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Sujet

Solutions positives pour certaines classes de problèmes aux limites associés à des équations différentielles du troisième ordre.

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INTRODUCTION

Our study is in the framework of the nonlinear differential equations with boundary conditions. The mathematical models try to reproduce in quantitative mode the experimental observations and make qualitatively possible to describe certain phenomena such as for example the transmission of information at the neural level. A neuron is an electrically active cell and its activity is manifested by the emission of variation in its membrane potential, called the action potential. The action potential results from intermembrane currents mainly made up of sodium ions and potassium ions. In 1952 A.L Hodgkin and A.F. Huxley developed a mathematical model which describes the initiation and the propagation of action potential. The Hodgkin and Huxley model is represented by a system of four equations with four unknowns which are the action potential, the activation function of potassium current, the activation function of sodium current and a variable which measures the inactivation of the sodium current. For more information in this subject, see [33, 57]. This model consists in reproducing the different behaviors of the action potential observed experimentally. Hodgkin and Huxley have obtained a solution by the numerical Euler's method. Many research led to the reduction of the number of variables and the one of the most famous reduced models is the FitzHugh-Nagumo (1961) model, with only two dimension. The scientist's goal is to establish a rigorous mathematical link between mathematical solving tools and the activity in a neuron through a model which reproduces the activity of a neuron to predict anomalies. As an example, Danziger and Elemergreen (see [31] p.133) have obtained the third-order linear differential equations:

$$\alpha_3\theta''' + \alpha_2\theta'' + \alpha_1\theta' + (1+k)\theta = kc, \quad \theta < c \quad \text{and}$$

$$\alpha_3\theta''' + \alpha_2\theta'' + \alpha_1\theta' + \theta = kc, \quad \theta > c$$

These equations describe the variation of thyroid hormone with time. Here $\theta = \theta(t)$ is the concentration of thyroid hormone at time t and $\alpha_3, \alpha_2, \alpha_1, k$ and c are constants. One of the reduced models of Hodgkin and Huxley model is that of Nagumo, he suggested a class of the third-order differential equations

$$u''' - cu'' + f'(u)u' - \frac{b}{c}u = 0.$$

And in the field of physical phenomena, we will quote the Kuramoto-Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + \frac{1}{2}u^2 = 0$$

which is introduced to describe pattern formulation in reaction-diffusion systems and to model the instability of flame front propagation (see Y.Kuramoto and T.Yamada [46] and D.Michelson [52]). A traveling wave solutions $u = \phi(x - ct)$ satisfies, after one integration, the third-order equation

$$\lambda\phi'''(x) + \phi'(x) + f(\phi) = 0,$$

where λ is a parameter and f is an even function. A three-layered beam is formed from parallel layers of different materials. For a loaded beam of this type, Krajcinovic in [44] has proved that the vector u is governed by the third-differential equation

$$-u''' + k^2u' = a$$

where k and a are the physical parameters which depend on the elasticity of the layers. Study of existence of positive solutions for third-order bvps has received a great deal of attention and was the subject of many articles, see, for instance, [29, 30, 32, 38, 37, 50, 58, 61, 65, 66, 67, 72], for third-order bvps posed on finite intervals and [1, 7, 16, 24, 25, 26, 27, 41, 43, 48, 49, 55, 60] for such bvp's posed on the half-line. Our goal in this work is to explain how the Hypothesis we have imposed on the nonlinearity term could have led to solve the third-order value problem in each of cases we studied with the same boundary conditions. For this, we divide our work in four chapters.

The first Chapter is devoted for the needed background, where we recall some basic facts of fixed point theory in cones, from the reminder of cones and properties, the positivity and compactness of operators, the spectral theory which we exhibit the importance of the first eigenvalue to the fixed point theorems to the bounded value problems under different conditions on the nonlinear operator.

In Chapter 2, we consider the problem

$$\begin{cases} -u'''(t) + k^2 u'(t) = f(t, u(t)),\ t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases} \tag{1}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. We use the technique of Strongly Index-Jump Property(SIJP for short) developed in [10], to prove under eigenvalue criteria existence and non existence of positive solution to the bvp (1), (see Theorems 1.2, 1.3 and 1.4 in [13]). More precisely, we first begin by proving in Proposition 1.1 in [13] existence of the positive eigenvalue μ of the linear problem associated to the bvp (1) under a suitable hypothesis. To prove the solvability results, namely Theorems 1.3 and 1.4 to the fixed point equation, we assume that the nonlinearity f is controlled by the limits

$$g^+_{i,\nu}(q) = \limsup_{u \to v} \left(\max_{t \geq 0} \frac{g(t, p_i(t)u)}{p_i(t)q(t)u} \right), \quad g^-_{i,\nu}(q) = \liminf_{u \to v} \left(\min_{t \geq 0} \frac{g(t, p_i(t)u)}{p_i(t)q(t)u} \right),$$

with respect to the positive eigenvalue μ, this leads to assumptions of Theorems 1.19 and 1.20 cited in Abstract background. In Theorem 1.2 in [13] we prove that bvp (1) has no solution. Particularly, we prove that depending on whether f takes a particular form given in Corollaries 1.5, 1.6 and 1.7, the nonexistence and existence of a positive solution. We use the main tool which is the SIJP of the positive compact operator to prove the existence of a positive solution to the bvp (1) in the Theorems cited above. Also, the additional interest in this work is to demonstrate under which condition, the problem (1) has the positive and bounded solution. We give an example of a nonlinearity f which satisfies the assumptions of the Theorems 1.3 and 1.4 and we discuss the different cases of obtaining a bounded solution and an unbounded solution.

In Chapter 3, we consider the case where the nonlinearity is positive and additionally depends on the derivative of the solution u. Namely, we consider the problem

$$\begin{cases} -u'''(t) + k^2 u'(t) = \phi(t) f(t, u(t), u'(t)),\ t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases} \tag{2}$$

where $\phi \in L^1(0, +\infty)$ and doesn't vanish identically on $(0, \infty)$ and the function f : $\mathbb{R}^+ \times (0, +\infty) \times (0, +\infty) \to \mathbb{R}^+$ may be singular at $u = 0$ and $u' = 0$. Naturally, in such boundary value problems, the nonlinearity may have a singular dependence on time or on the space variable. This was the case in the papers [8, 24, 25, 26, 49, 50, 61, 65, 66], which motivated this work. We use the Theorem "Fixed point theorem of cone expansion and compression" under conditions (6) in [11] about the nonlinearity f to prove existence of a positive solution to the bvp (2) (see Theorem 1 in [11]). We give a detail in Remark 1 in [11] to explain why we impose the two conditions separately on the limit and the integral on the function ϕ. Also the polynomial growth condition on f given in Remark 2, is the particular case where condition (6) is satisfied. About Remark 3 in [11], we prove that the integral of ϕ is finite. This result plays a role in the proof of the main Theorem in [11]. To illustrate our results, we give an example where the functions f and ϕ verify conditions of Theorem 1 and we tack under the calculations at the existence of a positive solution to the bvp (2). We complete this chapter by some comments at first by proving that our positive solution of the problem (2) is bounded and in other hand we discuss the possibility to find constants in order to optimize the interval of location of the limits respectively

$$f^0 = \lim_{|(w,z)|\to 0} \frac{f(t, e^{kt}w, e^{kt}z)}{w+z}, \quad f^\infty = \lim_{|(w,z)|\to \infty} \frac{f(t, e^{kt}w, e^{kt}z)}{w+z},$$

with as a reminder that we have found the constants which represent the bounds of these limits, are important to achieve existence of a positive solution for the problem (2).

In Chapter 4, we investigate the existence of a positive solution to the singular problem

$$\begin{cases} -u'''(t) + k^2u'(t) = f(t, u(t), u'(t)), \ t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases} \tag{3}$$

where the function $f : (0, +\infty) \times (0, +\infty) \times (0, +\infty) \to \mathbb{R}$ is a Carathéodory function, that is

- $f(., u, v)$ is mesurable function for all $u, v \in I(I := (0, +\infty)$, and
- $f(t, ., .)$ is continuous for a.e. $t \in I$,

semipositone and may be singular at $t = 0, u = 0$ and $u' = 0$. We use the Theorem 1.1 in [12]) namely the "Fixed point theorem of cone expansion and compression" to prove under a suitable Hypothesis of the nonlinearity f the existence of a positive solution to the bvp (3) (see Theorem 3.1 in [12]). We prove by the two Lemmas 2.3 and 2.4 the existence of the radii of the open bounded sets of the Banach space which are necessary to use the Theorem 1.1 in [12]), and by the Lemma 1.4 we demonstrate that the solution is positive and bounded. We prove that the bvp (3) admits a positive solution in the Corollary 3.1 under suitable Hypothesis which permit us to use the Theorem 3.1. To illustrate our results, we build an example and proved that Hypothesis in Corollary 3.1 are the sufficient conditions to demonstrate that the bvp (3) with a given nonlinearity f admits a positive solution with k large enough and a special radius R.

This thesis is ended by a conclusion.

Chapter 1

Abstract background

1.1 Preliminaries

1.1.1 Compactness

First, recall some basic facts of compactness whose importance will be seen throughout this work.

Definition 1.1. *Let E be a topological space. A subset $M \subset E$ is called compact if every open covering of M has an finite covering, i.e., if $M \subset \bigcup_{i\in I} V_i$, where V_i is an open subset of E for all $i \in I$, then there exist $i_j \in I, j = 1, 2, ..., k$, such that $M \subset \bigcup_{j=1}^{k} V_{i_j}$.*

In case of a normed space E, $M \subset E$ is compact if every sequence $(x_n) \subset M$ has a convergent subsequence with limit in M.

Let us recall by way of example the subsets in $\mathbb{R}$ which are compact.

Example 1.1.

A subset $A \subset \mathbb{R}$ is compact if and only if it is closed and bounded.

Therefore, $\mathbb{R}$ is not compact because not bounded.

Definition 1.2. *M is called relatively compact if $\overline{M}$ is compact.*

In what follows, we consider E and F are Banach spaces and Ω a subset of E.

In general, an application which maps bounded sets into relatively compact sets is not necessarily continuous, hence the following definitions.

Definition 1.3 ([21])**.** *Let $T : \Omega \to F$. T is said to be compact if it is continuous and such that $T(\Omega)$ is relatively compact.*

Definition 1.4. *Let $T : \Omega \to F$. T is said to be completely continuous if it is continuous and maps bounded subsets of Ω into relatively compact sets.*

Theorem 1.1. *If M is a compact set of E, then $Id : M \to M$ is compact.*

In other words, M is compact if and only if $Id : M \to M$ is compact map. This equivalence shows that the notion of compactness joins the sets and maps.

It is well known that if $L : E \to F$ is a linear operator, then L is continuous if and only if L is bounded.

We have grouped together some assertions in the following Remark which connects the definitions of the compactness cited above .

Remark 1.1. *Let $T : \Omega \to E$ an operator.*

1. If T is compact, then T is completely continuous.

2. If T is a linear operator and maps bounded subsets of Ω into relatively compact sets then T is continuous.

3. If T is a linear compact operator then T is continuous.

4. If T is a linear operator, the two concepts compactness and completely continuous coincide.

5. If Ω is a bounded subset of E, T is compact then T is completely continuous and vice versa.

Remark that the compact subsets in E with $\dim E = \infty$ are scarce or rare, hence the interest of the compactness criteria that we will state in the following Theorems which are based on the two notions of equicontinuity and equiconvergence. These are Cesar Arzela and Giulio Ascoli who introduced the notion of equicontinuity in the late 19th. The equicontinuity of a family of continuous functions is certainly important to prove these Theorems, this is why we recall here the Definition of this notion and we continue with Remarks.

Let (E, d) a compact metric space, F a Banach space and denote by $C(E, F)$ the Banach space of continuous functions f from E to F, endowed with the norm:

$$||f|| = \sup_{x \in E} ||f(x)||_F$$

and let H the subset of $C(E, F)$.

Let's look now from a topological point of view the definition of equicontinuity of H before stating it in metric space to avoid confusion with the well-known continuity notion.

Definition 1.5. *H is said to be equicontinuous at a point x_0 in E if,*

$$\forall \epsilon > 0, \exists U_\epsilon \in V(x_0), \forall x \in E,$$

$$(x \in U_\epsilon \Rightarrow f(x) \in B(f(x_0), \epsilon)), \forall f \in H.$$

H is equicontinuous if it is equicontinuous at every point $x_0 \in E$, where $V(x_0)$ is the neighborhood of x_0.

Note that continuity of the function f at x_0 means that given f and given $\epsilon > 0$, there exists a neighborhood U of x_0 such that $||f(x) - f(x_0)|| < \epsilon$ for $x \in U$. Equicontinuity of H means that a single neighborhood U can be chosen that will work for all the functions $f \in H$.

Remark 1.2. *H is equicontinuous if and only if (it is uniformly equicontinuous) :*

$$\forall \epsilon > 0, \quad \exists \delta, \quad \forall x, y \in E,$$

$$||x - y|| < \delta \Rightarrow ||f(x) - f(y)||_F < \epsilon), \forall f \in H.$$

Counterexample: The sequence of functions $f_n(x) = arctan(nx)$, is not equicontinuous because the definition is violated at $x_0 = 0$.

The Ascoli-Arzela Theorem which we will state hereafter, is a fundamental result of mathematical analysis to prove the compactness of subsets of $C(E, F)$. This Theorem characterize the relatively compact sets of continuous functions space from a compact space to a metric space. The equicontinuity of the family of functions is the main condition in this Theorem which is the basis of many proofs for instance the ordinary differential equations theory.

Theorem 1.2 (**Ascoli-Arzela's Theorem**[28]). *A subset H of $C(E, F)$ is relatively compact if and only if*
(1)H is equicontinuous,
(2) $\forall x \in E$, the set $H(x) = \{f(x), f \in H\}$ is relatively compact.

Example 1.2. *Set $f_n(t) = sin(nx), x \in [0, 2\pi]$ and $H = \{f_n(.) : n \in \mathbb{N}\}$. Then H is bounded; however it is not equicontinuous in $C([0, 2\pi], \mathbb{R})$ (for this, consider the sequence $x_n = x + \frac{\pi}{2n}$). Hence H is not relatively compact, i.e. we cannot extract a convergent subsequence.*

Denote by $C_b(\mathbb{R}, \mathbb{R})$ the vector space of all bounded and continuous functions defined on $\mathbb{R}$, $C_l([t_0, T), \mathbb{R}^n)$ the space of all continuous maps from $[t_0, T)$ to $\mathbb{R}^n$ such that

$$\lim_{t \to T} = x_T \in \mathbb{R}^n$$

and the subset C_l of $C_b(\mathbb{R}, \mathbb{R})$, where

$$C_l = \left\{ u \in C(\mathbb{R}, \mathbb{R}) : \lim_{t \to \pm\infty} u(t) \quad exist \right\}.$$

To the question of the compactness of a subset H of $C_l([t_0, T), \mathbb{R}^n)$, first, the idea was to identify an element $x(t) \in C_l([t_0, T), \mathbb{R}^n)$ to $\overline{x}(t) \in C_l([t_0, T], \mathbb{R}^n)$ as follows: $\overline{x}(t) = x(t)$ for $t \in [t_0, T)$ and $\overline{x}(T) = x_T$, and this leads to an isomorphism between the spaces $C_l([t_0, T), \mathbb{R}^n)$ and $C_l([t_0, T], \mathbb{R}^n)$, when $T < +\infty$. The same approach is used if $T = +\infty$. Hence the condition of equiconvergence of H

$$\lim_{t \to T} x(t) = x_T \quad \text{exists uniformly with respect to} \quad x \in H$$

which is the necessary and sufficient condition which, together with the two conditions already mentioned in the Ascoli-Arzela Theorem constitute, the compactness criterion in $C_l([t_0, T), \mathbb{R}^n)$, namely Corduneanu Compactness Criterion. Introduced by Constantin Corduneanu, this Theorem provide the characterization of the relatively compact sets of continuous and bounded functions space from the whole real line $\mathbb{R}$ to $\mathbb{R}$. Unlike the Ascoli-Arzela Theorem, the Corduneanu Compactness Criterion does not require that E be compact, it is an unavoidable way to prove the compactness of the subsets of $C(E, F)$. We recall here the definition of equiconvergence before to state the Corduneanu Theorem in C_l.

Definition 1.6. *A family $H \subset C_l$ is called equiconvergent if*

$$\forall \epsilon > 0, \exists T = T_\epsilon > 0, \forall t_1, t_2 \in \mathbb{R},$$

$$|t_1| > T, |t_2| > T \Rightarrow ||f(t_1) - f(t_2)|| < \epsilon, \forall f \in H.$$

Remark 1.3. *Equivalently, H is equiconvergent if for any $\epsilon > 0$, there exists some $T = T(\epsilon) > 0$ such that $||f(t) - l_f^+|| \leq \epsilon$ and $||f(t) - l_f^-|| \leq \epsilon$ for all $|t| \geq T$ for all $f \in H$; here $l_f^+ := \lim_{t\to+\infty} f(t)$ and $l_f^- := \lim_{t\to-\infty} f(t)$.*

Theorem 1.3 (Corduneanu's compactness criterion in $C_b(\mathbb{R}, \mathbb{R})$[19, 20]). *A nonempty subset H of C_l is relatively compact if the following conditions hold:*

(a) H is uniformly bounded in C_b.

(b) H is equicontinuous on every compact interval of $\mathbb{R}$.

(c) H is equiconvergent.

1.1.2 Elementary spectral theory

In this part, let E a Banach space, we denote by $\mathcal{L}(E)$ the set of all linear bounded self-mapping defined on E and Id is the Identity operator $x \to x$ in E.

It is well known that each linear operator on a finite dimensional space can be represented by a matrix $A = (a_{ij})_{i=,..,n;j=,..n}$. The spectrum of A is the set of μ such that $\mu Id - A$ is not invertible. The spectrum contains the roots of $det(\mu Id - A)$ which are the eigenvalues of A. In the infinite dimension, the definition of the spectrum of a linear bounded operator is not the same at in finite dimension. The object of the spectral theory is the study the properties of the inverse of $\mu Id - A$, if it exists, which called the resolvent operator. For this, we recall first some needed definitions and some results about the existence of the inverse of a linear bounded operator and its properties.

Definition 1.7. *An operator $L \in \mathcal{L}(E)$ is said to be continuously invertible (or just invertible) if there exists an operator $L^{-1} \in \mathcal{L}(E)$ (the inverse of L) such that*

$$L^{-1}L = LL^{-1} = Id.$$

In the case of the operator $Id - T$ with $T \in \mathcal{L}(E)$, the Theorem cited below shows under which condition the inverse of $Id - T$ exists. Before this, we remember the definition of the norm of a linear operator.

Definition 1.8. *The norm of linear operator $T : E \to F$ between normed spaces is nonnegative extended real number $||T||$ defined by*

$$||T|| = \sup_{||x|| \leq 1} ||Tx|| = \min\{M \geq 0 : ||Tx|| \leq M||x|| \quad \textit{for all} \quad x \in E\}.$$

Theorem 1.4 (Lemma 1.1 [59])**.** *Let $T \in \mathcal{L}(E)$, have a norm strictly less that 1, i.e. $||T|| < 1$. Then $A = Id - T$ is continuously invertible. Moreover,*

$$A^{-1}(= (Id - T)^{-1}) = \sum_{i=0}^{\infty} T^n \tag{1.1}$$

where the sum on the right-hand side is defined as the uniform limit of the polynomials $S_n = Id + T + T^2 + ... + T^n$. Moreover, $||(Id - T)^{-1}|| \leq \frac{1}{(1-||T||)}$.

We give hereafter the definition of the spectrum of an operator $L \in L \in \mathcal{L}(E)$ and fundamental properties.

Definition 1.9. *The spectrum of L is defined by*

$$\sigma(L) = \{\mu \in \mathbb{C}, \quad \mu Id - L \quad \textit{is not invertible}\} \tag{1.2}$$

Theorem 1.5. *The spectrum $\sigma(L)$ of L is a nonempty and closed subset of $\mathbb{C}$.*

Proposition 1.1 (Proposition 1.16. [59])**.** *Let $L \in \mathcal{L}(E)$. Then $\lambda \in \sigma(L)$ if and only if there is a sequence $\{x_n\} \subset E$, $||x_n|| = 1$, such that*

$$\lim_{n \to \infty} ||\lambda x_n - Lx_n|| = 0. \tag{1.3}$$

The complement Ω of $\sigma(L)$ is called the resolvent set of L; it consists of values of $\mathbb{C}$ for which the operator $(\mu Id - L)^{-1} =: R(\mu, L)$ is well defined and belongs to $\mathcal{L}(E)$. The operator $R(\mu, L)$ is called the resolvent of L. The study of the operator $R_\lambda(A)$ considerably simplifies that of A, so we will remember the definition of the resolvent operator as a map $\lambda \to R_\lambda(A)$ from $\mathbb{C}$ to the set of linear operator and the resolvent properties. These reminders lead us to the definition of the spectral radius which plays an important role in the fixed point theory.

Theorem 1.6 (**(The resolvent identity)** Theorem 1.5 [59]). *Let* $\lambda, \mu \notin \sigma(L)$*. Then*

$$(\lambda Id - L)^{-1} - (\mu Id - L)^{-1} = (\mu - \lambda)(\lambda Id - L)^{-1}(\mu Id - L)^{-1}. \quad (1.4)$$

The spectrum $\sigma(L)$ of L can be divided into three disjoints pieces:

$$\sigma(L) = P_{\sigma(L)} \cup C_{\sigma(L)} \cup R_{\sigma(L)} \quad (1.5)$$

where

- $P_\sigma(L)$ is the set of all complex numbers $\lambda \in \mathbb{C}$ such that $\lambda Id - L$ has no inverse (neither bounded or not bounded) on E; $P_\sigma(L)$ is called the point spectrum of L.
- $C_\sigma(L)$ is the set of $\lambda \in \mathcal{L}(E)$ such that $\lambda Id - L$ has an inverse operator which is defined on a dense subset of E, but the operator $(\lambda Id - L)^{-1}$ is not bounded; $C_\sigma(L)$ is called the continuous spectrum of L.
- $R_\sigma(L)$ is the set of $\lambda \in \mathcal{C}$ such that the operator $(\lambda Id - L)^{-1}$ is defined on a domain which is not dense in E; $R_\sigma(L)$ is the residual spectrum of L.

From the next result we will know what an eigenvalue of the operator is. The interest of the eigenvalue and its use will intervene throughout this study.

Proposition 1.2 (Proposition 1.15 [59]). *A complex number* λ *belongs to* $P_\sigma(L)$ *if and only if the equation*

$$Lx = \lambda x \quad (1.6)$$

has a nonzero solution $x \neq 0$ *in* E.

In this case the number $\lambda \in \mathbb{C}$ is called an eigenvalue of L and the solution of the equation $Lx = \lambda x$ is called an eigenvector of L corresponding to λ.

The subspace

$$N(\lambda Id - L) = \{x \in E; (\lambda Id - L)x = 0\} \quad (1.7)$$

is called an eigensubspace of L corresponding to the eigenvalue λ.

Note that λ is an eigenvalue of L if $N(\lambda Id - L) \neq \{0\}$. And if L is compact linear operator, the following proposition provide the results of the sets $N(Id - L)$ and the range of L $R(L)(:= \{Lx : x \in E\})$.

Proposition 1.3. *Let $L \in \mathcal{L}(E)$ be compact, the set $N(Id - L)$ has a finite dimension and the set $R(L)$ is closed.*

We illustrate these notions by examples on a concrete sets.

Example 1.3. *Let $E = L^2[0,1]$ be the Hilbert space of equivalence classes of complex-valued square integrable functions on $[0,1]$, and let $A : E \to E$ be defined by*

$$Ax(t) = tx(t), \quad x(= x(t)) \in E. \tag{1.8}$$

Then A has no eigenvalues, since the equation

$$(\lambda Id - A)x = (\lambda - t)x(t) = 0 \tag{1.9}$$

is satisfied for all $t \in [0,1]$ if and only if $x(t) = 0$ almost everywhere.

Thus $P_\sigma(A) = \emptyset$. Nevertheless, the spectrum $\sigma(A)$ is no empty and consists of the elements of the continuous spectrum, i.e.

$$\sigma(A) = C_\sigma(A) = [0,1]. \tag{1.10}$$

Indeed, if $\lambda \notin [0,1]$, then $(\lambda Id - A)^{-1} \in \mathcal{L}(E)$. On the other hand, if $\lambda \in [0,1]$, the one can show that there is a sequence $\{x_n\} \subset E, ||x_n|| = 1$, such that $(\lambda Id - A)x_n \to 0 \quad asn \to \infty$. Thus $(\lambda Id - A)^{-1}$ is not bounded for such λ.

And now, let give some properties which verified by the spectrum to introduce the spectral radius.

Proposition 1.4. *Let $L \in \mathcal{L}(E)$. Then $\sigma(L)$ is compact in $\mathbb{C}$ and*

$$\sigma(L) \subset [-||L||, ||L||].$$

Proof. Let $\lambda \in \mathbb{C}$ with $|\lambda| > ||L||$; let prove that $L - \lambda Id$ is bijective - which proves that $\sigma(L) \subset [-||L||, ||L||]$. Let $f \in E$ the equation

$$Lu - \lambda u = f \tag{1.11}$$

has a unique solution because (1.11) is the same equation as

$$u = \frac{1}{\lambda}(Lu - f) \tag{1.12}$$

and we can apply Banach Fixed point Theorem.Prove now that resolvent the set

$$\Omega = \{\lambda \in \mathbb{R}, (L - \lambda Id) \quad \text{is bijective from } E \quad \text{into} \quad E\}$$

is open. Let $\lambda_0 \in \rho(L)$. Let $\lambda \in \mathbb{R}$ (neighborhood of λ_0) and $f \in E$ we want to solve (1.11). And (1.11) is written $Lu - \lambda_0 u = f + (\lambda - \lambda_0)u$ .ie.

$$u = (L - \lambda_0 Id)^{-1}(f + (\lambda - \lambda_0)u) \tag{1.13}$$

We apply again Banach Fixed point Theorem and we see that (1.13) has a unique fixed point if $|(\lambda - \lambda_0)|||(L - \lambda_0 Id)^{-1}|| < 1$. □

The spectrum of L is bounded then we can set the definition of the spectral radius of $L \in \mathcal{L}(E)$.

Definition 1.10. *The spectral radius of L is the number*

$$r(L) = \sup\{|\lambda| : \quad \lambda \in \sigma(L)\}. \tag{1.14}$$

The following Theorem precise $r(T)$.

Theorem 1.7 (**Spectral radius formula**). *If $L \in \mathcal{L}(E)$, then*

$$r(L) = \lim_{n\to\infty} ||L^n||^{\frac{1}{n}} = \inf_{n\geq 1} ||L^n||^{\frac{1}{n}}. \tag{1.15}$$

Note that equation (1.15) implies that

$$r(L) \leq ||L||. \tag{1.16}$$

And in the case of a compact linear bounded operator, some properties which verified by the spectrum are collected in the following Proposition.

Proposition 1.5 (Proposition 3.24.[47]). *Let $L \in \mathcal{L}(E)$ be compact. Then*

(i) $0 \in \sigma(L)$,

(ii) *if $\lambda \neq 0$ then $\lambda \in \sigma(L)$ if and only if λ is an eigenvalue of L,*

(iii) *$\sigma(L)$ is finite or $\sigma(L)$ is a sequence which converge to* 0.

Proof. **(i)**By contrary, suppose that $0 \notin \sigma(L)$, then L is bijective and $Id = LoL^{-1}$ is compact (because L is compact) and the unit ball $B(0,1)$ is compact then $dimE < \infty$.

(ii) Let $\lambda \in \sigma(L), \lambda \neq .0$ Prove that λ is an eigenvalue of L. By contrary, suppose that $KerL - \lambda Id = \{0\}$. Then $R(L - \lambda Id) = E$ and $\lambda \in \rho(L)$ which is absurd.

For **(iii)**, let us prove this lemma:

Lemma 1.1 (Lemma VI.2[18]). *Let $(\lambda_n)n \geq 1$ a sequence of distinct reals with $\lambda_n \to \lambda$ and $\lambda_n \in \sigma(L)\backslash\{0\}\forall\ n$. Then $\lambda = 0$ which means that the points of $\sigma(L)\backslash\{0\}$ are clusters.*

Proof. We know that λ_n are eigenvalues of L. Let $e_n \neq 0$ such that $(L - \lambda_n)e_n = 0$. Let $E_n = span[e_1, e_2, ...e_n]$. Prove that $E_n \subsetneq E_{n+1}$ for all n. It sufficient to prove that all vectors $e_1, e_2, ...e_n$ are linearly independents. By induction with respect to n. We admit the result true at n and suppose $e_{n+1} = \sum_{i=1}^{n} \alpha_i e_i$.
Then $Le_{n+1} = \sum_{i=1}^{n} \alpha_i \lambda_i e_i = \sum_{i=1}^{n} \alpha_i \lambda_{n+1} e_i$. Hence, $\alpha_i(\lambda_i - \lambda_{n+1}) = 0$ for all $i = 1, 2..n$ and then $\alpha_i = 0$ for all $i = 1, 2...n$ which is absurd. Then $E_n \subsetneq E_{n+1}$ for all n.
In the other hand, it is clear that $(L - \lambda_n)E_n \subset E_{n-1}$. By the Riesz Theorem we construct a sequence $(u_n)_{n\geq 1}$ such that $u_n \in E_n, ||u_n|| = 1$ and $dist(u_n, E_{n-1}) \geq 1/2$ for all $n \geq 2$.
Let $2 \leq m < n$ such that

$$E_{m-1} \subset E_m \subset E_{n-1} \subset E_n$$

We have

$$\| \frac{Lu_n}{\lambda_n} - \frac{Lu_m}{\lambda_m} \| = || \frac{(Lu_n - \lambda_n u_n)}{\lambda_n} - \frac{(Lu_m - \lambda_m u_m)}{\lambda_m} + u_n - u_m|| \geq dist(u_n, E_{n-1}) \geq 1/2$$

If $\lambda_n \to \lambda \neq 0$ we end up with a contradiction since (Tu_n) has a convergent subsequence, ending Lemma. □

For all $n \geq 1$ the set $\sigma(L) \cap \{\lambda \in \mathbb{R}; |\lambda| \geq \frac{1}{n}\}$ is empty or finite (if the set contains an infinity of distinct points, we would have a cluster point-since $\sigma(L)$ is compact- and we end up with a contradiction.When $\sigma(L)\backslash\{0\}$ contains an infinity distinct points which we can tidy in a sequence which converge to 0. □

1.2 Topological degree

1.2.1 The Leray Schauder degree

Let E be a Banach space, $\Omega \subset E$ an open bounded and denote by $\overline{\Omega}$, $\partial\Omega$ be the closure and the boundary of $\Omega \subset E$ respectively, $f : \overline{\Omega} \to E$ a compact perturbation of Identity (f = Id - K). Set $y_0 \in E \backslash f(\partial\Omega)$ and we let $\delta = dist(y_0, f(\partial\Omega))$. Let $K_\epsilon : \overline{\Omega} \to X$ a continuous function, compact with values in a space N_ϵ which contains y_0 and $dim(N_\epsilon) < \infty$ such that $\sup ||K_\epsilon(x) - K(x)|| < \frac{\delta}{2}$. Then, we have

Proposition 1.6. *Let $F \subset X$ be a closed and bounded, A function $f : F \to X$. f is compact if and only if f is a limit of a sequence of compact functions (f_n) of finite rank.*

Recall the definition of the Brouwer degree in order to define the Leray-Schauder degree.

Definition 1.11. *The Brouwer degree $d((Id - K_\epsilon)|_{\overline{\Omega} \cap N_\epsilon}, \Omega \cap N_\epsilon, y_0)$ is well defined.*

Let K_ϵ the approximation of K (see Proposition 1.6), then the following definition of the Leray-Schauder degree.

Definition 1.12. $d(Id - K, \Omega, y_0) = d(Id - K_\epsilon, \Omega, y_0)$

This definition permits to state the following theorem to define the Leray-Schauder degree and some properties related to this degree.

Theorem 1.8 (Theorem 8.1.[21]). *Let A_c the set of triplets $(Id - A, \Omega, y)$ where Ω an open bounded subset of E, $y \in E$ and $A : \overline{\Omega} \to E$ is compact such that $y \notin (Id - A)(\partial\Omega)$. Then there exists exactly one function $d : A_c \to \mathbb{Z}$ such that:*

- ***Normality.*** *$d(Id, \Omega, y) = 1$ for any $y \in \Omega$.*

- ***Additivity.*** *If $y \notin (Id - A)(\overline{\Omega} \backslash (\Omega_1 \cup \Omega_2))$, whenever Ω_1, Ω_2 are two disjoint open subsets of Ω then,*

$$d(Id - A, \Omega, y) = d(Id - A, \Omega_1, y) + d(Id - A, \Omega_2, y)$$

- ***Homotopy invariance.*** *Let* $H : [0,1] \times \overline{\Omega} \to E$ *is a compact function and,* $y(t)$ *continuous function such that* $y : [0,1] \to E$ *and for all* $t \in [0,1], y(t) \notin (Id - H(t,.))(\partial\Omega)$ *then*

$$d(Id - H(0,.))\Omega, y(0)) = d(Id - H(1,.))\Omega, y(1))$$

This property is interest because we say that two functions are homotopic if they have the same degree.

- ***Translation invariance.*** $d(Id - A, \Omega, y) = d(Id - A - y, \Omega, 0)$,

d is called a topological degree of Leray-Schauder.

1.2.2 Some properties of the degree

The Leray Schauder degree verify the following properties:

1. **Solution property.** If $d(Id-A, \Omega, y) \neq 0$ then it exists $x \in \Omega$ such that $x - A(x) = y$.

2. **Excision property.** $d(Id - A, \Omega, y) = d(Id - A, \Omega_0, y)$ whenever Ω_0 is an open subset of Ω and $y \notin (Id - A)(\overline{\Omega} \backslash \Omega_0)$.

3. For all $z \in E, d(Id - A, \Omega, y) = d(Id - A, \Omega, y - z)$.

4. $d(Id - A, \Omega, .)$ is constant on the connected component of $E \backslash (Id - A)(\partial\Omega)$.

5. **Boundary value property.** $d(Id - A, \Omega, y) = d(Id - G, \Omega, y)$ whenever $A(x) = G(x)$ for any $x \in \partial\Omega$.

The following Proposition states en important property of the degree.

Proposition 1.7. *Let* $L \in \mathcal{L}(E)$ *and* L *is compact,* $\phi = Id - L$*. If* 1 *is not a characteristic value of* L *then, for all* $R > 0$,

$$d(\phi, B_R(0), 0) = (-1)^{\beta},$$

where β *is the sum algebraic multiplicities of the characteristic values of* L *which are between* 0 *and* 1.

