(\underline{x}).$$

Using the fact that

$$\begin{aligned}
\partial_{x_k} f(\underline{x}) &= \partial_{x_k} \left\{ \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\underline{x}\cdot\underline{\xi}} \widehat{f}(\underline{\xi}) dV(\underline{\xi}) \right\} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \partial_{x_k} \left\{ e^{i\underline{x}\cdot\underline{\xi}} \widehat{f}(\underline{\xi}) \right\} dV(\underline{\xi}) \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\underline{x}\cdot\underline{\xi}} \left\{ i\xi_k \widehat{f}(\underline{\xi}) \right\} dV(\underline{\xi}) \\
&= \mathcal{F}^{-1} \left[i\xi_k \widehat{f}(\bullet) \right] (\underline{x})
\end{aligned}$$

whence

$$\widehat{\partial_{x_k} f}(\underline{\xi}) = i\xi_k \widehat{f}(\underline{\xi}).$$

and so we have the norm equality

$$\left\| \widehat{\partial_{x_k} f} \right\|_2 = \left\| \xi_k \widehat{f} \right\|_2 .$$

In a similar way we can see that

$$\begin{aligned} \widehat{x_k f(\bullet)}(\underline{\xi}) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} x_k f(\underline{x}) e^{-i\underline{x}\cdot\underline{\xi}} dV(\underline{x}) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(\underline{x}) i\partial_{\xi_k} \left\{ e^{-i\underline{x}\cdot\underline{\xi}} \right\} dV(\underline{x}) \\ &= i\partial_{\xi_k} \mathcal{F}[f](\underline{\xi}). \end{aligned}$$

We have $A_k f(\underline{x}) = x_k f(\underline{x})$ and $B_k f(\underline{x}) = \partial_{x_k} f(\underline{x})$. So

$$\|A_k f\|_2 = \|x_k f\|_2$$

and

$$\|B_k f\|_2 = \left\| \xi_k \hat{f} \right\|_2$$

Applying theorem (4.2.1) to both operators A_k and B_k yields :

$$\|A_k f\|_2 \|B_k f\|_2 \geq \frac{1}{2} \left| < [A_k, B_k] f, f > \right| .$$

As the commutator

$$\begin{aligned} [A_k, B_k] f &= A_k B_k f - B_k A_k f \\ &= x_k \partial_{x_k} f(\underline{x}) - \partial_{x_k} \{ x_k f(\underline{x}) \} \\ &= x_k \partial_{x_k} f(\underline{x}) - f(\underline{x}) - x_k \partial_{x_k} f(\underline{x}) \\ &= -f(\underline{x}) \end{aligned}$$

we obtain finally

$$\|x_k f\|_2 \left\| \xi_k \widehat{f} \right\|_2 \geq \frac{1}{2} \|f\|_2^2 . \tag{4.3}$$

Theorem 4.2.2. For A and B two symmetric operators on a Hilbert space $\mathcal{H}$ and for $f \in L^2(\mathcal{H}, dx)$ we have

$$\|Af\|_2 \|Bf\|_2 \geq \frac{1}{2} \sqrt{ \left| \langle [A, B] f, f \rangle \right|^2 + \left| \langle [A, B]_+ f, f \rangle \right|^2 } \tag{4.4}$$

where $[A, B]_+ = AB + BA$ is the anti-commutator.

Proof. See [165] and [69]. □

As for Theorem 4.3.1, the first step is to apply this result to the Fourier transform. So we have

Theorem 4.2.3. [69] Let $f \in L^1 \cap L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$ and define $A_k f(\underline{x}) = x_k f(\underline{x})$ and $B_k f(\underline{x}) = \frac{1}{i}\partial_{x_k} f(\underline{x})$ then

$$\|x_k f\|_2 \left\|\xi_k \widehat{f}\right\|_2 \geq \sqrt{2}(\|f\|_2^2 + |2\langle x_k \partial_{x_k} f, f\rangle|).$$

Proof. We have by Theorem 4.2.2

$$\|A_k f\|_2 \|B_k f\|_2 \geq \frac{1}{2}\sqrt{|\langle [A_k, B_k] f, f\rangle|^2 + \left|\langle [A_k, B_k]_+ f, f\rangle\right|^2}. \tag{4.5}$$

Observe next that

$$\begin{aligned}
[A_k, B_k] f(\underline{x}) &= A_k B_k f(\underline{x}) - B_k A_k f(\underline{x}) \\
&= x_k \frac{1}{i}\partial_{x_k} f(\underline{x}) - \frac{1}{i}\partial_{x_k} x_k f(\underline{x}) \\
&= \frac{1}{i} x_k \partial_{x_k} f(\underline{x}) - \frac{1}{i} f(\underline{x}) - \frac{1}{i} x_k \partial_{x_k} f(\underline{x}) \\
&= -\frac{1}{i} f(\underline{x}).
\end{aligned}$$

By the same way

$$[A_k, B_k]_+ f(\underline{x}) = \frac{1}{i}(2x_k \partial_{x_k} f(\underline{x}) + f(\underline{x})).$$

Substituting in (4.5) we get

$$\|x_k f\|_2 \|\partial_{x_k} f\|_2 \geq \frac{1}{2}\sqrt{\|f\|_2^4 + |\langle 2x_k \partial_{x_k} f + f, f\rangle|^2}.$$

We know that

$$\widehat{\partial_{x_k} f}(\underline{\xi}) = i\xi_k \widehat{f}(\underline{\xi}).$$

So

$$\|\partial_{x_k} f\|_2 = \left\|\widehat{\partial_{x_k} f}\right\|_2 = \left\|\xi_k \widehat{f}\right\|_2.$$

Then (4.5) becomes

$$\begin{aligned}
\|x_k f\|_2 \left\|\xi_k \widehat{f}\right\|_2 &\geq \frac{1}{2}\sqrt{\|f\|_2^4 + |\langle 2x_k \partial_{x_k} f + f, f\rangle|^2} \\
&= \frac{1}{2}\sqrt{\|f\|_2^4 + \left|\|f\|_2^2 + 2\langle x_k \partial_{x_k} f, f\rangle\right|^2} \\
&\geq \frac{1}{2}\sqrt{\|f\|_2^4 + 4|\langle x_k \partial_{x_k} f, f\rangle|^2}.
\end{aligned}$$

Since for $a, b > 0$, $\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2}$ then $\frac{1}{2}\sqrt{a^2+b^2} \geq \sqrt{2}(a+b)$. So

$$\begin{aligned}
\|x_k f\|_2 \left\|\xi_k \widehat{f}\right\|_2 &\geq \frac{1}{2}\sqrt{\|f\|_2^4 + 4|\langle x_k \partial_{x_k} f, f\rangle|^2} \\
&\geq \sqrt{2}(\|f\|_2^2 + |2\langle x_k \partial_{x_k} f, f\rangle|).
\end{aligned}$$

□

4.3 Clifford wavelet uncertainty principle

In this section, we establish the main result of the thesis : a new Heisenberg uncertainty principle for the Clifford wavelet transform. This result have been published in [14].

Theorem 4.3.1. Let $\psi \in L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$ be an admissible Clifford mother wavelet. Then for $f \in L^2(\mathbb{R}^n, \mathbb{R}_n, dV(\underline{x}))$ the following inequality holds

$$\left(\int_{Spin(n)} \int_{\mathbb{R}^+} \|b_k T_\psi [f] (a, \bullet, s)\|_2^2 \frac{da}{a^{n+1}} ds\right)^{\frac{1}{2}} \left\|\xi_k \widehat{f}\right\|_2 \geq \frac{(2\pi)^{\frac{n}{2}}}{2} \sqrt{A_\psi} \|f\|_2^2,$$

where $k = 1, 2, \cdots, n$.

To prove this result we need the following lemma.

