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T H E S I S

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By: Abdelkrim MEHENNI

Algebraic operations on ordered structures

Publicly defended on Thursday, 17/11/2022, in front of the jury composed by:

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To the spirit of my mother who passed away, God have mercy on her.

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Abdelkrim MEHENNI

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Abstract

In this thesis, we study several classes of algebraic operations on ordered structures. More precisely, we study classes of associative and weakly associative operations on bounded lattices and bounded trellises. First, we generalize the notion of aggregation operator to f-aggregation operator with respect to an arbitrary function on a bounded lattice and we discuss its fundamental properties. Second, we study several classes of associative (resp. weakly associative) operations on trellises with additional properties, like; commutative, increasing and neutral elements.

Key-words: Lattice, Trellis, Binary Operation, f-aggregation, Uninorm, T-norm, T-conorm.

Résumé

Dans cette thèse, nous étudions des classes des opérations algébriques dans des structures ordonnées. Plus précisément, nous étudions des classes d'opérations associatives et faiblement associatives dans un treillis bornés et trellis bornés. Tout d'abord, nous généralisons la notion d'agrégation à f-agrégation par rapport à une fonction arbitraire sur un treillis borné et nous discutons ses propriétés. Deuxièmement, nous étudions des classes particulières d'opérations associatives (resp. faiblement associatives) dans des trellis borné avec des propriétés supplémentaires, comme; la commutativité, la croissance et les éléments neutres.

Mots-clés : Treillis, Trellis, Opération binaire, f-agrégation, Uninorme, T-norme, T-conorme.

Introduction

It is well-known that the transitivity of order relations are fundamental in a wide variety of mathematical theories [12, 49]. An important step in the theory of partial orderings was the postulation of greatest lower bounds (meets) and least upper bounds (joins) and the development of the lattice theory. Transitivity is a necessary property for the associativity of meet and join on lattices. Associativity has been regarded as essential of many theorems heavily dependent upon it.

Binary operations have become essential tools in lattice theory and its applications. Several notions and properties and the notion of the lattice itself can be expressed in terms of binary operations [12, 49]. Further, several classes of binary operations on bounded lattices with specific properties appear in various theoretical and application domains. Aggregation operators or aggregation functions as a particular class of binary operations have appeared in many theoretical and applied areas, for instance, in the fuzzy set theory [5, 23], in operations research, computer and information sciences, economics and social sciences [8, 15, 24, 28, 50].

Several subclasses of aggregation operators on the interval $[0, 1]$ or on bounded lattices (e.g. triangular norms (t-norms, for short) and conorms (t-conorms), uninorms, nullnorms, ...etc) were introduced, and discussed [29, 30, 32, 36, 37, 38, 48]. These subclasses play important role in the theory of fuzzy sets and fuzzy logics [3] as they generalize the basic connectives between fuzzy sets. Recently, t-norms and t-conorms have been used in multicriteria decision support and several branches of information sciences. On the interval $[0, 1]$ or on bounded lattices, the transitivity of the order relation $\leq$ and the associativity of meet $(\wedge)$ and join $(\vee)$ play an important role in the constructions, representations, and characterization of t-norms and t-conorms [30]. For instance, the meet $(\wedge)$ (resp. the join $(\vee)$) is a t-norm (resp. t-conorm).

However, many theoretical and practical developments warrant us to look beyond transitivity [56]. The notions of non-transitive relations arising in everyday observations such as games, the relation of closeness, and some from mathematical considerations such as the theory of graphs, and logic of non-transitive implications can not be overlooked. Maybe, the most common and illustrative example of a non transitive relation in our real life is the acquaintance relation: If A knows B, and B knows C, persons A and C don't need to be acquainted. The preference loop or cycle (A is preferred to B, B is preferred to C, and C is preferred to

A) is a non-transitive relation. For instance, the non-transitive relations also appear in the football tournament (e.g.; team A beats team B and team B beats team C, but it is not necessarily that team A will beat team C). Furthermore, the non-transitive relations appear in the different fields of pure and applied mathematics. In topology, the closeness relation is another example (e.g.; in the power set of $\mathbb{R}^2$ the closeness relation defined by: A is close to B if and only if $d(A, B) = 0$ is not-transitive relation). In geometry, (e.g.; let $\triangle$ be the set of straight lines in the plane $\mathbb{R}^2$ and we define the binary relation R on $\triangle$ as follows: $d_1 R d_2$ if and only if d_1 and d_2 are orthogonal, for any $d_1, d_2 \in \triangle$. The relation R is not-transitive). The theory of graphs is also concerned with non-transitive relations (e.g.; a vertex A is adjacent to B, and vertex B is adjacent to C, but A is not necessarily adjacent to C).

Fried obtained a generalized lattice called a T-lattice or a weakly associative lattice. In a series of his papers [17, 18, 19], he discussed the properties and various characterizations of weakly associative lattices and tournaments. In [20], E. Fried and G. Gratzer studied the non-associative extension of the class of distributive lattices. Skala [51] defined a reflexive and antisymmetric but not necessarily transitive relation on a given set and called it a pseudo-ordered set. Skala could extend many theorems from lattices to trellises. In [11], Chajda and Niederle studied ideals of weakly associative lattices. Gladstien [22] proved that trellises of finite length are complete if and only if every cycle of elements has a least upper bound and a greatest lower bound. Bhatta and Shashirekha [6, 42, 43] generalized this characterization in terms of joins of cycles and pseudo-chains in pseudo-ordered sets. Some fixed point theorems on lattices were extended to the case of pseudo-ordered sets and trellises by Bhatta and George [40], Stouti and Zedam [53].

In this thesis, inspired by the above extensions and developments of the notions of binary operations on lattices, we generalize specific classes of binary operations on lattices to the trellis trellises. Before that, we generalize the notion of aggregation operators to f-aggregation operators with respect to an arbitrary function on a bounded lattice and we discuss its fundamental properties. Moreover, we study particular classes of associative operations on trellises. More precisely, we study classes of binary operation that are associative, commutative increasing, and have 1 (resp. 0) as a neutral element. These classes generalize the classes of t-norms and t-conorms on bounded lattices. Furthermore, we extend the same classes on bounded trellises by considering a weakest associativity property.

This thesis is structured as follows:

- In Chapter 1, we present some preliminaries on lattices and trellises that will be needed in this thesis.
- In Chapter 2, we introduce some basic concepts on aggregation operators on bounded lattices and study the notion of f-aggregation operators with respect to a given function f on a bounded lattice. More precisely, we show some new properties of binary operations based on a given function on a lattice.
- In Chapter 3, we generalize the class of triangular norms and the class of triangular conorms on bounded lattices to the setting of bounded trellises.
- In Chapter 4, we study two classes of weakly associative operations and investigate its various properties. A class of weakly associative operations with neutral element 1 and a class of weakly associative operations with neutral element 0.
- Finally, we give a conclusion including the important results of this thesis and some future works.

