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Existence and stability of solutions for certain backward impulsive differential equations on Banach spaces

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"If I have seen further, it is by standing on the shoulders of giants"

Isaac Newton, 1676

Dedication

I dedicate this Doctorate Thesis to those who paved the way

In the memory of my father

To my mother

In the memory of my brother and my sisters

To all my family

To everyone I love and everyone who loves me.

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In the Name of Allah, the Most Gracious, the Most Merciful

All praise and thankfulness is due to Allah, I praise him and testify that there is no God but Allah, and that Mohammed is his slave and messenger (peace be upon him and all prophets and messengers).

Prophet Mohammed (Peace be upon him) said: "*He will not be thankful to Allah, he who would not be thankful to people*" (Corrected and Narrated by Al-Tirmidhi)

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May the Almighty God shower the above cited personalities with success and honour in their life and richly bless all of them.

Existence and stability of solutions for certain backward impulsive differential equations on Banach spaces

Abstract

In this thesis, we are interested by the study of some backward impulsive differential equations with local or nonlocal conditions in Banach spaces. Our main objective is to establish sufficient conditions for the existence and the Ulam stability of solutions to this kind of equations. The approach is to transform the research of solutions to the research of the existence of one or at least one fixed point of a suitable operator. In this process the main instruments are certain well-known fixed point theorems, such as Banach, Schaefer, and Krasnoselskii fixed point theorems, lemmas and theorems on compactness and other results of functional analysis. The study was done in both ordinary and fractional differential equation. Thus, this study can be considered as a comparison between these two types of differential equations. The obtained results in both cases are illustrated by some examples.

Keywords: Backward impulsive differential equations, fractional dérivative, fractional integral, local condition, nonlocal condition, fixed point, Ulam stability.

Mathematics Subject Classification 2020 : 34A08, 34A12, 34A37, 34D20, 34K20, 34K37, 34K45, 26A33.

وجود و استقرار الحلول لبعض المعادلات التفاضلية النبضية الإرتجاعية في فضاءات بناخ

ملخص

نهتم في هذه الأطروحة بدراسة بعض المعادلات التفاضلية النبضية الإرتجاعية مع الشروط المحلية أو غير المحلية في فضاءات بناخ. هدفنا الرئيسي هو إيجاد الشروط الكافية لوجود و استقرار أولام للحلول لهذا النوع من المعادلات التفاضلية. يتمثل النهج في تحويل البحث عن وجود الحلول إلى البحث عن وجود نقطة ثابتة واحدة أو على الأقل واحدة لمؤثر مناسب. في هذه العملية، الأدوات الرئيسية هي بعض نظريات النقطة الثابتة المعروفة مثل نظرية بناخ، نظرية شيفر و نظرية كراسنوزلسكي و نظريات حول التراص و نتائج أخرى من التحليل الدالي. أجريت الدراسة بمعادلات تفاضلية عادية ومعادلات تفاضلية كسرية وبالتالي يمكن اعتبار هذه الدراسة بمثابة مقارنة بين هذين النوعين من المعادلات التفاضلية. النتائج التي تم الحصول عليها في كلتا الحالتين موضحة ببعض الأمثلة.

الكلمات المفتاحية : المعادلات التفاضلية النبضية الإرتجاعية، الإشتقاق الكسري، التكامل الكسري، الشروط المحلية، الشروط غير المحلية، النقطة الثابتة، استقرار أولام.

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Existence et stabilité des solutions pour certaines équations différentielles impulsives rétrogrades dans des espaces de Banach

Résumé

Dans cette thèse, nous nous intéressons à l'étude de certaines équations différentielles impulsives rétrogrades avec des conditions locales ou non locales dans des espaces de Banach. Notre objectif principal est d'établir des conditions suffisantes pour l'existence et la stabilité d'Ulam des solutions de ce type d'équations. L'approche consiste à transformer la recherche de solutions en recherche de l'existence d'un ou d'au moins un point fixe d'un opérateur convenable. Dans ce processus, les principaux instruments sont certains théorèmes de point fixe bien connus, tels que les théorèmes de point fixe de Banach, Schaefer et Krasnoselskii, les lemmes et théorèmes sur la compacité et d'autres résultats d'analyse fonctionnelle. L'étude a été réalisée en équation différentielle ordinaire et fractionnaire. Ainsi, cette étude peut être considérée comme une comparaison entre ces deux types d'équations différentielles. Les résultats obtenus dans les deux cas sont illustrés par quelques exemples.

Mots-clés : Équations différentielles impulsives rétrogrades, dérivée fractionnaire, intégrale fractionnaire, condition locale, condition non locale, point fixe, stabilité d'Ulam.

Classification des matières mathématiques 2020 : 34A08, 34A12, 34A37, 34D20, 34K20, 34K37, 34K45, 26A33.

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Chapter 1

Introduction

Differential equations are used to model dynamical systems in various fields in sciences and engineering. These models, in general, are nonlinear and as such exact solutions are difficult to obtain. Among these equations, the impulsive differential equations, which are the differential equations involving impulse effect, appear as a natural description of observed evolution phenomena of several real world problems. We can meet this kind of problems in many physical phenomena that have a sudden change in their states such as mechanical systems with impact, biological systems such as heart beats, blood flows, population dynamics [8, 63, 64], theoretical physics, radiophysics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology processes, chemistry, engineering, control theory, medicine and so on.

There Are Many Examples

- Operation of a damper, subjected to the percussive effects.

- Change of the valve shutter speed in its transition from open to closed state.

- Fluctuations of pendulum system in the case of external impulsive effects.

- Percussive model of a clock mechanism.

- Percussive systems with vibrations.

- Relaxational oscillations of the electromechanical systems.

- Remittent oscillator, subjected to the impulsive effects.

- Dynamic of a system with automatic regulation.

- The passage of the solid body from a given fluid density to another fluid density.

- Control of the satellite orbit, using the radial acceleration.

- Change of the speed of a chemical reaction in the addition or removal of a catalyst.

- Disturbances in cellular neural networks.

- Impulsive external intervention and optimization problems in the dynamics of isolated populations.

- Death in the populations as a result of impulsive effects.

- Impulsive external interference and the optimization problems in population dynamics of the predator-prey types.

- "Shock" changes of the prices in the closed markets etc.

Adequate mathematical models of such processes are systems of differential equations with impulses and the use of mathematical apparatus in the form of modeling systems impulsive differential equations in all these cases is natural and binding as a rule.

The theory of impulsive differential equations is a new and important branch of differential equations. The first paper in this theory is related to A. D. Mishkis and V. D. Mil'man in 1960 ([54]). The last decades have seen major developments in this theory. In spite of its importance, the development of the theory has been quite slow due to special features possessed by impulsive differential equations in general, such as pulse phenomena, confluence, and loss of autonomy (see for instance, [30]). First and second order ordinary differential equations with impulses have been treated in several works (see [2, 3, 4, 5, 22, 31, 35, 76]). An impulsive differential equation is described by three components: a continuous-time differential equation, which governs the state of the system between impulses, an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs, and a jump criterion, which defines a set of jump events in which the impulse equation is active (see [68]).

Different Classes of Impulsive Differential Equations

There are three classes of impulsive differential equations:

Class 1: Equations with fixed moments of the impulse effect

$$\begin{cases} \frac{du}{dt} = f(t,u), & t \neq t_k \\ \Delta u = I_k(u), & t = t_k \end{cases} \tag{1.1}$$

The impulse is fixed beforehand by defining the sequence $t_k : t_k < t_{k+1} \quad (k \in K \subset Z)$.

For $t \in (t_k, t_{k+1}]$ the solution $u(t)$ of equation (1.1) satisfies the equation $\frac{du}{dt} = f(t,u)$, and for $t = t_k,\ u(t)$ satisfies the relation $I_k(u(t_k^-)) = u(t_k^+) - u(t_k^-)$.

Class 2: Equations with state-dependent moments of the impulse effect

$$\begin{cases} \frac{du}{dt} = f(t,u), & t \neq t_k(u) \\ \Delta u = I_k(u), & t = t_k(u) \end{cases} \tag{1.2}$$

where $t_k : \Omega \to \mathbb{R}$ and $t_k < t_{k+1} \quad (k \in K \subset Z, \quad u \in \Omega)$. The impulse occurs when the mapping point (t,u) meets some hypersurface σ_k of the equation $t = t_k(u)$.

Class 3: Autonomous impulsive equations

$$\begin{cases} \frac{du}{dt} = f(u), & t \notin \sigma \\ \Delta u = I_k(u), & t \in \sigma \end{cases} \tag{1.3}$$

where σ is an $(n-1)-$dimensional manifold contained in the phase space $\mathbb{R}^n$.

The impulse occurs when the solution $u(t)$ meets the manifold σ.

For the basic theory on impulsive differential equations, the reader can refer to the monographs of Lakshmikantham et al. [6] and Benchohra et al. [13]. There are many other good monographs on the impulsive differential equations [71, 74, 78].

Recently, fractional differential equations have been proved to be valuable tools in the modelling of many phenomena in various fields of engineering, physics and economics. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic models. Actually, fractional differential equations are considered as an alternative model to integer differential equations. For more details on fractional calculus theory, one can see the monographs of Diethelm [28], Kilbas et al. [38], Lakshmikantham et al. [40], Miller and Ross [53] and Podlubny [59]. Fractional differential equations (inclusion) and control problems involving the Riemann–Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attention.

Usually, the initial conditions for the differential equations are given in a forward manner, that is, starting at $t = 0$. But for some classes of problems in which the initial state set is unknown the procedure may be more convenient if one considers backward initial conditions, i.e., at $t = T$. This approach plays a vital role in many physical areas. A typical example of such a problem is the backward heat problem (BHP), also known as the final value problem. For application in stochastic differential equations, see, for example, [26].

The main objective of this thesis is to investigate certain fixed point theorems and some functional analysis results to prove the existence and the Ulam stability of solution to some backward differential equations under impulse effect, and in this way to fulfill the gap in the literature about integration of impulsive differential equations of this kind. In this process our main instruments are fixed point theorems, lemmas and theorems on compactness and other results of functional analysis. As far as we know, no papers exist in the literature devoted to such problems.

The thesis consists of six chapters. In Chapter 1, an introdution where we state the problem and a review of related literature. Chapter 2 contains preliminary concepts about necessary theorems and definitions from functional analysis and impulsive differential equations. In Chapter 3, we state a brief historical review of the theory of fractional calculus and its applications. A look at the historical development can in parts explain the absence of this field in today's standard mathematics textbooks on calculus and in addition give the reader not familiar with this field a good access to the topics addressed in this thesis. In this chapter we state some well known definitions and properties of fractional order derivatives and integrals. In chapter 4, new results, about the existence and the Ulam stability to backward impulsive ordinary differential equation. Several subsidiary examples are placed at the end of the chapter. In Chapter 5, new results, about the existence and the Ulam stability to backward impulsive fractional differential equation. Several subsidiary examples are placed at the end of the chapter. The last chapter contains a summary of the thesis and suggests some open problems for further studies.

Chapter 2

Overview of functional analysis

2.1 A few words about function spaces

A *function space* is what its name says it is, a vector space whose elements are functions with domain in some set, values in another. In the most common examples, all functions take either real or complex values. The set of functions is closed under addition (defined as usual) and scalar multiplication (defined as usual), in other words is a real or complex vector space. Because we are in analysis we need to have a notion of convergence for life to make sense so that these spaces are given a topology which may or may not be metric. We will see exclusively or almost exclusively topologies coming from a norm, in other words metric topologies. I'll stick to the real case, but most everything we do will also be valid for complex vector spaces with some obvious modifications.

Function spaces appear now all over in analysis. They play a very big role in the theory of differential equations, in numerical analysis, and in modern physics, for example in quantum mechanics.

2.2 Banach spaces

Definition 2.1 *A Banach space is a complete normed space. In other words, a space in which all Cauchy sequences converge. In even more words, the normed space V is a Banach space if whenever $\{x_n\}$ is a sequence of points of V such that for every $\varepsilon > 0$ there is N such* *that* $\|x_n - x_m\| < \varepsilon$ *whenever* $n, m > N$*, then there exists* $x \in V$ *such that* $\lim\limits_{n\to\infty} \|x_n - x\| = 0$.

The function space we have in mind in this work is $C(K)$. Let K be a compact metric space, then

$$C(K) = \{f : K \to \mathbb{R} /\ f \text{ is continuous}\}$$

The sup-norm in $C(K)$ is defined by

$$\|f\| = \sup_{x\in K} |f(x)| = \max_{x\in K} |f(x)|$$

It is very easy to see that this defines a norm, so $C(K)$ is a normed space. We have the following theorem

Theorem 2.2 *The space* $C([a, b])$ *of continuous, real-valued (or complex-valued) functions on* $[a, b]$ *with the sup-norm is a complete normed space, hence a Banach space. More generally, the space* $C(K)$ *of continuous functions on a compact metric space* K *equipped with the sup-norm is a Banach space.*

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from $J = [0, T]$ into $\mathbb{R}$ endowed with the norm

$$\|u\|_\infty = \sup_{t\in J} |u(t)|$$

The set of piecewise continuous functions

$$PC(J, \mathbb{R}) = \big\{u : J \to \mathbb{R} : u|_{(t_k, t_{k+1})} \in C((t_k, t_{k+1}), \mathbb{R}), k = 0, 1, ..., m,$$
$$u\left(t_k^-\right) = u(t_k) \text{ and } u\left(t_k^+\right) \text{ exists}\big\}$$

is a Banach space with the norm

$$\|u\|_{PC} = \sup_{t\in J} |u(t)|.$$

Define the set $B_r = \{u \in PC(J, \mathbb{R}) : \|u\| \le r\}$.

2.2.1 Bounded operators in a Banach space

Definition 2.3 *Let* X, Y *be Banach spaces (both over* $\mathbb{R}$ *or over* $\mathbb{C}$*). and* $F : D \subset X \to Y$*. The operator* F *is said to be bounded if it maps any bounded subset of* D *into a bounded subset of* Y*.*

2.2.2 Compact operator

Definition 2.4 *An operator $F : X \to Y$, where X, Y are normed vector spaces, is said to be compact if F is continuous and maps bounded parts of X into relatively compact part of Y. (When F is linear, the second condition suffices for it to be bounded, hence continuous).*

Notice that if $\dim(X) = \infty$ and $F : X \to Y$ is invertible, then F is not compact.

Definition 2.5 *Sequential Compactness*

In an Euclidean Space $\mathbb{R}^n$, a set is sequentially compact if and only if every infinite sequence has a convergent subsequence.

The notion of sequential compactness is largely characterized by the Bolzano-Weierstrass theorem. This is stated without proof, considering it to be known.

2.2.3 Completely-continuous operator

Definition 2.6 *Let X, Y be Banach spaces and $F : X \to Y$. The operator F is said to be completely continuous if it is continuous and for all bounded subset $D \subset X$ then $\overline{F(D)}$ is compact.*

Remark 2.7 *If F is linear then F completely continuous if and only if F is compact.*

Theorem 2.8 *(1) If the operators $F_1, F_2 : D \subset X \to Y$ are bounded (respectively, completely continuous) then for every $\alpha, \beta \in \mathbb{R}$ the operator $\alpha F_1 + \beta F_2$ is bounded (respectively, completely continuous).*
(2) Let X, Y, Z be Banach spaces and F_1, F_2 be two operators which are defined as follows:

$$F_1 : D_1 \to F_1(D_1) \subset D_2,\ F_2 : D_2 \to Z,\ D_1 \subset X, D_2 \subset Y.$$

If both operators F_1, F_2 are bounded then the composite operator F_2F_1 is also bounded. If one of the operators F_1, F_2 is bounded continuous and the other one is completely continuous, then F_2F_1 is completely continuous.

Proof. Both statements (1) and (2) are simple consequences of definitions. For (2) also use the fact that a continuous operator maps relatively compact sets into relatively compact sets. ■

As is actually true in every infinite dimensional space, the Heine Borel theorem fails to hold. That is, there are closed, bounded subsets that are not compact, in fact every set with non empty interior fails to be compact. We can prove that $C(K)$ is infinite dimensional.

Consider the vector space $C[a, b]$ consisting of continuous functions defined on the interval $a < t < b$. $C[a, b]$ is infinite-dimensional because it cannot be spanned by a finite number of linearly independent functions.

We can for instance observe that $C[a, b]$ contains the subspace of all polynomials, which is infinite-dimensional.

As it turns out, the Heine Borel theorem is ALWAYS false in an infinite dimensional normed vector space. But for many applications it is quite important to know exactly which are the compact subsets of a given normed vector space, the Ascoli-Arzela Theorem provides an answer for $C(K)$.

In this chapter, we begin to discuss the ways in which $\mathbb{R}^n$ differs from $C(K)$. In particular, we compare the characterization of compact subsets of $\mathbb{R}^n$ by Heine-Borel with the characterization of compact subsets of $C(K)$ by Ascoli-Arzela. We find that subsets of $C(K)$ must satisfy more conditions than subsets of $\mathbb{R}^n$ if they are to be compact.

Before stating the version that is most commonly used in applications, we recall two famous Analysis Theorems.

Theorem 2.9 (Bolzano-Weierstrass theorem) *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

That is, if a subset $A \in \mathbb{R}^n$ is closed and bounded, it is sequentially compact.

Theorem 2.10 (Heine-Borel theorem) *In $\mathbb{R}$ or more generally $\mathbb{R}^n$, K compact if and only if K is closed and bounded.*

Do these theorems hold in the space $C[a, b]$ of real valued continuous functions with domain $[a, b]$.

We know that in $\mathbb{R}^n$, closed and bounded sets are compact. Unfortunately, this is not true in $C([a,b])$.

In both situations, if we add the condition called equicontinuity, then both theorems hold in $C[a,b]$.