Proof. First, Recall the definition of the algebraic multiplicities of the eigenvalue μ:

$$m(\mu) := dim \bigcup_{k=0}^{\infty} \{x \in E : \quad (\mu Id - L)^k x = 0\}.$$

The degree is well defined because 0 is a unique solution of $\phi(u) = 0$ (1 is not a characteristic value of L). By Proposition 1.5 the set of characteristic values μ_i is at most countable and its unique cluster point is $+\infty$.

This implies that the number of characteristic values of L in the bounded set of $\mathbb{R}$ is finite. Say $\mu_1, \mu_2..., \mu_p$ as characteristic values of L in $(0, 1)$.

Let

$$N_i = \bigcup_{n=1}^{\infty} N(\mu_i L - Id)^n = N(\mu_i L - Id)^{\alpha_i}$$

where $\alpha_i \in \mathbb{N}, i = 1, 2..., p$ et let N is the direct sum of N_i. From the Fredholm alternative N is invariant by L and the $dim(N) < \infty$. Since $N \bigoplus F = E$ by the product formulation of degree (see proposition 8.4 [21])

$$d(Id - L, B_r(0), 0) = d(Id - L_{|N}, B_r(0) \cap N, 0) d(Id - L_{|F}, B_r(0) \cap F, 0)$$

In $B_r(0) \cap F$ we consider the deformation $Id - tL, 0 \le t \le 1$ (if $(Id - tL)(x) = 0, x \in F \quad \text{then} \quad x = 0)$. Then

$$d(Id - L, B_r(0), 0) = d(Id - L_{|N}, B_r(0) \cap N, 0) = (-1)^{\beta} \tag{1.17}$$

□

As an application of the degree we recall this Theorem.

Theorem 1.9 (**Schauder's fixed point**[21])**.** *Let E be a real Banach space. $K \subset E$ nonempty closed bounded and convex, and $A : K \to K$ compact. Then A has a fixed point: it exists $x \in K$ such that $A(x) = x$.*

Proof. If there exists a fixed point on ∂K then we are done. Otherwise we suppose that $A(x) \neq x$ for all $x \in \partial K$. Since A doesn't have a fixed point on ∂K, we have $d(Id - A, K, 0) \in A_c$ which A_c is defined in Theorem 1.8, we will prove that $d(Id - A, K, 0) = 1$ and by solvability property we conclude that $Id - A$ has at least one zero in K and then

A has fixed point in K. Let $H(t,x) = tA(x)$, which is compact on $[0,1] \times K$. If for $t_0 \in [0,1]$ and $x_0 \in \partial K$ we have $x_0 - H(t_0, x_0) = 0$ then $t_0 A(x_0) = x_0$; since $|x_0| = 1$ and $|A(x_0)| \leq 1$ $(A : K \to K)$ this implies that $t_0 = 1$ and $x_0 = A(x_0)$, then x_0 is a fixed point on ∂K contradicts our supposition. We can use normality and homotopy properties (for $t = 0$ and $t = 1$) of degree whose give

$$1 = d(Id - H(0,.), K, 0) = d(Id - H(1,.), K, 0)$$

since $(H(0,.) = 0$ and $H(1,.) = A)$ ending the proof. □

1.3 Cones and ordered Banach space

1.3.1 Cones

As the real numbers, the comparison between the functions needs an order on the set of functions. That is why we recall the definition of the order. First, we begin by the definition of the set which induces this order.

Definition 1.13. *A nonempty closed convex* $(\lambda x + \mu y \in K, \forall \lambda \geq 0, \mu \geq 0, x, y \in K)$ *subset of* E *such that* $K \cap (-K) = \{0\}$ *and* $tK \subset K$ *for all* $t \geq 0$ *is called a cone in* E.

A cone K induces a partial order in the Banach space E. We write for all $x, y \in E$, $x \preceq y$ if $y - x \in K$, $x \prec y$ if $y - x \in K$ and $x \neq y$, $x \npreceq y$ if $y - x \notin K$ and $x \prec\prec y$ represents $y - x \in int(K)$ if $int(K) \neq \emptyset$ ($int(K)$ is the interior of K). Notations $\succeq, \succ, \nsucceq$ and $\succ\succ$ denote the inverse situations.

We give an example of a cone.

Example 1.4. *Let* $X = C(\overline{M})$, *space of continuous functions on a bounded set* M *in* $\mathbb{R}^n$. *We set* $K = \{f \in X : f(x) \geq 0 \quad on \quad \overline{M}\}$. *Then* K *is an order cone in* X *and we have*

$$f \leq g \quad \textit{if and only if} \quad f(x) \leq g(x) \quad \textit{for all } x \in \overline{M} \tag{1.18}$$

$$f << g \quad \textit{if and only if} \quad f(x) < g(x) \quad \textit{for all } x \in \overline{M} \tag{1.19}$$

In what follows, K is a cone on the Banach space E.

We introduce in the following definitions some characteristics and the basic properties of the cone.

Definition 1.14. *K is said to be normal cone with a constant $n_K > 0$ if for all $u, v \in K, u \preceq v$ implies $||u|| \leq n_K ||v||$.*

Geometrically, normality means that the angle between two positive unit vectors has to be bounded away from π. In other words, a normal cone cannot be too large.

Recall the result which is useful in case of a normal cone K , for this let's begin by this definition.

Definition 1.15. *The set $[x, y] = \{z \in E, x \preceq z \preceq y\}$ is called the order interval.*

Proposition 1.8 (Proposition 7.11 [71])**.** *If K is normal, then every order interval $[x, y]$ is bounded.*

Proof. If $x \preceq w \preceq y$ then $0 \preceq w - x \preceq y - x$ and hence $||w - x|| \leq c||y - x||$. □

Definition 1.16. *K is said to be solid if $int(K) \neq \emptyset$ where $int(K)$ is the interior of K.*

We give some examples of a cone having some of these properties.

Example 1.5. *Let $E = L^p(\Omega)$, the space of Lebesgue mesurable functions which are pth power summable on $\Omega \subset \mathbb{R}^n$, where $p \geq 1$ and $0 < mes\Omega < +\infty$. Let $P = \{x(t) \in L^p(\Omega) \backslash x(t) \geq 0\}$ is a normal cone in $L^p(\Omega)$ but P is not solid.*

Example 1.6. *Let $E = C(G)$, space of continuous functions on a bounded closed set G in $\mathbb{R}^n$ and $P = \{x(t) \in E : x(t) \geq 0 \quad and \quad \int_{G_0} x(t)dt \geq \epsilon_0 ||x||_E\}$ where G_0 is a closed subset of G and ϵ_0 is a given number satisfying $0 < \epsilon_0 < 1$. P is a solid and normal cone in E.*

Definition 1.17. *K is said to be reproducing if $K - K = E$.*

If $int(K) \neq \emptyset$, then K is reproducing, in fact, take $x_0 \in int(K)$ and $r > 0$ such that $B(x_0, 2r) \subset K$, where $B(x_0, 2r) = \{y, ||y - x_0|| < 2r\}$. For any $x \in E$ with $x \neq 0$, we have $x_0 + r||x||^{-1}x \in K$. Moreover,

$$x = ||x||r^{-1}(x_0 + r||x||^{-1}x) - ||x||r^{-1}x_0 \in K - K,$$

thus K is reproducing, ending the proof.

Definition 1.18. *K is said to be total if $\overline{K-K}=E$, i.e., the set $\{x-y, \quad x,y\in K\}$ is dense in E.*

1.3.2 Positive operators

Let E, F two ordered Banach spaces and K is a cone on E.

Recall the definitions of the positive and monotone mapping $T: D(T)\subset E\to F$.

Definition 1.19. *T is said to be positive if and only if both $T(0)\succeq 0$ and for all $x\in D(T)$,*

$$x\succ 0 \quad \textit{implies} \quad Tx\succeq 0.$$

T is strongly positive as the symbol $\succeq$ is replaced by $\succ\succ$.

Definition 1.20. *T is said to be monotone increasing if and only if it is true for all $x,y\in D(T)$ that*

$$x\prec y \quad \textit{implies} \quad Tx\preceq Ty.$$

T is called strictly, strongly monotone increasing if and only if the symbol $\preceq$ is replaced by $\prec$ or $\prec\prec$ respectively.

Definition 1.21. *T is said to be monotone decreasing if and only if it is true for all $x,y\in D(T)$ that*

$$x\prec y \quad \textit{implies} \quad Tx\preceq Ty.$$

T is called strictly, strongly monotone decreasing if and only if the symbol $\succeq$ is replaced by $\succ$ or $\succ\succ$ respectively.

Remark that in case of linear operators, positivity is equivalent to monotonicity.

Definition 1.22. *Let $T: E\to E$ be a positive operator. T is said to be lower bounded on the cone K, if*

$$c=\inf\{\|Tu\| : u\in K\cap\partial B(0_E,1)\}>0.$$

In this case we have $||Tu||\geq c||u||$ for all $u\in K$.

We remember in the following definitions the minorant (resp. the majorant) for an operator, for this let K be a subset in E and $T_1, T_2 : K \to K$ be continuous mappings.

Definition 1.23. *We write $T_1 \preceq T_2$ if $T_1 x \preceq T_2 x$ for all $x \in K$.*

Definition 1.24. *If $T_1 x \preceq T_2 x$ for $x \in K$, then T_1 is called a minorant for T_2 on K, T_2 is called a majorant for T_1.*

The interest of the recalling of cones, properties of cone and positivity of operator materializes in the study of the equations

$$\lambda x - Tx = y, \quad y \succ 0, \tag{1.20}$$

and the corresponding homogeneous equation

$$Tx = \lambda x, \quad x \succ 0. \tag{1.21}$$

and the existence of a positive solution $x \neq 0$ for the homogeneous equation (1.21) was facilitated by the Krein-Rutman Theorem by proving under conditions of T and the cone K, existence of a positive eigenvalue of T which is the spectral radius of L. The notion of the spectral radius is evoked in the section Elementary spectral theory. The well known Krein-Rutman theorems, which we will state hereafter, are the famous result in the linear positive compact operator theory. Those theorems are a generalization of the Perron-Frobenius Theorem to infinite Banach spaces. They were proved in 1948 and since then they haven't stopped being mentioned. The various applications of those theorems are in Bifurcation theory, the investigation of nonlinear problems, stability analysis of solutions to elliptic equations and steady-state of the corresponding parabolic equations ...

Hereafter, the first version of those Theorems.

Theorem 1.10 (**Krein-Rutman Theorem**.Theorem 19.2. [21])**.** *Assume that the cone K is total and $L \in \mathcal{L}(E)$ compact and positive with $r(L) > 0$. Then $r(L)$ is a positive eigenvalue of L.*

The second version of the Krein-Rutman Theorem require more than the positivity of T with the cone which is not total.

Theorem 1.11 (**Krein-Rutman Theorem**.Theorem 19.3. [21])**.** *Assume that $K \subset E$ a cone with $int(K) \neq \emptyset$ and $L \in \mathcal{L}(E)$ compact and strongly positive. Then we have*
(a) $r(L) > 0, r(L)$ is a algebraically simple eigenvalue with an eigenvector $v \in int(K)$ and there is no other eigenvalue with a positive eigenvector.

(b) $|\lambda| < r(L)$ for all eigenvalues $\lambda \neq r(L)$.

The following result answers about existence of a positive solution of the inhomogeneous problem (1.20) in the form of consequence of the Theorems cited above.

Corollary 1.1 (Corollary 7.27 [71])**.** *For every $y > 0$, the equation* (1.20) *has exactly one solution $x > 0$ if $\lambda > r(T)$ and no solution if $\lambda \leq r(T)$.*

$\lambda x - Tx = \mu y$ and $x > 0, y > 0$ implies $sgn(\mu) = sgn(\lambda - r(T))$. Here λ and μ are real numbers.

1.3.3 Fixed point Index

The notion of the fixed point index was introduced and discussed by Nussbaum [54](see also [3, 5]). The purpose being to investigate the fixed points of some nonlinear operators, we combine the properties of cones with the fixed point index. We take back the properties of the Leray-Schauder degree for the fixed point index in the following Theorem because the definition of the index emanates from the Leray-Schauder degree theory.

A subset $K \subset E$ is called a retract of E if there exists a continuous mapping $r : E \to K$, and a retraction, when $r(x) = x, x \in K$. From a Theorem due to Dugundji (see Dugundji [l]), and particularly, every cone of E is a retract of E.

Theorem 1.12 (Theorem 2.3.1,[34])**.** *Let K be a retract of E, then for every relatively bounded open subset U of K and every compact operator $A : \overline{U} \to K$ which has no fixed point on ∂U, there exists an integer $i(A, U, K)$ satisfying the following conditions:*

1. ***Normality:*** *$i(A, U, K) = \mathbf{1}$ if $Ax = x_0 \in U$ for all $x \in \overline{U}$.*

2. ***Homotopy invariance:*** *Let $H : [0, 1] \times \overline{U} \to K$ be a compact mapping such that $H(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial U$. The integer $i(H(t, \cdot), U, K)$ is independent of t.*

3. ***Additivity***:

$$i\,(A,U,K) = i\,(A,U_1,K) + i\,(A,U_2,K)$$

whenever U_1 *and* U_2 *are two disjoint open subsets of* U *such that* A *has no fixed point in* $\overline{U} \setminus (U_1 \cup U_2)$.

4. ***Permanence***: *If* P *is a retract of* K *with* $A\left(\overline{U}\right) \subset P$ *then*

$$i\,(A,U,K) = i\,(A,U \cap P,P)\,.$$

5. ***Solution property***: *If* $i\,(A,U,K) \neq 0$ *then* A *admits a fixed point in* U.

6. ***Excision property***: $i\,(A,U,K) = i\,(A,U_0,K)$ *whenever* U_0 *is an open subset of* U *such that* A *has no fixed points in* $\overline{U}\backslash U_0$.

Moreover, let

$$\{(A,U,K) \quad \textit{where} \quad K \quad \textit{retract of } E, \quad U \quad \textit{open} \quad \textit{bounded} \quad \textit{in} \quad K,$$

$$A : \overline{U} \to K \quad \textit{compact and} \quad Ax \neq x \quad \textit{on} \quad \partial U\}.$$

Then there exists exactly one function $d : M \to \mathbb{Z}$ *satisfying* $1-4$. *In other words,* $i(A,U,K)$ *is uniquely defined.* $i(A,U,K)$ *is called the fixed point index of* A *on* U *with respect to* K.

Proof. First we prove the uniqueness of the fixed point index. Let $\{i(A,U,K)\}$ be any family satisfying conditions $1-4$. We define

$$d(Id - A, U, p) = i(A + p, U, E) \tag{1.22}$$

where U bounded open of E, $f(x) \neq p$ on ∂U, i.e. $A + p$ has no fixed point on ∂U. From the conditions $1-4$ and (1.22) it is easy to show that the function $d(f,U,p)$ has the four properties which characterize the Leray-Schauder degree and hence, by the uniqueness of the Leray-Schauder degree, we have

$$d(f,U,p) = d(Id - A, U, p) \tag{1.23}$$

Taking $p = 0$ in (1.22) and (1.23), we get

$$i(A, U, E) = d(Id - A, U, 0) \tag{1.24}$$

Suppose now that K is an arbitrary retract of E and denote by $r : E \to K$ an arbitrary retraction. For open subset U of K, we choose a ball $B_R = \{x \in E \quad \text{such that} \quad ||x|| < R\}$ such that $B_R \supset U$. Then, by the Permanence property and (1.24) we have

$$i(A, U, E) = i(Aor, B_R \cap r^{-1}(U), E) = d(Id - Aor, B_R \cap r^{-1}(U), 0) \tag{1.25}$$

Hence, (1.25) and the uniqueness of the Leray-Schauder degree imply the uniqueness of the fixed point index.

By the above uniqueness proof we are led to define

$$i(A, U, K) = d(Id - Aor, B_R \cap r^{-1}(U), 0) \tag{1.26}$$

where $r : E \to K$ an arbitrary retraction and $B_R \supset U$. Evidently, $B_R \cap r^{-1}(U)$ is a bounded open set of E and

$$\overline{B_R \cap r^{-1}(U)} \subset \overline{r^{-1}(U)} \subset r^{-1}(\overline{U}).$$

It is easy to see that

$$x_0 \in r^{-1}(\overline{U}), Aor(x_0) = x_0 \quad \text{implies } x_0 \in U, Ax_0 = x_0 \tag{1.27}$$

Now, we prove that $i(A, U, K)$ defined by (1.26) is independent of the choice of R and r. Let $R_1 > R$. Since

$$U \subset B_R \cap r^{-1}(U) \subset B_{R_1} \cap r^{-1}(U),$$

by (1.27) we know that Aor has no fixed point in $B_{R_1} \cap r^{-1}(U) \backslash (B_{R_1} \cap r^{-1}(U))$, and consequently, by the excision property of Leray-Schauder degree

$$d(Id - Aor, B_R \cap r^{-1}(U), 0) = d(Id - Aor, B_{R_1} \cap r^{-1}(U), 0),$$

i.e. $i(A, U, K)$ is independent of the choice of R. Next, let $r_1 : E \to K$ be another retraction of E and let $V = B_R \cap r^{-1}(U) \cap r_1^{-1}(U)$. Then V is a bounded open set of E

and $V \supset U$. By (1.27) we know that Aor has no fixed points in $\overline{B_R \cap r^{-1}(U)} \backslash V$ and Aor_1 has no fixed points in $\overline{B_R \cap r_1^{-1}(U)} \backslash V$. Hence,

$$d(Id - Aor, B_R \cap r^{-1}(U), 0) = d(Id - Aor, V, 0) \tag{1.28}$$

and

$$d(Id - Aor_1, B_R \cap r_1^{-1}(U), 0) = d(Id - Aor_1, V, 0) \tag{1.29}$$

Now, let $h(t, x) = x - H(t, x)$ where $H(t, x) = r(tAor(x) + (1 - t)Aor_1(x))$.
Clearly, $H : [0, 1] \times \overline{V} \to E$ is compact. We now prove $0 \notin h(t, \partial V)$ for any $t \in [0, 1]$. In fact, if there exists $t_0 \in [0, 1]$ and $x_0 \in \partial V$ such that $h(t_0, x_0) = 0$ then $x_0 = r(t_0 Aor(x_0) + (1 - t_0)Aor_1(x_0)) \in K$.
As a result, $r(x_0) = x_0, r_1(x_0) = x_0 \text{and} x_0 = Ax_0$.
And so, by $(1.27), x_0 \in U \subset V$, in contradiction with $x_0 \in \partial V$. Thus, using the homotopy invariance property of the Leray-Schauder degree and observing that that

$$H(0, x) = r(Aor_1(x)) = Aor_1(x)$$

and

$$H(1, x) = r(Aor(x)) = Aor(x)$$

we have

$$d(Id - Aor_1, V, 0) = d(Id - Aor, V, 0). \tag{1.30}$$

It follows from $(3.7), (1.29)$ and (1.30) that

$$d(Id - Aor, B_R \cap r^{-1}(U), 0) = d(Id - Aor_1, B_R \cap r_1^{-1}(U), 0) \tag{1.31}$$

which shows that $i(A, U, E)$ is dependent of the choice of r.
The conditions $1 - 4$ are the same basic conditions as the Leray-Schauder degree. We prove the conditions $5 - 6$.

Let $U_1 = U$ and $U_2 = \emptyset$ in additivity property; we get $i\,(A, \emptyset, K) = 0$. From this and setting $U_1 = U_0$ and $U_2 = \emptyset$ in additivity, we obtain $i\,(A, U, K) = i\,(A, U_0, K)$ then the property 6 is proved.

If A has no fixed point in U, letting $U_0 = \emptyset$ in excision property, we get

$$i\,(A, U, K) = i\,(A, \emptyset, K) = 0.$$

and hence property 5 is proved. □

Using properties of the index cited below, the following Lemmas provide computations of the index useful for the next Theorems. For this, let $A : \overline{K_R} \to K$ be compact, where $K_R = K \cap B(0,R)$, $B(0,R)$ the unit ball in E. K_R is bounded, its boundary is $\partial K_R = K \cap \partial B(0,R)$ and $\overline{K \cap B(0,R)} = K \cap \overline{B(0,R)}$.

Lemma 1.2 (Lemma 2.3.1[34])**.** *If $Ax \neq \lambda x$ for all $x \in \partial K_R$ and $\lambda \geq 1$ then*

$$i\,(A, K_R, K) = 1. \tag{1.32}$$

Proof. Let $H(t,x) = tAx$. Then $H : [0,1] \times (K \cap \overline{B(0,R)}) \to K$ is continuous, and the continuity of $H(t,x)$ in t is uniform with respect to $x \in K_R$. We have $H(t,.) : K_R \to K$ is compact for all $t \in [0,1]$ because A is compact. We have $H(t,x) \neq x$ for all $x \in K_R$ and $0 \leq t \leq 1$. Hence by the homotopy invariance and normality of fixed point, we have $i(A, K_R, K) = i(H(1,.), K_R, K) = i(H(0,.), K_R, K) = i(0, K_R, K) = 1$ □

Lemma 1.3 (Lemma 2.3.2[34])**.** *Suppose that $B : K \cap \partial B(0,R) \to K$ is compact and*
(a) $\inf_{x \in \partial K_R} ||Bx|| > 0$
(b)$x - Ax \neq tBx$ *for all* $x \in \partial K_R, t \geq 0,$
Then

$$i(A, K_R, K) = 0$$

Lemma 1.4 (see Corollary 2.3.1, page 91[34])**.** *Let operator $A : K_R \to K$ be compact. If there exists a $u_0 > 0$ such that*

$$x - Ax \neq tu_0 \qquad \forall x \in \partial K_R, \qquad t \geq 0, \tag{1.33}$$

then $i(A, K_R, K) = 0$

Proof. This Lemma follows directly from Lemma 1.3 by putting $Bx = u_0$ for any $x \in \partial K_R$. □

Lemma 1.5 (Lemma 2.3.3[34])**.** *Let $A : K_R \to K$ be compact and suppose that*
(i) $\inf_{x \in K_R} ||Ax|| > 0$

(ii)$Ax \neq \mu x$ *for all* $x \in K_R, t \geq 0$,

Then

$$i(A, K_R, K) = 0$$

Proof. Taking $B = A$ in Lemma1.3, we see that condition (a) of Lemma 1.3 is the same at condition (i) in Lemma1.5. Also, condition (b)of Lemma 1.3 is true. In fact, if there exist $x_0 \in K_R$ and $t \geq 0$ such that $x_0 - Ax_0 = t_0 Ax_0$, then $Ax_0 = \mu_0 x_0$ where $\mu_0 = (1 + t_0)^{-1}$. Evidently, $0 < \mu_0 \leq 1$, which contradicts the condition (ii). Thus,

$$i(A, K_R, K) = 0$$

follows from Lemma 1.3. □

Lemma 1.6. *If* $Ax \not\geq x$ *for all* $x \in \partial K_R$ *then* $i(A, K_R, K) = 1$.

Lemma 1.7. *If* $Ax \not\leq x$ *for all* $x \in \partial K_R$ *then* $i(A, K_R, K) = 0$.

The proof of these two Lemmas are part of the proof of theorem cited below.

1.3.4 Fixed point of cone expansion and compression

The two following theorems are the fundamental theorems in fixed point theory. The Theorem 1.13, namely The fixed point of cone expansion and compression, is due to Krasnoselskii (see [45]) and the Theorem 1.14, namely The fixed point of cone expansion and compression with norm type, is due to Guo (see [35, 36]). Those Theorems are used on the ordered Banach spaces to prove that a compact operator under conditions which called expansion and compression of the cone has a fixed point. The fixed point index computations have been used a lot to prove these Theorems. Known to be difficult to apply, it is an unavoidable way for a lot of fixed point on a cone problems also for the location of the fixed point in the space. To state these Theorems, let Ω_1 and Ω_2 be two bounded open sets in E such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let operator $A : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ be compact.

Theorem 1.13 (Theorem 2.3.3.[34])**.** *Suppose that one of the two conditions:*

$$(H_1) \quad Ax \not\geq x, \forall x \in K \cap \partial\Omega_1 \quad \textit{and} \quad Ax \not\leq x, \forall x \in K \cap \partial\Omega_2 \quad (\textit{expansion of the cone}) \tag{1.34}$$

(H_2) $\quad Ax \not\leq x, \forall x \in K \cap \partial\Omega_1$ *and* $Ax \not\geq x, \forall x \in K \cap \partial\Omega_2$ (*compression of the cone*).

$$(1.35)$$

is satisfied. Then A has at least one fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Proof. First we assume that (H_1) is satisfied, i.e., it is the case of cone expansion. It is easy to see that

$$Ax \neq \mu x \qquad \forall x \in K \cap \partial\Omega_1, \qquad \mu \geq 1 \tag{1.36}$$

since, otherwise, there exist $x_0 \in K \cap \partial\Omega_1$ and $\mu_0 \geq 1$ such that $Ax_0 = \mu_0 x_0 \geq x_0$, in contradiction with (H_1). Now from (1.36) and Lemma 1.2, we obtain

$$i(A, K \cap \Omega_1, K) = 1 \tag{1.37}$$

On the other hand, choosing an arbitrary $u_0 > 0$, we have

$$x - Ax \neq tu_0 \qquad \forall x \in K \cap \partial\Omega_2, \qquad t \geq 0. \tag{1.38}$$

In fact, if there exist $x_1 \in K \cap \partial\Omega_2$ and $t \geq 0$ such that $x_1 - Ax_1 = tu_0 \geq 0$ then $x_1 \geq Ax_1$ in contradiction with (H_2). Hence, by (1.38) and Lemma 1.4 we have

$$i(A, K \cap \Omega_2, K) = 0 \tag{1.39}$$

It follows therefore from additivity property of fixed point that

$$i(A, K \cap (\Omega_2 \backslash \overline{\Omega_1}), K) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = -1 \neq 0. \tag{1.40}$$

Hence, by the solution property of fixed point index, A has at least one fixed point in $(\Omega_2 \backslash \overline{\Omega_1})$.

Similarly, when H_2 is satisfied, instead of (1.37), (1.39) and (1.40), we have $i(A, K \cap \Omega_1, K) = 0, i(A, K \cap \Omega_2, K) = 1$, and $i(A, K \cap (\Omega_2 \backslash \overline{\Omega_1}), K) = 1$. As a result we also can assert that A has at least one fixed point in $\Omega_2 \backslash \overline{\Omega_1}$. □

Theorem 1.14 (Theorem 2.3.4.[34])**.** *Suppose that one of the two conditions:*

$$(H_3) \quad ||Ax|| \leq ||x||, \forall x \in K \cap \partial\Omega_1 \quad \textit{and} \quad ||Ax|| \geq ||x||, \forall x \in K \cap \partial\Omega_2. \tag{1.41}$$

$$(H_4) \quad ||Ax|| \geq ||x||, \forall x \in K \cap \partial\Omega_1 \quad \textit{and} \quad ||Ax|| \leq ||x||, \forall x \in K \cap \partial\Omega_2. \tag{1.42}$$

is satisfied. Then A has at least one fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Proof. We only need to prove this theorem under condition (H_3), since the proof is similar when (H_4) is satisfied. By the extension theorem, A can be extended to a compact operator from $(K \cap \overline{\Omega_2})$ into K. We may assume that A has no fixed points on $(K \cap \overline{\Omega_1})$ and $(K \cap \overline{\Omega_2})$. It is easy to see that (1.36) holds, since otherwise, there exists $x_0 \in (K \cap \overline{\Omega_1})$ and $\mu_0 > 1$ such that $Ax_0 = \mu_0 x_0$ and hence $||Ax_0|| = \mu_0 ||x_0|| > ||x_0||$ in contradiction with (H_3). Thus, by (1.36) and Lemma 1.2, (1.37) holds. On the other hand, it is also easy to verify

$$Ax \neq \mu x \qquad \forall x \in K \cap \partial\Omega_2, \qquad 0 \geq \mu \geq 1 \tag{1.43}$$

In fact, if there are $x_1 \in K \cap \partial\Omega_2$ and $0 < \mu_1 < 1$ such that $Ax_1 = \mu_1 x_1$, then $||Ax_1|| = ||\mu_1 x_1|| < ||x_1||$, in contradiction with (H_3). In addition, by (H_3) we have

$$\inf_{x \in K \cap \partial\Omega_2} ||Ax|| \geq \inf_{x \in K \cap \partial\Omega_2} ||x|| > 0 \tag{1.44}$$

It follows from (1.43), (1.44) and Lemma 2.3.2 in [34] that $i(A, K \cap \partial\Omega_2, K) = 0$ holds. As before,

$$i(A, K \cap (\overline{\Omega_2} \backslash \Omega_1), K) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = 0 - 1 = -1 \neq 0 \tag{1.45}$$

and therefore A has at least one fixed point in $\Omega_2 \backslash \overline{\Omega_1}$. □

1.3.5 Index jump property

The previous Theorems 1.13 and 1.14 are useful for demonstrating, by means of the concept of fixed point index and compactness of a nonlinear operator, existence of a positive solution in a cone of the boundary value problem, but their assumptions are difficult to show. In order to overcome this difficulty and drawing inspiration from the work of Webb in [64] (see Theorems 4.4, 4.5 and 4.7) which gives the different results of the fixed point index, assuming that in addition to the fact of the comparison between the nonlinear mapping T and the linear operator L, together with according to the position of $r(L)$ with respect to 1, Benmezai in his work [10](see Theorems 3.24 and 3.25) use the hypothesis of the comparison of T and the approximative L by introducing the condition of the Strongly Index-Jump Property (SIJP for short) that should check by the minorant L of T, with according the positive spectrums respectively of the minorant and the majorant of T, which located oppositely by 1 (not necessarily $r(L)$ as cited in previous Theorems in [64]) gives different results of fixed point index by exploiting its properties. In practice, it is often difficult to prove existence of SIJP of L, this is why we resort to prove that L is a limit for a nondecreasing sequence of operators having the SIJP (see the proof of Theorem 1.16), that is another advantage of using the SIJP. It turns out to be a powerful tool to prove for applications to the bvp's. Before developed the notion of the SIJP, let introduce the sets Γ_L and Λ_L for a linear compact operator on a Banach space E. For this, let $K,\ P$ be two cones in E with $P \subset K$ and the set

$$L_K^P(E) = \{L \in \mathcal{L}(E) \quad \text{compact operator} : L(K) \subset P\},$$

where for all $L \in L_K^P(E)$, $VP_{L,K}$ denotes the set of all positive eigenvalues of L and we set so,

$$\lambda_{L,K}^- = \inf VP_{L,K} \quad \text{ and } \quad \lambda_{L,K}^+ = \begin{cases} \sup VP_{L,K} \text{ if } VP_{L,K} \neq \emptyset, \\ 0 \text{ if } VP_{L,K} = \emptyset. \end{cases}$$

If $L \in L_K^P(E)$ then L is positive because $L(K) \subset P \subset K$. In the case of $P = K$, we denote $\mathcal{L}_K(E)$ instead of $L_K^P(E)$.

The cone K is a natural cone which is related to the space E and the cone P is related to the operator L and it represents in some manner, the regularity of L. The permanence property of the fixed point index allows to have for all compact mapping, $L : \overline{K_R} \to K$ with $L(x) \neq x$ for all $x \in \partial K_R,\ i(L, K_R, K) = i(L, P_R, P)$.

Also, for $L \in L_K^P(E)$, we define the subsets

$$\Gamma_L = \{\theta \geq 0 : \text{ there exists } u \in P \setminus \{0_E\} \text{ such that } Lu \preceq \theta u\},$$

$$\Lambda_L = \{\theta \geq 0 : \text{ there exists } u \in P \setminus \{0_E\} \text{ such that } Lu \succeq \theta u\}.$$

Observe that

- $0 \in \Lambda_L$ and if $\theta \in \Lambda_L$ then $[0, \theta] \subset \Lambda_L$.
- if $\theta \in \Gamma_L$ then $[\theta, +\infty[\subset \Gamma_L$.

When these two quantities exists, we set

$$\theta_{L,P}^+ = \inf \Gamma_L \text{ and } \theta_{L,P}^- = \sup \Lambda_L.$$

The following results show the properties of these quantities and the sets that contain them.

Lemma 1.8 (Lemma 3.1. [10]). *Assume that* $0 < \theta^-_{L,P}, \theta^+_{L,P} < \infty$ *then for all* $R > 0$ *we have*

$$i(\gamma L, K_R, K) = \begin{cases} 1, \; if \; \gamma\theta^-_{L,P} < 1, \\ 0, \; if \; \gamma\theta^+_{L,P} > 1. \end{cases}$$

Proof. Let $\gamma > 0$ be such that $\gamma\theta^-_{L,P} < 1$. Suppose that for some $u \in \partial P_R$, $\gamma L u \succeq u$ then we have $Lu \succeq u/\gamma$ and $1/\gamma \in \Lambda_L$, leading to the contradiction $\gamma\theta^-_{L,P} \geq 1$ because $1/\gamma \leq \theta^-_{L,P}$. So, we have proved that $\gamma L u \not\succeq u$ for all $u \in \partial P_R$ and Lemma 1.6 leads to, $i(\gamma L, K_R, K) = i(\gamma L, P_R, P) = 1$.

The case $\gamma\theta^+_{L,P} > 1$ is checked similarly by means of Lemma 1.7. □

Lemma 1.9 (Lemma 3.2. [10]). *For all* $L \in L^P_K(E)$ *we have*

$$\theta^+_{L,P} \leq \theta^-_{L,P}. \tag{1.46}$$

Proof. Indeed, if $\theta^+_{L,P} > \theta^-_{L,P}$ we have from Lemma 1.8, for $\gamma \in \left(1/\theta^+_{L,P}, 1/\theta^-_{L,P}\right)$, the contradiction

$$i(\gamma L, K_R, K) = \begin{cases} 1, \; \text{if } \gamma\theta^-_{L,P} < 1, \\ 0, \; \text{if } \gamma\theta^+_{L,P} > 1. \end{cases}$$

□

Remark 1.4. *Clearly, we have for all* $L \in L^P_K(E)$, $VP_{L,K} \subset \left[\theta^+_{L,P}, \theta^-_{L,P}\right]$.

Lemma 1.10 ([15]). *For all* $L \in L^P_K(E)$, *the set* Γ_L *is not empty.*

Proof. Let $\lambda > ||L|| = \sup_{||u||=1} ||Lu||$ and $e \in P\backslash\{0_E\}$ and consider the equation

$$u = L_\lambda(u, t) \tag{1.47}$$

where for all $u \in P$ and $t \in [0,1]$ $L_\lambda(u,t) = (t/\lambda)Lu + e$. Clearly, $L_\lambda(P \times [0,1]) \subset P$ and equation (1.47) has no solution in ∂P_R with $R > \max(\lambda||e||/\lambda - ||L||, ||e||)$. Thus, by homotopoy and normality properties of the fixed point index, we conclude that

$$i(L_\lambda(.,1), P_R, P) = i(L_\lambda(.,0), P_R, P) = 1 \tag{1.48}$$

then, equation $L_\lambda(u,1) = u$ admits a solution $u_0 \in P_R\backslash\{0\}$ and $\lambda \in \Gamma_L$. □

The following result shows once again the importance of $r(L)$.