Notations

1. $\leq$: Order relation.
2. $\trianglelefteq$: Pseudo-order relation.
3. $(P, \leq)$: Partially ordered set.
4. $(P, \trianglelefteq)$: Pseudo-ordered set.
5. $a \wedge b$: The meet operation of a and b.
6. $a \vee b$: The join operation of a and b.
7. $(L, \leq, \wedge, \vee)$: Lattice.
8. $(L, \trianglelefteq, \wedge, \vee)$: Trellis.
9. X^{rtr}: the set of all right-transitive elements of X.
10. X^{ltr}: the set of all left-transitive elements of X.
11. X^{tr}: the set of all transitive elements of X.
12. $X^{\wedge\text{-ass}}$: the set of all $\wedge$-associative elements of X. ‘
13. $X^{\vee\text{-ass}}$: the set of all $\vee$-associative elements of X.
14. X^{ass}: the set of all associative elements of X.
15. X^{dis}: the set of all distributive elements of X.
16. f^{-1}: The reciprocal function of f.
17. $\mathcal{A}_f(L)$: The set of all f-aggregation operators on L.
18. $T \downarrow L_1$: The restriction of the t-norm T to L_1.
19. $\mathcal{AO}_e(X)$: The class of all binary operations on X that are associative, commutative, increasing, and have an arbitrary element $e \in X$ as neutral element.
20. $Atom(X)$: The set of all atoms of X.
21. $Coatom(X)$: The set of all coatoms of X.
22. $\mathcal{WAO}_e(X)$: The class of all binary operations on X that are commutative, weakly-increasing, weakly-associative and have e as a neutral element.

1 Generalities on lattices and trellises

In this chapter, we recall the necessary basic concepts and properties of partially ordered sets and lattices. Also, we recall some basic notions of pseudo-ordered sets and trellises. Moreover, we present and study binary operations on those structures that will be needed throughout this thesis.

1.1. Partially ordered sets and lattices

This section contains the basic definitions and properties of partially ordered sets (posets, for short) and lattices. Further information can be found in [12, 31, 33, 41, 45, 49, 54].

1.1.1. Partially ordered sets

A *binary relation* R on a set X is a subset of the Cartesian product $X \times X$. A binary relation R thus contains all the pairs of points that are related to each other under R. For any binary relation R in this text we will write pRq instead of $(p, q) \in R$, whenever p is related to q via R.

An *order relation* (a *partial order*) $\leq$ on X is a binary relation on X that is reflexive (i.e., $x \leq x$, for any $x \in X$), antisymmetric (i.e., $x \leq y$ and $y \leq x$ implies $x = y$, for any $x, y \in X$) and transitive (i.e., $x \leq y$ and $y \leq z$ imply $x \leq z$, for any $x, y, z \in X$). A set P equipped with an order relation $\leq$ on P is called a *partially ordered set* (*poset*, for short) and denoted by $(P, \leq)$.

Let $(P, \leq)$ be a poset and A a subset of P. An element $x_0 \in P$ is called a *lower bound* of A if $x_0 \leq x$, for any $x \in A$. x_0 is called the *greatest lower bound* (or the *infimum*) of A if x_0 is a lower bound and $m \leq x_0$, for any lower bound m of A. The *upper bound* and *least upper bound* (or *supremum*) are defined dually.

Example 1.1. *Let $D(60)$ be the set of positive divisors of 60 and let $|$ be the divisibility relation. One can easily verify that the divisibility relation $|$ is a reflexive, antisymmetric and transitive binary relation on $D(60)$. Thus, $|$ is an order relation (a partial order) on $D(60)$ and the structure $(D(60), |)$ is a poset.*

A poset $(P, \leq)$ is called *bounded*, if it has a least and a greatest element, respectively denoted by 0 and 1, i.e., $0 \leq x \leq 1$, for any $x \in L$. Usually, the notation $(P, \leq, 0, 1)$ is used to describe a bounded poset.

Example 1.2. *The poset $(D(60), |)$ given in Example 1.1 has 1 as the least element and 60 as the greatest element. Indeed, 1 devises all the elements of $D(60)$ and any element of $D(60)$ devises 60. Thus, the structure $(D(60), |, 1, 60)$ is a bounded poset.*

For a poset $(P, \leq)$, we say that an element $y \in P$ covers an element $x \in P$ if $x < y$ (i.e., $x \leq y$ and $x \neq y$) and there is no element $z \in P$ such that $x < z < y$. The set of pairs (x, y) such that y covers x is called the *covering* relation of $(P, \leq)$.

Since an order is an example of a relation, we can draw it as a directed graph. But there is a more concise and attractive way to draw partial orders and linear orders, in which the reflexivity and transitivity of the order are implicit. A *Hasse diagram* for a partial order $\leq$ on a set X is a graph drawn in the plane, with vertices corresponding to the elements of X and edge going up from x to y if $x < y$ (so excluding $x = y$) and there is no element z with $x < z < y$. In general a poset can be conveniently represented by Hasse diagram, displaying the covering relation $<$. Note that $x < y$ if there is a sequence of connected lines upwards from x to y.

Example 1.3. *The Figure 1.1 presents the Hasse diagram of the bounded poset $(D(60), |, 1, 60)$ given in Example 1.1.*

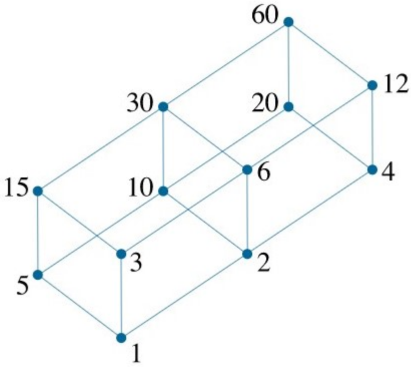


Figure 1.1: The Hasse diagram of the bounded poset $(D(60), |, 1, 60)$.

Throughout this thesis, for a given function $f : P \longrightarrow P$, we shortly write fx instead of $f(x)$.

Definition 1.1. *Let $(P, \leq)$ be a poset and f a function on P. Then f is called isotone (resp. antitone) if $x \leq y$ implies $fx \leq fy$ (resp. $fy \leq fx$), for any $x, y \in P$.*

1.1.2. Lattices

A poset $(P, \leq)$ is called a $\wedge$-*semilattice* if any two elements x and y have a greatest lower bound, denoted by $x \wedge y$ and called the *meet* (*infimum*) of x and y. Analogously, it is called a $\vee$-*semilattice* if any two elements x and y have a least upper bound, denoted by $x \vee y$ and called the *join* (*supremum*) of x and y.