Definition 2.11 *Let $\{f_n\}_1^\infty$ be a sequence of $\mathbb{R}$-valued continuous functions on a compact set E.*

1. *$\{f_n\}$ is pointwise bounded on E if for each $x_0 \in E$ the sequence of numbers $\{f_n(x_0)\}$ is bounded.*
2. *$\{f_n\}$ is uniformly bounded on E if there exists an $M \in \mathbb{R}$ such that $f_n(x) \leq M$, $\forall n \in \mathbb{N}$ and $\forall x \in E$.*

Definition 2.12 *Let $\mathcal{F}$ a family of functions $E \to \mathbb{R}$ (equally well, we can take the values to lie in $\mathbb{C}$), where E subset of a metric space $(\mathcal{X}, d)$. We say $\mathcal{F}$ is equicontinuous on E if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for all $x, y \in E$ with $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{F}$.*

This is the strongest sense of continuity. It is very similar to the definition for uniform continuity of a specific function f in A. The only difference is that, for a given ε, we must choose a $\delta > 0$ which works for all possible functions f in A. That is, we must choose the δ before we are allowed to look at the function, making this a property of a set rather than a function, whereas in proving uniform continuity, we choose our δ for a specific function.

Very roughly a set will be equicontinuous if the functions in the set do not have too many oscillations. Also very roughly, if we are working on a compact domain (as we will) there isn't enough room for a continuous function to have more than a finite number of oscillations, except if these oscillations get smaller and smaller. But if you have an infinite number of functions on a compact set, you can get each function to be more oscillatory than a preceding one, and end with a non-equicontinuous set.

Examples

Example 2.1 *Let* $\varphi(x) = \sin x, \quad 0 \leq x \leq \pi$ *and* $\varphi(x) = 0, \quad x \geq \pi$.

i) The functions $f_n(x) = n\varphi(nx)$ *converge pointwise to zero and are pointwise bounded on* $I = [0, \pi]$*, but neither uniformly bounded nor equicontinuous on* $I = [0, \pi]$.

ii) The functions $g_n(x) = \varphi(nx)$ are uniformly bounded but they are not equicontinuous on $I = [0, \pi]$.

iii) The functions $h_n(x) = n$ are equicontinuous but not uniformly bounded on $I = [0, \pi]$.

Example 2.2 *Suppose that* $M > 0$. *Let*

$$\mathcal{A} = \left\{ f \in C^1[0,1] \,/\, \|f'\|_{\sup} \leq M \right\}.$$

Then $\mathcal{A}$ *is equicontinuous.*

In effect, given $\varepsilon > 0$. Take $\delta = \frac{\varepsilon}{M}$. For $f \in \mathcal{A}, \quad |s - t| < \delta$.

By the mean value theorem, there exist τ between s and t such that

$$|f(s) - f(t)| = |f'(\tau)|\,|s - t| \leq M\,|s - t| < M\frac{\varepsilon}{M} = \varepsilon.$$

Example 2.3 *Give a subset of some function space which is closed and bounded but not compact.*

In effect, let

$$\mathcal{B} = \left\{ f \in C[0,1] \,/\, \|f\|_{\sup} \leq 1 \right\}$$

The set is clearly bounded. Since $\|.\|_{\sup} : C[0,1] \to [0, \infty]$ is continuous and $[0,1]$ is closed in $[0, \infty]$, $(\|.\|_{\sup})^{-1}([0,1]) = \mathcal{B}$ is closed.

Let

$$f_n(x) = x^n, x \in [0,1].$$

Then $f_n \in \mathcal{B}$. This sequence converges pointwise to the function

$$f(x) = \begin{cases} 0, & \text{if } x \in [0,1[\\ 1, & \text{if } x = 1 \end{cases}$$

But f is not even in $C[0,1]$, hence $\{f_n\}_{n=1}^{\infty}$ has no convergent subsequences which implies that $\mathcal{B}$ is not sequentially compact, and by Bolzano-Weierstrass theorem, it is not compact.

The Arzela-Ascoli theorem (or Ascoli-Arzela theorem, or Montel's theorem, or . . .) is one of the most useful theorems in functional analysis, because it's one of the easiest ways to produce a convergent sequence of functions. Aside from its numerous applications to Partial Differential Equations, it is also used as a tool in obtaining functional analysis results, such as the compactness for duals of compact operators, as presented for example [69]. The Ascoli-Arzela theorem gives necessary and sufficient conditions to decide whether every sequence of a given family of real-valued continuous functions defined on a closed and bounded interval has a uniformly convergent subsequence. The main condition is the equicontinuity of the family of functions. The theorem is a fundamental result in mathematics. In particular, it forms the basis for the proof of the Peano existence theorem in the theory of ordinary differential equations and Montel's theorem in complex analysis. It is the product of two mathematicians, Arzela and Ascoli[(1)], who were studying both equicontinuity and compactness at a similar time. It was first proven in a weaker form by Ascoli in 1883 then later, in 1893, the proof was completed by Arzela. It is a very important technical result, used in many branches of mathematics.

Until now, the Ascoli-Arzela theorem has been written in multiple ways. What matters is a good grasp of the central idea, and good judgment which version to use in which application. For example, textbooks in differential equations, even at otherwise quite sophisticated levels, often only state a simple version adapted to the specific needs. Here, we state and compare two versions of the Ascoli-Arzela theorem and then we state an advanced version that we will use later.

Theorem 2.13 (Ascoli-Arzela theorem version I) *Let $\{f_n\}_1^{\infty}$ be a sequence of $\mathbb{R}$-valued continuous functions on a compact set K. That is $f_n \in C(K)$ $\forall n \in \mathbb{N}$, $\{f_n\}$ is pointwise*

[(1)] Giulio Ascoli (1843–1896) and Cesare Arzela (1847–1912) were both Italian mathematicians *bounded on K and $\{f_n\}$ is equicontinuous implies that $\{f_n\}$ is uniformly bounded and $\{f_n\}$ has a uniformly convergent subsequence.*

Theorem 2.14 (Ascoli-Arzela theorem version II) *Let A be a family of functions in $C[K]$, where K is compact. Then A is compact if and only if A is closed, bounded, and equicontinuous.*

Proof. Assume S is closed, bounded and equicontinuous. By the version I of the Arzela-Ascoli Theorem, if $\{f_n\}$ is a sequence in S, it has a convergent subsequence. Because S is closed, the limit of the subsequence must be in S. Thus S is sequentially compact, hence compact. Conversely, assume S is compact. Then, of course, it is closed and bounded. To see it is equicontinuous, let $\varepsilon > 0$. There exist then, by compactness, a finite number of functions $f_1, ..., f_m \in S$ such that $S \subset \cup_{k=1}^{m} B\left(f_k, \frac{\varepsilon}{3}\right)$. Because we have only a finite number of functions, there is a common $\delta > 0$ such that $|x-y| < \delta$ implies $|f_k(x) - f_k(y)| < \frac{\varepsilon}{3}$ for all $x, y \in [a,b]$, $|x-y| < \varepsilon$. If now $x, y \in [a,b]$ and $|x-y| < \delta$ then, if $f \in S$, there will be k such that $f \in B(f_k, \frac{\varepsilon}{3})$. Then

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_k(x) + f_k(x) - f_k(y) + f_k(y) - f(y)| \\ &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\ &\leq \|f - f_k\| + |f_k(x) - f_k(y)| + \|f_k - f\| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

wich complete the proof of the theorem. ■

Theorem 2.15 (PC type-Ascoli-Arzela theorem[76]) *Let E be a Banach space and $W \subset PC(J,E)$ be such that*

(i) W is uniformly bounded subset of $PC(J,E)$,

(ii) W is equicontinuous in (t_k, t_{k+1}), $k = 0, 1, 2, \cdots, m$, where $t_0 = 0$ and $t_{m+1} = T$,

(iii) $W(t) = \{u(t)|\ u \in W,\ t \in J \backslash \{t_1, \cdots, t_m\}\}$, $W(t_k^+) = \{u(t_k^+)|u \in W\}$ and $W(t_k^-) = \{u(t_k^-)|\ u \in W\}$ are relatively compact subsets of E.

Then W is a relatively compact subset of $PC(J,E)$.

2.3 Fixed point

One of the most important instrument to treat nonlinear problems with the aid of functional analysis methods is the fixed point approach. This approach is an important part of functional analysis and is deeply connected to geometric methods of topology.

Theorem concerning the existence and properties of fixed points are known as fixed point theorem. There are many varieties of fixed point theorems. Some gives conditions for uniqueness or multiplicity of solutions. Fixed point theory is a beautiful mixture of analysis, topology and geometry. In particular fixed point theorem has been applied in such field as mathematics engineering, physics, economics, game theory, biology and chemistry etc. We consider in this thesis the famous theorems of Banach, Schaefer and Krasnoselskii. A more detailed description of the fixed point theory can be found for instance in [1, 32, 36].

Definition 2.16 *Let X be a set and let $F : X \to X$ be a function that maps X into itself. (Such a function is often called an operator, a transformation, or a transform on X, and the notation $F(x)$ or even Fx is often used). A fixed point of F is an element $x \in X$ for which $F(x) = x$.*

Remark 2.17 *Note that the definition of a fixed point requires no structure on either the set X or the function F.*

Notation 2.18 (Fixed point set) *The set $Fix(F)$ is the set of fixed points of the operator $F : X \to X$, that is,*

$$Fix(F) = \{x \in X \mid x = Fx\}.$$

For example, if $F = I$ is the identity, then $Fix(I) = X$. If $T = -I$, then $Fix(-I) = \{0\}$. In the above definition, X is any general space (topological, metric, normed, etc.), but in the theorem below, it must be a complete metric space.

Definition 2.19 (Contraction) *Let X be a metric space, and $f : X \to X$. We will say that f is a contraction if there exists some $0 < c < 1$ such that $d(f(x), f(y)) < cd(x, y)$ for all $x, y \in X$. The inf of such c's is called the contraction coefficient.*

We will also refer to the case $c \leq 1$ as being non-expansive.

Theorem 2.20 *Every contraction mapping is continuous.*

Proof. Let $F : X \to X$ be a contraction on a metric space (X, d), with modulus β, and let $\bar{x} \in X$. Let $\varepsilon > 0$, and let $\delta = \varepsilon$. Then $d(x, \bar{x}) < \delta \Rightarrow d(Fx, F\bar{x}) \leq \beta\delta < \varepsilon$. Therefore F is continuous at $\bar{x}$. Since $\bar{x}$ was arbitrary, F is continuous on X. ■

The Banach Fixed Point theorem is also called the contraction mapping theorem, and it is in general use to prove that a unique solution to a given equation exists. There are several examples of where Banach Fixed Point theorem can be used in Economics for more detail you can check ([1, 32]). For concreteness purposes let focus in one of the most known applications for Banach's theorem for economists, Bellman's functional equations.

Theorem 2.21 (Schauder's fixed point theorem) *Assume that K is a convex compact set in a Banach space X and that $F : K \to K$ is a continuous mapping. Then F has a fixed point.*

Theorem 2.22 (Banach fixed point theorem) [(1)] *Let $F : X \to X$ and let X be a complete metric space. If F is a strict contraction, then $Fix(F)$ consists of exactly one element x.*

The following short proof of Banach's Fixed Point Theorem was given by Richard S. Palais in 2007; see [57].

Proof. Let α denote the contraction constant of T. Then, according to the triangle inequality,

$$\begin{aligned} d(x_1, x_2) &\leq d(x_1, F(x_1)) + d(F(x_1), F(x_2)) + d(x_2, F(x_2)) \\ &\leq d(x_1, F(x_1)) + \alpha d(x_1, x_2) + d(x_2, F(x_2)), \end{aligned}$$

which means that

$$d(x_1, x_2) \leq \frac{d(x_1, F(x_1)) + d(x_2, F(x_2))}{1 - \alpha} \tag{2.1}$$

for all points $x_1, x_2 \in X$. This inequality immediately implies that F cannot have more than one fixed point.

[(1)] Contraction mapping theorem, due to Banach 1922; also known as Banach-Picard. It is the simplest fixed point theorem and is unusual because it is constructive, and implies the implicit function theorem.

Let F^n denote the composition of F with itself n times. It is easy to show that F^n is a contraction with contraction constant α^n. If we now apply (2.1) to the points $x_1 = F^m(x_0)$ and $x_2 = F^n(x_0)$, where $x_0 \in X$ is arbitrary, we obtain that

$$\begin{aligned} d(F^m(x_0), F^n(x_0)) &\leq \frac{d(F^m(x_0), F^m(F(x_0))) + d(F^n(x_0), F^n(F(x_0)))}{1-\alpha} \\ &\leq \frac{\alpha^m + \alpha^n}{1-\alpha} d(x_0, F(x_0)). \end{aligned}$$

Since $0 \leq \alpha < 1$, this implies that the sequence $(F^n(x_0))_{n=1}^{\infty}$ is a Cauchy sequence and therefore that $F^n(x_0) \to x$ for some $x \in X$. Finally, because F is continuous, then we can write $F(x) = F(\lim_{n\to\infty} F^n(x_0)) = \lim_{n\to\infty} F^{n+1}(x_0) = x$, so x is a fixed point of F. ■

Note that the proof of uniqueness did not require that the space be complete.

Theorem 2.23 (Schaefer's fixed point theorem) *Let E be a Banach space and $U \subset E$ a convex set such that $0 \in U$. Let F be an operator defined on E such that $F : U \to U$ is completely continuous. If*

$$\Omega = \{u \in U : u = \lambda F u,\ \lambda \in]0,1[\}$$

is bounded, then F admits at least one fixed point in E.

Proof. By hypothesis, we can choose a constant M so large that

$$\|x\| < M \text{ if } x = \lambda F(x) \text{ for some } \lambda \in [0,1].$$

Define a retraction $r : X \to B(0; M)$ by

$$r(x) = \begin{cases} x & \text{if } \|x\| \leq M \\ (M/\|x\|)\, x & \text{if } \|x\| > M \end{cases}$$

and observe that the composition $(r \circ F) : B(0;M) \to B(0;M)$ is compact since F is compact. Let K denote the closed convex hull of $(r \circ F)(B(0;M))$. The set K is convex by definition, and the compactness of $r \circ F$ implies K is compact. By Schauder's fixed point theorem, there exists a fixed point $x \in K$ of the restriction $(r \circ F)|_K : K \to K$. We claim that x is also a fixed point of F. To show this, it is sufficient to prove that $F(x) \in K$. Suppose not. Then $\|F(x)\| > M$ and

$$x = r(F(x)) = \frac{M}{\|F(x)\|} F(x) \tag{2.2}$$

which implies

$$\|x\| = \left\| \frac{M}{\|F(x)\|} F(x) \right\| = M.$$

On the other hand, $M/\|F(x)\| \in (0,1)$, so our choice of M and (2.2) also imply $\|x\| < M$, a contradiction. ■

Two main results of fixed-point theory are Schauder's theorem and the contraction mapping principle. Krasnoselskii combined them into the following result.

Theorem 2.24 (Krasnoselskii fixed point theorem) *Let S be a closed convex nonempty subset of a Banach space E. Let P and Q be two operators satisfying the following conditions:*

1. *$Px + Qy \in S$ whenever $x, y \in S$,*
2. *P is a contraction mapping,*
3. *Q is compact and continuous.*

Then there exist $z \in S$ such that $z = Pz + Qz$, i.e., the operator $P + Q$ admits a fixed point on S.

Lemma 2.25 (Gronwall inequality [10]) *Let for $F \geq t_0 \geq 0$ the following inequality holds*

$$x(t) \leq a(t) + b\int_{t_0}^{t} x(s)\, ds + \sum_{t_0 < t_k < t} \beta_k x(t_k),$$

where $x, a \in PC([t_0, \infty), \mathbb{R}^+)$, a is nondecreasing and b, $\beta_k > 0$. Then, for $t \geq t_0$, the following inequality is valid:

$$x(t) \leq a(t) \prod_{t_0 < t_k < t} (1 + \beta_k) \exp\left(\int_{t_0}^{t} b(s)\, ds\right).$$

For more integral inequalities of Gronwall type for piecewise continuous functions, see [9, 10].

Chapter 3

Background to fractional calculus

3.1 Historical overview

The concept of derivative is the main idea of calculus. It shows the sensitivity to change of a function i.e. the rate or slope of a quantity. The current definition of derivative was suggested by Newton in 1666. Newton with a physical viewpoint of derivative interpreted the instantaneous velocity [14]. The intuition of researchers from derivative and integral is based on their geometrical or physical meaning, for example the first and second order derivative of displacement is called velocity and acceleration respectively, also jerk and jounce for 3rd and 4th derivatives.

The intuitive idea of fractional calculus dates back to September 30, 1695. On that day, Leibniz wrote a letter to the French mathematician Guillaume de L'Hôpital, raising the following question: "*Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?*". L'Hôpital was some what curious about that question and replied by another question to Leibniz: "*What if the order will be* $\frac{1}{2}$*?* ". Leibniz replied that "*It will lead to a paradox, from which one day, useful consequences will be drawn*"

The question raised by Leibnitz for a fractional derivative was an ongoing topic for more than 300 years. Many known mathematicians contributed to this theory over the years, among them L. Euler in 1730 he mentioned interpolating between integral orders of a derivative, he,wrote: "*When* n *is a positive integer and* p *is a function of* x, $p = p(x)$, *the ratio of* $d^n p$ *to* dx^n *can always be expressed algebraically. But what kind of ratio can then be made if* *n be a fraction?"*. In 1812 Laplace defined a fractional derivative by means of an integral, and the first discussion of a derivative of fractional order appeared in a calculus written by Lacroix in 1819, and it has been developed progressively up to now. Within years the fractional calculus became a very attractive subject to mathematicians, includes P.S. Laplace (1812), S. F. Lacroix (1819), J. B. J. Fourier (1822), N. H. Abel (1823–1826), J. Liouville (1832–1873), B. Riemann (1847), H. Holmgren (1865–1867), A. K. Grunwald (1867–1872), A. V. Letnikov (1868–1872), H. Laurent (1884), P. A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892–1912), S. Pincherle (1902), G. H. Hardy and J. E. Littlewood (1917-1928), H. Weyl (1917), P. Lévy (1923), A. Marchaud (1927), H. T. Davis (1924-1936), E. L. Post (1930), A. Zygmund (1935-1945), E. R. Love (1938-1996), A. Erdelyi (1939- 1965), H. Kober (1940), D. V. Widder (1941), M. Riesz (1949), W. Feller (1952). Caputo (1967), (Hilfer 2000, Kilbas et al. 2006, Podlubny 1999, Samko et al. 1993) and the more recent notions of Cresson (2007), Katugampola (2011), Klimek (2005), Kilbas and Saigo (2004) or variable order fractional operators introduced by Samko and Ross (1993).