Lemma 4.1. *Given ψ and f as in Theorem (4.3.1) then*

$$\int_{Spin(n)} \int_{\mathbb{R}^+} \int_{\mathbb{R}^n} |\xi_k \widehat{T_\psi [f]}(a, \underline{\xi}, s)|^2 dV(\underline{\xi}) \frac{da}{a^{n+1}} ds = \frac{A_\psi}{(2\pi)^n} \left\|\xi_k \widehat{f}\right\|_2^2.$$

Proof. As the daughter wavelet $\psi^{a,\underline{b},s}$ has the following Fourier expression

$$\widehat{\psi^{a,\underline{b},s}}(\underline{\xi}) = a^{\frac{n}{2}} e^{-i<\underline{b},\underline{\xi}>} s\widehat{\psi}(a\overline{s}\underline{\xi}s)\overline{s},$$

we get

$$T_\psi [f] (a, \underline{b}, s) = a^{\frac{n}{2}} \mathcal{F}^{-1} \left[\overline{s} \left[\widehat{\psi}(a\overline{s} \bullet s)\right]^\dagger s\widehat{f}(\bullet)\right] (\underline{b})$$

and in the frequency domain

$$\widehat{T_\psi [f]}(a, \underline{\xi}, s) = a^{\frac{n}{2}} \overline{s} \left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger s\widehat{f}(\underline{\xi}). \tag{4.6}$$

Therefore by (3.8)

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\xi_k \widehat{T_\psi[f]}(a,\underline{\xi},s)|^2 dV(\underline{\xi}) \\
&= \int_{\mathbb{R}^n} |\xi_k a^{\frac{n}{2}} \overline{s} \left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger s\widehat{f}(\underline{\xi})|^2 dV(\underline{\xi}) \\
&= \int_{\mathbb{R}^n} \left\{\xi_k a^{\frac{n}{2}} \overline{s} \left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger s\widehat{f}(\underline{\xi})\right\}^\dagger \xi_k a^{\frac{n}{2}} \overline{s} \left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger s\widehat{f}(\underline{\xi}) dV(\underline{\xi}) \\
&= \int_{\mathbb{R}^n} \xi_k^2 a^n \overline{s} \left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger s\widehat{f}(\underline{\xi}) \left\{\overline{s} \left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger s\widehat{f}(\underline{\xi})\right\}^\dagger dV(\underline{\xi}) \\
&= \int_{\mathbb{R}^n} \xi_k^2 a^n \overline{s} \left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger s\widehat{f}(\underline{\xi}) \left[\widehat{f}(\underline{\xi})\right]^\dagger \overline{s}\widehat{\psi}(a\overline{s}\underline{\xi}s) s dV(\underline{\xi}) \\
&= \int_{\mathbb{R}^n} \xi_k^2 a^n \left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger \widehat{f}(\underline{\xi}) \left[\widehat{f}(\underline{\xi})\right]^\dagger \widehat{\psi}(a\overline{s}\underline{\xi}s) dV(\underline{\xi}) \\
&= \int_{\mathbb{R}^n} \xi_k^2 a^n \left\{\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger \widehat{\psi}(a\overline{s}\underline{\xi}s)\right\} \left\{\widehat{f}(\underline{\xi}) \left[\widehat{f}(\underline{\xi})\right]^\dagger\right\} dV(\underline{\xi}). \qquad (4.7)
\end{aligned}
$$

Using (4.7) we obtain

$$
\begin{aligned}
&\int_{Spin(n)} \int_{\mathbb{R}^+} \int_{\mathbb{R}^n} |\xi_k \widehat{T_\psi[f]}(a,\underline{\xi},s)|^2 dV(\underline{\xi}) \frac{da}{a^{n+1}} ds \\
&= \int_{Spin(n)} \int_{\mathbb{R}^+} \int_{\mathbb{R}^n} \xi_k^2 a^n \left\{\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger \widehat{\psi}(a\overline{s}\underline{\xi}s)\right\} \left\{\widehat{f}(\underline{\xi}) \left[\widehat{f}(\underline{\xi})\right]^\dagger\right\} dV(\underline{\xi}) \frac{da}{a^{n+1}} ds \\
&= \int_{Spin(n)} \int_{\mathbb{R}^+} \int_{\mathbb{R}^n} \xi_k^2 \left\{\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger \widehat{\psi}(a\overline{s}\underline{\xi}s)\right\} \left\{\widehat{f}(\underline{\xi}) \left[\widehat{f}(\underline{\xi})\right]^\dagger\right\} dV(\underline{\xi}) \frac{da}{a} ds \\
&= \int_{\mathbb{R}^n} \left\{ \int_{Spin(n)} \int_{\mathbb{R}^+} \frac{\left[\widehat{\psi}(a\overline{s}\underline{\xi}s)\right]^\dagger \widehat{\psi}(a\overline{s}\underline{\xi}s)}{a} da ds \right\} \xi_k^2 \left\{\widehat{f}(\underline{\xi}) \left[\widehat{f}(\underline{\xi})\right]^\dagger\right\} dV(\underline{\xi}).
\end{aligned}
$$

According to (3.30), we get finally

$$
\int_{Spin(n)} \int_{\mathbb{R}^+} \int_{\mathbb{R}^n} |\xi_k \widehat{T_\psi[f]}(a,\underline{\xi},s)|^2 dV(\underline{\xi}) \frac{da}{a^{n+1}} ds = \frac{A_\psi}{(2\pi)^n} \left\|\xi_k \widehat{f}\right\|_2^2 .
$$

□

Proof. of Theorem (4.3.1). Using the inequality (4.3) and setting $\underline{x} = \underline{b} \in \mathbb{R}^n$, we obtain

$$
\|b_k T_\psi[f](a,\bullet,s)\|_2 \left\|\xi_k \widehat{T_\psi[f]}(a,\bullet,s)\right\|_2 \geq \frac{1}{2} \|T_\psi[f](a,\bullet,s)\|_2^2 .
$$

Therefore

$$
\int_{Spin(n)} \int_{\mathbb{R}^+} \|b_k T_\psi[f](a,\bullet,s)\|_2 \left\|\xi_k \widehat{T_\psi[f]}(a,\bullet,s)\right\|_2 \frac{da}{a^{n+1}} ds
$$

$$\geq \frac{1}{2}\int_{Spin(n)}\int_{\mathbb{R}^+}\|T_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds$$

According to the Cauchy-Schwartz inequality (3.10), it follows that

$$\int_{Spin(n)\times\mathbb{R}^+}\|b_kT_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds\times\int_{Spin(n)\times\mathbb{R}^+}\left\|\xi_k\widehat{T_\psi[f]}(a,\bullet,s)\right\|_2^2\frac{da}{a^{n+1}}ds$$

$$\geq\left(\frac{1}{2}\int_{Spin(n)\times\mathbb{R}^+\times\mathbb{R}^n}|T_\psi[f](a,\underline{b},s)|^2dV(\underline{b})\frac{da}{a^{n+1}}ds\right)^2. \tag{4.8}$$

Now, using Lemma 4.1 and the fact that the Clifford wavelet transform is an isometry, we get by (3.32)

$$\int\limits_{Spin(n)}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^+}|T_\psi[f](a,\underline{b},s)|^2\frac{da}{a^{n+1}}dV(\underline{b})ds=\mathcal{A}_\psi\|f\|_2^2.$$

The inequality (4.8) becomes

$$\left(\int\limits_{Spin(n)}\int\limits_{\mathbb{R}^+}\|b_kT_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds\right)^{\frac{1}{2}}\left(\frac{A_\psi}{(2\pi)^n}\left\|\xi_k\widehat{f}\right\|_2^2\right)^{\frac{1}{2}}\geq\frac{1}{2}\mathcal{A}_\psi\|f\|_2^2. \tag{4.9}$$

Hence, we obtain

$$\left(\int\limits_{Spin(n)}\int\limits_{\mathbb{R}^+}\|b_kT_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds\right)^{\frac{1}{2}}\left\|\xi_k\widehat{f}\right\|_2\geq\frac{(2\pi)^{\frac{n}{2}}}{2}\sqrt{A_\psi}\|f\|_2^2.$$

□

4.4 A sharper Clifford wavelet uncertainty principle

In the present section we state and prove the second main result which concerns a sharper formulation of the Clifford-wavelet uncertainty principle.