Lemma 1.11 (Theorem 2.7 in [63]). *For all operator $L \in L_K^P(E)$ the set Λ_L is bounded from above by $r(L)$.*

Proof. Let $\theta > r(L)$ and $R_\theta = \sum_{k\in\mathbb{N}} \frac{L^k}{\theta^k}$ and note that $R_\theta = Id + R_\theta(L/\theta)$. Moreover, we have $R_\theta(P) \subset P$ since for all k, $L^k(P) \subset P$. Now, by the contrary, suppose that there exits $u \in \partial P$ such that $Lu \geq \theta u$ and set $v = \theta^{-1}Lu$. We have then the contradiction

$$R_\theta(v) \geq R_\theta(u) = u + R_\theta(v) > R_\theta(v). \tag{1.49}$$

This shows that Λ_L is bounded by $r(L)$. □

The Proposition below shows the relation between the sets Γ_{L_i} and Λ_{L_i} where $L_i \in L_K^P(E)(i = 1, 2)$.

Proposition 1.9. *Let $L_1, L_2 \in L_K^P(E)$ and assume that $L_1 \preceq L_2$. Then we have*

$$\Lambda_{L_1} \subset \Lambda_{L_2} \textit{ and } \Gamma_{L_2} \subset \Gamma_{L_1}.$$

and

$$\theta^+_{L_2,P} \leq \theta^+_{L_1,P} \textit{ and } \theta^-_{L_1,P} \leq \theta^-_{L_2,P}.$$

Proof. By Definition 1.23, $L_1 \preceq L_2$ means that $L_1 u \preceq L_2 u$ for all $u \in P$. If $\theta \in \Lambda_{L_1}$ then it exists $u \in P\backslash\{0\}$ such that $L_1 u \succeq \theta u$ and this implies $\theta u \preceq L_1 u \preceq L_2 u$ and we get $\theta \in \Lambda_{L_2}$. The same proof for $\Gamma_{L_2} \subset \Gamma_{L_1}$.

It exists $u \in P\backslash\{0\}$ such that $\theta^-_{L_1} u \preceq L_1 u \preceq L_2 u$, then $\theta^-_{L_1} \in \Lambda_{L_2}$ hence $\theta^-_{L_1,P} \leq \theta^-_{L_2,P}$. The same proof for $\theta^+_{L_2,P} \leq \theta^+_{L_1,P}$. □

The following proposition shows the importance of the constant $\theta^+_{L,P}$.

Proposition 1.10 (Proposition 3.6 [10]). *Let $L \in L_K^P(E)$ with $\theta^+_{L,P} > 0$ and consider for $y \in P \setminus \{0_E\}$ the equation*

$$\lambda u - Lu = y. \tag{1.50}$$

Then Equation (1.50) has no solution in $P \setminus \{0_E\}$ for all $\lambda \in \left(0, \theta^+_{L,P}\right)$.

The condition for nonexistence of positive solutions to Equation (1.50) in Proposition 1.10 is more naturel to that given in Theorem 2.16 in [45].

Remark 1.5. *Let $L \in L_K^P(E)$ and set*

$$\theta_{L,K}^{+} = \inf\{\theta \geq 0 \text{ there exists } u \succ 0_E \text{ such that } Lu \preceq \theta u\}$$

and

$$\theta_{L,K}^{-} = \sup\{\theta \geq 0 \text{ there exists } u \succ 0_E \text{ such that } Lu \succeq \theta u\}.$$

Note that if $\theta_{L,K}^{+} > 0$ then for all $y \succ 0_E$, Equation (1.50) has no positive solution.

Let $L \in L_K^P(E)$ and $\gamma \in (0, +\infty) \setminus VP_{L,K}$. The integer $i(\gamma L, K_R, K)$ is defined for all $R > 0$ and the excision property of the fixed point index, make it independent of R. This justifies the following definition.

Definition 1.25. *An operator $L \in L_K^P(E)$ is said to have the IJP if there exists $\mu_L > 0$ such that for all $R > 0$ and all $\gamma \in (0, +\infty) \setminus VP_{L,K}$, we have*

$$i(\gamma L, K_R, K) = \begin{cases} 1, & if\ \gamma\mu_L < 1, \\ 0, & if\ \gamma\mu_L > 1. \end{cases}$$

Clearly the real number μ_L in Definition 1.25 is unique.

The following Theorem give the condition under which L has an IJP.

Theorem 1.15 (Theorem 3.9.[10])**.** *Let $L \in L_K^P(E)$. Then L has the IJP if and only if $VP_{L,K} \neq \emptyset$. Moreover, we have that $\mu_L = \lambda_{L,K}^{+}$.*

Proof. Let $L \in L_K^P(E)$ having the IJP at μ_L and by the contrary suppose that μ_L is not an eigenvalue. Then $i\left(\frac{1}{\mu_L} L, K_R, K\right)$ is defined and from the continuity property of the fixed point index, yields the contradiction

$$\begin{aligned} i\left(\frac{1}{\mu_L} L, K_R, K\right) &= \lim_{\gamma \overset{<}{\to} 1/\mu_L} i(\gamma L, K_R, K) = 1 \\ i\left(\frac{1}{\mu_L} L, K_R, K\right) &= \lim_{\gamma \overset{>}{\to} 1/\mu_L} i(\gamma L, K_R, K) = 0. \end{aligned}$$

Thus, we have proved that $VP_{L,K} \neq \emptyset$.

Now, we need to prove that if μ_0 is a positive eigenvalue of L, Then $i(\gamma L, K_R, K) = 0$ for all $\gamma \in (1/\mu_0, +\infty) \setminus VP_{L,K}$ and $R > 0$. To this aim, let $e > 0_E$ be the eigenvector associated with the eigenvalue μ_0. We claim that for all $\lambda \in (0, \mu_0) \setminus \sigma(L)$ and all $t > 0$ equation

$$\lambda u - Lu = te \tag{1.51}$$

admits no positive solution. Indeed, from the Riesz-Schauder theory, there is two subspaces $N(\mu_0)$ and $R(\mu_0)$ such that

$$\begin{gathered}\dim(N(\mu_0)) < \infty,\ R(\mu_0)\ \text{is closed},\\ E = N(\mu_0) \oplus R(\mu_0),\\ L(N(\mu_0)) \subset N(\mu_0),\ L(R(\mu_0)) \subset R(\mu_0)\end{gathered}$$

and μ_0 is the unique eigenvalue of L_{μ_0}, the restriction of L to $N(\mu_0)$. Moreover, if P, Q are respectively the projections of E on $N(\mu_0)$ and $R(\mu_0)$, we have that $PL = LP$ and $QL = LQ$.

Thus, Equation (1.51) is equivalent to the system

$$\begin{cases}\lambda v - Lv = te\\ \lambda w - Lw = 0.\end{cases} \tag{1.52}$$

where $v = Pu$ and $w = Qu$. Since $\lambda \notin \sigma(L)$, the second equation in System (1.52) has $w = Qu = 0$ as a unique solution.

For the first equation in System (1.52), there exists a basis $B = \{e_i\}_{i=1}^{i=n}$ where $n = \dim(N(\mu_0))$ and $e_1 = e$ in which the matrix M_{μ_0} of L_{μ_0} has the Jordan form

$$\begin{pmatrix}\mu_0 & m_{1,2} & 0 & \cdot & \cdot & 0\\ 0 & \mu_0 & m_{2,3} & 0 & & \cdot\\ \cdot & 0 & \cdot & & \cdot & \cdot\\ \cdot & & \cdot & \cdot & & 0\\ \cdot & & & \cdot & \cdot & m_{n-1,n}\\ 0 & 0 & \cdot & \cdot & 0 & \mu_0\end{pmatrix}$$

where for $i = 1, \cdots, n-1$, $m_{i,i+1} = 1$ or 0.

Therefore, if X and b are respectively the coordinate matrices of $v = P(u)$ and te in the basis B, then, the first equation in System (1.52) take the matricial form

$$(\lambda I - M_{\mu_0})\, X = b$$

having the unique solution

$$X = \begin{pmatrix} t/\left(\lambda - \mu_0\right) \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

and so, $u = \frac{t}{\lambda - \mu_0} e \notin K$ is the unique solution of Equation (1.51). The claim is proved.

Let $\gamma \in (1/\mu_0, +\infty)$ with $1/\gamma \notin \Lambda_{L,K}$ and let us compute $i\,(\gamma L, K_R, K)$. We distinguish two cases:

-) $1/\gamma \in (0, \mu_0) \setminus \sigma\,(L)$, in this case if (T_n) is a sequence of positive operators such that $T_n(u) = \gamma L(u) + t_n e$ where (t_n) is a sequence of positive real numbers with $\lim t_n = 0$, then we have since the equation

$$u - \gamma L u = t_n e$$

has no solution in $\overline{K_R}$,

$$i\,(\gamma L, K_R, K) = \lim i\,(T_n, K_R, K) = 0.$$

=) $1/\gamma \in (\sigma\,(L) \setminus \Lambda_{L,K}) \cap (0, \mu_0)$, then there is a sequence (γ_n) such that $1/\gamma_n \in (0, \mu_0) \setminus \sigma\,(L)$ and $\lim \gamma_n = \gamma$; thus, we have

$$i\,(\gamma L, K_R, K) = \lim i\,(\gamma_n L, K_R, K) = 0.$$

Reciprocally, suppose that $VP_{L,K} \neq \emptyset$ and let $\gamma > 0$. We have from the above that $i\,(\gamma L, K_R, K) = 0$ if $1/\gamma \in \left(0, \lambda^+_{L,K}\right) \setminus \Lambda_{L,K}$, so, let us discuss the case $1/\gamma \in \left(\lambda^+_{L,K}, +\infty\right)$. Assume that for some $\lambda \geq 1$ and $u \in \partial K_R,\ \gamma L u = \lambda u$. Then λ/γ is a positive eigenvalue of L and we have the contradiction

$$1/\gamma \leq \lambda/\gamma \leq \lambda^+_{L,K} < 1/\gamma.$$

Therefore, Lemma 1.2 leads to $i(\gamma L, K_R, K) = 1$. Thus, we have proved that L has the IJP at $\lambda^+_{L,K}$ and by uniqueness, we have $\lambda^+_{L,K} = \mu_L$, ending the proof. □

Remark 1.6. *Let $L \in L^P_K(E)$ and assume that the cone K is total and $r(L) > 0$. We have from Lemma 1 in [53] that L has the IJP at $r(L)$. Clearly, Theorem 1.15 generalize this lemma to the case where the cone K is not total.*

Remark 1.7. *Let $L \in L^P_K(E)$, the Schauder index has the jump property (see Corollary 14.6 in [71]) and the jump happens at any eigenvalue of L having an odd algebraic multiplicity. This means that the Schauder index can jumps infinitely many times. However, for the fixed point index, the jump can happens at most one time and this happens only at the largest positive eigenvalue of L.*

Corollary 1.2 (Corollary 3.12 [10])**.** *Assume that $L \in L^P_K(E)$. Then $VP_{L,K} \neq \emptyset$ if and only if $\theta^-_{L,K} > 0$ (i.e. there exists $\theta > 0$ and $u \succ 0_E$ such that $Lu \succeq \theta u$).*

Proof. Let $\theta_0 > 0$ and $e \succ 0_E$ be such that $Le \succeq \theta_0 e$ and consider the cone

$$K^0 = \{u \in K : Lu \succeq \theta_0 u\}.$$

Since $K^0 \neq \{0_E\}$ and $L(K^0) \subset K^0$, the constants θ^+_{L,K^0}, θ^-_{L,K^0} are well defined and one can check easily that

$$0 < \theta_0 \leq \theta^+_{L,K^0} \leq \theta^-_{L,K^0} \leq r(L).$$

Thus, we understand from Lemma 1.8 that L has the IJP on the cone K^0, then we have from Theorem 1.15 that $\sigma_{K^0}(L) \neq \emptyset$. Ending the proof. □

The properties of the limit of a sequence of operators having an IJP are given in the Proposition below.

Proposition 1.11 (Proposition 3.13. [10])**.** *Let $(L_n) \subset L^P_K(E)$ be such that for all integer n, L_n has the IJP at μ_n and assume that $L_n \to L$ in operator norm. Then either*

- $\lim \mu_n = 0$ *or*
- *L has the IJP at some $\mu > 0$.*

Proof. First, since $\lim \|L_n\| = \|L\|$, there exists $c > 0$ such that

$$0 < \mu_n \leq \|L_n\| \leq \|L\| + c.$$

Clearly if $\lim \mu_n \neq 0$, the real number $\mu = \limsup \mu_n$ is positive. Assume that is the case and let (μ_{n_k}) be a subsequence of (μ_n) converging to μ. We have from Lemma 1.12 that μ is a positive eigenvalue of L. So, let us compute $i(\gamma L, P_R, P)$ for any $R > 0$ and $\gamma \in (0, +\infty) \smallsetminus \sigma_K(L)$. If $\gamma \in (0, 1/\mu) \smallsetminus \sigma_K(L)$, then there exists $k_0 \in \mathbb{N}$ such that $\gamma < 1/\mu_{n_k}$ for all $k \geq k_0$ and in this case, $i(\gamma L_{n_k}, P_R, P) = 1$ for all $k \geq k_0$ and we have

$$i(\gamma L, P_R, P) = \lim i(\gamma L_{n_k}, P_R, P) = 1.$$

If $\gamma \in (1/\mu, +\infty) \smallsetminus \sigma_K(L)$ then there exists $k_1 \in \mathbb{N}$ such that $\gamma > 1/\mu_{n_k}$ for all $k \geq k_0$ and in this case $i(\gamma L_{n_k}, P_R, P) = 0$ for all $k \geq k_0$ and we have

$$i(\gamma L, P_R, P) = \lim i(\gamma L_{n_k}, P_R, P) = 0.$$

So, L has the IJP at its largest positive eigenvalue μ and this ends the proof. □

In order to state the Theorems of existence and non existence of the fixed point of a non linear operator, we introduce the SIJP for this purpose.

Definition 1.26. *An operator $L \in L_K^P(E)$ is said to have the SIJP if $\theta_{L,P}^+ > 0$. In the particular case where $\theta_{L,P}^+ = \theta_{L,P}^- = \mu > 0$ we say that L has the SIJP at μ.*

Remark 1.8. *Clearly, If $L \in L_K^P(E)$ has the SIJP, then L has the IJP.*

In the following, we look for the sufficient conditions for operators $L_K^P(E)$ having the SIJP.

Proposition 1.12 (Proposition 3.16. [10])**.** *Let $L \in L_K^P(E)$ be strongly positive. Then L has the SIJP at $r(L)$.*

Proof. First, we have from Theorem 1.11, Remark 1.4 and Lemma 1.11 that

$$0 \leq \theta_{L,P}^+ \leq r(L) \leq \theta_{L,P}^- \leq r(L)$$

that is $0 < \theta_{L,P}^- = r(L)$.

Now, assume that $\theta^+_{L,P} < r(L)$ and let $\theta_0 \in \left(\theta^+_{L,P}, r(L)\right)$ and $u_0 \in P \setminus \{0_E\}$ be such that $L(u_0) \leq \theta_0 u_0$. In fact, we have that $L(u_0) < \theta_0 u_0$ indeed, if $L(u_0) = \theta_0 u_0$, then uniqueness in Theorem 1.11 leads to the contradiction $r(L) = \theta_0 < r(L)$. Thus, one has that the equation

$$\lambda u - Lu = y$$

has a positive solution for $\lambda = \theta_0 < r(L)$ and $y = \theta_0 u_0 - Lu_0$, contradicting Corollary 1.1. This completes the proof. □

Proposition 1.13 (Proposition 3.17. [10])**.** *Let $L \in L^P_K(E)$ and assume that L is lower bounded on the cone P. Then L has the SIJP.*

Proof. Because of definition of the SIJP, we have to show that $\theta^+_{L,P} > 0$. Set $c_{L,P} = \inf\{\|Lu\| : u \in \partial P\} > 0$ and suppose that there exists sequences (θ_n) and $(u_n) \subset P \setminus \{0_E\}$ with $\lim \theta_n = 0$ and $\|u_n\| = 1$ such that

$$Lu_n \preceq \theta_n u_n. \tag{1.53}$$

Since $\|\theta_n u_n\| = \theta_n$ we have that $\lim \theta_n u_n = 0_E$. Consequently up to a subsequence $\lim Lu_n = 0_E$. So the contradiction

$$0 < c_{L,P} \leq \lim \|Lu_n\| = 0.$$

This shows that $\theta^+_{L,P} > 0$, ending the proof. □

Proposition 1.14 (Proposition 3.18. [10])**.** *Let $L \in L^P_K(E)$ and assume that there exists $L_1, L_2 \in L^P_K(E)$ having the SIJP such that $L_1 \preceq L \preceq L_2$. Then L has the SIJP.*

Proof. Indeed, we have from Proposition 1.9 that

$$\Gamma_{L_2} \subset \Gamma_L \subset \Gamma_{L_1} \text{ and } \Lambda_{L_1} \subset \Lambda_L \subset \Lambda_{L_2}$$

and so

$$0 < \theta^+_{L_2,P} \leq \theta^+_{L,P} \leq \theta^+_{L_1,P} \text{ and } \theta^-_{L_1,P} \leq \theta^-_{L,P} \leq \theta^-_{L_2,P} < \infty.$$

□

A sufficient condition that a compact positive operator admits an eigenvalue is given in the Lemma below.

Lemma 1.12. *Let for all integer n, L_n be a compact operator in $\mathcal{L}(E)$ and positive having a positive eigenvalue λ_n. If $L_n \to L$ in operator norm and (λ_n) converges to some real number $\lambda > 0$ then λ is a positive eigenvalue of L.*

Proof. Let ϕ_n be the eigenvector associated with λ_n such that $\|\phi_n\| = 1$ and set $\psi_n = L\phi_n$. Since L is compact and the sequence (ϕ_n) is bounded, we have up to a subsequence $\psi_n \to \psi \in K$. Thus, we obtain the following estimates,

$$\begin{aligned} \|\lambda_n\phi_n - \psi\| &= \|L_n\phi_n - \psi\| \\ &\leq \|L_n\phi_n - L\phi_n\| + \|L\phi_n - \psi\| \\ &\leq \|L_n - L\| + \|\psi_n - \psi\| \end{aligned}$$

leading to

$$\lim \lambda_n\phi_n = \psi \text{ and } \|\psi\| = \lim \|\lambda_n\phi_n\| = \lim \lambda_n = \lambda > 0.$$

Also, we have

$$\left\|L_n\phi_n - \tfrac{1}{\lambda}L\psi\right\| = \left\|\tfrac{1}{\lambda_n}L_n(\lambda_n\phi_n) - \tfrac{1}{\lambda}L\psi\right\|$$
$$\leq \left\|\tfrac{1}{\lambda_n}L_n(\lambda_n\phi_n) - \tfrac{1}{\lambda}L_n(\lambda_n\phi_n)\right\| + \left\|\tfrac{1}{\lambda}L_n(\lambda_n\phi_n) - \tfrac{1}{\lambda}L(\lambda_n\phi_n)\right\| + \left\|\tfrac{1}{\lambda}L(\lambda_n\phi_n) - \tfrac{1}{\lambda}L\psi\right\|$$
$$\leq \left|\tfrac{1}{\lambda_n} - \tfrac{1}{\lambda}\right|\lambda_n\|L_n\| + \tfrac{\lambda_n}{\lambda}\|L_n - L\| + \tfrac{1}{\lambda}\|L\|\,\|\lambda_n\phi_n - \psi\|$$

leading to

$$\lim L_n\phi_n = \frac{1}{\lambda}L\psi.$$

Thus, letting $n \to \infty$ in equation $L_n\phi_n = \lambda_n\phi_n$ we obtain $L\psi = \lambda\psi$ that is λ is a positive eigenvalue of L. This ends the proof. □

In what remains, we let $\Gamma(E)$ be the class of operators $L \in L_K^P(E)$ such that there exists a sequence of cones (P^n) and an increasing sequence of operators (L_n), such that for all $n \in \mathbb{N}$, $P^n \subset P$, L_n has the SIJP at λ_n and $L_n \to L$ in operator norm.

Clearly, all the above classes of positive operators considered in Propositions 1.12, 1.13 are contained in $\Gamma(E)$. So, let us prove that operators in $\Gamma(E)$ have also the SIJP.

Theorem 1.16 (Theorem 3.23.[10])**.** *Assume that $L \in \Gamma(E)$. Then L has the SIJP and $\theta^{+}_{L,P}$ is the unique positive eigenvalue of L (at which it has the IJP). Moreover if the cone K is total then L has the SIJP at $r(L)$.*

Proof. Let $(P^n), (L_n)$ and (λ_n) be the sequences making of L an operator in the class $\Gamma(E)$ and let ϕ_n be the normalized eigenvector associated with λ_n.

First, we have that $\{\theta \geq 0 : \exists u \in P^n \backslash \{0\} \quad \text{such that} \quad L_n u \geq \theta u\} = \Lambda_{L_n}$. Indeed; it is obvious that $\{\theta \geq 0 : \exists u \in P^n \backslash \{0\} \quad \text{such that} \quad L_n u \geq \theta u\} \subset \Lambda_{L_n}$ and if $\theta > 0,\ u \in P \setminus \{0_E\}$ are such that $L_n u \geq \theta u$ then $L_n(u) \in P^n \setminus \{0_E\}$ and $L_n(L_n u) \geq \theta L_n u$. This shows that $\theta \in \{\theta \geq 0 : \exists u \in P^n \backslash \{0\} \quad \text{such that} \quad L_n u \geq \theta u\}$ and $\Lambda_{L_n} \subset \{\theta \geq 0 : \exists u \in P^n \backslash \{0\} \quad \text{such that} \quad L_n u \geq \theta u\}$.

Since L_n has the SIJP at λ_n, we have $\lambda_n = \theta^{-}_{L_n,P^n} = \theta^{-}_{L_n,P}$ then from Proposition 1.11, (λ_n) is a nondecreasing bounded sequence ($\lambda_n \leq \|L\| + C$ for some $C > 0$). Set $\lambda_L = \lim \lambda_n$. We have from Proposition 1.11 that λ_L is the largest positive eigenvalue of L. Also, we have from Proposition 1.10 that

$$\theta^{+}_{L_n,P} \leq \theta^{+}_{L_n,P^n} = \lambda_n \leq \theta^{+}_{L,P}$$

in which letting $n \to \infty$ we get since λ_L is an eigenvalue of L,

$$\theta^{+}_{L,P} \leq \lambda_L = \lim \lambda_n = \lim \theta^{+}_{L_n,P^n} \leq \theta^{+}_{L,P}$$

that is $\lambda_L = \theta^{+}_{L,P}$.

We conclude from all the above that

$$0 < \theta^{+}_{L,P} = \lambda_L \leq \theta^{-}_{L,P} \leq r(L)$$

that is L has the SIJP and $\theta^{+}_{L,P} = \lambda_L$ is the unique positive eigenvalue of L.

Moreover, if the cone K is total then we have from Theorem 1.10 that $r(L)$ is a positive eigenvalue of L and so,

$$0 < \theta^{+}_{L,P} = \lambda_L = \theta^{-}_{L,P} = r(L).$$

This ends the proof. □

Let introduce the following class of operators. Set

$$SIJP(E) = \left\{L \in L^{P}_{K}(E) : L \text{ has the SIJP}\right\}.$$

In the following two theorems, we make a new control of the mapping T which has asymptotically a majorant and a minorant in special classes of operators in $SIJP(E)$.

Theorem 1.17 (Theorem 3.24[10])**.** *Let $T : K \to K$ be a completely continuous mapping and assume that the cone K is normal and there exists three operators $L_1, L_2, L_3 \in L_K^P(E)$ and three continuous functions $F_1, F_2, F_3 : K \to K$ such that*

$$L_2 \in SIJP(E), \ \theta^-_{L_1,P} < 1 < \theta^+_{L_2,P}$$

and for all $u \in K$

$$Tu \preceq L_1u + F_1u,$$

$$L_2u - F_2u \preceq Tu \preceq L_3u + F_3u. \tag{1.54}$$

If either

$$F_1u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } F_iu = \circ(\|u\|) \text{ as } u \to \infty, \ i = 2, 3 \tag{1.55}$$

or

$$F_1u = \circ(\|u\|) \text{ as } u \to \infty \text{ and } F_iu = \circ(\|u\|) \text{ as } u \to 0, \ i = 2, 3, \tag{1.56}$$

then T has a positive fixed point.

Proof. We present the proof in the case where (1.55) holds, the other case is checked similarly. We have to prove existence of $0 < r < R$ such that

$$i(T, P_r, P) = 1 \text{ and } i(T, P_R, P) = 0.$$

In such a situation, additivity and solution properties of the fixed point index imply that

$$i(T, P_R \setminus \overline{P_r}, P) = i(T, P_R, P) - i(T, P_r, P) = -1$$

and T has a positive fixed point u with $r < \|u\| < R$.

Now, consider the function $H_1 : [0, 1] \times K \to K$ defined by $H_1(t, u) = (1-t)Tu + tL_2u$ and let us prove existence of $R > 0$ large enough, such that for all $t \in [0, 1]$ equation $H_1(t, u) = u$ has no solution in ∂P_R. By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0, 1]$ and $u_n \in \partial P_n$ such that

$$u_n = (1 - t_n)Tu_n + t_nL_2u_n.$$

Note that $v_n = u_n / \|u_n\| \in \partial P_1$ and satisfies

$$v_n = (1 - t_n)\frac{Tu_n}{\|u_n\|} + t_n L_2 v_n.$$

Thus, the inequalities

$$L_2 v_n - \frac{F_2 u_n}{\|u_n\|} \preceq \frac{Tu_n}{\|u_n\|} \preceq L_3 v_n + \frac{F_3 u_n}{\|u_n\|}$$

combined with the normality of the cone K and the fact that $F_i(u_n) = \circ\left(\|u_n\|\right)$ as $n \to \infty$ for $i = 2, 3$, implies that the sequence $(Tu_n / \|u_n\|)$ is bounded. This and because of the compactness of L_2, there exists a subsequence denoted also (v_n) converging to $v \in \partial P_1$, satisfying $v \succeq L_2 v$. Therefore, we have $1 \geq \theta^+_{L_2,P}$, contradicting $\theta^+_{L_2,P} > 1$.

For such a radius $R > 0$, homotopy property of the fixed point index leads to

$$i(T, P_R, P) = i(H_1(0, \cdot), P_R, P) = i(H_1(1, \cdot), P_R, P) = i(L_2, P_R, P) = 0.$$

In similar way, consider the function $H_2 : [0, 1] \times K \to K$ defined by $H_2(t, u) = (1 - t)Tu + tL_1 u$ and let us prove existence of $r > 0$ small enough, such that for all $t \in [0, 1]$ equation $H_2(t, u) = u$ has no solution in ∂P_r. By the contrary suppose that for all integer $n \geq 1$ there exist $t_n \in [0, 1]$ and $u_n \in \partial P_n$ such that

$$u_n = (1 - t_n)Tu_n + t_n L_1 u_n.$$

Note that $v_n = u_n / \|u_n\| \in \partial P_1$ and satisfies

$$v_n = (1 - t_n)\frac{Tu_n}{\|u_n\|} + t_n L_1 v_n.$$

Thus, the inequality

$$\frac{Tu_n}{\|u_n\|} \preceq L_1\left(v_n\right) + \frac{F_1 u_n}{\|u_n\|}$$

combined with the normality of the cone K and the fact that $F_1(u_n) = \circ\left(\|u_n\|\right)$ as $n \to \infty$ implies that the sequence $(F_1 u_n / \|u_n\|)$ is bounded. This and because of the compactness of L_1, there exists a subsequence denoted also (v_n) which converges to $v \in \partial P_1$ satisfying $v \preceq L_1 v$. Therefore, we have1 $\leq \theta^-_{L_1,P}$ contradicting $\theta^-_{L_1,P} < 1$.

For such a radius $r > 0$, homotopy property of the fixed point index leads to

$$i(T, P_r, P) = i(H_2(0, \cdot), P_r, P) = i(H_2(1, \cdot), P_r, P) = i(L, P_r, P) = 1.$$

This completes the proof □

Theorem 1.18 (Theorem 3.25[10]). *Let $T : K \to K$ be a completely continuous mapping and assume that there exists two operators $L_1, L_2 \in L_K^P(E)$ and two continuous functions $F_1, F_2 : K \to K$ such that*

$$L_1 \text{ is lower bounded on } P,$$
$$\theta^-_{L_2,P} < 1 < \theta^+_{L_1,P}$$

and for all $u \in K$

$$L_1u - F_1u \preceq Tu \preceq L_2u + F_2u.$$

If either

$$F_1u = \circ(\|u\|) \text{ as } u \to \infty \text{ and } F_2u = \circ(\|u\|) \text{ as } u \to 0 \tag{1.57}$$

or

$$F_1u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } F_2u = \circ(\|u\|) \text{ as } u \to \infty, \tag{1.58}$$

then T has a positive fixed point.

Proof. We present the proof in the case where (1.57) holds, the other case is checked similarly. We have to prove existence of $0 < r < R$ such that

$$i(T, P_r, P) = 1 \text{ and } i(T, P_R, P) = 0.$$

In such a situation, additivity and solution properties of the fixed point index imply that

$$i(T, P_R \setminus \overline{P_r}, P) = i(T, P_R, P) - i(T, P_r, P) = -1$$

and T has a positive fixed point u with $r < \|u\| < R$.

Now consider the function $H_3 : [0,1] \times K \to K$ defined by $H_3(t, u) = (1-t)Tu + tL_1u$ and let us prove existence of $R > 0$ large enough, such that for all $t \in [0,1]$, equation $H_3(t, u) = u$ has no solution in ∂P_R. By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial P_n$ such that

$$u_n = (1-t_n)Tu_n + t_nL_1u_n.$$

Note that $v_n = u_n / \|u_n\| \in \partial P_1$ satisfies

$$v_n = (1-t_n)\frac{Tu_n}{\|u_n\|} + t_nL_1v_n$$

then

$$L_1 v_n = (1-t_n) L_1 \left(\frac{T u_n}{\|u_n\|} \right) + t_n L_1 \left(L_1 v_n \right). \tag{1.59}$$

Because of the lower boundedness of L_1 we have

$$\|L_1 v_n\| \geq c_{L_1,P} > 0$$

where $c_{L_1,P} = \inf \{ \|L_1 u\|, u \in \partial P_1 \}$. We distinguish two cases.

Either (t_n) admits a subsequence denoted also (t_n) such that $t_n \to 1$. In this case letting $n \to \infty$ in (1.59), we get from the compactness of L_1 and the boundedness of $(L_1 v_n)$ that $v = \lim L_1 v_n$ satisfies

$$v = L_1 v \text{ and } \|v\| = \lim \|L_1 v_n\| \geq c_{L_1,P} > 0.$$

This leads to the contradiction

$$1 < \theta^+_{L_1,K} \leq \lambda^-_{L_1,K} \leq 1 \leq \lambda^+_{L_1,K}.$$

Or there exists $\epsilon \in (0,1)$ such that $t_n < 1 - \epsilon$ for all $n \in \mathbb{N}$. In this case we have from (1.59)

$$\left\| \frac{T u_n}{\|u_n\|} \right\| \leq (1-t_n)^{-1} \left(1 + t_n \|L_1\| \right) \leq \epsilon^{-1} \left(1 + \|L_1\| \right)$$

and the sequence $(T u_n / \|u_n\|)$ is bounded. As above, $v = \lim L_1 v_n$ satisfies

$$v \succeq L_1 v \text{ and } \|v\| = \lim \|L_1 v_n\| \geq c_{L_1,P} > 0$$

leading to $\theta^+_{L_1,P} \leq 1$ which contradicts the hypothesis $1 < \theta^+_{L_1,P}$.

Thus, there exists $R > 0$ large such that $H_5(t,u) \neq u$ for all $t \in [0,1]$ and $u \in \partial P_R$ and for such a radius $R > 0$, homotopy property of the fixed point index implies that

$$i(T, P_R, P) = i(H_3(0,\cdot), P_R, P) = i(H_3(1,\cdot), P_R, P) = i(L_1, P_R, P) = 0.$$

Arguing as in proof of Theorem 1.17, we prove existence of $r > 0$ small enough, such that $i(T, P_r, P) = 1$ and this completes the proof □

The following two theorems are respectively adapted versions of the two Theorems above. They provide solvability results to the equation $u = Tu$ under eigenvalue criteria.

Theorem 1.19. *Assume that the cone K is normal and there exists three operators L_1, L_2, L_3 in $\mathcal{L}_K(E)$ and three functions $F_1, F_2, F_3 : K \to K$ such that*

$$\begin{cases} L_1 \text{ has the SIJP at } r(L_1), \\ 0 < r(L_1) < 1 < r(L_2) \text{ and} \\ Tu \preceq L_1 u + F_1 u, \\ L_2 u - F_2 u \preceq Tu \preceq L_3 u + F_3 u \text{ for all } u \in K. \end{cases}$$

If either

$$F_1 u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } F_i u = \circ(\|u\|) \text{ as } u \to \infty \text{ for } i = 2, 3 \tag{1.60}$$

or

$$F_1 u = \circ(\|u\|) \text{ as } u \to \infty \text{ and } F_i u = \circ(\|u\|) \text{ as } u \to 0 \text{ for } i = 2, 3, \tag{1.61}$$

then T has a positive fixed point.

Theorem 1.20. *Assume that there exists two operators $L_1, L_2 \in$ in $\mathcal{L}_K(E)$ and two continuous functions $F_1, F_2 : K \to K$ such that*

$$\begin{cases} L_1 \text{ has the SIJP at } r(L_1) \\ L_1 \text{ is lower bounded on } K, \\ r(L_2) < 1 < r(L_1) \text{ and} \\ L_1 u - F_1 u \preceq Tu \preceq L_2 u + F_2 u \text{ for all } u \in K. \end{cases}$$

If either

$$F_1 u = \circ(\|u\|) \text{ as } u \to \infty \text{ and } F_2 u = \circ(\|u\|) \text{ as } u \to 0 \tag{1.62}$$

or

$$F_1 u = \circ(\|u\|) \text{ as } u \to 0 \text{ and } F_2 u = \circ(\|u\|) \text{ as } u \to \infty, \tag{1.63}$$

then T has a positive fixed point.

Let $T : K \to K$ be a completely continuous mapping. The following proposition provide under eigenvalue criteria a nonexistence result for fixed point to the mapping T.

