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Titre

Suites r-Fibonacci bi-périodiques et nombres de Horadam hyper-duaux

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By: **N. Rosa AIT-AMRANE**

Title

Bi-Periodic r-Fibonacci Sequences and Horadam Hyper-Dual Numbers

Publicly defended, on 27 / 03 /2022, in front of the jury composed of:

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In the name of Allah, Most Gracious and Merciful

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ABSTRACT

In this thesis, entitled " *Bi-Periodic r-Fibonacci Sequences and Horadam Hyper-Dual Numbers,*" related to Number Theory and Enumerative Combinatorics, we introduce new sequences, new numbers and polynomials and we study their algebraic properties. This work is based on our four papers [2, 3, 4, 79]. Many powerful methods are used to study these sequences, numbers and polynomials, like linear recurrence relations, generating functions, explicit formulas and Binet forms. Our manuscript is structured as follows.

First, for a positive integer r, we study bi-periodic r-Fibonacci sequence, and we define its family of companion sequences, each companion sequence is named bi-periodic r-Lucas sequence of type s, with $1 \leq s \leq r$. These sequences generalize the classical Fibonacci and Lucas sequences. This construction of the r-Lucas sequences of type s is one of our most important results. Moreover, we establish the link between the bi-periodic r-Fibonacci sequence and its companion sequences. Furthermore, we give their properties as linear recurrence relations, generating functions, explicit formulas and Binet forms [2].

Afterwards, we introduce the bi-periodic Horadam hybrid numbers and deduce particular cases: the bi-periodic Fibonacci hybrid numbers and the bi-periodic Lucas hybrid numbers, respectively. We establish the generating functions, the Binet forms and some basic properties of these new hybrid numbers [79].

Also, we define a bivariate r-Fibonacci hybrid polynomials and bivariate r-Lucas hybrid polynomials of type s and we obtain some properties of these polynomials [3].

Finally, we develop a new class of quaternions, called hyper-dual Horadam quaternions, which are constructed from the quaternions whose components are hyper-dual Horadam numbers. We investigate some basic properties of these

quaternions [4].

Keywords: Bi-periodic *r*-Fibonacci sequence; companion sequence; bi-periodic *r*-Lucas sequence; horadam numbers; hybrid numbers; quaternion numbers.

Résumé

Cette thèse, intitulée " *Suites r-Fibonacci bi-périodiques et nombres de Horadam hyper-duaux* ", s'inscrit dans les domaines de la Théorie des Nombres et de la Combinatoire Enumérative. Nous introduisons de nouvelles suites, de nouveaux nombres et polynômes et nous donnons leurs propriétés algèbriques. Ce travail est basé sur nos quatres articles [2, 3, 4, 79]. Plusieurs méthodes sont utilisées pour étudier ces suites, nombres et polynômes, comme les relations de récurrences linéaires, les fonctions génératrices, les formules explicites et les formes de Binet.

En premier lieu, pour un entier positif non nul r, nous étudions la suite r-Fibonacci bi-périodique. L'un de nos principaux résultats est la définition de la famille de suites compagnons associées ; chacune de ces suites est appelée suite bi-périodique r-Lucas de type s, avec s un entier tel que $1 \leq s \leq r$. Nous établissons, également, le lien entre la suite r-Fibonacci bi-périodique avec chacune de ses suites compagnons. En outre, on donne leurs propriétés comme : les relations de récurrences linéaires, les fonctions génératrices, les formules explicites et les formes de Binet [2].

Ensuite, nous introduisons les nombres bi-périodiques hybrides de Horadam, desquels on déduit les nombres bi-périodiques hybrides de Fibonacci et les nombres bi-périodiques hybrides de Lucas. Nous donnons la fonction génératrice, la forme de Binet et quelques propriétés de base de ces nouveaux nombres hybrides [79].

Nous définissons les polynômes r-Fibonacci hybrid bivariés et r-Lucas hybrid bivariés de type s et nous obtenons quelques propriétés de ces polynômes [3].

Finalement, nous développons aussi une nouvelle classe des quaternions, appelée les quaternions de Horadam hyper-duaux, dont les composantes sont les nombres de Horadam hyper-duaux et on donne quelques propriétés de base de ces nombres quaternions [4].

Mots clés : Suite r-Fibonacci bi-periodique ; suite compagnon ; suite r-Lucas bi-periodique ; nombres de Horadam ; nombres hybrid ; nombres quaternions.

Contents

List of contributions

1. **Bi-periodic r-Fibonacci sequence and bi-periodic r-Lucas sequence of type s.**
   N. R. Ait-Amrane and H. Belbachir. *Hacettepe Journal of Mathematics and Statistics*, Vol. (2022), No. , pp 1-20.
   https://doi.org/10.15672/hujms.825908.

2. **On a new generalization of Fibonacci hybrid numbers.**
   E. Tan and **N. R. Ait-Amrane**. *Indian Journal of Pure and Applied Mathematics* http://arxiv.org/abs/2006.09727. Accepted 2022.

3. **On generalized Fibonacci and Lucas hybrid polynomials.**
   N. R. Ait-Amrane, H. Belbachir and E. Tan. *Turkish Journal of Mathematics* Accepted, 2022.

4. **Hyper-Dual Horadam Quaternions.**
   N. R. Ait-Amrane, İ. Gök and E. Tan. *Miskolc Mathematical Notes*, Vol. 22 (2021), No. 2, pp. 903-913. http://doi.org/10.18514/MMN.2021.3747.

List of Tables

Notations

$\mathbb{C}$	field of complex numbers
$\mathbb{R}$	field of real numbers
$\mathbb{Q}$	field of rational numbers
$\mathbb{Z}$	ring of integers
$\mathbb{Z}^k$	k copy of $\mathbb{Z}$
$\mathbb{Z}^*$	set of nonzero integers
$\mathbb{N}$	set of the nonnegative integers
$\mathbb{N}^*$	set of the positive integers
$\mathbb{H}$	quaternion set numbers
$\mathbb{D}$	dual set numbers
$\mathbb{HD}$	hyper-dual set numbers
$\mathbb{K}$	hybrid set numbers
$\lfloor . \rfloor$	floor function
$\xi\,(.)$	parity function
$:=$	equality by definition (affectation)
$\binom{n}{k}$	binomial coefficient
$\binom{n}{k_1,k_2,\ldots,k_m}$	multinomial coefficient
det	determinant
$\lVert . \rVert$	euclidean norm

F_n	Fibonacci numbers
L_n	Lucas numbers
W_n	Horadam numbers
U_n	generalized (p,q)-Fibonacci numbers
V_n	generalized (p,q)-Lucas numbers
$\mathfrak{U}_n$	generalized $(p,1)$-Fibonacci numbers
$\mathcal{V}_n$	generalized $(p,1)$-Lucas numbers
$F_n(x)$	Fibonacci polynomials
$FH_n(x)$	Fibonacci hybrid polynomials
$T_n^{(r)}$	bivariate r-Fibonacci polynomials
$Z_n^{(r,s)}$	bivariate r-Lucas polynomials of type s
$\mathbb{K}_{T^{(r)},n}$	r-Fibonacci hybrid polynomials
$\mathbb{K}_{Z^{(r,s)},n}$	r-Lucas hybrid polynomials
p_n	bi-periodic Fibonacci numbers
q_n	bi-periodic Lucas numbers
u_n	generalized bi-periodic Fibonacci numbers
v_n	generalized bi-periodic Lucas numbers
w_n	bi-periodic Horadam numbers
$U_n^{(r)}$	bi-periodic r-Fibonacci numbers
$V_n^{(r,s)}$	bi-periodic r-Lucas numbers of type s
$\mathbb{K}_{W,n}$	Horadam hybrid numbers
$\mathbb{K}_{u,n}$	generalized bi-periodic Fibonacci hybrid numbers
$\mathbb{K}_{v,n}$	generalized bi-periodic Lucas hybrid numbers
$\mathbb{K}_{\widehat{u},n}$	modified generalized bi-periodic Fibonacci hybrid numbers
$\mathbb{K}_{\widehat{v},n}$	modified generalized bi-periodic Lucas hybrid numbers

$\mathbb{K}_{w,n}$	bi-periodic Horadam hybrid numbers
$Q_{W,n}$	Horadam quaternion numbers
$\widetilde{Q}_{W,n}$	dual Horadam quaternion numbers
$\widehat{W}_n$	hyper-dual Horadam numbers
$\widehat{Q}_{W,n}$	hyper-dual Horadam quaternion numbers

Introduction

This thesis, entitled " *Bi-Periodic r-Fibonacci Sequences and Horadam Hyper-Dual Numbers* " is related to Number Theory and Enumerative Combinatorics, we introduce new sequences, new numbers, and polynomials, we investigate their algebraic properties. [2, 3, 4, 79].

This thesis is structured into five chapters as follows:

In the first chapter, we give some definitions and results required to the comprehention of this thesis, such as the linear recurrence sequences of order m, the k-periodic recurrence sequences, companion sequences, generating functions, explicit formulas, Binet forms. We define some well-known numbers as hybrid numbers, quaternion numbers, dual and hyper-dual numbers and so on.

In the second chapter, for a positive integer r, we study bi-periodic r-Fibonacci sequence and its family of companion sequences, bi-periodic r-Lucas sequence of type s with $1 \leq s \leq r$, which extend the classical Fibonacci and Lucas sequences. Afterwards, we establish a link between the bi-periodic r-Fibonacci sequence and its companion sequences. Furthermore, we give their properties as linear recurrence relations, generating functions, explicit formulas and Binet forms.

In the third chapter, we introduce the bi-periodic Horadam hybrid numbers and deduce particular cases; the bi-periodic Fibonacci hybrid numbers and the bi-periodic Lucas hybrid numbers. We establish a relation between the bi-periodic Fibonacci hybrid numbers and the bi-periodic Lucas hybrid numbers then we give the generating function, the Binet form and some basic properties of these new hybrid numbers.

In the fourth chapter, we define a new generalization of Fibonacci and Lucas hybrid polynomials, it is called bivariate r-Fibonacci hybrid polynomials and bivariate r-Lucas hybrid polynomials of type s. We investigate some properties of these polynomials.

In the fifth, we develop a new class of quaternions, called hyper-dual Horadam quaternions, they are constructed from the quaternions whose components are hyper-dual Horadam numbers. The hyper-dual numbers extend the dual numbers as the quaternions extend the complexe numbers. We investigate some basic properties of these quaternions.

CHAPTER

1

PRELIMINARIES

In this chapter, we give some notions and definitions that will be useful for understanding the present thesis. We mention the definitions of linear recurrence sequences, multi-periodic sequences as **k**-periodic sequences, and the particular case the bi-periodic sequences. Also, we give some algebraic properties such as, characteristic polynomial, companion matrix, generating function, explicit formula. We give the definitions of hybrid numbers, quaternion numbers, dual and hyper-dual numbers. Moreover, we mention some well-known sequences. Throughout this thesis, $\mathbf{K}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$ and $\mathbf{K}[x]$ the ring of polynomials in one variable x with coefficients in $\mathbf{K}$.

1.1 Recurrence relation

A recurrence relation is an equation that recursively defines a sequence or multi-dimentional array of values, once one or more initial terms of the same function are given, each further term of the sequence or array is defined as a function of the preceding terms of the same function. More precisely, in the case where only the immediately preceding element is involved, a recurrence relation has the form

$$\mathrm{u}_n = \phi(n, \mathrm{u}_{n-1}), \qquad n \geq 1, \tag{1.1}$$

where

$$\phi : \mathbb{N} \times X \to X, \tag{1.2}$$

is a function, where X is a set to which the elements of the sequences must belong. For any $u_0 \in X$, this defines a unique sequence with u_0 as its first element called the initial condition. It is easy to modify the definition for getting sequences starting from the term of index 1 or higher. This defines the recurrence relation of first order.

The recurrence relation of order m has the form

$$u_n = \phi(n, u_{n-1}, u_{n-2}, \dots, u_{n-m}), \qquad n \geq m, \tag{1.3}$$

where

$$\phi : \mathbb{N} \times X^m \to X,$$

is a function that involves m consecutive elements of the sequence. In this case, m initial conditions are needed for defining such sequence.

1.2 Linear recurrence sequences

In the particular case when ϕ is a linear function, we deduce the following definition.

Definition 1. *A linear recurrence sequence $(u_n)_n$ with constant coefficients $a_1, \dots, a_m \in$ **K** and $a_m \neq 0$ is a sequence which satisfies the following relation*

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_m u_{n-m}, \qquad n \geq m. \tag{1.4}$$

This sequence is defined with m initial conditions $u_0, \dots, u_{m-1}$ and the integer m is called the order of the recurrence sequence.

Example 1. *The recurrence of order two satisfied by the Fibonacci numbers is the canonical example of a linear recurrence relation with constant coefficients. The Fibonacci numbers $(F_n)_n$ was first described in connection with computing the number of descendants of pair of rabbits in the book Liber Abaci in* 1202 *[68]. This sequence is probably one of the well-known recurrent sequences and it is defined by the second order recurrence, firstly used by Albert Girard in* 1634 *[25]. The Fibonacci sequence is defined using the recurrence*

$$F_n = F_{n-1} + F_{n-2}, \qquad n \geq 2, \tag{1.5}$$

with initial conditions $F_0 = 0, F_1 = 1$. *We obtain the sequence of Fibonacci numbers which begins* $0, 1; 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$.

1.3 Characteristic polynomial

The characteristic polynomial associated to the sequence (1.4) is the polynomial defined by

$$P(x) = x^m - a_1 x^{m-1} - a_2 x^{m-2} - \cdots - a_m \in \mathbf{K}[x]. \tag{1.6}$$

1.4 Characteristic equation

The characteristic equation of the sequence (1.4) is the equation obtained by equating its characteristic polynomial (1.6) to zero.

$$P(x) = x^m - a_1 x^{m-1} - a_2 x^{m-2} - \cdots - a_m = 0. \tag{1.7}$$

When $P(x)$ is a split polynomial over $\mathbf{K}[x]$ then we can write

$$P(x) = \prod_{i=1}^{\mathfrak{h}} (x - \alpha_i)^{r_i} \in \mathbf{K}[x], \tag{1.8}$$

where α_i for $i = 1, \ldots, \mathfrak{h}$ with multiplicity r_i are the distinct complex roots of the polynomial $P(x)$ and $\sum_{i=1}^{\mathfrak{h}} r_i = m$.

Example 2. *The characteristic equation of the Fibonacci sequence* (1.5) *is*

$$x^2 - x - 1 = 0, \tag{1.9}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ *and* $\beta = \dfrac{1-\sqrt{5}}{2}$ *are its roots. The positive root* α *is known as "golden ratio".*

1.5 Companion matrix

The companion matrix of the polynomial (1.6) associated to the sequence (1.4) is defined as

$$M := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_1 \end{pmatrix}, \tag{1.10}$$

in which the first superdiagonal consists entirely of ones and all other elements above the last row are zeros. The characteristic equation of M is $det(M - xI) = 0$, where I is the identity matrix.

In classical linear algebra, the eigenvalues of a matrix are sometimes defined as the roots of the corresponding characteristic polynomial. An algorithm to compute the roots of a polynomial by computing the eigenvalues of the corresponding companion matrix turns the tables on the usual definition. When the computation of the n^{th} power of the companion matrix (1.10) is not difficult, we can determine the n^{th} term u_n of the sequence $(u_n)_n$, as the following method.

In [69], Silvester shows that a number of the properties of the Fibonacci sequence can be derived from a matrix representation. For more details, we refer to [41]. Just as Silvester derived many interesting properties of Fibonacci numbers from a matrix representation.

Every linear recurrence sequence has a matrix formulation, so the linear relation (1.4) induces

$$\begin{pmatrix} u_{n-m+1} \\ u_{n-m+2} \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_1 \end{pmatrix} \begin{pmatrix} u_{n-m} \\ u_{n-m+1} \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix}, \tag{1.11}$$

for $n \geq m$, which is equivalent to:

$$\begin{pmatrix} u_{n+1} \\ u_{n+2} \\ \vdots \\ u_{n+m-1} \\ u_{n+m} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_1 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+m-2} \\ u_{n+m-1} \end{pmatrix}, \qquad (1.12)$$

for $n \geq 0$. By an inductive argument, we get

$$\begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+m-2} \\ u_{n+m-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_1 \end{pmatrix}^n \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-2} \\ u_{m-1} \end{pmatrix}. \qquad (1.13)$$

for $n \geq 0$. Thus, the n^{th} term u_n of the sequence $(u_n)_n$ is given by

$$\begin{aligned} u_n &= (1,0,\cdots,0,0) \begin{pmatrix} u_n \\ u_{n+1} \\ \vdots \\ u_{n+m-2} \\ u_{n+m-1} \end{pmatrix} \\ &= (1,0,\cdots,0,0) \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_m & a_{m-1} & a_{m-2} & \cdots & a_1 \end{pmatrix}^n \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-2} \\ u_{m-1} \end{pmatrix}, \end{aligned}$$

for $n \geq 0$, with initial conditions $u_0, u_1, \ldots, u_{m-1}$.

1.6 Generating function

A generating function is a way of encoding an infinite sequence of numbers $(u_n)_n$ by treating them as coefficients of formal power series. This serie is called the generating function of the sequence. The generating function provide a powerful tool for solving linear recurrence relation with constant coefficients. Generating functions were introduced for the first time by Abraham de Moivre in 1730 in order to solve the general linear recurrence problem [45]. Generating functions are one of the most suprising and useful inventions in Discrete mathematics. Roughly speaking, generating functions transform problems about sequences into problems about functions. This is great because we have piles of mathematical machinery for manipulating functions. The generationg function of the infinite sequence $(u_n)_n$ is the power series

$$\sum_{n\geq 0} u_n x^n.$$

Example 3. *The generating function of the Fibonacci sequence* (1.5) *is*

$$\sum_{n\geq 0} F_n x^n = \frac{x}{1-x-x^2}. \tag{1.14}$$

The technics calculation is given in [49].

1.7 Binet form

We remind the reader of the famous Binet form, also known as the "de Moivre formula", that can be used to calculate Fibonacci numbers. Binet form is used to obtain the n^{th} term of a sequence, using the roots of the characteristic equation. This form can be employed to derive a myriad of identities. Fibonacci numbers have a closed form expression for the computation of the n^{th} Fibonacci number without appealing to its recurrence. It is called Binet form in honor of Jacques Binet, who discovered this form in 1843 [12, 25].

A fundamental result in the theory of recurrence sequences asserts that:

Theorem 2. *Let $(\mathbf{u_n})_n$ be a linear recurrence sequence whose characteristic polynomial $P(x)$ splits as*

$$P(x) = (x-\alpha_1)^{r_1}\dots(x-\alpha_{\mathfrak{h}})^{r_{\mathfrak{h}}} = \prod_{i=1}^{\mathfrak{h}}(x-\alpha_i)^{r_i},$$

where α_i, for $i = 1,\dots,\mathfrak{h}$ with multiplicity r_i, are the distinct complex roots of the polynomial $P(x)$ and $\sum_{i=1}^{\mathfrak{h}} r_i = m$. Then, there exist uniquely determined non-zero polynomials

$P_1, \ldots, P_{\mathfrak{h}} \in \mathbb{Q}(\alpha_j)[x]$ *for* $j = 1, \ldots, \mathfrak{h}$, *with* $degP_j \leq r_j - 1$, *such that*

$$\mathbf{u_n} = P_1(n)\alpha_1^n + P_2(n)\alpha_2^n + + P_{\mathfrak{h}}(n)\alpha_{\mathfrak{h}}^n, \qquad n \geq 0. \tag{1.15}$$

The proof of this result can be found in [67], Theorem C.1.

Example 4. *Binet form of the Fibonacci sequence* (1.5) *is*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.16}$$

where α *and* β *are the roots of the characteristic equation* (1.9).

1.8 Companion sequence

The companion sequence of the Fibonacci sequence (1.5) is the well-known Lucas sequence. The latter was studied by Edouard Lucas (1842-1891), which satisfies the same recurrence relation as the Fibonacci sequence

$$L_n = L_{n-1} + L_{n-2}, \qquad n \geq 2, \tag{1.17}$$

with initial conditions $L_0 = 2, L_1 = 1$. We obtain the sequence of Lucas numbers which begins $2, 1; 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$.

The Lucas sequence (1.17) has a variety of relationships with the Fibonacci sequence (1.5), we cite one of them

$$L_n = F_{n-1} + F_{n+1}, \qquad n \geq 1. \tag{1.18}$$

Binet form of the Lucas sequence is

$$L_n = \alpha^n + \beta^n, \tag{1.19}$$

where α, β are defined in (1.9).
The generating function of the Lucas sequence (1.17) is

$$\sum_{n \geq 0} L_n x^n = \frac{2 - x}{1 - x - x^2}. \tag{1.20}$$

Fibonacci and Lucas sequences are well-known sequences among integer sequences. These sequences and their generalizations have many intresting properties and applications to almost every field of science and art.

1.9 Binomial coefficient

Binomial coefficient can be interpreted as the number of distinct ways to choose k elements from a set of n elements. They are also involved in the expansion of the expression

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \qquad n \geq 0, \tag{1.21}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \qquad 0 \leq k \leq n,$$

with the convention

$$\binom{n}{k} = 0, \quad \text{if} \quad k < 0 \quad \text{or} \quad k > n.$$

1.10 Multinomial coefficients

Multinomial coefficients have a direct combinatorial interpretation, as the number of ways of depositing n distinct objects into m distinct bins, with k_1 objects in the first bin, k_2 objects in the second bin, and so on. Multinomial coefficients $\binom{n}{k_1,k_2,\ldots,k_m}$ occur in the expansion of the polynomial $(x_1 + x_2 + \cdots + x_m)^n$ as follows:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m},$$

where

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1! k_2! \ldots k_m!}, \qquad k_1 + k_2 + \cdots + k_m = n,$$

with

$$\binom{n}{k_1, k_2, \ldots, k_m} = 0 \quad \text{if} \quad k_1 + k_2 + \cdots + k_m \neq n \quad \text{or} \quad k_i < 0, \quad i = 1, \ldots, m.$$

Also, multinomial coefficients can be expressed as binomial coefficients

$$\binom{n}{k_1, k_2, \ldots, k_m} = \binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-k_2-\cdots-k_{m-1}}{k_m}$$

and they satisfy the following recurrence relation

$$\binom{n}{k_1, k_2, \ldots, k_m} = \binom{n-1}{k_1-1, k_2, \ldots, k_m} + \binom{n-1}{k_1, k_2-1, \ldots, k_m} + \cdots + \binom{n-1}{k_1, k_2, \ldots, k_m-1}.$$

We can also use the notation for the multinomial coefficients, given in [8], for all $k_1, k_2, \ldots, k_m$ and $k \in \mathbb{Z}$

$$\binom{k}{k_1, k_2, \ldots, k_m} = \begin{cases} \dfrac{k!}{k_1!k_2!\ldots k_m!} & if \quad k_1+k_2+\cdots+k_m = k, \\ 0 & \text{otherwise} . \end{cases}$$

.

1.11 Explicit formula

In the following theorem, Belbachir and Bencherif [9] gave a formula expressing general term of a linear recurrence sequence.

Theorem 3. *Let $(u_n)_{n>-m}$ be the sequence of elements over an unitary ring $\mathcal{A}$, defined by*

$$\begin{cases} u_{-j} &= 0 & 1 \le j \le m-1, \\ u_0 &= 1, \\ u_n &= a_1u_{n-1} + a_2u_{n-2} + \cdots + a_mu_{n-m} & n \ge 1. \end{cases}$$

Then for all integers $n > -m$,

$$u_n = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1+k_2+\cdots+k_m}{k_1, k_2, \ldots, k_m} a_1^{k_1}a_2^{k_2}\ldots a_m^{k_m}. \tag{1.22}$$

1.12 Bivariate polynomials

This section is devoted to recall different well-known sequences of bivariate polynomials. Let $(U_n(x,y))_n$ and $(V_n(x,y))_n$ be the sequences of polynomials with two variables x and y with real coefficients defined by the following second order recurrence relations:

$$U_n(x,y) = h(x)U_{n-1}(x,y) + k(y)U_{n-2}(x,y), \qquad n \geq 2, \tag{1.23}$$

$$U_0(x,y) = 0, U_1(x,y) = 1$$

and

$$V_n(x,y) = h(x)V_{n-1}(x,y) + k(y)V_{n-2}(x,y), \qquad n \geq 2, \tag{1.24}$$

$$V_0(x,y) = 2, V_1(x,y) = h(x),$$

where $h(x), k(y)$ are polynomials with indeterminates x, y and real coefficients. The polynomials $U_n(x,y)$ and $V_n(x,y)$ are called generalized Fibonacci and Lucas bivariate polynomials, respectively. Many classical sequences, known in the literature, are derived from the sequences $(U_n(x,y))_n$ and $(V_n(x,y))_n$.

- For $h(x) = k(y) = 1$, we obtain the Fibonacci and Lucas sequences [49, 62]: $U_n(x,y) := F_n$ and $V_n(x,y) := L_n$ defined by

$$F_n = F_{n-1} + F_{n-2}, \qquad n \geq 2,$$

$$F_0 = 0, F_1 = 1$$

  and

$$L_n = L_{n-1} + L_{n-2}, \qquad n \geq 2,$$

$$L_0 = 2, L_1 = 1.$$

- For $h(x) = 2$ and $k(y) = 1$, we obtain the Pell and Pell-Lucas numbers [48]: $U_n(x,y) := P_n$ and $V_n(x,y) := Q_n$ defined by

$$P_n = 2P_{n-1} + P_{n-2}, \qquad n \geq 2,$$

$$P_0 = 0, P_1 = 1$$

  and

$$Q_n = 2Q_{n-1} + Q_{n-2}, \qquad n \geq 2,$$

$$Q_0 = 2, Q_1 = 2.$$

- For $h(x) = 2$ and $k(y) = q$, we obtain the q-Pell numbers [14]: $U_n(x,y) := P_{q,n}$

defined by

$$P_{q,n} = 2P_{q,n-1} + qP_{q,n-2}, \qquad n \geq 2,$$

$$P_{q,0} = 0, P_{q,1} = 1.$$

- For $h(x) = p$ and $k(y) = q$, we obtain the generalized Fibonacci and Lucas sequences also, called the Lucas sequences [53]: $U_n(x,y){:=}U_n(p,q) = U_n$ and $V_n(x,y){:=}V_n(p,q) = V_n$ defined by

$$U_n = pU_{n-1} + qU_{n-2}, \qquad n \geq 2,$$

$$U_0 = 0, U_1 = 1$$

and

$$V_n = pV_{n-1} + qV_{n-2}, \qquad n \geq 2,$$

$$V_0 = 2, V_1 = p.$$

- For $h(x) = x$ and $k(y) = 1$, we obtain the Fibonacci polynomials also, called Catalan polynomials and Lucas polynomials [48]: $U_n(x,y){:=}F_n(x)$ and $V_n(x,y){:=}L_n(x)$ defined by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \qquad n \geq 2,$$

$$F_0(x) = 0, F_1(x) = 1$$

and

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \qquad n \geq 2,$$

$$L_0(x) = 2, L_1(x) = x.$$

- For $h(x) = 1$ and $k(y) = y$, we obtain the Jacobsthal polynomials [49]: $U_n(x,y){:=}J_n(y)$ defined by

$$J_n(y) = J_{n-1}(y) + yJ_{n-2}(y), \qquad n \geq 2,$$

$$J_0(y) = 0, J_1(y) = 1,$$

also, we have the Jacobsthal polynomials defined by Horadam in [38] defined by

$$J_n(y) = J_{n-1}(y) + 2yJ_{n-2}(y), \qquad n \geq 2,$$

$$J_0(y) = 0, J_1(y) = 1$$

and

$$J_n(y) = J_{n-1}(y) + 2yJ_{n-2}(y), \qquad n \geq 2,$$

$$J_0(y) = 2, J_1(y) = 1,$$

the Jacobsthal-Lucas polynomials [16].

- For $h(x) = 2x$ and $k(y) = 1$, we obtain the polynomials studied by Byrd [13]: $U_n(x,y) := \phi_n(x)$ defined by

$$\phi_n(x) = 2x\phi_{n-1}(x) + \phi_{n-2}(x), \qquad n \geq 2,$$

$$\phi_0(x) = 0, \phi_1(x) = 1.$$

- The case $k(y) = 1$ and arbitrary h has been studied by A. Nalli and P. Haukkanen [54].

- For $h(x) = px$ and $k(y) = q$, we obtain the Horadam polynomials sequence [39]: $U_n(x,y) := W_n(x)$ defined by

$$W_n(x) = pxW_{n-1}(x) + qW_{n-2}(x), \qquad n \geq 2,$$

$$W_0(x) = W_0, W_1(x) = W_1x,$$

with the arbitrary values W_0, W_1.

1.13 k-periodic sequences

Generalizations of the Fibonacci numbers have been extensively studied. To generalize the Fibonacci sequence, some authors [34, 40, 44, 60, 91] have altered the starting values, while others [7, 27, 50, 51, 59, 64] have preserved the first two terms of the sequence but changed the recurrence relation. We note that **k**-periodic sequences, as multi-periodic sequences, satisfy a linear recurrence relation when considered modulo **k**, even though these sequences themselves do not. Then we employ this recurrence relation to determine the generating functions and Binet forms.

This generalization has its own Binet form and satisfies identities that are analogous to the identities satisfied by the classical Fibonacci sequence. Now, we introduce a further generalization of the Fibonacci sequence; we call it the **k**-periodic Fibonacci sequence [26]. This new generalization is defined using a non-linear recurrence relation that depends on $\mathbf{k}$ real parameter and is an extension of the generalized Fibonacci sequence.

The non-linear recurrence relation is given by linear recurrence relation with non-constant coefficients.

Definition 4. *For any $\mathbf{k}$-tuple $(a_1, a_2, \ldots, a_{\mathbf{k}}) \in \mathbb{Z}^{\mathbf{k}}$, we define recursively the $\mathbf{k}$-periodic Fibonacci sequence, denoted $\left(t_n^{(a_1,a_2,\ldots,a_{\mathbf{k}})}\right)_n = (t_n)_n$ by*

$$t_n = \begin{cases} a_1 t_{n-1} + t_{n-2}, & if \quad n \equiv 2 \pmod{\mathbf{k}}, \\ a_2 t_{n-1} + t_{n-2}, & if \quad n \equiv 3 \pmod{\mathbf{k}}, \\ \vdots & \\ a_{\mathbf{k}-1} t_{n-1} + t_{n-2}, & if \quad n \equiv 0 \pmod{\mathbf{k}}, \\ a_{\mathbf{k}} t_{n-1} + t_{n-2}, & if \quad n \equiv 1 \pmod{\mathbf{k}}, \end{cases}$$

for $n \geq 2$ with initial conditions $t_0 = 0, t_1 = 1$.

This new generalization is in fact a family of sequences where each new combination of $a_1, a_2, \ldots, a_{\mathbf{k}}$ produces a new sequence.