Theorem 4.4.1. [13]Let $\psi\in L^2(\mathbb{R}^n,\mathbb{R}_n,dV(\underline{x}))$ be an admissible Clifford mother wavelet. Then for $f\in L^2(\mathbb{R}^n,\mathbb{R}_n,dV(\underline{x}))$ the following inequality holds

$$(\int_{Spin(n)}\int_{\mathbb{R}^+}\|b_kT_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds)^{\frac{1}{2}}\left\|\xi_k\widehat{f}\right\|_2\geq\sqrt{2^{n+1}\pi^nA_\psi}\left\{\|f\|_2^2+2|\langle f_1,f_2\rangle|\right\}$$

where

$$\begin{cases} f_1(\underline{x}) & = \frac{1}{A_\psi}\int_{Spin(n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}+}\psi^{a,\underline{b},s}(\underline{x})\partial_{b_k}T_\psi[f](a,\underline{b},s)\frac{da}{a^{n+1}}dV(\underline{b})ds \\ f_2(\underline{x}) & = \frac{1}{A_\psi}\int_{Spin(n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}+}\psi^{a,\underline{b},s}(\underline{x})b_kT_\psi[f](a,\underline{b},s)\frac{da}{a^{n+1}}dV(\underline{b})ds \end{cases}$$

Proof. From Theorem 4.2.3 we have

$$\|x_k f\|_2\left\|\xi_k\widehat{f}\right\|_2 \geq \sqrt{2}(\|f\|_2^2+|2\langle x_k\partial_{x_k}f,f\rangle|).$$

We substitute $f(\bullet)$ by $T_\psi[f](a,\bullet,s)$, then

$$\|b_kT_\psi[f](a,\bullet,s)\|_2\left\|\xi_k\widehat{T_\psi[f](a,\bullet,s)}\right\|_2 \geq \sqrt{2}(\|T_\psi[f](a,\bullet,s)\|_2^2$$
$$+|2\langle b_k\partial_{b_k}T_\psi[f](a,\bullet,s),T_\psi[f](a,\bullet,s)\rangle|).$$

Hence

$$\int_{Spin(n)}\int_{\mathbb{R}+}\|b_kT_\psi[f](a,\bullet,s)\|_2\left\|\xi_k\widehat{T_\psi[f](a,\bullet,s)}\right\|_2\frac{da}{a^{n+1}}ds.$$
$$\geq\sqrt{2}\int_{Spin(n)}\int_{\mathbb{R}+}(\|T_\psi[f](a,\bullet,s)\|_2^2+|2\langle b_k\partial_{b_k}T_\psi[f](a,\bullet,s),T_\psi[f](a,\bullet,s)\rangle|)\frac{da}{a^{n+1}}ds.$$

By the inequality of Cauchy-Schwartz

$$\int_{Spin(n)}\int_{\mathbb{R}+}\|b_kT_\psi[f](a,\bullet,s)\|_2\left\|\xi_k\widehat{T_\psi[f](a,\bullet,s)}\right\|_2\frac{da}{a^{n+1}}ds$$
$$\leq(\int_{Spin(n)}\int_{\mathbb{R}+}\|b_kT_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds)^{\frac{1}{2}}$$
$$\times(\int_{Spin(n)}\int_{\mathbb{R}+}\left\|\xi_k\widehat{T_\psi[f](a,\bullet,s)}\right\|_2^2\frac{da}{a^{n+1}}ds)^{\frac{1}{2}}.$$

Then

$$(\int_{Spin(n)}\int_{\mathbb{R}+}\|b_kT_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds)^{\frac{1}{2}}\times(\int_{Spin(n)}\int_{\mathbb{R}+}\left\|\xi_k\widehat{T_\psi[f](a,\bullet,s)}\right\|_2^2\frac{da}{a^{n+1}}ds)^{\frac{1}{2}}$$
$$\geq\sqrt{2}\int_{Spin(n)}\int_{\mathbb{R}+}(\|T_\psi[f](a,\bullet,s)\|_2^2$$
$$+|2\langle b_k\partial_{b_k}T_\psi[f](a,\bullet,s),T_\psi[f](a,\bullet,s)\rangle|)\frac{da}{a^{n+1}}ds$$
$$=\sqrt{2}\int_{Spin(n)}\int_{\mathbb{R}+}(\|T_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds$$
$$+2\sqrt{2}\int_{Spin(n)}\int_{\mathbb{R}+}|\langle b_k\partial_{b_k}T_\psi[f](a,\bullet,s),T_\psi[f](a,\bullet,s)\rangle|)\frac{da}{a^{n+1}}ds.$$

Knowing that

$$\begin{cases}\int_{Spin(n)}\int_{\mathbb{R}+}\left\|\xi_k\widehat{T_\psi[f]}(a,\bullet,s)\right\|_2^2\frac{da}{a^{n+1}}ds & = \frac{A_\psi}{(2\pi)^n}\left\|\xi_k\widehat{f}\right\|_2^2 \\ \int_{Spin(n)}\int_{\mathbb{R}+}\|T_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds & = A_\psi\|f\|_2^2.\end{cases}$$

Then we have

$$(\int_{Spin(n)}\int_{\mathbb{R}^+}\|b_kT_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds)^{\frac{1}{2}}\times(\frac{A_\psi}{(2\pi)^n}\left\|\xi_k\widehat{f}\right\|_2^2)^{\frac{1}{2}}\geq\sqrt{2}A_\psi\|f\|_2^2.$$

$$+2\sqrt{2}\int_{Spin(n)}\int_{\mathbb{R}^+}|\langle b_k\partial_{b_k}T_\psi[f](a,\bullet,s),T_\psi[f](a,\bullet,s)\rangle|)\frac{da}{a^{n+1}}ds.$$

So

$$(\int_{Spin(n)}\int_{\mathbb{R}^+}\|b_kT_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds)^{\frac{1}{2}}\left\|\xi_k\widehat{f}\right\|_2\geq\sqrt{2^{n+1}\pi^nA_\psi}\|f\|_2^2$$

$$+\sqrt{\frac{2^{n+3}\pi^n}{A_\psi}}\int_{Spin(n)}\int_{\mathbb{R}^+}|\langle b_k\partial_{b_k}T_\psi[f](a,\bullet,s),T_\psi[f](a,\bullet,s)\rangle|)\frac{da}{a^{n+1}}ds$$

$$=\sqrt{2^{n+1}\pi^nA_\psi}\|f\|_2^2$$

$$+\sqrt{\frac{2^{n+3}\pi^n}{A_\psi}}\int_{Spin(n)}\int_{\mathbb{R}^+}|\langle b_k\partial_{b_k}T_\psi[f](a,\bullet,s),T_\psi[f](a,\bullet,s)\rangle|)\frac{da}{a^{n+1}}ds.$$

We know that

$$\int_{Spin(n)}\int_{\mathbb{R}^+}|\langle b_k\partial_{b_k}T_\psi[f](a,\bullet,s),T_\psi[f](a,\bullet,s)\rangle|)\frac{da}{a^{n+1}}ds$$

$$=\int_{Spin(n)}\int_{\mathbb{R}^+}\left|\int_{\mathbb{R}^n}[\partial_{b_k}T_\psi[f](a,\underline{b},s)]^\dagger b_kT_\psi[f](a,\underline{b},s)dV(\underline{b})\right|\frac{da}{a^{n+1}}ds$$

$$\geq\left|\int_{Spin(n)}\int_{\mathbb{R}^+}\int_{\mathbb{R}^n}[\partial_{b_k}T_\psi[f](a,\underline{b},s)]^\dagger b_kT_\psi[f](a,\underline{b},s)dV(\underline{b})\frac{da}{a^{n+1}}ds\right|$$

$$=|A_\psi[\partial_{b_k}T_\psi[f],b_kT_\psi[f]]|.$$

Then we have

$$(\int_{Spin(n)}\int_{\mathbb{R}^+}\|b_kT_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds)^{\frac{1}{2}}\left\|\xi_k\widehat{f}\right\|_2\geq\sqrt{2^{n+1}\pi^nA_\psi}\|f\|_2^2$$

$$+\sqrt{2^{n+3}\pi^nA_\psi}\,|[\partial_{b_k}T_\psi[f],b_kT_\psi[f]]|.$$

Since for $f,g\in L^2(\mathbb{R}^n,dV(\underline{x})$ (see for instance [56])

$$[T_\psi[f],T_\psi[g]]=\langle f,g\rangle$$

we may write

$$(\int_{Spin(n)}\int_{\mathbb{R}^+}\|b_kT_\psi[f](a,\bullet,s)\|_2^2\frac{da}{a^{n+1}}ds)^{\frac{1}{2}}\left\|\xi_k\widehat{f}\right\|_2\geq\sqrt{2^{n+1}\pi^nA_\psi}\left\{\|f\|_2^2+2\left|\langle f_1,f_2\rangle\right|\right\}$$

where

$$\begin{cases} f_1(\underline{x}) &= \frac{1}{A_\psi}\int_{Spin(n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^+}\psi^{a,\underline{b},s}(\underline{x})\partial_{b_k}T_\psi[f](a,\underline{b},s)\frac{da}{a^{n+1}}dV(\underline{b})ds \\ f_2(\underline{x}) &= \frac{1}{A_\psi}\int_{Spin(n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^+}\psi^{a,\underline{b},s}(\underline{x})b_kT_\psi[f](a,\underline{b},s)\frac{da}{a^{n+1}}dV(\underline{b})ds \end{cases}$$