In 2010, an interesting perspective to the subject, unifying all mentioned notions of fractional derivatives and integrals, was introduced in Agrawal (2010) and later studied in Bourdin et al. (2014), Klimek and Lupa (2013), Odzijewicz et al. (2012a, b, 2013a, b, c). Precisely, authors considered general operators, which by choosing special kernels, reduce to the standard fractional operators. However, other nonstandard kernels can also be considered as particular cases. Note that There are more than FIFTEEN definitions of fractional derivative operator.

Only since the Seventies has fractional calculus been the object of specialized conferences and treatises. For the first conference the merit is due to B. Ross who, shortly after his Ph.D. dissertation on fractional calculus, organized the First Conference on Fractional Calculus and its Applications at the University of New Haven in June 1974, and edited the proceedings. For the first monograph the merit is ascribed to K. B. Oldham and I. Spanier who, after a joint collaboration begun in 1968, published a book devoted to fractional calculus in 1974.

The application of fractional derivatives and integrals has been infrequent until recently. However, in recent years, advances in the theory of chaos and fractals revealed relationships with fractional derivatives and integrals, leading to renewed interest in this field.

Insofar as it concerns the application of its concepts, we can cite research in different areas such as viscoelastic damping [42, 52, 67], robotics and control [19, 41], signal processing [24], and electric circuits [81].

Some work has been carried out in the field of dynamical systems theory, but the proposed models and algorithms are still in the preliminary stage. It has been shown that the fractional order models of real systems are regularly more adequate than usually used integer order models.

Historically, fractional order calculus has been unexplored or its applications delayed in engineering because of its inherent complexity, the apparent selfsufficiency of integer order calculus, and the fact that it lacks a fully acceptable geometric or physical interpretation.

With these ideas in mind, this paragraph presents preliminary definitions and facts of fractional operators.

3.2 Elements of fractional calculus

Probably the most useful advance in the development of fractional calculus was due to a paper written by G. F. Bernhard Riemann during his student days. Unfortunately, the paper was published only posthumously in 1892. Seeking to generalize a Taylor series in 1853, Riemann derived different definition that involved a definite integral and was applicable to power series with non-integer exponents

$$D_{c,x}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{c}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau + \varphi(t)$$

Due to the ambiguity in the lower limit of integration c, Riemann added to his definition a "complementary" function $\varphi(t)$ where the present-day definition of fractional integration is without the troublesome complementary function. Since neither Riemann nor Liouville solved the problem of the complementary function, it is of historical interest how today's Riemann-Liouville definition was finally deduced.

The earliest work that ultimately led to what is now called the Riemann-Liouville definition appears to be the paper by N. Y. Sonin in 1869, [73] where he used Cauchy‘s integral formula as a starting point to reach differentiation with arbitrary index. A. V. Letnikov extended the idea of Sonin a short time later in 1872.

Finally, Laurent used a contour given as an open circuit (known as Laurent loop) instead of a closed circuit used by Sonin and Letnikov and thus produced today's definition of the Riemann-Liouville fractional integral

$$ {}_{RL}I_{a,x}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \mathrm{Re}(\alpha) > 0. \tag{3.1} $$

This expression is the most widely utilized definition of the fractional integration operator in use today. By choosing $a = 0$ in this one obtains the Riemann's formula without the problematic complementary function $\varphi(t)$ and by choosing $a = -\infty$, it is equivalent to Liouville's first definition. These two facts explain why equation (3.1) is called Riemann-Liouville fractional integral.

The two most frequently used definitions for the general fractional calculus are: the Grunwald-Letnikov (GL) and the Riemann-Liouville definitions. Also, the Caputo derivative, as a variation of the Riemann-Liouville differential operator, is used frequently.

The fractional Riemann-Liouville integral of the order α for the function $f(t)$ for $\alpha, a \in \mathbb{R}$ can be expressed as follows

$$ {}_{RL}I_a^{\alpha} f(t) = {}_{RL}D_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau. $$

Moreover, the left and the right Riemann-Liouville fractional integral are defined respectively as :

Definition 3.1 *The fractional left integral in the sense of Riemann-Liouville of order* $\alpha \in \mathbb{R}_+$ *of a function* $f : [a,b] \to \mathbb{R}$ *or* $\mathbb{C}, -\infty < a < b < +\infty$ *is formally defined by:*

$$ {}_{RL}I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \mathrm{Re}(\alpha) > 0, $$

and the right one is defined by

$$ {}_{RL}I_{b^-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\, d\tau, \quad \mathrm{Re}(\alpha) > 0. $$

We recall that all the definitions and properties we will use are "left", the right versions are rarely used because they depend on the future of the functions.

Example 3.1 *The Power function is very important in Mathematics since many functions can be derived from an infinite power series. First we will use the Riemann-Liouville fractional integral to compute the integral of order* $\alpha \in \mathbb{R}_+$ *of the power function* $(t-a)^{\beta}$.

Plugging this into the equation gives

$$_{RL}I_a^{\alpha} f\left(t\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_a^t \left(t-s\right)^{\alpha-1} f\left(s\right) ds.$$

If we make the substitution $\frac{s-a}{t-a} = \xi$ *from which it follows that* $ds = (t-a)d\xi$ *and the new interval of integration is* $[0,1]$, *we can rewrite the last expression as*

$$\begin{aligned} _{RL}I_a^{\alpha}(t-a)^{\beta} &= \frac{(t-a)^{\beta+\alpha}}{\Gamma\left(\alpha\right)} \int_0^1 \left(1-\xi\right)^{\alpha-1} \xi^{\beta} d\xi \\ &= \frac{(t-a)^{\beta+\alpha}}{\Gamma\left(\alpha\right)} B\left(\alpha, \beta+1\right) \\ &= \frac{\Gamma(\beta+1)}{\Gamma\left(\beta+\alpha+1\right)} (t-a)^{\beta+\alpha}, \quad \beta > -1. \end{aligned}$$

One can define the Riemann-Liouville fractional-order derivatives of a function f as well by :

Definition 3.2 *The left Riemann-Liouville fractional derivative is defined as*

$$_{RL}D_{a,t}^{\alpha} f\left(t\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \frac{d^n}{dt^n} \int_a^t \left(t-\tau\right)^{n-\alpha-1} f\left(\tau\right) d\tau.$$

and the right Riemann-Liouville fractional derivative is defined as

$$_{RL}D_{t,b}^{\alpha} f\left(t\right) = \frac{\left(-1\right)^n}{\Gamma\left(n-\alpha\right)} \frac{d^n}{dt^n} \int_t^b \left(\tau-t\right)^{n-\alpha-1} f\left(\tau\right) d\tau.$$

where $n-1 < \alpha < n$, a, b *are the terminal points of the interval* $[a,b]$, *which can also be* $-\infty, \infty$. *In the very important case of* $\alpha \in (0,1)$ *the above definition of the left Riemann-Liouville fractional derivative is reduced to*

$$_{RL}D_{a,t}^{\alpha} f\left(t\right) = \frac{1}{\Gamma\left(1-\alpha\right)} \frac{d}{dt} \int_a^t \left(t-\tau\right)^{-\alpha} f\left(\tau\right) d\tau.$$

A very important fact is that for integer values of order α the Riemann-Liouville derivative coincides with the classical, integer order one.

Example 3.2 *Now we will compute the derivative of order* $\alpha \in \mathbb{R}_+$ *of the same power function* $(t-a)^{\beta}$ *using the Riemann-Liouville fractional derivative.*

Again, as in the previous example, filling in $f(t) = (t-a)^{\beta}$ *gives*

$$\begin{aligned} _{RL}D_{a,t}^{\alpha}(t-a)^{\beta} &= \frac{1}{\Gamma\left(n-\alpha\right)} \frac{d^n}{dt^n} \int_a^t \left(t-\tau\right)^{n-\alpha-1} (t-a)^{\beta} d\tau \\ &= \frac{d^n}{dt^n} \left(I_a^{n-\alpha}(t-a)^{\beta}\right). \end{aligned}$$

Now we are able to use the integral of the power function we have just computed previously. If we replace the order α *by* $n-\alpha>0$ *we can rewrite the last expression as*

$$\begin{aligned} {}_{RL}D^{\alpha}_{a,t}(t-a)^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma\left(\beta+n-\alpha+1\right)}(t-a)^{\beta+n-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma\left(\beta-\alpha+1\right)}(t-a)^{\beta-\alpha}, \quad \beta>-1. \end{aligned}$$

If we put $\beta=0$ *in the previous equality, we get*

$${}_{RL}D^{\alpha}_{a,t}(1) = \frac{(t-a)^{-\alpha}}{\Gamma\left(1-\alpha\right)}.$$

Than the Riemann-Liouville derivative of a constant is not zero! It may be infinite if $t=a$.

Among the most significant modern contributions to fractional calculus are those made by the results of M. Caputo in 1967 [20]. He reformulated the more "classic" definition of the Riemann-Liouville fractional derivative in order to use classical initial conditions, then he gave the definition of fractional derivative and it is also known as second popular definition of the fractional derivatives.

$${}^{C}D^{\alpha}_{a}f\left(x\right) = \frac{1}{\Gamma\left(n-\alpha\right)}\int_a^x \left(x-t\right)^{n-\alpha-1}\left(\frac{d}{dt}\right)^n f\left(t\right)dt, \quad n-1<\alpha<n$$

This derivative is strongly connected to the Riemann-Liouville fractional derivative and is today frequently used in applications. It is interesting to note that Rabotnov [62] introduced the same differential operator into the Russian viscoelastic literature a year before Caputo's paper was published. Regardless of this fact, the proposed operator is in the present-day literature commonly named after Caputo. The left and the right Caputo fractional integral are defined respectively as :

Definition 3.3 *The left Caputo fractional derivative of order* $\alpha\in\mathbb{R}_+$ *of a function* $f:[a,b]\to\mathbb{R}$ *or* $\mathbb{C}, -\infty<a<b<+\infty$ *is formally defined by:*

$${}^{C}D^{\alpha}_{a,t}f\left(t\right) = \frac{1}{\Gamma\left(n-\alpha\right)}\int_a^t \left(t-\tau\right)^{n-\alpha-1} f^{(n)}\left(\tau\right)d\tau.$$

and the right Caputo fractional derivative is

$${}^{C}D^{\alpha}_{t,b}f\left(t\right) = \frac{(-1)^n}{\Gamma\left(n-\alpha\right)}\int_t^b \left(\tau-t\right)^{n-\alpha-1} f^{(n)}\left(\tau\right)d\tau. \tag{3.2}$$

where $f^{(n)}\left(\tau\right)=\frac{d^n f(\tau)}{d\tau^n}$ *and* $n-1<\alpha<n\in\mathbb{Z}^*$. *It is obvious from the definition 3.2 that the Caputo fractional derivative of a constant is zero.*

Example 3.3 *Let's take the example of the same power function* $(t-a)^{\beta}$ *as in the previous examples, than*

$$
\begin{aligned}
{}^{C}D_{a}^{\alpha}(t-a)^{\beta} &= \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{n-\alpha-1}\left(\frac{d}{dt}\right)^{n}(t-a)^{\beta}dt \\
&= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-\alpha+1)}\int_{a}^{x}(x-t)^{n-\alpha-1}(t-a)^{\beta-\alpha}dt \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)}\frac{\Gamma(\beta+n-\alpha+1)}{\Gamma(\beta+1-\alpha)}(x-a)^{\beta+n-\alpha} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-a)^{\beta-\alpha}.
\end{aligned}
$$

It is a well-known fact that the classical derivative is the left inverse of the classical integral. The similar relation holds for the Riemann-Liouville derivative and integral

$$
{}_{RL}D_{a,t}^{\alpha}f(t)_{RL}\, I_{a,t}^{\alpha}f(t) = f(t).
$$

The opposite, however, is not true (in both the fractional and integer order case)

$$
{}_{RL}I_{a,t}^{\alpha}f(t)_{RL}\, D_{a,t}^{\alpha}f(t)_{RL} = f(t) - \sum_{j=1}^{n}\frac{f_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha-j+1)}(t-a)^{\alpha-j}. \tag{3.3}
$$

Utilizing expression (3.3), similar expressions can be obtained relating the Riemann-Liouville integral and derivative of Caputo type. In particular, assuming that the integrand is continuous or, at least, essentially bounded function, Caputo derivative is also the left inverse of the fractional integral.

It is rather important to notice that the Caputo and the Riemann-Liouville formulations coincide when the initial conditions are zero. In fact, assuming that all initial conditions are zero, a number of relations between the fractional order operators is greatly simplified. In such a case, both fractional integral and fractional derivatives possess the semi-group property.

The following result is very interesting and it is widely used in the study of fractional order differential equations.

Lemma 3.4 ([80]) *Let* $u \in C^{n}[0,1]$. *then*

$$
I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1}t^{n-1}
$$

where $n = [\alpha]+1$ *and* $c_i \in \mathbb{R}$ $i = 0, 1, \ldots, n-1$.

3.3 Main properties of fractional calculus

The main properties of fractional integrals and derivatives are:

1. For the fractional integral $I_a^\alpha f(t)$ of a time function f to exist, it suffices that f be piecewise continuous over $]t_0, +\infty[$ and integrable over $[t_0, t]$ for all $t > t_0$.

2. The integral and derivative operators are linear i.e. $I_a^\alpha\left(\lambda f(x) + \mu g\left(x\right)\right) = \lambda I_a^\alpha f(x) + \mu I_a^\alpha g\left(x\right)$ and $D_a^\alpha\left(\lambda f(x) + \mu g\left(x\right)\right) = \lambda D_a^\alpha f(x) + \mu D_a^\alpha g\left(x\right)$. Linearity follows from just filling in the definitions of the fractional derivatives and integrals.

3. Similar to the classical integer-order integral, the Riemann-Liouville fractional integral satisfies the semigroup property, i.e. for any positive orders α and β

$$_{RL}I_{a,t}^{\alpha} f\left(t\right)_{RL}\,I_{a,t}^{\beta} f\left(t\right) =_{RL} I_{a,t}^{\beta} f\left(t\right)_{RL}\,I_{a,t}^{\alpha} f\left(t\right) = I_{a,t}^{\alpha+\beta} f\left(t\right).$$

4. The definition of fractional order derivation being based on that of fractional order integration,

5. A derivation of fractional order has a global character unlike an integer derivation. It turns out indeed that the fractional order derivative of a function $f(t)$ requires knowledge of $f(t)$ on the interval $[a, t]$, whereas in the integer case only local knowledge of f around t is necessary. This property allows fractional order systems to be interpreted as long-memory systems, whole systems then being interpretable as short-memory systems.

6. In general

$$_{RL}D_{a,t}^{\alpha} f\left(t\right)_{RL}\,D_{a,t}^{\beta} f\left(t\right) \neq_{RL} D_{a,t}^{\beta} f\left(t\right)_{RL}\,D_{a,t}^{\alpha} f\left(t\right) \neq_{RL} D_{a,t}^{\alpha+\beta} f\left(t\right).$$

   Fractional derivatives do not commute!

7. For $\alpha = n$, where n is an integer, the operation ${}^{c}D_a^\alpha f(x)$ produces the same result as the classical integer derivation.

8. For $\alpha = 0$, the operator is the identity operator, so ${}^{c}D_a^\alpha f(x) = f(x)$.

Chapter 4

Existence and Ulam stability of solution for some backward impulsive differential equations on Banach spaces

4.1 Introduction

The theory of impulsive differential equations is an adequate mathematical model for describing processes that experience a sudden change of their state at certain moments between intervals of continuous evolution. Because the duration of these changes is often negligible compared to the total extent of the process, they can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. Thus, such processes tend to be more suitably modelled by impulsive differential equations, which allow for discontinuities in the evolution of the state.

Processes of the described type often arise naturally, for example, in physics, chemical technology, population dynamics, aeronautics, biotechnology, chemotherapy, optimal control, ecology, economics, and engineering. The corresponding theory has seen significant development over the past decades. For more details, we refer the interested reader to the monographs of Lakshmikantham et al. [40] and Samoilenko and Perestyuk [71] for the case of ordinary impulsive system, works [33, 34, 63, 64] for partial differential and partial functional differential equations with impulses and the references therein.

Many authors were interested in the solvability of boundary value problems with impulses. Nieto [55] studied the existence of solutions to the first order periodic problem with impulse. Chen, Tisdell and Yuan [22] considered the solvability of the periodic problem with impulse. Lin and Jiang [43] studied the existence of positive solutions for the second order Dirichlet boundary value problem with impulse.

However, as far as we know, there are no results on the existence and stability of solutions to backward impulsive differential equations. In this chapter, using some well known classical fixed point theorems, we study the problem of the existence of solutions and their Ulam stability for the following backward impulsive differential equations in Banach spaces

$$\begin{cases} u'(t) = f(t, u(t)), \quad t \in J = [0, T], \ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), \qquad k = 1, \cdots, m, \\ u(T) = u_T, \end{cases} \tag{4.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $\Delta u|_{t=t_k} = u\left(t_k^+\right) - u\left(t_k^-\right)$ represents the jump of the function u at t_k, $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, ..., n$, are appropriate functions, and $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear real function. Our method of study is to convert the initial value problem (4.1) into an equivalent integral equation and apply Banach, Schaefer or Krasnoselskii fixed point theorem. Further, we prove the existence of a unique solution or at least one solution to this problem with local and nonlocal conditions. Consider the following nonlocal problem

$$\begin{cases} u'(t) = f(t, u(t)), \quad t \in J = [0, T], \ t \neq t_k, \qquad k = 1, \cdots, m, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), \qquad k = 1, \cdots, m, \\ u(T) - g(u) = u_T, \end{cases} \tag{4.2}$$

where f and I_k, $k = 1, ..., m$, are defined as in the previous paragraph and g is a continuous function defined from $PC(J, \mathbb{R})$ to $\mathbb{R}$. Nonlocal conditions were first investigated by Byszewski and Lakshmikantham [18]. Using the Banach fixed point theorem, they obtained conditions for the existence and uniqueness of mild solutions to nonlocal differential equations. Byszewski [17] proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problem.