- When $a_1 = a_2 = \cdots = a_{\mathbf{k}} = 1$, we have the classical Fibonacci sequence [49].
- When $a_1 = a_2 = \cdots = a_{\mathbf{k}} = 2$, we get the Pell numbers [48].
- When $a_1 = a_2 = \cdots = a_{\mathbf{k}} = p$, for some positive integer p, we get the p-Fibonacci numbers [29], also known as generalized Fibonacci numbers.
- When $a_1 = a, a_2 = b$, if $\mathbf{k} = 2$ we obtain the bi-periodic Fibonacci sequence [27], that will be seen in Section 1.14.

Example 5. *The sequence descriptions that follow give reference numbers found in Sloane's On-Line Encyclopedia of Integer Sequences [70].*
When $\mathbf{k} = 3$, for:

- $(a_1, a_2, a_3) = (1, 0, 1)$, *we obtain the sequence A092550.*
- $(a_1, a_2, a_3) = (2, 1, 1)$, *we obtain the sequence A179238.*
- $(a_1, a_2, a_3) = (1, -1, 2)$, *we obtain the sequence A011655.*

When $\mathbf{k} = 4$, for:

- $(a_1, a_2, a_3, a_4) = (2, 1, 2, 1)$, *we get the sequence A048788.*
- $(a_1, a_2, a_3, a_4) = (1, 2, 1, 2)$, *we get the sequence A002530.*

In order to describe the terms of the sequence $(t_n)_n$ explicitly using the generalization of Binet form, we have first show that for some constant ψ, the sequence $(t_n)_n$ satisfies the following recurrence relation

$$t_{m\mathbf{k}+j} = \psi t_{(m-1)\mathbf{k}+j} + (-1)^{\mathbf{k}-1} t_{(m-2)\mathbf{k}+j}, \qquad 0 \le j \le \mathbf{k}-1, \tag{1.25}$$

for $m \ge 2\mathbf{k}$. For the interested readers and more details we refer to [26].

For clarity, we record some remarks. Since there are currently several types of generalizations of the Fibonacci and Lucas sequences, it is very difficult for anyone to know exactly what type of sequence an author really means. We think that this notation is quite unclear as there are already many types of generalized $\mathbf{k}$-Fibonacci sequences. Many generalizations of the Fibonacci sequence have appeared in the literature. Probably the most well-known generalization is the $\mathbf{k}$-generalized Fibonacci sequence $(\mathfrak{F}(\mathbf{k})_n)_n, n \ge -(\mathbf{k}-2)$ (also known as the multi-bonacci, the $\mathbf{k}$-bonacci, the $\mathbf{k}$-fold Fibonacci or $\mathbf{k}^{th}$ order Fibonacci), satisfying

$$\mathfrak{F}(\mathbf{k})_n = \mathfrak{F}(\mathbf{k})_{n-1} + \mathfrak{F}(\mathbf{k})_{n-2} + \cdots + \mathfrak{F}(\mathbf{k})_{n-\mathbf{k}},$$

with initial conditions $\mathfrak{F}(\mathbf{k})_{-j} = 0$ for $j = 0, 1, 2, \ldots, \mathbf{k}-2$ and $\mathfrak{F}(\mathbf{k})_1 = 1$. Thus, we draw the reader's attention to these important different notations.

1.14 Bi-periodic Fibonacci and Lucas sequences

Edson and Yayenie [27] introduced the bi-periodic Fibonacci sequence using a non-linear recurrence relation depending on two real parameters which is a particular case of $(t_n)_n$ for $\mathbf{k} = 2$ as it is defined below.

Definition 5. *For any two nonzero real numbers a and b, the bi-periodic Fibonacci sequence $(p_n)_n$ is defined by*

$$p_n = \begin{cases} ap_{n-1} + p_{n-2}, & if \quad n \equiv 0 \pmod 2, \\ bp_{n-1} + p_{n-2}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \ge 2$, with initial conditions $p_0 = 0, p_1 = 1$.

For particular values of a, b we deduce some well-known sequences in Table 1.1.

$a = b = 1$	$0, 1; 1, 2, 3, 5, 8, \ldots$	Fibonacci sequence
$a = b = 2$	$0, 1; 2, 5, 12, 29, 70, \ldots$	Pell sequence
$a = b = 3$	$0, 1; 3, 10, 33, 109, \ldots$	3-Fibonacci sequence
$a = b = p$	$0, 1; p, p^2 + 1, p^3 + 2p, p^4 + 3p^2 + 1,$ $p^5 + 4p^3 + 3p, \ldots$	p-Fibonacci sequence [29]

Table 1.1: Classical sequences.

Bilgici [11] defined the generalization of Lucas sequence similar to the bi-periodic Fibonacci sequence using a non-linear recurrence relation depending on two nonzero real numbers as follows:

Definition 6. *For any two nonzero real numbers a and b, the bi-periodic Lucas sequence* $(q_n)_n$ *is defined by*

$$q_n = \begin{cases} bq_{n-1} + q_{n-2}, & if \quad n \equiv 0 \pmod 2, \\ aq_{n-1} + q_{n-2}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq 2$*, with initial conditions* $q_0 = 2, q_1 = a$.

For particular values of a, b we deduce some well-known companion sequences in Table 1.2.

$a = b = 1$	$2, 1; 3, 4, 7, 11, 18, 29, \ldots$	Lucas sequence
$a = b = 2$	$2, 2; 6, 14, 34, 82, 198, 478, \ldots$	Pell-Lucas sequence
$a = b = 3$	$2, 3; 11, 36, 119, 393, 1298, 4287, \ldots$	3-Lucas sequence
$a = b = p$	$2, p; p^2 + 2, p^3 + 3p, p^4 + 4p^2 + 2,$ $p^5 + 5p^3 + 5p, p^6 + 6p^4 + 9p^2 + 2, \ldots$	p-Lucas sequence [28]

Table 1.2: Classical companion sequences.

Binet forms of the sequences $(p_n)_n$ and $(q_n)_n$ are given by

$$p_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \tag{1.26}$$

and

$$q_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} (\alpha^n + \beta^n), \tag{1.27}$$

here $\alpha = \dfrac{ab + \sqrt{a^2b^2 + 4ab}}{2}$ and $\beta = \dfrac{ab - \sqrt{a^2b^2 + 4ab}}{2}$ are roots of the polynomial $x^2 - abx - ab$, with $\lfloor . \rfloor$ denotes the floor function and $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, i.e., $\xi(n) = 0$, when n is even and $\xi(n) = 1$ when n is odd.

The generating functions of the bi-periodic sequences $(p_n)_n$ and $(q_n)_n$ are

$$\sum_{n \geq 0} p_n x^n = \frac{x + ax^2 - x^3}{1 - (ab + 2)x^2 + x^4}$$

and

$$\sum_{n \geq 0} q_n x^n = \frac{2 + ax - (ab + 2)x^2 + ax^3}{1 - (ab + 2)x^2 + x^4}.$$

The Fibonacci conditional sequence is a further generalization introduced by

Sahin [63], it is defined as follows:

Definition 7. *For any nonzero numbers a, b, c and d, the bi-periodic Fibonacci sequence $(h_n)_n$ is defined by*

$$h_n = \begin{cases} ah_{n-1} + ch_{n-2}, & if \quad n \equiv 0 \pmod 2, \\ bh_{n-1} + dh_{n-2}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq 2$, with initial conditions $h_0 = 0, h_1 = 1$.

- If we take $a = b = p$, $c = d = q$ we get the generalized Fibonacci sequence $(U_n)_n = (U_n(p,q))_n$ [53].
- If we take $a = b = 1, c = d = 2$ we get the Jacobsthal sequence $(J_n)_n$ [37, 47].
- If we take $a = b = p$, $c = d = 2$ we get the p-Jacobsthal sequence $(\widehat{J}_n)_n$ [90].

Taking initial conditions $h_0 = 2$ and $h_1 = a$, authors gave some properties of the Lucas conditional sequence in [82] which is defined as follows:

Definition 8. *For any nonzero numbers a, b, c and d, the bi-periodic Lucas sequence $(\tau_n)_n$ is defined by*

$$\tau_n = \begin{cases} b\tau_{n-1} + d\tau_{n-2}, & if \quad n \equiv 0 \pmod 2, \\ a\tau_{n-1} + c\tau_{n-2}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq 2$, with initial conditions $\tau_0 = 2, \tau_1 = a$.

It should be noted that more results related to these sequences can be found in [11, 26, 58, 63, 80, 81, 82, 92]. In literature, these sequences are called the generalized Fibonacci sequences. Thus, we ask the reader to be careful and pay attention to generalizations.

- If we take $a = b = p$, $c = d = q$ we get the generalized Lucas sequence $(V_n)_n = (V_n(p,q))_n$ [53].
- If we take $a = b = 1, c = d = 2$ we get the Jacobsthal-Lucas sequence $(j_n)_n$ [37, 47].
- If we take $a = b = p$, $c = d = 2$ we get the p-Jacobsthal-Lucas sequence $(\widehat{j}_n)_n$ [90].

1.15 Horadam sequences

The main advantage of introducing the Horadam sequence is that many celebrated sequences such as Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas sequences can be deduced as particular cases of the Horadam sequence, either for the bi-periodic cases. The Horadam sequence $(W_n)_n$ is defined by Horadam [36] as

$$W_n = pW_{n-1} + qW_{n-2}, \qquad n \geq 2, \tag{1.28}$$

with initial conditions W_0, W_1 where W_0, W_1, p, q are arbitrary integers. It has considered a generalization of the classical Fibonacci and Lucas sequences.

In particular,

- If we take $W_0 = 0, W_1 = 1$, we obtain the generalized Fibonacci sequence $(U_n)_n := (U_n(p,q))_n$ [53].
- If we take $W_0 = 2, W_1 = p$, we obtain the generalized Lucas sequence $(V_n)_n := (V_n(p,q))_n$ [53].
- If we take $q = 1, W_0 = 0, W_1 = 1$, we obtain the generalized Fibonacci sequence $(\mathfrak{U}_n)_n = (U_n(p,1))_n$ [29].
- If we take $q = 1, W_0 = 2, W_1 = p$, we obtain the generalized Lucas sequence $(\mathcal{V}_n)_n = (V_n(p,1))_n$ [28].
- If we take $p = 1$, $q = 2$ and $W_0 = 0, W_1 = 1$, we obtain the Jacobsthal sequence $(J_n)_n$ [37, 47].
- If we take $p = 1, q = 2$ and $W_0 = 2, W_1 = 1$, we obtain the Jacobsthal-Lucas sequence $(j_n)_n$ [37, 47].
- If we take $p, q = 2$ and $W_0 = 0, W_1 = 1$, we obtain the p-Jacobsthal sequence $(\widehat{J}_n)_n$ [90].
- If we take p, $q = 2$ and $W_0 = 2, W_1 = 1$, we obtain the p-Jacobsthal-Lucas sequence $(\widehat{j}_n)_n$ [90].

The Binet form of the Horadam sequence is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where

$$A := W_1 - W_0\beta, \qquad \text{and} \qquad B := W_1 - W_0\alpha.$$

Here $\alpha = \frac{(p+\sqrt{p^2+4q})}{2}$ and $\beta = \frac{(p-\sqrt{p^2+4q})}{2}$ are the roots of the characteristic polynomial $x^2 - px - q$.

The generating function of the Horadam sequence $(W_n)_n$ is

$$\sum_{n\geq 0} W_n x^n = \frac{(1-px)W_0 + xW_1}{1-px-qx^2}.$$

1.16 Bi-periodic Horadam sequences

Similar to the Fibonacci and Lucas sequences that were generalized as the Horadam sequence, the bi-periodic Fibonacci and Lucas sequences were generalized as the bi-periodic Horadam sequence $(\delta_n)_n$ which is defined first in [27] as follows:

Definition 9. *For two nonzero real numbers a and b, the bi-periodic Horadam sequence $(\delta_n)_n$ is defined by*

$$\delta_n = \begin{cases} a\delta_{n-1} + \delta_{n-2}, & if \quad n \equiv 0 \pmod 2, \\ b\delta_{n-1} + \delta_{n-2}, & if \quad n \equiv 1 \pmod 2, \end{cases} \tag{1.29}$$

for $n \geq 2$ with arbitrary initial conditions δ_0, δ_1 where δ_0, δ_1 are nonzero values.

Motivating by Horadam's results in [36], Tan [78] gave some basic properties of the bi-periodic Horadam sequence and some identities for the bi-periodic Lucas sequences. Some sequences in the literature can be stated in terms of the sequence $(\delta_n)_n$ as:

- If we take $\delta_0 = 0$, $\delta_1 = 1$, in the sequence (1.29), we get the bi-periodic Fibonacci sequence $(p_n)_n$ [27].
- If we take $\delta_0 = 2$, $\delta_1 = b$, in the sequence (1.29), we get the bi-periodic Lucas sequence $(q_n)_n$ [82] with the case of $c = d = 1$.
- If we take $a = b = p$ and $\delta_0 = 0, \delta_1 = 1$, in the sequence (1.29), we get the generalized Fibonacci sequence $(\mathfrak{U}_n)_n = (U_n(p,1))_n$ [29].
- If we take $a = b = p$ and $\delta_0 = 2, \delta_1 = p$, in the sequence (1.29), we get the generalized Lucas sequence $(\mathcal{V}_n)_n = (V_n(p,1))_n$ [28].

The Binet form of the bi-periodic Horadam sequence is

$$\delta_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(A\alpha^n - B\beta^n \right), \tag{1.30}$$

where

$$A := \frac{\delta_1 - \frac{\beta}{a}\delta_0}{\alpha - \beta} \qquad \text{and} \qquad B := \frac{\delta_1 - \frac{\alpha}{a}\delta_0}{\alpha - \beta}. \tag{1.31}$$

Here $\alpha = \dfrac{ab + \sqrt{a^2b^2 + 4abc}}{2}$ and $\beta = \dfrac{ab - \sqrt{a^2b^2 + 4abc}}{2}$ are the roots of the polynomial $x^2 - abx - abc$.
The generating function of the bi-periodic Horadam sequence $(\delta_n)_n$ is

$$\sum_{n \geq 0} \delta_n x^n = \frac{\left(1 - (ab+1)\,x^2 + bx^3\right)\delta_0 + x\left(1 + ax - x^2\right)\delta_1}{1 - (ab+2)\,x^2 + x^4}.$$

Another generalization of bi-periodic Horadam sequence defined by

$$w_n := \begin{cases} aw_{n-1} + cw_{n-2}, & if \quad n \equiv 0 \pmod 2, \\ bw_{n-1} + cw_{n-2}, & if \quad n \equiv 1 \pmod 2, \end{cases} \tag{1.32}$$

for $n \geq 2$ with arbitrary initial conditions w_0, w_1 where w_0, w_1, a, b, c are nonzero real numbers is given in [84]. This sequence will be generalized in Chapter 3.

- If we take $a = b = p$, $c = q$ and $w_0 = 0, w_1 = 1$ in the sequence (1.29), we get the generalized Fibonacci sequence $(U_n)_n = (U_n(p,q))_n$ [53].
- If we take $a = b = p$, $c = q$ and $w_0 = 2, w_1 = 1$ in the sequence (1.29), we get the generalized Lucas sequence $(V_n)_n = (V_n(p,q))_n$ [53].
- If we take $a = b = 1$, $c = 2$ and $w_0 = 0, w_1 = 1$ in the sequence (1.29), we get the Jacobsthal sequence $(J_n)_n$ [37, 47].
- If we take $a = b = 1$, $c = 2$ and $w_0 = 2, w_1 = 1$ in the sequence (1.29), we get the Jacobsthal-Lucas sequence $(j_n)_n$ [37, 47].
- If we take $a = b = p$, $c = 2$ and $w_0 = 0, w_1 = 1$ in the sequence (1.29), we get the p-Jacobsthal sequence $(\widehat{J}_n)_n$ [90].
- If we take $a = b = p$, $c = 2$ and $w_0 = 2, w_1 = 1$ in the sequence (1.29), we get the p-Jacobsthal-Lucas sequence $(\widehat{j}_n)_n$ [90].

These sequences are studied by the French mathematician Edouard Lucas (1842-1891) in [53].

1.17 Quaternion numbers

Quaternions were defined by Hamilton (1866) as a generalization of complex numbers. Hamilton introduced a quaternion in the form $q = a + bi + cj + dk$ where a, b, c, d are real numbers or coefficients. The real quaternion algebra is the first noncommutative division algebra to be discovered and defined by

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\},$$

where the complex numbers i, j and k satisfy the following algebraic rules

$$i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k$$

which imply

$$ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j$$

and satisfy, for all real number x

$$ix = xi, \qquad jx = xj, \qquad kx = xk.$$

The conjugate of the quaternion q is defined by $\overline{q} = a - bi - cj - dk$. Quaternions form a 4-dimensional vector space over real numbers with basis $\{1, i, j, k\}$, which is an associative but noncommutative algebra over $\mathbb{R}$. Noncommutative algebra have broad applications in many areas, especially in physics and mathematics. Hamilton's book [33] serves as an excellent reference to the properties of quaternions.

1.18 Dual numbers

Dual numbers were invented by Clifford [20] as an extension of the real numbers. The set of dual numbers is defined as

$$\mathbb{D} = \{d = a + a^*\varepsilon \mid a, a^* \in \mathbb{R}\}, \tag{1.33}$$

where ε is the dual number with $\varepsilon \neq 0, \varepsilon^2 = 0$. Dual numbers have many interesting applications on mechanics, robotics, computer graphics, geometry and physics. The addition and multiplication of two dual numbers

$$d_1 = a + a^*\varepsilon \qquad \text{and} \qquad d_2 = b + b^*\varepsilon$$

are defined as

$$d_1 + d_2 = (a+b) + (a^* + b^*)\varepsilon \qquad \text{and} \qquad d_1 d_2 = ab + (ab^* + a^*b)\varepsilon,$$

respectively.

1.19 Dual quaternion numbers

Similar to the quaternions, dual quaternions are defined by taking dual numbers instead of real numbers as a coefficient. A dual quaternion $\widetilde{q}$ is defined as

$$\widetilde{q} = d_0 + d_1 i + d_2 j + d_3 k,$$

where $d_0, d_1, d_2, d_3 \in \mathbb{D}$, and the elements i, j, k satisfy the quaternion multiplication rule

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.34}$$

Since any dual quaternion can be written as a dual number with a real quaternion coefficient, it is constructed from 8 real parameters. For the detailed information related to these numbers and their applications, we refer to [20, 33, 71].

1.20 Hyper-dual numbers

Hyper-dual numbers can be seen as an extension of dual numbers in the same way that quaternions are an extension of complex numbers. To get an advantage on exact calculations of second (or higher) derivatives, Fike and Alanso [30, 31] introduced the hyper-dual numbers. The set of hyper-dual numbers is defined by

$$\mathbb{HD} = \{D = a_0 + a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_1\varepsilon_2 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}, \tag{1.35}$$

where the dual numbers ε_1 and ε_2 satisfy the following rules

$$\varepsilon_1^2 = \varepsilon_2^2 = 0, \quad \varepsilon_1 \neq \varepsilon_2, \quad \varepsilon_1 \neq 0, \quad \varepsilon_2 \neq 0, \quad \varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1 \neq 0. \tag{1.36}$$

Hyper-dual numbers form a 4-dimensional vector space over real numbers with basis $\{1, \varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2\}$.
Also, a hyper-dual number D can be written as

$$D = d + d^*\varepsilon^*,$$

where

$$d = a_0 + a_1\varepsilon, \qquad d^* = a_2 + a_3\varepsilon \in \mathbb{D}$$

and

$$\varepsilon_1 = \varepsilon, \qquad \varepsilon_2 = \varepsilon^*.$$

Let $D_1 = d_1 + d_1^*\varepsilon^*$ and $D_2 = d_2 + d_2^*\varepsilon^*$ be any two hyper-dual numbers.
The addition and the multiplication of hyper-dual numbers are defined as

$$D_1 + D_2 = (d_1 + d_2) + (d_1^* + d_2^*)\varepsilon^*$$

and

$$D_1 D_2 = d_1 d_2 + (d_1 d_2^* + d_1^* d_2)\varepsilon^*,$$

respectively. For applications of hyper-dual numbers, see [21, 22, 23].

1.21 Hybrid numbers

The hybrid numbers is a new noncommutative numbers introduced by Ozdemir [57] as a generalization of complex, dual and hyperbolic numbers. The geometry of this new numbers can be seen as a generalization of the geometries of the Euclidean, Lorentzian and Galilean, respectively. The set of hybrid numbers is defined as

$$\mathbb{K} = \{a + bi + c\varepsilon + dh \mid a, b, c, d \in \mathbb{R}\} \tag{1.37}$$

where the complex i, the dual ε and the hyperbolic h units satisfy the following rules

$$i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i.$$

The addition, substraction and multiplication of two hybrid numbers

$$k_1 = a_1 + b_1 i + c_1 \varepsilon + d_1 h \quad \text{and} \quad k_2 = a_2 + b_2 i + c_2 \varepsilon + d_2 h,$$

are defined as

$$k_1 \pm k_2 = (a_1 \pm a_2) + (b_1 \pm b_2)\, i + (c_1 \pm c_2)\, \varepsilon + (d_1 \pm d_2)\, h$$

and

$$\begin{aligned} k_1 k_2 &= a_1 a_2 - b_1 b_2 + d_1 d_2 + b_1 c_2 + c_1 b_2 \\ &\quad + (a_1 b_2 + b_1 a_2 + b_1 d_2 - d_1 b_2)\, i \\ &\quad + (a_1 c_2 + c_1 a_2 + b_1 d_2 - d_1 b_2 + d_1 c_2 - c_1 d_2)\, \varepsilon \\ &\quad + (a_1 d_2 + d_1 a_2 + c_1 b_2 - b_1 c_2)\, h. \end{aligned}$$

The multiplication of a hybrid number $k = a + bi + c\varepsilon + dh$ by a real scalar s is defined as

$$sk = sa + sbi + sc\varepsilon + sdh$$

and the norm of a hybrid number k is defined by

$$\|k\| := \sqrt{|C\,(k)|},$$

where $C\,(k) := k\bar{k}$ is the character of the hybrid number k and $\bar{k} := a - bi - c\varepsilon - dh$ is the conjugate of k. Ozdemir's paper [57] serves as an excellent reference to the algebraic and geometric properties of hybrid numbers.

CHAPTER

2

BI-PERIODIC r-FIBONACCI SEQUENCE AND r-LUCAS SEQUENCE OF TYPE s [2]

In the present chapter, for a positive integer r, we study bi-periodic r-Fibonacci sequence and its family of companion sequences, bi-periodic r-Lucas sequence of type s with $1 \leq s \leq r$ which extend the classical Fibonacci and Lucas sequences. Afterwards, we establish the link between the bi-periodic r-Fibonacci sequence and its companion sequences. Furthermore, we give their properties as linear recurrence relations, generating functions, explicit formulas and Binet forms.

2.1 Introduction

For a positive integer r and positive real numbers a, b, Yazlik et *al.* [93] introduced the sequences $(f_n)_n$ and $(l_n)_n$ as follows:

$$f_n = \begin{cases} af_{n-1} + f_{n-r-1}, & if \quad n \equiv 0 \pmod 2, \\ bf_{n-1} + f_{n-r-1}, & if \quad n \equiv 1 \pmod 2) \end{cases}$$

and

$$l_n = \begin{cases} bl_{n-1} + l_{n-r-1}, & if \quad n \equiv 0 \pmod 2, \\ al_{n-1} + l_{n-r-1}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq r+1$ with initial conditions

$$f_0 = 0, f_1 = 1, f_2 = a, \ldots, f_r = a^{\lfloor r/2 \rfloor} b^{\lfloor (r-1)/2 \rfloor}$$

and

$$l_0 = r+1, l_1 = a, l_2 = ab, \ldots, l_r = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor},$$

respectively.
It is clear to see that when $a = b = 1$ and $r = 1$, the sequences $(f_n)_n$ and $(l_n)_n$ reduce to the Fibonacci and Lucas sequences, respectively.

Raab [61] introduced the generalized r-Fibonacci sequence, for a positive integer r and real numbers x and y, by

$$T_n^{(r)} = xT_{n-1}^{(r)} + yT_{n-r-1}^{(r)}, \qquad n \geq r+1$$

and initial conditions

$$T_0^{(r)} = 0, T_k^{(r)} = x^{k-1}, \qquad 1 \leq k \leq r.$$

When $x = y = 1$, the numbers $T_n^{(r)}$ reduce to the r-Fibonacci numbers.

Abbad et *al.* [1] defined the family of companion sequences; the r-Lucas sequences of type s, for a positive integers r, s with $1 \leq s \leq r$ and real numbers x and y, by

$$Z_n^{(r,s)} = xZ_{n-1}^{(r,s)} + yZ_{n-r-1}^{(r,s)}, \qquad n \geq r+1$$

and initial conditions

$$Z_0^{(r)} = s+1, Z_k^{(r)} = x^k, \qquad 1 \leq k \leq r.$$

Our study consists of two aspects. The first one, is to introduce the parameters c and d in the expression of the recurrence sequences given by Yazlik et *al.* in [93]. The second one, is to define a family of companion sequences as introduced in [1] for the bi-periodic case.

2.2 Bi-periodic r-Fibonacci and r-Lucas sequences

In this section, we define bi-periodic r-Fibonacci sequence $(U_n^{(r)})_n$ and we introduce the family of its companion sequences, bi-periodic r-Lucas sequence of type

s, $(V_n^{(r,s)})_n$, then we express $V_n^{(r,s)}$ in terms of $U_n^{(r)}$ and we give their linear recurrence relations.

Definition 10. *For nonzero real numbers a, b, c, d and positive integer r, bi-periodic r-Fibonacci sequence $(U_n^{(r)})_n$ is defined by*

$$U_n^{(r)} = \begin{cases} aU_{n-1}^{(r)} + cU_{n-r-1}^{(r)}, & if \quad n \equiv 0 \pmod 2, \\ bU_{n-1}^{(r)} + dU_{n-r-1}^{(r)}, & if \quad n \equiv 1 \pmod 2, \end{cases} \tag{2.1}$$

for $n \geq r+1$ with initial conditions

$$U_0^{(r)} = 0, U_1^{(r)} = 1, U_2^{(r)} = a, \ldots, U_r^{(r)} = a^{\lfloor r/2 \rfloor} b^{\lfloor (r-1)/2 \rfloor}.$$

We give the first values of the bi-periodic r-Fibonacci sequence.

1. For $r = 1$,

$$\begin{aligned} U_0^{(1)} &= 0, U_1^{(1)} = 1, U_2^{(1)} = a, U_3^{(1)} = ab + d, U_4^{(1)} = a^2 b + a(d + c), \\ U_5^{(1)} &= a^2 b^2 + ab(2d + c) + d^2, U_6^{(1)} = a^3 b^2 + a^2 b(2d + 2c) + a(d^2 + dc + c^2). \end{aligned}$$

2. For $r = 2$,

$$\begin{aligned} U_0^{(2)} &= 0, U_1^{(2)} = 1, U_2^{(2)} = a, U_3^{(2)} = ab, U_4^{(2)} = a^2 b + c, \\ U_5^{(2)} &= a^2 b^2 + (bc + ad), U_6^{(2)} = a^3 b^2 + a(2bc + ad). \end{aligned}$$

The bi-periodic r-Fibonacci sequence can be expressed by the following linear recurrence relation.

Theorem 11. *Let a, b, c, d be nonzero real numbers and r be a positive integer. The bi-periodic r-Fibonacci sequence satisfies the following linear recurrence relation:*
For $n \geq 2r + 2$,

$$U_n^{(r)} = abU_{n-2}^{(r)} + (a^{\xi(r+1)} d + b^{\xi(r+1)} c) U_{n-r-1-\xi(r+1)}^{(r)} - (-1)^{r+1} cd U_{n-2r-2}^{(r)}, \tag{2.2}$$

with initial conditions

$$U_0^{(r)} = 0, U_1^{(r)} = 1, U_2^{(r)} = a, \ldots, U_r^{(r)} = a^{\lfloor r/2 \rfloor} b^{\lfloor (r-1)/2 \rfloor},$$

for $r+1 \le m \le 2r+1$,

$$U_m^{(r)} = \begin{cases} a^{\lfloor \frac{m}{2} \rfloor} b^{\lfloor \frac{m-1}{2} \rfloor} + \left(\lfloor \frac{m-r}{2} \rfloor d + \left\lfloor \frac{m-r-1}{2} \right\rfloor c \right) a^{\lfloor \frac{m-r-1}{2} \rfloor} b^{\lfloor \frac{m-r-2}{2} \rfloor}, & \text{if } r \text{ is odd}, \\ a^{\lfloor \frac{m}{2} \rfloor} b^{\lfloor \frac{m-1}{2} \rfloor} + \lfloor \frac{m-r}{2} \rfloor a^{\lfloor \frac{m-r-2}{2} \rfloor} b^{\lfloor \frac{m-r-1}{2} \rfloor} c + \left\lfloor \frac{m-r-1}{2} \right\rfloor a^{\lfloor \frac{m-r}{2} \rfloor} b^{\lfloor \frac{m-r-3}{2} \rfloor} d, & \text{if } r \text{ is even}, \end{cases} \tag{2.3}$$

where $\xi(k) = 2(k/2 - \lfloor k/2 \rfloor)$ *is the parity function and* $\lfloor\rfloor$ *is the floor function.*

Proof. Note that $\xi(n+m) = \xi(n) + \xi(m) - 2\xi(n)\xi(m)$.
Formula (2.1) can be rewritten as

$$\begin{aligned} U_n^{(r)} &= a^{1-\xi(n)} b^{\xi(n)} U_{n-1}^{(r)} + c^{1-\xi(n)} d^{\xi(n)} U_{n-r-1}^{(r)} \\ &= a^{1-\xi(n)} b^{\xi(n)} \left(a^{\xi(n)} b^{1-\xi(n)} U_{n-2}^{(r)} + c^{\xi(n)} d^{1-\xi(n)} U_{n-r-2}^{(r)} \right) \\ &+ c^{1-\xi(n)} d^{\xi(n)} \left(a^{\xi(n+r)} b^{1-\xi(n+r)} U_{n-r-2}^{(r)} + c^{\xi(n+r)} d^{1-\xi(n+r)} U_{n-2r-2}^{(r)} \right) \\ &= abU_{n-2}^{(r)} + \left(a^{1-\xi(n)} b^{\xi(n)} c^{\xi(n)} d^{1-\xi(n)} + c^{1-\xi(n)} d^{\xi(n)} a^{\xi(n+r)} b^{1-\xi(n+r)} \right) U_{n-r-2}^{(r)} \\ &+ c^{1-\xi(n)} d^{\xi(n)} c^{\xi(n+r)} d^{1-\xi(n+r)} U_{n-2r-2}^{(r)}. \end{aligned}$$

When r is odd, we get

$$\begin{aligned} U_n^{(r)} &= abU_{n-2}^{(r)} + \left(a^{1-\xi(n)} b^{\xi(n)} c^{\xi(n)} d^{1-\xi(n)} + c^{1-\xi(n)} d^{\xi(n)} a^{1-\xi(n)} b^{\xi(n)} \right) U_{n-r-2}^{(r)} \\ &+ c^{1-\xi(n)} d^{\xi(n)} c^{1-\xi(n)} d^{\xi(n)} U_{n-2r-2}^{(r)} \\ &= abU_{n-2}^{(r)} + a^{1-\xi(n)} b^{\xi(n)} (c+d) U_{n-r-2}^{(r)} + c^{2(1-\xi(n))} d^{2\xi(n)} U_{n-2r-2}^{(r)} \\ &= abU_{n-2}^{(r)} + (c+d) \left(U_{n-r-1}^{(r)} - c^{1-\xi(n)} d^{\xi(n)} U_{n-2r-2}^{(r)} \right) + c^{2(1-\xi(n))} d^{2\xi(n)} U_{n-2r-2}^{(r)} \\ &= abU_{n-2}^{(r)} + (c+d) U_{n-r-1}^{(r)} + \left(c^{2(1-\xi(n))} d^{2\xi(n)} - (c+d) c^{1-\xi(n)} d^{\xi(n)} \right) U_{n-2r-2}^{(r)} \\ &= abU_{n-2}^{(r)} + (c+d) U_{n-r-1}^{(r)} \\ &+ \left(c^{2(1-\xi(n))} d^{2\xi(n)} - c^{2-\xi(n)} d^{\xi(n)} - c^{1-\xi(n)} d^{1+\xi(n)} \right) U_{n-2r-2}^{(r)} \\ &= abU_{n-2}^{(r)} + (c+d) U_{n-r-1}^{(r)} - cdU_{n-2r-2}^{(r)}, \end{aligned}$$

when r is even, we get

$$\begin{aligned} U_n^{(r)} &= abU_{n-2}^{(r)} + \left(a^{1-\xi(n)} b^{\xi(n)} c^{\xi(n)} d^{1-\xi(n)} + c^{1-\xi(n)} d^{\xi(n)} a^{\xi(n)} b^{1-\xi(n)} \right) U_{n-r-2}^{(r)} \\ &+ c^{1-\xi(n)} d^{\xi(n)} c^{\xi(n)} d^{1-\xi(n)} U_{n-2r-2}^{(r)} \\ &= abU_{n-2}^{(r)} + (ad+bc) U_{n-r-2}^{(r)} + cdU_{n-2r-2}^{(r)}. \end{aligned}$$

□

Now, we introduce a family of companion sequences related to the bi-periodic r-Fibonacci sequence, called bi-periodic r-Lucas sequence of type s, $(V_n^{(r,s)})_n$.