□

This result is evidently more powerful than the one given in Theorem (4.3.1) (see [14]). For that, we notice that there is a slit increase in the lower bound :

$$\begin{aligned}\sqrt{2^{n+1}\pi^nA_\psi}\left\{\|f\|_2^2+2\left|\langle f_1,f_2\rangle\right|\right\} &= \sqrt{2}(2\pi)^{\frac{n}{2}}\sqrt{A_\psi}\left\{\|f\|_2^2+2\left|\langle f_1,f_2\rangle\right|\right\} \\ &\geq \frac{(2\pi)^{\frac{n}{2}}\sqrt{A_\psi}}{2}\|f\|_2^2\end{aligned}$$

where f_1 and f_2 are as given above.

Those result may be seen as an improvement of those given in [129, 135, 66] in the special case of the continuous wavelet transform defined on the quaternions algebra $\mathbb{H}$ as for the Clifford wavelet transform defined on the geometric algebra $Cl_{n,0}$ for $n = 2 \equiv [4]$ established in [137] and for $n = 2, 3 \equiv [4]$ given in [97, 92] and of the similar results obtained by E. Hitzer and M. Bahri for the $Cl_{3,0}$ in [130, 132] .

4.5 Conclusion

In this chapter, we could formulate and prove a new result on Clifford wavelet uncertainty principle stating that we can't know simultaneously the values of the Clifford-Fourier transform and the Clifford wavelet transform of a square integrable multi-vector valued function. The results are based on the generalizations of the uncertainty principle to Clifford-Fourier transform.

Chapter 5

Conclusion and perspectives

5.1 Conclusion

In this thesis, we answered the main problematic which is that the Clifford wavelet transform and the Clifford-Fourier transform of multivector valued function can't be both sharp an uncertainty principle associated with the continuous wavelet transform in the Clifford algebra's settings has been formulated and proved. Starting from the definition of real Clifford algebra and the real continuous wavelet transform, we have presented a continuous Clifford wavelet transform, displayed its properties and formulated an associated uncertainty principle. This research aimed to state a new uncertainty principle for the Clifford wavelet transform. Based on proprieties of Clifford algebra-valued monogenic admissible mother wavelets and harmonic analysis of the Clifford-Fourier transform, we concluded on the impossibility for a Clifford wavelet transform of a function and its Clifford-Fourier transform to be simultaneously sharply concentrated. which expresses the limitations on the simultaneous concentration of $T_\psi\left[f\right]$, and $\widehat{f}$. This results have been published in

- Banouh, H., Ben Mabrouk, A. and Kesri, M. Clifford-Wavelet Transform and the Uncertainty Principle, *Advances in Applied Clifford Algebras*, 2019, Vol. 29, pp. 1-23. DOI:10.1007/s00006-019-1026-4.

- Banouh, H., Ben Mabrouk. A. A Sharp Clifford-Wavelet Heisenberg-type Uncertainty

Principle, *Journal of Mathematical Physics*, 2020, Vol. 62, Issue 9. DOI:10.1063/5.0015989.

5.2 Perspectives

5.2.1 Practical Applications

We attend to apply the results of this thesis to some concrete Clifford wavelets such as Clifford-Hermite, Clifford-Bessel, Clifford-Laguerre and Clifford-Gegenbauer wavelets [20, 35, 33, 8, 7, 6, 10, 21] and other.

5.2.2 Donoho-Stark uncertainty principle for the Clifford wavelet transform

Following [67], [2] and [98] we may try to extend the Dohono-Stark uncertainty principle for ε-concentrated Clifford wavelet transforms. We recall that for $\Omega \subset \mathbb{R}^n$ the function $f : \Omega \longrightarrow \mathbb{R}_n$ is ε-concentrated in the L^p norm on Ω if there exists $\epsilon_\Omega > 0$ such

$$\left(\int_{\mathbb{R}^n \setminus \Omega} |f(\underline{x})|^p dV(\underline{x}) \right)^{\frac{1}{p}} \leq \varepsilon_\Omega \parallel f \parallel_p .$$

5.2.3 Continuous shearlet transform in Clifford algebra

As for the continuous wavelet transform, the shearlet transform [118, 50, 117, 51] can be derived from a square-integrable group representation of a specific group namely the shear group $\mathbb{R}^* \times \mathbb{R}^{n-1} \times \mathbb{R}^n$ with the operators defined the following way : we set for $a \in \mathbb{R}^*$ and $\underline{s} = (s_1, s_2, \cdots, s_{n-1}) \in \mathbb{R}^{n-1}$ the dilation matrix

$$A_a = \begin{pmatrix} a & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a \end{pmatrix}$$

and the shear matrix

$$S_s = \begin{pmatrix} 1 & s_1 & s_2 & \dots & s_{n-1} \\ 0 & 1 & s_1 & s_2 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & s_1 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

We may try to extend this transform into the Clifford algebra $\mathbb{R}_n$ and deduce a new uncertainty principle (see for example [167, 49, 178, 36, 147, 136]) and formulate new uncertainty principles .

5.2.4 New uncertainty principles for the Clifford wavelet transform

We may find a generalised result for the L^p-variance instead of the square integrable one and a logarithmic uncertainty principle based on [136] and [135].

5.2.5 Controllability of Clifford algebra valued Systems

One future perspective is to introduce the theory of controllability and observability in the settings of non-commutative Clifford algebra $\mathbb{R}_n$ and apply it to some practical problems as three dimensional movements of planes and space shuttles [185, 73, 184, 186, 104]

Bibliography

[1] A Geometric-algebraic View of Linear Systems. In John L. Casti, editor, *Mathematics in Science and Engineering*, chapter 9, pages 246–316. Elsevier, 1977.

[2] A. Abouelaz, A. Achak, R. Daher, and N. Safouane. Donoho-Stark's Uncertainty Principle for the Quaternion Fourier Transform. *Boletín de la Sociedad Matemática Mexicana*, jul 2019.

[3] B. Amri and L. T. Rachdi. Beckner logarithmic uncertainty principle for the Riemann-Liouville operator. *International Journal of Mathematics*, 24(09), 2013.

[4] B. Amri and L. T. Rachdi. Uncertainty Principle in Terms of Entropy for the Riemann–Liouville Operator. *Bulletin of the Malaysian Mathematical Sciences Society*, 39(1):457–481, 2015.

[5] J.-P. Antoine, R. Murenzi, P. Vandergheynst, and S. T. Ali. *Two-Dimensional Wavelets and their Relatives.* Cambridge University Press, 2004.

[6] S. Arfaoui. *Some Monogenic Gegenbauer-Jacobi Polynomials and Associated Spheroidal Wavelets.* PhD thesis, Faculté Des Sciences De Monastir, Universite De Monastir, 2017.

[7] S. Arfaoui and A. Ben Mabrouk. Some Old Orthogonal Polynomials Revisited and Associated Wavelets : Two-Parameters Clifford-Jacobi Polynomials and Associated Spheroidal Wavelets. *Acta Applicandae Mathematicae*, 155(1):177–195, December 2017.

[8] S. Arfaoui and A. Ben Mabrouk. Some Ultraspheroidal Monogenic Clifford-Gegenbauer-Jacobi Polynomials and Associated Wavelets. *Advances in Applied Clifford Algebras*, 27:2287–2306, May 2017.