A poset $(L, \leq)$ is called a *lattice* if it is both a $\wedge$- and a $\vee$-semilattice. Usually, the notation $(L, \leq, \wedge, \vee)$ is used for a lattice. A poset $(L, \leq)$ is called a *complete* lattice if every subset A of L has both a greatest lower bound, denoted by $\bigwedge A$ and called the infimum of A, and a least upper bound, denoted by $\bigvee A$ and called the supremum of A, in $(P, \leq)$.

Example 1.4. *(i) If X is a nonempty set, then the power set $\wp(X)$ is a complete lattice under the union and intersection;*

(ii) The set of real numbers $\mathbb{R}$ ordered by the usual order $\leq$ is a lattice, where min and max are its meet and join operations;

(iii) The set of positive natural numbers $\mathbb{N}^*$ with the divisibility order $|$ has a structure of a lattice, where pcm and gcd are its meet and join operations;

(iv) The poset given in Examples 1.1 is a complete lattice.

Proposition 1.1. *Let L be a finite lattice. Then L is complete.*

We introduced lattices as ordered sets of a special type. However, we may adopt an alternative viewpoint, and we view a lattice as an algebraic structure $\langle L; \wedge, \vee \rangle$. For a lattice $(L, \leq, \wedge, \vee)$, the binary operations meet and join on the non-empty set L defined by

$$a \wedge b := \inf\{a, b\} \quad \text{and} \quad a \vee b := \sup\{a, b\} \quad (a, b \in L)$$

satisfy the following algebraic properties:

(i) $x \wedge x = x \vee x = x$ (idempotency);

(ii) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ (commutativity);

(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$ (associativity);

(iv) $x \wedge (y \vee x) = x = x \vee (y \wedge x)$ (absorption-laws);

(v) the order relation $\leq$ and $\wedge$, $\vee$ are connected as:

$$x \leq y \text{ iff } x \wedge y = x \text{ iff } x \vee y = y, \text{ for any } x, y \in L.$$

A *bounded lattice* is a lattice that additionally has a greatest element 1 and a smallest element 0, which satisfy $0 \leq x \leq 1$, for any $x \in L$. A finite lattice is automatically bounded with $1 = \bigvee L$ and $0 = \bigwedge L$. Usually, the notation $(L, \leq, \wedge, \vee, 0, 1)$ is used to describe a bounded lattice.

A lattice $(L, \leq, \wedge, \vee)$ is *distributive* if, for any $x, y, z \in L$, the following additional conditions hold

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ and } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

A lattice $(L, \leq, \wedge, \vee)$ is *modular* if the following condition holds

$$x \leq z \text{ implies that } x \vee (y \wedge z) = (x \vee y) \wedge z, \text{ for any } x, y, z \in L.$$

Let $(L, \leq, \wedge, \vee)$ be a lattice and A is a proper non-empty subset of L (i.e., $A \subsetneq L$ and $A \neq \emptyset$). The set A is called:

(i) a $\wedge$-*sublattice* (resp. a $\vee$-*sublattice*) of L if $x \wedge y \in A$ (resp. $x \vee y \in A$), for any $x, y \in A$;

(ii) a *sublattice* of L if it is both a $\wedge$- and a $\vee$-sublattice of L.

Let $(L, \leq, \wedge, \vee)$ and $(M, \leq, \wedge, \vee)$ be two lattices. A function $\varphi : L \to M$ is called:

(i) a $\wedge$-*homomorphism* if $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$, for any $x, y \in L$;

(ii) a $\vee$-*homomorphism* if $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$, for any $x, y \in L$;

(iii) a *lattice-homomorphism* if it is both a $\wedge$- and a $\vee$-homomorphism;

(iv) a *lattice-isomorphism* if it is a bijective lattice-homomorphism.

If $L = M$, a lattice-homomorphism $\varphi : L \to L$ is called a *lattice-endomorphism*. Further, a lattice-isomorphism $\varphi : L \to L$ is called a *lattice-automorphism*.

Proposition 1.2. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and φ a function on L. If φ is a lattice-automorphism, then φ^{-1} is isotone. Moreover, $\varphi(0) = \varphi^{-1}(0) = 0$ and $\varphi(1) = \varphi^{-1}(1) = 1$.*

Definition 1.2. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. A function N on L is called a negation on L if it satisfies the following conditions:*

(i) $N(0) = 1$ *and* $N(1) = 0$;

(ii) N *is antitone.*

Additionally, N is called a strong negation, if it is also involutive (i.e., $N(N(x)) = x$, for any $x \in L$).

1.1.3. Binary operations on lattices

A *binary operation* on a non-empty set X is any function from the Cartesian product $X \times X$ into X (i.e., $F : X \times X \to X$). A binary operation F is called:

(i) *commutative*, if $F(x, y) = F(y, x)$, for any $x, y \in X$;

(ii) *associative*, if $F(x, F(y, z)) = F(F(x, y), z)$, for any $x, y, z \in X$.

An element $e \in X$ is called a *neutral* element of F, if $F(e, x) = F(x, e) = x$, for any $x \in X$.

Example 1.5. *On the set of real numbers $\mathbb{R}$, the addition $(+)$ and the multiplication $(\times)$ operations are commutative and associative operations. Where, 0 is the neural element of $(+)$ and 1 is the neural element of $(\times)$.*

Definition 1.3. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $F : L^2 \longrightarrow L$ a binary operation on L. A binary operation F on L is called:*

(i) idempotent, if $F(x, x) = x$, *for any* $x \in L$;

(ii) increasing, if $x_1 \leq x_2$ *and* $y_1 \leq y_2$ *imply* $F(x_1, y_1) \leq F(x_2, y_2)$, *for any* $x_1, x_2, y_1, y_2 \in L$;

(iii) conjunctive (resp. disjunctive), if $F(x, y) \leq x \wedge y$ *(resp.* $x \vee y \leq F(x, y)$*), for any* $x, y \in L$;

(iv) averaging, if $x \wedge y \leq F(x, y) \leq x \vee y$, *for any* $x, y \in L$.

1.2. Pseudo-ordered sets and trellises

This section contains the basic definitions and properties of pseudo-ordered sets, trellis structure and some notions that will be needed later. Moreover, we present and study binary operations on those structures that will be needed throughout this thesis. More information can be found in [17, 20, 21, 51, 52, 54].

1.2.1. Pseudo-ordered sets

A *pseudo-order* relation $\trianglelefteq$ on a set X is a binary relation on X that is reflexive (i.e., $x \trianglelefteq x$, for any $x \in X$) and antisymmetric (i.e., $x \trianglelefteq y$ and $y \trianglelefteq x$ implies $x = y$, for any $x, y \in X$). A set X equipped with a pseudo-order relation $\trianglelefteq$ is called a *pseudo-ordered set* (*psoset*, for short) and denoted by $(X, \trianglelefteq)$. For any two elements $a, b \in X$, if $a \trianglelefteq b$ and $a \neq b$, then we denote it as $a \triangleleft b$. If $a \trianglelefteq b$ does not hold, then we denote it by $a \ntrianglelefteq b$. Similarly to the setting of ordered sets, a finite pseudo-ordered sets can be represented by *Hasse diagram* with the convention that: x is below y and is joined to y by a line if $x \trianglelefteq y$. Otherwise, i.e., if x and y are not related, then x and y will be joined by a dashed curve.