Definition 12. *For nonzero real numbers a, b, c, d and integers r, s such that $1 \leq s \leq r$, bi-periodic r-Lucas sequence of type s is defined by*

$$V_n^{(r,s)} = \begin{cases} bV_{n-1}^{(r,s)} + dV_{n-r-1}^{(r,s)}, & if \quad n \equiv 0 \pmod 2, \\ aV_{n-1}^{(r,s)} + cV_{n-r-1}^{(r,s)}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq r+1$ with initial conditions

$$V_0^{(r,s)} = s+1, V_1^{(r,s)} = a, V_2^{(r,s)} = ab, \ldots, V_r^{(r,s)} = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor}.$$

We give the first values of the bi-periodic r-Lucas sequence of type s.

1. For $r = s = 1$,

$$\begin{aligned} V_0^{(1,1)} &= 2, V_1^{(1,1)} = a, V_2^{(1,1)} = ab + 2d, V_3^{(1,1)} = a^2b + 2ad + ac, \\ V_4^{(1,1)} &= a^2b^2 + 3abd + abc + 2d^2, \\ V_5^{(1,1)} &= a^3b^2 + 3a^2bd + 2a^2bc + 2ad^2 + 2adc + ac^2. \end{aligned}$$

2. For $r = 2$ and $s \in \{1, 2\}$,

$$\begin{aligned} V_0^{(2,s)} &= s+1, V_1^{(2,s)} = a, V_2^{(2,s)} = ab, V_3^{(2,s)} = a^2b + (s+1)c, \\ V_4^{(2,s)} &= a^2b^2 + (s+1)bc + ad, V_5^{(2,s)} = a^3b^2 + (s+2)abc + a^2d. \end{aligned}$$

The bi-periodic r-Fibonacci sequence $(U_n^{(r)})_n$ and the bi-periodic r-Lucas sequence of type s, $(V_n^{(r,s)})_n$ can be seen as a generalization of the Fibonacci and Lucas sequences, we list some particular cases.

- For $a = b = c = d = 1$ and $r = s = 1$, we get the classical Fibonacci and Lucas sequences.
- For $a = b = 2$, $c = d = 1$ and $r = s = 1$, we get the classical Pell and Pell-Lucas sequences.
- For a, b nonzero real numbers, $c = d = 1$ and $r = s = 1$, we get the bi-periodic Fibonacci and bi-periodic Lucas sequences.

- For a, b nonzero real numbers, $c = d = 2$ and $r = s = 1$, we get the Jacobsthal and the Jacobsthal-Lucas sequences.
- For $a = b$, $c = d$ nonzero real numbers, we get the r-Fibonacci sequence and the r-Lucas sequence of type s.

For more details on these sequences, we refer the reader to [1, 11, 27, 61, 93].

Each sequence in the family of companion sequences, the bi-periodic r-Lucas sequence of type s, satisfies the following linear recurrence relation:

Theorem 13. *Let a, b, c, d be nonzero real numbers and r, s be integers such that $1 \leq s \leq r$. The bi-periodic r-Lucas sequence of type s satisfies the following linear recurrence relation:*
For $n \geq 2r + 2$,

$$V_n^{(r,s)} = abV_{n-2}^{(r,s)} + (a^{\xi(r+1)}d + b^{\xi(r+1)}c)V_{n-r-1-\xi(r+1)}^{(r,s)} - (-1)^{r+1}cdV_{n-2r-2}^{(r,s)}, \quad (2.4)$$

with initial conditions

$$V_0^{(r,s)} = s + 1, V_1^{(r,s)} = a, V_2^{(r,s)} = ab, \ldots, V_r^{(r,s)} = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor},$$

for $r + 1 \leq m \leq 2r + 1$,

$$V_m^{(r,s)} = \begin{cases} a^{\lfloor \frac{m+1}{2} \rfloor} b^{\lfloor \frac{m}{2} \rfloor} + \left(\left(s + \left\lfloor \frac{m-r+1}{2} \right\rfloor \right) d + \left\lfloor \frac{m-r}{2} \right\rfloor c \right) a^{\lfloor \frac{m-r}{2} \rfloor} b^{\lfloor \frac{m-r-1}{2} \rfloor}, & \text{if } r \text{ is odd,} \\ a^{\lfloor \frac{m+1}{2} \rfloor} b^{\lfloor \frac{m}{2} \rfloor} + \left(s + \left\lfloor \frac{m-r+1}{2} \right\rfloor \right) a^{\lfloor \frac{m-r-1}{2} \rfloor} b^{\lfloor \frac{m-r}{2} \rfloor} c + \left\lfloor \frac{m-r}{2} \right\rfloor a^{\lfloor \frac{m-r+1}{2} \rfloor} b^{\lfloor \frac{m-r-2}{2} \rfloor} d, & \text{if } r \text{ is even.} \end{cases} \quad (2.5)$$

Proof. The proof is done using Definition 12. □

Theorem 14. *Let r and s be positive integers, such that $1 \leq s \leq r$, the bi-periodic r-Fibonacci sequence and the bi-periodic r-Lucas sequence of type s satisfy the following relationship:*

$$V_n^{(r,s)} = \begin{cases} U_{n+1}^{(r)} + sdU_{n-r}^{(r)}, & n \geq r, \quad \text{if } r \text{ is odd,} \\ U_{n+1}^{(r)} + scbU_{n-r-1}^{(r)} + scdU_{n-2r-1}^{(r)}, & n \geq 2r+1, \quad \text{if } r \text{ is even.} \end{cases} \quad (2.6)$$

Proof. We prove the theorem by induction on n, using Definition 12 and relations (2.3), (2.5) in Theorem 11 and Theorem 13, respectively.

□

2.3 Generating functions

In this section, we give the generating functions of the bi-periodic r-Fibonacci sequence and the bi-periodic r-Lucas sequence of type s.

Theorem 15. *Let r be a positive integer, the generating function of $(U_n^{(r)})_n$ is*

$$G(x) = \frac{x + ax^2 + (-1)^{\xi(r)} c x^{r+2}}{1 - abx^2 - (a^{\xi(r+1)}d + b^{\xi(r+1)}c)x^{r+1+\xi(r+1)} - (-1)^r cdx^{2r+2}}. \tag{2.7}$$

Proof. The formal power series representation of the generating function for $({U_n}^{(r)})_n$ gives

$$G(x) = \frac{\sum_{k=0}^{2r+1} U_k^{(r)} x^k - abx^2 \sum_{k=0}^{2r-1} U_k^{(r)} x^k - (a^{\xi(r+1)}d + b^{\xi(r+1)}c)x^{r+1+\xi(r+1)} \sum_{k=0}^{r-\xi(r+1)} U_k^{(r)} x^k}{1 - abx^2 - (a^{\xi(r+1)}d + b^{\xi(r+1)}c)x^{r+1+\xi(r+1)} - (-1)^r cdx^{2r+2}}.$$

Indeed, we suppose that r is odd, we write

$$G(x) = \sum_{k \geqslant 0} U_k^{(r)} x^k.$$

Then

$$\begin{aligned}
-abx^2 G(x) &= -ab \sum_{k \geqslant 0} U_k^{(r)} x^{k+2}. \\
(-(d+c))x^{r+1} G(x) &= -(d+c) \sum_{k \geqslant 0} U_k^{(r)} x^{k+r+1}. \\
(cd)x^{2r+2} G(x) &= cd \sum_{k \geqslant 0} U_k^{(r)} x^{k+2r+2}.
\end{aligned}$$

The relation (2.2) in Theorem 11 gives

$$\begin{aligned}
(1 - abx^2 - (d+c)x^{r+1} + cdx^{2r+2})G(x) &= U_0^{(r)} + U_1^{(r)} x^1 + \cdots + U_{2r+1}^{(r)} x^{2r+1} \\
&- abU_0^{(r)} x^2 - abU_1^{(r)} x^3 - \cdots - abU_{2r-1}^{(r)} x^{2r+1} \\
&- (d+c)U_0^{(r)} x^{r+1} - (d+c)U_1^{(r)} x^{r+2} - \cdots \\
&- (d+c)U_r^{(r)} x^{2r+1} \\
&= \sum_{k=0}^{2r+1} U_k^{(r)} x^k - abx^2 \sum_{k=0}^{2r-1} U_k^{(r)} x^k \\
&- (d+c)x^{r+1} \sum_{k=0}^{r} U_k^{(r)} x^k.
\end{aligned}$$

Using relation (2.3) given in Theorem 11, we obtain

$$G(x) = \frac{x + ax^2 - cx^{r+2}}{1 - abx^2 - (d+c)x^{r+1} + cdx^{2r+2}}.$$

Similary, if r is even, we get

$$G(x) = \frac{x + ax^2 + cx^{r+2}}{1 - abx^2 - (ad+bc)x^{r+2} - cdx^{2r+2}}.$$

□

Remark 16. If we take $r = 1$, we obtain the generating function of the bi-periodic Fibonacci sequence given by Sahin [63].

The following theorem express the generating function of $(V_n^{(r,s)})_n$.

Theorem 17. *Let r and s be positive integers, such that $1 \leq s \leq r$, the generating function of $(V_n^{(r,s)})_n$ is*

$$H(x) = \frac{(s+1) + ax - absx^2 + (-1)^{\xi(r)}(s+1)cx^{r+1} + (-1)^{\xi(r+1)}adsx^{r+2}}{1 - abx^2 - (a^{\xi(r+1)}d + b^{\xi(r+1)}c)x^{r+1+\xi(r+1)} - (-1)^r cdx^{2r+2}}. \quad (2.8)$$

Proof. For odd r, relation (2.6) gives

$$\begin{aligned}
H(x) &= \sum_{n\geqslant 0} V_n^{(r,s)} x^n \\
&= \sum_{n\geqslant 0} U_{n+1}^{(r)} x^n + sd \sum_{n\geqslant r} U_{n-r}^{(r)} x^n \\
&= \frac{1}{x} \sum_{n\geqslant 0} U_{n+1}^{(r)} x^{n+1} + sdx^r \sum_{n\geqslant r} U_{n-r}^{(r)} x^{n-r} \\
&= \frac{1}{x} \sum_{n\geqslant 0} U_n^{(r)} x^n + sdx^r \sum_{n\geqslant 0} U_n^{(r)} x^n \\
&= \frac{1 + ax - cx^{r+1}}{1 - abx^2 - (d+c)x^{r+1} + cdx^{2r+2}} + \frac{sd(x^{r+1} + ax^{r+2} - cx^{2r+2})}{1 - abx^2 - (d+c)x^{r+1} + cdx^{2r+2}} \\
&= \frac{1 + ax - cx^{r+1} + sdx^{r+1} + sadx^{r+2} - scdx^{2r+2}}{1 - abx^2 - (d+c)x^{r+1} + cdx^{2r+2}} \\
&= \frac{(s+1) + ax - absx^2 - (s+1)cx^{r+1} + adsx^{r+2}}{1 - abx^2 - (d+c)x^{r+1} + cdx^{2r+2}}.
\end{aligned}$$

For even r, relation (2.6) gives

$$\begin{aligned}
H(x) &= \sum_{n\geqslant 0} V_n^{(r,s)} x^n \\
&= \sum_{n\geqslant 0} U_{n+1}^{(r)} x^n + scb \sum_{n\geqslant r+1} U_{n-r-1}^{(r)} x^n + scd \sum_{n\geqslant 2r+1} U_{n-2r-1}^{(r)} x^n \\
&= \frac{1}{x} \sum_{n\geqslant 0} U_{n+1}^{(r)} x^{n+1} + scbx^{r+1} \sum_{n\geqslant r+1} U_{n-r-1}^{(r)} x^{n-r-1} \\
&\quad + scdx^{2r+1} \sum_{n\geqslant 2r+1} U_{n-2r-1}^{(r)} x^{n-2r-1} \\
&= \frac{1}{x} \sum_{n\geqslant 0} U_n^{(r)} x^n + scbx^{r+1} \sum_{n\geqslant 0} U_n^{(r)} x^n + scdx^{2r+1} \sum_{n\geqslant 0} U_n^{(r)} x^n \\
&= \left(\frac{1}{x} + scbx^{r+1} + scdx^{2r+1}\right) \sum_{n\geqslant 0} U_n^{(r)} x^n \\
&= \frac{(\frac{1}{x} + scbx^{r+1} + scdx^{2r+1})(x + ax^2 + cx^{r+2})}{1 - abx^2 - (ad+bc)x^{r+2} - cdx^{2r+2}} \\
&= \frac{1 + ax + cx^{r+1} + s - sabx^2 - sadx^{r+2} + scx^{r+1}(abx^2 + cbx^{r+2} + adx^{r+2} + cdx^{2r+2})}{1 - abx^2 - (ad+bc)x^{r+2} - cdx^{2r+2}} \\
&= \frac{(s+1) + ax - absx^2 + (s+1)cx^{r+1} - adsx^{r+2}}{1 - abx^2 - (ad+bc)x^{r+2} - cdx^{2r+2}}.
\end{aligned}$$

□

Remark 18. If we take $r = 1$ and $c = d = 1$, we obtain the generation function of the bi-periodic Lucas sequence given by Bilgici [11].

2.4 Explicit formulas

In this section, we will state explicit formulas for $(U_n^{(r)})_n$ and $(V_n^{(r,s)})_n$, to generalize the explicit formulas of bi-periodic Fibonacci and Lucas sequences.
Using Theorem 3, we give an explicit formula of the bi-periodic r-Fibonacci sequence.

Theorem 19. *For any integer $r \geq 1$, we have*

$$U_{n+1}^{(r)} = \begin{cases} \displaystyle\sum_{2i+(r+1)t+(r+1)k=n} \binom{i+t}{t}\binom{t}{k} (ab)^i (c+d)^{t-k} (-cd)^k, & \text{if } r \text{ is odd,} \\[2ex] \displaystyle\sum_{2i+(r+2)t+rk=n} \binom{i+t}{t}\binom{t}{k} (ab)^i (ad+bc)^{t-k} (cd)^k, & \text{if } r \text{ is even.} \end{cases}$$

Proof. Considering the sequence

$$\mathcal{W}_n^{(r)} = U_{n+1}^{(r)},$$

then $\mathcal{W}_0^{(r)} = 1, \mathcal{W}_{-j}^{(r)} = 0$ for $1 \leq j \leq 2r+1$, relation (2.2) gives

$$\mathcal{W}_n^{(r)} = ab\mathcal{W}_{n-2}^{(r)} + (a^{\xi(r+1)}d + b^{\xi(r+1)}c)\mathcal{W}_{n-r-1-\xi(r+1)}^{(r)} - (-1)^{r+1}cd\mathcal{W}_{n-2r-2}^{(r)}. \tag{2.9}$$

If r is odd, formula (2.9) reduces to

$$\mathcal{W}_n^{(r)} = ab\mathcal{W}_{n-2}^{(r)} + (c+d)\mathcal{W}_{n-r-1}^{(r)} - cd\mathcal{W}_{n-2r-2}^{(r)}. \tag{2.10}$$

Using Theorem 3, we get

$$\begin{aligned}
\mathcal{W}_{n+1}^{(r)} &= \sum_{2i+(r+1)j+2(r+1)k=n} \binom{i+j+k}{i,j,k} (ab)^i (c+d)^j (-cd)^k \\
&= \sum_{2i+(r+1)(j+k)+(r+1)k=n} \binom{i+j+k}{j+k} \binom{j+k}{k} (ab)^i (c+d)^j (-cd)^k \\
&= \sum_{2i+(r+1)t+(r+1)k=n} \binom{i+t}{t} \binom{t}{k} (ab)^i (c+d)^{t-k} (-cd)^k.
\end{aligned}$$

If r is even, formula (2.9) reduces to

$$\mathcal{W}_n^{(r)} = ab\mathcal{W}_{n-2}^{(r)} + (ad+bc)\mathcal{W}_{n-r-2}^{(r)} + cd\mathcal{W}_{n-2r-2}^{(r)}. \tag{2.11}$$

Using Theorem 3, we get

$$\begin{aligned}
\mathcal{W}_{n+1}^{(r)} &= \sum_{2i+(r+2)j+2(r+1)k=n} \binom{i+j+k}{i,j,k} (ab)^i (ad+bc)^j (cd)^k \\
&= \sum_{2i+(r+2)(j+k)+rk=n} \binom{i+j+k}{j+k} \binom{j+k}{k} (ab)^i (ad+bc)^j (cd)^k \\
&= \sum_{2i+(r+2)t+rk=n} \binom{i+t}{t} \binom{t}{k} (ab)^i (ad+bc)^{t-k} (cd)^k.
\end{aligned}$$

□

Now, we give an analogous result for the bi-periodic r-Lucas sequence of type s.

Theorem 20. *For any positive integers r and s, such that* $1 \leq s \leq r$*, we have*

$$\begin{aligned} V_n^{(r,s)} &= \sum_{2i+(r+1)t+(r+1)k=n} \binom{i+t}{t}\binom{t}{k}(ab)^i(c+d)^{t-k}(-cd)^k \\ &+ sd \sum_{2i+(r+1)t+(r+1)k=n-r-1} \binom{i+t}{t}\binom{t}{k}(ab)^i(c+d)^{t-k}(-cd)^k, \end{aligned}$$

if r is odd.

$$\begin{aligned} V_n^{(r,s)} &= \sum_{2i+(r+2)t+rk=n} \binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k \\ &+ sbc \sum_{2i+(r+2)t+rk=n-r-2} \binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k \\ &+ scd \sum_{2i+(r+2)t+rk=n-2r-2} \binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k, \end{aligned}$$

if r is even.

Proof. We get the proof by using Theorem 14. □

Remark 21. Theorems 19 and 20 generalize the explicit formulas given in [81, 92].

2.5 Binet forms

In order to obtain the Binet forms of the bi-periodic r-Fibonacci sequence and the bi-periodic r-Lucas sequence of type s, we first express the characteristic polynomial. Considering relations (2.2) and (2.4), we get the characteristic polynomial of $(U_n^{(r)})_n$ and $(V_n^{(r,s)})_n$

$$y^{2r+2} - aby^{2r} - (a^{\xi(r+1)}d + b^{\xi(r+1)}c)y^{r+\xi(r)} - (-1)^{\xi(r)}cd, \tag{2.12}$$

putting $x = y^2$, we obtain

$$x^{r+1} - abx^r - (a^{\xi(r+1)}d + b^{\xi(r+1)}c)x^{\lfloor \frac{r+1}{2} \rfloor} - (-1)^{\xi(r)}cd. \tag{2.13}$$

Before stating the main theorems of this section, the following lemma will be useful:

Lemma 22. *Let* $\mathbf{K}$ *be a field and* $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = \sum_{i=0}^{n} a_ix^i \in$

$\mathbf{K}[\mathbf{x}]$, a split polynomial on $\mathbf{K}$ with n roots, $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{K}$. The polynomial $P(x)$ can be written as $P(x) = a_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ and

$$\sigma_p = \sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_p} = (-1)^p \frac{a_{n-p}}{a_n}. \tag{2.14}$$

For any i, j, we put $\sigma_j = \alpha_i \tilde{\sigma}^i_{j-1} + \tilde{\sigma}^i_j$, where

$$\tilde{\sigma}^i_j = \sum_{\substack{1 \le k_1 < k_2 < \cdots < k_{r+1-j} \le r+1 \\ k_1, k_2, \ldots, k_{r+1-j} \ne i}} \alpha_{k_1}\alpha_{k_2}\ldots\alpha_{k_{r+1-j}}.$$

Theorem 23. *Let $\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ be the distinct roots of the characteristic polynomial (2.13) associated with $(U_n^{(r)})_n$ and $(V_n^{(r,s)})_n$, we have*

$$U_n^{(r)} = \sum_{i=1}^{r+1} \frac{\left(\sum_{j=1}^{r} (-1)^j \tilde{\sigma}^i_j \; U^{(r)}_{2r-2j+\xi(n)} + U^{(r)}_{2r+\xi(n)}\right)}{\prod_{\substack{1 \le k \le r+1 \\ k \ne i}} (\alpha_i - \alpha_k)} \alpha_i^{\lfloor n/2 \rfloor}$$

and

$$V_n^{(r,s)} = \begin{cases} \sum_{i=1}^{r+1} \frac{\left(\sum_{j=1}^{r} (-1)^j \tilde{\sigma}^i_j \; U^{(r)}_{2r-2j+\xi(n+1)} + U^{(r)}_{2r+\xi(n+1)}\right)}{\prod_{\substack{1 \le k \le r+1 \\ k \ne i}} (\alpha_i - \alpha_k)} \left(\alpha_i^{\lfloor (n+1)/2 \rfloor} + sd\alpha_i^{\lfloor (n-r)/2 \rfloor}\right), & \text{if } r \text{ is odd,} \\ \sum_{i=1}^{r+1} \frac{\left(\sum_{j=1}^{r} (-1)^j \tilde{\sigma}^i_j \; U^{(r)}_{2r-2j+\xi(n+1)} + U^{(r)}_{2r+\xi(n+1)}\right)}{\prod_{\substack{1 \le k \le r+1 \\ k \ne i}} (\alpha_i - \alpha_k)} \left(\alpha_i^{\lfloor (n+1)/2 \rfloor} + scb\alpha_i^{\lfloor (n-r-1)/2 \rfloor} + scd\alpha_i^{\lfloor (n-2r-1)/2 \rfloor}\right), & \text{if } r \text{ is even,} \end{cases}$$

with initial conditions

$$U_0^{(r)} = 0, U_1^{(r)} = 1, U_2^{(r)} = a, \ldots, U_r^{(r)} = a^{\lfloor r/2 \rfloor} b^{\lfloor (r-1)/2 \rfloor},$$
$$V_0^{(r,s)} = s+1, V_1^{(r,s)} = a, V_2^{(r,s)} = ab, \ldots, V_r^{(r,s)} = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor}.$$

Proof. As mentioned in [19], the general term of $(U_n^{(r)})_n$ is given by $U_n^{(r)} = \sum_{i=1}^{r+1} b_{i,n} \alpha_i^{\lfloor \frac{n}{2} \rfloor}$, where $b_{i,n}$'s are rational numbers. The system can be solved by Cramer's rule with Vandermonde determinant, for more details, we refer to

[41]. Using the initial terms of the sequence $(U_n^{(r)})_n$, for $n = 0, 2, 4, \ldots, 2r$, we get

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{r+1} \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \cdots & \alpha_{r+1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1^r & \alpha_2^r & \alpha_3^r & \cdots & \alpha_{r+1}^r \end{pmatrix}^{-1} \begin{pmatrix} U_0^{(r)} \\ U_2^{(r)} \\ U_4^{(r)} \\ \vdots \\ U_{2r}^{(r)} \end{pmatrix} = \begin{pmatrix} b_{1,n} \\ b_{2,n} \\ b_{3,n} \\ \vdots \\ b_{r+1,n} \end{pmatrix}$$

and for $n = 1, 3, 5, \ldots, 2r+1$, we get

$$\begin{pmatrix} \sqrt{\alpha_1} & \sqrt{\alpha_2} & \sqrt{\alpha_3} & \cdots & \sqrt{\alpha_{r+1}} \\ \sqrt{\alpha_1}^3 & \sqrt{\alpha_2}^3 & \sqrt{\alpha_3}^3 & \cdots & \sqrt{\alpha_{r+1}}^3 \\ \sqrt{\alpha_1}^5 & \sqrt{\alpha_2}^5 & \sqrt{\alpha_3}^5 & \cdots & \sqrt{\alpha_{r+1}}^5 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\alpha_1}^{2r+1} & \sqrt{\alpha_2}^{2r+1} & \sqrt{\alpha_3}^{2r+1} & \cdots & \sqrt{\alpha_{r+1}}^{2r+1} \end{pmatrix}^{-1} \begin{pmatrix} U_1^{(r)} \\ U_3^{(r)} \\ U_5^{(r)} \\ \vdots \\ U_{2r+1}^{(r)} \end{pmatrix} = \begin{pmatrix} b_{1,n} \\ b_{2,n} \\ b_{3,n} \\ \vdots \\ b_{r+1,n} \end{pmatrix},$$

it results that

$$b_{i,n} = \frac{\sum\limits_{j=1}^{r} (-1)^j \sum\limits_{\substack{1 \le k_1 < k_2 < \ldots < k_{r+1-j} \le r+1 \\ k_1, k_2, \ldots, k_{r+1-j} \ne i}} \alpha_{k_1} \alpha_{k_2} \ldots \alpha_{k_{r+1-j}} U_{2r-2j+\xi(n)}^{(r)} + U_{2r+\xi(n)}^{(r)}}{\prod\limits_{\substack{1 \le k \le r+1 \\ k \ne i}} (\alpha_i - \alpha_k)},$$

using Lemma 22, we obtain

$$b_{i,n} = \frac{\sum\limits_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i U_{2r-2j+\xi(n)}^{(r)} + U_{2r+\xi(n)}^{(r)}}{\prod\limits_{\substack{1 \le k \le r+1 \\ k \ne i}} (\alpha_i - \alpha_k)},$$

which gives

$$U_n^{(r)} = \sum_{i=1}^{r+1} \frac{\sum\limits_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i \ U_{2r-2j+\xi(n)}^{(r)} + U_{2r+\xi(n)}^{(r)}}{\prod\limits_{\substack{1 \le k \le r+1 \\ k \ne i}} (\alpha_i - \alpha_k)} \alpha_i^{\lfloor n/2 \rfloor}.$$

Using relation (2.6) in Theorem 14 for odd r, we get

$$\begin{aligned}
V_n^{(r,s)} &= U_{n+1}^{(r)} + sdU_{n-r}^{(r)} \\
&= \sum_{i=1}^{r+1} \frac{\sum_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i \; U_{2r-2j+\xi(n+1)}^{(r)} + U_{2r+\xi(n+1)}^{(r)}}{\prod\limits_{\substack{1\le k\le r+1 \\ k\neq i}} (\alpha_i - \alpha_k)} \alpha_i^{\lfloor (n+1)/2 \rfloor} \\
&+ \; sd \sum_{i=1}^{r+1} \frac{\sum_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i \; U_{2r-2j+\xi(n-r)}^{(r)} + U_{2r+\xi(n-r)}^{(r)}}{\prod\limits_{\substack{1\le k\le r+1 \\ k\neq i}} (\alpha_i - \alpha_k)} \alpha_i^{\lfloor (n-r)/2 \rfloor} \\
&= \sum_{i=1}^{r+1} \frac{\sum_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i \; U_{2r-2j+\xi(n+1)}^{(r)} + U_{2r+\xi(n+1)}^{(r)}}{\prod\limits_{\substack{1\le k\le r+1 \\ k\neq i}} (\alpha_i - \alpha_k)} \left(\alpha_i^{\lfloor (n+1)/2 \rfloor} + sd\alpha_i^{\lfloor (n-r)/2 \rfloor} \right)
\end{aligned}$$

and using relation (2.6) in Theorem 14 for even r, we get

$$\begin{aligned}
V_n^{(r,s)} &= U_{n+1}^{(r)} + scbU_{n-r-1}^{(r)} + scdU_{n-2r-1}^{(r)} \\
&= \sum_{i=1}^{r+1} \frac{\sum_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i \; U_{2r-2j+\xi(n+1)}^{(r)} + U_{2r+\xi(n+1)}^{(r)}}{\prod\limits_{\substack{1\le k\le r+1 \\ k\neq i}} (\alpha_i - \alpha_k)} \alpha_i^{\lfloor (n+1)/2 \rfloor} \\
&+ scb \sum_{i=1}^{r+1} \frac{\sum_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i \; U_{2r-2j+\xi(n-r-1)}^{(r)} + U_{2r+\xi(n-r-1)}^{(r)}}{\prod\limits_{\substack{1\le k\le r+1 \\ k\neq i}} (\alpha_i - \alpha_k)} \alpha_i^{\lfloor (n-r-1)/2 \rfloor} \\
&+ scd \sum_{i=1}^{r+1} \frac{\sum_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i \; U_{2r-2j+\xi(n-2r-1)}^{(r)} + U_{2r+\xi(n-2r-1)}^{(r)}}{\prod\limits_{\substack{1\le k\le r+1 \\ k\neq i}} (\alpha_i - \alpha_k)} \alpha_i^{\lfloor (n-2r-1)/2 \rfloor} \\
&= \sum_{i=1}^{r+1} \frac{\sum_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i \; U_{2r-2j+\xi(n+1)}^{(r)} + U_{2r+\xi(n+1)}^{(r)}}{\prod\limits_{\substack{1\le k\le r+1 \\ k\neq i}} (\alpha_i - \alpha_k)} \\
&\qquad \times \left(\alpha_i^{\lfloor (n+1)/2 \rfloor} + scb\alpha_i^{\lfloor (n-r-1)/2 \rfloor} + scd\alpha_i^{\lfloor (n-2r-1)/2 \rfloor} \right).
\end{aligned}$$

□

Remark 24. If we take $c = d = 1$, we obtain the Binet form for the sequence $(f_n)_n$ given by Yazlik et *al*. [93].