Proposition 1.15. *Assume that there exists $L \in \mathcal{L}_K(E)$ having the SIJP at μ such that one of the following conditions (1.64) and (1.65) holds true,*

$$\mu > 1 \text{ and } Tu \succeq Lu \text{ for all } u \in K \tag{1.64}$$

$$\mu < 1 \text{ and } Tu \preceq Lu \text{ for all } u \in K. \tag{1.65}$$

Then T has no positive fixed point.

Proof. We present the proof in the case of (1.64) holds, the other case is checked similarly. To the contrary, suppose there exists $u \succ 0_X$ such that $Tu = u$. In this case we have that $u = Tu \succeq Lu$,

$$1 \in \{\theta \geq 0 : \exists\, u \succ 0_X \text{ such that } Lu \preceq \theta u\}$$

and

$$\mu = \inf \Lambda_L \leq 1.$$

This contradicts the hypothesis $\mu > 1$ in (1.64). □

Chapter 2

Eigenvalue criteria for existence and nonexistence of bounded and unbounded positive solution to a third-order BVP on the half line

2.1 Introduction and main results

Because they arise in modeling various physical phenomenons, the study of existence of solutions to boundary value problems (bvp for short) associated with third-order ordinary differential equations, has becomed an important area of applied mathematics. For instance, Danziger and Elemergreen (see [31], p. 133) have obtained the following third-order linear differential equations:

$$\begin{aligned} &\alpha_3 u''' + \alpha_2 u'' + \alpha_1 u' + (1+k)\,u = kc,\ \theta < c, \text{ and} \\ &\alpha_3 u''' + \alpha_2 u'' + \alpha_1 u' + u = 0,\ \theta > c. \end{aligned} \tag{2.1}$$

These equations describe the variation of thyroid hormone with time. Here $u = u(t)$ is the concentration of thyroid hormone at time t and $\alpha_3, \alpha_2, \alpha_2, k$ and c are constants.

In [42], Jackiewicz et al. have investigated the asymptotic behaviour of the solutions of Volterra integro-differential equations of the form

$$u'(t) = \gamma u(t) + \int_0^1 (\lambda + \mu t + \vartheta s)\, u(s) ds \; t \geq 0,$$
$$u(0) = 1,$$

with the help of third-order differential equations of the type

$$u''' = \gamma u'' + (\lambda + (\mu + \vartheta)\, t)\, u' + (2\mu + \vartheta)\, u, \tag{2.2}$$

where λ, γ, μ and ϑ are real parameters and $\mu + \vartheta = 0$.

A reduced version of the Hodgkin–Huxley model was proposed by Nagumo. He suggested the class of third-order differential equation

$$u''' - cu'' + f'(u)u' - \frac{b}{c}u = 0 \tag{2.3}$$

as a model exhibiting many of the features of the Hodgkin–Huxley equations, where f is a regular function. Recall that the Hodgkin–Huxley model describes the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axon.The model has played a vital role in biophysics and neuronal modeling. For more details of Nagumo's equations, we refer to the paper by McKeen [51].

The Kuramoto–Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + \frac{1}{2}u^2 = 0$$

arises in a wide variety of physical phenomena. It was introduced to describe pattern formulation in reaction diffusion systems, and to model the instability of flame front propagation (see Y. Kuramoto and T. Yamada [46] and D. Michelson [52]). A traveling wave solutions $u = \phi(x - ct)$ satisfies, after one integration, the third-order equation

$$\lambda\phi'''(x) + \phi'(x) + f(\phi) = 0, \tag{2.4}$$

where λ is a parameter depend on the constant c and f is an even function.

A three-layer beam is formed by parallel layers of different materials. For an equally loaded beam of this type, Krajcinovic in [44] proved that the deflection u is governed by the third order differential equation

$$-u''' + k^2u' = a, \tag{2.5}$$

where k and a are physical parameters depending on the elasticity of the layers.

Especially, study of existence of positive solutions for third-order bvps has received a great deal of attention and was the subject of many articles, see, for instance, [29, 30, 32, 38, 37, 50, 58, 61, 65, 66, 67, 72], for third-order bvps posed on finite intervals and [1, 7, 16, 24, 25, 26, 27, 41, 43, 48, 49, 55, 60] for such bvp's posed on the half-line.

In this chapter, we establish under eigenvalue criteria, nonexistence and existence results for positive solutions to the third-order bvp:

$$\begin{cases} -u'''(t) + k^2 u'(t) = f(t, u(t)), \ t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases} \tag{2.6}$$

where k is a positive constant, the function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous ($\mathbb{R}^+ := [0, +\infty)$) and observe that the form of the differential equation in (2.6) is more general to those of (2.1)-(2.4). Here the constant k which may have a physical signification as in (2.5), will play an important role in finding a suitable framework for a fixed point formulation of bvp (2.6).

By a positive solution to the bvp (2.6), we mean a function $u \in C^3(\mathbb{R}^+, \mathbb{R}^+)$ with $u(t_*) > 0$ for some $t_* > 0$ satisfying all equations in the bvp (2.6).

When looking for positive solutions by means of the fixed point theory in cones, authors often make use of the compression and expansion of a cone principle in a Banach space. This principle states that if P is a cone in a Banach space $(B, \|\cdot\|)$, $T : P_{r,R} \to P$ is a compact mapping where $P_{r,R} = \{u \in P : r \leq \|u\| \leq R\}$ and one of the following situations a) and b) holds:

a) $\|Tu\| \geq \|u\|$ for all $u \in P$, $\|u\| = r$ and $\|Tu\| \leq \|u\|$ for all $u \in P$, $\|u\| = R$,

b) $\|Tu\| \leq \|u\|$ for all $u \in P$, $\|u\| = r$ and $\|Tu\| \geq \|u\|$ for all $u \in P$, $\|u\| = R$,

then T has a fixed point w such that $r \leq \|w\| \leq R$.

This principle has advantage to be applicable on any region of the cone P and it has the flaw that the realization of the inequality $\|Tu\| \geq \|u\|$ requires a specific cone, see, for instance [30, 32, 50, 66, 67].

Also we will use in this work the fixed point theory in cones. The operator of our fixed point formulation associated to bvp (2.6) is defined on the Banach space of continuous functions u satisfying $\lim_{t\to+\infty} \frac{u(t)}{t} = 0$. Notice that this space is imposed by the boundary condition in (2.6) $\lim_{t\to+\infty} u'(t) = 0$, since by the L'Hopital's rule $\lim_{t\to+\infty} \frac{u(t)}{t} = \lim_{t\to+\infty} u'(t) = 0$. Unfortunately, the cone of nonnegative function lying in the above space does not offer the possibility to realize the inequality $\|Tu\| \geq \|u\|$. To overcome this difficulty we use the approach exposed in Section 3. This approach gives a necessary condition for existence of positive solution (see Proposition 1.15), and has the advantage to be applicable in any cone. However, it has the inconvenient that the radii r and R must be taken near 0 and $+\infty$ respectively. In other words we loss the localization established in the compression and expansion of a cone principle in a Banach space, $r \leq \|w\| \leq R$.

Since a function u satisfying $\lim_{t\to+\infty} \frac{u(t)}{t} = 0$ may be bounded or unbounded (as $u(t) = \ln(1+t)$), we provide in each existence result established in this paper sufficient conditions for the boundedness or unboundedness of the obtained positive solution.

In all this paper, we let :

$$\begin{aligned}
&\Gamma = \{q \in C(\mathbb{R}^+, \mathbb{R}^+) : q(s) > 0 \text{ a.e. } s > 0\}, \\
&\Gamma_0 = \left\{q \in \Gamma : \sup_{s\geq 0} q(s) < \infty\right\}, \\
&\Gamma_1 = \{q \in \Gamma : \lim_{s\to+\infty} q(s) = 0\}, \\
&\Gamma_2 = \left\{q \in \Gamma : \lim_{s\to+\infty} q(s) = 0 \text{ and } \int_0^{+\infty} q(s)ds < \infty\right\}, \\
&\Delta_i = \{q \in \Gamma : qp_i \in \Gamma_i\} \text{ for } i = 0, 1, 2, \\
&\Delta_3 = \{q \in \Gamma : qp_3 \in \Gamma_1\}, \\
&\Delta = \Delta_1 \cup \Delta_2,
\end{aligned}$$

where

$$p_1(t) = 1 + t \quad p_0(t) = p_2(t) = 1 \quad p_3(t) = e^{kt}.$$

Notice that $\Gamma_2 \subset \Gamma_1 \subset \Gamma_0$, $\Delta_2 = \Gamma_2$, $\Delta_3 \subset \Delta_1 \cap \Delta_2$, $\Delta_1 \setminus \Delta_2 \neq \emptyset$ and $\Delta_2 \setminus \Delta_1 \neq \emptyset$. Indeed, for

$$q_1(s) = \frac{1}{(1+s)\ln(4+s)},\; q_2(s) = \frac{m(s)}{1+s},$$

where

$$m(s) = \begin{cases} 2n^4 s - n(2n^4 - 1) \text{ if } s \in \left[n - \frac{1}{2n^3}, n\right], \\ -2n^4 s + n(2n^4 + 1) \text{ if } s \in \left[n, n + \frac{1}{2n^3}\right], \\ 0 \text{ if not}, \end{cases}$$

we have $q_1 \in \Delta_1 \setminus \Delta_2$ and $q_2 \in \Delta_2 \setminus \Delta_1$.

A continuous mapping $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is said to be a:

- Γ_i-Caratheodory function for $i = 0, 1, 2$, if for all $r > 0$ there exists a function $\psi_r \in \Gamma_i$ such that

$$|g(t, p_i(t)u)| \leq \psi_r(t) \text{ for all } t \geq 0 \text{ and } u \in [-r, r].$$

- Γ_{2+i}-Caratheodory function for $i = 1, 2$, if for all $r > 0$ there exists a function $\psi_r \in \Gamma_i$ such that

$$|g(t, p_3(t)u)| \leq \psi_r(t) \text{ for all } t \geq 0 \text{ and } u \in [-r, r].$$

Consider for $q \in \Delta$, the linear eigenvalue problem associated with the bvp (2.6)

$$\begin{cases} -u'''(t) + k^2 u'(t) = \mu q(t) u(t), \ t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases} \tag{2.7}$$

where μ is a real parameter.

A positive real number μ_0 is said to be a positive eigenvalue of the bvp (2.7), if there exists a function $\phi \in C^3(\mathbb{R}^+, \mathbb{R}^+)$ such that $\phi(t_0) > 0$ for some $t_0 > 0$ and the pair (μ_0, ϕ) satisfies all equations in the bvp (2.7).

The first result of this paper concerns existence of the positive eigenvalue of the bvp (2.7).

Proposition 2.1. *For all $q \in \Delta$, the eigenvalue problem (2.7) admits a unique positive eigenvalue $\mu(q) > 0$ having an eigenfunction ϕ. Moreover, if $q \in \Delta_2$ then ϕ is bounded and if not (i.e. $\int_0^{+\infty} q(s)ds = +\infty$), then ϕ is unbounded, i.e. $\lim_{t\to+\infty} \phi(t) = +\infty$.*

Theorem 2.1. *Assume for $i = 1$ or 2, the nonlinearity f is a Γ_i-Caratheodory function and there exists a function q in Δ_i such that either*

$$\inf \left\{ \frac{f(t, p_i(t)u)}{p_i(t)q(t)u} : t, u > 0 \right\} > \mu(q) \tag{2.8}$$

or

$$\sup \left\{ \frac{f(t, p_i(t)u)}{p_i(t)q(t)u} : t, u > 0 \right\} < \mu(q). \tag{2.9}$$

Then the bvp (2.6) admits no positive solution.

The statements of the following existence results need additional notations. Let $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a function. If g is a Γ_i-Caratheodory function we set for $q \in \Delta_i$ with $i \in \{0,1,2,3\}$ and $\nu = 0, +\infty$,

$$g^+_{i,\nu}(q) = \limsup_{u\to\nu} \left(\max_{t\geq 0} \frac{g(t,p_i(t)u)}{p_i(t)q(t)u} \right),$$
$$g^-_{i,\nu}(q) = \liminf_{u\to\nu} \left(\min_{t\geq 0} \frac{g(t,p_i(t)u)}{p_i(t)q(t)u} \right).$$

Theorem 2.2. *Assume for $i = 1$ or 2 the nonlinearity f is a Γ_i-Caratheodory function and there exist two functions q_0 and q_∞ in Δ_i such that either*

$$\frac{f^+_{i,\infty}(q_\infty)}{\mu(q_\infty)} < 1 < \frac{f^-_{i,0}(q_0)}{\mu(q_0)} \leq \frac{f^+_{i,0}(q_0)}{\mu(q_0)} < \infty \tag{2.10}$$

or

$$\frac{f^+_{i,0}(q_0)}{\mu(q_0)} < 1 < \frac{f^-_{i,+\infty}(q_\infty)}{\mu(q_\infty)} \leq \frac{f^+_{i,\infty}(q_\infty)}{\mu(q_\infty)} < \infty. \tag{2.11}$$

Then the bvp (2.6) admits a solution u in K_i. Moreover, if $i = 2$ then u is bounded and if $i = 1$ and

$$\begin{cases} \lim_{t\to+\infty} \int_1^t f(s,p_1(s)\lambda)ds = +\infty \text{ uniformly} \\ \text{for } \lambda \text{ in compact intervals of } (0,+\infty), \end{cases} \tag{2.12}$$

then u is unbounded.

In Theorem 2.2 Conditions (2.10) and (2.11) impose to the nonlinearity f to be sub linear at $+\infty$, that is there is a positive constants d and a function $c \in \Gamma_i$ such that $f(t,u) \leq c(t)\,u$ for all $u \geq d$ and $t \geq 0$. To avoid such a condition, we have been led to look for positive solutions in a largest Banach space. We have obtained then the following result.

Theorem 2.3. *Assume that the nonlinearity f is a Γ_3-Caratheodory function and there exist two functions q_0 and q_∞ in Δ_3 such that either*

$$\frac{f^+_{3,\infty}(q_\infty)}{\mu(q_\infty)} < 1 < \frac{f^-_{3,0}(q_0)}{\mu(q_0)}, \tag{2.13}$$

or

$$\frac{f^+_{3,0}(q_0)}{\mu(q_0)} < 1 < \frac{f^-_{3,\infty}(q_\infty)}{\mu(q_\infty)}. \tag{2.14}$$

Then the bvp (2.6) admits a positive solution u. Moreover, if the nonlinearity f is a Γ_4*-Caratheodory function then the solution u is bounded, and if*

$$\begin{cases} \lim_{t\to+\infty} \int_1^t f(s, p_3(s)\lambda)ds = +\infty \text{ uniformly} \\ \text{for } \lambda \text{ in compact intervals of } (0, +\infty), \end{cases} \tag{2.15}$$

then u is unbounded.

Consider now, the particular version of the bvp (2.6) where the nonlinearity f takes the form $f(t, u) = q_*(t)h(t, u)$; Namely, we consider the bvp

$$\begin{cases} -u'''(t) + k^2u'(t) = q_*(t)h(t, u(t)), \ t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases} \tag{2.16}$$

where $q_* \in \Gamma$ and $h : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function.

If h/p_i is a Γ_0-Caratheodory function for $i = 1, 2$ or 3, we set then for $\nu = 0, +\infty$

$$h^+_{i,\nu} = h^+_{i,\nu}(1) \quad h^-_{i,\nu} = h^-_{i,\nu}(1).$$

We obtain respectively from Theorems 2.1, 2.2 and 2.3 the following corollaries:

Corollary 2.1. *Assume for $i = 1$ or 2 that $q_* \in \Delta_i$, the function h/p_i is Γ_0-Caratheodory and either*

$$\inf \left\{ \frac{h(t, p_i(t)u)}{p_i(t)u} : t, u > 0 \right\} > \mu(q),$$

or

$$\sup \left\{ \frac{f(t, p_i(t)u)}{p_i(t)u} : t, u > 0 \right\} < \mu(q).$$

Then the bvp (2.16) has no positive solution.

Corollary 2.2. *Assume for $i = 1$ or 2 that $q_* \in \Delta_i$, the function h/p_i is Γ_0-Caratheodory and either*

$$h^+_{i,\infty} < \mu(q_*) < h^-_{i,0} \le h^+_{i,0} < \infty,$$

or

$$h^+_{i,0} < \mu(q_*) < h^-_{i,\infty} \le h^+_{i,\infty} < \infty.$$

Then the bvp (2.16) admits a positive solution. Moreover, if $i = 2$ then u is bounded and if $i = 1$ and

$$\begin{cases} \lim_{t\to+\infty} \int_1^t q_*(s)h(s, p_1(s)\lambda)ds = +\infty \text{ uniformly} \\ \text{for } \lambda \text{ in compact intervals of } (0, +\infty), \end{cases}$$

then u is unbounded.

Corollary 2.3. *Suppose that $q_* \in \Delta_3$, the function h/p_3 is Γ_0-Caratheodory and either*

$$h^+_{3,\infty} < \mu(q_*) < h^-_{3,0},$$

or

$$h^+_{3,0} < \mu(q_*) < h^-_{3,\infty}.$$

Then the bvp (2.16) admits a positive solution. Moreover, if $q_* \in \Delta_2$ then u is bounded and if and

$$\begin{cases} \lim_{t\to+\infty} \int_1^t q_*(s)h(s, p_3(s)\lambda)ds = +\infty \text{ uniformly} \\ \text{for } \lambda \text{ in compact intervals of } (0, +\infty), \end{cases}$$

then u is unbounded.

2.2 Example

Consider for $i = 1, 2, 3$ the bvp (2.6) with

$$f(t, u) = F_i(t, u) = Aq_0(t)\frac{p_i(t)u}{(p_i(t))^2 + u^2} + Bq_\infty(t)\frac{u^2}{p_i(t) + u},$$

where A and B are positive real numbers and $q_0, q_\infty \in \Delta_i$.

It is easy to see that F_i is a Γ_i-Caratheodory function and if

$$0 < \inf_{t\geq 0} \frac{q_\infty(t)}{q_0(t)} \leq \sup_{t\geq 0} \frac{q_\infty(t)}{q_0(t)} < \infty,$$

then

$$f^-_{i,0}(q_0) = f^+_{i,0}(q_0) = A \text{ and } f^-_{i,\infty}(q_\infty) = f^+_{i,\infty}(q_\infty) = B.$$

We deduce from Theorems 2.2 and 2.3 that for such a nonlinearity f, the bvp (2.6) admits a solution if either

$$A < \mu(q_0) \text{ and } B > \mu(q_\infty)$$

or

$$A > \mu(q_0) \text{ and } B < \mu(q_\infty).$$

Evidently for $i = 2$, the obtained solution u is bounded and for $i = 1$, if $\int_0^{+\infty} q_0 p_1 ds = +\infty$ then u is unbounded. Indeed, for any interval $[a, b] \subset (0, +\infty)$ we have

$$\begin{aligned} \int_1^t f(s, p_2(s)\lambda)ds &\geq A \int_1^t q_0(s)p_1(s)\frac{\lambda}{1+\lambda^2}ds \\ &\geq \frac{Aa}{1+a^2}\int_1^t q_0(s)p_1(s)ds \to +\infty \text{ as } t \to +\infty. \end{aligned}$$

For instance if $q_0(t) = q_\infty(t) = (1+t)^{-2}$ the obtained solution is unbounded.

In the case $i = 3$, if $\int_1^{+\infty} q_0(s)p_3(s)ds < +\infty$ then the solution is bounded and if $\int_1^{+\infty} q_0(s)p_3(s)ds = +\infty$, the same computations as above lead to u is unbounded. For example if $q_0(t) = q_\infty(t) = (1+t)^{-1} e^{-kt}$, then the obtained solution is unbounded.

2.3 Abstract background

Remark 2.1. *We have from Proposition 3.14 and Proposition 3.15 in [14] that if* $L \in \mathcal{L}_K(X)$ *has the SIJP at* μ *then* μ *is the unique positive eigenvalue of* L.

Remark 2.2. *It is easy to see that if* $L \in \mathcal{L}_K(X)$ *has the SIJP at* μ *and* $L(K) \subset P \subset K$ *where* P *is a cone in* E, *then* $L \in \mathcal{L}_P(X)$ *has the SIJP at* μ.

In this work, the problem of existence and nonexistence of positive solutions to the bvp (2.6) will be converted to that of existence and nonexistence of fixed point for a completely continuous mapping defined on a cone of an appropriate functional space.

2.4 Fixed point formulation

In all this paper we let

$$\begin{aligned} E_0 &= \{u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} u(t) = 0\}, \\ E_1 &= \{u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} \tfrac{u(t)}{1+t} = 0\}, \\ E_2 &= \{u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} u(t) = l \in \mathbb{R}\}, \\ E_3 &= \{u \in C(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} e^{-kt}u(t) = 0\}. \end{aligned}$$

Endowed respectively with the norms

$$\|u\|_1 = \sup_{t\geq 0} \frac{|u(t)|}{1+t}, \ \ \|u\|_2 = \sup_{t\geq 0} |u(t)| \ \text{ and } \ \|u\|_3 = \sup_{t\geq 0} e^{-kt} |u(t)| \,,$$

E_1, E_2 and E_3 become Banach spaces.

We let also, K_1, K_2 and K_3 be respectively the cones in E_1, E_2 and E_3 defined by

$$\begin{aligned} K_1 &= \{u \in E_1 : u(t) \geq 0 \text{ for all } t \geq 0 \text{ and } u \text{ is nondecreasing}\}, \\ K_2 &= \{u \in E_2 : u(t) \geq 0 \text{ for all } t \geq 0\}, \\ K_3 &= \{u \in E_3 : u(t) \geq \gamma(t)||u||_3 \text{ for all } t \geq 0\} \end{aligned}$$

where

$$\gamma(t) = \frac{1}{3k}\left(e^{-3kt} - 3e^{-kt} + 2\right).$$

Let $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function given by

$$G(t,s) = \frac{1}{k^2} \begin{cases} e^{-ks}\left(\cosh(kt) - 1\right) \text{ if } t \leq s \\ -e^{-kt}\sinh(ks) + (1 - e^{-ks}) \text{ if } s \leq t. \end{cases}$$

The functions G and $\dfrac{\partial G}{\partial t}$ are continuous and they have the following properties:

$$G(t,s) > 0 \text{ for all } t, s > 0, \tag{2.17}$$

$$\frac{\partial G}{\partial t}(t,s) > 0 \text{ for all } t, s > 0, \tag{2.18}$$

$$G(0,s) = \frac{\partial G}{\partial t}(0,s) = 0 \text{ for all } s \in \mathbb{R}^+ \tag{2.19}$$

$$\lim_{t\to+\infty} G(t,s) = \frac{1}{k^2}(1 - e^{-ks}) \text{ for all } s \in \mathbb{R}^+ \tag{2.20}$$

$$\int_0^{+\infty} G(t,s)ds = \frac{1}{k^2}t - \frac{1}{k^3}(1 - e^{-kt}) \text{ for all } t \geq 0, \tag{2.21}$$

$$\sup_{t\geq 0} \frac{1}{1+t} \int_0^{+\infty} G(t,s)ds = \frac{1}{k^2}, \tag{2.22}$$

$$\int_0^{+\infty} |G(t_2,s) - G(t_1,s)|\, ds \leq \frac{2}{k^2}\,|t_2 - t_1| \ \text{ for all } t_2, t_1 \geq 0, \tag{2.23}$$

Properties (2.17)-(2.21) and (2.22) are obvious and Property (2.23) is obtained from Property (2.21) for each of the cases $t_2 \geq t_1$ and $t_2 \leq t_1$.

Lemma 2.1. *For all functions v in E_0, $u(t)=\int_0^{+\infty}G(t,s)v(s)ds$ is the unique solution of the bvp*

$$\begin{cases} -u'''(t)+k^2u'=v, \ \ in \ \ (0,+\infty) \\ u(0)=u'(0)=u'(+\infty)=0. \end{cases} \tag{2.24}$$

Moreover u belongs to E_1.

Proof. Let $v\in E_0$. For any $t\geq 0$ we have by Property (2.21),

$$|u(t)|=\left|\int_0^{+\infty}G(t,s)v(s)ds\right|\leq \|v\|_2\int_0^{+\infty}G(t,s)ds<\infty.$$

Furthermore, for any $t_1,t_2\geq 0$, we have by Property (2.23),

$$\begin{aligned} |u(t_2)-u(t_1)| &= \left|\int_0^{+\infty}G(t_2,s)v(s)ds-\int_0^{+\infty}G(t_1,s)v(s)ds\right| \\ &\leq \int_0^{+\infty}|G(t_2,s)-G(t_1,s)|\,ds\,\|v\|_2 \\ &\leq \frac{2\|v\|_2}{k^2}|t_2-t_1|. \end{aligned}$$

The above estimates show that u is well defined and u is continuous on $\mathbb{R}^+$.

Differentiating three times in the identity

$$u(t)=-\frac{e^{-kt}}{k^2}\int_0^t\sinh(ks)v(s)\,ds+\frac{1}{k^2}\int_0^t(1-e^{-ks})v(s)\,ds+\frac{\cosh(kt)-1}{k^2}\int_t^{+\infty}e^{-ks}v(s)\,ds,$$

we find

$$u'(t)=\frac{1}{k}\left(e^{-kt}\int_0^t\sinh(ks)v(s)\,ds+\sinh(kt)\int_t^{+\infty}e^{-ks}v(s)\,ds\right),$$

$$u''(t)=-e^{-kt}\int_0^t\sinh(ks)v(s)\,ds+\cosh(kt)\int_t^{+\infty}e^{-ks}v(s)\,ds,$$

$$\begin{aligned} u'''(t) &= k\left(e^{-kt}\int_0^t\sinh(ks)v(s)\,ds+\sinh(kt)\int_t^{+\infty}e^{-ks}v(s)\,ds\right)-v(t) \\ &= k^2u'(t)-v(t). \end{aligned}$$

Hence, u satisfies $-u'''(t)+k^2u'=v$. Since (4.4) gives $u(0)=u'(0)=0$, it remains to prove that $\lim_{t\to+\infty}u'(t)=\lim_{t\to+\infty}\frac{u(t)}{1+t}=0$. We have

$$u'(t)=\int_0^{+\infty}\frac{\partial G}{\partial t}(t,s)v(s)ds=\frac{1}{k}e^{-kt}\int_0^t\sinh(ks)v(s)ds+\frac{1}{k}\sinh(kt)\int_t^{+\infty}e^{-ks}v(s)ds.$$

Using L'Hopital's formula, we obtain

$$\lim_{t\to+\infty} e^{-kt}\int_0^t \sinh(ks)v(s)ds = \lim_{t\to+\infty}\frac{\int_0^t \sinh(ks)v(s)ds}{e^{kt}} = \lim_{t\to+\infty}\frac{\sinh(kt)}{ke^{kt}}v(t) = 0$$

and

$$\begin{aligned}\lim_{t\to+\infty}\left(\sinh(kt)\int_t^{+\infty} e^{-ks}v(s)ds\right) &= \lim_{t\to+\infty}\frac{\sinh(kt)}{e^{kt}}\frac{\int_t^{+\infty} e^{-ks}v(s)ds}{e^{-kt}}\\ &= \lim_{t\to+\infty}\frac{\int_t^{+\infty} e^{-ks}v(s)ds}{e^{-kt}} = \lim_{t\to+\infty}\frac{v(t)}{k} = 0.\end{aligned}$$

This completes the proof. □

Lemma 2.2. *Assume for $i = 1$ or 2 the function $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a Γ_i-Caratheodory. Then the operator $T_g^i : E_i \to E_i$ where for $u \in E_i$, $T_g^i u(t) = \int_0^{+\infty} G(t,s)g(s,u(s))ds$, is well defined and if $g(t,x) \geq 0$ for all $t, x \geq 0$ then $T_g^i(K_i) \subset K_i$. Moreover, if $u \in E_i$ is a fixed point of T_g^i then u is a solution to the bvp*

$$\begin{cases} -u'''(t) + k^2u' = g(t,u), \text{ in } (0,+\infty) \\ u(0) = u'(0) = u'(+\infty) = 0. \end{cases} \tag{2.25}$$

Proof. Since $\Gamma_2 \subset \Gamma_1$, in both the cases $i = 1$ or 2, g is a Γ_1-Caratheodory function. Hence for any $u \in E_i$, $g(t,u)$ belongs to E_0 and $T_g^i u$ belongs to E_1 and satisfies the bvp (2.24) within $v = g(t,u)$. In the case $i = 2$, for $u \in E_2$ we have $g(t,u)$ belongs to Γ_2 (i.e. $\int_0^{+\infty} g(s,u(s))ds < \infty$). Therefore, Lebesgues convergence theorem and Property (2.20) lead to

$$\lim_{t\to+\infty} T_g^2 u(t) = \frac{1}{k^2}\int_0^{+\infty}\left(1-e^{-ks}\right)g(s,u(s))ds \leq \frac{1}{k^2}\int_0^{+\infty} g(s,u(s))ds < \infty.$$

This shows that T_g^2 is well defined.

At the end, it follows from Lemma 2.1 that if $u \in E_i$ is a fixed point of T_g^i then u is a solution to the bvp (2.25) and it is easy to see that if g is nonnegative then $T_g^i(K_i) \subset K_i$ for $i = 1, 2$. □

Lemma 2.3. *Assume for $i = 1$ or 2 the function $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a Γ_3-Caratheodory. Then the operator $T_g^3 : E_3 \to E_3$ where for $u \in E_3$, $T_g^3 u(t) = \int_0^{+\infty} G(t,s)g(s,u(s))ds$, is well defined and if $g(t,x) \geq 0$ for all $t, x \geq 0$ then $T_g^3(K_3) \subset K_3$. Moreover, if $u \in E_3$ is a fixed point of T_g^3 then u is a solution to the bvp (2.25).*

Proof. Since g is a Γ_3-Caratheodory function, for any $u \in E_3$ we have $|g(t,u)|$ belongs to Γ_1 (i.e. $\lim_{s\to+\infty} g(s,u(s)) = 0$). Hence Lemma 2.1 guarantees that $T_g^3 u \in E_1$ and and satisfies the bvp (2.24) within $v = g(t,u)$. Furthermore, for any $u \in E_3$ we have

$$e^{-kt}\left|T_g^3 u(t)\right| \leq \sup_{s\geq 0}|g(s,u(s))|\left(e^{-kt}\int_0^{+\infty} G(t,s)ds\right) \to 0 \text{ as } t \to +\infty.$$

This shows that T_g^3 is well defined.

Clearly, if $u \in E_3$ is a fixed point of T_g^3 then u is a solution to the bvp (2.25). So let us prove that if g is nonnegative then $T_g^3(K_3) \subset K_3$.

Let $u \in E_3$, taking in consideration Lemma 2.3 in [27], we obtain

$$\begin{aligned} T_g^3 u(t) &= \int_0^t \frac{dT_g^3 u}{dt}(\xi)d\xi = \int_0^t \int_0^{+\infty} \frac{\partial G}{\partial t}(\xi,s)g(s,u(s)dsd\xi \\ &= \int_0^t e^{k\xi}\int_0^{+\infty} e^{-k\xi}\frac{\partial G}{\partial t}(\xi,s)g(s,u(s)dsd\xi \\ &\geq \int_0^t \int_0^{+\infty} e^{k\xi}\widetilde{\gamma}(\xi)e^{-k\tau}\frac{\partial G}{\partial t}(\tau,s)g(s,u(s)dsd\xi \\ &\geq \left(\int_0^t e^{k\xi}\widetilde{\gamma}(\xi)d\xi\right)\left(e^{-k\tau}\int_0^{+\infty}\frac{\partial G}{\partial t}(\tau,s)g(s,u(s)ds\right) \end{aligned}$$

where $\widetilde{\gamma}(\xi) = \left(e^{2k\xi}-1\right)e^{-4k\xi}$. This leads to

$$T_g^3 u(t) \geq \left(\int_0^t e^{k\xi}\widetilde{\gamma}(\xi)d\xi\right)\left\|\frac{dT_g^3 u}{dt}\right\|_3. \tag{2.26}$$

Because $\frac{dT_g^3 u}{dt} \in E_3$, we have

$$\begin{aligned} T_g^3 u(t) = \int_0^t \tfrac{dT_g^3 u}{dt}(\xi)d\xi = \int_0^t e^{k\xi}\left(e^{-k\xi}\tfrac{dT_g^3 u}{dt}(\xi)\right)d\xi \leq \int_0^t e^{k\xi}d\xi\left\|\tfrac{dT_g^3 u}{dt}\right\|_3 \\ \leq \tfrac{\left(e^{kt}-1\right)}{k}\left\|\tfrac{dT_g^3 u}{dt}\right\|_3 \leq \tfrac{e^{kt}}{k}\left\|\tfrac{dT_g^3 u}{dt}\right\|_3. \end{aligned}$$

Leading to

$$\left\|\frac{dT_g^3 u}{dt}\right\|_3 \geq k\left\|T_g^3 u\right\|_3. \tag{2.27}$$

Combining (2.26) with (2.27), we obtain

$$T_g^3 u(t) \geq \gamma(t)\left\|T_g^3 u\right\|_3.$$

Ending the proof. □

The following lemma is an adapted version for the case of the space $E_i,\ i=1,2,3,$ of Cordeneanu's compactness criterion ([19], p. 62). It will be used in this work to prove that some operators are completely continuous.

Lemma 2.4. *A nonempty subset M of E_i $i=1,2,3,$ is relatively compact if the following conditions hold:*

(a) M is bounded in E_i,

(b) the set $\left\{u : u(t)=\dfrac{x(t)}{p_i\left(t\right)},\ x\in M\right\}$ is locally equicontinuous on $[0,+\infty)$, and

(c) the set $\left\{u : u(t)=\dfrac{x(t)}{p_i\left(t\right)},\ x\in M\right\}$ is equiconvergent at $+\infty$.