Remark 1.1. *It is easily seen that any order relation on a set X is a pseudo-order relation on X.*

Example 1.6. *(i) Let $X = \{a, b, c, d, e\}$ and $\trianglelefteq$ a binary relation defined on X as follows:*

$$\trianglelefteq = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, d), (a, e),$$

$$(b, c), (b, e), (c, d), (c, e), (d, e)\}.$$

The couple $(X, \trianglelefteq)$ is a pseudo-ordered set. We note that $\trianglelefteq$ is not transitive because (b, c) and $(c, d) \in \trianglelefteq$, while $(b, d) \notin \trianglelefteq$.

(ii) Let $X = \mathbb{R}$ be the set of real numbers. The relation $\trianglelefteq$ on X defined, for any $x, y \in X$ as:

$$x \trianglelefteq y \text{ if and only if } 0 \leqslant y - x \leqslant a, \text{ where } a \text{ is a constant element of } \mathbb{R}^+$$

is a pseudo order relation on X.

(iii) Let $X = \{a, b, c, d, e, f\}$ be a set and $\trianglelefteq$ a pseudo-order relation on X. The Hasse diagram of X is depicted by the following figure:

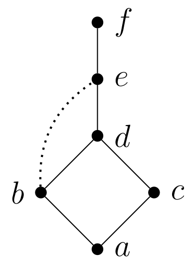


Figure 1.2: Hasse diagram of $(X, \trianglelefteq)$.

For a subset $A \subseteq X$, the notions of a lower bound, an upper bound, greatest lower bound and least upper bound are defined analogously to the corresponding notion in a poset. Now, we will only introduce the new concepts related to the pseudo-order relation.

According to Skala [52], we have the following definition.

Definition 1.4. *Let $(X, \trianglelefteq)$ be a psoset and A is a nonempty subset of X. A transitive and reflexive, but not necessarily antisymmetric relation $\trianglelefteq_A$ can be defined on A by setting $x \trianglelefteq_A x'$, for any $x, x' \in A$ if and only if there exists a finite sequence $(x_1, \ldots, x_n)$ of elements from A such that $x \trianglelefteq x_1 \trianglelefteq \ldots, x_n \trianglelefteq x'$. If one of the relations $x \trianglelefteq_A x'$ or $x' \trianglelefteq_A x$ holds, for any $x, x' \in A$, then $(A, \trianglelefteq)$ is called a pseudo-chain. A subset A is called a cycle if, for each pair of elements x and $\grave{x}$ of A both the relations $x \trianglelefteq_A \grave{x}$ and $\grave{x} \trianglelefteq_A x$ hold. Empty set is a cycle, any single element set on psoset is also a cycle. A non-trivial cycle is a cycle having more than one element and it contains at least three elements. A psoset is called acyclic if it does not contain a non-trivial cycle.*

1.2.2. Trellises

The notion of trellis was introduced by Fried [17] and Skala [51, 52] as one of the most important algebraic structures in order theory. Which is considered as an extension of the notion of lattice by dropping the property of transitivity. A trellis is defined as a psoset $(X, \trianglelefteq)$ in which pair of elements has a least upper bound and a greatest lower bound. If A is a subset of X and has a greatest lower bound or a least upper bound, then they are unique and will be denoted by $\bigwedge A$ and $\bigvee A$, respectively. If A is a pair $\{a, b\}$, we also write $\bigwedge A = a \wedge b$ and $\bigvee A = a \vee b$.

Definition 1.5. *Let $(X, \trianglelefteq)$ be a non-empty pseudo-ordered set.*

(i) $(X, \trianglelefteq)$ *is called a $\wedge$-semitrellis if $x \wedge y$ exists, for any $x, y \in X$.*

(ii) $(X, \trianglelefteq)$ *is called a $\vee$-semitrellis if $x \vee y$ exists, for any $x, y \in X$.*

(iii) $(X, \trianglelefteq)$ *is called a trellis if it is both a $\wedge$- and a $\vee$-semitrellis.*

In other words, a trellis is an algebra $(X, \wedge, \vee)$, where the binary operations $\wedge$ and $\vee$ satisfy the following properties, for any $x, y, z \in X$.

(i) $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$ *(commutativity);*

(ii) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ *(absorption identity);*

(iii) $x \wedge ((x \vee y) \wedge (x \vee z)) = x = x \vee ((x \wedge y) \vee (x \wedge z))$ *(weak-associativity).*

For a given trellis $(X, \trianglelefteq, \wedge, \vee)$ and $x, y \in X$, the following statements are equivalent:

(i) $x \trianglelefteq y$;

(ii) $x \wedge y = x$;

(iii) $x \vee y = y$.

Theorem 1.1. *[51, 52] A set X with two commutative, absorptive, and weak-associative binary operations $\wedge$ and $\vee$ is a trellis if $a \trianglelefteq b$ is defined as $a \wedge b = a$ and/or $a \vee b = b$. These operations are also idempotent.*

Theorem 1.2. *[52] Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis. The following statements are equivalent:*

(i) $\trianglelefteq$ *is transitive;*

(ii) The meet $(\wedge)$ and the join $(\vee)$ operations are associative;

(iii) One of the operations $(\wedge)$ or $(\vee)$ is associative.

Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis and A a proper non-empty subset of X (i.e., $A \subsetneq X$ and $A \neq \emptyset$). The set A is called:

(i) a $\wedge$-subtrellis (resp. a $\vee$-subtrellis) of X if $x \wedge y \in A$ (resp. $x \vee y \in A$), for any $x, y \in A$, where $\wedge$ and $\vee$ are taken in X;

(ii) a sublattice of X if it is both a $\wedge$- and a $\vee$-subtrellis of X.

The notions of complete trellis, distributive and modular trellis are defined analogously to the corresponding notions in lattices.

Remark 1.2. *It is well-known that every finite lattice is complete, but it is not true in the finite trellis. Indeed, let us consider the trellis* $(X, \trianglelefteq, \wedge, \vee)$ *given in the following table and Figure 1.3 .*

$\vee$ \ $\wedge$	a	b	c	d	e
a		b	c	d	a
b	a		e	e	e
c	a	a		e	e
d	a	a	a		e
e	e	b	c	d	

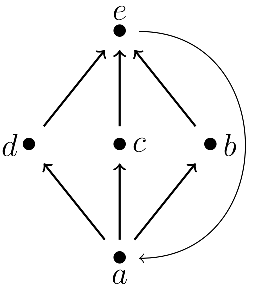


Figure 1.3: Finite trellis not complete.