Equivalently, we can express the Binet forms of $(U_n^{(r)})_n$ and $(V_n^{(r,s)})_n$ as follows.

Theorem 25. *For any integer $r \geq 1$, we have*

$$U_n^{(r)} = \sum_{i=1}^{r+1} A_i^{(n)} \alpha_i^{\lfloor n/2 \rfloor}$$

and

$$V_n^{(r,s)} = \begin{cases} \sum_{i=1}^{r+1} A_i^{(n+1)} \left(\alpha_i^{\lfloor (n+1)/2 \rfloor} + sd\alpha_i^{\lfloor (n-r)/2 \rfloor} \right), & \text{if } r \text{ is odd,} \\ \sum_{i=1}^{r+1} A_i^{(n+1)} \left(\alpha_i^{\lfloor (n+1)/2 \rfloor} + scb\alpha_i^{\lfloor (n-r-1)/2 \rfloor} + scd\alpha_i^{\lfloor (n-2r-1)/2 \rfloor} \right), & \text{if } r \text{ is even,} \end{cases}$$

where

$$A_i^{(n)} = \frac{\sum_{j=1}^{r} -\alpha_i^{j-1}(ab-\alpha_i)U_{2r-2j+\xi(n)}^{(r)} + \sum_{j=\lfloor (r+2)/2 \rfloor}^{r} -\alpha_i^{j-\lfloor (r+2)/2 \rfloor}(a^{\xi(r+1)}d + b^{\xi(r+1)}c)\ U_{2r-2j+\xi(n)}^{(r)} + U_{2r+\xi(n)}^{(r)}}{\alpha_i^{\lfloor \frac{r-1}{2} \rfloor}\left((r+1)\alpha_i^{\lfloor \frac{r+2}{2} \rfloor} - rab\alpha_i^{\lfloor \frac{r}{2} \rfloor} - \left\lfloor \frac{r+1}{2} \right\rfloor (a^{\xi(r+1)}d + b^{\xi(r+1)}c) \right)},$$

with initial conditions

$$U_0^{(r)} = 0, U_1^{(r)} = 1, U_2^{(r)} = a, \ldots, U_r^{(r)} = a^{\lfloor r/2 \rfloor} b^{\lfloor (r-1)/2 \rfloor},$$
$$V_0^{(r,s)} = s+1, V_1^{(r,s)} = a, V_2^{(r,s)} = ab, \ldots, V_r^{(r,s)} = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor}.$$

Proof. Considering

$$\begin{aligned} P(x) &= x^{r+1} - abx^r - (a^{\xi(r+1)}d + b^{\xi(r+1)}c)x^{\lfloor \frac{r+1}{2} \rfloor} - (-1)^{\xi(r)}cd \\ &= (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{r+1}), \end{aligned}$$

then for $1 \leq i \leq r+1$, we get

$$P'(\alpha_i) = (r+1)\alpha_i^r - rab\alpha_i^{r-1} - \left\lfloor \frac{r+1}{2} \right\rfloor (a^{\xi(r+1)}d + b^{\xi(r+1)}c)\alpha_i^{\lfloor \frac{r-1}{2} \rfloor}$$

$$= \alpha_i^{\lfloor \frac{r-1}{2} \rfloor} \left((r+1)\alpha_i^{\lfloor \frac{r+2}{2} \rfloor} - rab\alpha_i^{\lfloor \frac{r}{2} \rfloor} - \left\lfloor \frac{r+1}{2} \right\rfloor (a^{\xi(r+1)}d + b^{\xi(r+1)}c) \right)$$
$$= \prod_{\substack{1 \le k \le r+1 \\ k \ne i}} (\alpha_i - \alpha_k).$$

On the other hand, using Lemma 22 and formula (2.13) for odd r, we get

$$\begin{aligned}
\sigma_1 &= \sum_{1 \le i_1 \le r+1} \alpha_{i_1} = -a_r = ab, \\
\sigma_2 &= \sum_{1 \le i_1 < i_2 \le r+1} \alpha_{i_1}\alpha_{i_2} = -a_{r-1} = 0, \\
&\vdots \\
\sigma_{(r-1)/2} &= \sum_{1 \le i_1 < i_2 < \cdots < i_{(r-1)/2} \le r+1} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-1)/2}} = (-1)^{(r-1)/2}a_{r-1} = 0, \\
\sigma_{(r+1)/2} &= \sum_{1 \le i_1 < i_2 < \cdots < i_{(r+1)/2} \le r+1} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r+1)/2}} = (-1)^{(r+1)/2}a_{(r+1)/2} \\
&= (-1)^{(r+1)/2+1}(c+d), \\
&\vdots \\
\sigma_r &= \sum_{1 \le i_1 < i_2 < \cdots < i_r \le r+1} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_r} = (-1)^r a_1 = 0, \\
\sigma_{r+1} &= \prod_{1 \le i_1 < i_2 < \cdots < i_{r+1} \le r+1} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{r+1}} \\
&= (-1)^{(r+1)}a_0 = (-1)^{\xi(r+1)+r+1}cd = cd.
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{\sigma}_1^i &= \sum_{\substack{1 \le i_1 \le r+1 \\ i_1 \ne i}} \alpha_{i_1}, \\
\tilde{\sigma}_2^i &= \sum_{\substack{1 \le i_1 < i_2 \le r+1 \\ i_1, i_2 \ne i}} \alpha_{i_1}\alpha_{i_2}, \\
&\vdots \\
\tilde{\sigma}_{(r-1)/2}^i &= \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{(r-1)/2} \le r+1 \\ i_1, i_2, \ldots, i_{(r-1)/2} \ne i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-1)/2}}, \\
\tilde{\sigma}_{(r+1)/2}^i &= \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{(r+1)/2} \le r+1 \\ i_1, i_2, \ldots, i_{(r+1)/2} \ne i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r+1)/2}}, \\
&\vdots \\
\tilde{\sigma}_r^i &= \prod_{\substack{1 \le i_1 < i_2 < \cdots < i_r \le r+1 \\ i_1, i_2, \ldots, i_r \ne i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_r},
\end{aligned}$$

thus

$$
\begin{aligned}
\tilde{\sigma}_1^i &= ab - \alpha_i, \\
\tilde{\sigma}_2^i &= (-\alpha_i)\tilde{\sigma}_1^i = (-\alpha_i)(ab - \alpha_i), \\
&\vdots \\
\tilde{\sigma}_j^i &= (-\alpha_i)^{j-1}(ab - \alpha_i), \\
&\vdots \\
\tilde{\sigma}_{(r-1)/2}^i &= (-\alpha_i)^{(r-1)/2-1}(ab - \alpha_i), \\
\tilde{\sigma}_{(r+1)/2}^i &= (-1)^{(r+1)/2+1}(c+d) + (-\alpha_i)^{(r+1)/2-1}(ab - \alpha_i), \\
\tilde{\sigma}_{(r+1)/2+1}^i &= (-\alpha_i)(-1)^{(r+1)/2+1}(c+d) + (-\alpha_i)(-\alpha_i)^{(r+1)/2-1}(ab - \alpha_i), \\
\tilde{\sigma}_{(r+1)/2+2}^i &= (-\alpha_i)^2(-1)^{(r+1)/2+1}(c+d) + (-\alpha_i)^2(-\alpha_i)^{(r+1)/2-1}(ab - \alpha_i), \\
\tilde{\sigma}_{(r+1)/2+3}^i &= (-\alpha_i)^3(-1)^{(r+1)/2+1}(c+d) + (-\alpha_i)^3(-\alpha_i)^{(r+1)/2-1}(ab - \alpha_i), \\
&\vdots \\
\tilde{\sigma}_r^i &= {\alpha_i}^{\frac{r-1}{2}}(-1)^{r+1}(c+d) + (-\alpha_i)^{r-1}(ab - \alpha_i).
\end{aligned}
$$

Using Lemma 22 and formula (2.13) for even r, we get

$$
\begin{aligned}
\sigma_1 &= \sum_{1 \le i_1 \le r+1} \alpha_{i_1} = -a_r = ab, \\
\sigma_2 &= \sum_{1 \le i_1 < i_2 \le r+1} \alpha_{i_1}\alpha_{i_2} = -a_{r-1} = 0, \\
&\vdots \\
\sigma_{r/2} &= \sum_{1 \le i_1 < i_2 < \cdots < i_{r/2} \le r+1} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{r/2}} = (-1)^{r/2}a_{r/2+1} = 0, \\
\sigma_{(r+2)/2} &= \sum_{1 \le i_1 < i_2 < \cdots < i_{(r+2)/2} \le r+1} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r+1)/2}} = (-1)^{(r+2)/2}a_{r/2} \\
&= (-1)^{(r+2)/2+1}(ad + bc), \\
&\vdots \\
\sigma_r &= \sum_{1 \le i_1 < i_2 < \cdots < i_r \le r+1} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_r} = (-1)^r a_1 = 0, \\
\sigma_{r+1} &= \prod_{1 \le i_1 < i_2 < \cdots < i_{r+1} \le r+1} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{r+1}} = (-1)^{(r+1)}a_0 \\
&= (-1)^{\xi(r+1)+r+1}cd = cd,
\end{aligned}
$$

then

$$
\begin{aligned}
\tilde{\sigma}_1^i &= ab - \alpha_i, \\
\tilde{\sigma}_2^i &= (-\alpha_i)\tilde{\sigma}_1^i = (-\alpha_i)(ab - \alpha_i),
\end{aligned}
$$

$$
\begin{aligned}
&\vdots \\
\tilde{\sigma}^i_j &= (-\alpha_i)^{j-1}(ab-\alpha_i), \\
&\vdots \\
\tilde{\sigma}^i_{r/2} &= (-\alpha_i)^{r/2-1}(ab-\alpha_i), \\
\tilde{\sigma}^i_{(r+2)/2} &= (-1)^{(r+2)/2+1}(ad+bc)+(-\alpha_i)^{r/2}(ab-\alpha_i), \\
\tilde{\sigma}^i_{(r+2)/2+1} &= (-\alpha_i)(-1)^{(r+2)/2+1}(ad+bc)+(-\alpha_i)^{r/2+1}(ab-\alpha_i), \\
\tilde{\sigma}^i_{(r+2)/2+2} &= (-\alpha_i)^2(-1)^{(r+2)/2+1}(ad+bc)+(-\alpha_i)^{r/2+2}(ab-\alpha_i), \\
\tilde{\sigma}^i_{(r+2)/2+3} &= (-\alpha_i)^3(-1)^{(r+2)/2+1}(ad+bc)+(-\alpha_i)^{r/2+3}(ab-\alpha_i), \\
&\vdots \\
\tilde{\sigma}^i_r &= (-\alpha_i)^{(r-2)/2}(-1)^{(r+2)/2+1}(ad+bc)+(-\alpha_i)^{r-1}(ab-\alpha_i).
\end{aligned}
$$

□

Considering that $r \geq 2$ and $\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ are nonzero roots, the Binet forms of the sequences $(U_n^{(r)})_n$ and $(V_n^{(r,s)})_n$ have two equivalent expressions given in the following corollaries.

Corollary 26. *For any integer $r \geq 2$, we have*

$$U_n^{(r)} = \sum_{i=1}^{r+1} B_i^{(n)} \alpha_i^{\lfloor n/2 \rfloor}$$

and

$$
V_n^{(r,s)} = \begin{cases} \sum\limits_{i=1}^{r+1} B_i^{(n+1)} \left(\alpha_i^{\lfloor (n+1)/2 \rfloor} + sd\alpha_i^{\lfloor (n-r)/2 \rfloor} \right), & \text{if } r \text{ is odd}, \\ \sum\limits_{i=1}^{r+1} B_i^{(n+1)} \left(\alpha_i^{\lfloor (n+1)/2 \rfloor} + scb\alpha_i^{\lfloor (n-r-1)/2 \rfloor} + scd\alpha_i^{\lfloor (n-2r-1)/2 \rfloor} \right), & \text{if } r \text{ is even}, \end{cases}
$$

where

$$
B_i^{(n)} = \frac{\sum\limits_{j=1}^{\lfloor r/2 \rfloor} -\alpha_i^{j-1}(ab-\alpha_i) U^{(r)}_{2r-2j+\xi(n)} + \sum\limits_{j=\lfloor (r+2)/2 \rfloor}^{r} (-1)^j \frac{cd}{\alpha_i(-\alpha_i)^{r-j}} U^{(r)}_{2r-2j+\xi(n)} + U^{(r)}_{2r+\xi(n)}}{\alpha_i^{\left\lfloor \frac{r-1}{2} \right\rfloor} \left((r+1)\alpha_i^{\left\lfloor \frac{r+2}{2} \right\rfloor} - rab\alpha_i^{\left\lfloor \frac{r}{2} \right\rfloor} - \left\lfloor \frac{r+1}{2} \right\rfloor (a^{\xi(r+1)}d + b^{\xi(r+1)}c) \right)},
$$

with initial conditions

$$U_0^{(r)} = 0, U_1^{(r)} = 1, U_2^{(r)} = a, \ldots, U_r^{(r)} = a^{\lfloor r/2 \rfloor} b^{\lfloor (r-1)/2 \rfloor},$$
$$V_0^{(r,s)} = s+1, V_1^{(r,s)} = a, V_2^{(r,s)} = ab, \ldots, V_r^{(r,s)} = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor}.$$

Proof. Assume that r is odd, then

$$\begin{aligned}
\tilde{\sigma}_1^i &= \sum_{\substack{1 \le i_1 \le r+1 \\ i_1 \ne i}} \alpha_{i_1} = ab - \alpha_i, \\
\tilde{\sigma}_2^i &= \sum_{\substack{1 \le i_1 < i_2 \le r+1 \\ i_1, i_2 \ne i}} \alpha_{i_1}\alpha_{i_2} = (-\alpha_i)(ab - \alpha_i), \\
&\vdots \\
\tilde{\sigma}_j^i &= \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_j \le r+1 \\ i_1, i_2, \ldots, i_j \ne i}} \alpha_{i_1}\alpha_{i_2} \ldots \alpha_{i_j} = (-\alpha_i)^{j-1}(ab - \alpha_i), \\
&\vdots \\
\tilde{\sigma}_{(r-1)/2}^i &= \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{(r-1)/2} \le r+1 \\ i_1, i_2, \ldots, i_{(r-1)/2} \ne i}} \alpha_{i_1}\alpha_{i_2} \ldots \alpha_{i_{(r-1)/2}} = (-\alpha_i)^{(r-1)/2-1}(ab - \alpha_i), \\
\tilde{\sigma}_{(r+1)/2}^i &= \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{(r+1)/2} \le r+1 \\ i_1, i_2, \ldots, i_{(r+1)/2} \ne i}} \alpha_{i_1}\alpha_{i_2} \ldots \alpha_{i_{(r+1)/2}} = (-1)^{(r+1)/2+1}(c+d) \\
&+ \ (-\alpha_i)^{(r+1)/2-1}(ab - \alpha_i) = \tfrac{cd}{\alpha_i} \frac{1}{(-\alpha_i)^{\frac{r-1}{2}}}, \\
&\vdots \\
\tilde{\sigma}_{r-t}^i &= \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{(r-t)} \le r+1 \\ i_1, i_2, \ldots, i_{(r-t)} \ne i}} \alpha_{i_1}\alpha_{i_2} \ldots \alpha_{i_{(r-t)}} = \tfrac{cd}{\alpha_i} \tfrac{1}{(-\alpha_i)^t}, \\
&\vdots \\
\tilde{\sigma}_{r-2}^i &= \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{(r-2)} \le r+1 \\ i_1, i_2, \ldots, i_{(r-2)} \ne i}} \alpha_{i_1}\alpha_{i_2} \ldots \alpha_{i_{(r-2)}} = \tfrac{cd}{\alpha_i} \tfrac{1}{(-\alpha_i)^2}, \\
\tilde{\sigma}_{r-1}^i &= \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{(r-1)} \le r+1 \\ i_1, i_2, \ldots, i_{(r-1)} \ne i}} \alpha_{i_1}\alpha_{i_2} \ldots \alpha_{i_{(r-1)}} = \tfrac{cd}{\alpha_i} \tfrac{1}{(-\alpha_i)}, \\
\tilde{\sigma}_r^i &= \prod_{\substack{1 \le i_1 < i_2 < \cdots < i_r \le r+1 \\ i_1, i_2, \ldots, i_r \ne i}} \alpha_{i_1}\alpha_{i_2} \ldots \alpha_{i_r} = \tfrac{cd}{\alpha_i}.
\end{aligned}$$

Assume that r is even, then

$$
\begin{aligned}
\tilde{\sigma}_1^i &= \sum_{\substack{1\le i_1\le r+1\\ i_1\neq i}} \alpha_{i_1} = ab-\alpha_i,\\
\tilde{\sigma}_2^i &= \sum_{\substack{1\le i_1<i_2\le r+1\\ i_1,i_2\neq i}} \alpha_{i_1}\alpha_{i_2} = (-\alpha_i)(ab-\alpha_i),\\
&\vdots\\
\tilde{\sigma}_j^i &= \sum_{\substack{1\le i_1<i_2<\cdots<i_j\le r+1\\ i_1,i_2,\ldots,i_j\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_j} = (-\alpha_i)^{j-1}(ab-\alpha_i),\\
&\vdots\\
\tilde{\sigma}_{r/2}^i &= \sum_{\substack{1\le i_1<i_2<\cdots<i_{r/2}\le r+1\\ i_1,i_2,\ldots,i_{r/2}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{r/2}} = (-\alpha_i)^{r/2-1}(ab-\alpha_i),\\
\tilde{\sigma}_{(r+2)/2}^i &= \sum_{\substack{1\le i_1<i_2<\cdots<i_{(r+2)/2}\le r+1\\ i_1,i_2,\ldots,i_{(r+2)/2}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r+2)/2}} = (-1)^{(r+2)/2+1}(ad+bc)\\
&+ (-\alpha_i)^{(r+2)/2-1}(ab-\alpha_i) = \tfrac{cd}{\alpha_i}\tfrac{1}{(-\alpha_i)^{(r+2)/2}},\\
&\vdots\\
\tilde{\sigma}_{r-t}^i &= \sum_{\substack{1\le i_1<i_2<\cdots<i_{(r-t)}\le r+1\\ i_1,i_2,\ldots,i_{(r-t)}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-t)}} = \tfrac{cd}{\alpha_i}\tfrac{1}{(-\alpha_i)^t},\\
&\vdots\\
\tilde{\sigma}_{r-2}^i &= \sum_{\substack{1\le i_1<i_2<\cdots<i_{(r-2)}\le r+1\\ i_1,i_2,\ldots,i_{(r-2)}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-2)}} = \tfrac{cd}{\alpha_i}\tfrac{1}{(-\alpha_i)^2},\\
\tilde{\sigma}_{r-1}^i &= \sum_{\substack{1\le i_1<i_2<\cdots<i_{(r-1)}\le r+1\\ i_1,i_2,\ldots,i_{(r-1)}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-1)}} = \tfrac{cd}{\alpha_i}\tfrac{1}{(-\alpha_i)},\\
\tilde{\sigma}_r^i &= \prod_{\substack{1\le i_1<i_2<\cdots<i_r\le r+1\\ i_1,i_2,\ldots,i_r\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_r} = \tfrac{cd}{\alpha_i}.
\end{aligned}
$$

□

Corollary 27. *For any integer $r\geq 2$, we have*

$$U_n^{(r)} = \sum_{i=1}^{r+1} C_i^{(n)}\alpha_i^{\lfloor n/2\rfloor}$$

and

$$
V_n^{(r,s)} = \begin{cases} \sum_{i=1}^{r+1} C_i^{(n+1)}\left(\alpha_i^{\lfloor (n+1)/2\rfloor} + sd\alpha_i^{\lfloor (n-r)/2\rfloor}\right), & \text{if } r \text{ is odd},\\[2ex] \sum_{i=1}^{r+1} C_i^{(n+1)}\left(\alpha_i^{\lfloor (n+1)/2\rfloor} + scb\alpha_i^{\lfloor (n-r-1)/2\rfloor} + scd\alpha_i^{\lfloor (n-2r-1)/2\rfloor}\right), & \text{if } r \text{ is even},\end{cases}
$$

where

$$C_i^{(n)} = \frac{\sum_{j=1}^{r}(-1)^j \frac{cd}{\alpha_i(-\alpha_i)^{r-j}} U^{(r)}_{2r-2j+\xi(n)} + \sum_{j=1}^{\lfloor r/2\rfloor}(-1)^{j+\lfloor r/2\rfloor}\frac{(a^{\xi(r+1)}d+b^{\xi(r+1)}c)}{\alpha_i(-\alpha_i)^{\lfloor r/2\rfloor-j}}\; U^{(r)}_{2r-2j+\xi(n)} + U^{(r)}_{2r+\xi(n)}}{\alpha_i^{\lfloor\frac{r-1}{2}\rfloor}\left((r+1)\alpha_i^{\lfloor\frac{r+2}{2}\rfloor} - rab\alpha_i^{\lfloor\frac{r}{2}\rfloor} - \left\lfloor\frac{r+1}{2}\right\rfloor(a^{\xi(r+1)}d + b^{\xi(r+1)}c)\right)},$$

with initial conditions

$$U_0^{(r)} = 0, U_1^{(r)} = 1, U_2^{(r)} = a, \ldots, U_r^{(r)} = a^{\lfloor r/2\rfloor}b^{\lfloor (r-1)/2\rfloor},$$
$$V_0^{(r,s)} = s+1, V_1^{(r,s)} = a, V_2^{(r,s)} = ab, \ldots, V_r^{(r,s)} = a^{\lfloor (r+1)/2\rfloor}b^{\lfloor r/2\rfloor}.$$

2.6 Examples

In this section, we present some numerical results, for specific values of r and s.

1. For $s = r = 1$, we derive the bi-periodic 1-Fibonacci sequence $(U_n^{(1)})_n$ and its companion sequence the bi-periodic 1-Lucas sequence of type 1, $(V_n^{(1,1)})_n$

$$U_n^{(1)} = \begin{cases} aU^{(1)}_{n-1} + cU^{(1)}_{n-2}, & if \quad n \equiv 0 \pmod 2, \\ bU^{(1)}_{n-1} + dU^{(1)}_{n-2}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq 2$ with

$$U_0^{(1)} = 0, \qquad U_1^{(1)} = 1$$

and nonzero real numbers a, b, c and d.
Its linear recurrence relation is given by

$$U_n^{(1)} = (ab + c + d)U^{(1)}_{n-2} - cdU^{(1)}_{n-4}, \qquad n \geq 4,$$

with

$$U_0^{(1)} = 0, \quad U_1^{(1)} = 1, \quad U_2^{(1)} = a, \quad U_3^{(1)} = ab + d.$$

Its generating function is

$$G(x) = \frac{x + ax^2 - cx^3}{1 - (ab + c + d)x^2 + cdx^4}.$$

Its Binet form is

$$\begin{aligned} U_n^{(1)} &= \left(\frac{U_{2+\xi(n)}^{(1)} + (\alpha - ab - d - c) U_{\xi(n)}^{(1)}}{2\alpha - ab - d - c} \right) \alpha^{\lfloor n/2 \rfloor} \\ &+ \left(\frac{U_{2+\xi(n)}^{(1)} + (\beta - ab - d - c) U_{\xi(n)}^{(1)}}{2\beta - ab - d - c} \right) \beta^{\lfloor n/2 \rfloor}, \end{aligned}$$

with

$$\begin{cases} U_{\xi(n)}^{(1)} = 0, \; U_{2+\xi(n)}^{(1)} = a, & if \quad n \equiv 0 \pmod 2, \\ U_{\xi(n)}^{(1)} = 1, \; U_{2+\xi(n)}^{(1)} = ab + d, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

where α and β are the roots of the quadratic equation

$$x^2 - (ab + c + d)x + cd = 0.$$

The bi-periodic 1-Lucas sequence of type 1, $(V_n^{(1,1)})_n$

$$V_n^{(1,1)} = \begin{cases} bV_{n-1}^{(1,1)} + dV_{n-2}^{(1,1)}, & if \quad n \equiv 0 \pmod 2, \\ aV_{n-1}^{(1,1)} + cV_{n-2}^{(1,1)}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq 2$ with

$$V_0^{(1,1)} = 2, \quad V_1^{(1,1)} = a.$$

Its linear recurrence relation is given by

$$V_n^{(1,1)} = (ab + c + d)V_{n-2}^{(1,1)} - cdV_{n-4}^{(1,1)}, \qquad n \geq 4,$$

with

$$V_0^{(1,1)} = 2, \quad V_1^{(1,1)} = a, \quad V_2^{(1,1)} = ab + 2d, \quad V_3^{(1,1)} = a^2b + 2ad + ac.$$

The link between $U_n^{(1)}$ and $V_n^{(1,1)}$ is

$$V_n^{(1,1)} = U_{n+1}^{(1)} + dU_{n-1}^{(1)}, \qquad n \geq 1.$$

Its generating function is given by

$$H(x) = \frac{2 + ax - (ab + 2c)x^2 + adx^3}{1 - (ab + c + d)x^2 + cdx^4}.$$

Its Binet form is

$$\begin{aligned} V_n^{(1,1)} &= \left(\frac{V_{2+\xi(n)}^{(1,1)} + (\alpha - ab - d - c) V_{\xi(n)}^{(1,1)}}{2\alpha - ab - d - c} \right) \alpha^{\lfloor n/2 \rfloor} \\ &+ \left(\frac{V_{2+\xi(n)}^{(1,1)} + (\beta - ab - d - c) V_{\xi(n)}^{(1,1)}}{2\beta - ab - d - c} \right) \beta^{\lfloor n/2 \rfloor}, \end{aligned}$$

with

$$\begin{cases} V_{\xi(n)}^{(1,1)} = 2, \quad V_{2+\xi(n)}^{(1,1)} = ab + 2d, & if \quad n \equiv 0 \pmod{2}, \\ V_{\xi(n)}^{(1,1)} = a, \quad V_{2+\xi(n)}^{(1,1)} = a^2 b + 2ad + ac, & if \quad n \equiv 1 \pmod{2}. \end{cases}$$

We can also write

$$\begin{aligned} V_n^{(1,1)} &= \left(\frac{U_{2+\xi(n+1)}^{(1)} + (\alpha - ab - d - c) U_{\xi(n+1)}^{(1)}}{2\alpha - ab - d - c} \right) \left(\alpha^{\lfloor (n+1)/2 \rfloor} + d\alpha^{\lfloor (n-1)/2 \rfloor} \right) \\ &+ \left(\frac{U_{2+\xi(n+1)}^{(1)} + (\beta - ab - d - c) U_{\xi(n+1)}^{(1)}}{2\beta - ab - d - c} \right) \left(\beta^{\lfloor (n+1)/2 \rfloor} + d\beta^{\lfloor (n-1)/2 \rfloor} \right). \end{aligned}$$

An explicit formula of $(U_n^{(1)})_n$ is given by

$$U_{n+1}^{(1)} = \sum_{2i+2t+2k=n} \binom{i+t}{t} \binom{t}{k} (ab)^i (c+d)^{t-k} (-cd)^k$$

and an explicit formula of $(V_n^{(1,1)})_n$ is given by

$$\begin{aligned} V_n^{(1,1)} &= \sum_{2i+2t+2k=n} \binom{i+t}{t} \binom{t}{k} (ab)^i (c+d)^{t-k} (-cd)^k \\ &+ sd \sum_{2i+2t+2k=n-2} \binom{i+t}{t} \binom{t}{k} (ab)^i (c+d)^{t-k} (-cd)^k. \end{aligned}$$

2. For $r = 2$, we derive the bi-periodic 2-Fibonacci sequence $(U_n^{(2)})_n$ and its two companion sequences, the bi-periodic 2-Lucas sequence of type s, $(V_n^{(2,s)})_n$ with $s \in \{1, 2\}$

$$U_n^{(2)} = \begin{cases} aU_{n-1}^{(2)} + cU_{n-3}^{(2)}, & if \quad n \equiv 0 \pmod{2}, \\ bU_{n-1}^{(2)} + dU_{n-3}^{(2)}, & if \quad n \equiv 1 \pmod{2}, \end{cases}$$

for $n \geq 3$ with

$$U_0^{(2)} = 0, \quad U_1^{(2)} = 1, \quad U_2^{(2)} = a$$

and nonzero real numbers a, b, c and d.
Its linear recurrence relation is

$$U_n^{(2)} = abU_{n-2}^{(2)} + (ad+bc)U_{n-4}^{(2)} - cdU_{n-6}^{(2)}, \qquad n \geq 6,$$

with

$$\begin{aligned} &U_0^{(2)} = 0, \quad U_1^{(2)} = 1, \quad U_2^{(2)} = a, \quad U_3^{(2)} = ab, \\ &U_4^{(2)} = a^2b + c, \quad U_5^{(2)} = a^2b^2 + bc + ad. \end{aligned}$$

Its generating function is

$$G(x) = \frac{x + ax^2 + cx^4}{1 - abx^2 - (ad+bc)x^4 - cdx^6}.$$

Its Binet form is

$$\begin{aligned} U_n^{(2)} &= \left(\frac{U_{4+\xi(n)}^{(2)} - (ab-\alpha)U_{2+\xi(n)}^{(2)} + (\alpha^2 - \alpha ab - ad - bc)U_{\xi(n)}^{(2)}}{3\alpha^2 - 2ab\alpha - ad - bc} \right) \alpha^{\lfloor n/2 \rfloor} \\ &+ \left(\frac{U_{4+\xi(n)}^{(2)} - (ab-\beta)U_{2+\xi(n)}^{(2)} + (\beta^2 - \beta ab - ad - bc)U_{\xi(n)}^{(2)}}{3\beta^2 - 2ab\beta - ad - bc} \right) \beta^{\lfloor n/2 \rfloor} \\ &+ \left(\frac{U_{4+\xi(n)}^{(2)} - (ab-\gamma)U_{2+\xi(n)}^{(2)} + (\gamma^2 - \gamma ab - ad - bc)U_{\xi(n)}^{(2)}}{3\gamma^2 - 2ab\gamma - ad - bc} \right) \gamma^{\lfloor n/2 \rfloor}, \end{aligned}$$

with

$$\begin{cases} U_{\xi(n)}^{(2)} = 0, \; U_{2+\xi(n)}^{(2)} = a, \; U_{4+\xi(n)}^{(2)} = a^2b + c, & if \quad n \equiv 0 \pmod 2, \\ U_{\xi(n)}^{(2)} = 1, \; U_{2+\xi(n)}^{(2)} = ab, \; U_{4+\xi(n)}^{(2)} = a^2b^2 + bc + ad, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

where α, β and γ are the roots of the equation

$$x^3 - abx^2 - (ad+bc)x - cd = 0.$$

If the roots are nonzero, we can write

$$\begin{aligned} U_n^{(2)} &= \left(\frac{U_{4+\xi(n)}^{(2)} - (ab-\alpha)U_{2+\xi(n)}^{(2)} + \frac{cd}{\alpha}U_{\xi(n)}^{(2)}}{3\alpha^2 - 2ab\alpha - ad - bc} \right) \alpha^{\lfloor n/2 \rfloor} \\ &+ \left(\frac{U_{4+\xi(n)}^{(2)} - (ab-\beta)U_{2+\xi(n)}^{(2)} + \frac{cd}{\beta}U_{\xi(n)}^{(2)}}{3\beta^2 - 2ab\beta - ad - bc} \right) \beta^{\lfloor n/2 \rfloor} \\ &+ \left(\frac{U_{4+\xi(n)}^{(2)} - (ab-\gamma)U_{2+\xi(n)}^{(2)} + \frac{cd}{\gamma}U_{\xi(n)}^{(2)}}{3\gamma^2 - 2ab\gamma - ad - bc} \right) \gamma^{\lfloor n/2 \rfloor}. \end{aligned}$$