The nonlocal problem was motivated by physical problems. Also, it was demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems. Let us mention, for example, nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution, and nonlocal combustion. Particularly, in 1999, Byszewski [16] obtained conditions for the existence and uniqueness of classical solution to a class of abstract functional differential equations with nonlocal conditions of the form

$$\begin{cases} u'(t) = f(t, u(t), u\left(a\left(t\right)\right)), \quad t \in I, \\ u\left(t_0\right) + \sum\limits_{k=1}^{p} c_k u\left(t_k\right) = x_0, \end{cases}$$

where $I := [t_0, t_0 + T]$, $t_0 < t_1 < ... < t_p \leq t_0 + T,\ T > 0,\ f : I \times E^2 \to E$ and $a : I \to I$ are given functions, E is a Banach space, $x_0 \in E$, $c_k \neq 0,\ (k = 1, 2, ..., p)$, and $p \in \mathbb{N}$. The author pointed out that if $c_k \neq 0,\ k = 1, 2, ..., p$, then the results of the paper can be applied in kinematics to determine the evolution $t \to u(t)$ of a location of a physical object, for which we do not know the positions $u(0), u(t_1), ..., u(t_p)$, but we know that the nonlocal condition holds. To check the Ulam stability, we proceed as J. R. Wang et all [75].

4.2 Existence results

In this section, our attention is focused on the main results on the existence of a solution to the problem (4.1). We discuss conditions under which this problem has exactly one solution or at least one solution.

In the study of the problem (4.1), we will work with the following assumptions:

(H1) The function $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(H2) There exists a positive constant λ such that for any $t \in [0, T]$ and $x, y \in \mathbb{R}$,

$$\left|f\left(t, x\right) - f\left(t, y\right)\right| \leq \lambda \left|x - y\right|.$$

(H3) There exists a positive constant θ such that $\left|f\left(t, x\right)\right| < \theta$ for any $t \in [0, T]$ and $x \in \mathbb{R}$.

(H4) $\left|f\left(t, x\right)\right| \leq r$ for any $t \in [0, T]$ and $x \in B_r,\ r \in \mathbb{R}_+$.

(H5) There exists a constant $\mu > 0$, such that $\left|I_k\left(x\right) - I_k\left(y\right)\right| \leq \mu \left|x - y\right|$ for any $x, y \in \mathbb{R}$, $k = 1, ..., m$.

(H6) The functions $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and there exists a positive constant γ such that $|I_k(x)| < \gamma$ for any $x \in \mathbb{R}$, $k = 1, ..., m$.

A function $u \in PC(J, \mathbb{R})$ will be called a solution to (4.1) if its derivative exists on the set $J' = J - \{t_k, k = 1, 2, 3, ..., n\}$ and u satisfies the equation

$$u'(t) = f(t, u(t)), \quad t \in J',$$

and the conditions

$$\Delta u|_{t=t_k} = I_k(u(t_k^-)), \qquad k = 1, \cdots, m,$$
$$u(T) = u_T.$$

Lemma 4.1 *A function u is a solution to the integral equation:*

$$u(t) = u_T - \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \int_t^T h(s)\,ds \tag{4.3}$$

for $t \in (t_{m-k}, t_{m-k+1})$, $k = 0, ..., m$, if and only if u is a solution of the backward impulsive equation:

$$\begin{cases} u'(t) = h(t), \quad t \in J = [0, T], \ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), \qquad k = 1, \cdots, m, \\ u(T) = u_T. \end{cases} \tag{4.4}$$

Proof. Assume u satisfies (4.4). Then for $t \in (t_m, T)$, we have

$$u(t) = u_T - \int_{t_m}^T h(s)\,ds + \int_{t_m}^t h(s)\,ds.$$

We will proceed by induction on m. For $t \in (t_{m-1}, t_m)$, we can write

$$\begin{aligned} u(t) &= u\left(t_m^-\right) - \int_{t_{m-1}}^{t_m} h(s)\,ds + \int_{t_{m-1}}^t h(s)\,ds \\ &= -\triangle\left(u\left(t_m\right)\right) + u\left(t_m^+\right) - \int_{t_{m-1}}^{t_m} h(s)\,ds + \int_{t_{m-1}}^t h(s)\,ds \\ &= -I_m\left(u\left(t_m^-\right)\right) + u_T - \int_{t_m}^T h(s)\,ds - \int_{t_{m-1}}^{t_m} h(s)\,ds + \int_{t_{m-1}}^t h(s)\,ds. \end{aligned}$$

Further, for any $k = 0, 1, ..., m$ and $t \in (t_{m-k}, t_{m-k+1})$, we obtain

$$\begin{aligned} u(t) = u_T &- \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} h(s)\,ds + \int_{t_{m-k}}^t h(s)\,ds \\ = u_T &- \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \int_{t_{m-k}}^T h(s)\,ds + \int_{t_{m-k}}^t h(s)\,ds \\ = u_T &- \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \int_t^T h(s)\,ds. \end{aligned}$$

Conversely, assume that u satisfies the impulsive integral equation (4.3). If $t \in (t_m, T)$, then $u(T) = u_T$. If $t \in (t_{m-k}, t_{m-k+1})$, $k = 0, ..., m$, by differentiation (4.3), we get

$$u'(t) = h(t).$$

It remains to note that

$$\Delta u|_{t=t_k} = I_k(u(t_k^-)), \qquad k = 1, \cdots, m.$$

■

First, we discuss conditions under which the problem (4.1) has a unique solution. The following result is based on the Banach fixed point theorem.

Theorem 4.2 *Assume that the conditions* (**H1**), (**H2**) *and* (**H5**) *are verified and*

$$m\mu + \lambda T < 1. \tag{4.5}$$

Then the problem (4.1) *has a unique solution in* $PC(J, \mathbb{R})$.

Proof. We transform the problem (4.1) into a fixed point problem. Consider the operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by

$$Fu(t) = u_T - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \int_t^T f(s, u(s))ds.$$

Clearly, if the operator F has a fixed point, then it is a solution to the problem (4.1). Let $u, v \in PC(J, \mathbb{R})$. Then, for each $t \in J$, we have

$$\begin{aligned} |Fu(t) - Fv(t)| &\le \sum_{t<t_k<T} \left|I_k\left(u\left(t_k^-\right)\right) - I_k\left(v\left(t_k^-\right)\right)\right| \\ &+ \int_t^T |f(s, u(s)) - f(s, v(s))|\, ds \\ &\le \mu \sum_{t<t_k<T} \left|u\left(t_{m-p}^-\right) - v\left(t_{m-p}^-\right)\right| \\ &+ \lambda \int_t^T |u(s) - v(s)|\, ds \\ &\le m\mu\, |u(t) - v(t)| + \lambda T\, |u(t) - v(t)| \\ &= (m\mu + \lambda T)\, |u(t) - v(t)|. \end{aligned}$$

Hence, by (4.5), F is a contraction. Then, by the Banach contraction principle, F has a unique fixed point, which is a solution to the problem (4.1) ■

The following result provides sufficient conditions for the existence of at least one solution to the problem (4.1). It is based on the Schaefer's fixed point theorem.

Theorem 4.3 *If the conditions* (**H1**), (**H2**) *and* (**H4**) $-$ (**H6**) *are satisfied, then the problem* (4.1) *has at least one solution in* $PC(J, \mathbb{R})$.

Proof. For the sake of convenience, the proof of this result is divided into four steps.

Step1: The operator F is continuous. Let (u_n) be such a sequence that $u_n \to u$ on J. Then, for all $t \in [0, T]$,

$$\begin{aligned} |Fu_n(t) - Fu(t)| \leq & \sum_{t<t_k<T} \left| I_k\left(u_n\left(t_k^-\right)\right) - I_k\left(u\left(t_k^-\right)\right)\right| \\ & + \int_t^T |f(s, u_n(s)) - f(s, u(s))|\, ds. \end{aligned}$$

Since f and I_k, $k = 1, ..., m$, are continuous functions, we have

$$\|Fu_n - Fu\|_\infty \to 0 \text{ as } n \to \infty,$$

which implies that F is continuous.

Step2: F maps bounded sets into bounded sets in $PC(J, \mathbb{R})$. For all $u \in B_r$, we have

$$\begin{aligned} |Fu(t)| &= \left| u_T - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \int_t^T f(s, u(s))ds \right| \\ &\leq |u_T| + \left| \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right)\right| + \left| \int_t^T f(s, u(s))ds \right| \\ &\leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right)\right| + \int_t^T |f(s, u(s))|\, ds \\ &\leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right)\right| + r\int_t^T ds \\ &\leq |u_T| + m\gamma + rT = \rho. \end{aligned}$$

Hence, the operator F maps the bounded set B_r into the bounded set B_ρ.

Step3: F maps bounded sets into the equicontinuous sets of $PC(J, \mathbb{R})$.

Let $\tau_1, \tau_2 \in [0, T]$, $t_k < \tau_1 < \tau_2 < t_{k+1}$, $k = 0, 1, ..., m-1$, and let $u \in B_r$. Then

$$|Fu(\tau_2) - Fu(\tau_1)| \leq \sum_{\tau_1 < t_k < \tau_2} \left| I_k\left(u\left(t_k^-\right)\right) \right| + \int_{\tau_1}^{\tau_2} |f(s, u(s))|\, ds = \int_{\tau_1}^{\tau_2} |f(s, u(s))|\, ds.$$

The right-hand side of this inequality tends to zero when τ_1 tends to τ_2. By the precedent steps, together with the Ascoli-Arzela theorem, we conclude that F is completely continuous on interval $[t_k, t_{k+1}]$.

Thus, by the PC-type Arzela-Ascoli theorem, we conclude that $F : B_r \to B_\rho$ is continuous and completely continuous.

Step4: The set $\Omega = \{u \in PC(J, \mathbb{R}) : u = \lambda F(u), \ \ 0 < \lambda < 1\}$ is bounded. Since for any $u \in \Omega$, we have $u = \lambda F(u)$ for some $0 < \lambda < 1$, for all $t \in [0, T]$, we can write

$$\begin{aligned}
|u(t)| &= \lambda \left| u_T - \sum_{t < t_k < T} I_k\left(u\left(t_k^-\right)\right) - \int_t^T f(s, u(s)) ds \right| \\
&\leq |u_T| + \left| \sum_{t < t_k < T} I_k\left(u\left(t_k^-\right)\right) \right| + \int_t^T |f(s, u(s))|\, ds \\
&\leq |u_T| + \sum_{t < t_k < T} \left| I_k\left(u\left(t_k^-\right)\right) \right| + |f(t, u(t))| \int_t^T ds \\
&\leq |u_T| + m\gamma + \theta T.
\end{aligned}$$

This proves that Ω is bounded. Hence, by the Schaefer's fixed point theorem, F has a fixed point which is a solution to the problem (4.1). ■

4.3 Nonlocal backward impulsive differential equations

In this section, we generalize the results of the previous section to nonlocal impulsive differential equations 4.2.

Let us introduce the following assumptions:

(H7) There exists a positive constant C such that $|g(x) - g(y)| \leq C \|x - y\|$ for any $x, y \in PC(J, \mathbb{R})$.

(H8) There exists a positive constant κ such that $|g(u)| \leq \kappa$ for any function $u \in PC(J, \mathbb{R})$.

The equation (4.2) is equivalent to the following integral equation

$$u(t) = u_T + g(u) - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \int_t^T f(s,u(s))ds.$$

Theorem 4.4 *Assume that the conditions* (**H1**), (**H2**), *and* (**H5**) *are satisfied, and*

$$C + m\mu + \lambda T < 1. \tag{4.6}$$

Then the problem (4.2) *has a unique solution in* $PC(J,\mathbb{R})$.

Proof. Consider the operator $F : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ defined by

$$Fu(t) = u_T + g(u) - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \int_t^T f(s,u(s))ds.$$

First, we show that F is a contraction. Let $u, v \in PC(J,\mathbb{R})$. Then, for each $t \in J$, we have

$$\begin{aligned} |Fu(t) - Fv(t)| &\le |g(u) - g(v)| \\ &+ \sum_{t<t_k<T} \left|I_k\left(u\left(t_k^-\right)\right) - I_k\left(v\left(t_k^-\right)\right)\right| \\ &+ \int_t^T |f(s,u(s)) - f(s,v(s))|\, ds \\ &\le C\,|u - v| + \mu \sum_{t<t_k<T} \left|u\left(t_{m-p}^-\right) - v\left(t_{m-p}^-\right)\right| \\ &+ \lambda \int_t^T |u(s) - v(s)|\, ds \\ &\le C\,\|u - v\| + m\mu\,\|u - v\| \\ &+ \lambda T\,\|u - v\| \\ &= (C + m\mu + \lambda T)\,\|u - v\|. \end{aligned}$$

Hence, by (4.6), F is a contraction. Then, by the Banach contraction principle, we deduce that F has a unique fixed point which is a solution to the problem (4.2). ■

Theorem 4.5 *If* (**H1**), (**H3**) *and* (**H6**) − (**H8**) *are satisfied and* $C < 1$, *then the problem* (4.2) *has at least one solution in* $PC(J,\mathbb{R})$.

Proof. Let

$$r \ge \frac{|u_T| + \kappa}{1 - (m\gamma + \theta T)}, \tag{4.7}$$

and define the operators P and Q on the compact set $B_r \subset PC(J,\mathbb{R})$ by

$$\begin{aligned} Pu(t) &= u_T + g(u), \\ Qu(t) &= -\sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \int_t^T f(s,u(s))ds. \end{aligned}$$

For all $u \in B_r$, we have

$$|Pu(t)| = |u_T + g(u)| \leq |u_T| + |g(u)| \leq |u_{_T}| + \kappa \leq r\left(1 - (m\gamma + \theta T)\right) \leq r.$$

Hence, the operator P maps B_r into itself. Further, for all $u, v \in PC(J,\mathbb{R})$,we can write

$$|Pu(t) - Pv(t)| = |g(u) - g(v)| \leq C\|u - v\|,$$

and hence, the operator P satisfies the contraction property. Since

$$\begin{aligned} |Qv(t)| &\leq \sum_{t<t_k<T} \left|I_k\left(v\left(t_k^-\right)\right)\right| + \int_t^T |f(s,v(s))|\, ds \\ &\leq (m\gamma + \theta T)\, |v(t)|, \end{aligned}$$

we can write

$$\begin{aligned} |Pu(t) + Qv(t)| &\leq |Pu(t)| + |Qv(t)| \\ &\leq |u_{_T}| + \kappa + (\gamma m + \theta T)\, |v(t)| \\ &\leq |u_{_T}| + \kappa + (m\gamma + \theta T)\, r \\ &\leq r. \end{aligned}$$

Therefore, if $u, v \in B_r$, then $Pu + Qv \in B_r$. By (**H1**), Q is continuous and by the inequality (4.7), it is uniformly bounded on B_r. The equicontinuity of $Qv(t)$ follows from Theorem 4.3. Hence, by the Arzela Ascoli theorem, $Q(B_r)$ is relatively compact, which implies that Q is compact. Therefore, using Krasnoselskii theorem, we conclude that there exists a solution to the equation (4.2). ■

4.4 Ulam stability for some nonlinear backward impulsive differential equations

In this section, we study the Ulam stability of the solution to the problem (4.1).