One easily verifies that X *is not complete, since for example* $\bigvee\{a, b, d\}$ *does not exist.*

Definition 1.6. [52] *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis. We say* X *has a greatest element if there exists* $u \in X$ *such that* $x \trianglelefteq u$*, for any* $x \in X$*. Dually, we say* X *has a smallest element if there exists* $v \in X$ *such that* $v \trianglelefteq x$*, for any* $x \in X$*. A trellis* $(X, \trianglelefteq, \wedge, \vee)$ *possessing* 0 *and* 1 *is called a bounded trellis.*

Proposition 1.3. *[51] Let* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *be a bounded modular trellis. The following implications hold for any* $x, y, z \in X$*:*

(i) If $x \trianglelefteq y$ *and* $y \wedge z = 0$*, then* $x \trianglelefteq y \vee z$*;*

(ii) If $y \trianglelefteq x$ *and* $y \vee z = 1$*, then* $y \wedge z \trianglelefteq x$*.*

Definition 1.7. [52] *A function* φ *from a trellis* $(X, \trianglelefteq, \wedge, \vee)$ *into a trellis* $(Y, \trianglelefteq, \wedge, \vee)$ *(i.e.,* $\varphi : X \to Y$*) is called*

(i) a $\wedge$*-homomorphism if* $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$*, for any* $x, y \in X$*;*

(ii) a $\vee$*-homomorphism if* $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$*, for any* $x, y \in X$*;*

(iii) a homomorphism if φ is both $\wedge$- and $\vee$-homomorphism;

(iv) a isomorphism if φ is both one to one and onto homomorphism.

A homomorphism φ of a trellis $(X, \trianglelefteq, \wedge, \vee)$ into itself is called an endomorphism.

Next, we introduce some specific elements of a trellis.

Definition 1.8. *[52] Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis. An element $\alpha \in X$ is called:*

(i) right-transitive, if $\alpha \trianglelefteq x \trianglelefteq y$ implies $\alpha \trianglelefteq y$, for any $x, y \in X$;

(ii) left-transitive, if $x \trianglelefteq y \trianglelefteq \alpha$ implies $x \trianglelefteq \alpha$, for any $x, y \in X$;

(iii) middle-transitive, if $x \trianglelefteq \alpha \trianglelefteq y$ implies $x \trianglelefteq y$, for any $x, y \in X$;

(iv) transitive, if it is right-, left- and middle-transitive.

Example 1.7. *In the trellis $(X, \trianglelefteq, \wedge, \vee)$ given in Figure 1.2, note that $b \trianglelefteq d \trianglelefteq e$ and $b \ntrianglelefteq e$, then b isn't right-transitive. Otherwise, since if $x \trianglelefteq y \trianglelefteq d$ implies $x \trianglelefteq d$, for any $x, y \in X$, then it holds that d is a left-transitive element.*

Definition 1.9. *[52] Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis.*

(i) A sequence $(x, y, z) \in X^3$ is called $\wedge$-associative (resp. $\vee$-associative), if $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (resp. $(x \vee y) \vee z = x \vee (y \vee z)$);

(ii) An element $\alpha \in X$ is called $\wedge$-associative (resp. $\vee$-associative), if any sequence of three elements of X including α is $\wedge$-associative (resp. $\vee$-associative);

(iii) α is called associative, if it is both $\wedge$- and $\vee$-associative.

Notice that for the notion of associative element $\alpha \in X$, the commutativity of the meet and the join operations is sufficient to consider only the sequence $(\alpha, x, y) \in X^3$.

Theorem 1.3. *[52] Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis. Then any $\wedge$-associative (resp. $\vee$-associative) element is transitive.*

Remark 1.3. *The converse of the Theorem 1.3 does not hold in general. Indeed, let $(X, \trianglelefteq, \wedge, \vee)$ the trellis given by the Hasse diagram in Figure 1.2. Note that c is transitive but not $\vee$-associative since $(c \vee b) \vee e = e$ but $c \vee (b \vee e) = f$.*

The following results show two cases that the notion of associative elements and transitive elements are equivalent.

Theorem 1.4. *[52] Let $(X, \trianglelefteq, \wedge, \vee)$ be a modular trellis. Then any element is associative if and only if it is transitive.*

Proposition 1.4. *[52] Let $(X, \trianglelefteq, \wedge, \vee)$ be a pseudo-chain. Then any element is associative if and only if it is transitive.*

1.2.3. Binary operations on trellises

A binary operation F on a psoset $(X, \trianglelefteq)$ is called:

(i) *commutative*, if $F(x, y) = F(y, x)$, for any $x, y \in X$;

(ii) *associative*, if $F(x, F(y, z)) = F(F(x, y), z)$, for any $x, y, z \in X$;

(iii) *right-increasing*, if $x \trianglelefteq y$ implies $F(z, x) \trianglelefteq F(z, y)$, for any $x, y, z \in X$;

(iv) *left-increasing*, if $x \trianglelefteq y$ implies $F(x, z) \trianglelefteq F(y, z)$, for any $x, y, z \in X$;

(v) *increasing*, if $x \trianglelefteq y$ and $z \trianglelefteq t$ implies $F(x, z) \trianglelefteq F(y, t)$, for any $x, y, z, t \in X$.

If a binary operation F on X is increasing, then it is right- and left-increasing. The converse holds if $\trianglelefteq$ is transitive.

A binary operation F on a trellis $(X, \trianglelefteq, \wedge, \vee)$ is called:

(i) *conjunctive*, if $F(x, y) \trianglelefteq x \wedge y$, for any $x, y \in X$;

(ii) *disjunctive*, if $x \vee y \trianglelefteq F(x, y)$, for any $x, y \in X$.

Remark 1.4. *Consider a trellis $(X, \trianglelefteq, \wedge, \vee)$.*

(i) The meet $\wedge$ (resp. join $\vee$) is conjunctive (resp. disjunctive);

(ii) If a binary operation F on X satisfies $F(x, y) \trianglelefteq x$ and $F(x, y) \trianglelefteq y$ (resp. $x \trianglelefteq F(x, y)$ and $y \trianglelefteq F(x, y)$) for any $x, y \in X$, then it is conjunctive (resp. disjunctive). The converse holds if $\trianglelefteq$ is transitive (i.e., $(X, \trianglelefteq, \wedge, \vee)$ is a lattice) or F is (right and left)-increasing and $(X, \trianglelefteq, \wedge, \vee)$ has a greatest element that is the neutral element of F.