The bi-periodic 2-Lucas sequence of type s, $(V_n^{(2,s)})_n$ is defined by

$$V_n^{(2,s)} = \begin{cases} bV_{n-1}^{(2,s)} + dV_{n-3}^{(2,s)}, & if \quad n \equiv 0 \pmod 2, \\ aV_{n-1}^{(2,s)} + cV_{n-3}^{(2,s)}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq 3$ with

$$V_0^{(2,s)} = s+1, \quad V_1^{(2,s)} = a, \quad V_2^{(2,s)} = ab.$$

Its linear recurrence relation is given by

$$V_n^{(2,s)} = abV_{n-2}^{(2,s)} + (ad+bc)V_{n-4}^{(2,s)} + cdV_{n-6}^{(2,s)}, \qquad n \geq 6,$$

with

$$\begin{aligned} &V_0^{(2,s)} = s+1, \quad V_1^{(2,s)} = a, \quad V_2^{(2,s)} = ab, \quad V_3^{(2,s)} = a^2b + (s+1)c, \\ &V_4^{(2,s)} = a^2b^2 + (s+1)bc + ad, \quad V_5^{(2,s)} = a^3b^2 + (s+2)abc + a^2d. \end{aligned}$$

The link between $U_n^{(2)}$ and $V_n^{(2,s)}$ is

$$V_n^{(2,s)} = U_{n+1}^{(2)} + scbU_{n-3}^{(2)} + scdU_{n-5}^{(2)}, \qquad n \geq 5.$$

Its generating function is

$$H(x) = \frac{(s+1) + ax - absx^2 + (s+1)cx^3 - adsx^4}{1 - abx^2 - (ad+bc)x^4 - cdx^6}.$$

Its Binet form is

$$
\begin{aligned}
V_n^{(2,s)} &= \left(\frac{V_{4+\xi(n)}^{(2,s)} - (ab-\alpha)V_{2+\xi(n)}^{(2,s)} + (\alpha^2 - \alpha ab - ad - bc)V_{\xi(n)}^{(2,s)}}{3\alpha^2 - 2ab\alpha - ad - bc}\right)\alpha^{\lfloor n/2\rfloor} \\
&+ \left(\frac{V_{4+\xi(n)}^{(2,s)} - (ab-\beta)V_{2+\xi(n)}^{(2,s)} + (\beta^2 - \beta ab - ad - bc)V_{\xi(n)}^{(2,s)}}{3\beta^2 - 2ab\beta - ad - bc}\right)\beta^{\lfloor n/2\rfloor} \\
&+ \left(\frac{V_{4+\xi(n)}^{(2,s)} - (ab-\gamma)V_{2+\xi(n)}^{(2,s)} + (\gamma^2 - \gamma ab - ad - bc)V_{\xi(n)}^{(2,s)}}{3\gamma^2 - 2ab\gamma - ad - bc}\right)\gamma^{\lfloor n/2\rfloor},
\end{aligned}
$$

with

$$
\begin{cases}
V_{\xi(n)}^{(2,s)} = s+1, V_{2+\xi(n)}^{(2,s)} = ab, & V_{4+\xi(n)}^{(2,s)} = a^2b^2 + (s+1)bc + ad, \quad if \quad n \equiv 0 \,(\mathrm{mod}\ 2), \\
V_{\xi(n)}^{(2,s)} = a, V_{2+\xi(n)}^{(2,s)} = a^2b + (s+1)c, & V_{4+\xi(n)}^{(2,s)} = a^3b^2 + (s+2)abc + a^2d, \quad if \quad n \equiv 1 \,(\mathrm{mod}\ 2).
\end{cases}
$$

If the roots are nonzero, we can also write

$$
\begin{aligned}
V_n^{(2,s)} = &\left(\frac{U_{4+\xi(n+1)}^{(2)} - (ab-\alpha)U_{2+\xi(n+1)}^{(2)} + (\alpha^2 - \alpha ab - ad - bc)U_{\xi(n+1)}^{(2)}}{3\alpha^2 - 2ab\alpha - ad - bc}\right) \\
&\times \left(\alpha^{\lfloor (n+1)/2\rfloor} + scb\alpha^{\lfloor (n-3)/2\rfloor} + scd\alpha^{\lfloor (n-5)/2\rfloor}\right) \\
+ &\left(\frac{U_{4+\xi(n+1)}^{(2)} - (ab-\beta)U_{2+\xi(n+1)}^{(2)} + (\beta^2 - \beta ab - ad - bc)U_{\xi(n+1)}^{(2)}}{3\beta^2 - 2ab\beta - ad - bc}\right) \\
&\times \left(\beta^{\lfloor (n+1)/2\rfloor} + scb\beta^{\lfloor (n-3)/2\rfloor} + scd\beta^{\lfloor (n-5)/2\rfloor}\right) \\
+ &\left(\frac{U_{4+\xi(n+1)}^{(2)} - (ab-\gamma)U_{2+\xi(n+1)}^{(2)} + (\gamma^2 - \gamma ab - ad - bc)U_{\xi(n+1)}^{(2)}}{3\gamma^2 - 2ab\gamma - ad - bc}\right) \\
&\times \left(\gamma^{\lfloor (n+1)/2\rfloor} + scb\gamma^{\lfloor (n-3)/2\rfloor} + scd\gamma^{\lfloor (n-5)/2\rfloor}\right).
\end{aligned}
$$

An explicit formula of $(U_n^{(2)})_n$ is given by

$$
U_n^{(2)} = \sum_{2i+4t+2k=n} \binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k
$$

and an explicit formula of $(V_n^{(2,s)})_n$ is given by

$$\begin{aligned} V_n^{(2,s)} &= \sum_{2i+4t+2k=n} \binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k \\ &+ sbc \sum_{2i+4t+2k=n-4} \binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k \\ &+ scd \sum_{2i+4t+2k=n-6} \binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k. \end{aligned}$$

CHAPTER

3

BI-PERIODIC HORADAM HYBRID NUMBERS [79]

The hybrid numbers were introduced by Ozdemir in [57] as a new generalization of complex, dual and hyperbolic numbers. A hybrid number is defined by $k = a + bi + c\varepsilon + dh$, where a, b, c, d are real numbers and i, ε, h are numbers such that $i^2 = -1, \varepsilon^2 = 0, h^2 = 1$ and $ih = -hi = \varepsilon + i$. This work was intended as an attempt to introduce the bi-periodic Horadam hybrid numbers which generalized the classical Horadam hybrid numbers. We give the generating function, the Binet form and some basic properties of these new hybrid numbers. Also, we investigate some relationships between generalized bi-periodic Fibonacci hybrid numbers and generalized bi-periodic Lucas hybrid numbers.

3.1 Introduction

Recently, many studies have been devoted to hybrid numbers whose components are taken from special integer sequences such as Fibonacci, Lucas, Pell, Jacobsthal sequences, etc. In particular, Szynal-Liana [72] introduced the Horadam hybrid numbers as

$$\mathbb{K}_{W,n} = W_n + W_{n+1}i + W_{n+2}\varepsilon + W_{n+3}h, \qquad n \geq 0, \tag{3.1}$$

where $(W_n)_n$ is the Horadam sequence defined by $W_n = pW_{n-1} + qW_{n-2}$ with arbitrary initial values W_0, W_1. In [72, 73, 74, 75], the authors studied some basic properties of special type of hybrid numbers. The basic properties of q-Pell hybrid numbers were investigated by Catarino [15]. Also, Morales [18] considered the (p, q)-Fibonacci and (p, q)-Lucas hybrid numbers and gave many relations between them. Recently motivated by the Szynal-Liana's paper, Senturk et al. [65] derived summation formulas, matrix representations, general bilinear formula, Honsberger formula, etc. regarding to the Horadam hybrid numbers.

This work has been intended as an attempt to introduce a new generalization of Horadam hybrid numbers, called as, bi-periodic Horadam hybrid numbers. The bi-periodic Horadam hybrid numbers generalize the well-known hybrid numbers in the literature, such as Horadam hybrid numbers, Fibonacci and Lucas hybrid numbers, q-Pell hybrid numbers, Pell and Pell-Lucas hybrid numbers, Jacobsthal and Jacobsthal-Lucas hybrid numbers, etc. The components of the bi-periodic Horadam hybrid numbers belong to the bi-periodic Horadam sequence $(w_n)_n$ which is defined by the recurrence relation

$$w_n = \chi(n) w_{n-1} + c w_{n-2}, \qquad n \geq 2 \tag{3.2}$$

where $\chi(n) = a$ if n is even, $\chi(n) = b$ if n is odd with arbitrary initial conditions w_0, w_1 and nonzero real numbers a, b and c. It is clear that if we take $a = b = p$ and $c = q$, then it reduces to the classical Horadam sequence. For the details of the bi-periodic Horadam sequences see [11, 27, 58, 63, 92].

We should note that for the case of $c = 1$, the generalized bi-periodic Fibonacci quaternions and the generalized bi-periodic Fibonacci dual quaternions were investigated in [83, 87, 88, 89]. For a survey on these researches we refer to [32, 35].

The Binet Form for the bi-periodic Horadam sequence $(w_n)_n$ is

$$w_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(A\alpha^n - B\beta^n\right), \tag{3.3}$$

where

$$A := \frac{w_1 - \frac{\beta}{a} w_0}{\alpha - \beta} \quad \text{and} \quad B := \frac{w_1 - \frac{\alpha}{a} w_0}{\alpha - \beta}. \tag{3.4}$$

Here $\alpha = \dfrac{ab + \sqrt{a^2b^2 + 4abc}}{2}$ and $\beta = \dfrac{ab - \sqrt{a^2b^2 + 4abc}}{2}$ are the roots of the polynomial

$$x^2 - abx - abc,$$

that is

$$\alpha\beta = -abc, \qquad \alpha + \beta = ab$$

and

$$\Delta := \alpha - \beta = \sqrt{a^2b^2 + 4abc},$$

with

$$a^2b^2 + 4abc > 0.$$

If we take the initial conditions $w_0 = 0$ and $w_1 = 1$, we get the Binet form of the generalized bi-periodic Fibonacci sequence $(u_n)_n$ as

$$u_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \tag{3.5}$$

and by taking the initial conditions $w_0 = 2$ and $w_1 = b$, we get the Binet form of the generalized bi-periodic Lucas sequence $(v_n)_n$ as

$$v_n = \frac{a^{-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\alpha^n + \beta^n \right). \tag{3.6}$$

The bi-periodic Horadam numbers for negative subscripts is defined as

$$(-c)^n w_{-n} = \left(\frac{b}{a} \right)^{\xi(n)} w_0 u_{n+1} - w_1 u_n. \tag{3.7}$$

Also we have

$$\alpha^m = a^{-1} a^{\frac{m+\xi(m)}{2}} b^{\frac{m-\xi(m)}{2}} \alpha u_m + c a^{\frac{m-\xi(m)}{2}} b^{\frac{m+\xi(m)}{2}} u_{m-1} \tag{3.8}$$

and

$$\beta^m = a^{-1} a^{\frac{m+\xi(m)}{2}} b^{\frac{m-\xi(m)}{2}} \beta u_m + c a^{\frac{m-\xi(m)}{2}} b^{\frac{m+\xi(m)}{2}} u_{m-1}. \tag{3.9}$$

For details see [84, 85].

3.2 Bi-periodic Horadam hybrid numbers

Definition 28. *For $n \geq 0$, the bi-periodic Horadam hybrid number $\mathbb{K}_{w,n}$ is defined by the recurrence relation*

$$\mathbb{K}_{w,n} = w_n + w_{n+1} i + w_{n+2} \varepsilon + w_{n+3} h, \qquad n \geq 0,$$

where w_n is the n^{th} bi-periodic Horadam number.

From the definition of bi-periodic Horadam hybrid numbers, we have

$$\begin{aligned}\mathbb{K}_{w,0} &= w_0 + w_1 i + (aw_1 + cw_0)\,\varepsilon + ((ab+c)\,w_1 + bcw_0)h,\\ \mathbb{K}_{w,1} &= w_1 + (aw_1 + cw_0)\,w_1 i + ((ab+c)\,w_1 + bcw_0)\varepsilon\\ &+ (a\,(ab+2c)\,w_1 + c\,(ab+c)\,w_0)h.\end{aligned}$$

In the following table we state several hybrid numbers in terms of the bi-periodic Horadam hybrid numbers $\mathbb{K}_{w,n}$ according to the initial conditions w_0, w_1 and the related coefficients a, b, c.

$\mathbb{K}_{w,n}$	$(w_0,w_1;a,b,c)$	bi-periodic Horadam hybrid numbers
$\mathbb{K}_{u,n}$	$(0,1;a,b,c)$	gen. bi-periodic Fibonacci hybrid numbers
$\mathbb{K}_{v,n}$	$(2,b;a,b,c)$	gen. bi-periodic Lucas hybrid numbers
$\mathbb{K}_{\widehat{u},n}$	$(0,1;b,a,c)$	modified gen. bi-periodic Fibonacci hybrid numbers
$\mathbb{K}_{\widehat{v},n}$	$(2,a;b,a,c)$	modified gen. bi-periodic Lucas hybrid numbers
$\mathbb{K}_{W,n}$	$(W_0,W_1;p,p,q)$	Horadam hybrid numbers [72, 65]
$\mathbb{K}_{U,n}$	$(0,1;p,p,q)$	(p,q)-Fibonacci hybrid numbers [18]
$\mathbb{K}_{V,n}$	$(2,p;p,p,q)$	(p,q)-Lucas hybrid numbers [18]
$\mathbb{K}_{F,n}$	$(0,1;1,1,1)$	Fibonacci hybrid numbers [74]
$\mathbb{K}_{L,n}$	$(2,1;1,1,1)$	Lucas hybrid numbers [72]
$\mathbb{K}_{P,n}$	$(0,1;2,2,1)$	Pell hybrid numbers [73]
$\mathbb{K}_{Q,n}$	$(2,2;2,2,1)$	Pell-Lucas hybrid numbers [73]
$\mathbb{K}_{kP,n}$	$(0,1;2,2,k)$	k-Pell hybrid numbers [15]
$\mathbb{K}_{J,n}$	$(0,1;1,1,2)$	Jacobsthal hybrid numbers [75]
$\mathbb{K}_{j,n}$	$(2,1;1,1,2)$	Jacobsthal-Lucas hybrid numbers [75]

Table 3.1: Special cases of the sequence $(\mathbb{K}_{w,n})_n$.

The norm of the n^{th} bi-periodic Horadam hybrid number $\mathbb{K}_{w,n}$ is $\|\mathbb{K}_{w,n}\| := \sqrt{|C\,(\mathbb{K}_{w,n})|}$. Here the character of the n^{th} bi-periodic Horadam hybrid number $\mathbb{K}_{w,n}$ is

$$C\,(\mathbb{K}_{w,n}) = \mathbb{K}_{w,n}\overline{\mathbb{K}_{w,n}} = w_n^2 + (w_{n+1} - w_{n+2})^2 - w_{n+2}^2 - w_{n+3}^2 \tag{3.10}$$

where $\overline{\mathbb{K}_{w,n}} := w_n - w_{n+1}i - w_{n+2}\varepsilon - w_{n+3}h$ is the conjugate of the bi-periodic Horadam hybrid number.

3.3 Generating function

We give the generating function for the bi-periodic Horadam hybrid numbers in the following theorem.

Theorem 29. *The generating function for the bi-periodic Horadam hybrid sequence,* $\mathbb{G}(x)$ *is given by*

$$\left(1-(ab+2c)\,x^2+c^2x^4\right)\mathbb{G}(x)=\left(1-(ab+c)\,x^2+bcx^3\right)\mathbb{K}_{w,0}+x\left(1+ax-cx^2\right)\mathbb{K}_{w,1}.$$

Proof. Let

$$\mathbb{G}(x)=\sum_{n=0}^{\infty}\mathbb{K}_{w,n}x^n=\mathbb{K}_{w,0}+\mathbb{K}_{w,1}x+\mathbb{K}_{w,2}x^2+\cdots+\mathbb{K}_{w,n}x^n+\cdots.$$

Since $K_{w,n}=(ab+2c)\,\mathbb{K}_{w,n-2}-c^2\mathbb{K}_{w,n-4}$, for $n\geq 4$, we get

$$\begin{aligned}
&(1-(ab+2c)\ x^2+c^2x^4\Big)\,\mathbb{G}(x)\\
&=\sum_{n=0}^{\infty}\mathbb{K}_{w,n}x^n-(ab+2c)\sum_{n=0}^{\infty}\mathbb{K}_{w,n}x^{n+2}+c^2\sum_{n=0}^{\infty}\mathbb{K}_{w,n}x^{n+4}\\
&=\sum_{n=0}^{\infty}\mathbb{K}_{w,n}x^n-(ab+2c)\sum_{n=2}^{\infty}\mathbb{K}_{w,n-2}x^{n}+c^2\sum_{n=4}^{\infty}\mathbb{K}_{w,n-4}x^{n}\\
&=\mathbb{K}_{w,0}+\mathbb{K}_{w,1}x+(\mathbb{K}_{w,2}-(ab+2c)\,\mathbb{K}_{w,0})\,x^2\\
&+(\mathbb{K}_{w,3}-(ab+2c)\,\mathbb{K}_{w,1})\,x^3\\
&+\sum_{n=4}^{\infty}\left(\mathbb{K}_{w,n}-(ab+2c)\,\mathbb{K}_{w,n-2}+c^2\mathbb{K}_{w,n-4}\right)x^n\\
&=\mathbb{K}_{w,0}+\mathbb{K}_{w,1}x+((a\mathbb{K}_{w,1}+c\mathbb{K}_{w,0})-(ab+2c)\,\mathbb{K}_{w,0})\,x^2\\
&+(((ab+c)\,\mathbb{K}_{w,1}+bc\mathbb{K}_{w,0})-(ab+2c)\,\mathbb{K}_{w,1})\,x^3\\
&=\left(1-(ab+c)\,x^2+bcx^3\right)\mathbb{K}_{w,0}+x\left(1+ax-cx^2\right)\mathbb{K}_{w,1}.
\end{aligned}$$

□

3.4 Binet form

Next, we state the Binet form for the bi-periodic Horadam hybrid numbers and so derive some well-known mathematical properties.

Theorem 30. *The Binet form for the bi-periodic Horadam hybrid numbers is*

$$\mathbb{K}_{w,n} = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(A\alpha_{\xi(n)}\alpha^n - B\beta_{\xi(n)}\beta^n \right),$$

where $\alpha_{\xi(n)}, \beta_{\xi(n)}$ *are defined as*

$$\begin{aligned} \alpha_{\xi(n)} &:= 1 + \frac{1}{a}\left(\frac{a}{b}\right)^{\xi(n)} \alpha i + \frac{1}{ab}\alpha^2 \varepsilon + \frac{1}{a^2 b}\left(\frac{a}{b}\right)^{\xi(n)} \alpha^3 h, \\ \beta_{\xi(n)} &:= 1 + \frac{1}{a}\left(\frac{a}{b}\right)^{\xi(n)} \beta i + \frac{1}{ab}\beta^2 \varepsilon + \frac{1}{a^2 b}\left(\frac{a}{b}\right)^{\xi(n)} \beta^3 h. \end{aligned}$$

Proof. By using the definition of the sequence $(\mathbb{K}_{w,n})_n$ and the Binet form of $(w_n)_n$, we have

$$\begin{aligned} \mathbb{K}_{w,n} &= w_n + w_{n+1}i + w_{n+2}\varepsilon + w_{n+3}h \\ &= \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} (A\alpha^n - B\beta^n) + \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(A\alpha^{n+1} - B\beta^{n+1} \right) i \\ &+ \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n+2}{2} \rfloor}} \left(A\alpha^{n+2} - B\beta^{n+2} \right) \varepsilon + \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+3}{2} \rfloor}} \left(A\alpha^{n+3} - B\beta^{n+3} \right) h \\ &= A\alpha^n \left(\frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} + \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + \xi(n)}} \alpha i + \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1}} \alpha^2 \varepsilon + \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1 + \xi(n)}} \alpha^3 h \right) \\ &- B\beta^n \left(\frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} + \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + \xi(n)}} \beta i + \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1}} \beta^2 \varepsilon + \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1 + \xi(n)}} \beta^3 h \right) \\ &= \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left[A\alpha^n \left(1 + \frac{1}{a^{\xi(n+1)} b^{\xi(n)}} \alpha i + \frac{1}{ab} \alpha^2 \varepsilon + \frac{1}{a^{\xi(n+1)} b^{\xi(n)} ab} \alpha^3 h \right) \right. \\ &\left. -B\beta^n \left(1 + \frac{1}{a^{\xi(n+1)} b^{\xi(n)}} \beta i + \frac{1}{ab} \beta^2 \varepsilon + \frac{1}{a^{\xi(n+1)} b^{\xi(n)} ab} \beta^3 h \right) \right], \end{aligned}$$

which gives the desired result. □

Remark 31. If we take $a = b = p$ and $c = q$, we obtain the Binet form of the classical Horadam hybrid numbers in [72].

Lemma 32. *We have*

$$\alpha_{\xi(n)}\beta_{\xi(n)} = \begin{cases} \mathbb{K}_{v,0} - \theta + \frac{\Delta}{a} c \left(\mathbb{K}_{u,0} - \eta \right), & \text{if } n \text{ is even}, \\ \mathbb{K}_{\widehat{v},0} - \widehat{\theta} + \frac{\Delta}{b} c \left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta} \right), & \text{if } n \text{ is odd} \end{cases} \tag{3.11}$$

and

$$\beta_{\xi(n)}\alpha_{\xi(n)} = \begin{cases} \mathbb{K}_{v,0} - \theta - \frac{\Delta}{a}c\left(\mathbb{K}_{u,0} - \eta\right), & \text{if } n \text{ is even,} \\ \mathbb{K}_{\widehat{v},0} - \widehat{\theta} - \frac{\Delta}{b}c\left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta}\right), & \text{if } n \text{ is odd,} \end{cases} \tag{3.12}$$

where

$$\begin{aligned} \eta &:= (1-b)\,i + (a-b-c)\,\varepsilon + (1+ab+c)\,h, \\ \widehat{\eta} &:= (1-a)\,i + (b-a-c)\,\varepsilon + (1+ab+c)\,h, \\ \theta &: = 1 - \frac{bc}{a} + bc + \frac{bc^3}{a}, \\ \widehat{\theta} &: = 1 - \frac{ac}{b} + ac + \frac{ac^3}{b}. \end{aligned}$$

And the sequences $(\mathbb{K}_{\widehat{u},0})_n$ *and* $(\mathbb{K}_{\widehat{v},0})_n$ *are the auxiliary sequences that are obtained from* $(\mathbb{K}_{u,0})_n$ *and* $(\mathbb{K}_{v,0})_n$ *just only switching* $a \leftrightarrow b$.
That is, $\widehat{u_n} = \left(\frac{b}{a}\right)^{\xi(n+1)} u_n$ *and* $\widehat{v_n} = \left(\frac{a}{b}\right)^{\xi(n)} v_n$.

Proof. By using the definition of multiplication of two hybrid numbers, we have

$$\begin{aligned} \alpha_{\xi(n)}\beta_{\xi(n)} &= 1 + \frac{bc}{a}\left(\frac{a}{b}\right)^{2\xi(n)} - \frac{bc^3}{a}\left(\frac{a}{b}\right)^{2\xi(n)} - \left(\frac{a}{b}\right)^{\xi(n)} bc \\ &+ \left(b\left(\frac{a}{b}\right)^{\xi(n)} + \left(\frac{a}{b}\right)^{2\xi(n)} \frac{bc}{a}\Delta\right) i \\ &+ \left((ab+2c) + \frac{bc}{a}\left(\frac{a}{b}\right)^{2\xi(n)}\Delta + \frac{c^2}{a}\left(\frac{a}{b}\right)^{\xi(n)}\Delta\right)\varepsilon \\ &+ \left(\left(\frac{a}{b}\right)^{\xi(n)}\left(ab^2+3bc\right) - \frac{c}{a}\left(\frac{a}{b}\right)^{\xi(n)}\Delta\right) h \\ &= 1 + \left(\frac{a}{b}\right)^{\xi(n)} bc\left(\frac{1}{a}\left(\frac{a}{b}\right)^{\xi(n)} - \frac{c^2}{a}\left(\frac{a}{b}\right)^{\xi(n)} - 1\right) \\ &+ \left(\frac{a}{b}\right)^{\xi(n)}\left(\left(\frac{b}{a}\right)^{\xi(n)} 2 + bi + \left(\frac{b}{a}\right)^{\xi(n)}(ab+2c)\,\varepsilon + (b\,(ab+3c))\,h\right) - 2 \\ &+ \Delta\left(\frac{a}{b}\right)^{\xi(n)}\frac{c}{a}\left(\left(\frac{a}{b}\right)^{\xi(n)} bi + \left(b\left(\frac{a}{b}\right)^{\xi(n)} + c\right)\varepsilon - h\right). \end{aligned}$$

After some necessary simplifications, we get the result (3.11). Similarly we can obtain $\beta_{\xi(n)}\alpha_{\xi(n)}$. □

By using the Lemma 32, we have

$$\alpha_{\xi(n)}\beta_{\xi(n)} + \beta_{\xi(n)}\alpha_{\xi(n)} = \begin{cases} 2\left(\mathbb{K}_{v,0} - \theta\right), & \text{if } n \text{ is even,} \\ 2\left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta}\right), & \text{if } n \text{ is odd} \end{cases} \tag{3.13}$$

and

$$\alpha_{\xi(n)}\beta_{\xi(n)} - \beta_{\xi(n)}\alpha_{\xi(n)} = \begin{cases} 2\Delta\frac{c}{a}\left(\mathbb{K}_{u,0} - \eta\right), & \text{if } n \text{ is even,} \\ 2\Delta\frac{c}{b}\left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta}\right), & \text{if } n \text{ is odd.} \end{cases} \tag{3.14}$$

Lemma 33. *We have*

$$\alpha_{\xi(n)}\alpha_{\xi(n)} = \begin{cases} \mathbb{K}_{v,0} + \mu_e + \frac{\Delta}{a}\left(\mathbb{K}_{u,0} + \gamma_e\right), & \text{if } n \text{ is even,} \\ \mathbb{K}_{\widehat{v},0} + \mu_o + \frac{\Delta}{b}\left(\mathbb{K}_{\widehat{u},0} + \gamma_o\right), & \text{if } n \text{ is odd} \end{cases} \tag{3.15}$$

and

$$\beta_{\xi(n)}\beta_{\xi(n)} = \begin{cases} \mathbb{K}_{v,0} + \mu_e - \frac{\Delta}{a}\left(\mathbb{K}_{u,0} + \gamma_e\right), & \text{if } n \text{ is even,} \\ \mathbb{K}_{\widehat{v},0} + \mu_o - \frac{\Delta}{b}\left(\mathbb{K}_{\widehat{u},0} + \gamma_o\right), & \text{if } n \text{ is odd,} \end{cases} \tag{3.16}$$

where

$$\begin{aligned} \mu_e &: = -1 + \frac{b}{a}c\left(u_5 + 2u_2 - u_1\right) + b\gamma_e, \\ \mu_o &: = -1 + \frac{a}{b}c\left(u_5 + 2\frac{b}{a}u_2 - u_1\right) + a\gamma_o \end{aligned}$$

and

$$\begin{aligned} \gamma_e &: = \frac{1}{2}\left(\frac{b}{a}u_6 + 2u_3 - \frac{b}{a}u_2\right), \\ \gamma_o &: = \frac{1}{2}\left(u_6 + 2u_3 - u_2\right). \end{aligned}$$

Proof. By considering the relations

$$\alpha_{\xi(n)}\alpha_{\xi(n)} = 2\alpha_{\xi(n)} - C\left(\alpha_{\xi(n)}\right)$$

and

$$\beta_{\xi(n)}\beta_{\xi(n)} = 2\beta_{\xi(n)} - C\left(\beta_{\xi(n)}\right),$$

where $C\left(\alpha_{\xi(n)}\right)$ is the character of the hybrid number $\alpha_{\xi(n)}$ and using the relations (3.8) and (3.9), we get the desired result. □

Remark 34. If we take $a = b = p$ and $c = q$, we obtain the analogous relations for (p,q)-Fibonacci hybrid numbers given in [18].