Now, we introduce Ulam's type stability concepts for the equation (4.1). Let $\varepsilon > 0,\ \psi \geq 0$ and $\varphi \in PC(J, \mathbb{R}^+)$ is nondecreasing. Consider the following inequalities:

$$\begin{cases} |y'(t) - f(t, y(t))| \leq \varepsilon, & t \in J', \\ \left|\Delta y|_{t=t_k} - I_k(y(t_k^-))\right| \leq \varepsilon, & k = 1, \cdots, m, \end{cases} \tag{4.8}$$

$$\begin{cases} |y'(t) - f(t, y(t))| \leq \varphi(t), & t \in J', \\ \left|\Delta y|_{t=t_k} - I_k(y(t_k^-))\right| \leq \psi, & k = 1, \cdots, m, \end{cases} \tag{4.9}$$

and

$$\begin{cases} |y'(t) - f(t, y(t))| \leq \varepsilon\varphi(t), & t \in J', \\ \left|\Delta y|_{t=t_k} - I_k(y(t_k^-))\right| \leq \varepsilon\psi, & k = 1, \cdots, m. \end{cases} \tag{4.10}$$

Definition 4.6 *Equation (4.1) is Ulam–Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in PC^1(J, \mathbb{R})$ of the inequality (4.8), there exists a solution $u \in PC^1(J, \mathbb{R})$ of the equation (4.1) with*

$$|y(t) - u(t)| \leq c_{f,m}\varepsilon, \quad t \in J'.$$

Definition 4.7 *Equation (4.1) is generalized Ulam–Hyers stable if there exists $\theta_{f,m} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\theta_{f,m}(0) = 0$ such that for each solution $y \in PC^1(J, \mathbb{R})$ of the inequality (4.8), there exists a solution $u \in PC^1(J, \mathbb{R})$ of the equation (4.1) with*

$$|y(t) - u(t)| \leq \theta_{f,m}(\varepsilon), t \in J'.$$

Definition 4.8 *Equation (4.1) is Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,m,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in PC^1(J, \mathbb{R})$ of the inequality (4.10), there exists a solution $u \in PC^1(J, \mathbb{R})$ of the equation (4.1) with*

$$|y(t) - u(t)| \leq c_{f,m,\varphi}\varepsilon(\varphi(t) + \psi), t \in J'.$$

Definition 4.9 *Equation (4.1) is said to be generalized Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,m,\varphi} > 0$ such that for each solution $y \in PC^1(J, \mathbb{R})$ of inequality (4.9), there exists a solution $u \in PC^1(J, \mathbb{R})$ of equation (4.1) with*

$$|y(t) - u(t)| \leq c_{f,m,\varphi}(\varphi(t) + \psi), t \in J'.$$

Remark 4.10 *It is clear that (i) Definition 4.6 implies Definition 4.7, (ii) Definition 4.8 implies Definition 4.9, (iii) Definition 4.8 for* $\varphi(t)=\psi=1$ *implies Definition 4.6.*

Proposition 4.11 *A function* $y\in PC^1(J,\mathbb{R})$ *is a solution of inequality (4.8) if and only if there is* $g\in PC(J,\mathbb{R})$ *and a sequence* $g_k,\ k=1,2,...,m$ *(which depend on y) such that*

(i) $|g(t)|\le\varepsilon,\ t\in J$ *and* $|g_k|\le\varepsilon,\ k=1,2,...,m,$

(ii) $y'(t)=f(t,y(t))+g(t),\ t\in J',$

(iii) $\Delta y(t_k)=I_k(y(t_k^-))+g_k,\ k=1,2,...,m.$

Proposition 4.12 *If* $y\in PC^1(J,\mathbb{R})$ *is a solution of inequality (4.8), then y is a solution of the following inequality*

$$\left|y\left(t\right)-u_T+\sum_{p=0}^{k-1}I_{m-p}\left(y\left(t_{m-p}^-\right)\right)+\int_t^T f(s,y(s))ds\right|\le\left(m+t-T\right)\varepsilon,\ t\in J'.$$

Proof. Indeed, by proposition 4.11, we have

$$\begin{cases} y'(t)=f(t,y(t))+g(t), & t\in J',\\ \Delta y(t_k)=I_k(y(t_k^-))+g_k, & k=1,2,...,m.\end{cases}$$

Then, for $t\in(t_{m-k},t_{m-k+1})$ and $k=0,...,m$.

$$\begin{aligned} y\left(t\right)=u_T-\sum_{p=0}^{k-1}I_{m-p}\left(y\left(t_{m-p}^-\right)\right)-\sum_{p=0}^{k-1}g_i\\ -\int_t^T f(s,y(s))ds-\int_t^T g(s)ds.\end{aligned}$$

From here it follows that

$$\begin{aligned} &\left|y\left(t\right)-u_T+\sum_{p=0}^{k-1}I_{m-p}\left(y\left(t_{m-p}^-\right)\right)+\int_t^T f(s,y(s))ds\right|\\ &\le\sum_{i=1}^{m}|g_i|+\int_t^T|g(s)|\,ds\le m\varepsilon+\varepsilon\int_T^t ds\\ &\le m\varepsilon-\varepsilon T+\varepsilon t=\left(m+t-T\right)\varepsilon.\end{aligned}$$

■

Similar remarks or propositions hold true for the solutions of the inequalities (4.9) and (4.10).

Note that the Ulam stabilities of the impulsive differential equations are some special types of data dependence of the solutions of impulsive differential equations.

Theorem 4.13 *Let the assumptions* **(H1)**, **(H2)** *and* **(H5)** *hold and suppose there exists* $\lambda_\varphi > 0$ *such that*

$$\int_t^T \varphi(s)\,ds \le \lambda_\varphi \varphi(t)$$

for each $t \in J$ *where* $\varphi \in PC^1(J, \mathbb{R}^+)$ *is nondecreasing. Then the equation* (4.1) *is generalized Ulam-Hyers-Rassias stable with respect to* (φ, ψ).

Proof. Let $y \in PC^1(J, \mathbb{R})$ be a solution to the inequality (4.9). Denote by u the unique solution of the backward impulsive problem

$$\begin{cases} u'(t) = f(t, u(t)), \quad t \in J' = [0, T],\ t \ne t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), \qquad k = 1, \cdots, m, \\ u(T) = u_T. \end{cases}$$

Then we have

$$u(t) = u_T - \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \int_t^T f(s, u(s))ds,$$

where $t \in (t_{m-k}, t_{m-k+1})$ for $k = 0, ..., m$. Differentiating the inequality (4.9) (see Proposition 4.12), for each t in (t_{m-k}, t_{m-k+1}), we obtain

$$\left| y(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^-\right)\right) + \int_t^T f(s, y(s))ds \right| \le \sum_{i=1}^{m} |g_i| + \int_t^T \varphi(s)\,ds$$

$$\le m\psi + \lambda_\varphi \varphi(t) \le (\varphi(t) + \psi)(\lambda_\varphi + m).$$

Hence, for each $t \in (t_{m-k}, t_{m-k+1})$ and $k = 0, ..., m$, we can write

$$|y(t) - u(t)| \le \left| y(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^-\right)\right) + \int_t^T f(s, y(s))ds \right|$$
$$+ \sum_{p=0}^{k-1} \left| I_{m-p}\left(y\left(t_{m-p}^-\right)\right) - I_{m-p}\left(u\left(t_{m-p}^-\right)\right) \right|$$
$$+ \int_t^T |f(s, y(s)) - f(s, u(s))|\,ds$$

$$\leq (\varphi(t)+\psi)(\lambda_\varphi+m)+\sum_{p=0}^{k-1}\mu_k\left|\left(y\left(t_{m-p}^-\right)\right)-\left(u\left(t_{m-p}^-\right)\right)\right|$$
$$+\lambda\int_t^T|y(s)-u(s)|\,ds.$$

Finally, by Lemma 2.25, we obtain

$$|y(t)-u(t)|\leq(\varphi(t)+\psi)(\lambda_\varphi+m)\prod_{t<t_k<T}(1+\mu_k)\exp(\lambda(T-t)).$$

Thus, equation (4.1) is generalized Ulam–Hyers–Rassias stable with respect to (φ,ψ). The proof is completed. ■

Remark 4.14

Using the approach developed in [75], one can prove the validity of the following statements.

1. *Under the assumptions of Theorem 4.13, if we consider the equation (4.1) and inequality (4.8), by the same process we can verify that the equation (4.1) is Ulam–Hyers stable.*
2. *Under the assumptions of Theorem 4.13, if we consider the equation (4.1) and inequality (4.10), we can use the same process to verify that the equation (4.1) is Ulam–Hyers–Rassias stable with respect to (φ,ψ).*
3. *The above results can be extended to the case of the equation (4.2).*

4.5 Applications

Example 4.1

Consider the backward impulsive ordinary differential equation

$$\begin{cases} u'(t)=0, \quad t\in J'=[0,1]-\{1/2\}, \\ \Delta u|_{t=\frac{1}{2}}=\dfrac{\left|u\left(\frac{1}{2}^-\right)\right|}{1+\left|u\left(\frac{1}{2}^-\right)\right|}, \\ u(1)=1. \end{cases} \tag{4.11}$$

and the inequalities

$$\begin{cases} |y'(t)| \leq \varepsilon, & t \in J' = [0,1] - \{1/2\}, \\ \left| \Delta y|_{t=\frac{1}{2}} - \dfrac{\left|y\left(\frac{1}{2}^-\right)\right|}{1+\left|y\left(\frac{1}{2}^-\right)\right|} \right| \leq \varepsilon, & \varepsilon > 0. \end{cases} \tag{4.12}$$

Let $y \in PC^1([0,1],\mathbb{R})$ be a solution to the inequality (4.12). Then there exist $g \in PC^1([0,1],\mathbb{R})$ and $g_1 \in \mathbb{R}$ such that:

$$|g(t)| \leq \varepsilon, \quad t \in [0,1], \text{ and } |g_1| \leq \varepsilon, \tag{4.13}$$

$$y'(t) = g(t), \quad t \in J' = [0,1] - \{1/2\}, \tag{4.14}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{\left|y\left(\frac{1}{2}^-\right)\right|}{1+\left|y\left(\frac{1}{2}^-\right)\right|} + g_1. \tag{4.15}$$

Integrating (4.14) from t to 1 via (4.15), we obtain

$$\begin{aligned} y(t) &= y(1) - \left(I_{\frac{1}{2}}\left(y\left(t_{\frac{1}{2}}^-\right)\right) + g_1\right) - \int_t^1 g(s)ds \\ &= 1 - \left(\frac{\left|y\left(\frac{1}{2}^-\right)\right|}{1+\left|y\left(\frac{1}{2}^-\right)\right|} + g_1\right) - \int_t^1 g(s)ds. \end{aligned}$$

Let us consider the solution u of (4.11) given by

$$u(t) = 1 - \frac{\left|u\left(\frac{1}{2}^-\right)\right|}{1+\left|u\left(\frac{1}{2}^-\right)\right|}.$$

Then we can write

$$\begin{aligned} |y(t) - u(t)| &= \left| \frac{\left|u\left(\frac{1}{2}^-\right)\right|}{1+\left|u\left(\frac{1}{2}^-\right)\right|} - \frac{\left|y\left(\frac{1}{2}^-\right)\right|}{1+\left|y\left(\frac{1}{2}^-\right)\right|} - g_1 - \int_t^1 g(s)ds \right| \\ &\leq \left| \frac{\left|u\left(\frac{1}{2}^-\right)\right|}{1+\left|u\left(\frac{1}{2}^-\right)\right|} - \frac{\left|y\left(\frac{1}{2}^-\right)\right|}{1+\left|y\left(\frac{1}{2}^-\right)\right|} \right| + |g_1| + \left|\int_t^1 g(s)ds\right| \\ &\leq \left| \left|u\left(\frac{1}{2}^-\right)\right| - \left|y\left(\frac{1}{2}^-\right)\right| \right| + |g_1| + \left|\int_t^1 g(s)ds\right| \\ &\leq \left| u\left(\frac{1}{2}^-\right) - y\left(\frac{1}{2}^-\right) \right| + |g_1| + \int_t^1 |g(s)|\, ds \\ &\leq \left| u\left(\frac{1}{2}^-\right) - y\left(\frac{1}{2}^-\right) \right| + \varepsilon + \varepsilon \int_t^1 ds, \quad t \in [0,1] \\ &\leq \left| u\left(\frac{1}{2}^-\right) - y\left(\frac{1}{2}^-\right) \right| + \varepsilon + \varepsilon(1-t), \quad t \in [0,1] \\ &\leq 3\varepsilon + \varepsilon(1-t) = \varepsilon(4-t), \quad t \in [0,1]. \end{aligned}$$

Thus, Equation (4.11) is generalized Ulam–Hyers stable, which is a special case of generalized Ulam–Hyers–Rassias stable.

Example 4.2

Consider the backward impulsive ordinary differential equation

$$\begin{cases} u'(t) = \dfrac{1}{5+e^t+u^2(t)}, & t \in J' = [0,\pi],\ t \neq t_k, \\ \Delta u|_{t=t_k} = \dfrac{1}{5k+u^2\left(t_k^-\right)}, & k = 1,\cdots,5, \quad t_k = k\frac{\pi}{5}, \\ u(\pi) = 1. \end{cases} \tag{4.16}$$

Set

$$f(t,x) = \frac{1}{5+e^t+x^2}, \quad (t,x) \in J \times [0,+\infty),$$

and

$$I_k(x) = \frac{1}{5k+x^2}.$$

Let $x, y \in [0,+\infty)$ and $t \in J$. By the mean value theorem, we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= \frac{2\xi}{(5+e^t+\xi^2)^2}|x-y| \\ &\leq \frac{2\xi}{(6+\xi^2)^2}|x-y|, \quad x<\xi<y. \end{aligned}$$

Then

$$|f(t,x) - f(t,y)| \leq \frac{\sqrt{2}}{32}|x-y|.$$

Hence, the condition (**H2**) holds with $\lambda = \sqrt{2}/32$.

Let $x, y \in [0,+\infty)$. By the mean value theorem, we have

$$|I_k(x) - I_k(y)| = \frac{2\xi}{(5k+\xi^2)^2}|x-y|, \quad x<\xi<y.$$

Then

$$|I_k(x) - I_k(y)| \leq \frac{3\sqrt{15}}{200}|x-y|.$$

Hence, the condition (**H5**) holds with $\mu = 3\sqrt{15}/200$.

Since

$$5 \times \frac{3\sqrt{15}}{200} + \frac{\pi\sqrt{2}}{32} = \frac{3\sqrt{15}}{50} + \frac{\pi\sqrt{2}}{32} = \frac{48\sqrt{15}+25\pi\sqrt{2}}{800} < 1.$$

the condition (4.5) is satisfied. Thus, by Theorem 4.2, the problem (4.16) has a unique solution.

Now, we proceed as in the proof of Theorem 4.13. Let $u \in PC^1(J, \mathbb{R})$ be a solution of the inequality (4.9). Denote by x the unique solution to the backward impulsive problem (4.16). Then we obtain

$$x(t) = 1 - \sum_{p=0}^{k-1} I_{5-p}\left(x\left(t_{5-p}^-\right)\right) - \int_t^\pi \frac{ds}{5 + e^s + x^2(s)},$$

where $t \in (t_{5-k}, t_{5-k+1})$ for $k = 0, ..., 5$.

Like in proposition 4.12, by differentiation of the inequality, for each $t \in (t_{5-k}, t_{5-k+1})$, we have

$$\begin{aligned}\left| u(t) - 1 + \sum_{p=0}^{k-1} I_{5-p}\left(u\left(t_{5-p}^-\right)\right) + \int_t^\pi \frac{ds}{5 + e^s + u^2(s)} \right| \\ \leq \sum_{i=1}^{5} |g_i| + \int_t^\pi \varphi(s)\, ds \\ \leq 5\psi + \lambda_\varphi \varphi(t) \\ \leq (\varphi(t) + \psi)(\lambda_\varphi + 5)\end{aligned}$$

Hence, for each $t \in (t_{5-k}, t_{5-k+1})$ and $k = 0, ..., 5$, we can write

$$\begin{aligned}|u(t) - x(t)| \leq & \left| u(t) - 1 + \sum_{p=0}^{k-1} I_{5-p}\left(u\left(t_{5-p}^-\right)\right) + \int_t^\pi \frac{ds}{5 + e^s + u^2(s)} \right| \\ & + \sum_{p=0}^{k-1} \left| I_{5-p}\left(u\left(t_{5-p}^-\right)\right) - I_{5-p}\left(x\left(t_{5-p}^-\right)\right) \right| \\ & + \int_t^\pi |f(s, u(s)) - f(s, x(s))|\, ds \\ \leq & (\varphi(t) + \psi)(\lambda_\varphi + 5) + \sum_{p=0}^{k-1} \mu_k \left| \left(u\left(t_{5-p}^-\right)\right) - \left(x\left(t_{5-p}^-\right)\right) \right| \\ & + \lambda \int_t^\pi |u(s) - x(s)|\, ds\end{aligned}$$

By Lemma 2.25, we obtain

$$|u(t) - x(t)| \leq (\varphi(t) + \psi)(\lambda_\varphi + 5) \prod_{t < t_k < \pi} (1 + \mu_k) \exp(\lambda(\pi - t)).$$

Recalling that $\lambda = \sqrt{2}/32$ and $\mu_k = 3\sqrt{15}/200$, we find

$$\begin{aligned} |u(t) - x(t)| &\leq (\varphi(t) + \psi)(\lambda_\varphi + 5) \prod_{t<t_k<\pi} \left(1 + \frac{3\sqrt{15}}{200}\right) \\ &\quad \times \exp\left(\frac{\sqrt{2}}{32}(\pi - t)\right) \\ &= (\varphi(t) + \psi)(\lambda_\varphi + 5)\left(1 + \frac{3\sqrt{15}}{200}\right)^k \exp\left(\frac{\sqrt{2}}{32}(\pi - t)\right) \\ &\leq (\varphi(t) + \psi)(\lambda_\varphi + 5)\left(1 + \frac{3\sqrt{15}}{200}\right)^k e^{\pi - t}. \end{aligned}$$

This implies that the problem (4.16) is generalized Ulam-Hyers-Rassias stable with respect to (φ, ψ).

Chapter 5

Existence and Ulam stability of solutions of some backward impulsive fractional differential equations on Banach space

5.1 Introduction

The theory of impulsive differential equations describes evolutionary processes characterized by the fact that at certain moments of time they experience a change of state between intervals of continuous evolution. Because the duration of these changes are often negligible compared to the total duration of the process, such changes can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses.

Such processes are naturally seen in biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control model in economics, natural disasters and in some other processes and phenomena in science and technology, see, for instance the monographs [3, 6, 40, 58, 78].

Therefore, the study of this class of dynamical systems has been investigated in the last few years by many authors in several directions. So, a great deal of techniques and methods have been used in the study of different type of impulsive differential equations to obtain some quantitative or qualitative results regarding the solutions of such new problems, see for instance [7, 12, 31, 60, 68]. For the general theory and applications of impulsive differential equations, we refer the reader to the references [13, 58, 71, 74, 82].

Fractional differentiation and integration are the generalization of the ordinary differentiation and integration to an arbitrary non-integer order. One knows that the fractional derivatives (Riemann-Liouville fractional derivative, Caputo fractional derivative and Hadamard fractional derivative and other type see [38, 53, 56, 59, 71] are actually nonlocal operators because integrals are nonlocal operators. Moreover, calculating time fractional derivatives of a function at some time requires all the past history and hence fractional derivatives can be used for modeling systems with memory. Using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modeling real life phenomena. For more details, we refer to the books by Oldham and Spanier [56] and by Miller and Ross [54].

Due to their significance, it is important to study the solvability of impulsive fractional differential equations.

It is worthwhile mentioning that impulsive differential equations of fractional order have not been much studied and many aspects of these equations are yet to be explored. This field is becoming a very important research area at the present time due to its many applications in engineering, physics and economics. For more details, we refer the reader to [13, 29, 65, 71].