Example 1.8. *Consider the trellis $(X = \{0, a, b, c, d, 1\}, \trianglelefteq, \wedge, \vee)$ with the Hasse diagram displayed in Figure 1.4. The binary operation F on X defined by the following table:*

$F(x,y)$	0	a	b	c	d	1
0	0	a	b	c	c	1
a	0	a	b	c	c	1
b	0	a	c	d	d	1
c	a	a	c	d	d	1
d	b	c	c	d	1	1
1	c	c	c	1	1	1

is right- and left-increasing, but it is not increasing since $a \trianglelefteq b$ *and* $b \trianglelefteq c$*, while* $F(a,b) = b \not\trianglelefteq d = F(b,c)$.

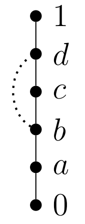


Figure 1.4: The Hasse diagram of the trellis $(X = \{0, a, b, c, d, 1\}, \trianglelefteq)$.

The binary operations $\wedge$ and $\vee$ of a trellis $(X, \trianglelefteq, \wedge, \vee)$ are not necessarily right- and left-increasing.

Example 1.9. *Consider again the trellis* $(X = \{0, a, b, c, d, 1\}, \trianglelefteq, \wedge, \vee)$ *with the Hasse diagram displayed in Figure 1.4. Since* $c \trianglelefteq d$ *and* $b \wedge c = c \wedge b = b \not\trianglelefteq a = b \wedge d = d \wedge b$*, it holds that the meet* $\wedge$ *is neither right-, nor left-increasing. A similar observation holds for the join operation* $\vee$.

In fact, the increasingness of $\wedge$ and $\vee$ are reserved for the case of lattices.

Proposition 1.5. *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis. Then it holds that the binary operation* $\wedge$ *(or* $\vee$*) is increasing if and only if* $\trianglelefteq$ *is transitive (i.e.,* $(X, \trianglelefteq, \wedge, \vee)$ *is a lattice).*

Proof. We only give the proof for $\wedge$. Suppose that $\wedge$ is increasing and $\trianglelefteq$ is not transitive (i.e., $x \trianglelefteq y \trianglelefteq z$ and $x \not\trianglelefteq z$, for some $x, y, z \in X$). Since $\wedge$ is increasing and $y \trianglelefteq z$, it follows that $x = x \wedge y \trianglelefteq x \wedge z$. Hence, $x = x \wedge z$. Thus, $x \trianglelefteq z$, a contradiction. The proof of the converse implication is immediate. □

Combining Theorem 1.2 and Proposition 1.5 leads to the following corollary.

Corollary 1.1. *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis. The following statements are equivalent:*

(i) $\wedge$ *(resp.* $\vee$*) is increasing;*

(ii) $\trianglelefteq$ *is transitive;*

(iii) $\wedge$ *(resp.* $\vee$*) is associative.*

1.2.4. Specific subsets on trellises

Next, we need to recall the following notations on a given trellis and some related results.

Notation 1.1. [56] *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis. We denote by:*

(i) X^{rtr}*: the set of all right-transitive elements of* X*;*

(ii) X^{ltr}*: the set of all left-transitive elements of* X*;*

(iii) X^{tr}*: the set of all transitive elements of* X*;*

(iv) $X^{\wedge\text{-ass}}$*: the set of all* $\wedge$*-associative elements of* X*;*

(v) $X^{\vee\text{-ass}}$*: the set of all* $\vee$*-associative elements of* X*;*

(vi) X^{ass}*: the set of all associative elements of* X*;*

(vii) X^{dis}*: the set of all distributive elements of* X*.*

Proposition 1.6. [56] *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis. It holds that*

(i) $X^{\mathrm{dis}} \subseteq X^{\mathrm{ass}} \subseteq X^{\wedge\text{-ass}} \subseteq X^{\mathrm{tr}} \subseteq X^{\mathrm{rtr}}$*;*

(ii) $X^{\mathrm{dis}} \subseteq X^{\mathrm{ass}} \subseteq X^{\vee\text{-ass}} \subseteq X^{\mathrm{tr}} \subseteq X^{\mathrm{ltr}}$*.*

The following corollary is an immediate result of Propositions 1.4 and 1.6.

Corollary 1.2. *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a pseudo-chain or a modular trellis. It holds that*

$$X^{\mathrm{ass}} = X^{\wedge\text{-ass}} = X^{\vee\text{-ass}} = X^{\mathrm{tr}}\,.$$

Next, we need to recall the following results related to the above subsets.

Proposition 1.7. *[52] Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis and* $x, a \in X$*. It holds that*

(i) if a *is right-transitive, then*

(a) $a \trianglelefteq x_1 \trianglelefteq \cdots \trianglelefteq x_k$*, implies* $a \trianglelefteq x_k$*;*

(b) $a \trianglelefteq x$*, implies that* $a \vee y \trianglelefteq x \vee y$*, for any* $y \in X$*.*

(ii) if a *is left-transitive, then*

(a) $x_1 \trianglelefteq \cdots \trianglelefteq x_k \trianglelefteq a$*, implies* $x_1 \trianglelefteq a$*;*

(b) $x \trianglelefteq a$, *implies that* $x \wedge y \trianglelefteq a \wedge y$, *for any* $y \in X$.

(iii) *if* a *is* $\wedge$*-associative, then* $x \trianglelefteq y$ *implies that* $a \wedge x \trianglelefteq a \wedge y$, *for any* $x, y \in X$.

(iv) *if* a *is* $\vee$*-associative, then* $x \trianglelefteq y$ *implies that* $a \vee x \trianglelefteq a \vee y$, *for any* $x, y \in X$.

The following results provide two subsets of a trellis that have the structure of lattice with respect to the operations of that trellis.

Proposition 1.8. [52] *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a pseudo-chain or a modular trellis. Then the set of all transitive (resp. associative) elements is a sublattice.*

Proposition 1.9. [52] *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis. Then the set of all distributive elements of* X *is a distributive sublattice.*

Proposition 1.10. [56] *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis. It holds that*

(i) $(X^{\mathrm{rtr}}, \trianglelefteq, \vee)$ *(resp.* $(X^{\mathrm{ltr}}, \trianglelefteq, \wedge)$*) is a* $\vee$*-semitrellis (resp.* $\wedge$*-semitrellis);*

(ii) $(X^{\wedge\text{-ass}}, \trianglelefteq, \wedge)$ *(resp.* $(X^{\vee\text{-ass}}, \trianglelefteq, \vee)$*) is a* $\wedge$*-semitrellis (resp.* $\vee$*-semitrellis).*

Proposition 1.11. [56] *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a pseudo-chain.*

If X^{rtr} *is finite, then it is a subtrellis on* $(X, \trianglelefteq, \wedge, \vee)$.

In the same line, we obtain the following propositions.

Proposition 1.12. *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis. The following implications hold:*

(i) *If* $\alpha \in \{\mathrm{dis}, \mathrm{ass}, \wedge\text{-ass}, \vee\text{-ass}, \mathrm{tr}, \mathrm{rtr}\}$, *then* $x \vee (y \vee z) = (x \vee y) \vee z$, *for any* $x, y, z \in X^{\alpha}$;

(ii) *If* $\alpha \in \{\mathrm{dis}, \mathrm{ass}, \wedge\text{-ass}, \vee\text{-ass}, \mathrm{tr}, \mathrm{ltr}\}$, *then* $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, *for any* $x, y, z \in X^{\alpha}$.