Lemma 2.5. *Let $g:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ be a Γ_1-Caratheodory function. The operator T_g^1 is completely continuous.*

Proof. Let us prove first that the operator T_g^1 is continuous. To this aim let (u_n) be a sequence in E_1 with $\lim u_n=u$ in E_1, and let $R>0$ and $\psi_R\in\Gamma_2\subset\Gamma_0$ be such that $\|u_n\|_1\le R$ for all $n\ge 1$ and

$$\left|g\left(t,p_1(t)\left(\frac{u}{p_1(t)}\right)\right)\right|\le\psi_R(t)\text{ for all }t\ge 0\text{ and }\left(\frac{u}{p_1(t)}\right)\in[-R,R]\,.$$

We have then

$$\left\|T_g^1u_n-T_g^1u\right\|_1=\sup_{t\ge 0}\frac{\left|T_g^1u_n\left(t\right)-T_g^1u\left(t\right)\right|}{p_1(t)}\le\sup_{t\ge 0}\Phi_n(t),$$

where

$$\begin{aligned}
\Phi_n(t) &= \frac{1}{p_1(t)}\int_0^{+\infty}G(t,s)\left|g(s,u_n(s))-g(s,u(s))\right|ds\\
&= \frac{1}{1+t}\int_0^{+\infty}G(t,s)\left|g\left(s,p_1(s)\left(\frac{u_n\left(s\right)}{p_1(s)}\right)\right)-g\left(s,p_1(s)\left(\frac{u\left(s\right)}{p_1(s)}\right)\right)\right|ds\\
&\le \frac{2}{p_1(t)}\int_0^{+\infty}G(t,s)\psi_R(s)ds\\
&\le \left\|\psi_R\right\|_2\sup_{t\ge 0}\left(\frac{2}{p_1(t)}\int_0^{+\infty}G(t,s)ds\right)=\frac{2\left\|\psi_R\right\|_2}{k^2}.
\end{aligned}$$

Let (t_n) be such that $\Phi_n(t_n)=\sup_{t\ge 0}\Phi_n(t)$ and let (t_{n_l}) be such that $\lim\Phi_{n_l}(t_{n_l})=\limsup\Phi_n(t_n)$. Therefore, we have to prove that $\lim\Phi_{n_l}(t_{n_l})=0$. We distinguish then two cases:

$\cdot$) (t_{n_l}) is bounded by $c > 0$; In this case we have

$$\begin{aligned}\Phi_{n_l}(t_{n_l}) &= \left(\frac{1}{p_1(t_{n_l})}\int_0^{+\infty} G(t_{n_l},s)\left|g(s,u_{n_l}(s)) - g(s,u(s))\right| ds\right)\\ &\leq \int_0^{+\infty} G(c,s)\left|g(s,u_{n_l}(s)) - g(s,u(s))\right| ds,\end{aligned}$$

$$\lim_{n\to+\infty} G(c,s)\left|g(s,u_n(s)) - g(s,u(s))\right| = 0,$$

$$\begin{aligned}\left|g(s,u_n(s)) - g(s,u(s))\right| &= \left|g\left(t, p_1(s)\left(\frac{u_n(s)}{p_1(s)}\right)\right) - g\left(t, p_1(s)\left(\frac{u(s)}{p_1(s)}\right)\right)\right|\\ &\leq 2\psi_R(s),\end{aligned}$$

for all $s > 0$ and by (2.21) $\int_0^{+\infty} G(c,s)\psi_R(s)ds < \infty$. Hence the dominated convergence theorem leads to $\lim \Phi_{n_l}(t_{n_l}) = \limsup \Phi_n(t_n) = 0$.

$\cdot\cdot$) $\lim t_{n_l} = +\infty$ (up to a subsequence); In this case we have from Lemma 2.2,

$$\begin{aligned}\Phi_{n_l}(t_{n_l}) &= \left(\frac{1}{p_1(t_{n_l})}\int_0^{+\infty} G(t_{n_l},s)\left|g(s,u_{n_l}(s)) - g(s,u(s))\right| ds\right)\\ &\leq \frac{2}{p_1(t_{n_l})}\int_0^{+\infty} G(t_{n_l},s)\psi_R(s)ds \to 0 \text{ as } l\to\infty.\end{aligned}$$

Thus, we have proved that $\lim T_g^1 u_{n_l} = T_g^1 u$ in E_1 and T_g^1 is continuous.

Now we prove by means of Lemma 2.4 that T_g^1 maps bounded sets of E_1 into relatively compact sets of E_1. To this aim, let Ω be a subset of E_1 bounded by $R > 0$ and let $\psi_R \in \Gamma_1$ be such that

$$|g(s,p_1(s)u)| \leq \psi_R(s) \text{ for all } s \geq 0 \text{ and all } u \in [-R,R]\,.$$

For any $u \in \Omega$ we have by Property (2.22),

$$\begin{aligned}\left\|T_g^1 u\right\|_1 &= \sup_{t\geq 0}\left|\frac{T_g^1 u(t)}{p_1(t)}\right| = \sup_{t\geq 0}\left(\frac{1}{p_1(t)}\int_0^{+\infty} G(t,s)\left|g\left(s, p_1(s)\left(\frac{u(s)}{p_1(s)}\right)\right)\right| ds\right)\\ &\leq \sup_{t\geq 0}\left(\frac{1}{p_1(t)}\int_0^{+\infty} G(t,s)\psi_R(s)ds\right)\\ &\leq \sup_{t\geq 0}\left(\frac{1}{p_1(t)}\int_0^{+\infty} G(t,s)ds\right)\left\|\psi_R\right\|_1 = \frac{1}{k^2}\left\|\psi_R\right\|_1.\end{aligned}$$

Hence $T_g^1(\Omega)$ is bounded in E_1.

Let $t_1, t_2 \in [\eta, \zeta] \subset \mathbb{R}^+$ with $t_1 \leq t_2$. For all $u \in \Omega$ we have

$$\begin{aligned}\left|\frac{T_g^1 u(t_2)}{p_1(t_2)} - \frac{T_g^1 u(t_1)}{p_1(t_1)}\right| \leq\ & \int_0^{t_1} \left|\frac{G(t_2,s)}{p_1(t_2)} - \frac{G(t_1,s)}{p_1(t_1)}\right| \psi_R(s)ds \\ & + \int_{t_1}^{t_2} \left|\frac{G(t_2,s)}{p_1(t_2)} - \frac{G(t_1,s)}{p_1(t_1)}\right| \psi_R(s)ds \\ & + \int_{t_2}^{+\infty} \left|\frac{G(t_2,s)}{p_1(t_2)} - \frac{G(t_1,s)}{p_1(t_1)}\right| \psi_R(s)ds,\end{aligned}$$

$$\begin{aligned}& \int_0^{t_1} \left|\frac{G(t_2,s)}{p_1(t_2)} - \frac{G(t_1,s)}{p_1(t_1)}\right| \psi_R(s)ds \\ \leq\ & \frac{1}{k^2}\left(\frac{e^{-kt_1}}{p_1(t_1)} - \frac{e^{-kt_2}}{p_1(t_2)}\right) \int_0^{\zeta} \sinh(ks)\psi_R(s)ds \\ & + \frac{1}{k^2}\left(\frac{1}{p_1(t_1)} - \frac{1}{p_1(t_2)}\right) \int_0^{\zeta} (1-e^{-ks})\psi_R(s)ds \\ \leq\ & \frac{C_1(k)}{k^2}\left(\int_0^{\zeta} \psi_R(s)ds\right)(t_2 - t_1)\end{aligned}$$

$$\begin{aligned}& \int_{t_1}^{t_2} \left|\frac{G(t_2,s)}{p_1(t_2)} - \frac{G(t_1,s)}{p_1(t_1)}\right| \psi_R(s)ds \\ \leq\ & \frac{1}{k^2}\int_{t_1}^{t_2} \left(\frac{e^{-kt_2}}{p_1(t_2)}\sinh(ks) + \frac{1-e^{-ks}}{p_1(t_2)} + \frac{\cosh(kt_1)-1}{p_1(t_1)} e^{-ks}\right) \psi_R(s)ds \\ \leq\ & \frac{C_2(k)}{k^2}(t_2 - t_1)\end{aligned}$$

and

$$\begin{aligned}& \int_{t_2}^{+\infty} \left|\frac{G(t_2,s)}{p_1(t_2)} - \frac{G(t_1,s)}{p_1(t_1)}\right| \psi_R(s)ds \\ \leq\ & \frac{1}{k^2}\left|\frac{\cosh(kt_2)-1}{p_1(t_2)} - \frac{\cosh(kt_1)-1}{p_1(t_1)}\right| \int_{\eta}^{+\infty} e^{-ks}\psi_R(s)e^{-ks}ds \\ \leq\ & \frac{C_3(k)}{k^2}(t_2 - t_1),\end{aligned}$$

where

$$C_1(k) = (k+1)\sinh(k\zeta) + 1,$$

$$C_2(k) = \left(\frac{\sinh(k\zeta)e^{-k\eta}}{1+\eta} + 1 + \frac{\cosh(k\zeta)-1}{1+\zeta}\right) \sup_{s\in[\eta,\zeta]} \psi_R(s),$$

$$C_3(k) = \sup_{t\in[\eta,\zeta]} \left(\frac{\cosh(kt)-1}{1+t}\right)'.$$

All the above calculations lead to

$$\left|\frac{T_g^1 u\left(t_2\right)}{p_1(t_2)}-\frac{T_g^1 u\left(t_1\right)}{p_1(t_1)}\right| \leq \frac{C_1(k)+C_2(k)+C_3(k)}{k^2}\left(t_2-t_1\right).$$

Hence $T_g^1(\Omega)$ is equicontinuous on compact intervals of $\mathbb{R}^+$.

We have for all $u \in \Omega$ and $t \geq 0$

$$\begin{aligned} \frac{\left|T_g^1 u(t)\right|}{1+t} &\leq \int_0^{+\infty} \frac{G(t,s)}{1+t}\,|g(s,u(s))|\,ds \\ &\leq \frac{1}{1+t}\int_0^{+\infty} G(t,s)\psi_R(s)ds := \widetilde{H}(t). \end{aligned}$$

Since Lemma 2.2 guarantees that $\lim_{t\to+\infty}\widetilde{H}(t)=0$, we conclude that $T_g^1(\Omega)$ is equiconvergent at $+\infty$. This ends the proof. □

Lemma 2.6. *Let $g:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ be a Γ_2-Caratheodory function. Then the operator T_g^2 is completely continuous.*

Proof. First, let us prove that T_g^2 is continuous. To this aim let (u_n) be a sequence in E_2 with $\lim u_n = u$ in E_2, and let $R>0$ and ψ_R be such that $\|u_n\|_2 \leq R$ for all $n\geq 1$ and $|g(t,p_2(t)u)| \leq \psi_R(t)$ for all $t\geq 0$ and $u\in[-R,R]$. Hence we have

$$\left\|T_g^2 u_n - T_g^2 u\right\|_2 = \sup_{t\geq 0}\left|T_g^2 u_n\left(t\right)-T_g^2 u\left(t\right)\right| \leq \int_0^{+\infty} G(\infty,s)\left|g(s,u_n(s))-g(s,u(s))\right|ds$$

with

$$\lim_{n\to+\infty}|g(s,u_n(s))-g(s,u(s))|=0$$

and

$$|g(s,u_n(s))-g(s,u(s))| = |g(s,p_2(s)u_n(s))-g(s,p_2(s)u(s))| \leq 2\psi_R(s).$$

for all $s>0$. Since $\psi_R\in L^1(\mathbb{R}^+)$, we conclude by means of the dominated convergence theorem that $\lim T_g^2 u_n = T_g^2 u$ in E_2, proving the continuity of T_g^2.

Now we prove by means of Lemma 2.4 that T_g^2 maps bounded sets of E_2 into relatively compact sets of E_2. To this aim, let Ω be a subset of E_2 bounded by a constant $R>0$ and let $\psi_R\in\Gamma_2$ be such that

$$|g(s,p_2(s)u)| \leq \psi_R(s) \text{ for all } s\geq 0 \text{ and all } u\in[-R,R].$$

Hence for all $u \in \Omega$, we have by Property (2.18) and (2.20)

$$\left\|T_g^2 u\right\|_2 \leq \sup_{t\geq 0} \int_0^{+\infty} G(t,s)\left|g(s,u(s))\right| ds = \sup_{t\geq 0} \int_0^{+\infty} G(t,s)\left|g(s,p_2(s)u(s))\right| ds$$
$$\leq \int_0^{+\infty} G(\infty,s)\psi_R(s)ds < \infty.$$

This estimate proves that $T_g^2(\Omega)$ is bounded in E_2.

Let $t_1, t_2 \in [\eta, \zeta] \subset \mathbb{R}^+$ and $u \in \Omega$. We obtain from Property (2.23) of the function G that

$$\left|T_g^2 u(t_2) - T_g^2 u(t_1)\right| \leq \int_0^{+\infty} \left|G(t_2,s) - G(t_1,s)\right| ds \left\|\psi_R\right\|_1 \leq \frac{2\left\|\psi_R\right\|_1}{k^2}\left|t_2 - t_1\right|.$$

Proving that $T_g^2(\Omega)$ is equicontinuous on compact intervals of $\mathbb{R}^+$.

We have for all $u \in \Omega$ and $t \geq 0$

$$\left|T_g^2 u(\infty) - T_g^2 u(t)\right| \leq \int_0^{+\infty} \left(G(\infty,s) - G(t,s)\right)\psi_R(s)ds := H(t).$$

Taking in account Property (2.20) and the fact that

$$\left(G(\infty,s) - G(t,s)\right)\psi_R(s) \leq \frac{1}{k^2}\psi_R(s) \text{ for all } s > 0,$$

where $\psi_R \in L^1(\mathbb{R}^+)$, we obtain by means of the dominated convergence theorem that $\lim_{t\to+\infty} H(t) = 0$. Thus $T_g^2(\Omega)$ is equiconvergent at $+\infty$ and the proof is complete. □

Lemma 2.7. *Let $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a Γ_3-Caratheodory function with $i = 1$ or 2. Then the operator T_g^3 is completely continuous.*

Proof. Observe that since g is Γ_3-Caratheodory, for all $u \in E_3$ we have $T_g^3 u \in E_1$. Therefore considering the operator $T_g^{1,3} : E_3 \to E_1$ with $T_g^{1,3}u(t) = T_g^3 u(t)$ and arguing as in the proofs of Lemmas 2.5, we obtain that $T_g^{1,3}$ is completely continuous. Since $T_g^3 = I_1 \circ T_g^{1,3}$, where I_1 is the continuous embeding of E_1 in E_3, we have that T_g^3 is completely continuous. □

We obtain from Lemmas 2.5, 2.6 and 2.7 the following fixed point formulation for the bvp (2.6).

Corollary 2.4. *Assume that the nonlinearity f is a Γ_i-Caratheodory function for some $i \in \{1,2,3\}$. Then $u_i \in E_i$ is a positive solution to the bvp (2.6) if and only if u_i is a fixed point of T_f^i where $T_f^i : K_i \to K_i$ is completely continuous.*

2.5 Proofs of main results

2.5.1 Auxilliary results

Let for $q \in \Delta_i$ with $i = 1, 2, 3$, $L_q^i : E_i \to E_i$ be the linear operator defined by

$$L_q^i u(t) = \int_0^{+\infty} G(t,s)q(s)u(s)ds \text{ for } u \in E_i.$$

We have from Lemmas 2.5, 2.6 and 2.7 that for $i = 1, 2, 3$, the linear operator L_q^i is compact. The main goal of this subsection is to prove that for $i = 1, 2, 3$, the operator L_q^i has the SIJP at its spectral radius $r(L_q^i)$ and in particular, L_q^3 is lower bounded on K_3. These results are requirement of Proposition 1.15, Theorem 1.19 and Theorem 1.20, and so are needed for the proofs of the main results of this article. We start by introducing some notations.

Let for $T > 0$, $G_T : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$G_T(t,s) = \begin{cases} G(t,s) \text{ if } t \leq T \\ G(T,s) \text{ if } t \geq T. \end{cases}$$

and for $i = 1, 2$,

$$\begin{aligned} E_T &= \left\{u \in C\left(\mathbb{R}^+\right) : u(0) = 0 \text{ and } u(t) = u(T) \text{ for } t \geq T\right\}, \\ X_T &= \left\{u \in E_T \cap C^2[0,T] : u'(0) = 0\right\}, \\ Y_T &= X_T \cap C^3[0,T]. \end{aligned}$$

Equipped respectively with the norms

$$\|u\|_T = \sup_{t\in[0,T]} |u(t)| \text{ for all } u \in E_T,$$
$$\|u\|_X = max(\|u\|_T, \|u'\|_T, \|u''\|_T) \text{ for all } u \in X_T \text{ and}$$
$$\|u\|_Y = max(\|u\|_X, \|u'''\|_T) \text{ for all } u \in Y_T,$$

E_T, X_T and Y_T become Banach spaces.

In what follows E_T^+ and X_T^+ denote respectively the cones of nonnegative functions in the Banach spaces E_T and X_T.

Let for $q \in \Delta$ and $T > 0$, $L^i_{q,T} : E_i \to E_i$, $L_{q,T} : E_T \to E_T$, $A_{q,T} : X_T \to X_T$, $\widetilde{L}_{q,T} : E_T \to Y_T$, and $\widetilde{A}_{q,T} : X_T \to Y_T$ be the linear bounded operators defined by

$$\begin{aligned}
&L^i_{q,T}u(t) = \textstyle\int_0^{+\infty} G_T(t,s)q(s)u(s)ds \text{ for } u \in E_i,\\
&\widetilde{L}_{q,T}u = L_{q,T}u = L^i_{q,T}u \text{ for } u \in E_T \text{ and}\\
&A_{q,T}u(t) = \widetilde{A}_{q,T}u = L_{q,T}u \text{ for } u \in X_T.
\end{aligned}$$

Let I, J be respectively the compact embedding of Y_T into E_T and Y_T into X_T. Since $L_{q,T} = I \circ \widetilde{L}_{q,T}$ and $A_{q,T} = J \circ \widetilde{A}_{q,T}$, we have that $L_{q,T}$ and $A_{q,T}$ are compact operators. Moreover, arguing as in the proofs of Lemmas 2.5 and 2.6, we obtain that for $i = 1, 2$, $L^i_{q,T}$ is a compact operator.

Lemma 2.8. *The set O_T defined by*

$$O_T = \{u \in X_T : u' > 0 \textit{ in } (0,T] \textit{ and } u''(0) > 0\},$$

is open in the Banach space X_T.

Proof. We have $O_T^c = F_1 \cup F_2$ where

$$\begin{aligned}
F_1 &= \{u \in X_T : \ u'(t_0) \leq 0 \text{ for some } t_0 \in (0,T]\},\\
F_2 &= \{u \in X_T : u''(0) \leq 0\}.
\end{aligned}$$

Since F_2 is a closed set in X_T, we have to show that $\overline{F_1} \subset F_1 \cup F_2$. To this aim, let $(u_n) \subset F_1$ with $\lim u_n = u$ and let $(x_n) \subset (0,T]$ be such $u'(x_n) \leq 0$ and $\lim x_n = \overline{x}$. We distinguish the following two cases:

Case 1. $\overline{x} \in (0,T]$; In this case we have

$$u'(\overline{x}) = \lim u'_n(x_n) \leq 0,$$

proving that $u \in F_1$.

Case 2. $\overline{x} = 0$; In this case we have

$$u''(0) = \lim_{n\to\infty} \frac{u'_n(x_n)}{x_n} \leq 0,$$

proving that $u \in F_2$. □

Lemma 2.9. *For $i = 1$ or 2, q in Δ_i and $T > 0$, the operator $L^i_{q,T}$ has the SIJP at its spectral radius $r(L^i_{q,T})$.*

Proof. First, we prove that the operator $A_{q,T}$ is strongly positive. Let $u \in X_T^+ \setminus \{0\}$ and $v = A_{q,T}u$, we have from Property (2.18) of the function G that

$$v'(t) = \int_0^T \frac{\partial G_T}{\partial t}(t,s)q(s)u(s)ds > 0 \text{ for all } t \in (0,T). \tag{2.28}$$

Moreover, we have

$$v''(0) = \int_0^T \frac{\partial^2 G_T}{\partial t^2}(t,s)q(s)u(s)ds > 0. \tag{2.29}$$

Clearly, (2.28) and (2.29) show that $v = A_{q,T}u \in O_T \subset int\left(X_T^+\right)$, proving that $A_{q,T}\left(X_T^+ \setminus \{0\}\right) \subset O_T \subset int\left(X_T^+\right)$ and $A_{q,T}$ is strongly positive. Therefore, we conclude from Proposition 1.12 that the operator $A_{q,T}$ has the SIJP at $r(A_{q,T})$.

Now, we are able to prove that the operator $L_{q,T}$ has the SIJP at $r(L_{q,T})$. Let $\mu_0 > 0$ and $u \in E_T^+ \setminus \{0\}$ such that $L_{q,T}u \geq \mu_0 u$, then $U = L_{q,T}u \in X_T^+ \setminus \{0\}$ and satisfies $L_{q,T}U = A_{q,T}U \geq \mu_0 U$. Hence, we have that $\mu_0 \in \Lambda_{A_{q,T}}$ and $\mu_0 \leq \sup \Lambda_{A_{q,T}} = r(A_{q,T})$.

Similarly if $\eta_0 \geq 0$ and $v \in E_T^+ \setminus \{0\}$ are such that $L_{q,T}v \leq \eta_0 v$, then $V = L_{q,T}v \in X_T^+ \setminus \{0\}$ and satisfies $L_{q,T}V = A_{q,T}V \leq \eta_0 V$. Therefore, we have that $\eta_0 \in \Gamma_{A_{q,T}}$ and $\eta_0 \geq \inf \Gamma_{A_{q,T}} = r(A_{q,T})$.

Therefore, we have proved that

$$\sup \Lambda_{L_{q,T}} \leq r\left(A_{q,T}\right) = \inf \Gamma_{A_{q,T}} = \sup \Lambda_{A_{q,T}} \leq \inf \Gamma_{L_{q,T}}$$

and this combined with (1.46) leads to $\inf \Gamma_{L_{q,T}} = \sup \Lambda_{L_{q,T}} = r(A_{q,T})$ and $L_{q,T}$ has the SIJP at $r(A_{q,T})$. Since the cone E_T^+ is total in the Banach space E_T, we have that $r\left(L_{q,T}\right)$ is a positive eigenvalue. Hence taking in consideration Remark 2.1, we obtain that $r\left(L_{q,T}\right) = r\left(A_{q,T}\right)$ and $L_{q,T}$ has the SIJP at $r(L_{q,T})$.

Noticing that for all $u \in K_i \setminus \{0\}$, $U = L^i_{q,T}u \in E_T^+ \setminus \{0\}$ and $L^i_{q,T}U = L_{q,T}U$, then arguing as above we obtain that $L^i_{q,T}$ has the SIJP at $r(L^i_{q,T})$. Ending the proof. □

Theorem 2.4. *For $i = 1$ or 2 and q in Δ_i the operator L^i_q has the SIJP at its spectral radius $r(L^i_q)$.*

Proof. In order to make the use of Theorem 1.16 possible we prove that for a function q in Δ_i, $T \to L^i_{q,T}$ is increasing and $\lim_{T\to+\infty} L^i_{q,T} = L^i_q$. Let q in Δ_i and T_1, T_2 be such that $0 < T_1 < T_2 < \infty$. For $u \in K_i$ we have

$$L^i_{q,T_2}u(t) - L^1_{q,T_1}u(t) = \begin{cases} \int_0^{+\infty} (G(t,s) - G(t,s))\, q(s)u(s)ds = 0 \text{ if } t \le T_1 \\ \int_0^{T_1} (G(t,s) - G(T_1,s))\, q(s)u(s)ds \ge 0 \text{ if } T_1 < t \le T_2 \\ \int_0^{T_1} (G(T_2,s) - G_{T_1}(T_1,s))\, q(s)u(s)ds \ge 0 \text{ if } T_2 < t, \end{cases}$$

proving that $L^i_{q,T_2}u - L^i_{q,T_1}u \in K_i$ and $L^i_{q,T_1} \le L^i_{q,T_2}$.

If $i = 1$, for $u \in E_1$ with $\|u\|_1 = 1$ we have

$$\begin{aligned} \left|\tfrac{L^1_q u(t) - L^1_{q,T}u(t)}{p_1(t)}\right| &\le \tfrac{1}{1+t}\int_0^{+\infty} (G(t,s) - G_T(t,s))\, q(s)ds \\ &= \begin{cases} 0 \text{ if } t \le T \\ \tfrac{1}{1+t}\int_0^{+\infty} (G(t,s) - G(T,s))\, q(s)ds \text{ if } t \ge T. \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{t\ge0}\left|\tfrac{L^1_q u(t) - L^1_{q,T}u(t)}{1+t}\right| &= \sup_{t\ge T}\left(\tfrac{1}{1+t}\int_0^{+\infty} (G(t,s) - G(T,s))\, q(s)ds\right) \\ &\le \sup_{t\ge T}\left(\tfrac{1}{1+t}\int_0^{+\infty} G(t,s)q(s)ds\right). \end{aligned}$$

Since

$$\lim_{t\to+\infty}\left(\tfrac{1}{1+t}\int_0^{+\infty} G(t,s)q(s)ds\right) = 0,$$

we have

$$\begin{aligned} \lim_{T\to+\infty}\left(\sup_{\|u\|_1=1}\left\|L^1_q u - L^1_{q,T}u\right\|_1\right) &= \lim_{T\to+\infty}\left(\sup_{\|u\|_1=1}\left(\sup_{t\ge0}\left|\tfrac{L^1_q u(t) - L^1_{q,T}u(t)}{1+t}\right|\right)\right) \\ &\le \lim_{T\to+\infty}\left(\sup_{t\ge T}\left(\tfrac{1}{1+t}\int_0^{+\infty} G(t,s)q(s)ds\right)\right) = 0. \end{aligned}$$

Hence we obtain by Theorem 1.16 that the operator L^1_q has the SIJP at its spectral radius $r(L^1_q)$.

If $i = 2$, for $u \in E_2$ with $\|u\|_2 = 1$ we have

$$\begin{aligned} \left|L^2_q u(t) - L^2_{q,T}u(t)\right| &\le \int_0^{+\infty} (G(t,s) - G_T(t,s))\, q(s)ds \\ &= \begin{cases} 0 \text{ if } t \le T \\ \int_0^{+\infty} (G(t,s) - G(T,s))\, q(s)ds \text{ if } t \ge T. \end{cases} \end{aligned}$$

Hence we have

$$\left\| L_q^2 - L_{q,T}^2 \right\| = \sup_{\|u\|_2=1} \left\| L_q^2 u - L_{q,T}^2 u \right\|_2 \leq \int_0^{+\infty} \left(G(t,s) - G(T,s)\right) q(s) ds,$$

then by Lebesgue dominated convergence theorem we conclude that $L_{q,T}^2 \to L_q^2$ as $T \to +\infty$. By Theoerem 1.16 we obtain that the operator L_q^2 has the SIJP at its spectral radius $r(L_q^2)$. $\square$

Theorem 2.5. *For $i = 1$ or 2 and q in Δ_3 the operator L_q^3 has the SIJP at its spectral radius $r(L_q^3)$ and L_q^3 is lower bounded on the cone K_3.*

Proof. Notice first that for all $u \in K_3$, $L_q^3 u \in K_1$. Indeed, we have for $u \in K_3$ and for all $t > 0$

$$\frac{L_q^3 u(t)}{1+t} \leq \frac{||u||_3}{1+t} \int_0^{+\infty} G(t,s) \left(e^{ks} q(s)\right) ds \to 0 \text{ as } t \to +\infty,$$

since $\lim_{s \to +\infty} e^{ks} q(s) = 0$, and

$$\left(L_q^3 u\right)'(t) = \int_0^{+\infty} \frac{\partial G}{\partial t}(t,s) q(s) u(s) ds > 0.$$

Let now, $\lambda_0 > 0$ and $u \in K_3 \setminus \{0\}$ be such that $L_q^3 u \leq \lambda_0 u$. Then $U = L_q^3 u$ satisfies $L_q^1 U = L_q^3 U \leq \lambda_0 U$ and we have $\lambda_0 \geq \inf \Gamma_{L_q^1} = r(L_q^1)$. Similarly if $\theta_0 > 0$ and $u \in K_3 \setminus \{0\}$ are such that $L_q^3 u \geq \theta_0 u$ then $U = L_q^3 u \in K_1 \setminus \{0\}$ and satisfies $L_q^1 U = L_q^3 U \geq \theta_0 U$ and we have $\theta_0 \leq \sup \Lambda_{L_q^1} = r(L_q^1)$.

The above leads to $r(L_q^1) = \inf \Gamma_{L_q^1} = \sup \Lambda_{L_q^1}$and the operator L_q^3 has the SIJP at $r(L_q^1)$. Since the cone K_3 is total in the Banach space E_3 and Remark 2.1 claims that $r\left(L_q^1\right)$ is the unique positive eigenvalue of the positive operator L_q^3, we have that $r\left(L_q^3\right) = r\left(L_q^1\right)$ and L_q^3 has the SIJP at $r(L_q^3)$.

It remains to show that L_q^3 is lower bounded on K_3. Let $u \in K_3$, with $\|u\|_3 = 1$, we have then for all $t \geq 0$,

$$L_q^3 u(t) = \int_0^{+\infty} G(t,s) q(s) u(s) ds \geq \int_0^{+\infty} G(t,s) q(s) \gamma(s) ds,$$

leading to

$$\inf \left\{ \left\| L_q^3 u \right\|_3 : u \in K_3 \cap \partial B(0_{E_3}, 1) \right\} \geq \sup_{t \geq 0} e^{-kt} \int_0^{+\infty} G(t,s) q(s) \gamma(s) ds > 0$$

and the operator L_q^3 is lower bounded on the cone K_3. This ends the proof. $\square$

2.5.2 Proof of Proposition 2.1

Let $q \in \Delta$, we have from Lemma 2.2 that μ is a positive eigenvalue of the linear eigenvalue problem (2.7) if and only if μ^{-1} is a positive eigenvalue of the compact operator L_q^i for $i = 1$ or 2. Since Theorem 2.4 claims that L_q^i has the SIJP at $r(L_q^i)$, we have from Remark 2.1 that $r(L_q^i)$ is the unique positive eigenvalue of L_q^i. Therefore, we have that $\mu(q) = 1/r(L_q^i)$ is the unique positive eigenvalue of the linear eigenvalue problem (2.7).

Now, let ϕ be the eigenfunction associated with $\mu(q)$. Clearly if $q \in \Delta_2$ then ϕ is bounded and if not then ϕ satisfies

$$\begin{aligned}\phi(t) = \int_0^{+\infty} G(t,s)q(s)\phi(s)ds &\geq \tfrac{1}{k^2}\int_1^t \left(-e^{-kt}\sinh(ks) + (1-e^{-ks})\right) q(s)\phi(s)ds \\ &\geq \tfrac{\left(1-e^{-k}\right)^2}{2k^2}\int_1^t q(s)\phi(s)ds \\ &\geq \tfrac{\left(1-e^{-k}\right)^2}{2k^2}\phi(1)\int_1^t q(s)ds.\end{aligned} \tag{2.30}$$

Thus, by the contrary if ϕ is bounded then passing to the limits in (2.30), we obtain the contradiction

$$+\infty > \lim_{t\to+\infty}\phi(t) = \lim_{t\to+\infty}\frac{\left(1-e^{-k}\right)^2}{2k^2}\phi(1)\int_1^t q(s)ds = +\infty.$$

Ending the proof.

2.5.3 Proof of Theorem 2.1

Assume that Hypothesis (2.8) holds true (the case where (2.9) holds is checked similarly). Let $\epsilon > 0$ be small such that for $i = 1, 2$,

$$\inf\left\{\frac{f(t,p_i(t)u)}{p_i(t)q(t)u} : t, u > 0\right\} \geq (\mu(q)+\epsilon).$$

Hence for all $u \in K_i$, we have

$$\begin{aligned} T_f^i u(t) &= \int_0^{+\infty} G(t,s)f(s,u(s))ds \\ &= \int_0^{+\infty} G(t,s)f(s,p_i(s)\frac{u(s)}{p_i(s)})ds \\ &\geq (\mu(q)+\epsilon)\int_0^{+\infty} G(t,s)q(s)u(s)ds \\ &= (\mu(q)+\epsilon)\, L_q^i u(t) := \widehat{L}_q^i u(t), \end{aligned}$$

and

$$r(\widehat{L}_q^i) = \frac{\mu(q) + \epsilon}{\mu(q)} > 1.$$

Since Theorems 2.4 and 2.5 state that the operator $\widehat{L}_q^i$ has the SIJP at $r(\widehat{L}_q^i)$, Hypothesis (1.64) holds and Proposition 1.15 guarantees that the operator T_f^i has no fixed point in K_i. At end, we conclude by Corollary 2.4 that the bvp (2.6) has no positive solution.

2.5.4 Proof of Theorem 2.2

Step 1. Existence in the case where (2.10) is satisfied:

Let $\epsilon \in \left(0, \mu(q_\infty) - f_{i,+\infty}^{+}(q_\infty)\right)$ there is R large such that

$$f(t, p_i(t)u) \leq (\mu(q_\infty) - \epsilon)\, p_i(t)q_\infty(t)u \text{ for all } t \geq 0 \text{ and } u \geq R.$$

Since the nonlinearity f is a Γ_i-Caratheodory function, there is $\psi_R \in \Gamma_i$ such that

$$f(t, p_i(t)u) \leq (\mu(q_\infty) - \epsilon)\, p_i(t)q_\infty(t)u + \psi_R(t) \text{ for all } t, u \geq 0,$$

and this leads to

$$f(t, u) \leq (\mu(q_\infty) - \epsilon)\, q_\infty(t)u + \psi_R(t) \text{ for all } t, u \geq 0. \tag{2.31}$$

Let $\varepsilon \in \left(0, f_{i,0}^{-}(q_0) - \mu(q_\infty)\right)$ there is $r > 0$ such that for all $t \geq 0$ and $u \in [0, r]$

$$\left(f_{i,0}^{-}(q_0) + \varepsilon\right) p_i(t)q_0(t)u \geq f(t, p_i(t)u) \geq (\mu(q_\infty) + \varepsilon)\, p_{1i}(t)q_0(t)u,$$

leading to

$$\left(f_{i,0}^{-}(q_0) + \varepsilon\right) q_0(t)u \geq f(t, u) \geq (\mu(q_\infty) + \varepsilon)\, q_0(t)u \text{ for all } t \geq 0 \text{ and } u \in [0, r].$$

Therefore, for all $t, u \geq 0$ we have

$$\left(f_{i,0}^{-}(q_0) + \varepsilon\right) q_0(t)u + \widehat{f}(t, u) \geq f(t, u) \geq (\mu(q_0) + \varepsilon)\, q_0(t)u - \widetilde{f}(t, u) \tag{2.32}$$

where

$$\begin{aligned} \widetilde{f}(t, u) &= \sup\left(0, (\mu(q_\infty) + \varepsilon)\, q_0(t)u - f(t, u)\right), \\ \widehat{f}(t, u) &= \sup\left(0, f(t, u) - \left(f_{i,0}^{-}(q_0) + \varepsilon\right) q_0(t)u\right). \end{aligned}$$

Therefore, we obtain from (2.31) and (2.32) that

$$T_f^i u \leq L_{q_\infty}^i u + F_\infty u \text{ for all } u \in K_i$$

and

$$L_{q_0}^i u - F_0 u \leq T_f^i u \leq L_{q_0}^i u + \widehat{F}_0 u \text{ for all } u \in K_i$$

where

$$F_0 u(t) = \int_0^{+\infty} G(t,s)\widetilde{f}(t,u(s))ds,$$
$$\widehat{F}_0 u(t) = \int_0^{+\infty} G(t,s)\widehat{f}(t,u(s))ds,$$
$$F_\infty u(t) = \int_0^{+\infty} G(t,s)\psi_R(s)\,ds,$$
$$r\left(L_{q_\infty}^i\right) = \frac{(\mu(q_\infty)-\epsilon)}{\mu(q_\infty)} < 1 < r\left(L_{q_0}^i\right) = \frac{(\mu(q_0)+\varepsilon)}{\mu(q_0)}.$$

We conclude from Theorem 2.4, Theorem 1.19 and Corollary 2.4 that the bvp (2.6) admits a positive solution $u \in K_i$.