Theorem 35. *(Vajda's like identity) For nonnegative integers n, r and s, we have*

$$\begin{aligned}
&\mathbb{K}_{w,n+2r}\mathbb{K}_{w,n+2s} - \mathbb{K}_{w,n}\mathbb{K}_{w,n+2(r+s)} \\
= &\begin{cases} (-c)^n AB\Delta^2 u_{2r}\left(\left(\mathbb{K}_{v,0}-\theta\right)u_{2s} - c\left(\mathbb{K}_{u,0}-\eta\right)v_{2s}\right), & \text{if } n \text{ is even,} \\ (-c)^n AB\Delta^2 u_{2r}\left(\left(\mathbb{K}_{\widehat{v},0}-\widehat{\theta}\right)\frac{b}{a}u_{2s} - c\left(\mathbb{K}_{\widehat{u},0}-\widehat{\eta}\right)v_{2s}\right), & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Proof. From the Binet form of the bi-periodic Horadam hybrid numbers, we get

$$\begin{aligned}
&\mathbb{K}_{w,n+2r}\mathbb{K}_{w,n+2s} - \mathbb{K}_{w,n}\mathbb{K}_{w,n+2(r+s)} \\
= &\frac{a^{\xi(n+2r+1)}}{(ab)^{\left\lfloor \frac{n+2r}{2}\right\rfloor}}\left(A\alpha_{\xi(n)}\alpha^{n+2r} - B\beta_{\xi(n)}\beta^{n+2r}\right)\frac{a^{\xi(n+2s+1)}}{(ab)^{\left\lfloor \frac{n+2s}{2}\right\rfloor}} \\
&\times\left(A\alpha_{\xi(n)}\alpha^{n+2s} - B\beta_{\xi(n)}\beta^{n+2s}\right) \\
- &\frac{a^{\xi(n+1)}}{(ab)^{\left\lfloor \frac{n}{2}\right\rfloor}}\left(A\alpha_{\xi(n)}\alpha^{n} - B\beta_{\xi(n)}\beta^{n}\right)\frac{a^{\xi(n+2(r+s)+1)}}{(ab)^{\left\lfloor \frac{n+2(r+s)}{2}\right\rfloor}} \\
&\times\left(A\alpha_{\xi(n)}\alpha^{n+2(r+s)} - B\beta_{\xi(n)}\beta^{n+2(r+s)}\right) \\
= &\frac{a^{2\xi(n+1)}}{(ab)^{2\left\lfloor \frac{n}{2}\right\rfloor+r+s}}\left[-AB\alpha_{\xi(n)}\beta_{\xi(n)}\alpha^{n+2r}\beta^{n+2s} - AB\beta_{\xi(n)}\alpha_{\xi(n)}\alpha^{n+2s}\beta^{n+2r}\right. \\
&\left.+AB\alpha_{\xi(n)}\beta_{\xi(n)}\alpha^{n}\beta^{n+2(r+s)} + AB\beta_{\xi(n)}\alpha_{\xi(n)}\alpha^{n+2(r+s)}\beta^{n}\right] \\
= &\frac{a^{2\xi(n+1)}}{(ab)^{2\left\lfloor \frac{n}{2}\right\rfloor+r+s}}AB(\alpha\beta)^n\left[\alpha_{\xi(n)}\beta_{\xi(n)}\beta^{2s}\left(\beta^{2r}-\alpha^{2r}\right) + \beta_{\xi(n)}\alpha_{\xi(n)}\alpha^{2s}\left(\alpha^{2r}-\beta^{2r}\right)\right] \\
= &\frac{a^{2\xi(n+1)}}{(ab)^{2\left\lfloor \frac{n}{2}\right\rfloor+r+s}}AB(\alpha\beta)^n\left(\alpha^{2r}-\beta^{2r}\right)\left[\beta_{\xi(n)}\alpha_{\xi(n)}\alpha^{2s} - \alpha_{\xi(n)}\beta_{\xi(n)}\beta^{2s}\right].
\end{aligned}$$

If n is even, by considering the relations (3.11) and (3.12), we obtain

$$\begin{aligned}
&\mathbb{K}_{w,n+2r}\mathbb{K}_{w,n+2s} - \mathbb{K}_{w,n}\mathbb{K}_{w,n+2(r+s)} \\
= &\frac{a^{2\xi(n+1)}}{(ab)^{2\left\lfloor \frac{n}{2}\right\rfloor+r+s}}AB(\alpha\beta)^n\left(\alpha^{2r}-\beta^{2r}\right)\left(\beta_{\xi(n)}\alpha_{\xi(n)}\alpha^{2s} - \alpha_{\xi(n)}\beta_{\xi(n)}\beta^{2s}\right) \\
= &\frac{a^2(-c)^n}{(ab)^{r+s}}AB\left(\alpha^{2r}-\beta^{2r}\right)\left[\left(\mathbb{K}_{v,0}-\theta\right)\left(\alpha^{2s}-\beta^{2s}\right)\right. \\
&\left.-\frac{\Delta}{a}c\left(\mathbb{K}_{u,0}-\eta\right)\left(\alpha^{2s}+\beta^{2s}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{a^2(-c)^n}{(ab)^{r+s}} AB \frac{(ab)^r}{a} \Delta u_{2r} \left[(\mathbb{K}_{v,0} - \theta) \left(\frac{(ab)^s}{a} u_{2s} \Delta \right) \right. \\
&\quad \left. - \frac{\Delta}{a} c (\mathbb{K}_{u,0} - \eta) \left((ab)^s v_{2s} \right) \right] \\
&= (-c)^n ABu_{2r} \Delta^2 \left[(\mathbb{K}_{v,0} - \theta) u_{2s} - c (\mathbb{K}_{u,0} - \eta) v_{2s} \right].
\end{aligned}
$$

If n is odd

$$
\begin{aligned}
&\mathbb{K}_{w,n+2r}\mathbb{K}_{w,n+2s} - \mathbb{K}_{w,n}\mathbb{K}_{w,n+2(r+s)} \\
&= \frac{a^{2\xi(n+1)}}{(ab)^{2\lfloor \frac{n}{2} \rfloor + r + s}} AB (\alpha\beta)^n \left(\alpha^{2r} - \beta^{2r} \right) \left(\beta_{\xi(n)} \alpha_{\xi(n)} \alpha^{2s} - \alpha_{\xi(n)} \beta_{\xi(n)} \beta^{2s} \right) \\
&= \frac{(-c)^n ab}{(ab)^{r+s}} AB \left(\alpha^{2r} - \beta^{2r} \right) \left(\beta_{\xi(n)} \alpha_{\xi(n)} \alpha^{2s} - \alpha_{\xi(n)} \beta_{\xi(n)} \beta^{2s} \right) \\
&= \frac{(-c)^n ab}{(ab)^{r+s}} AB \left(\alpha^{2r} - \beta^{2r} \right) \left[\left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta} \right) \left(\alpha^{2s} - \beta^{2s} \right) \right. \\
&\quad \left. - \frac{\Delta}{b} c \left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta} \right) \left(\alpha^{2s} + \beta^{2s} \right) \right] \\
&= \frac{(-c)^n ab}{(ab)^{r+s}} AB \left(\frac{(ab)^r}{a} u_{2r} \Delta \right) \left[\left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta} \right) \left(\frac{(ab)^s}{a} u_{2s} \Delta \right) \right. \\
&\quad \left. - \frac{\Delta}{b} c \left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta} \right) \left((ab)^s v_{2s} \right) \right] \\
&= (-c)^n ABu_{2r} \Delta^2 \left[\left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta} \right) \frac{b}{a} u_{2s} - c \left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta} \right) v_{2s} \right].
\end{aligned}
$$

Thus we get the desired result. □

Corollary 36. *If we take $s = -r$, we get the Catalan's like identity:*

$$
\begin{aligned}
&\mathbb{K}_{w,n+2r}\mathbb{K}_{w,n-2r} - \mathbb{K}_{w,n}^2 \\
&= \begin{cases} (-1)^{n+1} c^{n-2r} AB\Delta^2 u_{2r} \left((\mathbb{K}_{v,0} - \theta) u_{2r} + c (\mathbb{K}_{u,0} - \eta) v_{2r} \right), & \text{if } n \text{ is even}, \\ (-1)^{n+1} c^{n-2r} AB\Delta^2 u_{2r} \left(\left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta} \right) \frac{b}{a} u_{2r} + c \left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta} \right) v_{2r} \right), & \text{if } n \text{ is odd}. \end{cases}
\end{aligned}
$$

Corollary 37. *If we take $s = -r$ and $r = 1$, we get the Cassini's like identity:*

$$
\begin{aligned}
&\mathbb{K}_{w,n+2r}\mathbb{K}_{w,n-2r} - \mathbb{K}_{w,n}^2 \\
&= \begin{cases} (-1)^{n+1} ac^{n-2} AB\Delta^2 \left((\mathbb{K}_{v,0} - \theta) a + c (ab + 2c) (\mathbb{K}_{u,0} - \eta) \right), & \text{if } n \text{ is even}, \\ (-1)^{n+1} ac^{n-2} AB\Delta^2 \left(\left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta} \right) b + c (ab + 2c) \left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta} \right) \right), & \text{if } n \text{ is odd}. \end{cases}
\end{aligned}
$$

Note that for even case, the Cassini's like identity can be stated as by means of the following matrix identity:

$$\left[\begin{array}{cc} \mathbb{K}_{w,2n+2} & -c^2\mathbb{K}_{w,2n} \\ \mathbb{K}_{w,2n} & -c^2\mathbb{K}_{w,2n-2} \end{array}\right] = \left[\begin{array}{cc} \mathbb{K}_{w,4} & -c^2\mathbb{K}_{w,2} \\ \mathbb{K}_{w,2} & -c^2\mathbb{K}_{w,0} \end{array}\right] \left[\begin{array}{cc} ab+2c & -c^2 \\ 1 & 0 \end{array}\right]^{n-1}. \quad (3.17)$$

By taking determinant from above to down below of both sides of the matrix equality (3.17), we get

$$\mathbb{K}_{w,2n+2}\mathbb{K}_{w,2n-2} - \mathbb{K}^2_{w,2n} = c^{2n-2}\left(\mathbb{K}_{w,4}\mathbb{K}_{w,0} - \mathbb{K}^2_{w,2}\right). \quad (3.18)$$

By taking determinant from down below to above of both sides of the matrix equality (3.17), we get

$$\mathbb{K}_{w,2n-2}\mathbb{K}_{w,2n+2} - \mathbb{K}^2_{w,2n} = c^{2n-2}\left(\mathbb{K}_{w,0}\mathbb{K}_{w,4} - \mathbb{K}^2_{w,2}\right). \quad (3.19)$$

Theorem 38. *For $n \geq 1$, we have*

$$\sum_{r=1}^{n} \mathbb{K}_{w,r} = \frac{c^2\left(\mathbb{K}_{w,n} + \mathbb{K}_{w,n-1} - \mathbb{K}_{w,0} - \mathbb{K}_{w,-1}\right) - \mathbb{K}_{w,n+2} - \mathbb{K}_{w,n+1} + \mathbb{K}_{w,2} + \mathbb{K}_{w,1}}{c^2 - ab - 2c + 1}.$$

Proof. First note that by considering the formula in (3.7), the bi-periodic Horadam hybrid numbers for negative subscripts can be defined as

$$\mathbb{K}_{w,-n} = w_{-n} + w_{-n+1}i + w_{-n+2}\varepsilon + w_{-n+3}h,$$

where

$$(-c)^n w_{-n} = \left(\frac{b}{a}\right)^{\xi(n)} w_0 u_{n+1} - w_1 u_n.$$

If n is odd, we have

$$\sum_{r=1}^{n} \mathbb{K}_{w,r} = \sum_{r=1}^{\frac{n-1}{2}} \mathbb{K}_{w,2r} + \sum_{r=1}^{\frac{n+1}{2}} \mathbb{K}_{w,2r-1}$$

$$= \sum_{r=1}^{\frac{n-1}{2}} \frac{a}{(ab)^r}\left(A\alpha_{\xi(2r)}\alpha^{2r} - B\beta_{\xi(2r)}\beta^{2r}\right) + \sum_{r=1}^{\frac{n-1}{2}+1} \frac{ab}{(ab)^r}\left(A\alpha_{\xi(2r-1)}\alpha^{2r-1} - B\beta_{\xi(2r-1)}\beta^{2r-1}\right)$$

$$
\begin{aligned}
&= aA\alpha_{\xi(2r)}\sum_{r=1}^{\frac{n-1}{2}}\left(\frac{\alpha^2}{ab}\right)^r - aB\beta_{\xi(2r)}\sum_{r=1}^{\frac{n-1}{2}}\left(\frac{\beta^2}{ab}\right)^r \\
&+ \frac{ab}{\alpha}A\alpha_{\xi(2r-1)}\sum_{r=1}^{\frac{n-1}{2}+1}\left(\frac{\alpha^2}{ab}\right)^r - \frac{ab}{\beta}B\beta_{\xi(2r-1)}\sum_{r=1}^{\frac{n-1}{2}+1}\left(\frac{\beta^2}{ab}\right)^r \\
&= aA\alpha_{\xi(2r)}\left(\frac{\left(\frac{\alpha^2}{ab}\right)^{\frac{n-1}{2}+1}-\frac{\alpha^2}{ab}}{\frac{\alpha^2}{ab}-1}\right) - aB\beta_{\xi(2r)}\left(\frac{\left(\frac{\beta^2}{ab}\right)^{\frac{n-1}{2}+1}-\frac{\beta^2}{ab}}{\frac{\beta^2}{ab}-1}\right) \\
&+ \frac{ab}{\alpha}A\alpha_{\xi(2r-1)}\left(\frac{\left(\frac{\alpha^2}{ab}\right)^{\frac{n-1}{2}+2}-\frac{\alpha^2}{ab}}{\frac{\alpha^2}{ab}-1}\right) - \frac{ab}{\beta}B\beta_{\xi(2r-1)}\left(\frac{\left(\frac{\beta^2}{ab}\right)^{\frac{n-1}{2}+2}-\frac{\beta^2}{ab}}{\frac{\beta^2}{ab}-1}\right).
\end{aligned}
$$

Since $\xi(2r) = 0$ and $\xi(2r-1) = 1$, we have

$$
\begin{aligned}
\sum_{r=1}^{n}\mathbb{K}_{w,r} &= \frac{a}{(ab)^{\frac{n-1}{2}}}\left(\frac{A\alpha_0\alpha^{n+1}-A\alpha_0\alpha^2(ab)^{\frac{n-1}{2}}}{\alpha^2-ab}+\frac{-B\beta_0\beta^{n+1}+B\beta_0\beta^2(ab)^{\frac{n-1}{2}}}{\beta^2-ab}\right) \\
&+ \frac{ab}{(ab)^{\frac{n+1}{2}}}\left(\frac{A\alpha_1\alpha^{n+2}-A\alpha_1\alpha(ab)^{\frac{n+1}{2}}}{\alpha^2-ab}+\frac{-B\beta_1\beta^{n+2}+B\beta_1\beta(ab)^{\frac{n+1}{2}}}{\beta^2-ab}\right) \\
&= \frac{a}{(ab)^{\frac{n-1}{2}}(\alpha^2-ab)(\beta^2-ab)}\times \\
&\quad \Big[(\alpha\beta)^2\left(A\alpha_0\alpha^{n-1}-B\beta_0\beta^{n-1}\right)-ab\left(A\alpha_0\alpha^{n+1}-B\beta_0\beta^{n+1}\right) \\
&\quad +(ab)^{\frac{n-1}{2}}\left(-(\alpha\beta)^2(A\alpha_0-B\beta_0)+ab\left(A\alpha_0\alpha^2-B\beta_0\beta^2\right)\right)\Big] \\
&+ \frac{ab}{(ab)^{\frac{n+1}{2}}(\alpha^2-ab)(\beta^2-ab)}\times \\
&\quad \Big[(\alpha\beta)^2(A\alpha_1\alpha^n-B\beta_1\beta^n)-ab\left(A\alpha_1\alpha^{n+2}-B\beta_1\beta^{n+2}\right) \\
&\quad +(ab)^{\frac{n+1}{2}}\left(-(\alpha\beta)^2\left(A\alpha_1\alpha^{-1}-B\beta_1\beta^{-1}\right)+ab(A\alpha_1\alpha-B\beta_1\beta)\right)\Big] \\
&= \frac{c^2(\mathbb{K}_{w,n-1}-\mathbb{K}_{w,0}+\mathbb{K}_{w,n}-\mathbb{K}_{w,-1})-\mathbb{K}_{w,n+1}-\mathbb{K}_{w,n+2}+\mathbb{K}_{w,2}+\mathbb{K}_{w,1}}{c^2-ab-2c+1}.
\end{aligned}
$$

If n is even, we have

$$
\sum_{r=1}^{n}\mathbb{K}_{w,r} = \sum_{r=1}^{\frac{n}{2}}\mathbb{K}_{w,2r} + \sum_{r=1}^{\frac{n}{2}}\mathbb{K}_{w,2r-1}.
$$

In a similar manner, we get the desired result. □

Theorem 39. *For nonnegative even integer n and nonnegative integer r, we have*

$$\sum_{i=0}^{n}\binom{n}{i}(-c)^{n-i}\,\mathbb{K}_{w,2i+r}=(ab)^{\frac{n}{2}}\,\mathbb{K}_{w,n+r}.$$

Proof. From the Binet form of the bi-periodic Horadam hybrid numbers, we get

$$\sum_{i=0}^{n}\binom{n}{i}(-c)^{n-i}\,\mathbb{K}_{w,2i+r}$$

$$=\sum_{i=0}^{n}\binom{n}{i}(-c)^{n-i}\frac{a^{\xi(2i+r+1)}}{(ab)^{\lfloor\frac{2i+r}{2}\rfloor}}\left(A\alpha_{\xi(r)}\alpha^{2i+r}-B\beta_{\xi(r)}\beta^{2i+r}\right)$$

$$=\frac{a^{\xi(r+1)}}{(ab)^{\lfloor\frac{r}{2}\rfloor}}A\alpha_{\xi(r)}\alpha^{r}\sum_{i=0}^{n}\binom{n}{i}(-c)^{n-i}\left(\frac{\alpha^2}{ab}\right)^{i}-\frac{a^{\xi(r+1)}}{(ab)^{\lfloor\frac{r}{2}\rfloor}}B\beta_{\xi(r)}\beta^{r}\sum_{i=0}^{n}\binom{n}{i}(-c)^{n-i}\left(\frac{\beta^2}{ab}\right)^{i}$$

$$=\frac{a^{\xi(r+1)}}{(ab)^{\lfloor\frac{r}{2}\rfloor}}A\alpha_{\xi(r)}\alpha^{r}\left(\frac{\alpha^2}{ab}-c\right)^{n}-\frac{a^{\xi(r+1)}}{(ab)^{\lfloor\frac{r}{2}\rfloor}}B\beta_{\xi(r)}\beta^{r}\left(\frac{\beta^2}{ab}-c\right)^{n}$$

$$=\frac{a^{\xi(r+1)}}{(ab)^{\lfloor\frac{r}{2}\rfloor}}\left(A\alpha_{\xi(r)}\alpha^{n+r}-B\beta_{\xi(r)}\beta^{n+r}\right)=\frac{a^{\xi(r+1)}}{(ab)^{\lfloor\frac{r}{2}\rfloor}}\frac{(ab)^{\lfloor\frac{n+r}{2}\rfloor}}{a^{\xi(n+r+1)}}\mathbb{K}_{w,n+r}$$

$$=\frac{a^{-\xi(r)+\xi(n+r)}}{(ab)^{\lfloor\frac{r}{2}\rfloor-\lfloor\frac{n+r}{2}\rfloor}}\mathbb{K}_{w,n+r}=(ab)^{\frac{n}{2}}\,\mathbb{K}_{w,n+r}.$$

□

3.5 Links between bi-periodic Fibonacci and Lucas hybrid numbers

Now, we state some relations between generalized bi-periodic Fibonacci numbers and generalized bi-periodic Lucas hybrid numbers. We also give some relations between generalized bi-periodic Fibonacci numbers and modified generalized bi-periodic Lucas hybrid numbers. To do this, we consider the generalized bi-periodic Fibonacci hybrid numbers $\mathbb{K}_{u,n}$, the generalized bi-periodic Lucas hybrid numbers $\mathbb{K}_{v,n}$, and the modified generalized bi-periodic Lucas hybrid numbers $\mathbb{K}_{\widehat{v},n}$ which are stated in Table 3.1.

The Binet form of $\mathbb{K}_{u,n}$ is

$$\mathbb{K}_{u,n} = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha_{\xi(n)}\alpha^n - \beta_{\xi(n)}\beta^n}{\alpha - \beta} \right) \tag{3.20}$$

and the Binet form of $\mathbb{K}_{v,n}$ is

$$\mathbb{K}_{v,n} = \frac{a^{-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\alpha_{\xi(n)}\alpha^n + \beta_{\xi(n)}\beta^n \right). \tag{3.21}$$

Also, the Binet form of $\mathbb{K}_{\widehat{v},n}$ is

$$\mathbb{K}_{\widehat{v},n} = \frac{b^{-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\alpha_{\xi(n+1)}\alpha^n + \beta_{\xi(n+1)}\beta^n \right). \tag{3.22}$$

Theorem 40. *For any natural number m, n with $n > m$, we have*

(i) $\mathbb{K}_{u,n+1} + c\mathbb{K}_{u,n-1} = \mathbb{K}_{\widehat{v},n}$.

(ii) $\mathbb{K}_{\widehat{v},n+1} + c\mathbb{K}_{\widehat{v},n-1} = (ab + 4c)\,\mathbb{K}_{u,n}$.

(iii) $\mathbb{K}_{u,n}\mathbb{K}_{v,m} - \mathbb{K}_{u,m}\mathbb{K}_{v,n} = \begin{cases} 2(-c)^m (\mathbb{K}_{v,0} - \theta)\, u_{n-m}, & \text{if } n \text{ and } m \text{ are both even,} \\ 2(-c)^m \frac{b}{a} \left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta} \right) u_{n-m}, & \text{if } n \text{ and } m \text{ are both odd.} \end{cases}$

(iv) $\mathbb{K}_{v,n}\mathbb{K}_{v,n} - \dfrac{\Delta^2}{a^2}\mathbb{K}_{u,n}\mathbb{K}_{u,n} = \begin{cases} 4(-c)^n (\mathbb{K}_{v,0} - \theta), & \text{if } n \text{ is even,} \\ 4(-c)^n \frac{b}{a} \left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta} \right), & \text{if } n \text{ is odd.} \end{cases}$

(v) $\mathbb{K}_{v,n}\mathbb{K}_{v,n} + \dfrac{\Delta^2}{a^2}\mathbb{K}_{u,n}\mathbb{K}_{u,n} = \begin{cases} 2\left((\mathbb{K}_{v,0} + \mu_e)\, v_{2n} + \frac{\Delta^2}{a^2} (\mathbb{K}_{u,0} + \gamma_e)\, u_{2n} \right), & \text{if } n \text{ is even} \\ 2\left(\frac{b}{a} (\mathbb{K}_{\widehat{v},0} + \mu_o)\, v_{2n} + \frac{\Delta^2}{a^2} (\mathbb{K}_{\widehat{u},0} + \gamma_o)\, u_{2n} \right), & \text{if } n \text{ is odd.} \end{cases}$

Proof. (i) From the relations (3.20) and (3.22), we have,

$$\begin{aligned} &\mathbb{K}_{u,n+1} + c\mathbb{K}_{u,n-1} \\ &= \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\frac{\alpha_{\xi(n+1)}\alpha^{n+1} - \beta_{\xi(n+1)}\beta^{n+1}}{\alpha - \beta} \right) + c\frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} \left(\frac{\alpha_{\xi(n-1)}\alpha^{n-1} - \beta_{\xi(n-1)}\beta^{n-1}}{\alpha - \beta} \right) \\ &= \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + \xi(n)}} \left(\frac{\alpha_{\xi(n+1)}\alpha^{n+1} - \beta_{\xi(n+1)}\beta^{n+1} - \alpha_{\xi(n+1)}\alpha^n\beta + \beta_{\xi(n+1)}\beta^n\alpha}{\alpha - \beta} \right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + \xi(n)}} \left(\frac{\alpha_{\xi(n+1)}\alpha^n (\alpha-\beta) + \beta_{\xi(n+1)}\beta^n (\alpha-\beta)}{\alpha-\beta} \right) \\
&= \frac{a^{\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + \xi(n)}} \left(\alpha_{\xi(n+1)}\alpha^n + \beta_{\xi(n+1)}\beta^n \right) = \mathbb{K}_{\widehat{v},n}.
\end{aligned}
$$

(ii) By using the relations (3.20) and (3.22), we have

$$
\begin{aligned}
& \mathbb{K}_{\widehat{v},n+1} + c\mathbb{K}_{\widehat{v},n-1} \\
&= \frac{b^{-\xi(n+1)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\alpha_{\xi(n)}\alpha^{n+1} + \beta_{\xi(n)}\beta^{n+1} \right) + c\frac{b^{-\xi(n-1)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} \left(\alpha_{\xi(n)}\alpha^{n-1} + \beta_{\xi(n)}\beta^{n-1} \right) \\
&= \frac{b^{-\xi(n+1)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\alpha_{\xi(n)}\alpha^{n+1} + \beta_{\xi(n)}\beta^{n+1} \right) - \frac{b^{-\xi(n-1)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\alpha_{\xi(n)}\alpha^{n}\beta + \beta_{\xi(n)}\beta^{n}\alpha \right) \\
&= \frac{b^{-\xi(n+1)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\alpha_{\xi(n)}\alpha^{n+1} + \beta_{\xi(n)}\beta^{n+1} - \alpha_{\xi(n)}\alpha^{n}\beta - \beta_{\xi(n)}\beta^{n}\alpha \right) \\
&= \frac{b^{-\xi(n+1)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\alpha_{\xi(n)}\alpha^{n} (\alpha-\beta) - \beta_{\xi(n)}\beta^{n} (\alpha-\beta) \right) \\
&= \frac{b^{-\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor + \xi(n)}} (\alpha-\beta)^2 \left(\frac{\alpha_{\xi(n)}\alpha^{n} - \beta_{\xi(n)}\beta^{n}}{\alpha-\beta} \right) = \frac{\Delta^2}{ab}\mathbb{K}_{u,n} = (ab+4c)\,\mathbb{K}_{u,n}.
\end{aligned}
$$

(iii) By using the Binet forms for $\mathbb{K}_{u,n}$ and $\mathbb{K}_{v,n}$, and considering the relation (3.13), we get the desired result.

For even n and m, we have $\xi(n) = \xi(m) = 0$. Then we get

$$
\begin{aligned}
& \mathbb{K}_{u,n}\mathbb{K}_{v,m} - \mathbb{K}_{u,m}\mathbb{K}_{v,n} \\
&= \frac{a}{(ab)^{\frac{n+m}{2}}(\alpha-\beta)} \left((\alpha_0\alpha^n - \beta_0\beta^n)(\alpha_0\alpha^m + \beta_0\beta^m) - (\alpha_0\alpha^m - \beta_0\beta^m)(\alpha_0\alpha^n + \beta_0\beta^n) \right) \\
&= \frac{a}{(ab)^{\frac{n+m}{2}}(\alpha-\beta)} \begin{pmatrix} \alpha_0\alpha_0\alpha^{n+m} + \alpha_0\beta_0\alpha^n\beta^m - \beta_0\alpha_0\beta^n\alpha^m - \beta_0\beta_0\beta^{n+m} \\ -\alpha_0\alpha_0\alpha^{n+m} - \alpha_0\beta_0\beta^n\alpha^m + \beta_0\alpha_0\alpha^n\beta^m + \beta_0\beta_0\beta^{n+m} \end{pmatrix} \\
&= \frac{a}{(ab)^{\frac{n+m}{2}}(\alpha-\beta)} \left(\alpha_0\beta_0 (\alpha^n\beta^m - \beta^n\alpha^m) - \beta_0\alpha_0 (\beta^n\alpha^m - \alpha^n\beta^m) \right) \\
&= \frac{a}{(ab)^{\frac{n+m}{2}}(\alpha-\beta)} (\alpha_0\beta_0 + \beta_0\alpha_0)(\alpha^n\beta^m - \beta^n\alpha^m) \\
&= \frac{a}{(ab)^{\frac{n+m}{2}}} 2(\mathbb{K}_{v,0} - \theta) \frac{(\alpha\beta)^m(\alpha^{n-m} - \beta^{n-m})}{\alpha-\beta}
\end{aligned}
$$

$$\begin{aligned}
&= \frac{2a}{(ab)^{\frac{n+m}{2}}}\left(\mathbb{K}_{v,0}-\theta\right)\frac{(-abc)^{m}u_{n-m}(ab)^{\frac{n-m}{2}}}{a} \\
&= 2(-c)^{m}\left(\mathbb{K}_{v,0}-\theta\right)u_{n-m}.
\end{aligned}$$

For odd n and m, we have $\xi(n)=\xi(m)=1$. Then we get

$$\begin{aligned}
&\mathbb{K}_{u,n}\mathbb{K}_{v,m}-\mathbb{K}_{u,m}\mathbb{K}_{v,n} \\
&= \frac{1}{a(ab)^{\frac{n+m}{2}-1}(\alpha-\beta)}\left(\left(\alpha_1\alpha^n-\beta_1\beta^n\right)\left(\alpha_1\alpha^m+\beta_1\beta^m\right)-\left(\alpha_1\alpha^m-\beta_1\beta^m\right)\left(\alpha_1\alpha^n+\beta_1\beta^n\right)\right) \\
&= \frac{1}{a(ab)^{\frac{n+m}{2}-1}(\alpha-\beta)}\begin{pmatrix}\alpha_1\alpha_1\alpha^{n+m}+\alpha_1\beta_1\alpha^n\beta^m-\beta_1\alpha_1\beta^n\alpha^m-\beta_1\beta_1\beta^{n+m} \\ -\alpha_1\alpha_1\alpha^{n+m}-\alpha_1\beta_1\beta^n\alpha^m+\beta_1\alpha_1\alpha^n\beta^m+\beta_1\beta_1\beta^{n+m}\end{pmatrix} \\
&= \frac{1}{a(ab)^{\frac{n+m}{2}-1}(\alpha-\beta)}\left(\alpha_1\beta_1\left(\alpha^n\beta^m-\beta^n\alpha^m\right)-\beta_1\alpha_1\left(\beta^n\alpha^m-\alpha^n\beta^m\right)\right) \\
&= \frac{1}{a(ab)^{\frac{n+m}{2}-1}(\alpha-\beta)}\left(\alpha_1\beta_1+\beta_1\alpha_1\right)\left(\alpha^n\beta^m-\beta^n\alpha^m\right) \\
&= \frac{1}{a(ab)^{\frac{n+m}{2}-1}}2\left(\mathbb{K}_{\widehat{v},0}-\widehat{\theta}\right)\frac{(\alpha\beta)^m\left(\alpha^{n-m}-\beta^{n-m}\right)}{\alpha-\beta} \\
&= \frac{2}{a(ab)^{\frac{n+m}{2}-1}}\left(\mathbb{K}_{\widehat{v},0}-\widehat{\theta}\right)\frac{(-abc)^{m}u_{n-m}(ab)^{\frac{n-m}{2}}}{a} \\
&= (-c)^{m}\frac{2b}{a}\left(\mathbb{K}_{\widehat{v},0}-\widehat{\theta}\right)u_{n-m}.
\end{aligned}$$

(iv) By using the Binet forms for $\mathbb{K}_{u,n}$ and $\mathbb{K}_{v,n}$, we have

$$\begin{aligned}
&\mathbb{K}_{v,n}\mathbb{K}_{v,n}-\frac{\Delta^2}{a^2}\mathbb{K}_{u,n}\mathbb{K}_{u,n} \\
&= \frac{a^{-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}\left(\alpha_{\xi(n)}\alpha^n+\beta_{\xi(n)}\beta^n\right)^2-\frac{\Delta^2}{a^2}\frac{a^{2\xi(n+1)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha_{\xi(n)}\alpha^n-\beta_{\xi(n)}\beta^n}{\alpha-\beta}\right)^2 \\
&= \frac{a^{-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}\left(\left(\alpha_{\xi(n)}\alpha^n+\beta_{\xi(n)}\beta^n\right)^2-\left(\alpha_{\xi(n)}\alpha^n-\beta_{\xi(n)}\beta^n\right)^2\right) \\
&= \frac{a^{-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}\left[\alpha_{\xi(n)}\alpha_{\xi(n)}\alpha^{2n}+\beta_{\xi(n)}\beta_{\xi(n)}\beta^{2n}+(\alpha\beta)^n\left(\alpha_{\xi(n)}\beta_{\xi(n)}+\beta_{\xi(n)}\alpha_{\xi(n)}\right)\right. \\
&\quad\left.-\alpha_{\xi(n)}\alpha_{\xi(n)}\alpha^{2n}-\beta_{\xi(n)}\beta_{\xi(n)}\beta^{2n}+(\alpha\beta)^n\left(\alpha_{\xi(n)}\beta_{\xi(n)}+\beta_{\xi(n)}\alpha_{\xi(n)}\right)\right] \\
&= \frac{a^{-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}2(\alpha\beta)^n\left(\alpha_{\xi(n)}\beta_{\xi(n)}+\beta_{\xi(n)}\alpha_{\xi(n)}\right)
\end{aligned}$$

$$= \begin{cases} 4\frac{a^{-2\xi(n)}}{(ab)^n}(\alpha\beta)^n(\mathbb{K}_{v,0}-\theta), & \text{if } n \text{ is even} \\ 4\frac{a^{-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}(\alpha\beta)^n\left(\mathbb{K}_{\widehat{v},0}-\widehat{\theta}\right), & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} 4(-c)^n(\mathbb{K}_{v,0}-\theta), & \text{if } n \text{ is even} \\ 4(-c)^n\frac{b}{a}\left(\mathbb{K}_{\widehat{v},0}-\widehat{\theta}\right), & \text{if } n \text{ is odd} \end{cases}$$