However, as far as we know, there are no results on the existence and stability of solutions to backward impulsive fractional differential equations. In this chapter, using some well known classical fixed point theorems, we study the problem of the existence of solutions and their Ulam stability for the following backward impulsive differential equations in Banach spaces

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t)), \quad t \in J = [0, T]\,, \ t \neq t_k \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), \qquad k = 1, \cdots, m \\ u(T) = u_T \end{cases} \tag{5.1}$$

where D^{α} is a the Caputo fractional derivative, $0 < \alpha < 1$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $\Delta u|_{t=t_k} = u\left(t_k^+\right) - u\left(t_k^-\right)$ represents the jump of the function u at t_k, $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, ..., n$, are appropriate functions, and $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear real function. Our method of study is to convert the initial value problem (5.1) into an equivalent integral equation and apply Schaefer, Banach or Krasnoselskii fixed point theorem. Further, we prove the existence of a unique solution or at least one solution to this problem with local and nonlocal conditions. Consider the following nonlocal problem

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t)), \quad t \in J = [0, T], \; t \neq t_k, \quad k = 1, \cdots, m, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), \quad k = 1, \cdots, m, \\ u(T) - g(u) = u_T, \end{cases} \tag{5.2}$$

where f and I_k, $k = 1, ..., m$, are defined as in the previous paragraph and $g : PC(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function. Nonlocal conditions were first investigated by Byszewski and Lakshmikantham [18]. Using the Banach fixed point theorem, they obtained conditions for the existence and uniqueness of mild solutions to nonlocal differential equations. Byszewski [17] proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problem.

The nonlocal problem was motivated by physical problems. Also, it was demonstrated that the nonlocal problems have better effects in applications than the classical Cauchy problems. Let us mention, for example, nonlocal neural networks, nonlocal pharmacokinetics, nonlocal pollution, and nonlocal combustion. Particularly, in 1999, Byszewski [16] obtained conditions for the existence and uniqueness of classical solution to a class of abstract functional differential equations with nonlocal conditions of the form

$$\begin{cases} u'(t) = f(t, u(t), u(a(t))), \quad t \in I, \\ u(t_0) + \sum_{k=1}^{p} c_k u(t_k) = x_0, \end{cases}$$

where $I := [t_0, t_0 + T]$, $t_0 < t_1 < ... < t_p \leq t_0 + T$, $T > 0$, $f : I \times E^2 \to E$ and $a : I \to I$ are given functions, E is a Banach space, $x_0 \in E$, $c_k \neq 0$, $(k = 1, 2, ..., p)$, and $p \in \mathbb{N}$. The author pointed out that if $c_k \neq 0$, $k = 1, 2, ..., p$, then the results of the paper can be applied in kinematics to determine the evolution $t \to u(t)$ of a location of a physical object, for which we do not know the positions $u(0), u(t_1), ..., u(t_p)$, but we know that the nonlocal condition holds. To check the Ulam stability, we proceed as J.R. Wang et all [75].

5.2 Existence results

In this section, our attention is focused on the main results on the existence of a solution to the problem (5.1). We discuss conditions under which this problem has exactly one solution or at least one solution.

In the study of the problem (5.1), we will work with the same assumptions as in the previous chapter.

A function $u \in PC(J, \mathbb{R})$ will be called a solution to (5.1) if its derivative exists on $J' = J - \{t_k, k = 1, 2, 3, ..., n\}$ and u satisfies the equation

$$D^\alpha u(t) = f(t, u(t)), \quad t \in J',$$

and the conditions

$$\begin{aligned} &\Delta u|_{t=t_k} = I_k(u(t_k^-)), \qquad k = 1, \cdots, m, \\ &u(T) = u_T. \end{aligned}$$

Lemma 5.1 *A function u is a solution of the integral equation*

$$u(t) = \begin{cases} u_T - \frac{1}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1} h(s)\, ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} h(s)\, ds, \quad \text{if } t \in (t_m, T) \\ u_T - \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) \\ - \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} (t_{m-p+1}-s)^{\alpha-1} h(s)\, ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} (t-s)^{\alpha-1} h(s)\, ds, \quad \text{if } t \in (t_{m-k}, t_{m-k+1}) \end{cases} \tag{5.3}$$

if and only if u is a solution of the backward impulsive fractional differential equation

$$\begin{cases} D^\alpha u(t) = h(t), \quad t \in I = [0, T], \ t \neq t_k \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), \qquad k = 1, \cdots, m \\ u(T) = u_T. \end{cases} \tag{5.4}$$

Proof. Assume u satisfies (5.4). Then, according to the lemma (3.4), for $t \in (t_m, t_{m+1})$, where $t_{m+1} = T$; we have

$$u(t) = u_T - \frac{1}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1} h(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} h(s)\, ds$$

For $t \in (t_{m-1}, t_m)$

$$\begin{aligned}
u(t) &= u\left(t_m^-\right) - \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\alpha-1} h(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t} (t-s)^{\alpha-1} h(s)\, ds \\
&= -\left(-u\left(t_m^-\right)\right) - \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\alpha-1} h(s)\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t} (t-s)^{\alpha-1} h(s)\, ds \\
&= -\triangle\left(u(t_m)\right) + u\left(t_m^+\right) - \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\alpha-1} h(s)\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t} (t-s)^{\alpha-1} h(s)\, ds \\
&= -I_m\left(u\left(t_m^-\right)\right) + u_T - \frac{1}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1} h(s)\, ds \\
&- \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\alpha-1} h(s)\, ds + \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t} (t-s)^{\alpha-1} h(s)\, ds
\end{aligned}$$

For $t \in (t_{m-2}, t_{m-1})$

$$\begin{aligned}
u(t) &= u\left(t_{m-1}^-\right) - \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t_{m-1}} (t_{m-1} - s)^{\alpha-1} h(s)\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t} (t-s)^{\alpha-1} h(s)\, ds \\
&= -\left(-u\left(t_{m-1}^-\right)\right) - \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t_{m-1}} (t_{m-1} - s)^{\alpha-1} h(s)\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t} (t-s)^{\alpha-1} h(s)\, ds \\
&= -\triangle\left(u(t_{m-1})\right) + u\left(t_{m-1}^+\right) - \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t_{m-1}} (t_{m-1} - s)^{\alpha-1} h(s)\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t} (t-s)^{\alpha-1} h(s)\, ds \\
&= -I_m\left(u\left(t_m^-\right)\right) - I_{m-1}\left(u\left(t_{m-1}^-\right)\right) + u_T - \frac{1}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1} h(s)\, ds \\
&- \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t_m} (t_m - s)^{\alpha-1} h(s)\, ds - \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t_{m-1}} (t_{m-1} - s)^{\alpha-1} h(s)\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-2}}^{t} (t-s)^{\alpha-1} h(s)\, ds
\end{aligned}$$

Then, by induction, we obtain : For $t \in (t_{m-k}, t_{m-k+1})$, $\quad k = 0, ..., m$

$$\begin{aligned}
u(t) &= u_T - \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} (t_{m-p+1} - s)^{\alpha-1} h(s)\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} (t-s)^{\alpha-1} h(s)\, ds.
\end{aligned}$$

for $k=1,2,...,m$. Conversely, assume that u satisfies the impulsive integral equation (5.3). If $t\in(t_m,T)$, then $u(T)=u_T$. If $t\in(t_{m-k},t_{m-k+1})$, $k=0,...,m$, by differentiation, we get

$$D^{\alpha}u(t)=h(t).$$

Also, we can easily show that

$$\Delta u|_{t=t_k}=I_k(u(t_k^-)),\qquad k=1,\cdots,m.$$

■

First, we discuss conditions under which the problem (5.1) has a unique solution. The following result is based on the Banach fixed point theorem.

Theorem 5.2 *Assume that the function f verifies the conditions* (**H1**), (**H2**) *and* (**H5**), *and*

$$m\mu+\frac{\lambda(m+1)}{\Gamma(\alpha+1)}T^{\alpha}<1 \tag{5.5}$$

then the problem (5.1) *has a unique solution in $PC(J,\mathbb{R})$.*

Proof. We transform the problem (5.1) into a fixed point problem. Consider the operator $F:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ defined by

$$Fu(t)=u_T-\sum_{t<t_k<T}I_k\left(u\left(t_k^-\right)\right)-\frac{1}{\Gamma(\alpha)}\sum_{t<t_k<T}\int_{t_k}^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}f(s,u(s))ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,u(s))ds.$$

Clearly, a fixed point of the operator F is a solution of the problem (5.1). We use the Banach contraction principle to prove that F has a fixed point. We shall show that F is a contraction. Let $u,v\in PC(J,\mathbb{R})$. Then, for each $t\in J$, we have

$$|Fu(t)-Fv(t)|\leq\sum_{t<t_k<T}\left|I_k\left(u\left(t_k^-\right)\right)-I_k\left(v\left(t_k^-\right)\right)\right|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{t<t_k<T}\int_{t_k}^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}|f(s,u(s))-f(s,v(s))|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,u(s))-f(s,v(s))|ds$$

$$\begin{aligned}
&\leq \mu \sum_{t<t_k<T} \left|u\left(t_{m-p}^{-}\right) - v\left(t_{m-p}^{-}\right)\right| \\
&+ \frac{\lambda}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(t_{k+1}-s\right)^{\alpha-1} \left|u(s)-v(s)\right| ds \\
&+ \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left|u(s)-v(s)\right| ds \\
&\leq m\mu \left\|u(t)-v(t)\right\| + \frac{m\lambda T^{\alpha}}{\Gamma(\alpha+1)} \left\|u(t)-v(t)\right\| \\
&+ \frac{\lambda T^{\alpha}}{\Gamma(\alpha+1)} \left\|u(t)-v(t)\right\| \\
&= \left[m\mu + \frac{\lambda(m+1)}{\Gamma(\alpha+1)} T^{\alpha}\right] \left\|u(t)-v(t)\right\|.
\end{aligned}$$

Hence, by (5.5), F is a contraction. Then, by the Banach contraction principle, we deduce that F has a unique fixed point which is a solution of the problem (5.1). ■

The following result provides sufficient conditions for the existence of at least one solution to the problem (5.1). It is based on the Schaefer's fixed point theorem.

Theorem 5.3 *If the conditions* (**H1**), (**H2**), (**H3**), (**H5**) *and* (**H6**), *are satisfied then the problem* (5.1) *has at least one solution in* $PC(J, \mathbb{R})$.

Proof. *The proof of this result is divided in several steps*

Step1: *The operator F is continuous. Let (u_n) a sequence such that $u_n \to u$ on J. For all $t \in [0, T]$ then*

$$\begin{aligned}
\left|Fu_n(t) - Fu(t)\right| \leq & \sum_{t<t_k<T} \left|I_k\left(u_n\left(t_k^{-}\right)\right) - I_k\left(u\left(t_k^{-}\right)\right)\right| \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} \left(t_{k+1}-s\right)^{\alpha-1} \left|f(s,u_n(s)) - f(s,u(s))\right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \left|f(s,u_n(s)) - f(s,u(s))\right| ds.
\end{aligned}$$

Since f and I_k, $k = 1, ..., m$ are continuous functions, then

$$\left\|Fu_n - Fu\right\|_{\infty} \to 0 \text{ as } n \to \infty$$

which implies that F is continuous.

Step2: *F maps bounded sets into the bounded sets in $PC(J,\mathbb{R})$. For all $u(t)\in B_r$ we have*

$$\begin{aligned}
|Fu(t)| &= \left| u_T - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} f(s,u(s))ds \right.\\
&\left. +\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s,u(s))ds \right|.\\
&\leq |u_T| + \left| \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) \right| + \left| \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} f(s,u(s))ds \right|\\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s,u(s))ds \right|\\
&\leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right) \right| + \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} |f(s,u(s))| ds\\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |f(s,u(s))| ds\\
&\leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right) \right| + \frac{1}{\Gamma(\alpha)} \|f(s,u(s))\| \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} ds\\
&+ \frac{1}{\Gamma(\alpha)} \|f(s,u(s))\| \int_{t_k}^{t} (t-s)^{\alpha-1} ds\\
&\leq |u_T| + m\gamma + \frac{m\theta T^{\alpha}}{\Gamma(\alpha+1)} + \frac{\theta T^{\alpha}}{\Gamma(\alpha+1)}\\
&= |u_T| + m\gamma + \frac{\theta(m+1)}{\Gamma(\alpha+1)} T^{\alpha} = \rho.
\end{aligned}$$

Hence, the operator F maps the bounded set B_r into a bounded set B_ρ.

Step3: *F maps bounded sets into the equicontinuous sets of $PC(J,\mathbb{R})$ .Let $\tau_1,\tau_2\in[0,T]$, $\tau_1<\tau_2$ and let $u\in B_r$, then*

$$\begin{aligned}
|Fu(\tau_2) - Fu(\tau_1)| \leq & \sum_{\tau_1<t_k<\tau_2} \left| I_k\left(u\left(t_k^-\right)\right) \right|\\
& + \int_{\tau_1}^{\tau_2} |f(s,u_n(s))| ds
\end{aligned}$$

which tends to zero when τ_1 tends to τ_2. By the precedent steps, together with the Ascoli-Arzela theorem, therefore F is completely continuous on interval $[t_k,t_{k+1}]$.
As a consequence of Step 1-3 together with the PC-type Arzela-Ascoli theorem, we conclude that $F: B_r\to B_\rho$ is continuous and completely continuous.

Step4: *We show that the set $\Omega=\{u\in PC(J,\mathbb{R}) : u=\lambda F(u),\quad 0<\lambda<1\}$ is bounded.*

Let $u \in \Omega$, *then* $u = \lambda F(u)$, *for some* $0 < \lambda < 1$, *then for all* $t \in [0,T]$ *we have*

$$
\begin{aligned}
|u(t)| &= \lambda \left| u_T - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} f(s,u(s))ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s,u(s))ds \right| \\
&\leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right)\right| + \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} |f(s,u(s))| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |f(s,u(s))| ds \\
&\leq |u_T| + \sum_{t<t_k<T} \left| I_k\left(u\left(t_k^-\right)\right)\right| + \frac{1}{\Gamma(\alpha)} \|f(s,u(s))\| \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \|f(s,u(s))\| \int_{t_k}^{t} (t-s)^{\alpha-1} ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq |u_T| + m\gamma + \frac{\theta m T^\alpha}{\Gamma(\alpha+1)} + \frac{\theta T^\alpha}{\Gamma(\alpha+1)} \\
&= |u_T| + m\gamma + \frac{\theta(m+1)}{\Gamma(\alpha+1)} T^\alpha,
\end{aligned}
$$

which prove that Ω *is bounded. By the Schaefer's fixed point theorem,* F *has a fixed point which is a solution of the problem* (5.1). ■

5.3 Nonlocal backward impulsive fractional differential equations

In this section, we generalize the results of the previous section to nonlocal impulsive differential equations 5.2.

The equation (5.2) is equivalent to the following integral equation

$$
\begin{aligned}
Fu(t) &= u_T + g(u) - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} f(s,u(s))ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s,u(s))ds.
\end{aligned}
$$

Theorem 5.4 *Assume that the function* f *verifies the conditions* (**H2**), (**H5**), *and*

$$C + m\mu + \frac{\lambda(m+1)T^\alpha}{\Gamma(\alpha+1)} < 1 \tag{5.6}$$

then the problem (5.2) *has a unique solution in* $PC(J,\mathbb{R})$.

Proof. We transform the problem (5.2) into a fixed point problem. Consider the operator $F : PC(J,R) \to PC(J,R)$ defined by

$$Fu(t) = u_T + g(u) - \sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} f(s,u(s))ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s,u(s))ds.$$

As in the last paragraph, it is clear that the fixed points of the operator F are solutions of the problem (5.2). We use the Banach contraction principle to prove that F has a fixed point. We shall show that F is a contraction. Let $u, v \in PC(J,R)$. Then, for each $t \in J$, we have

$$\begin{aligned}
|Fu(t) - Fv(t)| &\leq |g(u) - g(v)| + \sum_{t<t_k<T} \left|I_k\left(u\left(t_k^-\right)\right) - I_k\left(v\left(t_k^-\right)\right)\right| \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} |f(s,u(s)) - f(s,v(s))| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |f(s,u(s)) - f(s,v(s))| \, ds \\
&\leq C|u-v| + \mu \sum_{t<t_k<T} \left|u\left(t_k^-\right) - v\left(t_k^-\right)\right| \\
&+ \frac{\lambda}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} |u(s) - v(s)| \, ds \\
&+ \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |u(s) - v(s)| \, ds \\
&\leq C\|u-v\| + m\mu\|u-v\| + \frac{\lambda m T^\alpha}{\Gamma(\alpha+1)} \|u-v\| \\
&+ \frac{\lambda T^\alpha}{\Gamma(\alpha+1)} \|u-v\| \\
&= \left(C + m\mu + \frac{\lambda m T^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda T^\alpha}{\Gamma(\alpha+1)}\right) \|u-v\| \\
&= \left(C + m\mu + \frac{\lambda(m+1)T^\alpha}{\Gamma(\alpha+1)}\right) \|u-v\|.
\end{aligned}$$

Hence, by (5.6), F is a contraction. Then, by the Banach contraction principle, we deduce that F has a unique fixed point which is a solution of the problem (5.2). ■

Theorem 5.5 *If* **(H1)**, **(H3)**, **(H6)**, **(H7)** *and* **(H8)** *are satisfied, and if* $C < 1$, *then the Problem* (5.2) *has at least a solution in* $PC(J,\mathbb{R})$.

Proof. Let

$$\frac{|u_T|}{1-\left(C+m\gamma+\frac{(m+1)\theta T^{\alpha}}{\Gamma(\alpha+1)}\right)}\leq r \tag{5.7}$$

and define the operators P and Q on the compact set $B_r \subset PC(J,\mathbb{R})$ by :

$$Pu(t)=u_T+g(u)$$

and

$$\begin{aligned} Qu(t) &= -\sum_{t<t_k<T} I_k\left(u\left(t_k^-\right)\right) - \frac{1}{\Gamma(\alpha)}\sum_{t<t_k<T}\int_{t_k}^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}f(s,u(s))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,u(s))ds. \end{aligned}$$

Firstly, P maps B_r into itself i.e. $PB_r \subset B_r$.