[9] S. Arfaoui, A. Ben Mabrouk, and C. Cattani. New Type of Gegenbauer-Hermite Monogenic Polynomials and Associated Clifford Wavelets. *Journal of Mathematical Imaging and Vision*, October 2019.

[10] S. Arfaoui, A. Ben Mabrouk, and C. Cattani. New Type of Gegenbauer-Jacobi-Hermite Monogenic Polynomials and Associated Continuous Clifford Wavelet Transform. *Acta Applicandae Mathematicae*, mar 2020.

[11] A. Axelsson, R. Grognard, J. Hogan, and A. McIntosh. Harmonic Analysis of Dirac Operators on Lipschitz Domains. In *Clifford Analysis and Its Applications*, pages 231–246. Springer Netherlands, 2001.

[12] H. Banouh. Uncertainty Principle Associated with the Clifford Continuous Wavelet Transform. University of Science and Technology of China, Hefei, China, August 2020.

[13] H. Banouh and A. Ben Mabrouk. A sharp Clifford wavelet Heisenberg-type uncertainty principle. *J. Math. Phys.*, 61(9):093502, sep 2020.

[14] H. Banouh, A. Ben Mabrouk, and M. Kesri. Clifford-Wavelet Transform and the Uncertainty Principle. *Advances in Applied Clifford Algebras*, 29(5):1–23, October 2019.

[15] S. Bernstein, editor. *Topics in Clifford Analysis*. Springer International Publishing, 2019.

[16] L. E. Blumenson. A Derivation of n-Dimensional Spherical Coordinates. *The American Mathematical Monthly*, 67(1):63, jan 1960.

[17] A. Boulkhemair, A. Chakib, and A Sadik. On a Shape Derivative Formula for a Family of Star-Shaped Domains. working paper or preprint, January 2020.

[18] R. N. Bracewell. *The Fourier Transform and Its Applications*. Electrical engineering series. McGraw Hill, 2000.

[19] F. Brackx, J. S. R. Chisholm, and V. Souček, editors. *Clifford Analysis and Its Applications*. Springer Netherlands, 2001.

[20] F. Brackx, N. De Schepper, and F. Sommen. The Clifford-Laguerre continuous wavelet transform. *Bulletin of The Belgian Mathematical Society-Simon Stevin*, 11(2):201–215, 2004.

[21] F. Brackx, N. De Schepper, and F. Sommen. The Clifford-Gegenbauer Polynomials and the Associated Continuous Wavelet Transform. *Integral Transforms and Special Functions*, 15(5):387–404, October 2004.

[22] F. Brackx, N. De Schepper, and F. Sommen. The Clifford-Fourier transform. *Journal of Fourier Analysis and Applications*, 11(6):669–681, 2005.

[23] F. Brackx, N. De Schepper, and F. Sommen. Clifford-Hermite and Two-dimensional Clifford-Gabor Filters for Early Vision. In K. Gürlebeck and C. Könke, editors, *Proceedings 17th International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering*, Bauhaus-Universität Weimar, 2006.

[24] F. Brackx, N. De Schepper, and F. Sommen. Clifford-Hermite Monogenic Operators. *Czechoslovak Mathematical Journal*, 56(4):1301–1322, dec 2006.

[25] F. Brackx, N. De Schepper, and F. Sommen. The Two-dimensional Clifford-Fourier Transform. *J. Math. Imaging*, 26:5–18, 2006.

[26] F. Brackx, N. De Schepper, and F. Sommen. The Fourier Transform in Clifford analysis. In *Advances in Imaging and Electron Physics*, volume 156, pages 55–201. Elsevier, 2009.

[27] F. Brackx, N. De Schepper, and F. Sommen. The Clifford-Fourier Integral Kernel in Even Dimensional Euclidean Space. *Journal of Mathematical Analysis and Applications*, 365(2):718–728, may 2010.

[28] F. Brackx, R. Delanghe, and F. Sommen. *Clifford Analysis*, volume 76 of *Research Notes in Mathematics*. Pitman Advanced Publishing Program, 1982.

[29] F. Brackx, R. Delanghe, and F. Sommen. Cauchy-Kowalewski theorems in Clifford analysis: a survey. *Proceedings of the 11th Winter School on Abstract Analysis*, pages 55–70, 1984.

[30] F. Brackx, E. Hitzer, and S. J. Sangwine. History of Quaternion and Clifford-Fourier Transforms and Wavelets. In *Quaternion and Clifford Fourier Transforms and Wavelets*, pages 12–28. Springer Basel, 2013.

[31] F. Brackx, H. De Schepper, R. Lávička, and V. Souček. The Cauchy-Kovalevskaya Extension Theorem in Hermitian Clifford Analysis. *Journal of Mathematical Analysis and Applications*, 381(2):649–660, September 2011.

[32] F. Brackx, T. E. Simos, G. Psihoyios, and C. Tsitouras. Two Powerful Theorems in Clifford Analysis. In *AIP Conference Proceedings*, volume 1281, pages 3–7. American Institute of Physics, AIP, 2010.

[33] F. Brackx and F. Sommen. Clifford-Hermite wavelets in Euclidean space. *J. Fourier Anal. Appl.*, 6(3):299–310, 2000.

[34] F. Brackx and F. Sommen. The Generalized Clifford-Hermite Continuous Wavelet Transform. *Advances in Applied Clifford Algebras*, 11(1):219–231, 2001.

[35] F. Brackx and F. Sommen. Clifford–Bessel wavelets in Euclidean space. *Mathematical methods in the applied sciences*, 25(16-18):1479–1491, 2002.

[36] K. Brahim, B. Nefzi, and E. Tefjeni. Uncertainty Principles For The Continuous Quaternion Shearlet Transform. *Advances in Applied Clifford Algebras*, 29(3), April 2019.

[37] R. Bujack, H. De Bie, N. De Schepper, and G. Scheuermann. Convolution Products for Hypercomplex Fourier Transforms. *J Math Imaging Vis*, 48(3):606–624, mar 2013.

[38] R. Bujack, G. Scheuermann, and E. Hitzer. A General Geometric Fourier Transform. In E. Hitzer and S. J. Sangwine, editors, *Quaternion and Clifford Fourier transforms and wavelets, Trends in Mathematics 27*, pages 155–176. Birkhauser, Basel, 2013.

[39] R. Bujack, G. Scheuermann, and E. Hitzer. A General Geometric Fourier Transform Convolution Theorem. *Adv. Appl. Clifford Alg.*, 23(1):15–38, 2013.

[40] P. Carré and M. Berthier. *Advanced Color Image Processing and Analysis*, chapter Chapter 6, Color Representation and Processes with Clifford Algebra, pages 147–179. Springer, New York, 2013.

[41] A.-L. Cauchy. *Oeuvres Complètes. Série 1. Tome 7.* Gauthier-Villars (Paris, 1882-1974), 2018.

[42] C. K. Chui. *An Introduction to Wavelets.* Academic Press Professional, Inc., San Diego, CA, USA, 1992.

[43] W. K. Clifford. Applications of Grassmann's Extensive Algebra. *American Journal of Mathematics*, 1(4):350, 1878.

[44] W. K. Clifford. On the Classification of Geometric Algebras. *Mathematical Papers*, pages 397–401, 1882.

[45] J. Cnops. The Wavelet Transform in Clifford Analysis. *Computational Methods and Function Theory*, 1(2):353–374, November 2001.

[46] J. Cnops. *An Introduction to Dirac Operators on Manifolds.* Birkhäuser Boston, 2002.

[47] J. M. Combes, A. Grossmann, Ph. Tchamitchian, and Allan D. Pierce. *Wavelets Time-Frequency Methods and Phase Space.* Springer Berlin Heidelberg, 1989.

[48] A. K. Common and F. Sommen. Axial monogenic functions from holomorphic functions.

Journal of Mathematical Analysis and Applications, 179(2):610–629, 1993.

[49] S. Dahlke, G. Kutyniok, P. Maass, C. Sagiv, H.-G. Stark, and G. Teschke. The Uncertainty Principle Associated with the Continuous Shearlet Transform. *International Journal of Wavelets, Multiresolution and Information Processing*, 6(02):157–181, 2008.

[50] S. Dahlke, G. Steidl, and G. Teschke. The continuous shearlet transform in arbitrary space dimensions. *Journal of Fourier Analysis and Applications*, 16(3):340–364, October 2009.