Proof. (i) We only give the proof for $\alpha = \mathrm{rtr}$, as the other cases proved from Proposition 1.6. Let $x, y, z \in X^{\mathrm{rtr}}$, since $x \trianglelefteq x \vee y \trianglelefteq (x \vee y) \vee z$, $y \trianglelefteq x \vee y \trianglelefteq (x \vee y) \vee z$ and $z \trianglelefteq (x \vee y) \vee z$, it follows that $x \trianglelefteq (x \vee y) \vee z$, $y \trianglelefteq (x \vee y) \vee z$ and $z \trianglelefteq (x \vee y) \vee z$. Moreover, $y \vee z \trianglelefteq (x \vee y) \vee z$. Hence, $x \vee (y \vee z) \trianglelefteq (x \vee y) \vee z$. In a similar way, we prove that $(x \vee y) \vee z \trianglelefteq x \vee (y \vee z)$. Thus, $x \vee (y \vee z) = (x \vee y) \vee z$.

(ii) The proof is dual to that of (i).

□

For a given trellis X, the following proposition shows that if a specific subsets X^{α} is a subtrellis of X, then it is a sublattice of X.

Proposition 1.13. *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis and* $\alpha \in \{\mathrm{ass}, \wedge\text{-ass}, \vee\text{-ass}, \mathrm{tr}, \mathrm{ltr}, \mathrm{rtr}\}$. *If* X^α *is subtrellis of* X*, then it is a sublattice of* X*.*

Proof. Let $\alpha \in \{\mathrm{ass}, \wedge\text{-ass}, \vee\text{-ass}, \mathrm{tr}, \mathrm{ltr}, \mathrm{rtr}\}$. Suppose that X^α is subtrellis of X. If $\alpha \in \{\mathrm{ass}, \wedge\text{-ass}, \vee\text{-ass}, \mathrm{tr}\}$, then Theorem 1.2 and Propositions 1.12 and 1.3 guarantee that X^α is sublattice of X. If $\alpha \in \{\mathrm{ltr}, \mathrm{rtr}\}$, then from Proposition 1.12 it follows that $\vee$ is associative on X^{rtr} and $\wedge$ is associative on X^{ltr}. Next, we show that $\wedge$ is associative on X^{rtr} and $\vee$ is associative on X^{ltr}. Let $x, y, z \in X^{\mathrm{rtr}}$, then the fact that X^{rtr} is a subtrellis implies that $x \wedge (y \wedge z) \in X^{\mathrm{rtr}}$. Moreover, $x \wedge (y \wedge z) \trianglelefteq x$, $x \wedge (y \wedge z) \trianglelefteq y \wedge z \trianglelefteq y$ and $x \wedge (y \wedge z) \trianglelefteq y \wedge z \trianglelefteq z$. This implies that $x \wedge (y \wedge z) \trianglelefteq x$, $x \wedge (y \wedge z) \trianglelefteq y$ and $x \wedge (y \wedge z) \trianglelefteq x$. Hence, $x \wedge (y \wedge z) \trianglelefteq x \wedge y$ and $x \wedge (y \wedge z) \trianglelefteq z$. Thus, $x \wedge (y \wedge z) \trianglelefteq (x \wedge y) \wedge z$. In a similar way, we obtain that $(x \wedge y) \wedge z \trianglelefteq x \wedge (y \wedge z)$. Hence, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$. Thus, X^{rtr} is a sublattice of X. Analogously, we prove that X^{ltr} is a sublattice of X. □

For further use, we recall the following results.

Proposition 1.14. [56] *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis. The following statements are equivalent:*

(i) $(X, \trianglelefteq, \wedge, \vee)$ *is a lattice;*

(ii) $X^{\mathrm{ltr}} = X$ *;*

(iii) $X^{\mathrm{rtr}} = X$ *;*

(iv) $X^{\wedge\text{-ass}} = X$ *;*

(v) $X^{\vee\text{-ass}} = X$ *.*

Theorem 1.5. [52] *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis. It holds that*

(i) if $x_1, \ldots, x_k$ *are right-transitive, then the join of* $\{x_1, \ldots, x_k\}$ *exists and equals* $x_{i_1} \vee \ldots \vee x_{i_k}$ *for any permutation* $i_1, \ldots, i_k$ *of* $1, \ldots, k$*. Moreover,* $\bigvee \{x_1, \ldots, x_k\}$ *is right-transitive. Similarly for the left-transitive elements* $x_1, \ldots, x_k$ *and the meet of* $\{x_1, \ldots, x_k\}$*.*

(ii) if $x_1, \ldots, x_k$ *are* $\wedge$*-associative, then the meet of* $\{x_1, \ldots, x_k\}$ *exists and equals* $x_{i_1} \wedge \ldots \wedge x_{i_k}$ *for any permutation* $i_1, \ldots, i_k$ *of* $1, \ldots, k$*. Moreover,* $\bigwedge \{x_1, \ldots, x_k\}$ *is* $\wedge$*-associative. Similarly for the* $\vee$*-associative elements* $x_1, \ldots, x_k$ *and the join of* $\{x_1, \ldots, x_k\}$*.*

Next, we need to show the following result. This result completes the cases to that given by Skala in the above Theorem 1.5.

Proposition 1.15. *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis and* $\alpha \in \{\text{ass}, \wedge\text{-ass}, \text{tr}\}$*. If* X^α *is subtrellis of* X*, then for any finite subset* $\{x_1, \ldots, x_k\} \subseteq X^\alpha$ *it holds that* $\bigvee\{x_1, \ldots, x_k\}$ *exists and equals* $x_{i_1} \vee \ldots \vee x_{i_k}$ *for any permutation* $i_1, \ldots, i_k$ *of* $1, \ldots, k$*. Moreover,* $\bigvee\{x_1, \ldots, x_k\} \in X^\alpha$*.*

Proof. We only give the proof for $\alpha = \text{tr}$, as the other cases can be proved similarly. Let $\{x_1, \ldots, x_k\} \subseteq X^{\text{tr}}$. Since $X^{\text{tr}} \subseteq X^{\text{rtr}}$, it follows from Theorem 1.5 that $\bigvee\{x_1, \ldots, x_k\}$ exists and $\bigvee\{x_1, \ldots, x_k\} = x_{i_1} \vee \ldots \vee x_{i_k}$ for any permutation $i_1, \ldots, i_k$ of $1, \ldots, k$. Now, we prove that $\bigvee\{x_1, \ldots, x_k\}$ is transitive. Since X^{tr} is a subtrellis, it follows that $x_i \vee x_j$ is transitive for any two elements $x_i, x_j \in X^{tr}$. The fact $\bigvee\{x_1, \ldots, x_k\} = x_{i_1} \vee \ldots \vee x_{i_k}$ for any permutation $i_1, \ldots, i_k$ of $1, \ldots, k$, implies that $\bigvee\{x_1, \ldots, x_k\} \in X^\alpha$. □