(v)

$$\mathbb{K}_{v,n}\mathbb{K}_{v,n}+\frac{\Delta^2}{a^2}\mathbb{K}_{u,n}\mathbb{K}_{u,n}$$

$$= \frac{a^{-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}\left(\alpha_{\xi(n)}\alpha^n+\beta_{\xi(n)}\beta^n\right)^2+\frac{\Delta^2}{a^2}\frac{a^{2\xi(n+1)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}\left(\frac{\alpha_{\xi(n)}\alpha^n-\beta_{\xi(n)}\beta^n}{\alpha-\beta}\right)^2$$

$$= \frac{a^{-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}\left(\left(\alpha_{\xi(n)}\alpha^n+\beta_{\xi(n)}\beta^n\right)^2+\left(\alpha_{\xi(n)}\alpha^n-\beta_{\xi(n)}\beta^n\right)^2\right)$$

$$= \frac{a^{-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}\Big[\alpha_{\xi(n)}\alpha_{\xi(n)}\alpha^{2n}+\beta_{\xi(n)}\beta_{\xi(n)}\beta^{2n}+(\alpha\beta)^n\left(\alpha_{\xi(n)}\beta_{\xi(n)}+\beta_{\xi(n)}\alpha_{\xi(n)}\right)$$
$$+\alpha_{\xi(n)}\alpha_{\xi(n)}\alpha^{2n}+\beta_{\xi(n)}\beta_{\xi(n)}\beta^{2n}-(\alpha\beta)^n\left(\alpha_{\xi(n)}\beta_{\xi(n)}+\beta_{\xi(n)}\alpha_{\xi(n)}\right)\Big]$$

$$= \frac{a^{-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}2\left(\alpha_{\xi(n)}\alpha_{\xi(n)}\alpha^{2n}+\beta_{\xi(n)}\beta_{\xi(n)}\beta^{2n}\right)$$

$$= \begin{cases} \frac{a^{-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}2\left(\left(\mathbb{K}_{v,0}+\mu_e+\frac{\Delta}{a}(\mathbb{K}_{u,0}+\gamma_e)\right)\alpha^{2n}\right. \\ \qquad\left.+\left(\mathbb{K}_{v,0}+\mu_e-\frac{\Delta}{a}(\mathbb{K}_{u,0}+\gamma_e)\right)\beta^{2n}\right), & \text{if } n \text{ is even} \\ \frac{a^{-2\xi(n)}}{(ab)^{2\lfloor\frac{n}{2}\rfloor}}2\left(\left(\mathbb{K}_{\widehat{v},0}+\mu_o+\frac{\Delta}{b}(\mathbb{K}_{\widehat{u},0}+\gamma_o)\right)\alpha^{2n}\right. \\ \qquad\left.+\left(\mathbb{K}_{\widehat{v},0}+\mu_o-\frac{\Delta}{b}(\mathbb{K}_{\widehat{u},0}+\gamma_o)\right)\beta^{2n}\right), & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{2}{(ab)^n}\left((\mathbb{K}_{v,0}+\mu_e)\left(\alpha^{2n}+\beta^{2n}\right)+\frac{\Delta}{a}(\mathbb{K}_{u,0}+\gamma_e)\left(\alpha^{2n}-\beta^{2n}\right)\right), & \text{if } n \text{ is even} \\ \frac{2a^{-2}}{(ab)^{n-1}}\left((\mathbb{K}_{\widehat{v},0}+\mu_o)\left(\alpha^{2n}+\beta^{2n}\right)+\frac{\Delta}{b}(\mathbb{K}_{\widehat{u},0}+\gamma_o)\left(\alpha^{2n}-\beta^{2n}\right)\right), & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} 2\left((\mathbb{K}_{v,0}+\mu_e)v_{2n}+\frac{\Delta^2}{a^2}(\mathbb{K}_{u,0}+\gamma_e)u_{2n}\right), & \text{if } n \text{ is even} \\ 2\left(\frac{b}{a}(\mathbb{K}_{\widehat{v},0}+\mu_o)v_{2n}+\frac{\Delta^2}{a^2}(\mathbb{K}_{\widehat{u},0}+\gamma_o)u_{2n}\right), & \text{if } n \text{ is odd.} \end{cases}$$

By considering the relations (3.13), (3.15), and (3.16), we get the desired result. □

CHAPTER

4

BIVARIATE r-FIBONACCI AND r-LUCAS HYBRID POLYNOMIALS [3]

In this chapter, we introduce a new generalization of Fibonacci and Lucas hybrid polynomials. We investigate some properties of these polynomials [3].

4.1 Introduction

Similar to the quaternion multiplication, the hybrid number multiplication is noncommutative. The set of hybrid numbers form a noncommutative algebra. For more details of hybrid numbers, see Ozdemir's paper [57].

Recently, Szynal-Liana [74] introduced the Fibonacci hybrid polynomials (alias hybrinomials) as

$$FH_n(x) = F_n(x) + F_{n+1}(x)\,i + F_{n+2}(x)\,\varepsilon + F_{n+3}(x)\,h, \qquad n \geq 0, \tag{4.1}$$

where $F_n(x)$ is the n^{th} Fibonacci polynomial (see [49]) defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \qquad n \geq 2,$$

with initial values $F_0(x) = 0, F_1(x) = 1$. In [43], Kizilates defined the Horadam hybrinomials which generalize the Fibonacci hybrinomials. Several studies related to hybrid numbers with generalized Fibonacci number coefficients can be found in [18, 46, 65, 66, 72, 73, 75, 76, 77, 79] and for a recent study related to the generalized Fibonacci numbers and polynomials we refer to [5]. It is also worth noting that, in the literature there exist another type of hybrid polynomials which are related to the families of special functions such as the Laguerre and the Hermite polynomials, see [24]. We should note that our approach will be different from that polynomials.

This work has been intended as an attempt to introduce a new class of hybrid polynomials which are so-called "r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials of type s". They give a natural generalization of the Fibonacci and Lucas hybrinomials. We give the generating functions, the Binet forms, matrix representations and several basic properties of these hybrid polynomials. A relation between r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials is also given.

Now we start by recalling some basic results concerning to the r-Fibonacci polynomials and r-Lucas polynomials of type s. For the detailed information related to these polynomials, we refer to [1, 61].

Let $r \geq 1$ be any integer, and let $s = 1, 2, \ldots, r$. The bivariate r-Fibonacci polynomials $\left(T_n^{(r)}\right)_n := \left(T_n^{(r)}(x, y)\right)_n$ are defined by

$$T_n^{(r)} = xT_{n-1}^{(r)} + yT_{n-r-1}^{(r)}, \qquad n \geq r+1 \tag{4.2}$$

with initial conditions $T_0^{(r)} = 0, T_k^{(r)} = x^{k-1}$ for $k = 1, 2, \ldots, r$. Their companion sequences, the bivariate r-Lucas polynomials of type s, $\left(Z_n^{(r,s)}\right)_n := \left(Z_n^{(r,s)}(x, y)\right)_n$ are defined by

$$Z_n^{(r,s)} = xZ_{n-1}^{(r,s)} + yZ_{n-r-1}^{(r,s)}, \qquad n \geq r+1 \tag{4.3}$$

with initial conditions $Z_0^{(r,s)} = s+1, Z_k^{(r,s)} = x^k$ for $k = 1, 2, \ldots, r$. It is clear that if we take $r = 1, s = 1$, then these polynomials respectively reduce to the classical bivariate Fibonacci and Lucas polynomials, see [10].
If we take $x = y = 1$, they reduce to the r-Fibonacci and r-Lucas numbers.

The Binet forms for the bivariate r-Fibonacci polynomials and the bivariate r-Lucas polynomials of type s are

$$T_n^{(r)} = \sum_{k=1}^{r+1} \frac{\alpha_k^n}{(r+1)\alpha_k - rx} \tag{4.4}$$

and

$$Z_n^{(r,s)} = \sum_{k=1}^{r+1} \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx}, \tag{4.5}$$

respectively. Here α_k are the distinct roots of the polynomial $R(t) = t^{r+1} - xt^r - y$. For details see [1].

In this study, we introduce bivariate r-Fibonacci hybrid polynomials and bivariate r-Lucas hybrid polynomials of type s. We give the generating functions, the Binet forms, matrix representations and several basic properties of these hybrid polynomials. Some relationships between the r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials are also given.

4.2 Bivariate hybrid polynomials

In this section, we give the definition of r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials of type s. We give the generating functions, the Binet forms, the summation formulas of these polynomials. Also, we establish a relation between the r-Fibonacci hybrid polynomials and the r-Lucas hybrid polynomials of type s.

Definition 41. *For $n \geq 0$, the r-Fibonacci hybrid polynomials $\mathbb{K}_{T^{(r)},n}$ is defined by the recurrence relation*

$$\mathbb{K}_{T^{(r)},n} = T_n^{(r)} + T_{n+1}^{(r)} i + T_{n+2}^{(r)} \varepsilon + T_{n+3}^{(r)} h, \qquad n \geq 0, \tag{4.6}$$

where $T_n^{(r)}$ is the n^{th} r-Fibonacci polynomial. The r-Lucas hybrid polynomials of type s, $\mathbb{K}_{Z^{(r,s)},n}$ is defined by the recurrence relation

$$\mathbb{K}_{Z^{(r,s)},n} = Z_n^{(r,s)} + Z_{n+1}^{(r,s)} i + Z_{n+2}^{(r,s)} \varepsilon + Z_{n+3}^{(r,s)} h, \qquad n \geq 0, \tag{4.7}$$

where $Z_n^{(r,s)}$ is the n^{th} r-Lucas polynomial of type s.

In the following table, we state some special cases of r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials of type 1.

x	y	r	s	$\mathbb{K}_{U^{(r)},n}$	$\mathbb{K}_{V^{(r,s)},n}$
x	y	1	1	bivariate Fibonacci hybrid polynomials	bivariate Lucas hybrid polynomials
x	1	1	1	Fibonacci hybrid polynomials [74]	Lucas hybrid polynomials [74]
$2x$	1	1	1	Pell hybrid polynomials [77]	Pell-Lucas hybrid polynomials [77]
m	1	p	p	gen. hybrid Fibonacci p-numbers [46]	gen. hybrid Lucas p-numbers [46]
1	1	p	p	hybrid Fibonacci p-numbers [46]	hybrid Lucas p-numbers [46]
1	1	1	1	Fibonacci hybrid numbers [73]	Lucas hybrid numbers [72]
2	1	1	1	Pell hybrid numbers [75]	Pell-Lucas hybrid numbers [75]
1	2	1	1	Jacobsthal hybrid numbers [76]	Jacobsthal-Lucas hybrid numbers [7

Table 4.1: Special cases of bivarite polynomials.

4.3 Generating functions

We state the following lemma, which is useful to obtain the generating functions of r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials of type s.

Lemma 42. *The r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials of type s satisfy the following relations*

$$\mathbb{K}_{T^{(r)},n} = x\mathbb{K}_{T^{(r)},n-1} + y\mathbb{K}_{T^{(r)},n-r-1} \qquad for \qquad n \geq r+1. \tag{4.8}$$

and

$$\mathbb{K}_{Z^{(r,s)},n} = x\mathbb{K}_{Z^{(r,s)},n-1} + y\mathbb{K}_{Z^{(r,s)},n-r-1} \qquad for \qquad n \geq r+1. \tag{4.9}$$

Proof. By using the definition of r-Fibonacci polynomials, we have

$$\begin{aligned}
\mathbb{K}_{T^{(r)},n} &= T_n^{(r)} + T_{n+1}^{(r)}i + T_{n+2}^{(r)}\varepsilon + T_{n+3}^{(r)}h \\
&= xT_{n-1}^{(r)} + yT_{n-r-1}^{(r)} + (xT_n^{(r)} + yT_{n-r}^{(r)})i + (xT_{n+1}^{(r)} + yT_{n-r+1}^{(r)})\varepsilon \\
&+ (xT_{n+2}^{(r)} + yT_{n-r+2}^{(r)})h \\
&= x(T_{n-1}^{(r)} + T_n^{(r)}i + T_{n+1}^{(r)}\varepsilon + T_{n+2}^{(r)}h) + y(T_{n-r-1}^{(r)} + T_{n-r}^{(r)}i + T_{n-r+1}^{(r)}\varepsilon + T_{n-r+2}^{(r)}h) \\
&= x\mathbb{K}_{T^{(r)},n-1} + y\mathbb{K}_{T^{(r)},n-r-1}.
\end{aligned}$$

Thus, we get the desired result.
The relation for the r-Lucas hybrid polynomials of type s can be proven similarly. So we omit it here. □

Theorem 43. *The generating functions for the r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials of type s,* $\widehat{\mathbb{G}}(z)$ *and* $\widehat{\mathbb{H}}(z)$ *are given by*

$$\left(1 - xz - yz^{r+1}\right)\widehat{\mathbb{G}}(z) = \mathbb{K}_{T^{(r)},0} + \sum_{n=1}^{r}\left(\mathbb{K}_{T^{(r)},n} - x\mathbb{K}_{T^{(r)},n-1}\right)z^n \tag{4.10}$$

and

$$\left(1 - xz - yz^{r+1}\right)\widehat{\mathbb{H}}(z) = \mathbb{K}_{Z^{(r,s)},0} + \sum_{n=1}^{r}\left(\mathbb{K}_{Z^{(r,s)},n} - x\mathbb{K}_{Z^{(r,s)},n-1}\right)z^n. \tag{4.11}$$

respectively.

Proof. Let

$$\widehat{\mathbb{G}}(z) = \sum_{n=0}^{\infty}\mathbb{K}_{T^{(r)},n}z^n = \mathbb{K}_{T^{(r)},0} + \mathbb{K}_{T^{(r)},1}z + \mathbb{K}_{T^{(r)},2}z^2 + \cdots + \mathbb{K}_{T^{(r)},n}z^n + \cdots.$$

Since $\mathbb{K}_{T^{(r)},n} = x\mathbb{K}_{T^{(r)},n-1} + y\mathbb{K}_{T^{(r)},n-r-1}$ for $n \geq r+1$, we get

$$\begin{aligned}
\left(1 - xz - yz^{r+1}\right)\widehat{\mathbb{G}}(z) &= \sum_{n=0}^{\infty}\mathbb{K}_{T^{(r)},n}z^n - x\sum_{n=0}^{\infty}\mathbb{K}_{T^{(r)},n}z^{n+1} + y\sum_{n=0}^{\infty}\mathbb{K}_{T^{(r)},n}z^{n+r+1} \\
&= \sum_{n=0}^{\infty}\mathbb{K}_{T^{(r)},n}z^n - x\sum_{n=1}^{\infty}\mathbb{K}_{T^{(r)},n-1}z^n - y\sum_{n=r+1}^{\infty}\mathbb{K}_{T^{(r)},n-r-1}z^n \\
&= \sum_{n=0}^{r}\mathbb{K}_{T^{(r)},n}z^n - x\sum_{n=1}^{r}\mathbb{K}_{T^{(r)},n-1}z^n \\
&+ \sum_{n=r+1}^{\infty}\left(\mathbb{K}_{T^{(r)},n} - x\mathbb{K}_{T^{(r)},n-1} - y\mathbb{K}_{T^{(r)},n-r-1}\right)z^n \\
&= \sum_{n=0}^{r}\mathbb{K}_{T^{(r)},n}z^n - x\sum_{n=1}^{r}\mathbb{K}_{T^{(r)},n-1}z^n \\
&= \mathbb{K}_{T^{(r)},0} + \sum_{n=1}^{r}\left(\mathbb{K}_{T^{(r)},n} - x\mathbb{K}_{T^{(r)},n-1}\right)z^n.
\end{aligned}$$

The generating function for r-Lucas hybrid polynomials of type s can be proven similarly. So we omit it here. □

4.4 Binet forms

Next, we state the Binet forms for r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials of type s and by using these forms, we derive some properties of them.

Theorem 44. *The Binet forms for the r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials of type s are*

$$\mathbb{K}_{T^{(r)},n} = \sum_{k=1}^{r+1} \frac{\alpha_k^* \alpha_k^n}{(r+1)\alpha_k - rx} \tag{4.12}$$

and

$$\mathbb{K}_{Z^{(r,s)},n} = \sum_{k=1}^{r+1} \alpha_k^* \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx}, \tag{4.13}$$

respectively, where $\alpha_k^* = 1 + \alpha_k i + \alpha_k^2 \varepsilon + \alpha_k^3 h$.

Proof. By using the definitions of the sequences $(\mathbb{K}_{T^{(r)},n})_n$, $(\mathbb{K}_{Z^{(r,s)},n})_n$ and the Binet forms of $(T_n^{(r)})_n$ and $(Z_n^{(r,s)})_n$, we have

$$\begin{aligned}
\mathbb{K}_{T^{(r)},n} &= T_n^{(r)} + T_{n+1}^{(r)} i + T_{n+2}^{(r)} \varepsilon + T_{n+3}^{(r)} h \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^n}{(r+1)\alpha_k - rx} + \sum_{k=1}^{r+1} \frac{\alpha_k^{n+1}}{(r+1)\alpha_k - rx} \quad i + \sum_{k=1}^{r+1} \frac{\alpha_k^{n+2}}{(r+1)\alpha_k - rx} \quad \varepsilon \\
&+ \sum_{k=1}^{r+1} \frac{\alpha_k^{n+3}}{(r+1)\alpha_k - rx} \quad h \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^n + \alpha_k^{n+1} i + \alpha_k^{n+2} \varepsilon + \alpha_k^{n+3} h}{(r+1)\alpha_k - rx} \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^n (1 + \alpha_k i + \alpha_k^2 \varepsilon + \alpha_k^3 h)}{(r+1)\alpha_k - rx} \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^n}{(r+1)\alpha_k - rx} \left[1 + \alpha_k i + \alpha_k^2 \varepsilon + \alpha_k^3 h\right] \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^* \alpha_k^n}{(r+1)\alpha_k - rx}.
\end{aligned}$$

For the r-Lucas hybrid polynomials, we have

$$\mathbb{K}_{Z^{(r,s)},n} = Z_n^{(r,s)} + Z_{n+1}^{(r,s)} i + Z_{n+2}^{(r,s)} \varepsilon + Z_{n+3}^{(r,s)} h$$

$$
\begin{aligned}
&= \sum_{k=1}^{r+1} \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx} + \sum_{k=1}^{r+1} \alpha_k^{n+1} \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx} \quad i + \sum_{k=1}^{r+1} \alpha_k^{n+2} \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx} \quad \varepsilon \\
&+ \sum_{k=1}^{r+1} \alpha_k^{n+3} \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx} \quad h \\
&= \sum_{k=1}^{r+1} \frac{(\alpha_k^n + \alpha_k^{n+1} i + \alpha_k^{n+2}\varepsilon + \alpha_k^{n+3} h)((s+1)\alpha_k - sx)}{(r+1)\alpha_k - rx} \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^n (1 + \alpha_k i + \alpha_k^2 \varepsilon + \alpha_k^3 h)((s+1)\alpha_k - sx)}{(r+1)\alpha_k - rx} \\
&= \sum_{k=1}^{r+1} \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx} \left[1 + \alpha_k i + \alpha_k^2 \varepsilon + \alpha_k^3 h\right] \\
&= \sum_{k=1}^{r+1} \alpha_k^* \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx}.
\end{aligned}
$$

Which gives the desired results. □

The link between the r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials of type s can be given in the following result.

Theorem 45. *The r-Lucas hybrid polynomials of type s can be expressed in term of r-Fibonacci hybrid polynomials as*

$$
\mathbb{K}_{Z^{(r,s)},n} = \mathbb{K}_{T^{(r)},n+1} + sy\mathbb{K}_{T^{(r)},n-r}, \qquad n \geq r+1. \tag{4.14}
$$

Proof. Using the Binet forms of r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials of type s, we have

$$
\begin{aligned}
\mathbb{K}_{T^{(r)},n+1} + sy\mathbb{K}_{T^{(r)},n-r} &= \sum_{k=1}^{r+1} \frac{\alpha_k^* \alpha_k^{n+1}}{(r+1)\alpha_k - rx} + sy \sum_{k=1}^{r+1} \frac{\alpha_k^* \alpha_k^{n-r}}{(r+1)\alpha_k - rx} \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^* \alpha_k^{n+1} + sy\alpha_k^* \alpha_k^{n-r}}{(r+1)\alpha_k - rx} \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^* \alpha_k^n (\alpha_k + sy\alpha_k^{-r})}{(r+1)\alpha_k - ra} \\
&= \sum_{k=1}^{r+1} \alpha_k^* \alpha_k^n \frac{(s+1)\alpha_k - sx}{(r+1)\alpha_k - rx} \\
&= \mathbb{K}_{Z^{(r,s)},n}.
\end{aligned}
$$

Because $\alpha_k = x + y\alpha_k^{-r}$ then $\alpha_k + sy\alpha_k^{-r} = (s+1)\alpha_k - sx$. □

Next, we give some summation formulas for $\mathbb{K}_{T^{(r)},n}$ and $\mathbb{K}_{Z^{(r,s)},n}$ in the following theorem.

Theorem 46. *For $m \geq 0$, we have*

$$\sum_{n=0}^{m} \mathbb{K}_{T^{(r)},n} = \sum_{k=1}^{r+1} \frac{\alpha_k^*(\alpha_k^{m+1}-1)}{(r+1)\alpha_k^2-(r(x+1)+1)\alpha_k+rx} \tag{4.15}$$

and

$$\sum_{n=0}^{m} \mathbb{K}_{Z^{(r,s)},n} = \sum_{k=1}^{r+1} \alpha_k^* \frac{((s+1)\alpha_k-sx)(\alpha_k^{m+1}-1)}{(r+1)\alpha_k^2-(r(x+1)+1)\alpha_k+rx}. \tag{4.16}$$

Proof. Using the Binet form of $\mathbb{K}_{T^{(r)},n}$, we get

$$\begin{aligned}
\sum_{n=0}^{m} \mathbb{K}_{T^{(r)},n} &= \sum_{n=0}^{m}\sum_{k=1}^{r+1} \frac{\alpha_k^*\alpha_k^n}{(r+1)\alpha_k-rx} \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^*}{(r+1)\alpha_k-rx}\frac{\alpha_k^{m+1}-1}{\alpha_k-1} \\
&= \sum_{k=1}^{r+1} \frac{\alpha_k^*(\alpha_k^{m+1}-1)}{(r+1)\alpha_k^2-(r(x+1)+1)\alpha_k+rx}.
\end{aligned}$$

And from Binet form of $\mathbb{K}_{Z^{(r,s)},n}$, we get

$$\begin{aligned}
\sum_{n=0}^{m} \mathbb{K}_{Z^{(r,s)},n} &= \sum_{n=0}^{m}\sum_{k=1}^{r+1} \alpha_k^*\alpha_k^n \frac{(s+1)\alpha_k-sx}{(r+1)\alpha_k-rx} \\
&= \sum_{k=1}^{r+1} \alpha_k^* \frac{(s+1)\alpha_k-sx}{(r+1)\alpha_k-rx} \sum_{n=0}^{m} \alpha_k^n \\
&= \sum_{k=1}^{r+1} \alpha_k^* \frac{((s+1)\alpha_k-sx)(\alpha_k^{m+1}-1)}{(r+1)\alpha_k^2-(r(x+1)+1)\alpha_k+rx}.
\end{aligned}$$

□

4.5 Matrix representation

In this section, we give a matrix representation of r-Fibonacci hybrid polynomials.

Let $Q_r := \begin{bmatrix} x & 0 & \cdots & 0 & y \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$ be a matrix of size $(r+1) \times (r+1)$.
For $n \geq r$, it can be verified that

$$Q_r^n = \begin{bmatrix} T_{n+1}^{(r)} & yT_{n-r+1}^{(r)} & \cdots & yT_{n-1}^{(r)} & yT_n^{(r)} \\ T_n^{(r)} & yT_{n-r}^{(r)} & & yT_{n-2}^{(r)} & yT_{n-1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{n-r+1}^{(r)} & yT_{n-2r+1}^{(r)} & \cdots & yT_{n-r-1}^{(r)} & yT_{n-r}^{(r)} \end{bmatrix}$$

with $T_j^{(r)} = 0$, for $j = -1, -2, \ldots$.
Now, let's define the matrix

$$K(n) := \begin{bmatrix} \mathbb{K}_{T^{(r)},n+1} & y\mathbb{K}_{T^{(r)},n-r+1} & \cdots & y\mathbb{K}_{T^{(r)},n-1} & y\mathbb{K}_{T^{(r)},n} \\ \mathbb{K}_{T^{(r)},n} & y\mathbb{K}_{T^{(r)},n-r} & & y\mathbb{K}_{T^{(r)},n-2} & y\mathbb{K}_{T^{(r)},n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{K}_{T^{(r)},n-r+1} & y\mathbb{K}_{T^{(r)},n-2r+1} & \cdots & y\mathbb{K}_{T^{(r)},n-r-1} & y\mathbb{K}_{T^{(r)},n-r} \end{bmatrix}.$$

For any nonnegative integer $n \geq r$, we have

$$K(n) = K(0)\, Q_r^n. \tag{4.17}$$

By taking the determinant of both sides of the matrix equality (4.17), we get the generalized Cassini's identity for the r-Fibonacci hybrid polynomials as

$$\det K(n) = (-1)^{nr} y^n \det K(0). \tag{4.18}$$

Remark 47. If $r = 1$ in (4.18), we get

$$\mathbb{K}_{T^{(1)},n+1}\mathbb{K}_{T^{(1)},n-1} - \mathbb{K}^2_{T^{(1)},n} = (-1)^n \left(\mathbb{K}_{T^{(1)},1}\mathbb{K}_{T^{(1)},-1} - \mathbb{K}^2_{T^{(1)},0}\right). \tag{4.19}$$

By using the matrix identity (4.17), we get the following theorem which can be seen as a generalization of Honsberger formula.

Theorem 48. *For $n, s, t \geq r$, we have*

$$\mathbb{K}_{T^{(r)},s+t} = \mathbb{K}_{T^{(r)},s}\mathbb{K}_{T^{(r)},t+1} + y\sum_{j=0}^{r-1}\mathbb{K}_{T^{(r)},s-r+j}\mathbb{K}_{T^{(r)},t-j}. \tag{4.20}$$

$$\mathbb{K}_{T^{(r)},s+t} = T_s^{(r)}\mathbb{K}_{T^{(r)},t+1} + y\sum_{j=1}^{r-1}T_{s-r+j}^{(r)}\mathbb{K}_{T^{(r)},t-j}. \tag{4.21}$$

Proof. Let $K := K(0)$, considering the matrix equalities

$$\left(KQ_r^{s+t}\right)K = \left(KQ_r^s\right)\left(Q_r^tK\right)$$

and

$$\left(KQ_r^{s+t}\right) = \left(KQ_r^s\right)Q_r^t,$$

then equating the corresponding entries, we get the desired results respectively. □

Remark 49. If $r = 1$, the identities (4.20) and (4.21) reduce to the classical bivariate hybrid Fibonacci polynomials as

$$\mathbb{K}_{T^{(1)},s+t} = \mathbb{K}_{T^{(1)},s}\mathbb{K}_{T^{(1)},t+1} + y\mathbb{K}_{T^{(1)},s-1}\mathbb{K}_{T^{(1)},t}. \tag{4.22}$$

$$\mathbb{K}_{T^{(1)},s+t} = T_s^{(1)}\mathbb{K}_{T^{(1)},t+1} + yT_{s-1}^{(r)}\mathbb{K}_{T^{(1)},t}. \tag{4.23}$$

CHAPTER

5

HYPER-DUAL HORADAM QUATERNIONS [4]

This chapter deals with developing a new class of quaternions, called hyper-dual Horadam quaternions which are constructed from the quaternions whose components are hyper-dual Horadam numbers. We investigate some basic properties of these quaternions. The main advantage of introducing the hyper-dual Horadam quaternions is that many hyper-dual numbers with celebrated numbers such as Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers can be deduced as particular cases of the hyper-dual Horadam quaternions.

5.1 Introduction

Horadam [35] introduced the quaternions whose components are Fibonacci numbers. More generally, by using Horadam's approach, Halıcı and Karatas [32] defined the n^{th} Horadam quaternion as

$$Q_{W,n} = W_n + W_{n+1}i + W_{n+2}j + W_{n+3}k, \qquad n \geq 0,$$

where $(W_n)_n$ is the Horadam sequence [36] and is defined by

$$W_n = pW_{n-1} + qW_{n-2}, \qquad n \geq 2,$$

with the arbitrary initial values W_0, W_1 and nonzero integers p, q. It is clear to see that the Horadam sequence $(W_n)_n := (W_n(W_0, W_1; p, q))_n$ generalizes many well-known integer sequences such as Fibonacci sequence $(F_n)_n = (W_n(0,1;1,1))_n$, Lucas sequence $(L_n)_n = (W_n(2,1;1,1))_n$, generalized Fibonacci sequence $(U_n)_n = (W_n(0,1;p,q))_n$, and the generalized Lucas sequence $(V_n)_n = (W_n(2,p;p,q))_n$. The Binet form of the Horadam sequence $(W_n)_n$ is

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where

$$A := W_1 - W_0\beta \qquad \text{and} \qquad B := W_1 - W_0\alpha.$$

Here $\alpha = \frac{(p+\sqrt{p^2+4q})}{2}$ and $\beta = \frac{(p-\sqrt{p^2+4q})}{2}$ are the roots of the characteristic polynomial

$$x^2 - px - q,$$

that is

$$\alpha\beta = -q, \qquad \alpha + \beta = p$$

and

$$\Delta := \alpha - \beta = \sqrt{p^2 + 4q},$$

with

$$p^2 + 4q > 0.$$

The Binet form for the Horadam quaternions is

$$Q_{W,n} = \frac{A\alpha^*\alpha^n - B\beta^*\beta^n}{\alpha - \beta}, \tag{5.1}$$

where $\alpha^* = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ and $\beta^* = 1 + \beta i + \beta^2 j + \beta^3 k$. For details, see [17, 84]. Also, Nurkan and Guven [55] introduced the dual Fibonacci quaternions by taking dual Fibonacci numbers instead of real numbers as coefficients. These numbers can also be seen as dual numbers with Fibonacci quaternion coefficients. A generalization of dual Fibonacci quaternions can be found in [83]. In [42], the author introduced the dual Horadam quaternions as

$$\widetilde{Q}_{W,n} = \widetilde{W}_n + \widetilde{W}_{n+1}i + \widetilde{W}_{n+2}j + \widetilde{W}_{n+3}k, \qquad n \geq 0,$$

where $\widetilde{W}_n = W_n + W_{n+1}\varepsilon$ is the n^{th} dual Horadam number. Recently, in [56], the authors defined the hyper-dual numbers whose coefficients are from the sequences $(\mathfrak{U}_{kn})_n$ and $(\mathcal{V}_{kn})_n$ which reduce to the sequences $(W_n(0,1;p,1))_n$ and $(W_n(2,p;p,1))_n$ for $k=1$, respectively.