[51] S. Dahlke and G. Teschke. *The continuous shearlet transform in higher dimensions variations of a theme.* Citeseer, March 2009.

[52] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Communications on Pure and Applied Mathematics*, 41(7):909–996, oct 1988.

[53] I. Daubechies and Bruce J. Bates. *Ten Lectures on Wavelets*, volume 61. SIAM, March 1993.

[54] H. De Bie. Clifford algebras, Fourier transforms, and quantum mechanics. *Mathematical Methods in the Applied Sciences*, 35(18):2198–2228, September 2012.

[55] H. De Bie and Y. Xu. On the Clifford-Fourier Transform. *International Mathematics Research Notices*, 2011(22):5123–5163, jan 2011.

[56] N. De Schepper. *Multi-dimensional Continuous Wavelet Transforms and Generalized Fourier Transforms in Clifford Analysis.* PhD Thesis, Ghent University, 2006.

[57] L. Debnath and F. A. Shah. *Wavelet Transforms and Their Applications.* Birkhauser Boston, 2015.

[58] L. Debnath and F. A. Shah. *Lecture Notes on Wavelet Transforms.* Springer International Publishing, 2017.

[59] R. Delanghe. On Regular-Analytic Functions with Values in a Clifford Algebra. *Math. Ann.*, 185:91–111, 1970.

[60] R. Delanghe. Clifford Analysis: History and Perspective. *Computational Methods and Function Theory*, 1(1):107–153, September 2001.

[61] R. Delanghe, F. Sommen, and V. Souček. *Clifford Algebra and Spinor-valued Functions.* Springer Netherlands, 1992.

[62] P. A. M. Dirac. The quantum theory of the electron part ii. *Proceedings of the Royal Society of London*, A118:351–361, 1928.

[63] P. A. M. Dirac. The Quantum Theory of the Electron Part I. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, A117(778):610–624, 1928.

[64] J. Dubinskii and M. Reissig. Variational problems in Clifford analysis. *Mathematical Methods in the Applied Sciences*, 25(14):1161–1176, 2002.

[65] D. Eelbode and E. Hitzer. Operator exponentials for the Clifford Fourier transform on multivector fields in detail. *Advances in Applied Clifford Algebras*, 26(3):953–968, October 2016.

[66] Y. El Haoui and S. Fahlaoui. The continuous quaternion algebra-valued wavelet transform and the associated uncertainty principle.

[67] Y. El Haoui and S. Fahlaoui. Donoho-stark's uncertainty principles in real clifford algebras. *Advances in Applied Clifford Algebras*, 29(5):94, September 2019.

[68] Y. El Haoui, S. Fahlaoui, and E. Hitzer. Generalized uncertainty principles associated with the quaternionic offset linear canonical transform. *ArXiv:1807.04068v2 [math.CA]*, 2019.

[69] M. Fei, Y. Pan, and Y. Xu. Some Shaper Uncertainty Principles for Multivector-Valued Functions. *International Journal of Wavelets, Multiresolution and Information Processing*, 14(06):1650043–1650043, 2016.

[70] H. G. Feichtinger and K. Gröchenig. Gabor Wavelets and the Heisenberg Group: Gabor Expansions and Short Time Fourier Transform from the Group Theoretical Point of View. *Wavelets*, pages 359–397, 1992.

[71] Y. Fu and L. Li. Uncertainty Principle for Multivector-Valued Functions. *International Journal of Wavelets, Multiresolution and Information Processing*, 13(01):1550005, January 2015.

[72] R. Fueter. Zur Theorie Der Regularen Funktionen Einer Quaternionenvariablen. *Monat. fur Math. und Phys.*, 43:69–74, 1935.

[73] K. Fujimoto, T. Takeuchi, and Y. Matsumoto. On Port-Hamiltonian Modeling and Control of Quaternion Systems. *IFAC-PapersOnLine*, 48(13):39–44, 2015.

[74] D. Gabor. Theory of communication. Part 1: The analysis of information. *Journal of the Institution of Electrical Engineers-Part III: Radio and Communication Engineering*, 93(26):429–441, 1946.

[75] T. Gao, Z.-G. Liu, S.-H. Yue, J.-Q. Mei, and J. Zhang. Traffic video-based moving vehicle detection and tracking in the complex environment. *Cybernetics and Systems: An International Journal*, 40(7):569–588, 2009.

[76] J. E. Gilbert and M. A. M. Murray. *Clifford algebras and Dirac operators in harmonic analysis*, volume 26 of *Cambridge studies in advanced mathematics*. Press Syndicate of the University of Cambridge, cambridge university press edition, 1991.

[77] P. Goupillaud, A. Grossmann, and J. Morlet. Cycle-octave and related transforms in seismic signal analysis. *Geoexploration*, 23(1):85–102, oct 1984.

[78] H. Grassmann. *Die lineale Ausdehnungslehre ein neuer Zweig der Mathematik*. His Die Wissenschaft der extensiven Grösse, oder die Ausdehnungslehre. O. Wigand, 1844.

[79] A. W. Green. *The Uncertainty Principle in Control Theory for Partial Differential Equations*. PhD thesis, Graduate School of Clemson Univesity, 2020.

[80] A. Grossmann and J. Morlet. Decomposition of Hardy Functions into Square Integrable Wavelets of Constant Shape. *SIAM Journal on Mathematical Analysis*, 15(4):723–736, July 1984.

[81] A. Grossmann, J. Morlet, and T. Paul. Transforms Associated to Square Integrable Group Representations. I. General Results. *Journal of Mathematical Physics*, 26(10):2473–2479, October 1985.

[82] A. Grossmann, J. Morlet, and T. Paul. Transforms Associated to Square Integrable Group Representations. II. Examples. *Ann. Inst. Henri Poincaré*, 45(3):293–309, 1986.

[83] A. Haar. Zur theorie der orthogonalen funktionen-systeme. *Mathematische Annalen*, 69:331–371, 1910.

[84] A. Haar. Der Massbegriff in der Theorie der kontinuierlichen Gruppen. *Annals of Mathematics*, 34(1):147–169, 1933.

[85] W. R. Hamilton. On a New Species of Imaginary Quantities Connected with a Theory of Quaternions. In *Proceedings of the Royal Irish Academy*, volume 2, pages 424–234, 1844.

[86] W. R. Hamilton. *Elements of quaternions*. Longmans, Green, & Company, 1866.

[87] C. Heil and D. Walnut. Continuous and discrete wavelet transforms. *SIAM Review*, 31(4):628–666, December 1989.

[88] W. Heisenberg. Uber Den Anschaulichen Inhalt Der Quantentheoretischen Kinematik Und

Mechanik. *Zeitschrift fur Physik*, 43:172–198, 1927.

[89] W. Heisenberg. *Die Physik der Atomkerne.* Vieweg+Teubner Verlag, Wiesbaden, 1949.

[90] D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus.* Springer Netherlands, 1984.

[91] E. Hitzer. Tutorial on fourier transformations and wavelet transformations in clifford geometric algebra. In *Lecture notes of the International Workshop for 'Computational Science with Geometric Algebra'(FCSGA2007)*, pages 65–87, 2007.

[92] E. Hitzer. Clifford (Geometric) Algebra Wavelet Transform. volume 2, pages 94–101. IEEE, 2009.

[93] E. Hitzer. Directional Uncertainty Principle for Quaternion Fourier Transform. *Advances in Applied Clifford Algebras*, 20(2):271–284, July 2009.

[94] E. Hitzer. New Developments in Clifford Fourier Transforms. *Advances in Applied and Pure Mathematics*, pages 19–25, 2014.

[95] E. Hitzer and B. Mawardi. Clifford Fourier Transform on Multivector Fields and Uncertainty Principles for Dimensions $n = 2(mod4)$ and $n = 3(mod4)$. *Advances in Applied Clifford Algebras*, 18(3-4):715–736, May 2008.

[96] E. Hitzer and B. Mawardi. Uncertainty Principle for the Clifford-Geometric Algebra $Cl_{3,0}$ based on Clifford Fourier Transform. *ArXiv:1306.2089v1 [math.RA]*, June 2013.