For all $u(t)\in B_r$ we have

$$|Pu(t)|=|u_T+g(u)|\leq|u_T|+|g(u)|\leq r.$$

Hence, the operator P maps B_r into itself. We prove that P is a contraction map. Let $u,v\in PC(J,\mathbb{R})$,then

$$|Pu(t)-Pv(t)|=|g(u)-g(v)|\leq C|u(t)-v(t)|$$

then the operator P satisfies the contraction property, and

$$\begin{aligned} |Qv(t)| &\leq \sum_{t<t_k<T}\left|I_k\left(v\left(t_k^-\right)\right)\right| + \frac{1}{\Gamma(\alpha)}\sum_{t<t_k<T}\int_{t_k}^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}|f(s,u(s))|\,ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,u(s))|\,ds \\ &\leq \sum_{t<t_k<T}\left|I_k\left(v\left(t_k^-\right)\right)\right| + \frac{1}{\Gamma(\alpha)}\|f(s,u(s))\|\sum_{t<t_k<T}\int_{t_k}^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}ds \\ &+ \frac{1}{\Gamma(\alpha)}\|f(s,u(s))\|\int_{t_k}^{t}(t-s)^{\alpha-1}ds \\ &\leq \left(m\gamma+\frac{m\theta T^{\alpha}}{\Gamma(\alpha+1)}+\frac{\theta T^{\alpha}}{\Gamma(\alpha+1)}\right)\|v(s)\| \\ &= \left(m\gamma+\frac{(m+1)\,\theta T^{\alpha}}{\Gamma(\alpha+1)}\right)\|v(s)\| \end{aligned}$$

hence

$$\begin{aligned}
|Pu(t)+Qv(t)| &\le |Pu(t)|+|Qv(t)| \\
&\le |u_{T}|+C|u|+\left(m\gamma+\frac{(m+1)\,\theta T^{\alpha}}{\Gamma(\alpha+1)}\right)|v(s)| \\
&\le |u_{T}|+Cr+\left(m\gamma+\frac{(m+1)\,\theta T^{\alpha}}{\Gamma(\alpha+1)}\right)r \\
&= |u_{T}|+\left(C+m\gamma+\frac{(m+1)\,\theta T^{\alpha}}{\Gamma(\alpha+1)}\right)r \\
&\le r.
\end{aligned}$$

Therefore, if $u, v \in B_r$, then $Pu+Qv \in B_r$. Obviously, in view of the condition (**H1**), Q is continuous and by the inequality (5.7), it is uniformly bounded on B_r. Evidently, the equicontinuity of $Qv(t)$ follows from Theorem 5.3. Hence, by the Arzela Ascoli Theorem, $Q(B_r)$ is relatively compact which implies that Q is compact. Therefore, using Krasnoselkii Theorem, there exists a solution to equation (5.2). ■

5.4 Ulam stability results for some nonlinear backward impulsive fractional differential equations

In this section, we study the Ulam stability of the solution of the problem (5.1). We start by introducing Ulam's type stability concepts for equation (5.1). Let $\varepsilon>0$, $\psi\ge 0$ and $\varphi\in PC(J,\mathbb{R}^{+})$ is nondecreasing. We consider the following inequalities

$$\begin{cases} |D^{\alpha}u(t)-f(t,u(t))|\le\varepsilon, & t\in J',\ t\ne t_k \\ \left|\Delta u|_{t=t_k}-I_k(u(t_k^{-}))\right|\le\varepsilon, & k=1,\cdots,m \end{cases} \tag{5.8}$$

$$\begin{cases} |D^{\alpha}u(t)-f(t,u(t))|\le\varphi(t), & t\in J',\ t\ne t_k \\ \left|\Delta u|_{t=t_k}-I_k(u(t_k^{-}))\right|\le\psi, & k=1,\cdots,m \end{cases} \tag{5.9}$$

and

$$\begin{cases} |D^{\alpha}u(t)-f(t,u(t))|\le\varepsilon\varphi(t), & t\in J',\ t\ne t_k \\ \left|\Delta u|_{t=t_k}-I_k(u(t_k^{-}))\right|\le\varepsilon\psi, & k=1,\cdots,m \end{cases} \tag{5.10}$$

Definition 5.6 *Equation (5.1) is Ulam–Hyers stable if there exists a real number $c_{f,m}>0$ such that for each $\varepsilon>0$ and for each solution $y\in PC^{1}(J,\mathbb{R})$ of inequality (5.8) there exists* *a solution* $x \in PC^1(J,\mathbb{R})$ *of equation* (5.1) *with*

$$|y(t) - x(t)| \leq c_{f,m}\varepsilon, t \in J.$$

Definition 5.7 *Equation* (5.1) *is generalized Ulam–Hyers stable if there exists* $\theta_{f,m} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\theta_{f,m}(0) = 0$ *such that for each solution* $y \in PC^1(J,\mathbb{R})$ *of inequality* (5.8) *there exists a solution* $x \in PC^1(J,\mathbb{R})$ *of equation* (5.1) *with*

$$|y(t) - x(t)| \leq \theta_{f,m}(\varepsilon), t \in J.$$

Definition 5.8 *Equation* (5.1) *is Ulam–Hyers–Rassias stable with respect to* (φ, ψ) *if there exists* $c_{f,m,\varphi} > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $y \in PC^1(J,\mathbb{R})$ *of inequality* (5.10) *there exists a solution* $x \in PC^1(J,\mathbb{R})$ *of equation* (5.1) *with*

$$|y(t) - x(t)| \leq c_{f,m,\varphi}\varepsilon(\varphi(t) + \psi), t \in J.$$

Definition 5.9 *Equation* (5.1) *is generalized Ulam–Hyers–Rassias stable with respect to* (φ, ψ) *if there exists* $c_{f,m,\varphi} > 0$ *such that for each solution* $y \in PC^1(J,\mathbb{R})$ *of inequality* (5.9) *there exists a solution* $x \in PC^1(J,\mathbb{R})$ *of equation* (5.1) *with*

$$|y(t) - x(t)| \leq c_{f,m,\varphi}(\varphi(t) + \psi), t \in J.$$

Remark 5.10 *It is clear that: (i) Definition.5.6 implies Definition 5.7; (ii) Definition 5.8 implies Definition 5.9; (iii) Definition 5.8. for* $\varphi(t) = \psi = 1$ *implies Definition 5.6.*

Remark 5.11 *A function* $y \in PC^1(J,\mathbb{R})$ *is a solution of inequality* (5.8) *if and only if there is* $g \in PC(J,\mathbb{R})$ *and a sequence* g_k, $k = 1, 2, ..., m$ *(which depend on y) such that :*

(i) $|g(t)| \leq \varepsilon, t \in J$ *and* $|g_k| \leq \varepsilon, \ k = 1, 2, ..., m,$

(ii) $D^\alpha u(t) = f(t, u(t)) + g(t), \ t \in J',$

(iii) $\Delta y(t_k) = I_k(y(t_k^-)) + g_k, k = 1, 2, ..., m.$

We can have similar remarks for the inequalities (5.9) and (5.10).

So, the Ulam stabilities of the impulsive differential equations are some special types of data dependence of the solutions of impulsive differential equations.

Proposition 5.12 *If $y \in PC^1(J,\mathbb{R})$ is a solution of inequality (5.8), then y is a solution of the following inequality*

$$\left| y(t) - u_T + \sum_{t<t_k<T} I_k\left(y\left(t_k^-\right)\right) + \frac{1}{\Gamma(\alpha)} \sum_{t<t_k<T} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} f(s,y(s))ds \right.$$
$$\left. -\frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} f(s,y(s))ds \right| \le \frac{(\alpha m\Gamma(\alpha) + mT^{\alpha} + (t-T)^{\alpha})}{\Gamma(\alpha+1)}\varepsilon, \quad t \in J.$$

Proof. Indeed, by remark (5.11), we have that

$$\begin{cases} D^{\alpha}y(t) = f(t,y(t)) + g(t), t \in J'; \\ \Delta y(t_k) = I_k(y(t_k^-)) + g_k, k = 1,2,...,m. \end{cases}$$

Then, for $t \in (t_{m-k}, t_{m-k+1})$ for $k = 0,...,m$.

$$\left| y(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^-\right)\right) + \sum_{p=0}^{k-1} g_i + \tfrac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} (t_{m-p+1}-s)^{\alpha-1} f(s,y(s))ds \right.$$
$$+\tfrac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} (t_{m-p+1}-s)^{\alpha-1} g(s)ds - \tfrac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} (t-s)^{\alpha-1} f(s,y(s))ds$$
$$\left. -\tfrac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} (t-s)^{\alpha-1} g(s)ds \right| \le \left| y(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^-\right)\right) \right.$$
$$\left. +\tfrac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} (t_{m-p+1}-s)^{\alpha-1} f(s,y(s))ds - \tfrac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} (t-s)^{\alpha-1} f(s,y(s))ds \right|$$
$$+\sum_{p=0}^{k-1} |g_i| + \tfrac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} (t_{m-p+1}-s)^{\alpha-1} |g(s)|\, ds + \tfrac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} (t-s)^{\alpha-1} g(s)ds.$$

From this it follows

$$\left| y(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(y\left(t_{m-p}^-\right)\right) + \tfrac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} (t_{m-p+1}-s)^{\alpha-1} f(s,y(s))ds \right.$$
$$\left. -\tfrac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} (t-s)^{\alpha-1} f(s,y(s))ds \right| \le \sum_{i=1}^{m} \varepsilon + \tfrac{\varepsilon}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{\alpha-1} ds$$
$$+\tfrac{\varepsilon}{\Gamma(\alpha)} \int_{T}^{t} (t-s)^{\alpha-1} ds$$
$$\le \sum_{i=1}^{m} \varepsilon - \tfrac{\varepsilon}{\alpha\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{\alpha-1} ds + \tfrac{\varepsilon}{\alpha\Gamma(\alpha)} \int_{T}^{t} (t-s)^{\alpha-1} ds$$
$$\le m\varepsilon + \tfrac{\varepsilon}{\alpha\Gamma(\alpha)} \left(\sum_{i=1}^{m} (t_{i+1}-t_i)^{\alpha} + (t-T)^{\alpha} \right)$$
$$\le m\varepsilon + \tfrac{\varepsilon}{\alpha\Gamma(\alpha)} (mT^{\alpha} + (t-T)^{\alpha}) = \tfrac{(\alpha m\Gamma(\alpha) + mT^{\alpha} + (t-T)^{\alpha})}{\Gamma(\alpha+1)}\varepsilon.$$

Thus, the proof of the proposition is complete. ■

We can obtain similar propositions for the solutions of the inequalities (5.9) and (5.10).

Theorem 5.13 *If the assumptions* **(H1)**, **(H2)** *and* **(H5)** *hold. Then equation* (5.1) *is generalized Ulam-Hyers stable.*

Proof. Let $u \in PC^1(J,\mathbb{R})$ be a solution of inequality (5.9). Denote by x the unique solution of the backward impulsive problem

$$\begin{cases} D^\alpha u(t) = f(t,u(t)), & t \in J = [0,T], \ t \neq t_k \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), & k = 1, \cdots, m \\ u(T) = u_T \end{cases}$$

Then we have

$$u(t) = u_T - \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} (t_{m-p+1}-s)^{\alpha-1} f(s,u(s))ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} (t-s)^{\alpha-1} f(s,u(s))ds,$$

where $t \in (t_{m-k}, t_{m-k+1})$ for $k = 0, ..., m$. Like in proposition (5.12), by differential inequality, for each $t \in (t_{m-k}, t_{m-k+1})$, we have

$$\Bigg| u(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right) + \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} (t_{m-p+1}-s)^{\alpha-1} f(s,u(s))ds$$
$$- \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} (t-s)^{\alpha-1} f(s,u(s))ds \Bigg|$$
$$\leq \sum_{i=1}^{m} |g_i| + \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} (t_{i+1}-s)^{\alpha-1} \varepsilon ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} (t-s)^{\alpha-1} \varepsilon ds$$
$$\leq m\varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)} \sum_{i=0}^{m} (t_{i+1}-t_i)^\alpha + \frac{\varepsilon}{\alpha\Gamma(\alpha)} (t-t_{m-k})^\alpha$$
$$\leq m\varepsilon + \frac{\varepsilon}{\Gamma(\alpha+1)} mT^\alpha + \frac{\varepsilon}{\Gamma(\alpha+1)} T^\alpha = \frac{\varepsilon}{\Gamma(\alpha+1)} (m\varepsilon + (m+1)T^\alpha)$$

Hence for each $t \in (t_{m-k}, t_{m-k+1})$ for $k = 0, ..., m$, it follows

$$|u(t) - x(t)| \leq \Bigg| u(t) - u_T + \sum_{p=0}^{k-1} I_{m-p}\left(u\left(t_{m-p}^-\right)\right)$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{m-p}}^{t_{m-p+1}} (t_{m-p+1}-s)^{\alpha-1} f(s,u(s))ds$$
$$- \frac{1}{\Gamma(\alpha)} \int_{t_{m-k}}^{t} (t-s)^{\alpha-1} f(s,u(s))ds \Bigg|$$
$$+ \sum_{p=0}^{k-1} \left| I_{m-p}\left(u\left(t_{m-p}^-\right)\right) - I_{m-p}\left(x\left(t_{m-p}^-\right)\right) \right|$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{p=0}^{k}\int_{t_{m-p}}^{t_{m-p+1}}(t_{m-p+1}-s)^{\alpha-1}\left|f(s,u(s))-f(s,x(s))\right|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}(t-s)^{\alpha-1}\left|f(s,u(s))-f(s,x(s))\right|ds$$

$$\leq\frac{\varepsilon}{\Gamma(\alpha+1)}\left(m\varepsilon+(m+1)T^{\alpha}\right)+\sum_{p=0}^{k-1}\mu_{k}\left|u\left(t_{m-p}^{-}\right)-x\left(t_{m-p}^{-}\right)\right|$$
$$+\frac{\lambda}{\Gamma(\alpha)}\sum_{p=0}^{k}\int_{t_{m-p}}^{t_{m-p+1}}(t_{m-p+1}-s)^{\alpha-1}\left|u(s)-x(s)\right|ds$$
$$+\frac{\lambda}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}(t-s)^{\alpha-1}\left|u(s)-x(s)\right|ds$$
$$\leq\frac{\varepsilon}{\Gamma(\alpha+1)}\left(m\varepsilon+(m+1)T^{\alpha}\right)+\left\|u-x\right\|_{PC}\sum_{p=0}^{k-1}\mu_{k}$$
$$+\frac{\lambda\left\|u-x\right\|_{PC}}{\Gamma(\alpha)}\sum_{p=0}^{k}\int_{t_{m-p}}^{t_{m-p+1}}(t_{m-p+1}-s)^{\alpha-1}ds$$
$$+\frac{\lambda\left\|u-x\right\|_{PC}}{\Gamma(\alpha)}\int_{t_{m-k}}^{t}(t-s)^{\alpha-1}ds$$
$$\leq\frac{\varepsilon}{\Gamma(\alpha+1)}\left(m\varepsilon+(m+1)T^{\alpha}\right)+\left\|u-x\right\|_{PC}\sum_{p=0}^{k-1}\mu_{k}$$
$$+\frac{\lambda\left\|u-x\right\|_{PC}}{\alpha\Gamma(\alpha)}\sum_{p=0}^{k}(t_{m-p+1}-t_{m-p})^{\alpha}+\frac{\lambda\left\|u-x\right\|_{PC}}{\alpha\Gamma(\alpha)}(t-t_{m-k})^{\alpha}$$
$$\leq\frac{\varepsilon}{\Gamma(\alpha+1)}\left(m\varepsilon+(m+1)T^{\alpha}\right)+\left\|u-x\right\|_{PC}\sum_{p=0}^{k-1}\mu_{k}$$
$$+\frac{\lambda\left\|u-x\right\|_{PC}}{\Gamma(\alpha+1)}mT^{\alpha}+\frac{\lambda\left\|u-x\right\|_{PC}}{\Gamma(\alpha+1)}(t-t_{m-k})^{\alpha}$$

From which we have

$$\left\|u-x\right\|_{PC}\leq\frac{\varepsilon}{\Gamma(\alpha+1)}\left(m\varepsilon+(m+1)T^{\alpha}\right)+\left\|u-x\right\|_{PC}\sum_{p=0}^{k-1}\mu_{k}$$
$$+\frac{\lambda\left\|u-x\right\|_{PC}}{\Gamma(\alpha+1)}mT^{\alpha}+\frac{\lambda\left\|u-x\right\|_{PC}}{\Gamma(\alpha+1)}(t-t_{m-k})^{\alpha}$$

which implies that

$$\left\|u-x\right\|_{PC}\leq\frac{1}{1-\left[\sum_{p=0}^{k-1}\mu_{k}+\frac{\lambda}{\Gamma(\alpha+1)}mT^{\alpha}+\frac{\lambda}{\Gamma(\alpha+1)}(t-t_{m-k})^{\alpha}\right]}\frac{\varepsilon}{\Gamma(\alpha+1)}\left(m\varepsilon+(m+1)T^{\alpha}\right)$$

Then

$$\left\|u-x\right\|_{PC}\leq\frac{\left(m\varepsilon+(m+1)T^{\alpha}\right)\varepsilon}{\Gamma(\alpha+1)-\left[\Gamma(\alpha+1)\sum_{p=0}^{k-1}\mu_{k}+\lambda mT^{\alpha}+\lambda(t-t_{m-k})^{\alpha}\right]}$$

Thus, equation (5.1) is generalized Ulam–Hyers stable. The proof is completed. ■

Remark 5.14 *By similar process we can extend the above results to the case of the equation* (5.2) *.*