In this chapter, motivating the definition of hyper-dual split quaternions in [6], we consider the quaternions whose coefficients are taken from hyper-dual Horadam numbers. To do this, first we define the hyper-dual Horadam numbers, then we introduce the quaternions whose coefficients are taken from those numbers. We give the generating function and the Binet form for hyper-dual Horadam quaternions. Some algebraic properties of these quaternions such as Vajda's identity, Catalan's identity, Cassini's identity and d'Ocagne's identity are derived with the aid of the Binet form. Moreover, we develop some matrix identities involving the hyper-dual Horadam quaternions which allow us to obtain some properties of these quaternions.

5.2 Hyper-Dual Horadam quaternions

In this section, first we define hyper-dual Horadam numbers, then by using these numbers we introduce hyper-dual Horadam quaternions and investigate the basic properties of these quaternions.

Definition 50. *The n^{th} hyper-dual Horadam number is defined as*

$$\widehat{W}_n = W_n + W_{n+1}\varepsilon_1 + W_{n+2}\varepsilon_2 + W_{n+3}\varepsilon_1\varepsilon_2, \qquad n \geq 0,$$

where W_n is the n^{th} Horadam number and $\varepsilon_1, \varepsilon_2$ are dual numbers satisfying the multiplication rules in (1.36).

Definition 51. *The n^{th} hyper-dual Horadam quaternion is defined as*

$$\widehat{Q}_{W,n} = \widehat{W}_n + \widehat{W}_{n+1}i + \widehat{W}_{n+2}j + \widehat{W}_{n+3}k, \qquad n \geq 0,$$

where $\widehat{W}_n$ is the n^{th} hyper-dual Horadam number and i, j, k satisfy the quaternion multiplication rules in (1.34).

In the following table, we give the types of quaternions which are mentioned in this chapter.

Type of quaternion	Definition
Horadam quaternion [32]	$Q_{W,n} = W_n + W_{n+1}i + W_{n+2}j + W_{n+3}k$, W_n is the n^{th} Horadam number
Dual Horadam quaternion [42]	$\widetilde{Q}_{W,n} = \widetilde{W}_n + \widetilde{W}_{n+1}i + \widetilde{W}_{n+2}j + \widetilde{W}_{n+3}k$, $\widetilde{W}_n$ is the n^{th} dual Horadam number
Hyper-dual Horadam quaternion	$\widehat{Q}_{W,n} = \widehat{W}_n + \widehat{W}_{n+1}i + \widehat{W}_{n+2}j + \widehat{W}_{n+3}k$, $\widehat{W}_n$ is the n^{th} hyper-dual Horadam number

Table 5.1: Type of quaternions.

Note that the n^{th} hyper-dual Horadam quaternion can be expressed as

$$\widehat{Q}_{W,n} = Q_{W,n} + Q_{W,n+1}\varepsilon_1 + Q_{W,n+2}\varepsilon_2 + Q_{W,n+3}\varepsilon_1\varepsilon_2, \qquad n \geq 0,$$

where $Q_{W,n}$ is the n^{th} Horadam quaternion, and $\varepsilon_1, \varepsilon_2$ are dual numbers. The addition and the multiplication of two hyper-dual Horadam quaternions $\widehat{Q}_{W,n}$ and $\widehat{Q}_{W,m}$ are defined as

$$\begin{aligned}
\widehat{Q}_{W,n} + \widehat{Q}_{W,m} &= (Q_{W,n} + Q_{W,m}) + (Q_{W,n+1} + Q_{W,m+1})\,\varepsilon_1 \\
&\quad + (Q_{W,n+2} + Q_{W,m+2})\,\varepsilon_2 + (Q_{W,n+3} + Q_{W,m+3})\,\varepsilon_1\varepsilon_2, \\
\widehat{Q}_{W,n}\widehat{Q}_{W,m} &= Q_{W,n}Q_{W,m} + (Q_{W,n}Q_{W,m+1} + Q_{W,n+1}Q_{W,m})\,\varepsilon_1 \\
&\quad + (Q_{W,n}Q_{W,m+2} + Q_{W,n+2}Q_{W,m})\,\varepsilon_2 \\
&\quad + (Q_{W,n}Q_{W,m+3} + Q_{W,n+1}Q_{W,m+2} + Q_{W,n+2}Q_{W,m+1} \\
&\quad + Q_{W,n+3}Q_{W,m})\,\varepsilon_1\varepsilon_2,
\end{aligned}$$

respectively.

The norm of a hyper-dual Horadam quaternion $\widehat{Q}_{W,n}$ is defined as

$$N(\widehat{Q}_{W,n}) := \widehat{Q}_{W,n}\overline{\widehat{Q}}_{W,n} = \overline{\widehat{Q}}_{W,n}\widehat{Q}_{W,n} = \widehat{W}_n^2 + \widehat{W}_{n+1}^2 + \widehat{W}_{n+2}^2 + \widehat{W}_{n+3}^2,$$

where $\overline{\widehat{Q}}_{W,n} := \widehat{W}_n - \widehat{W}_{n+1}i - \widehat{W}_{n+2}j - \widehat{W}_{n+3}k$ is the conjugate of $\widehat{Q}_{W,n}$. Also the norm of $\widehat{Q}_{W,n}$ can be obtained by the determinant of the matrix

$$\begin{pmatrix} \widehat{W}_n + \widehat{W}_{n+1}i & -\widehat{W}_{n+2}j - \widehat{W}_{n+3}i \\ \widehat{W}_{n+2}j - \widehat{W}_{n+3}i & \widehat{W}_n - \widehat{W}_{n+1}i \end{pmatrix}.$$

Theorem 52. *The hyper-dual Horadam quaternions satisfy the following relation:*

$$\widehat{Q}_{W,n} = p\widehat{Q}_{W,n-1} + q\widehat{Q}_{W,n-2}, \qquad n \geq 2.$$

Proof. From the definition of the hyper-dual Horadam quaternions and the Horadam quaternions, we have

$$\begin{aligned} p\widehat{Q}_{W,n-1} + q\widehat{Q}_{W,n-2} &= p\left(Q_{W,n-1} + Q_{W,n}\varepsilon_1 + Q_{W,n+1}\varepsilon_2 + Q_{W,n+2}\varepsilon_1\varepsilon_2\right) \\ &\quad + q\left(Q_{W,n-2} + Q_{W,n-1}\varepsilon_1 + Q_{W,n}\varepsilon_2 + Q_{W,n+1}\varepsilon_1\varepsilon_2\right) \\ &= \left(pQ_{W,n-1} + qQ_{W,n-2}\right) + \left(pQ_{W,n} + qQ_{W,n-1}\right)\varepsilon_1 \\ &\quad + \left(pQ_{W,n+1} + qQ_{W,n}\right)\varepsilon_2 + \left(pQ_{W,n+2} + qQ_{W,n+1}\right)\varepsilon_1\varepsilon_2 \\ &= Q_{W,n} + Q_{W,n+1}\varepsilon_1 + Q_{W,n+2}\varepsilon_2 + Q_{W,n+3}\varepsilon_1\varepsilon_2. \end{aligned}$$

□

5.3 Generating function

In the following theorem, we state the generating function for the hyper-dual Horadam quaternions.

Theorem 53. *The generating function for hyper-dual Horadam quaternions, $\widehat{G}(x)$ is given by*

$$\left(1 - px - qx^2\right)\widehat{G}(x) = \widehat{Q}_{W,0} + \left(\widehat{Q}_{W,1} - p\widehat{Q}_{W,0}\right)x\,.$$

Proof. Let

$$\widehat{G}(x) := \sum_{n=0}^{\infty} \widehat{Q}_{W,n}x^n = \widehat{Q}_{W,0} + \widehat{Q}_{W,1}x + \sum_{n=2}^{\infty} \widehat{Q}_{W,n}x^n.$$

From Theorem 52, we have

$$\begin{aligned}
\left(1 - px - qx^2\right) \widehat{G}(x) &= \widehat{Q}_{W,0} + \widehat{Q}_{W,1}x + \sum_{n=2}^{\infty} \widehat{Q}_{W,n}x^n \\
&- p\widehat{Q}_{W,0}x - p\sum_{n=2}^{\infty} \widehat{Q}_{W,n-1}x^n - q\sum_{n=2}^{\infty} \widehat{Q}_{W,n-2}x^n \\
&= \widehat{Q}_{W,0} + \widehat{Q}_{W,1}x - p\widehat{Q}_{W,0}x \\
&+ \sum_{n=2}^{\infty} \left(\widehat{Q}_{W,n} - p\widehat{Q}_{W,n-1} - q\widehat{Q}_{W,n-2}\right) x^n \\
&= \widehat{Q}_{W,0} + \left(\widehat{Q}_{W,1} - p\widehat{Q}_{W,0}\right) x.
\end{aligned}$$

Thus, we get the desired result. □

5.4 Binet form

The Binet form of the hyper-dual Horadam quaternions is given in the following theorem.

Theorem 54. *The Binet form of hyper-dual Horadam quaternions is*

$$\widehat{Q}_{W,n} = \frac{A\alpha^* \underline{\alpha}\alpha^n - B\beta^* \underline{\beta}\beta^n}{\alpha - \beta},$$

where

$$\begin{aligned}
\alpha^* &= 1 + \alpha i + \alpha^2 j + \alpha^3 k, & \beta^* &= 1 + \beta i + \beta^2 j + \beta^3 k, \\
\underline{\alpha} &= 1 + \alpha\varepsilon_1 + \alpha^2\varepsilon_2 + \alpha^3\varepsilon_1\varepsilon_2, & \underline{\beta} &= 1 + \beta\varepsilon_1 + \beta^2\varepsilon_2 + \beta^3\varepsilon_1\varepsilon_2.
\end{aligned}$$

Proof. From the Binet form of Horadam quaternions in (5.1), we have

$$\begin{aligned}
\widehat{Q}_{W,n} &= Q_{W,n} + Q_{W,n+1}\varepsilon_1 + Q_{W,n+2}\varepsilon_2 + Q_{W,n+3}\varepsilon_1\varepsilon_2 \\
&= \left(\frac{A\alpha^*\alpha^n - B\beta^*\beta^n}{\alpha - \beta}\right) + \left(\frac{A\alpha^*\alpha^{n+1} - B\beta^*\beta^{n+1}}{\alpha - \beta}\right)\varepsilon_1 \\
&+ \left(\frac{A\alpha^*\alpha^{n+2} - B\beta^*\beta^{n+2}}{\alpha - \beta}\right)\varepsilon_2 + \left(\frac{A\alpha^*\alpha^{n+3} - B\beta^*\beta^{n+3}}{\alpha - \beta}\right)\varepsilon_1\varepsilon_2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{A\alpha^*\alpha^n}{\alpha-\beta}\left(1+\alpha\varepsilon_1+\alpha^2\varepsilon_2+\alpha^3\varepsilon_1\varepsilon_2\right) - \frac{B\beta^*\beta^n}{\alpha-\beta}\left(1+\beta\varepsilon_1+\beta^2\varepsilon_2+\beta^3\varepsilon_1\varepsilon_2\right) \\
&= \frac{A\alpha^*\underline{\alpha}\alpha^n - B\beta^*\underline{\beta}\beta^n}{\alpha-\beta}.
\end{aligned}
$$

□

From Theorem 54, the Binet forms of the hyper-dual generalized Fibonacci and Lucas quaternions can be obtained as

$$
\widehat{Q}_{U,n} = \frac{\alpha^*\underline{\alpha}\alpha^n - \beta^*\underline{\beta}\beta^n}{\alpha-\beta} \quad \text{and} \quad \widehat{Q}_{V,n} = \alpha^*\underline{\alpha}\alpha^n + \beta^*\underline{\beta}\beta^n,
$$

respectively.

By using (5.4), we obtain the following relation between the hyper-dual Fibonacci quaternions and the hyper-dual Lucas quaternions.

Theorem 55. *Let n be a positive integer. For hyper-dual Fibonacci quaternions and hyper-dual Lucas quaternions, the following equality holds:*

$$
\widehat{Q}_{V,n} = \widehat{Q}_{U,n+1} + q\widehat{Q}_{U,n-1}, \qquad n \geq 1.
$$

Now, we need the following lemma which allows us a remarkable simplification for obtaining the properties of hyper-dual Horadam quaternions.

Lemma 56. *Let $\theta := 1-q+q^2-q^3$ and $\omega := (1-q)\,i + \left(1+p^2+q\right)k$. Then we have*

$$
\begin{aligned}
\underline{\alpha}\underline{\beta} &= \widehat{V}_0 - (1+pq\varepsilon_1\varepsilon_2), \\
\alpha^*\beta^* &= Q_{V,0} - \theta - \Delta q\,(Q_{U,0} - \omega)\,, \\
\beta^*\alpha^* &= Q_{V,0} - \theta + \Delta q\,(Q_{U,0} - \omega)\,.
\end{aligned}
$$

Proof. The proof can be done by using the multiplication rules in (1.34) and (1.36). We should note that the set of hyper-dual numbers form a commutative algebra. Therefore we have $\underline{\alpha}\underline{\beta} = \underline{\beta}\underline{\alpha}$. But since the quaternion multiplication is noncommutative, $\alpha^*\beta^*$ need not be equal to $\beta^*\alpha^*$. The results for $\alpha^*\beta^*$ and $\beta^*\alpha^*$ can also be found in [17, Lemma 1]. □

Theorem 57. *(Vajda's Identity) For integers n, r and s, we have*

$$\begin{aligned}&\widehat{Q}_{W,n+r}\widehat{Q}_{W,n+s}-\widehat{Q}_{W,n}\widehat{Q}_{W,n+r+s}\\&\quad=AB\left(-q\right)^{n}\left(\widehat{V}_{0}-\left(1+pq\varepsilon_{1}\varepsilon_{2}\right)\right)U_{r}\left(\left(Q_{V,0}-\theta\right)U_{s}+q\left(Q_{U,0}-\omega\right)V_{s}\right).\end{aligned}$$

Proof. From the Binet form of hyper-dual Horadam quaternions, we have

$$\begin{aligned}&\Delta^{2}\left(\widehat{Q}_{W,n+r}\widehat{Q}_{W,n+s}-\widehat{Q}_{W,n}\widehat{Q}_{W,n+r+s}\right)\\&=\left(A\alpha^{*}\underline{\alpha}\alpha^{n+r}-B\beta^{*}\underline{\beta}\beta^{n+r}\right)\left(A\alpha^{*}\underline{\alpha}\alpha^{n+s}-B\beta^{*}\underline{\beta}\beta^{n+s}\right)\\&-\left(A\alpha^{*}\underline{\alpha}\alpha^{n}-B\beta^{*}\underline{\beta}\beta^{n}\right)\left(A\alpha^{*}\underline{\alpha}\alpha^{n+r+s}-B\beta^{*}\underline{\beta}\beta^{n+r+s}\right)\\&=A^{2}\left(\alpha^{*}\underline{\alpha}\right)^{2}\alpha^{2n+r+s}-AB\alpha^{*}\beta^{*}\underline{\alpha}\underline{\beta}\alpha^{n+r}\beta^{n+s}-AB\beta^{*}\alpha^{*}\underline{\beta}\underline{\alpha}\alpha^{n+s}\beta^{n+r}\\&+B^{2}\left(\beta^{*}\underline{\beta}\right)^{2}\beta^{2n+r+s}-A^{2}\left(\alpha^{*}\underline{\alpha}\right)^{2}\alpha^{2n+r+s}+AB\alpha^{*}\beta^{*}\underline{\alpha}\underline{\beta}\alpha^{n}\beta^{n+r+s}\\&+AB\beta^{*}\alpha^{*}\underline{\beta}\underline{\alpha}\beta^{n}\alpha^{n+r+s}-B^{2}\left(\beta^{*}\underline{\beta}\right)^{2}\beta^{2n+r+s}\\&=AB\left(\alpha\beta\right)^{n}\underline{\alpha}\underline{\beta}\left(\alpha^{*}\beta^{*}\left(-\alpha^{r}\beta^{s}+\beta^{r+s}\right)+\beta^{*}\alpha^{*}\left(-\alpha^{s}\beta^{r}+\alpha^{r+s}\right)\right).\end{aligned}$$

By using Lemma 56, we have

$$\begin{aligned}&\widehat{Q}_{W,n+r}\widehat{Q}_{W,n+s}-\widehat{Q}_{W,n}\widehat{Q}_{W,n+r+s}\\&=\frac{AB}{\Delta^{2}}\left(-q\right)^{n}\underline{\alpha}\underline{\beta}\left(-\alpha^{*}\beta^{*}\beta^{s}\left(\alpha^{r}-\beta^{r}\right)+\beta^{*}\alpha^{*}\alpha^{s}\left(\alpha^{r}-\beta^{r}\right)\right)\\&=\frac{AB}{\Delta}\left(-q\right)^{n}\underline{\alpha}\underline{\beta}U_{r}\left(\beta^{*}\alpha^{*}\alpha^{s}-\alpha^{*}\beta^{*}\beta^{s}\right)\\&=\frac{AB}{\Delta}\left(-q\right)^{n}\underline{\alpha}\underline{\beta}U_{r}\left(Q_{V,0}-\theta+\Delta q\left(Q_{U,0}-\omega\right)\right)\alpha^{s}\\&-\frac{AB}{\Delta}\left(-q\right)^{n}\underline{\alpha}\underline{\beta}U_{r}\left(Q_{V,0}-\theta-\Delta q\left(Q_{U,0}-\omega\right)\right)\beta^{s}\\&=AB\left(-q\right)^{n}\left(\widehat{V}_{0}-\left(1+pq\varepsilon_{1}\varepsilon_{2}\right)\right)U_{r}\left(\left(Q_{V,0}-\theta\right)U_{s}+q\left(Q_{U,0}-\omega\right)V_{s}\right).\end{aligned}$$

□

From Theorem 57, we have the following results:

If we set $r\to -s$, we get the following Catalan's identity for hyper-dual Horadam quaternions:

$$\begin{aligned}&\widehat{Q}_{W,n-s}\widehat{Q}_{W,n+s}-\widehat{Q}_{W,n}^{2}\\&\quad=-AB\left(-q\right)^{n-s}\left(\widehat{V}_{0}-\left(1+pq\varepsilon_{1}\varepsilon_{2}\right)\right)U_{s}\left(\left(Q_{V,0}-\theta\right)U_{s}+q\left(Q_{U,0}-\omega\right)V_{s}\right).\end{aligned}$$

Here note that $U_{-n} = \frac{-U_n}{(-q)^n}$.

If we set $s = -r = 1$, we get the following Cassini's identity for hyper-dual Horadam quaternions:

$$\widehat{Q}_{W,n-1}\widehat{Q}_{W,n+1} - \widehat{Q}^2_{W,n} = -AB(-q)^{n-1}\left(\widehat{V}_0 - (1 + pq\varepsilon_1\varepsilon_2)\right)(Q_{V,0} - \theta + pq(Q_{U,0} - \omega)). \quad (5.2)$$

If we set $s \to m - n$, and fix $r = 1$, we get the following d'Ocagne's identity for hyper-dual Horadam quaternions:

$$\widehat{Q}_{W,n+1}\widehat{Q}_{W,m} - \widehat{Q}_{W,n}\widehat{Q}_{W,m+1} = AB(-q)^n\left(\widehat{V}_0 - (1 + pq\varepsilon_1\varepsilon_2)\right)((Q_{V,0} - \theta)U_{m-n} + q(Q_{U,0} - \omega)V_{m-n}).$$

Next, we give some summation formulas for hyper-dual Horadam quaternions.

Theorem 58. *For $n \geq 2$, we have*

$$\sum_{r=1}^{n-1}\widehat{Q}_{W,r} = \frac{\widehat{Q}_{W,n} - \widehat{Q}_{W,1} + q\left(\widehat{Q}_{W,n-1} - \widehat{Q}_{W,0}\right)}{p + q - 1}.$$

Proof. From the Binet form for hyper-dual Horadam quaternions, we have

$$\begin{aligned}
\sum_{r=1}^{n-1}\widehat{Q}_{W,r} &= \sum_{r=1}^{n-1}\frac{A\alpha^*\underline{\alpha}\alpha^r - B\beta^*\underline{\beta}\beta^r}{\alpha - \beta} = \frac{A\alpha^*\underline{\alpha}}{\alpha - \beta}\sum_{r=1}^{n-1}\alpha^r - \frac{B\beta^*\underline{\beta}}{\alpha - \beta}\sum_{r=1}^{n-1}\beta^r \\
&= \frac{A\alpha^*\underline{\alpha}}{\alpha - \beta}\left(\frac{\alpha^n - \alpha}{\alpha - 1}\right) - \frac{B\beta^*\underline{\beta}}{\alpha - \beta}\left(\frac{\beta^n - \beta}{\beta - 1}\right) \\
&= \frac{1}{(\alpha - \beta)(1 - p - q)}\left(-\left(A\alpha^*\underline{\alpha}\alpha^n - B\beta^*\underline{\beta}\beta^n\right)\right. \\
&\quad - q\left(A\alpha^*\underline{\alpha}\alpha^{n-1} - B\beta^*\underline{\beta}\beta^{n-1}\right) + q\left(A\alpha^*\underline{\alpha} - B\beta^*\underline{\beta}\right) \\
&\quad \left.+\left(A\alpha^*\underline{\alpha}\alpha - B\beta^*\underline{\beta}\beta\right)\right) \\
&= \frac{-\widehat{Q}_{W,n} - q\widehat{Q}_{W,n-1} + q\widehat{Q}_{W,0} + \widehat{Q}_{W,1}}{1 - p - q}.
\end{aligned}$$

□

Theorem 59. *For nonnegative integers n and r, we have*

$$\sum_{m=0}^{n} \binom{n}{m} q^{n-m} p^m \widehat{Q}_{W,m+r} = \widehat{Q}_{W,2m+r}.$$

Proof. From the Binet form for hyper-dual Horadam quaternions, we have

$$\begin{aligned}
\sum_{m=0}^{n} \binom{n}{m} q^{n-m} p^m \widehat{Q}_{W,m+r} &= \sum_{m=0}^{n} \binom{n}{m} q^{n-m} p^m \left(\frac{A\alpha^* \underline{\alpha} \alpha^{m+r} - B\beta^* \underline{\beta} \beta^{m+r}}{\alpha - \beta} \right) \\
&= \frac{A\alpha^* \underline{\alpha} \alpha^r}{\alpha - \beta} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} (p\alpha)^m - \frac{B\beta^* \underline{\beta} \beta^r}{\alpha - \beta} \sum_{m=0}^{n} \binom{n}{m} q^{n-m} (p\beta)^m \\
&= \frac{A\alpha^* \underline{\alpha} \alpha^r}{\alpha - \beta} (q + p\alpha)^n - \frac{B\beta^* \underline{\beta} \beta^r}{\alpha - \beta} (q + p\beta)^n \\
&= \frac{A\alpha^* \underline{\alpha} \alpha^{2n+r} - B\beta^* \underline{\beta} \beta^{2n+r}}{\alpha - \beta} = \widehat{Q}_{W,2m+r}.
\end{aligned}$$

□

Finally, we give some matrix representations for hyper-dual Horadam quaternions and derive some properties of hyper-dual Horadam quaternions by using matrix approach.

Let's define the matrices $\mathbf{U} := \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$, $\mathbf{W}_n := \begin{bmatrix} W_{n+2} & qW_{n+1} \\ W_{n+1} & qW_n \end{bmatrix}$. It is well-known that for the Horadam numbers, we have the matrix equality:

$$\mathbf{W}_0 \mathbf{U}^n = \mathbf{W}_n. \tag{5.3}$$

For details, see [84]. Now, let's define the matrix $\mathbf{M}_{W,n} := \begin{bmatrix} \widehat{Q}_{W,n+2} & q\widehat{Q}_{W,n+1} \\ \widehat{Q}_{W,n+1} & q\widehat{Q}_{W,n} \end{bmatrix}$. Considering the relation (5.3), we have the matrix equalities

$$\mathbf{M}_{W,0} \mathbf{U}^n = \mathbf{M}_{W,n} \tag{5.4}$$

and

$$\mathbf{M}_{U,0} \left(\mathbf{W}_0 \mathbf{U}^{n-1} \right) = \mathbf{M}_{W,n}, \tag{5.5}$$

which can be proven by using induction.

Theorem 60. *For integers $n, m \geq 1$, we have the following equalities:*

$$\widehat{Q}_{W,m+n} = \widehat{Q}_{W,m+1}U_n + q\widehat{Q}_{W,m}U_{n-1}, \tag{5.6}$$
$$\widehat{Q}_{W,m+n} = W_n\widehat{Q}_{U,m+1} + qW_{n-1}\widehat{Q}_{U,m}. \tag{5.7}$$

Proof. From the matrix equality $\mathbf{M}_{W,0}\mathbf{U}^{m+n-2} = \left(\mathbf{M}_{W,0}\mathbf{U}^{m-1}\right)\mathbf{U}^{n-1}$, we have

$$\begin{bmatrix} \widehat{Q}_{W,m+n} & q\widehat{Q}_{W,m+n-1} \\ \widehat{Q}_{W,m+n-1} & q\widehat{Q}_{W,m+n-2} \end{bmatrix} = \begin{bmatrix} \widehat{Q}_{W,m+1} & q\widehat{Q}_{W,m} \\ \widehat{Q}_{W,m} & q\widehat{Q}_{W,m-1} \end{bmatrix} \begin{bmatrix} U_n & qU_{n-1} \\ U_{n-1} & qU_{n-2} \end{bmatrix}.$$

By equating the corresponding entries of both sides of the matrix equation, we get the identity (5.6).

Now, consider the matrix equality $\mathbf{M}_{U,0}\left(\mathbf{W}_0\mathbf{U}^{m+n-2}\right) = \left(\mathbf{W}_0\mathbf{U}^{n-1}\right)\mathbf{U}^{m-1}\mathbf{M}_{U,0}$. Then we have

$$\begin{bmatrix} \widehat{Q}_{W,m+n+1} & q\widehat{Q}_{W,m+n} \\ \widehat{Q}_{W,m+n} & q\widehat{Q}_{W,m+n-1} \end{bmatrix} = \begin{bmatrix} W_{n+1} & qW_n \\ W_n & qW_{n-1} \end{bmatrix} \begin{bmatrix} \widehat{Q}_{U,m+1} & q\widehat{Q}_{U,m} \\ \widehat{Q}_{U,m} & q\widehat{Q}_{U,m-1} \end{bmatrix}.$$

Similarly, by equating the corresponding entries of both sides of the above matrix equation, we get the desired result in (5.7). □

From the matrix equalities (5.4) and (5.5), one can obtain several results for the hyper-dual Horadam quaternions. For example, if we take the determinant of the both side of this matrix identity (5.4), then we get the Cassini's identity in terms of hyper-dual Horadam quaternions as

$$\widehat{Q}_{W,n-1}\widehat{Q}_{W,n+1} - \widehat{Q}^2_{W,n} = (-q)^{n-1}\left(\widehat{Q}_{W,0}\widehat{Q}_{W,2} - \widehat{Q}^2_{W,1}\right). \tag{5.8}$$

Note that different from the identity (5.2), here the right hand side of the equation (5.8) is expressed in terms of only the hyper-dual Horadam quaternions.

Conclusion and Perspectives

Along this thesis, we first gave some necessary definitions and mathematical preliminaries, which are required. Then, for r a positive integer, we studied genereal-ized bi-periodic r-Fibonacci sequence and defined the family of their companion sequences named the bi-periodic r-Lucas sequence of type s, with s an integer such that $1 \leq s \leq r$. After that, we gave their algebraic properties.

Afterwards, we introduced the bi-periodic Horadam hybrid numbers and gave the generating function, the Binet form, matrix representations and several basic properties of these hybrid numbers such as Catalan's identity, Cassini's identity, etc. In addition, we developed some relationships between the generalized bi-periodic Fibonacci hybrid numbers and the generalized bi-periodic Lucas hybrid numbers. Furthermore, we introduced r-Fibonacci hybrid polynomials and r-Lucas hybrid polynomials as a generalization of the bivariate r-Fibonacci polynomials and bivariate r-Lucas polynomials of type s. We derived several intresting properties. As an application of matrix method, we have derived a generalization of Honsberger formula. Finally, we defined quaternions whose components are hyper-dual Horadam numbers. The main advantage of introducing hyper-dual Horadam quaternions is that many hyper-dual numbers and celebrated numbers such as Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers can be deduced as particular cases of hyper-dual Horadam quaternions. We gave the generating function and the Binet form for these quaternions. With the help of the Binet form of hyper-dual Horadam quaternions, we derived many properties of these quaternions, such as summation formulas, binomial sum identities, Vajda's identity, Catalan's identity, Cassini identity and d'Ocagne's identity. Also, by means of the matrix representation of hyper-dual Horadam quaternions, we examined several identities for these quaternions. The algebra of quaternions is noncommutative, whereas the algebra of hyper-dual numbers is commutative.

Therefore, it was interesting to study a special type of numbers involving both quaternionic and hyper-dual units. For the interested readers, the results could be applied for higher order hyper-dual numbers which were given in [23].

Some challenging questions are part of interest, we put some perspectives which can be studied in our future research works.

1. Extending the sequences $(U_n^{(r)})_n$ and $(V_n^{(r,s)})_n$ defined and studied in the chapter 2 to the negative subscripts.

2. Defining a tri-periodic sequence and study further multi-periodic general cases. For any nonzero real numbers a, b, c, the tri-periodic Fibonacci sequence $(\mathfrak{S}_n)_n$ is defined by

$$\mathfrak{S}_n = \begin{cases} a\mathfrak{S}_{n-1} + \mathfrak{S}_{n-3}, & if \quad n \equiv 0 \pmod 3, \\ b\mathfrak{S}_{n-1} + \mathfrak{S}_{n-3}, & if \quad n \equiv 1 \pmod 3, \\ c\mathfrak{S}_{n-1} + \mathfrak{S}_{n-3}, & if \quad n \equiv 2 \pmod 3, \end{cases}$$

   for $n \geq 3$, with initial conditions $\mathfrak{S}_0, \mathfrak{S}_1, \mathfrak{S}_2$. We are intrested to estabilish its linear recurrence relation, generating function, Binet forms and some properties.

3. Estabilishing other linear recurrent sequences of higher order and define further new numbers, using the matrix representation to study more algebraic properties.

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