[97] E. Hitzer, T. E. Simos, G. Psihoyios, and C. Tsitouras. Real Clifford Algebra $Cl_{n,0}, n = 2, 3(mod4)$ Wavelet Transform. In *AIP Conference Proceedings.* AIP, 2009.

[98] K. Hleili. Continuous wavelet transform and uncertainty principle related to the Weinstein operator. *Integral Transforms and Special Functions*, 29(4):252–268, jan 2018.

[99] K. Hleili, S. Omri, and L. T. Rachdi. Uncertainty principle for the Riemann-Liouville operator. *CUBO A Mathematical Journal*, 13(03):91–115, October 2011.

[100] S. Huang, Y. Y. Qiao, and G. Wen. *Real and Complex Clifford Analysis.* Kluwer Academic Publishers, 2006.

[101] Z. J. Huang, G. H. Huang, and L. l. Cheng. Medical Image Segmentation of Blood Vessels Based on Clifford Algebra and Voronoi Diagram. *Journal of Software*, 13(6):361–373, 2018.

[102] R. Jday. Heisenberg and Hardy Uncertainty Principles in Real Clifford Algebras. *Integral Transforms and Special Functions*, 29(8):663–677, June 2018.

[103] B. Jefferies and A. Mcintosh. The Weyl Calculus and Clifford Analysis. *Bulletin of the Australian Mathematical Society*, 57(2):329–341, April 1991.

[104] B. Jiang, Y. Liu, K.-I. Kou, and Z. Wang. Controllability and Observability of Linear Quaternion-valued Systems. *Acta Mathematica Sinica*, 06 2020.

[105] P. Jorgensen and F. Tian. *Non-commutative analysis.* World Scientific, 2017.

[106] G. Kaiser. *A Friendly Guide to Wavelets.* Birkhauser Boston Inc., Cambridge, MA, USA, 1994.

[107] M. Kesri. Solutions globales du système intégro-différentiel non linéaire de Liouville–Boltzmann–Poisson et l'équation spectrale. *Colloquium Mathematicum*, 106(2):265–281, 2006.

[108] M. Kesri. An Explicit Method for Optimal Control Problems. *Applied Mathematics and Computation*, 219(15):8213–8221, apr 2013.

[109] M. Kesri. On the Class of Infinite Horizon Optimal Control Problems Affine, Which Are Structurally Stable. *Mathematical Methods in the Applied Sciences*, 39(11):2799–2807, oct 2015.

[110] M. Kesri. Continuous Dependence of the Value Function on a Class of One-Dimensional Autonomous Optimal Control Problems and Existence of Solutions. *Indian Journal of Pure and Applied Mathematics*, 50(1):269–281, mar 2019.

[111] A. Kheirabadi, A. M. Vaziri, and S. Effati. Solving Optimal Control Problem Using Hermite Wavelet. *Numerical Algebra, Control & Optimization*, 9(1):101–112, 2019.

[112] J. King. Uncertainty Principle in Control Theory, Part I: Analysis of Performance Limitations. *ArXiv pre-print : 1402.5468v1.*

[113] V. Kisil. Wavelets in Banach Spaces. *Acta Applicandae Mathematicae*, 59(1):79–109, 1999.

[114] K. I. Kou, J.-Y. Ou, and J. Morais. On uncertainty principle for quaternionic linear canonical transform. *Abstract and Applied Analysis*, 2013:1–15, 2013.

[115] K.-I. Kou and T. Qian. The Paley-Wiener Theorem in $\mathbb{R}^n$ with the Clifford Analysis Setting. *Journal of Functional Analysis*, 189(1):227–241, feb 2002.

[116] S. V. Kovalevskaja. Zur Theorie Der Partiellen Differentialgleichungen. *J. Fur Die Reine Und Angew. Mathem.*, 80:1–32, 1875.

[117] D. Kumar. Multi dimensional continuous shearlet transform and multi scale wavelet shrink-

age with integro differential equations. *American Review of Mathematics and Statistics*, 4(2):39–47, December 2016.

[118] G. Kutyniok and D. Labate, editors. *Shearlets*. Birkhäuser Boston, 2012.

[119] G. Laville. Some Topics in Clifford Analysis. *Advances in Applied Clifford Algebras*, 19(3-4):721–775, nov 2009.

[120] Z. Li, H. Yang, and Y. Qiao. A New Cauchy Integral Formula in the Complex Clifford Analysis. *Advances in Applied Clifford Algebras*, 28(4), jul 2018.

[121] P. Lounesto. *Clifford Algebras and Spinors*. Cambridge University Press, 2001.

[122] W. Luo and J. Du. Generalized Cauchy Theorem in Clifford Analysis and Boundary Value Problems for Regular Functions. *Advances in Applied Clifford Algebras*, 27(3):2531–2583, may 2017.

[123] G. Ma and J. Zhao. Quaternion Ridgelet Transform and Curvelet Transform. *Adv. Appl. Clifford Algebras*, 28(80), 2018.

[124] S. Mallat. A Theory for Multiresolution Signal Decomposition: The Wavelet Representation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 11(7):674–693, jul 1989.

[125] S. Mallat. *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way*. Academic Press, Inc., Orlando, Fl, Usa, 2008.

[126] M. Martin. Function Spaces in Quaternionic and Clifford Analysis. In *Operator Theory*, pages 1–26. Springer Basel, 2014.

[127] B. Mawardi. Construction of quaternion-valued wavelets. *MATEMATIKA*, 26(1):107–114, 2010.

[128] B. Mawardi and R. Ashino. Logarithmic Uncertainty Principle for Quaternion Linear Canonical Transform. In *Proceedings of the 2016 International Conference on Wavelet Analysis and Pattern Recognition*. IEEE, July 2016.

[129] B. Mawardi, R. Ashino, and R. Vaillancourt. Two-dimensional quaternion wavelet transform. *Applied Mathematics and Computation*, 218(1):10–21, 2011.

[130] B. Mawardi and E. Hitzer. Clifford Algebra $Cl_(3,0)$ valued Wavelets and Uncertainty Inequality for Clifford Gabor Wavelet Transformation. *International Journal of Wavelets, Multiresolution and Information Processing*, 05(06):997–1019, November 2006.

[131] B. Mawardi and E. Hitzer. Clifford Fourier Transformation and Uncertainty Principle for the Clifford Geometric Algebra $Cl_{3,0}$. *Advances in Applied Clifford Algebras*, 16(1):41–61, February 2006.

[132] B. Mawardi and E. Hitzer. Clifford Algebra Cl(3,0)-valued Wavelet Transformation, Clifford Wavelet Uncertainty Inequality and Clifford Gabor Wavelets. *International Journal of Wavelets, Multiresolution and Information Processing*, 5(6):997–1019, 2007.

[133] B. Mawardi, E. Hitzer, A. Hayashi, and R. Ashino. An Uncertainty Principle for Quaternion Fourier Transform. *Computers & Mathematics with Applications*, 56(9):2398–2410, November 2008.

[134] B. Mawardi and A. Ryuichi. A simplified proof of uncertainty principle for quaternion linear canonical transform. *Abstract and Applied Analysis*, 2016, 2016.

[135] B. Mawardi and A. Ryuichi. A Variation on Uncertainty Principle and Logarithmic Uncertainty Principle for Continuous Quaternion Wavelet Transforms. *Abstract and Applied Analysis*, 2017:1–11, 2017.

[136] B. Mawardi, Firdous A. Shah, and Azhar Y. Tantary. Uncertainty principles for the continuous shearlet transforms in arbitrary space dimensions. *Integral Transforms and Special Functions*, pages 1–18, jan 2020.

[137] B. Mawardi, A. Sriwulan, and J. Zhao. Clifford Algebra-Valued Wavelet Transform on Multivector Fields. *Advances in Applied Clifford Algebras*, 21(1):13–30, July 2010.

[138] H. Mejjaoli, N. Ben Hamadi, and S. Omri. Localization operators, time frequency concentration and quantitative-type uncertainty for the continuous wavelet transform associated with spherical mean operator. *International Journal of Wavelets, Multiresolution and Information Processing*, 17(4), 2019.

[139] Y. Meyer. Ondelettes, fonctions splines et analyses graduées. *Lectures given at the University of Torino, Italy*, 9, 1986.

[140] M. Mitrea. *Clifford Wavelets Singular Integrals, and Hardy Spaces.* Springer-Verlag, 1994.