Step 2. Existence in the case where (2.11) is satisfied:

Let $\epsilon \in \left(0, \mu_i(q_0) - f_{i,0}^+(q_0)\right)$ there is $r > 0$ small such that

$$f(t, p_i(t)u) \leq (\mu(q_\infty) - \epsilon)\, p_i(t) q_\infty(t) u \text{ for all } t \geq 0 \text{ and } u \leq r,$$

leading to

$$f(t,u) \leq (\mu(q_0) - \epsilon)\, q_0(t) u \text{ for all } t \geq 0 \text{ and } u \leq r.$$

Therefore, for all $t, u \geq 0$ we have

$$f(t,u) \leq (\mu(q_0) - \epsilon)\, q_0(t) u + \widehat{f}(t,u) \tag{2.33}$$

with

$$\widehat{f}(t,u) = \sup\left(0, f(t,u) - (\mu(q_0) - \epsilon)\, q_0(t) u\right).$$

Let $\varepsilon \in \left(0, f_{i,\infty}^-(q_\infty) - \mu_i(q_\infty)\right)$ there is $R > 0$ such that for all $t \geq 0$ and $u \geq R$,

$$(\mu(q_\infty) + \varepsilon)\, p_i(t) q_\infty(t) u \leq f(t, p_i(t)u) \leq \left(f_{i,\infty}^+(q_\infty) + \varepsilon\right) p_i(t) q_\infty(t) u,$$

Since the nonlinearity f is a Γ_i-Caratheodory function, there is $\psi_R \in \Gamma_i$ such that

$$f(t,u) \leq \left(f_{i,\infty}^+(q_\infty) + \varepsilon\right) q_\infty(t) p_i(t) u + \psi_R(t) \text{ for all } t, u \geq 0\ .$$

Therefore, for all $t, u \geq 0$ we have

$$\left(\mu_i\left(q_\infty\right)+\varepsilon\right) q_\infty(t) u-\widetilde{f}(t, u) \leq f(t, u) \leq\left(f_{i, \infty}^{+}\left(q_\infty\right)+\varepsilon\right) q_\infty(t) u+\psi_R(t) \tag{2.34}$$

where

$$\widetilde{f}(t, u)=\sup \left(0,\left(\mu\left(q_\infty\right)+\varepsilon\right) q_\infty(t) u-f(t, u)\right).$$

Therefore, we obtain from (2.33) and (2.34) that

$$T_f^i u \leq L_{q_0}^i u+F_0 u \text{ for all } u \in K_i$$

and

$$L_{q_\infty}^i u-F_\infty u \leq T_f^i u \leq L_{q_\infty}^i u+\widehat{F}_\infty u \text{ for all } u \in K_i$$

where

$$\begin{gathered} F_0 u(t)=\int_0^{+\infty} G(t, s) \widehat{f}(t, u(s)) d s, \\ \widehat{F}_\infty u(t)=\int_0^{+\infty} G(t, s) \psi_R(s) d s, \\ F_\infty u(t)=\int_0^{+\infty} G(t, s) \widetilde{f}(t, u(s)) d s, \\ r\left(L_{q_0}^i\right)=\frac{\left(\mu\left(q_\infty\right)-\epsilon\right)}{\mu\left(q_\infty\right)}<1<r\left(L_{q_\infty}^i\right)=\frac{\left(\mu\left(q_0\right)+\varepsilon\right)}{\mu\left(q_0\right)}. \end{gathered}$$

We conclude from Theorem 2.4, Theorem 1.19 and Corollary 2.4 that the bvp (2.6) admits a positive solution $u \in K_i$.

Step 3. Boundedness and unboundedness of the solution:

Evidently, if $i=1$ the solution u is bounded. If $i=2$ and Hypothesis (2.12) is fulfilled, then the solution u satisfies

$$\begin{aligned} u(t)=\int_0^{+\infty} G(t, s) f(s, u(s)) d s \geq & \frac{\left(1-e^{-k}\right)^2}{2 k^2} \int_1^t f(s, u(s)) d s \\ & =\frac{\left(1-e^{-k}\right)^2}{2 k^2} \int_1^t f\left(s, p_1(s)\left(\frac{u(s)}{p_1(s)}\right)\right) d s . \end{aligned} \tag{2.35}$$

Thus, by the contrary if the solution u is bounded then passing to the limits in (2.35), we obtain the contradiction

$$+\infty>\lim _{t \rightarrow+\infty} u(t)=\lim _{t \rightarrow+\infty} \frac{\left(1-e^{-k}\right)^2}{2 k^2} \int_1^t f\left(s, p_1(s)\left(\frac{u(s)}{p_1(s)}\right)\right) d s=+\infty .$$

Ending the proof.

2.5.5 Proof of Theorem 2.3

Step 1. Existence in the case where (2.13) is satisfied:

Let $\epsilon \in \left(0, \mu(q_\infty) - f^{+}_{i,3,\infty}(q_\infty)\right)$, there is R large such that

$$f(t, p_3(t)u) \leq (\mu\,(q_\infty) - \epsilon)\, p_3(t) q_\infty(t) u \text{ for all } t \geq 0 \text{ and } u \geq R.$$

Since the nonlinearity f is a Γ_3-Caratheodory function, there is $\psi_R \in \Gamma_1$ such that

$$f(t, p_3(t)u) \leq (\mu\,(q_\infty) - \epsilon)\, p_3(t) q_\infty(t) u + \psi_R\,(t) \;\text{ for all } t, u \geq 0,$$

and this leads to

$$f(t, u) \leq (\mu\,(q_\infty) - \epsilon)\, q_\infty(t) u + \psi_R\,(t) \;\text{ for all } t, u \geq 0 \tag{2.36}$$

Also, we have from $f^{-}_{3,0}(q_0) > \mu(q_0)$ that for $\varepsilon \in \left(0, f^{-}_{3,0}(q_0) - \mu(q_\infty)\right)$ there is $r > 0$ such that

$$f(t, p_3(t)u) \geq (\mu\,(q_\infty) + \varepsilon)\, p_3(t) q_0(t) u \text{ for all } t \geq 0 \text{ and } u \in [0, r]\,,$$

leading to

$$f(t, u) \geq (\mu\,(q_\infty) + \varepsilon)\, q_0(t) u \text{ for all } t \geq 0 \text{ and } u \in [0, r]\,.$$

Therefore we have

$$f(t, u) \geq (\mu\,(q_0) + \varepsilon)\, q_0(t) u - \widetilde{f}(t, u) \text{ for all } t, u \geq 0 \tag{2.37}$$

where

$$\widetilde{f}(t, u) = \sup\left(0, (\mu\,(q_\infty) + \varepsilon)\, q_0(t) u - f(t, u)\right).$$

Hence, we obtain from (2.36) and (2.37) that

$$L^3_{q_0} u - F_0 u \leq T^3_f u \leq L^3_{q_\infty} u + F_\infty u \text{ for all } u \in K_3$$

where

$$F_0 u(t) = \textstyle\int_0^{+\infty} G(t, s) \widetilde{f}(t, u\,(s)) ds,$$

$$F_\infty u(t) = \textstyle\int_0^{+\infty} G(t, s) \psi_R\,(s)\, ds,$$

$$r\left(L^3_{q_\infty}\right) = \frac{(\mu\,(q_\infty) - \epsilon)}{\mu\,(q_\infty)} < 1 < r\left(L^3_{q_0}\right) = \frac{(\mu\,(q_0) + \varepsilon)}{\mu\,(q_0)}.$$

We conclude from Theorem 2.5, Theorem 1.20 and Corollary 2.4 that the bvp (2.6) admits a positive solution.

Step 2. Existence in the case where (2.14) is satisfied:

Let $\epsilon \in \left(0, \mu(q_0) - f^{+}_{3,0}(q_0)\right)$, there is $r > 0$ small such that

$$f(t, p_3(t)u) \leq (\mu\,(q_0) - \epsilon)\, p_3(t)q_0(t)u \text{ for all } t \geq 0 \text{ and } u \leq r.$$

Hence for all $t, u \geq 0$ we have

$$f(t, u) \leq (\mu\,(q_0) - \epsilon)\, q_0(t)u + \widetilde{f}(t, u) \tag{2.38}$$

where

$$\widetilde{f}(t, u) = \sup\,(0, (f(t, u) - (\mu\,(q_0) - \epsilon)\, q_0(t)u)\,.$$

Let $\varepsilon \in \left(0, f^{-}_{3,\infty}(q_0) - \mu(q_\infty)\right)$ there is $R > 0$ largr such that

$$f(t, p_3(t)u) \geq (\mu\,(q_\infty) + \varepsilon)\, p_3(t)q_\infty(t)u \text{ for all } t \geq 0 \text{ and } u \geq R,$$

leading to

$$f(t, u) \geq (\mu\,(q_\infty) + \varepsilon)\, q_\infty(t)u \text{ for all } t \geq 0 \text{ and } u \geq R.$$

Therefore, we have

$$f(t, u) \geq (\mu\,(q_\infty) + \varepsilon)\, q_\infty(t)u - \widehat{f}(t, u) \text{ for all } t, u \geq 0 \tag{2.39}$$

where

$$\widehat{f}(t, u) = \sup(0, (\mu\,(q_\infty) + \varepsilon)\, q_\infty(t)u - f(t, u)).$$

Hence, we obtain from (2.38) and (2.39) that

$$L^3_{q_\infty}u - F_\infty u \leq T^3_f u \leq L^3_{q_0}u + F_0 u \text{ for all } u \in K_3$$

where

$$F_0 u(t) = \int_0^{+\infty} G(t, s)\widetilde{f}(t, u\,(s))ds,$$
$$F_\infty u(t) = \int_0^{+\infty} G(t, s)\widehat{f}(t, u\,(s))ds,$$
$$r\left(L^3_{q_0}\right) = \frac{(\mu\,(q_0) - \epsilon)}{\mu\,(q_0)} < 1 < r\left(L^3_{q_\infty}\right) = \frac{(\mu\,(q_\infty) + \varepsilon)}{\mu\,(q_\infty)}.$$

We conclude from Theorem 2.5, Theorem 1.20 and Corollary 2.4 that the bvp (2.6) admits a positive solution.

Step 3. Boundedness and unboundedness of the solution:

Evidently, if f is a Γ_4-Caratheodory function the solution u is bounded. If Hypothesis (2.15) is fulfilled, then the solution u satisfies

$$\begin{aligned} u(t) = \int_0^{+\infty} G(t,s) f(s,u(s))ds \geq \frac{\left(1-e^{-k}\right)^2}{2k^2} \int_1^t f(s,u(s))ds \\ = \frac{\left(1-e^{-k}\right)^2}{2k^2} \int_1^t f(s, p_3(s)\left(\frac{u(s)}{p_3(s)}\right))ds. \end{aligned} \tag{2.40}$$

Thus, by the contrary if the solution u is bounded then passing to the limits in (2.40) we obtain the contradiction

$$\lim_{t\to+\infty} u(t) = \lim_{t\to+\infty} \frac{\left(1-e^{-k}\right)^2}{2k^2} \int_1^t f(s, p_3(s)\left(\frac{u(s)}{p_3(s)}\right))ds = +\infty.$$

Ending the proof.

Chapter 3

Positive solution for singular third-order BVPs on the half line with first-order derivative dependence

3.1 Introduction and main results

Boundary value problems for third-order differential equations arise in many branches of physics and engineering where, for physical considerations, the positivity of the solution is required. For instance, Danziger and Elemergreen (see [31], p. 133) have studied the following third-order linear differential equations:

$$\begin{aligned} &\alpha_3 u''' + \alpha_2 u'' + \alpha_1 u' + (1+k)\, u = kc,\ \theta < c \text{ and} \\ &\alpha_3 u''' + \alpha_2 u'' + \alpha_1 u' + u = 0,\ \theta > c. \end{aligned} \tag{3.1}$$

These equations describe the variation of thyroid hormone with time. Here $u = u(t)$ is the concentration of thyroid hormone at time t and $\alpha_3, \alpha_2, \alpha_2, k$ and c are constants.

A reduced version of the Hodgkin–Huxley model was proposed by Nagumo. He suggested the class of third-order differential equation

$$u''' - cu'' + f'(u)u' - \frac{b}{c}u = 0 \tag{3.2}$$

as a model exhibiting many of the features of the Hodgkin–Huxley equations, where f is a regular function. The Hodgkin–Huxley model is a system of nonlinear differential equations that approximates the electrical characteristics of excitable cells such as neurons and cardiac myocytes. Recall that the Hodgkin–Huxley model describes the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axon.The model has played a vital role in biophysics and neuronal modelling. For more details of Nagumo's equations, we refer to the paper by McKeen [51].

The Kuramoto–Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + \frac{1}{2}u^2 = 0$$

arises in a wide variety of physical phenomena. It was introduced to describe pattern formulation in reaction diffusion systems, and to model the instability of flame front propagation (see Y. Kuramoto and T. Yamada [46] and D. Michelson [52]). A traveling wave solutions $u = \phi(x - ct)$ satisfies, after one integration, the third-order equation

$$\lambda\phi'''(x) + \phi'(x) + f(\phi) = 0, \tag{3.3}$$

where λ is a parameter depend on the constant c and f is an even function.

A three-layer beam is formed by parallel layers of different materials. For an equally loaded beam of this type, Krajcinovic in [44] proved that the deflection u is governed by the third order differential equation

$$-u''' + k^2u' = a, \tag{3.4}$$

where k and a are physical parameters depending on the elasticity of the layers.

Study of existence of positive solutions for third-order bvps has received a great deal of attention and was the subject of many articles, see, for instance, [29, 30, 32, 38, 37, 50, 58, 61, 65, 66, 67, 72], for the case of finite intervals and [1, 7, 8, 16, 24, 25, 26, 27, 41, 48, 49, 55, 60] for the case posed on the half-line. Naturally, in such boundary value problems, the nonlinearity may have a singular dependence on time or on the space variable. This was the case in the papers [8, 24, 25, 26, 49, 50, 61, 65, 66], which motivated this work.

We are concerned in this chapter by existence of a positive solution to the boundary value problem (bvp for short),

$$\begin{cases} -u'''(t) + k^2u'(t) = \phi\,(t)\,f(t, u(t), u'(t)),\ \ t > 0 \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases} \tag{3.5}$$

where k is a positive constant, $\phi : (0, +\infty) \to \mathbb{R}^+$ is a measurable function, $f : \mathbb{R}^+ \times (0, +\infty) \times (0, +\infty) \to \mathbb{R}^+$ is a continuous function and observe that the form of the differential equation in (3.5) is more general to those of (3.1)-(3.4). Here the constant k which may have a physical signification as in (3.4), will play an important role in finding a suitable framework for a fixed point formulation of bvp (3.5).

By positive solution to the bvp (3.5), we mean a function $u \in C^2(\mathbb{R}^+) \cap W^{3,1}(0, +\infty)$ such that $u > 0$ in $(0, +\infty)$ and $u(0) = u'(0) = \lim_{t\to+\infty} u'(t) = 0$, satisfying the differential equation in (3.5).

In all this chapter, we let

$$\begin{aligned}
&\gamma_1(t) = (e^{2kt} - 1)e^{-4kt},\\
&\widetilde{\gamma}(t) = k^* e^{kt}\gamma_1(t) = k^*\left(1 - e^{-kt}\right)(1 + e^{-kt})e^{-kt},\\
&\gamma(t) = \textstyle\int_0^t \widetilde{\gamma}(s)ds = \dfrac{k^*}{3k}\left(2 - 3e^{-kt} + e^{-3kt}\right) = \dfrac{k^*}{3k}\left(1 - e^{-kt}\right)^2(2 + e^{-kt})
\end{aligned}$$

where $k^* = \min(1, k)/2$ and we assume that the functions ϕ and f satisfy the following condition:

$$\begin{cases}
\text{for all } R > 0 \text{ there exists a function } \Psi_R : (0, +\infty) \times (0, +\infty) \to (0, +\infty)\\
\text{such that } \Psi_R \text{ nonincreasing following its two variables,}\\
f(t, e^{kt}w, e^{kt}z) \le \Psi_R(w, z) \text{ for all } t, w, z \ge 0 \text{ with } |(w, z)| \le R,\\
\lim_{s\to+\infty} \phi(s)\,\Psi_R\left(re^{-ks}\gamma(s), re^{-ks}\widetilde{\gamma}(s)\right) = 0 \text{ and}\\
\int_0^{+\infty} \phi(s)\,\Psi_R\left(re^{-ks}\gamma(s), re^{-ks}\widetilde{\gamma}(s)\right) ds < \infty \text{ for all } r \in (0, R].
\end{cases} \tag{3.6}$$

Remark 3.1. *Notice that functions m in $L^1(0, +\infty)$ do not satisfy $\lim_{t\to+\infty} m(t) = 0$. Indeed, the function*

$$m_0(t) = \begin{cases}
2n^4t - n(2n^4 - 1) \text{ if } t \in \left[n - \frac{1}{2n^3}, n\right]\\
-2n^4t + n(2n^4 + 1) \text{ if } t \in \left[n, n + \frac{1}{2n^3}\right]\\
0 \text{ if not}
\end{cases}$$

is integrable since $\int_0^{+\infty} m_0(t)dt \le \sum_{n\ge1} \frac{1}{n^2} < \infty$, and $\lim_{n\to+\infty} m_0(n) = \lim_{n\to+\infty} n = +\infty$.

Hence, the condition $\int_0^{+\infty} \phi(s)\,\Psi_R\left(re^{-ks}\gamma(s), re^{-ks}\widetilde{\gamma}(s)\right) ds < \infty$ in Hypothesis (3.6) does not imply that $\lim_{s\to+\infty} \phi(s)\,\Psi_R\left(re^{-ks}\gamma(s), re^{-ks}\widetilde{\gamma}(s)\right) = 0$.

Remark 3.2. *Observe that the case where the nonlinearity f satisfies the polynomial growth condition*

$$f(t,u,v) \leq C\left(1+u^{\sigma}+v^{\mu}\right)$$

with $c,\sigma,\mu>0$, $\lim_{s\to+\infty}\phi(s)=0$ and $\int_0^{+\infty}\phi(s)\,ds<\infty$, is a particular case where Condition (3.6) is satisfied.

Remark 3.3. *Notice that if Hypothesis (3.6) holds then $|\phi|_1=\int_0^{+\infty}\phi(s)\,ds<\infty$. Indeed, for $R=1$ we have*

$$\infty > \int_0^{+\infty}\phi(s)\,\Psi_1\left(e^{-ks}\gamma(s),e^{-ks}\widetilde{\gamma}(s)\right)ds \geq \Psi_1\left(\gamma^+,\gamma^+\right)|\phi|_1\,,$$

where $\gamma^+=\max_{s>0}\left(e^{-ks}\left(\gamma(s)+\widetilde{\gamma}(s)\right)\right)$.

The statement of the main result needs to introduce the following notations. Let

$$f^0=\limsup_{|(w,z)|\to 0}\left(\sup_{t\geq 0}\frac{f(t,e^{kt}w,e^{kt}z)}{w+z}\right),\qquad f^{\infty}=\limsup_{|(w,z)|\to+\infty}\left(\sup_{t\geq 0}\frac{f(t,e^{kt}w,e^{kt}z)}{w+z}\right),$$

$$f_0(\theta)=\liminf_{|(w,z)|\to 0}\left(\min_{t\in I_\theta}\frac{f(t,e^{kt}w,e^{kt}z)}{w+z}\right),\quad f_{\infty}(\theta)=\liminf_{|(w,z)|\to+\infty}\left(\min_{t\in J_\theta}\frac{f(t,e^{kt}w,e^{kt}z)}{w+z}\right),$$

where $|(w,z)|=|w|+|z|$, for $\theta>0$ $I_\theta=[0,\theta]$ and for $\theta>1$ $J_\theta=[1/\theta,\theta]$.

Let also,

$$\begin{aligned}
&\Gamma=\left(\Gamma_1+\Gamma_2\right)^{-1},\\
&\Theta_0(\theta)=\left(\Theta_{1,0}(\theta)+\Theta_{2,0}(\theta)\right)^{-1}\ \text{ if } \theta>0,\\
&\Theta_{\infty}(\theta)=\left(\Theta_{1,\infty}(\theta)+\Theta_{2,\infty}(\theta)\right)^{-1}\ \text{ if } \theta>1,
\end{aligned}$$

where

$$\begin{aligned}
&\Gamma_1=\sup_{t>0}\left(e^{-kt}\textstyle\int_0^{+\infty}G(t,s)\phi(s)ds\right),\\
&\Gamma_2=\sup_{t>0}\left(e^{-kt}\textstyle\int_0^{+\infty}\widetilde{G}(t,s)\phi(s)ds\right),\\
&\Theta_{1,0}(\theta)=\sup_{t>0}\left(e^{-kt}\textstyle\int_0^{\theta}G(t,s)\phi(s)e^{-ks}\gamma(s)\,ds\right),\\
&\Theta_{2,0}(\theta)=\sup_{t>0}\left(e^{-kt}\textstyle\int_0^{\theta}\widetilde{G}(t,s)\phi(s)e^{-ks}\gamma(s)\,ds\right),\\
&\Theta_{1,\infty}(\theta)=\sup_{t>0}\left(e^{-kt}\textstyle\int_{1/\theta}^{\theta}G(t,s)\phi(s)e^{-ks}\gamma(s)\,ds\right),\\
&\Theta_{2,\infty}(\theta)=\sup_{t>0}\left(e^{-kt}\textstyle\int_{1/\theta}^{\theta}\widetilde{G}(t,s)\phi(s)e^{-ks}\gamma(s)\,ds\right),
\end{aligned}$$

and notice that Remark 3.3 guarantees that the constants Γ_1 and Γ_2 are finite.

Theorem 3.1. *Assume that Hypothesis (3.6) holds and one of the following conditions*

$$f^0 < \Gamma, \quad \Theta_\infty(\theta) < f_\infty(\theta) \text{ for some } \theta > 1 \tag{3.7}$$

$$f^\infty < \Gamma, \quad \Theta_0(\theta) < f_0(\theta) \text{ for some } \theta > 0 \tag{3.8}$$

is satisfied. Then the bvp (3.5) admits at least one positive solution.

Remark 3.4. *For the particular case where* $f(t,u,v) = \left(e^{-kt}(u+v)\right)^\sigma$ *with* $\sigma > 0$ *and* $\sigma \neq 1$, *we have* $f^0 = 0$ *and* $f_\infty(\theta) = +\infty$ *for all* $\theta > 0$ *if* $\sigma > 1$, *and* $f^\infty = 0$ *and* $f_0(\theta) = +\infty$ *for all* $\theta > 0$ *if* $\sigma < 1$. *Hence, Conditions (3.7) and (3.8) in Theorem 3.1 correspond to the superlinear case and the sublinear case of the nonlinearity* f, *respectively.*

3.2 Example

Consider the case of the bvp (3.5) where $\phi(t) = e^{-\alpha t}$, $\alpha > 0$ and

$$f(t,u,v) = A\left(\frac{u+v}{e^{kt}+u+v}\right)^p + B\left(\frac{u+v}{e^{kt}}\right)^q,$$

with $A, B > 0$, $p \leq 1$ and $q \geq 1$.

Thus, for all $t, w, z > 0$ we have

$$f(t, e^{kt}w, e^{kt}z) = A\left(\frac{w+z}{1+w+z}\right)^p + B\,(w+z)^q,$$

and if $|(w,z)| = w+z < R$, then

$$f(t, e^{kt}w, e^{kt}z) = A\left(\frac{w+z}{1+w+z}\right)^p + B\,(w+z)^q \leq \Psi_R(w,z),$$

where

$$\Psi_R(w,z) = \begin{cases} AR^p + BR^q \text{ if } p \geq 0, \\ A\,(w+z)^p\,(1+R)^{-p} + BR^q \text{ if } p < 0. \end{cases}$$

Thus, if $p \geq 0$ then

$$\lim_{s\to+\infty} \phi(s)\psi_R\left(Re^{-ks}\gamma(s), Re^{-ks}\widetilde{\gamma}(s)\right) = (AR^p + BR^q) \lim_{s\to+\infty} e^{-\alpha s} = 0,$$

$$\int_0^{+\infty} \phi(s)\psi_R\left(Re^{-ks}\gamma(s), Re^{-ks}\widetilde{\gamma}(s)\right) ds = \frac{AR^p + BR^q}{\alpha} < \infty,$$

and if $p < 0$ then

$$\phi(s)\psi_R\left(Re^{-ks}\gamma(s), Re^{-ks}\widetilde{\gamma}(s)\right) = BR^q e^{-\alpha s} + A(1+R)^{-p}(k^*R)^p e^{-(\alpha+pk)s}\left(1-e^{-ks}\right)^p \rho(s),$$

where

$$\rho(s) = \left(\frac{1}{3k}\left(1-e^{-ks}\right)\left(2+e^{-ks}\right) + e^{-ks}\left(1+e^{-ks}\right)\right)^p$$

satisfies

$$\left(\max\left(2, \frac{2}{3k}\right)\right)^p \leq \rho(s) \leq \left(\min\left(2, \frac{2}{3k}\right)\right)^p.$$

Therefore, we have

$$\lim_{s\to+\infty} \phi(s)\psi_R\left(Re^{-ks}\gamma(s), Re^{-ks}\widetilde{\gamma}(s)\right) = 0 \text{ if and only if } \alpha > -pk$$

and

$$\int_0^{+\infty} \phi(s)\psi_R\left(Re^{-ks}\gamma(s), Re^{-ks}\widetilde{\gamma}(s)\right) ds < \infty \text{ if and only if } \alpha > -pk \text{ and } p > -1.$$

Straightforward computations lead to

$$f^\infty = f_\infty(\theta) = f_\infty = \begin{cases} +\infty \text{ si } q > 1, \\ B \text{ si } q = 1, \end{cases} \quad \text{for all } \theta > 1$$

$$f^0 = f_0(\theta) = f_0 = \begin{cases} +\infty \text{ si } p < 1, \\ A \text{ si } p = 1 < q, \\ A + B \text{ si } p = q = 1, \end{cases} \quad \text{for all } \theta > 0.$$

We conclude from Theorem 3.1 and all the above calculations that this case of the bvp (3.5) admits a positive solution in each of the following situations:

1. $p = 1,\ q = 1,\ B < \Gamma$ and $A + B > \Theta_0(\theta)$ for some $\theta > 0$,

2. $p = 1,\ q > 1,$ and $A > \Theta_0(\theta)$ for some $\theta > 0$,

3. $p \in [0, 1),\ q = 1$ and $B < \Gamma$,

4. $p \in (-1, 0),\ q = 1,\ B < \Gamma$ and $\alpha > -pk$.

3.3 Fixed point formulation

In all this paper, we let

$$E = \{u \in C^1(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} e^{-kt}u(t) = 0,\ \ \lim_{t\to+\infty} e^{-kt}u'(t) = 0\}.$$

Endowed with the norm $\|u\| = \|u\|_k + \|u'\|_k$ where $\|u\|_k = \sup_{t\geq 0}\left(e^{-kt}|u(t)|\right)$, E becomes a Banach space.

The following lemma is an adapted version for the case of the space E of Corduneanu's compactness criterion ([19], p. 62). It will be used in this work to prove that some operator is completely continuous.

Lemma 3.1. *A nonempty subset M of E is relatively compact if the following conditions hold:*

(a) *M is bounded in E,*

(b) *the sets $\{u : u(t) = e^{-kt}x(t),\ x \in M\}$ and $\{u : u(t) = e^{-kt}x'(t),\ x \in M\}$ are locally equicontinuous on $[0, +\infty)$, and*

(c) *the sets $\left\{u : u(t) = e^{-kt}x(t),\ x \in M\right\}$ and $\left\{u : u(t) = e^{-kt}x'(t),\ x \in M\right\}$ are equiconvergent at $+\infty$.*

In all this work, P denotes the cone in E defined by

$$P = \{u \in E :\ u'(t) \geq \widetilde{\gamma}(t)||u|| \text{ and } u(t) \geq \gamma(t)||u|| \text{ for all } t > 0\}.$$

Let $G, \widetilde{G} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be the functions defined by

$$G(t,s) = \frac{1}{k^2}\begin{cases} e^{-ks}\left(\cosh(kt) - 1\right) & \text{if } t \leq s, \\ -e^{-kt}\sinh(ks) + \left(1 - e^{-ks}\right) & \text{if } s \leq t, \end{cases}$$

$$\widetilde{G}(t,s) = \frac{\partial G}{\partial t}(t,s) = \frac{1}{k}\begin{cases} e^{-ks}\sinh(kt) & \text{if } t \leq s, \\ e^{-kt}\sinh(ks) & \text{if } s \leq t. \end{cases}$$

Lemma 3.2. *The functions G and $\widetilde{G}$ satisfy:*

(a) *For all $t, s \in \mathbb{R}^+$ we have $G(t,s) \geq 0$ and $\widetilde{G}(t,s) \geq 0$.*

(b) *The functions G and $\widetilde{G}$ are continuous and for all $s \geq 0$, we have*

$$G(0,s) = \widetilde{G}(0,s) = 0. \tag{3.9}$$

(c) *For all $t, s \geq 0$, we have*

$$G(t,s) \leq \frac{1}{k^2}(1-e^{-ks}) \leq \frac{1}{k^2}, \quad \widetilde{G}(t,s) \leq \widetilde{G}(s,s) \leq \frac{1}{2k}.$$

(d) *For all $s, t, \tau \geq 0$, we have*

$$\widetilde{G}(t,s)e^{-kt} \geq \gamma_1(t)\widetilde{G}(\tau,s)e^{-k\tau}.$$

(e) *For all $t_2, t_1 \geq 0$, we have*

$$\left|e^{-kt_2}G(t_2,s) - e^{-kt_1}G(t_1,s)\right| \leq \frac{3}{2k}\left|t_2 - t_1\right| \tag{3.10}$$

$$\left|e^{-kt_2}\widetilde{G}(t_2,s) - e^{-kt_1}\widetilde{G}(t_1,s)\right| \leq \left|t_2 - t_1\right| \tag{3.11}$$

Proof. Assertions (a), (b) and (c) are easy to prove, Assertion (d) is proved in [23]. Assertion (e) is obtained by the mean value theorem. □

Lemma 3.3. *Assume that Hypothesis (3.6) holds, then there exists a continuous operator $T : P \setminus \{0\} \to P$ such that for all r, R with $0 < r < R$, $T\left(P \cap \left(\overline{B}(0,R) \setminus B(0,r)\right)\right)$ is relatively compact and fixed points of T are positive solutions to the bvp (3.5).*

Proof. The proof is divided into four steps.

Step 1. Existence of the operator T. To this aim let $u \in P \setminus \{0\}$. By means of Hypothesis (3.6) with $R = \|u\|$, for all $t > 0$ we have

$$\begin{aligned}
&\int_0^{+\infty} G(t,s)\phi(s)\, f(s,u(s),u'(s))ds \\
&\qquad \leq \tfrac{1}{k^2}\int_0^{+\infty}\phi(s)\, f(s,u(s),u'(s))ds \\
&\qquad = \tfrac{1}{k^2}\int_0^{+\infty}\phi(s)\, f(s,e^{ks}\left(e^{-ks}u(s)\right), e^{ks}\left(e^{-ks}u'(s)\right))ds \\
&\qquad \leq \tfrac{1}{k^2}\int_0^{+\infty}\phi(s)\,\Psi_R\left(Re^{-ks}\gamma(s), Re^{-ks}\widetilde{\gamma}(s)\right)ds < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{+\infty}\widetilde{G}(t,s)\phi(s)\, f(s,u(s),u'(s))ds \leq \tfrac{1}{2k}\int_0^{+\infty}\phi(s)\, f(s,u(s),u'(s))ds \\
&\qquad \leq \tfrac{1}{2k}\int_0^{+\infty}\phi(s)\,\Psi_R\left(Re^{-ks}\gamma(s), Re^{-ks}\widetilde{\gamma}(s)\right)ds < \infty.
\end{aligned}$$

Thus, let v and w be the functions defined by

$$\begin{aligned} v(t) &= \int_0^{+\infty} G(t,s)\phi(s)\, f(s,u(s),u'(s))ds \\ w(t) &= \int_0^{+\infty} \widetilde{G}(t,s)\phi(s)\, f(s,u(s),u'(s))ds. \end{aligned}$$

Since for all $t > 0$,

$$\begin{aligned} v(t) \ &= -\tfrac{e^{-kt}}{k^2} \int_0^t \sinh(ks)\phi(s)\, f(s,u(s),u'(s))ds \\ &+\tfrac{1}{k^2} \int_0^t (1-e^{-ks})\phi(s)\, f(s,u(s),u'(s))ds \\ &+\tfrac{\cosh(kt)-1}{k^2} \int_0^t e^{-ks}\phi(s)\, f(s,u(s),u'(s))ds, \end{aligned}$$

we see that v is differentiable on $\mathbb{R}^+$ and for all $t \geq 0$,

$$\begin{aligned} v'(t) \ &= \tfrac{e^{-kt}}{k} \int_0^t \sinh(ks)\phi(s)\, f(s,u(s),u'(s))ds \\ &+\tfrac{\sinh(kt)}{k} \int_t^{+\infty} e^{-ks}\phi(s)\, f(s,u(s),u'(s))ds \qquad , \\ &= \int_0^{+\infty} \widetilde{G}(t,s)\phi(s)\, f(s,u(s),u'(s))ds = w(t) \end{aligned}$$

with w continuous on $\mathbb{R}^+$.

At this stage we have proved that v belongs to $C^1(\mathbb{R}^+,\mathbb{R})$ and we need to prove that $v \in E$. Thus, we have to show that $\lim_{t\to+\infty} e^{-kt}v(t) = \lim_{t\to+\infty} e^{-kt}v'(t) = 0$. Clearly for all $t > 0$, $v(t), v'(t) > 0$ and we have

$$\begin{aligned} e^{-kt}v(t) \ &= e^{-kt} \int_0^{+\infty} G(t,s)\phi(s)\, f(s,u(s),u'(s))ds \\ &\leq \frac{e^{-kt}}{k^2} \int_0^{+\infty} \phi(s)\, \Psi_R\left(Re^{-ks}\gamma(s), Re^{-ks}\widetilde{\gamma}(s)\right) ds \end{aligned}$$

and

$$\begin{aligned} e^{-kt}v'(t) \ &= e^{-kt} \int_0^{+\infty} \widetilde{G}(t,s)\phi(s)\, f(s,u(s),u'(s))ds \\ &\leq \frac{e^{-kt}}{2k} \int_0^{+\infty} \phi(s)\, \Psi_R\left(Re^{-ks}\gamma(s), Re^{-ks}\gamma(s)\right) ds. \end{aligned}$$

The above two estimates prove that $\lim_{t\to+\infty} e^{-kt}v(t) = \lim_{t\to+\infty} e^{-kt}v'(t) = 0$.