5.5 Applications

Example 5.1

Consider the backward impulsive fractional differential equation

$$\begin{cases} D^{\alpha}u(t) = 0, \quad t \in J' = [0,1] - \{1/2\}, \\ \Delta u|_{t=\frac{1}{2}} = \dfrac{\left|u\left(\frac{1}{2}^{-}\right)\right|}{1+\left|u\left(\frac{1}{2}^{-}\right)\right|}, \\ u(1) = 1. \end{cases} \tag{5.11}$$

where $0 < \alpha < 1$, and the inequalities

$$\begin{cases} |D^{\alpha}y(t)| \leq \varepsilon, \quad t \in J' = [0,1] - \{1/2\}, \\ \left|\Delta y|_{t=\frac{1}{2}} - \dfrac{\left|y\left(\frac{1}{2}^{-}\right)\right|}{1+\left|y\left(\frac{1}{2}^{-}\right)\right|}\right| \leq \varepsilon, \quad \varepsilon > 0. \end{cases} \tag{5.12}$$

Let $y \in PC^1([0,1],\mathbb{R})$ be a solution to the inequality (5.12). Then there exist $g \in PC^1([0,1],\mathbb{R})$ and $g_1 \in \mathbb{R}$ such that:

$$|g(t)| \leq \varepsilon, \quad t \in [0,1], \text{ and } |g_1| \leq \varepsilon, \tag{5.13}$$

$$D^{\alpha}y(t) = g(t), \quad t \in J' = [0,1] - \{1/2\}, \tag{5.14}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{\left|y\left(\frac{1}{2}^{-}\right)\right|}{1+\left|y\left(\frac{1}{2}^{-}\right)\right|} + g_1. \tag{5.15}$$

Integrating (5.14) from t to 1 via (5.15), we obtain

$$\begin{aligned} y(t) &= y(1) - \left(I_{\frac{1}{2}}\left(y\left(t_{\frac{1}{2}}^{-}\right)\right) + g_1\right) - \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{1}(1-s)^{\alpha-1}g(s)ds + \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{t}(t-s)^{\alpha-1}g(s)ds \\ &= 1 - \left(I_{\frac{1}{2}}\left(y\left(t_{\frac{1}{2}}^{-}\right)\right) + g_1\right) - \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{1}(1-s)^{\alpha-1}g(s)ds + \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{t}(t-s)^{\alpha-1}g(s)ds. \end{aligned}$$

Let us consider the solution u of (5.11) given by

$$u(t) = 1 - \frac{\left|u\left(\frac{1}{2}^-\right)\right|}{1 + \left|u\left(\frac{1}{2}^-\right)\right|}.$$

Then we can write

$$\begin{aligned}
|y(t) - u(t)| &= \left| \frac{\left|u\left(\frac{1}{2}^-\right)\right|}{1 + \left|u\left(\frac{1}{2}^-\right)\right|} - \frac{\left|y\left(\frac{1}{2}^-\right)\right|}{1 + \left|y\left(\frac{1}{2}^-\right)\right|} - g_1 \right. \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{2}}^{1} (1-s)^{\alpha-1} g(s)ds + \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{2}}^{t} (t-s)^{\alpha-1} g(s)ds \right| \\
&\le \left| \frac{\left|u\left(\frac{1}{2}^-\right)\right|}{1 + \left|u\left(\frac{1}{2}^-\right)\right|} - \frac{\left|y\left(\frac{1}{2}^-\right)\right|}{1 + \left|y\left(\frac{1}{2}^-\right)\right|} \right| + |g_1| + \left| \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{2}}^{1} (1-s)^{\alpha-1} g(s)ds \right| \\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{2}}^{t} (t-s)^{\alpha-1} g(s)ds \right| \\
&\le \left| \left|u\left(\frac{1}{2}^-\right)\right| - \left|y\left(\frac{1}{2}^-\right)\right| \right| + |g_1| + \left| \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{2}}^{1} (1-s)^{\alpha-1} g(s)ds \right| \\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{2}}^{t} (t-s)^{\alpha-1} g(s)ds \right| \\
&\le \left| u\left(\frac{1}{2}^-\right) - y\left(\frac{1}{2}^-\right) \right| + |g_1| + \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{2}}^{1} (1-s)^{\alpha-1} |g(s)|\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{\frac{1}{2}}^{t} (t-s)^{\alpha-1} |g(s)|\, ds \\
&\le \left| u\left(\frac{1}{2}^-\right) - y\left(\frac{1}{2}^-\right) \right| + \varepsilon + \frac{\varepsilon}{\Gamma(\alpha)} \int_{\frac{1}{2}}^{1} (1-s)^{\alpha-1} ds \\
&+ \frac{\varepsilon}{\Gamma(\alpha)} \int_{\frac{1}{2}}^{t} (t-s)^{\alpha-1} ds \\
&= \left| u\left(\frac{1}{2}^-\right) - y\left(\frac{1}{2}^-\right) \right| + \varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)} - (1-s)^{\alpha}\Big|_{\frac{1}{2}}^{1} + \frac{\varepsilon}{\alpha\Gamma(\alpha)} - (t-s)^{\alpha}\Big|_{\frac{1}{2}}^{t} \\
&= \left| u\left(\frac{1}{2}^-\right) - y\left(\frac{1}{2}^-\right) \right| + \varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)} \frac{1}{2^{\alpha}} + \frac{\varepsilon}{\alpha\Gamma(\alpha)} \left(t - \frac{1}{2}\right)^{\alpha} \\
&= \left| u\left(\frac{1}{2}^-\right) - y\left(\frac{1}{2}^-\right) \right| + \varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)\, 2^{\alpha}} + \frac{\varepsilon}{\alpha\Gamma(\alpha)} \left(t - \frac{1}{2}\right)^{\alpha} \\
&\le \left| u\left(\frac{1}{2}^-\right) - y\left(\frac{1}{2}^-\right) \right| + \varepsilon + \frac{\varepsilon}{\alpha\Gamma(\alpha)\, 2^{\alpha}} + \frac{\varepsilon}{\alpha\Gamma(\alpha)} \left(t - \frac{1}{2}\right)^{\alpha}, \quad t \in [0,1] \\
&\le 3\varepsilon + \frac{\varepsilon}{\Gamma(\alpha+1)\, 2^{\alpha}} + \frac{\varepsilon}{\Gamma(\alpha+1)} \left(t - \frac{1}{2}\right)^{\alpha} \\
&= \varepsilon \left(3 + \frac{1}{\Gamma(\alpha+1)\, 2^{\alpha}} + \frac{1}{\Gamma(\alpha+1)} \left(t - \frac{1}{2}\right)^{\alpha} \right), \quad t \in [0,1].
\end{aligned}$$

Thus, Equation (5.11) is generalized Ulam–Hyers stable, which is a special case of generalized Ulam–Hyers–Rassias stable.

Example 5.2

Consider the backward impulsive fractional differential equation

$$\begin{cases} D^{\alpha}u(t) = \dfrac{1}{15+e^{t}+u^{2}(t)}, & t \in J' = [0,\pi], \ t \neq t_k, \\ \Delta u|_{t=t_k} = \dfrac{1}{5k+u^{2}\left(t_k^{-}\right)}, & k = 1,\cdots,5, \quad t_k = k\frac{\pi}{5}, \\ u(\pi) = 1. \end{cases} \tag{5.16}$$

where $0 < \alpha < 1$.

Set

$$f(t,x) = \frac{1}{15+e^{t}+x^{2}}, \quad (t,x) \in J \times [0,+\infty),$$

and

$$I_k(x) = \frac{1}{5k+x^{2}}.$$

Let $(x,y) \in [0,+\infty)$ and $t \in J$. By the mean value theorem, we have

$$\begin{aligned} |f(t,x) - f(t,y)| &= \frac{2\xi}{\left(15+e^{t}+\xi^{2}\right)^{2}}|x-y| \\ &\leq \frac{2\xi}{\left(15+\xi^{2}\right)^{2}}|x-y|, \quad x<\xi<y. \end{aligned}$$

Then

$$|f(t,x) - f(t,y)| \leq \frac{\sqrt{5}}{200}|x-y|.$$

Hence, the condition (**H2**) holds with $\lambda = \sqrt{5}/200$.

Let $(x,y) \in [0,+\infty)$. By the mean value theorem, we have

$$|I_k(x) - I_k(y)| = \frac{2\xi}{\left(5k+\xi^{2}\right)^{2}}|x-y| \leq \frac{2\xi}{\left(5+\xi^{2}\right)^{2}}|x-y|, \quad x<\xi<y.$$

Then

$$|I_k(x) - I_k(y)| \leq \frac{3\sqrt{15}}{200}|x-y|.$$

Hence, the condition (**H5**) holds with $\mu = 3\sqrt{15}/200$.

Since

$$m\mu+\frac{\lambda\left(m+1\right)}{\Gamma\left(\alpha+1\right)}T^{\alpha}=5\times\frac{3\sqrt{15}}{200}+\frac{\sqrt{5}}{200}\frac{\left(5+1\right)}{\Gamma\left(\alpha+1\right)}\pi^{\alpha}=\frac{3\sqrt{15}}{40}+\frac{\sqrt{5}}{200}\frac{6}{\Gamma\left(\alpha+1\right)}\pi^{\alpha}$$
$$=\frac{3\sqrt{5}\left(2\pi^{\alpha}+5\sqrt{3}\Gamma\left(\alpha+1\right)\right)}{200\Gamma\left(\alpha+1\right)}<1.$$

the condition (5.5) is satisfied. Thus, by Theorem 5.2, the problem (5.16) has a unique solution.

Now, we proceed as in the proof of Theorem 5.13. Let $u\in PC^{1}(J,\mathbb{R})$ be a solution of the inequality (5.9). Denote by x the unique solution to the backward impulsive problem (5.16). Then we obtain

$$x\left(t\right)=1-\sum_{p=0}^{k-1}I_{5-p}\left(x\left(t_{5-p}^{-}\right)\right)-\frac{1}{\Gamma\left(\alpha\right)}\sum_{p=0}^{k}\int_{t_{5-p}}^{t_{5-p+1}}\left(t_{5-p+1}-s\right)^{\alpha-1}f(s,x(s))ds$$
$$+\frac{1}{\Gamma\left(\alpha\right)}\int_{t_{5-k}}^{t}\left(t-s\right)^{\alpha-1}f(s,x(s))ds,$$

where $t\in\left(t_{5-k},t_{5-k+1}\right)$ for $k=0,...,5$.

Like in proposition 5.12, by differentiation of the inequality, for each $t\in\left(t_{5-k},t_{5-k+1}\right)$, we have

$$\Bigg|x\left(t\right)-1+\sum_{p=0}^{k-1}I_{5-p}\left(x\left(t_{5-p}^{-}\right)\right)+\frac{1}{\Gamma\left(\alpha\right)}\sum_{p=0}^{k}\int_{t_{5-p}}^{t_{5-p+1}}\left(t_{5-p+1}-s\right)^{\alpha-1}\frac{ds}{15+e^{s}+u^{2}\left(s\right)}$$
$$-\frac{1}{\Gamma\left(\alpha\right)}\int_{t_{5-k}}^{t}\left(t-s\right)^{\alpha-1}\frac{ds}{15+e^{s}+u^{2}\left(s\right)}\Bigg|$$

$$\leq\sum_{i=1}^{5}\left|g_{i}\right|+\frac{1}{\Gamma\left(\alpha\right)}\sum_{p=0}^{k}\int_{t_{5-p}}^{t_{5-p+1}}\left(t_{5-p+1}-s\right)^{\alpha-1}g\left(s\right)ds-\frac{1}{\Gamma\left(\alpha\right)}\int_{t_{5-k}}^{t}\left(t-s\right)^{\alpha-1}g\left(s\right)ds$$
$$\leq\sum_{i=1}^{5}\left|g_{i}\right|+\frac{1}{\Gamma\left(\alpha\right)}\sum_{p=0}^{5}\int_{0}^{\pi}\left(\pi-s\right)^{\alpha-1}g\left(s\right)ds-\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-s\right)^{\alpha-1}g\left(s\right)ds$$
$$\leq\sum_{i=1}^{5}\left|g_{i}\right|+\frac{1}{\Gamma\left(\alpha\right)}\sum_{p=0}^{5}\int_{0}^{\pi}\left(\pi-s\right)^{\alpha-1}\varepsilon ds-\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}\left(t-s\right)^{\alpha-1}\varepsilon ds$$
$$\leq 5\varepsilon+\frac{5\varepsilon}{\alpha\Gamma\left(\alpha\right)}\pi^{\alpha}-\frac{\varepsilon}{\alpha\Gamma\left(\alpha\right)}t^{\alpha}=\frac{5\Gamma\left(\alpha+1\right)+5\pi^{\alpha}-t^{\alpha}}{\Gamma\left(\alpha+1\right)}\varepsilon.$$

Hence, for each $t \in (t_{5-k}, t_{5-k+1})$ and $k = 0, ..., 5$, we can write

$$
\begin{aligned}
|u(t) - x(t)| \leq & \left| x(t) - 1 + \sum_{p=0}^{k-1} I_{5-p}\left(x\left(t_{5-p}^{-}\right)\right) + \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{5-p}}^{t_{5-p+1}} (t_{5-p+1} - s)^{\alpha-1} \frac{ds}{15 + e^s + u^2(s)} \right. \\
& \left. - \frac{1}{\Gamma(\alpha)} \int_{t_{5-k}}^{t} (t-s)^{\alpha-1} \frac{ds}{5 + e^s + u^2(s)} \right| + \sum_{p=0}^{k-1} \left| I_{5-p}\left(u\left(t_{5-p}^{-}\right)\right) - I_{5-p}\left(x\left(t_{5-p}^{-}\right)\right) \right| \\
& + \frac{1}{\Gamma(\alpha)} \sum_{p=0}^{k} \int_{t_{5-p}}^{t_{5-p+1}} (t_{5-p+1} - s)^{\alpha-1} \left| \frac{1}{15 + e^s + u^2(s)} - \frac{1}{15 + e^s + x^2(s)} \right| ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_{5-k}}^{t} (t-s)^{\alpha-1} \left| \frac{1}{15 + e^s + u^2(s)} - \frac{1}{15 + e^s + x^2(s)} \right| ds \\
& \leq \frac{5\Gamma(\alpha+1) + 5\pi^\alpha - t^\alpha}{\Gamma(\alpha+1)} \varepsilon + \|u - x\|_{PC} \sum_{p=0}^{k-1} \mu_k + \frac{\lambda \|u - x\|_{PC}}{\Gamma(\alpha+1)} \sum_{p=0}^{k} (t_{5-p+1} - t_{5-p})^\alpha \\
& + \frac{\lambda \|u - x\|_{PC}}{\Gamma(\alpha+1)} (t - t_{5-k})^\alpha \\
& \leq \frac{5\Gamma(\alpha+1) + 5\pi^\alpha - t^\alpha}{\Gamma(\alpha+1)} \varepsilon + \|u - x\|_{PC} \sum_{p=0}^{k-1} \mu_k + \frac{\lambda \|u - x\|_{PC}}{\Gamma(\alpha+1)} 5\pi^\alpha + \frac{\lambda \|u - x\|_{PC}}{\Gamma(\alpha+1)} t^\alpha.
\end{aligned}
$$

From which we have

$$
\begin{aligned}
\|u - x\|_{PC} &\leq \frac{1}{1 - \left[\sum_{p=0}^{k-1} \mu_k + \frac{\lambda}{\Gamma(\alpha+1)} 5\pi^\alpha + \frac{\lambda}{\Gamma(\alpha+1)} t^\alpha\right]} \frac{5\Gamma(\alpha+1) + 5\pi^\alpha - t^\alpha}{\Gamma(\alpha+1)} \varepsilon \\
&= \frac{(5\Gamma(\alpha+1) + 5\pi^\alpha - t^\alpha)\varepsilon}{\Gamma(\alpha+1) - \left[\Gamma(\alpha+1) \sum_{p=0}^{k-1} \mu_k + \lambda(5\pi^\alpha + t^\alpha)\right]}.
\end{aligned}
$$

Recalling that $\lambda = \sqrt{5}/200$ and $\mu_k = 3\sqrt{15}/200$, we find

$$
\begin{aligned}
\|u - x\|_{PC} &\leq \frac{(5\Gamma(\alpha+1) + 5\pi^\alpha - t^\alpha)\varepsilon}{\Gamma(\alpha+1) - \left[\Gamma(\alpha+1) 5\frac{3\sqrt{15}}{200} + \frac{\sqrt{5}}{200}(\pi^\alpha + t^\alpha)\right]} \\
&= \frac{(5\Gamma(\alpha+1) + 5\pi^\alpha - t^\alpha)\varepsilon}{\Gamma(\alpha+1) - \left[\Gamma(\alpha+1) \frac{3\sqrt{15}}{40} + \frac{\sqrt{5}}{200}(\pi^\alpha + t^\alpha)\right]}.
\end{aligned}
$$

Thus

$$
\|u - x\|_{PC} \leq \theta(\varepsilon).
$$

This implies that the problem (5.16) is generalized Ulam-Hyers stable.

Chapter 6

Conclusion and future perspectives

In this last chapter we will give a summary of the contributions in this thesis work and present proposals for extending the study to other problems in different directions, using techniques similar to those used in this work.

6.1 Conclusion

In this thesis, we investigated the existence of solutions and the Ulam stability for backward impulsive differential equations on Banach spaces. The main challenges were overcome by using different classical fixed point theorems to establish the existence of solutions to our problem by adding suitable conditions on the nonlinear term. We succeeded to obtain a unique solution by using the Banach contraction principle and at least one solution by using other fixed point theorems such as Schaefer's and Krasnosel'skii fixed point theorem, both in the cases of local and nonlocal conditions. Further, we obtain generalized-Ulam-Hyers-Rassias stability results for our problem, and we presented examples to illustrate the consistency of our theoretical findings. The study was done in both ordinary and fractional differential equation. Thus, this study can be considered as a comparison between these two types of differential equations.

It is important to note that there are many applications of backward impulsive differential equations and their stability.

6.2 Future research

Many different adaptations have been left for the future due to lack of time. The following ideas could be tested:

1. It could be interesting to consider the existence and the stability of solution for backward impulsive stochastic ordinary or fractional differential equations.

2. It could be also interesting to consider the existence and the stability of solution for backward impulsive Katugampola fractional differential equations in different form.

3. It could be also interesting to consider a similar problem with the one dimensional p-laplacian operator in both ordinary and fractional differential equations.

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