[141] J. Morlet, G. Arens, E. Fourgeau, and D. Giard. Wave propagation and sampling theory Part II: Sampling theory and complex waves. *Geophysics*, 47(2):222–236, 1982.

[142] A. J. Morris. *Fourier and Wavelet Analysis of Clifford-Valued Functions.* PhD Thesis, University of Newcastle, Australia, Callaghan, NSW, 2308, Australia, February 2014.

[143] R. A. Mosna, D. Miralles, and J. Vaz. $\mathbb{Z}_2$-gradings of Clifford Algebras and Multivector Structures. *Journal of Physics A: Mathematical and General*, 36(15):4395–4405, April 2003.

[144] N. Msehli and L. T. Rachdi. Beurling-Hörmander uncertainty principle for the spherical mean Operator. *J. Inequal. Pure Appl. Math.*, 10(2), 2009.

[145] N. Msehli and L. T. Rachdi. Heisenberg-Pauli-Weyl Uncertainty Principle for the Spherical Mean Operator. *Mediterranean Journal of Mathematics*, 7(2):169–194, 2010.

[146] K. Nagata and T. Nakamura. Violation of Heisenberg's Uncertainty Principle. *Open Access Library Journal*, 2015.

[147] B. Nefzi, K. Brahim, and A. Fitouhi. Uncertainty principles for the multivariate continuous shearlet transform. *Journal of Pseudo-Differential Operators and Applications*, April 2019.

[148] C. A. Nolder. Conjugate Harmonic Functions and Clifford Algebras. *Journal of Mathematical Analysis and Applications*, 302(1):137–142, feb 2005.

[149] H. M. Ozaktas, M. A. Kutay, and D. Mendlovic. Introduction to the Fractional Fourier Transform and Its Applications. In *Advances in Imaging and Electron Physics*, pages 239–291. Elsevier, 1999.

[150] T. Qian. Generalization of Fueter's result to $\mathbb{R}^{n+1}$. *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni,Serie 9*, 8(2):111–117, 1997.

[151] T. Qian. Derivation of monogenic functions and applications. In *International Conference on Harmonic Analysis and Related Topics*, pages 118–127. Centre for Mathematics and its Applications, Mathematical Sciences Institute, 2003.

[152] T. Qian. Paley-Wiener Theorems and Shannon Sampling in the Clifford Analysis Setting. In *Clifford Algebras*, pages 115–124. Birkhauser Boston, 2004.

[153] L. T. Rachdi, B. Amri, and A. Hammami. Uncertainty principles and time frequency analysis related to the Riemann–Liouville operator. *Annali Dell'universita' Di Ferrara*, 2018.

[154] L. T. Rachdi and H. Herch. Uncertainty Principles for Continuous Wavelet Transforms Related to the Riemann–Liouville Operator. *Ricerche Di Matematica*, 66(2):553–578, feb 2017.

[155] L. T. Rachdi and F. Meherzi. Continuous wavelet transform and uncertainty principle related to the spherical mean operator. *Mediterranean Journal of Mathematics*, 14(1), dec 2016.

[156] M. Razzaghi and S. Yousefi. Legendre Wavelets Method for Constrained Optimal Control Problems. *Mathematical Methods in the Applied Sciences*, 25(7):529–539, 2002.

[157] N. Ricker. Wavelet Contraction, Wavelet Expansion, and the Control of Seismic Resolution. *GEOPHYSICS*, 18(4):769–792, oct 1953.

[158] M. Riesz. *Clifford Numbers and Spinors.* Springer Netherlands, 1993.

[159] D. Rizo-Rodríguez, H. Mendez-Vazquez, and E. Garcia-Reyes. Illumination invariant face recognition using quaternion-based correlation filters. *J. Math. Imaging Vis.*, 45:164–175, 2013.

[160] J. Ryan. Cauchy-Green type formulae in Clifford analysis. *Transactions of the American Mathematical Society*, 347(4):1331–1341, 1995.

[161] J. Ryan. Basic Clifford Analysis. *CUBO Math. Educ.*, 2:226–256, 2000.

[162] J. Ryan. *Clifford Algebras in Analysis and Related Topics.* CRC Press, March 2018.

[163] K. Sau, R. K. Basaka, and A. Chanda. Image Compression based on Block Truncation Coding Using Clifford Algebra. *Procedia Technology*, 10:699–706, 2013.

[164] M. Sce. Osservazioni Sulle Serie Di Potenze Nei Moduli Quadratici. *Atti Accad. Naz.Lincei. Rend. Cl. Sci. Fis. Mat. Nat.*, 8(23):220–225, 1957.

[165] K. K. Selig. Uncertainty Principles Revisited. *Electron. Trans. Numer. Anal.*, 14:165–177, 2002.

[166] D. Sen. The Uncertainty Relations in Quantum Mechanics. *Current Science*, 107(2):203–2018, July 2014.

[167] J. Sharma and S. K. Singh. Clifford Valued Shearlet Transform. *Advances in Applied Clifford Algebras*, 30(3), jun 2020.

[168] I. N. Sneddon. *Fourier Transforms.* Dover Publications Inc., 2003.

[169] F. Sommen. Special Functions in Clifford Analysis and Axial Symmetry. *Journal of Mathematical Analysis and Applications*, 130(1):110–133, February 1988.

[170] F. Sommen. Clifford analysis and integral geometry. In *Clifford Algebras and their Applications in Mathematical Physics*, pages 293–311. Springer Netherlands, 1992.

[171] F. Sommen. Clifford Analysis in Two and Several Vector Variables. *Applicable Analysis*, 73(1-2):225–253, oct 1999.

[172] F. Sommen and H. De Schepper. Introductory Clifford Analysis. *Operator Theory*, pages 1–27, 2015.

[173] F. Sommen and W. Sprößig. Introduction to clifford analysis. *Mathematical Methods in the Applied Sciences*, 25(16-18):1337–1342, nov 2002.

[174] P. A. Stabnikov. Geometric interpretation of the uncertainty principle. *Natural Science*, 11(5):146–148, 2019.

[175] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Monographs in harmonic analysis. Princeton University Press, 1970.

[176] E. M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces (PMS-32)*. Princeton University Press, dec 1972.

[177] D. C. Struppa. Quaternionic and Clifford Analysis in Several Variables. In *Operator Theory*, pages 1549–1578. Springer Basel, 2015.

[178] Y. Su. Heisenberg type uncertainty principle for continuous shearlet transform. *Journal of Nonlinear Science and Applications*, 9(3):778–786, 2016.

[179] W. Swaidan and A. Hussin. Feedback Control Method Using Haar Wavelet Operational Matrices for Solving Optimal Control Problems. *Abstract and Applied Analysis*, 2013:1–8, 2013.

[180] A. Teolis. *Computational Signal Processing with Wavelets*. Birkhauser Boston, 1998.

[181] L. Wang, R. Hu, J. Zhang, and Y. Ma. On the vortex detection method using continuous wavelet transform with application to propeller wake analysis. *Mathematical Problems in Engineering*, 2015.

[182] H. Weyl. *The Theory of Groups and Quantum Mechanics*. New York, second edition, 1950.

[183] P. Wojtaszczyk. *A mathematical introduction to wavelets*, volume 37. Cambridge University Press, 1997.

[184] D. Xu, J. He, A. Sheng, Z. Chen, and D. Wang. Robust Attitude Tracking Control of a Rigid Spacecraft Based on Nonlinearly Controlled Quaternions. In *2017 11th Asian Control Conference (ASCC)*. IEEE, dec 2017.

[185] Y. Yang. Spacecraft Attitude Determination and Control: Quaternion Based Method.

Annual Reviews in Control, 36(2):198–219, dec 2012.

[186] Y. Yang. *Spacecraft Modeling, Attitude Determination, and Control Quaternion-based Approach.* CRC Press, feb 2019.

[187] Y. Yang, P. Dang, and T. Qian. Stronger Uncertainty Principles for Hypercomplex Signals. *Complex Variables and Elliptic Equations*, 60(12):1696–1711, May 2015.

[188] Y. Yang and K. I. Kou. Uncertainty principles for hypercomplex signals in the linear canonical transform domains. *Signal Processing*, 95:67–75, February 2014.