Now for all $t, \tau > 0$, we have from Assertion (d) in Lemma 3.2

$$\begin{aligned} v'(t) \ &= e^{kt} \int_0^{+\infty} e^{-kt}\widetilde{G}(t,s)\phi(s) f(s,u(s),u'(s))ds \\ &\geq e^{kt}\gamma_1(t) \int_0^{+\infty} e^{-k\tau}\widetilde{G}(\tau,s)\phi(s) f(s,u(s),u'(s))ds \\ &= e^{kt}\gamma_1(t) e^{-k\tau} v'(\tau). \end{aligned}$$

Passing to the supremum on τ, we obtain

$$v'(t) \geq e^{kt}\gamma_1(t) \left\| v' \right\|_k \ \text{ for all } t > 0. \tag{3.12}$$

Since for all $t > 0$

$$v(t) = \int_0^t e^{k\xi} \left(e^{-k\xi} v'(\xi)\right) d\xi \leq \int_0^t e^{k\xi} d\xi \left\| v' \right\|_k \leq \frac{e^{kt}}{k} \left\| v' \right\|_k ,$$

we have

$$\left\| v' \right\|_k \geq k \left\| v \right\|_k . \tag{3.13}$$

Therefore, (3.12) combined with (3.13) leads to

$$v'(t) \geq k e^{kt} \gamma_1(t) \left\| v \right\|_k \text{ for all } t > 0,$$

then to

$$v'(t) \geq \widetilde{\gamma}(t) \left\| v \right\| \text{ for all } t > 0. \tag{3.14}$$

Integrating (3.14), yields $v(t) \geq \gamma(t) \left\| v \right\|$ for all $t > 0$.

Thus, we have proved that $v \in P$ and the operator $T : P \setminus \{0\} \to P$ where for $u \in P \setminus \{0\}$

$$Tu(t) = \int_0^{+\infty} G(t,s) \phi(s) f(s, u(s), u'(s)) ds,$$

is well defined.

Step 2. The operator T is continuous. Let (u_n) be a sequence in $P \setminus \{0\}$ such that $\lim_{n\to\infty} u_n = u_\infty$ in E with u_∞ in $P \setminus \{0\}$ and let $R > r > 0$ be such that $(u_n) \subset B(0,R) \setminus B(0,r)$. If Ψ_R is the function given by Hypothesis (3.6), then for all $n \geq 1$ we have

$$\begin{aligned} \left\| Tu_n - Tu_\infty \right\|_k &= \sup_{t\geq 0} \left| Tu_n(t) - Tu_\infty(t) \right| \\ &\leq \tfrac{1}{k^2} \int_0^{+\infty} \phi(s) \left| f(s, u_n(s), u_n'(s)) - f((s, u_\infty(s), u_\infty'(s)) \right| ds \end{aligned}$$

and

$$\begin{aligned} \left\| (Tu_n)' - (Tu_\infty)' \right\|_k &= \sup_{t\geq 0} \left| (Tu_n)'(t) - (Tu_\infty)'(t) \right| \\ &\leq \tfrac{1}{2k} \int_0^{+\infty} \phi(s) \left| f(s, u_n(s), u_n'(s)) - f((s, u_\infty(s), u_\infty'(s)) \right| ds. \end{aligned}$$

Because of

$$\left| f(s, u_n(s), u_n'(s)) - f((s, u_\infty(s), u_\infty'(s))) \right| \to 0, \text{ as } n \to +\infty$$

for all $s > 0$ and

$$\begin{aligned} &\phi(s) \left| f(s, u_n(s), u_n'(s)) - f((s, u_\infty(s), u_\infty'(s)) \right| \\ \leq\ & 2\phi(s) \Psi_R \left(re^{-ks} \gamma(s), re^{-ks} \gamma(s) \right) \end{aligned}$$

with $\int_0^{+\infty} \phi(s)\, \Psi_R \left(re^{-ks}\gamma(s), re^{-ks}\gamma(s)\right) ds < \infty$, the Lebesgue dominated convergence theorem guarantees that $\lim_{n\to\infty} \|Tu_n - Tu_\infty\| = 0$. Hence, we have proved that T is continuous.

Step 3. For $R > r > 0$**,** $T\left(P \cap \left(\overline{B}(0,R) \smallsetminus B(0,r)\right)\right)$ **is relatively compact**. Set $\Omega = P \cap \left(\overline{B}(0,R) \smallsetminus B(0,r)\right)$ and let $\Phi_{r,R}$ be defined by

$$\Phi_{r,R}(s) = \Psi_R \left(re^{-ks}\gamma(s), re^{-ks}\gamma(s)\right)$$

where Ψ_R is the function given by Hypothesis (3.6). For all $u \in \Omega$, we have

$$\|Tu\| \leq \left(\frac{1}{k^2} + \frac{1}{2k}\right) \int_0^{+\infty} \phi(s)\, \Phi_{r,R}(s) ds < \infty,$$

proving that $T\Omega$ is bounded in E.

Now, let $t_1, t_2 \in [\eta, \xi]$, for all $u \in \Omega$, we have from (3.10) and (3.11) the estimates

$$\begin{aligned} |e^{-kt_1}Tu(t_1) - e^{-kt_2}Tu(t_2)| &\leq \int_0^{+\infty} |e^{-kt_1}G(t_1,s) - e^{-kt_2}G(t_2,s)| \phi(s)\, \Phi_{r,R}(s) ds \\ &\leq \tfrac{3}{2k} |t_2 - t_1| \int_0^{+\infty} \phi(s)\, \Phi_{r,R}(s) ds \end{aligned}$$

and

$$\begin{aligned} |e^{-kt_1}(Tu)'(t_1) - e^{-kt_2}(Tu)'(t_2)| &\leq \int_0^{+\infty} |e^{-kt_1}\widetilde{G}(t_1,s) - e^{-kt_2}\widetilde{G}(t_2,s)| \phi(s)\, \Phi_{r,R}(s) ds \\ &\leq |t_2 - t_1| \int_0^{+\infty} \phi(s)\, \Phi_{r,R}(s) ds. \end{aligned}$$

Proving the equicontinuity of $T\Omega$ on bounded intervals.

For all $u \in \Omega$ and $t > 0$, we have

$$|e^{-kt}Tu(t)| \leq \frac{e^{-kt}}{k^2} \int_0^{+\infty} \phi(s)\, \Phi_{r,R}(s) ds$$

and

$$|e^{-kt}(Tu)'(t)| \leq \frac{e^{-kt}}{k} \int_0^{+\infty} \phi(s)\, \Phi_{r,R}(s) ds.$$

Thus, the equiconvergence of $T\Omega$ follows from the fact that $\lim_{t\to\infty} e^{-kt} = 0$. In view of Lemma 3.1, $T\Omega$ is relatively compact in E.

Step 4. Fixed points of T **are positive solutions to the bvp (3.5).** Let $u \in P \smallsetminus \{0\}$ be a fixed point of T, then for all $t > 0$ we have

$$\begin{aligned} u(t) &= \int_0^{+\infty} G(t,s)\phi(s)\, f((s,u(s),u'(s)) ds \text{ and} \\ u'(t) &= \int_0^{+\infty} \widetilde{G}(t,s)\phi(s)\, f((s,u(s),u'(s)) ds. \end{aligned}$$

These with (3.9) lead to $u(0) = u'(0) = 0$.

Differentiating twice in

$$\begin{aligned} u'(t) &= \int_0^{+\infty} \widetilde{G}(t,s)\phi(s) f((s,u(s),u'(s))ds \\ &= \frac{e^{-kt}}{k}\int_0^t \sinh(ks)\phi(s) f(s,u(s),u'(s))ds \\ &+ \frac{\sinh(kt)}{k}\int_t^{+\infty} e^{-ks}\phi(s) f(s,u(s),u'(s))ds, \end{aligned}$$

we see that $-u'''(t) + ku'(t) = \phi(t) f(t,u(t),u'(t))$ for all $t > 0$.

It remains to prove that $\lim_{t\to+\infty} u'(t) = 0$. We have

$$\begin{aligned} u'(t) &= \frac{1}{ke^{kt}}\int_0^t \sinh(ks)\phi(s) f(s,u(s),u'(s))ds \\ &+ \frac{\sinh(kt)}{k}\int_t^{+\infty} e^{-ks}\phi(s) f(s,u(s),u'(s))ds. \end{aligned}$$

By means of Hypothesis (3.6) with $R = \|u\|$ and the L'Hopital's rule, we obtain

$$\begin{aligned} &\lim_{t\to+\infty} \frac{1}{ke^{kt}}\int_0^t \sinh(ks)\phi(s) f(s,u(s),u'(s))ds \\ &\leq \lim_{t\to+\infty} \frac{1}{ke^{kt}}\int_0^t \sinh(ks)\phi(s)\Psi_R\left(Re^{-ks}\gamma(s), Re^{-ks}\gamma(s)\right) ds \\ &= \lim_{t\to+\infty} \frac{\sinh(kt)}{ke^{kt}}\phi(t)\Psi_R\left(Re^{-kt}\gamma(t), Re^{-kt}\gamma(t)\right) ds = 0. \end{aligned}$$

Also, we have

$$\begin{aligned} &\frac{\sinh(kt)}{k}\int_t^{+\infty} e^{-ks}\phi(s) f(s,u(s),u'(s))ds \\ &\leq \frac{\sinh(kt)e^{-kt}}{k}\int_t^{+\infty} \phi(s) f(s,u(s),u'(s))ds \to 0 \text{ as } t\to+\infty. \end{aligned}$$

The above calculations show that $\lim_{t\to+\infty} u'(t) = 0$, completing the proof of the lemma. □

3.4 Proof of Theorem 3.1

Step 1. Existence in the case where (3.7) holds

Let $\epsilon > 0$ be such that $(f^0 + \epsilon) < \Gamma$. For such a ϵ, there exists $R_1 > 0$ such that $f(t, e^{kt}w, e^{kt}z) \leq (f^0 + \epsilon)(w + z)$ for all w, z with $|(w,z)| \leq R_1$ and let $\Omega_1 = \{u \in E, \|u\| < R_1\}$.

Therefore, for all $u \in P \cap \partial\Omega_1$ and all $t > 0$, we have

$$\begin{aligned} e^{-kt}Tu(t) &= e^{-kt}\int_0^{+\infty} G(t,s)\phi(s)f(s, e^{ks}\left(e^{-ks}u(s)\right), e^{ks}\left(e^{-ks}u'(s)\right))ds \\ &\leq (f^0+\epsilon)\, e^{-kt}\int_0^{+\infty} G(t,s)\phi(s)e^{-ks}\left(u(s)+u'(s)\right))ds \\ &\leq \|u\|\,(f^0+\epsilon)\, e^{-kt}\int_0^{+\infty} G(t,s)\phi(s)ds \\ &\leq \Gamma_1 (f^0+\epsilon)\|u\|, \end{aligned}$$

leading to

$$\|Tu\|_k = \sup_{t>0}\left(e^{-kt}Tu(t)\right) \le \left(f^0+\epsilon\right)\Gamma_1 \|u\|. \tag{3.15}$$

Similarly, we have

$$\begin{aligned} e^{-kt}\left(Tu\right)'(t) &= e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\phi(s)f(s,e^{ks}\left(e^{-ks}u(s)\right),e^{ks}\left(e^{-ks}u'(s)\right))ds \\ &\le \left(f^0+\epsilon\right)e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\phi(s)e^{-ks}\left(u(s)+u'(s)\right))ds \\ &\le \|u\|\left(f^0+\epsilon\right)e^{-kt}\int_0^{+\infty}\widetilde{G}(t,s)\phi(s)ds \\ &\le \left(f^0+\epsilon\right)\Gamma_2\|u\|, \end{aligned}$$

leading to

$$\left\|(Tu)'\right\|_k = \sup_{t>0}\left(e^{-kt}\left(Tu\right)'(t)\right) \le \left(f^0+\epsilon\right)\Gamma_2\|u\|. \tag{3.16}$$

Summing (3.15) with (3.16), we obtain

$$\|Tu\| \le \|u\|\left(f^0+\epsilon\right)\Gamma^{-1} \le \|u\|.$$

Now, suppose that $f_\infty(\theta) > \Theta_\infty(\theta)$ for some $\theta > 1$ and let $\varepsilon > 0$ be such that $(f_\infty(\theta)-\varepsilon) > \Theta_\infty(\theta)$. There exists $\widetilde{R}_2 > R_1$ such that $f(t,e^{kt}w,e^{kt}z) > (f_\infty(\theta)-\varepsilon)(w+z)$ for all $t\in J_\theta$ and all w,z with $|(w,z)| \ge \widetilde{R}_2$. Let $\gamma_\theta = \min\left\{\gamma(s)e^{-ks} : s\in J_\theta\right\}$, $R_2 = \widetilde{R}_2/\gamma_\theta$ and $\Omega_2 = \{u\in E : \|u\| < R_2\}$. For all $u\in P\cap\partial\Omega_2$, and all $t>0$ we have

$$\begin{aligned} \|Tu\|_k &\ge e^{-kt}Tu(t) \ge e^{-kt}\int_{1/\theta}^{\theta}G(t,s)\phi(s)f(s,e^{ks}\left(e^{-ks}u(s)\right),e^{ks}\left(e^{-ks}u'(s)\right))ds \\ &\ge (f_\infty(\theta)-\varepsilon)e^{-kt}\int_{1/\theta}^{\theta}G(t,s)\phi(s)\left(e^{-ks}u(s)+e^{-ks}u'(s)\right)ds \\ &\ge (f_\infty(\theta)-\varepsilon)e^{-kt}\int_{1/\theta}^{\theta}G(t,s)\phi(s)e^{-ks}u(s)ds \\ &\ge \|u\|(f_\infty(\theta)-\varepsilon)e^{-kt}\int_{1/\theta}^{\theta}G(t,s)\phi(s)e^{-ks}\gamma(s)\,ds \end{aligned}$$

and

$$\begin{aligned} \left\|(Tu)'\right\|_k &\ge e^{-kt}\int_{1/\theta}^{\theta}\widetilde{G}(t,s)\phi(s)f(s,e^{ks}\left(e^{-ks}u(s)\right),e^{ks}\left(e^{-ks}u'(s)\right))ds \\ &\ge (f_\infty(\theta)-\varepsilon)e^{-kt}\int_{1/\theta}^{\theta}\widetilde{G}(t,s)\phi(s)\left(e^{-ks}u(s)+e^{-ks}u'(s)\right)ds \\ &\ge (f_\infty(\theta)-\varepsilon)e^{-kt}\int_{1/\theta}^{\theta}\widetilde{G}(t,s)\phi(s)e^{-ks}u(s)ds \\ &\ge \|u\|(f_\infty(\theta)-\varepsilon)e^{-kt}\int_{1/\theta}^{\theta}\widetilde{G}(t,s)\phi(s)e^{-ks}\gamma(s)\,ds. \end{aligned}$$

The above estimates lead to

$$\begin{aligned} \|Tu\|_k &\ge (f_\infty(\theta)-\varepsilon)\Theta_{1,\infty}(\theta)\|u\|, \\ \left\|(Tu)'\right\|_k &\ge (f_\infty(\theta)-\varepsilon)\Theta_{2,\infty}(\theta)\|u\| \end{aligned}$$

then to

$$\|Tu\| \geq (f_\infty(\theta) - \varepsilon)(\Theta_\infty(\theta))^{-1} \|u\| \geq \|u\|.$$

We deduce from Assertion $(H_3)$of Theorem 1.14that T admits a fixed point $u \in P$ with $R_1 \leq \|u\|_1 \leq R_2$ which is, by Lemma 3.3, a positive solution to Problem (3.5).

Step 2. Existence in the case where (3.8) holds

Suppose that $f_0(\theta) > \Theta_0(\theta)$ for some $\theta > 0$ and let $\varepsilon > 0$ be such that $(f_0(\theta) - \varepsilon) > \Theta_0(\theta)$. There exists R_1 such that $f(t, e^{kt}w, e^{kt}z) > (f_0(\theta) - \varepsilon)(w + z)$ for all w, z with $|(w, z)| \leq R_1$. Let $\Omega_1 = \{u \in E : \|u\| < R_1\}$, for all $u \in P \cap \partial\Omega_1$ and all $t > 0$, we have

$$\begin{aligned}
\|Tu\|_k &\geq e^{-kt}Tu(t) \geq e^{-kt} \textstyle\int_0^\theta G(t,s)\phi(s) f(s, e^{ks}\left(e^{-ks}u(s)\right), e^{ks}\left(e^{-ks}u'(s)\right))ds \\
&\geq (f_0(\theta) - \varepsilon)e^{-kt} \textstyle\int_0^\theta G(t,s)\phi(s)\left(e^{-ks}u(s) + e^{-ks}u'(s)\right) ds \\
&\geq (f_0(\theta) - \varepsilon)e^{-kt} \textstyle\int_0^\theta G(t,s)\phi(s)e^{-ks}u(s)ds \\
&\geq \|u\| (f_0(\theta) - \varepsilon)e^{-kt} \textstyle\int_0^\theta G(t,s)\phi(s)e^{-ks}\gamma(s)\, ds
\end{aligned}$$

and

$$\begin{aligned}
\left\|(Tu)'\right\|_k &\geq e^{-kt}Tu(t) \geq e^{-kt} \textstyle\int_0^\theta \widetilde{G}(t,s)\phi(s) f(s, e^{ks}\left(e^{-ks}u(s)\right), e^{ks}\left(e^{-ks}u'(s)\right))ds \\
&\geq (f_0(\theta) - \varepsilon)e^{-kt} \textstyle\int_0^\theta \widetilde{G}(t,s)\phi(s)\left(e^{-ks}u(s) + e^{-ks}u'(s)\right) ds \\
&\geq (f_0(\theta) - \varepsilon)e^{-kt} \textstyle\int_0^\theta \widetilde{G}(t,s)\phi(s)e^{-ks}u(s)ds \\
&\geq \|u\| (f_0(\theta) - \varepsilon)e^{-kt} \textstyle\int_0^\theta \widetilde{G}(t,s)\phi(s)e^{-ks}\gamma(s)\, ds.
\end{aligned}$$

The above estimates lead to

$$\begin{aligned}
\|Tu\|_k &\geq (f_0(\theta) - \varepsilon)\Theta_{1,0}(\theta)\|u\|, \\
\left\|(Tu)'\right\|_k &\geq (f_0(\theta) - \varepsilon)\Theta_{2,0}(\theta)\|u\|
\end{aligned}$$

then to

$$\|Tu\| \geq (f_0(\theta) - \varepsilon)(\Theta_0(\theta))^{-1} \|u\| \geq \|u\|.$$

Let $\epsilon > 0$ be such that $(f^\infty + \epsilon) < \Gamma$, there exists $R_\epsilon > 0$ such that

$$f(t, e^{kt}w, e^{kt}z) \leq (f^\infty + \epsilon)(w + z) + \Psi_{R_\epsilon}(w, z), \text{ for all } t, w, z > 0,$$

where Ψ_{R_ϵ} is the functions given by Hypothesis (3.6) for $R = R_\epsilon$.

Let

$$\Phi_\epsilon(t) = \Psi_{R_\epsilon}\left(R_\epsilon e^{-ks}\gamma(s), R_\epsilon e^{-ks}\widetilde{\gamma}(s)\right)$$
$$\widetilde{R}_2 = \frac{2\overline{\Psi_\epsilon}\Gamma}{\Gamma - (f^\infty + \epsilon)}$$
$$\text{with } \overline{\Phi_\epsilon} = \sup_{t\geq 0}\left(e^{-kt}\int_0^{+\infty} G(t,s)\Phi_\epsilon(s)\,ds\right)$$

and notice that $\Gamma^{-1}(f^\infty+\epsilon)R + 2\overline{\Phi_\epsilon} \leq R$ for all $R \geq \widetilde{R}_2$.

Let $R_2 > \max(R_1, \widetilde{R}_2, R_\epsilon)$ and $\Omega_2 = \{u \in E,\ \|u\| < R_2\}$. For all $u \in P \cap \partial\Omega_2$ and all $t > 0$, we have

$$\begin{aligned} e^{-kt}Tu(t) &= \int_0^{+\infty} G(t,s)\phi(s)f(s, e^{ks}\left(e^{-ks}u(s)\right), e^{ks}\left(e^{-ks}u'(s)\right))ds \\ &\leq e^{-kt}\int_0^{+\infty} G(t,s)\phi(s)\left((f^\infty+\epsilon)\left(e^{-ks}u(s) + e^{-ks}u'(s)\right)\right. \\ &\left.+\Psi_\epsilon\left(e^{-ks}u(s), e^{-ks}u'(s)\right)\right)ds \\ &\leq (f^\infty+\epsilon)\|u\|\, e^{-kt}\int_0^{+\infty} G(t,s)\phi(s)ds + \overline{\Psi_\epsilon} \\ &\leq (f^\infty+\epsilon)\|u\|\,\Gamma_1 + \overline{\Psi_\epsilon}, \end{aligned}$$

leading to

$$\|Tu\|_k \leq (f^\infty+\epsilon)\|u\|\,\Gamma_1 + \overline{\Psi_\epsilon}. \tag{3.17}$$

Similarly, we have

$$\begin{aligned} e^{-kt}(Tu)'(t) &= \int_0^{+\infty} \widetilde{G}(t,s)\phi(s)f(s, e^{ks}\left(e^{-ks}u(s)\right), e^{ks}\left(e^{-ks}u'(s)\right))ds \\ &\leq e^{-kt}\int_0^{+\infty} \widetilde{G}(t,s)\phi(s)\left((f^\infty+\epsilon)\left(e^{-ks}u(s) + e^{-ks}u'(s)\right)\right. \\ &\left.+\Psi_\epsilon\left(e^{-ks}u(s), e^{-ks}u'(s)\right)\right)ds \\ &\leq (f^\infty+\epsilon)\|u\|\, e^{-kt}\int_0^{+\infty} \widetilde{G}(t,s)\phi(s)ds + \overline{\Psi_\epsilon} \\ &\leq (f^\infty+\epsilon)\|u\|\,\Gamma_2 + \overline{\Psi_\epsilon}, \end{aligned}$$

leading to

$$\left\|(Tu)'\right\|_k \leq (f^\infty+\epsilon)\Gamma_2\|u\| + \overline{\Psi_\epsilon}. \tag{3.18}$$

Summing (3.17) with (3.18), we obtain

$$\|Tu\| \leq (f^\infty+\epsilon)\Gamma^{-1}\|u\| + 2\overline{\Psi_\epsilon} \leq \|u\|.$$

We deduce from Assertion $(H_4)$of Theorem 1.14that T admits a fixed point $u \in P$ with $R_1 \leq \|u\| \leq R_2$ which is, by Lemma 3.3, a positive solution to Problem (3.5).

Thus, the proof of Theorem 3.1 is complete.

3.5 Comments

1. Notice that the obtained positive solution in Theorem 3.1 is nondecreasing and bounded. Indeed, if $u \in P \setminus \{0\}$ is a fixed point of T with $\|u\| = R$, then for all $t > 0$

$$u'(t) = (Tu)'(t) = \int_0^{+\infty} \widetilde{G}(t,s)\phi(s)f(s,u(s),u'(s))ds > 0$$

and Hypothesis (3.6) leads to

$$\begin{aligned} u(t) &= Tu(t) = \int_0^{+\infty} G(t,s)\phi(s)f(s,u(s),u'(s))ds \\ &\leq \int_0^{+\infty} G(t,s)\phi(s)\Psi_R(\left(e^{-ks}u(s)\right), \left(e^{-ks}u'(s)\right))ds \\ &\leq \frac{1}{k^2}\int_0^{+\infty} \phi(s)\Psi_R(Re^{-ks}\gamma(s), Re^{-ks}\widetilde{\gamma}(s))ds < \infty. \end{aligned}$$

2. From the above comment arise the following question. Why we looked for solutions in the space E instead of looking for them in the natural space
$F = \{u \in C^1(\mathbb{R}^+) : \max\left(\sup_{t>0}|u(t)|, \sup_{t>0}|u'(t)|\right) < \infty\}$?

   The answer is: There is no cone in F where we can realize the inequality $\|Tu\| \geq \|u\|$ in Theorem 1.14.

3. Notice that for $\theta > 1$, $\Gamma < \Theta_0(\theta) < \Theta_\infty(\theta)$ and let the interval $\mathcal{I} = (\Gamma, \Theta_\infty(\theta))$. In the particular case where the limits

$$f^0 = \lim_{|(w,z)|\to 0} \frac{f(t,e^{kt}w,e^{kt}z)}{w+z}, \qquad f^\infty = \lim_{|(w,z)|\to\infty} \frac{f(t,e^{kt}w,e^{kt}z)}{w+z}$$

exist, then Theorem 3.1 claims that the bvp (3.5) admits a positive solution if f^0 and f^∞ are oppositely located relatively to the interval $\mathcal{I}$, that is the ratio $\left(f(t,e^{kt}w,e^{kt}z)/w+z\right)$ crosses the interval $\mathcal{I}$. Two questions arise from this observation; what happens if $\left(f(t,e^{kt}w,e^{kt}z)/w+z\right) > \Theta_\infty(\theta)$ or $\left(f(t,e^{kt}w,e^{kt}z)/w+z\right) < \Gamma$ for all $t,w,z > 0$?

   The second question is: are the constants $\Gamma, \Theta_0(\theta), \Theta_\infty(\theta)$ the best ones? In an other manner, does exist two positive constants α and β with $\Gamma < \alpha < \beta < \Theta_0(\theta)$ such that if f^0 and f^∞ are oppositely located relatively to the interval (α,β), then the bvp (3.5) admits a positive solution?

4. Let $p > 1$ and consider the case where E is equipped with the norm $\|u\|_p = \sqrt[p]{\|u\|_k^p + \|u\|_k^p}$. In this case, under Hypothesis (3.6), we prove by the same arguments that the bvp (3.5) admits a positive solution if $f^0 < \Gamma_p < \Theta^p_\infty(\theta) < f_\infty(\theta)$ for some $\theta > 1$ or $f^\infty < \Gamma < \Theta^p_0(\theta) < f_0(\theta)$ for some $\theta > 0$, where

$$\Gamma_p = ((\Gamma_1)^p + (\Gamma_2)^p)^{-1/p},$$
$$\Theta^p_0(\theta) = ((\Theta_{1,0}(\theta))^p + (\Theta_{2,0}(\theta))^p)^{-1/p} \text{ for } \theta > 0,$$
$$\Theta^p_\infty(\theta) = ((\Theta_{1,\infty}(\theta))^p + (\Theta_{2,\infty}(\theta))^p)^{-1/p} \text{ for } \theta > 0.$$

Noticing that $\Gamma_p > \Gamma$, $\Theta^p_0(\theta) > \Theta_0(\theta)$ and $\Theta^p_\infty(\theta) > \Theta_{p,\infty}(\theta)$ we understand that the problem posed in the above comment is a serious problem.

Chapter 4

Positive solution for third-order singular semipositone BVPs on the half line with first-order derivative dependence

4.1 Introduction

This article deals with existence of positive solutions to the third-order boundary value problem (bvp for short),

$$\begin{cases} -u'''(t) + k^2 u'(t) = f(t, u(t), u'(t)), \text{ a.e. } t \in I \\ u(0) = u'(0) = u'(+\infty) = 0, \end{cases} \tag{4.1}$$

where k is a positive constant, $I = (0, +\infty)$ and $f : I^3 \to \mathbb{R}$ is a Carathéodory function, that is

- $f(\cdot, u, v)$ is a measurable function for all $u, v \in I$, and
- $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in I$.

Throughout, we assume that

$$\begin{cases} \text{There exists a measurable function } q : I \to \mathbb{R}^+ \text{ such that} \\ \int_0^{+\infty} e^{ks} q(s)ds < \infty \text{ and } f(t, u, v) + q(t) \geq 0 \text{ for all } t, u, v > 0, \end{cases} \tag{4.2}$$

$$\begin{cases} \text{for all } \rho > 0 \text{ there exist two functions } \omega_\rho : (0, +\infty) \to \mathbb{R}^+ \\ \text{and } \Psi_\rho : (0, +\infty) \times (0, +\infty) \to (0, +\infty) \ \text{ such that} \\ \Psi_\rho \text{ is nonincreasing following its two variables,} \\ \left| f(t, e^{kt}w, e^{kt}z) \right| \le \omega_\rho(t) \, \Psi_\rho(w, z) \ \text{ for all } t, w, z \ge 0 \text{ with } \ |(w, z)| \le \rho, \\ \text{for all } r \in (0, \rho] \,, \ \lim_{s \to +\infty} \omega_\rho(s) \, \Psi_\rho\left(re^{-ks}\gamma(s), re^{-ks}\widetilde{\gamma}(s)\right) = 0 \text{ and} \\ \int_0^{+\infty} \omega_\rho(s) \, \Psi_\rho\left(re^{-ks}\gamma(s), re^{-ks}\widetilde{\gamma}(s)\right) ds < \infty, \end{cases} \tag{4.3}$$

where

$$\begin{aligned} &\gamma_1(t) = (e^{2kt} - 1)e^{-4kt}, \\ &\widetilde{\gamma}(t) = k^* e^{kt} \gamma_1(t) = k^* \left(1 - e^{-kt}\right)(1 + e^{-kt})e^{-kt}, \\ &\gamma(t) = \textstyle\int_0^t \widetilde{\gamma}(s) ds = \dfrac{k^*}{3k}\left(2 - 3e^{-kt} + e^{-3kt}\right) = \dfrac{k^*}{3k}\left(1 - e^{-ks}\right)^2 (2 + e^{-ks}), \\ &\text{and } k^* = \min(1, k)/2. \end{aligned}$$

By positive solution to the bvp (4.1), we mean a function $u \in C^2(\mathbb{R}^+) \cap W^{3,1}(I)$ such that $u > 0$ in I and $u(0) = u'(0) = \lim_{t \to +\infty} u'(t) = 0$, satisfying the differential equation in (4.1).

Notice that the nonlinearity f may exhibit singular at the solution and at its derivative. It is well known that the bvp (4.1) is called positone if $q(t) = 0$ a.e. $t \in I$, and semipositone if $q(t) > 0$ a.e. t in some interval of I.

BVPs for third-order differential equation originate from many applications in physics and engineering. For example, the deflection of a curved beam having a constant or varying cross section, a three layer beam, electromagnetic waves, gravity driven flows produce third-order boundary-value problems. During the last two decades, there has been many works dealing with several aspects of such BVPs, see, [1, 16, 30, 39, 56, 62] and the references therein. Often, for physical considerations, the positivity of the solution is required and many authors established existence and multiplicity results for positive solutions to such bvps posed on bounded intervals, where the nonlinear term is positive and satisfies either superlinear or sublinear growth conditions, see [17, 29, 32, 58, 70, 65, 66, 68, 69] and the references therein.

Because of a lack of compactness, the case where such bvps are posed on unbounded intervals is somewhat complicated and they has not been so extensively investigated. This case have been considered in [7, 8, 16, 24, 25, 26, 27, 41, 48, 49, 55, 60] and, to the authors' knowledge, there are no papers in the literature considering the singular semipositone version of such bvps. Thus, the purpose of this paper is to fill in the gap in this area.

Our approach in this work is based on a fixed point formulation of the bvp (4.1) and the main existence result in this work is then proved by the Guo-Krasnoselskii's version of expansion and compression of a cone principal in a Banach space.

4.2 Fixed point formulation

In all this chapter, we let

$$E = \{u \in C^1(\mathbb{R}^+, \mathbb{R}) : \lim_{t\to+\infty} e^{-kt}u(t) = 0,\ \lim_{t\to+\infty} e^{-kt}u'(t) = 0\}.$$

Endowed with the norm $\|u\| = \|u\|_k + \|u'\|_k$ where $\|u\|_k = \sup_{t\geq 0} e^{-kt}|u(t)|$, E becomes a Banach space.

The following lemma is an adapted version to the case of the space E of Corduneanu's compactness criterion ([19], p. 62). It will be used in this work to prove that some operator is compact.

Lemma 4.1. *A nonempty subset M of E is relatively compact if the following conditions hold:*

(a) M is bounded in E,

(b) the sets $\{u : u(t) = e^{-kt}x(t),\ x \in M\}$ and $\{u : u(t) = e^{-kt}x'(t),\ x \in M\}$ are locally equicontinuous on $[0, +\infty)$, and

(c) the sets $\{u : u(t) = e^{-kt}x(t),\ x \in M\}$ and $\{u : u(t) = e^{-kt}x'(t),\ x \in M\}$ are equiconvergent at $+\infty$.

Throughout, P denotes the cone in E defined by

$$P = \{u \in E :\ u'(t) \geq \widetilde{\gamma}(t)||u|| \text{ and } u(t) \geq \gamma(t)||u|| \text{ for all } t > 0\}$$

Let $G, \widetilde{G} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be the functions defined by

$$G(t,s) = \frac{1}{k^2} \begin{cases} e^{-ks}\left(\cosh(kt) - 1\right) & \text{if } t \le s, \\ -e^{-kt}\sinh(ks) + \left(1 - e^{-ks}\right) & \text{if } s \le t, \end{cases}$$

$$\widetilde{G}(t,s) = \frac{\partial G}{\partial t}(t,s) = \frac{1}{k} \begin{cases} e^{-ks}\sinh(kt) & \text{if } t \le s, \\ e^{-kt}\sinh(ks) & \text{if } s \le t. \end{cases}$$

Lemma 4.2. *The functions G and $\widetilde{G}$ satisfy:*

(a) *For all $t, s \in \mathbb{R}^+$ we have $G(t,s) \ge 0$ and $\widetilde{G}(t,s) \ge 0$.*

(b) *The functions G and $\widetilde{G}$ are continuous and for all $s \ge 0$, we have*

$$G(0,s) = \widetilde{G}(0,s) = 0. \tag{4.4}$$

(c) *For all $t, s \ge 0$, we have*

$$G(t,s) \le \frac{1}{k^2}(1 - e^{-ks}) \le \frac{1}{k^2}, \quad \widetilde{G}(t,s) \le \widetilde{G}(s,s) \le \frac{1}{2k}.$$

(d) *For all $s, t, \tau \ge 0$, we have*

$$e^{-ks}\widetilde{G}(s,s) \ge \widetilde{G}(t,s)e^{-kt} \ge \widetilde{\gamma}(t)\widetilde{G}(\tau,s)e^{-k\tau}.$$

(e) *For all $t_2, t_1 \ge 0$, we have*

$$\left|e^{-kt_2}G(t_2,s) - e^{-kt_1}G(t_1,s)\right| \le \frac{3}{2k}\left|t_2 - t_1\right| \tag{4.5}$$

$$\left|e^{-kt_2}\widetilde{G}(t_2,s) - e^{-kt_1}\widetilde{G}(t_1,s)\right| \le \left|t_2 - t_1\right| \tag{4.6}$$

Proof. Assertions (a), (b) and (c) are easy to prove, Assertion (d) is proved in [26]. Assertion (e) is obtained by the mean value theorem. □

Lemma 4.3. *Assume that Hypothesis (4.2) holds, then the function ϕ where for $t \in I$, $\phi(t) = \int_0^{+\infty} G(t,s)q(s)ds$, satisfies the following upper bound:*

$$\phi(t) \le \phi^*\gamma(t), \quad \phi'(t) \le \phi^*\widetilde{\gamma}(t) \;\; \textit{for all } t \in I$$

where

$$\phi^* = \max\left(\sup_{t>0}\frac{\phi(t)}{\gamma(t)}, \sup_{t>0}\frac{\phi'(t)}{\widetilde{\gamma}(t)}\right).$$

Proof. For all $t > 0$, we have

$$\begin{aligned}
\frac{\phi'(t)}{\widetilde{\gamma}(t)} &= \frac{3k}{k^*}\frac{\int_0^{+\infty}\widetilde{G}(t,s)q(s)ds}{(1-e^{-kt})(1+e^{-kt})e^{-kt}} \leq \frac{3k}{k^*}\frac{\int_0^{+\infty}\widetilde{G}(t,s)q(s)ds}{(1-e^{-kt})e^{-kt}} \\
&= \frac{3}{2k^*}\left(\frac{\int_0^t \sinh(ks)\,q(s)ds}{(1-e^{-kt})} + \frac{\sinh(kt)\int_t^{+\infty} e^{-ks}q(s)ds}{(1-e^{-kt})e^{-kt}}\right) \\
&= \frac{3}{2k^*}\left(\frac{\int_0^t \sinh(ks)\,e^{-ks}e^{ks}q(s)ds}{(1-e^{-kt})} + \frac{\sinh(kt)\int_t^{+\infty} e^{-2ks}e^{ks}q(s)ds}{(1-e^{-kt})e^{-kt}}\right) \\
&\leq \frac{3}{2k^*}\frac{\sinh(kt)\,e^{-kt}}{(1-e^{-kt})}\int_0^{+\infty} e^{ks}q(s)ds = \frac{3}{2k^*}\left(1+e^{-kt}\right)\int_0^{+\infty} e^{ks}q(s)ds \\
&\leq \frac{3}{k^*}\int_0^{+\infty} e^{ks}q(s)ds := \overline{\phi}.
\end{aligned}$$

This proves that $\sup_{t>0}\left(\phi'(t)/\widetilde{\gamma}(t)\right) < \infty$.

Therefore, we have

$$\frac{\phi(t)}{\gamma(t)} = \frac{\int_0^t \phi'(s)ds}{\gamma(t)} \leq \frac{\int_0^t \overline{\phi}\widetilde{\gamma}(s)ds}{\gamma(t)} = \overline{\phi},$$

leading to $\sup_{t>0}\left(\phi(t)/\gamma(t)\right) < \infty$. The proof is complete. □

Lemma 4.4. *Assume that Hypothesis (4.2) and (4.3) hold. Then for all $r, R \in \mathbb{R}$ with $R > r > \phi^*$ there exists a compact operator $T_{r,R} : P \cap \left(\overline{B}(0,R) \setminus B(0,r)\right) \to P$ such that if v is a fixed point of $T_{r,R}$ then $u = v - \phi$ is a positive solution to the bvp (4.1).*

Proof. Let r, R be two real numbers such that $R > r > \phi^*$ and set $\Omega = P \cap \left(\overline{B}(0,R) \setminus B(0,r)\right)$. In all this proof, we let by Φ the function defined by

$$\Phi(s) = \omega_R(s)\,\Psi_R\left((r-\phi^*)\,e^{-ks}\gamma(s), (r-\phi^*)\,e^{-ks}\widetilde{\gamma}(s)\right),$$

where ω_R and Ψ_R are the functions given by Hypothesis (4.3) for $\rho = R$ and ϕ^* is the constant given by Lemma 4.3. The proof is divided into four steps.

Step 1. Existence of the operator $T_{r,R}$. We have from the definition of the cone P and Lemma 4.3 that, for all $v \in \Omega$ and all $t > 0$,

$$v(t) - \phi(t) \geq (\|v\| - \phi^*)\,\gamma(t) \geq (r-\phi^*)\,\gamma(t) > 0.$$

$$v'(t)-\phi'(t)\geq(\|v\|-\phi^*)\widetilde{\gamma}(t)\geq(r-\phi^*)\widetilde{\gamma}(t)>0.$$

Therefore, for all $v\in\Omega$ the expression

$$f_{r,R}v(t)=f(t,v(t)-\phi(t),v'(t)-\phi'(t))+q(t) \tag{4.7}$$

is well defined.

Let $v\in\Omega$, for all $s>0$ we have

$$\begin{aligned} f_{r,R}v(s) &= f(s,e^{ks}\left(e^{-ks}(v(s)-\phi(s))\right),e^{ks}\left(e^{-ks}(u'(s)-\phi'(s))\right))+q(s) \\ &\leq \Phi(s)+q(s), \end{aligned}$$

then

$$\int_0^{+\infty}G(t,s)f_{r,R}v(t)\,ds\leq\frac{1}{k^2}\int_0^{+\infty}f_{r,R}v(s)\,ds\leq\frac{1}{k^2}\int_0^{+\infty}(\Phi(s)+q(s))\,ds<\infty$$

and

$$\int_0^{+\infty}\widetilde{G}(t,s)f_{r,R}v(s)ds\leq\frac{1}{2k}\int_0^{+\infty}f_{r,R}v(s)ds\leq\frac{1}{2k}\int_0^{+\infty}(\Phi(s)+q(s))\,ds<\infty.$$

Thus, let w and z be the function defined by

$$w(t)=\textstyle\int_0^{+\infty}G(t,s)f_{r,R}v(s)ds\quad,\qquad z(t)=\int_0^{+\infty}\widetilde{G}(t,s)f_{r,R}v(s)ds.$$

Since for all $t>0$,

$$\begin{aligned} w(t) &= -\frac{e^{-kt}}{k^2}\int_0^t\sinh(ks)f_{r,R}v(s)ds+\frac{1}{k^2}\int_0^t(1-e^{-ks})f_{r,R}v(s)ds \\ &\quad+\frac{\cosh(kt)-1}{k^2}\int_0^t e^{-ks}f_{r,R}v(s)ds, \end{aligned}$$

we see that w is differentiable on $\mathbb{R}^+$ and for all $t\geq0$

$$\begin{aligned} w'(t) &= \frac{e^{-kt}}{k}\int_0^t\sinh(ks)f_{r,R}v(s)ds+\frac{\sinh(kt)}{k}\int_t^{+\infty}e^{-ks}f_{r,R}v(s)ds \\ &= \int_0^{+\infty}\widetilde{G}(t,s)f_{r,R}v(s)ds=z(t) \end{aligned}$$

with z continuous on $\mathbb{R}^+$.

At this stage, we have proved that w belongs to $C^1(\mathbb{R}^+, \mathbb{R})$ and we need to prove that $w \in E$. Thus, we have to prove that $\lim_{t\to+\infty} e^{-kt}v(t) = \lim_{t\to+\infty} e^{-kt}w'(t) = 0.$. Clearly for all $t > 0$, $w(t), w'(t) > 0$ and we have

$$\begin{aligned} e^{-kt}w(t) &= e^{-kt}\int_0^{+\infty} G(t,s)f_{r,R}v(s)ds \leq \frac{e^{-kt}}{k^2}\int_0^{+\infty} (\Phi(s)+q(s))\,ds \\ e^{-kt}w'(t) &= e^{-kt}\int_0^{+\infty} \widetilde{G}(t,s)f_{r,R}v(s)ds \leq \frac{e^{-kt}}{2k}\int_0^{+\infty} (\Phi(s)+q(s))\,ds. \end{aligned}$$

The above two estimates show that $\lim_{t\to+\infty} e^{-kt}w(t) = \lim_{t\to+\infty} e^{-kt}w'(t) = 0$.

Now for all $t, \tau > 0$, we have from Assertion (c) in Lemma 4.2

$$\begin{aligned} w'(t) &= e^{kt}\int_0^{+\infty} e^{-kt}\widetilde{G}(t,s)f_{r,R}v(s)ds \\ &\geq e^{kt}\gamma_1(t)\int_0^{+\infty} e^{-k\tau}\widetilde{G}(\tau,s)f_{r,R}v(s)ds \\ &= e^{kt}\gamma_1(t)e^{-k\tau}w'(\tau). \end{aligned}$$

Passing to the supremum on τ, we obtain

$$w'(t) \geq e^{kt}\gamma_1(t)\left\|w'\right\|_k \text{ for all } t > 0. \tag{4.8}$$

Since for all $t > 0$

$$w(t) = \int_0^t e^{k\xi}\left(e^{-k\xi}w'(\xi)\right)d\xi \leq \int_0^t e^{k\xi}d\xi\left\|w'\right\|_k \leq \frac{e^{k\xi}}{k}\left\|w'\right\|_k$$

we have

$$\left\|w'\right\|_k \geq k\left\|w\right\|_k. \tag{4.9}$$

Therefore, (4.8) Combined with (4.9) leads to

$$w'(t) \geq ke^{kt}\gamma_1(t)\left\|w'\right\|_k \text{ for all } t > 0$$

then to

$$w'(t) \geq \widetilde{\gamma}(t)\left\|w\right\| \text{ for all } t > 0. \tag{4.10}$$

Integrating (4.10), yields $w(t) \geq \gamma(t)\left\|w\right\|$ for all $t > 0$.

Thus, we have proved that $w \in P$ and the operator $T_{r,R} : \Omega \to P$ where for $v \in \Omega$

$$T_{r,R}u(t) = \int_0^{+\infty} G(t,s)f_{r,R}v(s)ds,$$

is well defined.

Step 2. The operator $T_{r,R}$ is continuous. Let (v_n) be a sequence in Ω such that $\lim_{n\to\infty} v_n = v$ in E. For all $n \geq 1$, we have

$$\begin{aligned} \|T_{r,R}v_n - T_{r,R}v\|_k &= \sup_{t>0} e^{-kt} \left|T_{r,R}v_n(t) - Tv(t)\right| \\ &\leq \frac{1}{k^2} \int_0^{+\infty} |f_{r,R}v_n(s) - f_{r,R}v(s)|\, ds \end{aligned}$$

and

$$\begin{aligned} \left\|(T_{r,R}v_n)' - (T_{r,R}v)'\right\|_k &= \sup_{t>0} e^{-kt} \left|(T_{r,R}v_n)'(t) - (T_{r,R}v)'(t)\right| \\ &\leq \frac{1}{2k} \int_0^{+\infty} |f_{r,R}v_n(s) - f_{r,R}v(s)|\, ds. \end{aligned}$$

Because of

$$|f_{r,R}v_n(s) - f_{r,R}v(s)| \to 0, \text{ as } n \to +\infty$$

for all $s > 0$ and

$$|f_{r,R}v_n(s) - f_{r,R}v(s)| \leq 2\Phi(s) \text{ with } \int_0^{+\infty} \Phi(s)\, ds < \infty,$$

Lebesgues dominated convergence theorem guarantees that
$\lim_{n\to\infty} \|T_{r,R}v_n - T_{r,R}v\| = 0$. Hence, we have proved that T is continuous.

Step 3. $T_{r,R}$ is compact. For all $v \in \Omega$, we have

$$\|T_{r,R}u\| \leq \max\left(\frac{1}{k^2}, \frac{1}{2k}\right) \int_0^{+\infty} (\Phi(s) + q(s)ds) < \infty,$$

proving that $T(\Omega)$ is bounded in E.

Now, let $t_1, t_2 \in [\eta, \xi] \subset \mathbb{R}^+$, for all $v \in \Omega$, we have from (4.5) and (4.6) the estimates

$$\begin{aligned} |e^{-kt_1}T_{r,R}v(t_1) - e^{-kt_2}T_{r,R}v(t_2)| &\leq \int_0^{+\infty} |e^{-kt_1}G(t_1,s) - e^{-kt_2}G(t_2,s)|\Phi(s)\, ds \\ &\leq \frac{3}{2k}|t_2 - t_1| \int_0^{+\infty} \Phi(s)\, ds \end{aligned}$$

and

$$\begin{aligned} |e^{-kt_1}(T_{r,R}v)'(t_1) - e^{-kt_2}(T_{r,R}v)'(t_2)| &\leq \int_0^{+\infty} |e^{-kt_1}\widetilde{G}(t_1,s) - e^{-kt_2}\widetilde{G}(t_2,s)|\Phi\,(s)\,ds \\ &\leq |t_2 - t_1| \int_0^{+\infty} \Phi\,(s)\,ds. \end{aligned}$$

Proving the equicontinuity of $T\,(\Omega)$ on bounded intervals.

For all $v \in \Omega$ and $t > 0$, we have

$$|e^{-kt}Tv(t)| \leq \frac{e^{-kt}}{k^2} \int_0^{+\infty} (\Phi\,(s) + q(s))\,ds$$

and

$$|e^{-kt}\,(Tu)'\,(t)| \leq \frac{e^{-kt}}{k} \int_0^{+\infty} (\Phi\,(s) + q(s))\,ds.$$

Thus, the equiconvergence of $T\,(\Omega)$ follows from the fact that $\lim_{t\to\infty} e^{-kt} = 0$.

In view of Lemma 4.1, the operator is compact.

Step 4. if v is a fixed point of T then $u = v - \phi$ is a positive solution to the bvp (4.1). Let $v \in \Omega$ be a fixed point of T, then for all $t > 0$

$$\begin{aligned} u(t) = v(t) - \phi\,(t) \geq (\|v\| - \phi^*)\,\gamma\,(t) \geq (r - \phi^*)\,\gamma\,(t) > 0, \\ u'(t) = v'(t) - \phi'\,(t) \geq (\|v\| - \phi^*)\,\widetilde{\gamma}\,(t) \geq (r - \phi^*)\,\widetilde{\gamma}\,(t) > 0, \end{aligned}$$

and $u = v - \phi$ satisfies

$$\begin{aligned} u(t) &= -\phi\,(t) + \int_0^{+\infty} G(t,s)\,(f((s,u(s),u'(s)) + q(s))\,ds \\ &= -\int_0^{+\infty} G(t,s)q(s)ds + \int_0^{+\infty} G(t,s)\,(f((s,u(s),u'(s)) + q\,(s))\,ds \\ &= \int_0^{+\infty} G(t,s)f((s,u(s),u'(s))ds \end{aligned}$$

and

$$u'(t) = \int_0^{+\infty} \widetilde{G}(t,s)f((s,u(s),u'(s))ds.$$

These with (4.4) lead to $u(0) = u'(0) = 0$.

Differentiating twice in

$$\begin{aligned} u'(t) &= \int_0^{+\infty} \widetilde{G}(t,s)\phi(s) f((s,u(s),u'(s))ds \\ &= \frac{e^{-kt}}{k}\int_0^t \sinh(ks)\phi(s) f(s,u(s),u'(s))ds \\ &\quad +\frac{\sinh(kt)}{k}\int_t^{+\infty} e^{-ks}\phi(s) f(s,u(s),u'(s))ds, \end{aligned}$$

we see that $-u'''(t)+ku'(t)=f(t,u(t),u'(t))$ for all $t>0$.

It remains to prove that $\lim\limits_{t\to+\infty} u'(t)=0$. We have

$$\begin{aligned} u'(t) &\leq \frac{1}{ke^{kt}}\int_0^t \sinh(ks)\left|f(s,u(s),u'(s))\right| ds+\frac{\sinh(kt)}{k}\int_t^{+\infty} e^{-ks}\left|f(s,u(s),u'(s))\right| ds \\ &\leq \frac{1}{ke^{kt}}\int_0^t \sinh(ks)\Phi_R(s)\, ds+\frac{\sinh(kt)}{k}\int_t^{+\infty} e^{-ks}\Phi_R(s)\, ds, \end{aligned}$$

$$\lim_{t\to+\infty}\frac{\sinh(kt)}{k}\int_t^{+\infty} e^{-ks}\Phi_R(s)\, ds \leq \frac{1}{k}\lim_{t\to+\infty}\frac{\sinh(kt)}{e^{-kt}}\int_t^{+\infty} \Phi_R(s)\, ds = 0$$

and the L'Hopital's rule leads to

$$\begin{aligned} \lim_{t\to+\infty}\frac{1}{ke^{kt}}\int_0^t \sinh(ks)\Phi_R(s)\, ds &= \lim_{t\to+\infty}\frac{\sinh(kt)}{ke^{kt}}\frac{\int_t^{+\infty} e^{-ks}\Phi_R(s)\, ds}{e^{-kt}} \\ &= \frac{1}{k}\lim_{t\to+\infty}\frac{\sinh(kt)}{ke^{kt}}\lim_{t\to+\infty}\Phi_R(t)=0. \end{aligned}$$

The above calculations show that $\lim_{t\to+\infty} u'(t) = 0$, completing the proof of the lemma. □

4.3 Main result

The main result of this paper needs to introduce the following notations. For $\alpha \in L^1(I)$ with $\alpha(t)\geq 0$ a.e. $t>0$ and $\sigma>1$, we let

$$\Gamma(\alpha)=\sup_{t>0} e^{-kt}\int_0^{+\infty} G(t,s)\alpha(s)\, ds+\sup_{t>0} e^{-kt}\int_0^{+\infty} \widetilde{G}(t,s)\alpha(s)\, ds,$$

$$\Delta(\alpha,\sigma)=\sup_{t>0} e^{-kt}\int_{1/\sigma}^{\sigma} G(t,s)\alpha(s)\ ds+\sup_{t>0} e^{-kt}\int_{1/\sigma}^{\sigma} \widetilde{G}(t,s)\alpha(s)\ ds.$$

Theorem 4.1. *Suppose that Hypotheses (4.2) and (4.3) hold and*

(a) *there exist a function* $\alpha \in L^1(I)$ *and* $R_1 > \max(\phi^*, \Gamma(\alpha))$ *such that*

$$f(t, e^{kt}u, e^{kt}v) + q(t) \leq \alpha(t)$$

for a.e. $t > 0$ *and all* $u, v \in I$ *with* $|(u, v)| \leq R_1$,

(b) *there exist* $\sigma > 1$, *a function* $\beta \in L^1(I)$ *and* $R_2 \in (\phi^*, \Delta(\beta, \sigma))$ *with* $R_2 \neq R_1$ *such that*

$$f(t, e^{kt}u, e^{kt}v) + q(t) \geq \beta(t),$$

for a.e. $t \in [1/\sigma, \sigma]$, *all* $u \in [\gamma_\sigma (R_2 - \phi^*), R_2]$ *and all* $v \in [\widetilde{\gamma}_\sigma (R_2 - \phi^*), R_2]$, *where* $\gamma_\sigma = \min_{s\in[1/\sigma,\sigma]} \left(e^{-ks}\gamma(s)\right)$ *and* $\widetilde{\gamma}_\sigma = \min_{s\in[1/\sigma,\sigma]} \left(e^{-ks}\widetilde{\gamma}(s)\right)$.

Then, the bvp (4.1) admits a bounded positive solution.

Proof. Without loss of generality, assume that $R_1 < R_2$ and let $T = T_{R_1,R_2}$ be the operator given by Lemma 4.4. The following estimates hold, for all $v \in P \cap \partial B(0, R_1)$ and all $t > 0$,

$$\begin{aligned}
&e^{-kt}Tv(t) = \\
&\quad e^{-kt}\textstyle\int_0^{+\infty} G(t,s) f(s, e^{ks}(v(s) - \phi(s))\, e^{-ks}, e^{ks}(v'(s) - \phi'(s))\, e^{-ks}) ds \\
&+e^{-kt}\textstyle\int_0^{+\infty} G(t,s) q(s) ds \\
&\quad \leq e^{-kt}\textstyle\int_0^{+\infty} G(t,s)\,\alpha(s)\, ds \\
&\quad \leq \sup_{t>0} e^{-kt}\textstyle\int_0^{+\infty} G(t,s)\,\alpha(s)\, ds.
\end{aligned}$$

Passing to the supremum in the above estimates, we get

$$\|Tv\|_k \leq \sup_{t>0} e^{-kt} \int_0^{+\infty} G(t,s)\,\alpha(s)\, ds. \tag{4.11}$$

Similarly, we have

$$\begin{aligned}
e^{-kt}(Tv)'(t) &= e^{-kt}\int_0^{+\infty} \widetilde{G}(t,s)\left(f(s, v(s) - \phi(s), v'(s) - \phi'(s)) + q(s)\right) ds \\
&\leq e^{-kt}\int_0^{+\infty} \widetilde{G}(t,s)\,\alpha(s)\, ds \leq \sup_{t>0} e^{-kt}\int_0^{+\infty} \widetilde{G}(t,s)\,\alpha(s)\, ds,
\end{aligned}$$

leading to

$$\|(Tv)'\|_k \leq \sup_{t>0} e^{-kt}\int_0^{+\infty} \widetilde{G}(t,s)\,\alpha(s)\, ds. \tag{4.12}$$

Summing (4.11) with (4.12), we obtain

$$\begin{aligned} \|Tv\| &\leq \sup_{t>0} e^{-kt} \int_0^{+\infty} \widetilde{G}(t,s)\,\alpha(s)\,ds + \sup_{t>0} e^{-kt} \int_0^{+\infty} \widetilde{G}(t,s)\,\alpha(s)\,ds \\ &= \Gamma(\alpha) \leq R_1 = \|v\|\,. \end{aligned}$$

For all $v \in P \cap \partial B(0, R_2)$ and $s \in [1/\sigma, \sigma]$,

$$\begin{aligned} R_2 &\geq (v(s) - \phi(s))\, e^{-ks} \geq (R_2 - \phi^*)\, \gamma(s) e^{-ks} = (R_2 - \phi^*)\, \gamma_\sigma \\ R_2 &\geq (v'(t) - \phi'(s))\, e^{-ks} \geq (R_2 - \phi^*)\, \widetilde{\gamma}(s) e^{-ks} = (R_2 - \phi^*)\, \widetilde{\gamma}_\sigma \end{aligned} \tag{4.13}$$

Assumption **(b)** and (4.13) lead to the following estimates

$$\begin{aligned} &\|Tu\|_k \geq \\ &\sup_{t>0} \left(e^{-kt} \int_{1/\sigma}^{\sigma} G(t,s) f(s, e^{ks}\,(v(s) - \phi(s))\, e^{-ks}, e^{ks}\,(v'(s) - \phi'(s))\, e^{-ks}) ds \right. \\ &\qquad \left. + e^{-kt} \int_{1/\sigma}^{\sigma} G(t,s) q(s) ds \right) \\ &\qquad \geq \sup_{t>0} \left(e^{-kt} \int_{1/\sigma}^{\sigma} G(t,s) \beta(s)\ ds \right) \end{aligned}$$

and similarly

$$\|\,(Tv)'\,\|_k \geq \sup_{t \in >0} \left(e^{-kt} \int_{1/\sigma}^{\sigma} \widetilde{G}(t,s) \beta(s)\ ds \right).$$

Summing the above inequalities, we obtain

$$\begin{aligned} \|Tv\| &\geq \sup_{t>0} \left(e^{-kt} \int_{1/\sigma}^{\sigma} \widetilde{G}(t,s) \beta(s)\ ds \right) + \sup_{t>0} \left(e^{-kt} \int_{1/\sigma}^{\sigma} \widetilde{G}(t,s) \beta(s)\ ds \right) \\ &= \Delta(\beta, \sigma) \geq R_2 = \|v\|\,. \end{aligned}$$

Thus, it follows from Assertion (H_3) in Theorem 1.14 that T_{R_1,R_2} admits a fixed point v such that $R_1 \leq \|v\| \leq R_2$. Then by Lemma 4.4, $u = v - \phi$ is a positive solution to the bvp (4.1).

Now, we have to prove that u is bounded. Since for all $t > 0$,

$$\begin{aligned} \|v\| + \|\phi\| &\geq e^{-kt} u(t) = e^{-kt}\,(v(t) - \phi(t)) \geq (\|v\| - \phi^*)\, e^{-kt} \gamma(t), \\ \|v\| + \|\phi\| &\geq e^{-kt} u'(t) = e^{-kt}\,(v'(t) - \phi'(t)) \geq (\|v\| - \phi^*)\, e^{-kt} \widetilde{\gamma}(t), \end{aligned}$$

we obtain from Hypothesis (4.3) for $\rho = \|u\|\,,$

$$\begin{aligned} u(t) &= Tu(t) \leq \int_0^{+\infty} G(t,s)\, |f(s, u(s), u'(s))|\, ds \\ &\leq \frac{1}{k^2} \int_0^{+\infty} \omega_\rho(s)\, \Psi_\rho(\left(e^{-ks} u(s)\right), \left(e^{-ks} u'(s)\right)) ds \\ &\leq \frac{1}{k^2} \int_0^{+\infty} \omega_\rho(s)\, \Psi_R((\|v\| - \phi^*)\, e^{-ks} \gamma(s), (\|v\| - \phi^*)\, e^{-ks} \widetilde{\gamma}(s)) ds < \infty. \end{aligned}$$

The proof is complete. □

Set for $\sigma > 1$

$$f_\sigma = \liminf_{|(w,z)|\to+\infty} \left(\min_{t\in[1/\sigma,\sigma]} \frac{f(t, e^{kt}w, e^{kt}z)}{w+z} \right).$$

We obtain from Theorem 4.1 the following corollary:

Corollary 4.1. *Suppose that Hypotheses (4.2) and (4.3) hold and*

(c) *there exists $R_1 > \phi^*$ such that $\Gamma(\alpha_1) < R_1$ where*

$$\alpha_1 = \omega_{R_1}(s)\,\Psi_{R_1}\left((R_1 - \phi^*)\,e^{-ks}\gamma(s), (R_1 - \phi^*)\,e^{-ks}\widetilde{\gamma}(s)\right) + q(s),$$

(d) *there exists $\sigma > 1$, such that $f_\sigma \Delta(\beta_0, \sigma) > 1$, where $\beta_0(s) = e^{-ks}\gamma(s)$.*

Then, the bvp (4.1) admits a positive solution.

Proof. Clearly, Condition (a) of in Theorem 4.1 is satisfied for $\alpha = \alpha_1$. We have to prove that Condition (b) also is satisfied. Let $\varepsilon > 0$ be such that $(f_\sigma - \varepsilon)\Delta(\beta_0, \sigma) > 1$. There exists R_∞ such that $f(t, e^{kt}w, e^{kt}z) > (f_\sigma - \varepsilon)(w+z)$ for all $t \in [1/\sigma, \sigma]$ and all w, z with $|(w,z)| \geq R_\infty$. Let

$$R_2 = 1 + \sup\left(R_1, \phi^* + \frac{R_\infty}{\gamma_\sigma}, \frac{\phi^*(f_\sigma - \varepsilon)\Delta(\beta_0, \sigma)}{(f_\sigma - \varepsilon)\Delta(\beta_0, \sigma) - 1} \right)$$

and

$$\beta(t) = (f_\sigma - \varepsilon)(R_2 - \phi^*)\gamma(s)\,e^{-ks} + q(s).$$

where $\gamma_\sigma = \min_{s\in[1/\sigma,\sigma]}\left(e^{-ks}\gamma(s)\right)$ and notice that

$$(f_\sigma - \varepsilon)\Delta(\beta_0, \sigma)(R_2 - \phi^*) > R_2.$$

We have then

$$\begin{aligned} \Delta(\beta, \sigma) = &\sup_{t>0}\left(e^{-kt}\int_{1/\sigma}^{\sigma} G(t,s)\left((f_\sigma - \varepsilon)(R_2 - \phi^*)\gamma(s)\,e^{-ks} + q(s)\right)ds\right) \\ &+ \sup_{t>0}\left(e^{-kt}\int_{1/\sigma}^{\sigma} \widetilde{G}(t,s)\left((f_\sigma - \varepsilon)(R_2 - \phi^*)\gamma(s)\,e^{-ks} + q(s)\right)ds\right) \\ &\geq (f_\sigma - \varepsilon)\Delta(\beta_0, \sigma)(R_2 - \phi^*) > R_2. \end{aligned}$$

The proof is complete. □

4.4 Example

Consider the case of the bvp (4.1) where

$$f(t,u,v) = e^{-\delta t}\left(\left(\frac{u+v}{e^{kt}}\right)^p + \frac{B\,(u+v)^2}{Be^{kt}+u+v} - 1\right)$$

where $\delta > (1-p)\,k$, $p \in (-1,0)$ and $B > 0$.

Clearly, Hypothesis (4.2) is satisfied for $q(t) = e^{-\delta t}$ and we have

$$f(t, e^{kt}w, e^{kt}z) = e^{-\delta t}\left((w+z)^p + \frac{Be^{kt}\,(w+z)^2}{B+w+z} - 1\right),$$

leading to

$$\begin{aligned}\left|f(t, e^{kt}w, e^{kt}z)\right| &= \left|e^{-(\delta-k)t}\left(e^{-kt}\,(w+z)^p + \frac{B\,(w+z)^2}{B+w+z} - e^{-kt}\right)\right| \\ &\leq e^{-(\delta-k)t}\left((w+z)^p + \frac{B\,(w+z)^2}{B+w+z} + 1\right).\end{aligned}$$

Therefore, Hypothesis (4.3) is satisfied for all $\rho > 0$ with

$$\omega_\rho\,(s) = e^{-(\delta-k)s} \quad \text{and} \quad \Psi_\rho\,(w,z) = (w+z)^p + \frac{B\rho^2}{B+\rho} + 1$$

for all $s > 0$ and all $w, z > 0$ with $|(w,z)| = w+z \leq \rho$.

We have then

$$\begin{aligned}\omega_\rho\,(s)\,\psi_\rho\left(\rho e^{-ks}\gamma\,(s)\,, \rho e^{-ks}\widetilde{\gamma}\,(s)\right) &= e^{-(\delta-k)s}\left(1+\tfrac{B\rho^2}{B+\rho}\right) \\ &\quad + (k^*\rho)^p\,e^{-(\delta+pk-2k)s}\left(1-e^{-ks}\right)^p\theta(s)\end{aligned}$$

where

$$\theta(s) = \left(\frac{1}{3k}\left(1-e^{-ks}\right)\left(2+e^{-ks}\right) + e^{-ks}\left(1+e^{-ks}\right)\right)^p$$

and satisfies

$$\left(2+\frac{1}{k}\right)^p \leq \theta(s) \leq 2^p < 1.$$

Because of $\delta > (1-p)\,k$ and $p \in (-1,0)$, we have

$$\lim_{s\to+\infty}\omega_R(s)\psi_R\left(Re^{-ks}\gamma\,(s)\,, Re^{-ks}\widetilde{\gamma}\,(s)\right) = 0 \text{ and}$$
$$\int_0^{+\infty}\omega_R(s)\psi_R\left(Re^{-ks}\gamma\,(s)\,, Re^{-ks}\widetilde{\gamma}\,(s)\right)ds < \infty.$$

For

$$\alpha_1(t) = \omega_R(t)\psi_R\left((R-\phi^*)\,e^{-ks}\gamma(t),(R-\phi^*)\,e^{-ks}\widetilde{\gamma}(t)\right)+q(t)$$

straightforward computations lead to

$$\Gamma(\alpha_1) \leq \Lambda(R) = \widetilde{k}\left(\lambda(p,\delta,k)(R-\phi^*)^p + \mu(\delta,k)\frac{B(R-\phi^*)^2}{B+(R-\phi^*)} + \eta(\delta,k)\right)$$

where

$$\widetilde{k} = \frac{1}{k^2}+\frac{1}{2k}, \quad \mu(\delta,k) = \int_0^{+\infty}\omega_R(s)ds = \frac{1}{\delta-k},$$
$$\eta = \int_0^{+\infty}(\omega_R(s)+q(s))\,ds = \frac{1}{\delta-k}+\frac{1}{\delta}.$$

and

$$\begin{aligned}\lambda(p,\delta,k) &= (k^*)^p\int_0^{+\infty}e^{-(\delta+pk-k)s}\left(1-e^{-ks}\right)^p\theta(s)ds\\ &\leq (k^*)^p\int_0^{+\infty}e^{-(\delta+pk-k)s}\left(1-e^{-ks}\right)^p ds\\ &\leq (k^*)^p\left(\int_0^{1/k}\left(1-e^{-ks}\right)^p ds + \left(1-e^{-1}\right)^p\int_{1/k}^{+\infty}e^{-(\delta+pk-k)s}ds\right)\\ &\leq (k^*)^p\left(2^{-p}\int_0^{1/k}(ks)^p(2-ks)^p ds + \frac{\left(1-e^{-1}\right)^p}{(\delta+pk-k)}\right)\\ &\leq (k^*)^p\left(2^{-p}k^p\int_0^{1/k}s^p ds + \frac{\left(1-e^{-1}\right)^p}{(\delta+pk-k)}\right) \leq (k^*)^p\left(\frac{1}{2^p k(p+1)}+\frac{\left(1-e^{-1}\right)^p}{(\delta+pk-k)}\right).\end{aligned}$$

We have

$$\begin{aligned}\Lambda(1+\phi^*) &= \widetilde{k}\left(\lambda(p,\delta,k)+\mu(\delta,k)\frac{B}{B+1}+\eta(\delta,k)\right)\\ &\leq \widetilde{k}\left(\lambda(\delta,k)+\mu(\delta,k)+\eta(\delta,k)\right).\end{aligned}$$

The above calculations show that for k large enough we have

$$\Lambda(1+\phi^*) \leq 1 \leq 1+\phi^*$$

and Condition (c) in Corollary 4.1 is satisfied for $R = 1+\phi^*$.

Clearly, we have $f_\sigma = +\infty$ for all $\sigma > 1$. Therefore, we conclude from Corollary 4.1 and all the above calculations that if k is large enough then this case of the bvp (4.1) admits a positive solution.

CONCLUSION

The problem we studied is the existence of the positive solution for a certain class of the third-order differential equations with the same boundary conditions. This problem is converted to the problem of the fixed point and our work allowed us to give contributions in the fixed point theory. We have imposed assumptions on the nonlinearity depending on wether it depends on which variables and in order to extend the study to the class of the third-order differential equations, other suitable hypothesis in the case of continuity and singularity of the nonlinearity. In addition to the problem of the existence of the positive solution and under suitable Hypotheses of the nonlinearity, we are interested to the question of boundedness of the solution. The eigenvalue criteria used to prove existence of a positive solution and the the behavior of the nonlinearity are the basis of our results. This work will permit us to investigate of solution which is not necessarily positive because the aim of any work is to develop other techniques to solve the differential equations. This certainly will help to the future study for another class.

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