2 Aggregation operators on bounded lattices

In this chapter, we recall the necessary basic concepts and properties of aggregation operators on bounded lattices. Further, we introduce and study the notion of aggregation operator with respect to a given function f (f-aggregation operator, for short) on bounded lattices. This new notion is a natural generalization of the aggregation operators on bounded lattices. More precisely, we show some new properties of binary operations based on a given function on a lattice, and study their composition with respect to a given aggregation operator. Also, we investigate the transformation of f-aggregation operators based on a lattice-automorphism and a strong negation. Moreover, under some conditions on a given function f, we give the smallest (resp. the greatest) f-aggregation operator on a given bounded lattice.

2.1. Definitions and examples

This section contains the basic definitions and properties of Aggregation operators and some illustrative examples on bounded lattices. More information can be found in [8, 14, 28, 54].

Definition 2.1. *[38] Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and* $n \in \mathbb{N}$. *An (n-ary) aggregation operator on* L *is a function* $A : \bigcup_{k \leq n} L^k \longrightarrow L$ *such that:*

(i) $A(x_1, \ldots, x_n) \leq A(y_1, \ldots, y_n)$ *whenever* $x_i \leq y_i$, *for any* $i \in \{1, \ldots, n\}$;

(ii) $A(0, \ldots, 0) = 0$ *and* $A(1, \ldots, 1) = 1$.

Example 2.1. *(i) Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and let* χ *be an unary operator on* L *(i.e.,* $\chi : L \longrightarrow L$*) defined for any* $a \in L$ *by*

$$\chi_a(x) = \begin{cases} 1 & \text{if } x \geq a, x \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, χ_a *is an unary aggregation operator, for any* $a \in L$. *Moreover, it represents a characteristic function of the principal filter* $F(a) = \{x \in L : x \geq a\}$ *generated by* a, *provided* $a \neq 0$.

(ii) Consider the diamond lattice $D = \{0, a, b, 1\}$ given by the Hasse diagram in Figure 2.1. Then a function $A : D \to D$ is an unary aggregation operator on D if and only if $A(0) = 0$ and $A(1) = 1$ (i.e., the values $A(a)$ and $A(b)$ can be chosen arbitrarily). Let $B : D^2 \to D$ be a commutative binary operation defined by

$$B(x, y) = \begin{cases} 0, & \text{if } 0 \in \{x, y\}; \\ 1, & \text{if } 1 \in \{x, y\} \text{ and } 0 \notin \{x, y\}; \\ x, & \text{if } x = y. \end{cases}$$

Then B is well defined and it is an aggregation operator on D.

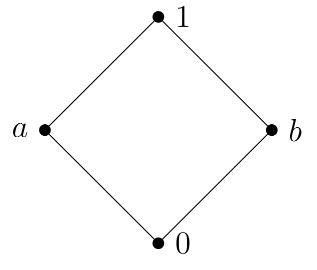


Figure 2.1: The Hasse diagram of diamond lattice D

Using the same notion of (n-ary) aggregation operator for $n = 2$ leads to the following definition.

Definition 2.2. *[38] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. An aggregation operator on L is a binary operation A on L which is increasing and it fulfills the boundary conditions $A(0, 0) = 0$ and $A(1, 1) = 1$.*

Denote by $\mathcal{A}(L)$ the set of all aggregation operator and consider $\mathcal{A}(L)$ with the following order: For $A, B \in \mathcal{A}(L)$,

$$A \leq B \text{ whenever } A(x, y) \leq B(x, y), \text{ for any } x, y \in L.$$

Proposition 2.1. *[32] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. Then the smallest and the greatest aggregation operators in $\mathcal{A}(L)$ are, respectively, defined by*

$$A_{\perp}(\mathbf{x}) = \begin{cases} 1 & \text{if } x = y = 1; \\ 0 & \text{otherwise;} \end{cases}$$

and

$$A_\top(\mathbf{x}) = \begin{cases} 0 & \text{if } x = y = 0; \\ 1 & \text{otherwise.} \end{cases}$$

Thus, we have $A_\perp \leq A \leq A_\top$, for any aggregation operator $A \in \mathcal{A}(L)$.

2.2. Function-based new generalized properties of binary operations on lattices

In this section, we discuss some new generalized properties of binary operations on lattice with respect to a given function on that lattice. These new properties are generalization of known properties of binary operations on a lattice with respect to a function f, and coincide with them when f is the identity function. As an application, with respect to a given function we study the relationships between some interesting properties of binary operations on a lattice and their extensions.

2.2.1. New generalized properties of binary operations on a lattice

In this subsection, we introduce some new properties of binary operations on lattices with respect to a given function and we present an illustrative example. More precisely, we introduce the notions of f-increasing (resp. f-decreasing), f-conjunctive (resp. f-disjunctive) and f-idempotent binary operations on lattices and investigate their properties.

Definition 2.3. *Let $(L, \leq, \wedge, \vee)$ be a lattice, A a binary operation on L and f a function on L. Then A is called:*

(i) left increasing (resp. left decreasing) with respect to f (left f-increasing (resp. left f-decreasing), for short), if $x \leq y$ implies $A(fx, z) \leq A(fy, z)$ (resp. $A(fy, z) \leq A(fx, z)$), for any $x, y, z \in L$;

(ii) right increasing (resp. right decreasing) with respect to f (right f-increasing (resp. right f-decreasing), for short), if $x \leq y$ implies $A(z, fx) \leq A(z, fy)$ (resp. $A(z, fy) \leq A(z, fx)$), for any $x, y, z \in L$;

(iii) increasing (resp. decreasing) with respect to f (f-increasing (resp. f-decreasing), for short), if A is both left and right f-increasing (resp. f-

decreasing).

Definition 2.4. *Let* $(L, \leq, \wedge, \vee)$ *be a lattice,* A *a binary operation on* L *and* f *a function on* L*. Then* A *is called:*

(i) left (resp. right) f*-conjunctive if it satisfies* $A(fx, y) \leq x$ *(resp.* $A(x, fy) \leq y$*), for any* $x, y \in L$*;*

(ii) f*-conjunctive if it is both left and right* f*-conjunctive;*

(iii) left (resp. right) f*-disjunctive if it satisfies* $x \leq A(fx, y)$ *(resp.* $y \leq A(x, fy)$*), for any* $x, y \in L$*;*

(iv) f*-disjunctive if it is both left and right* f*-disjunctive.*

In the following, we give an illustrative example of the above new generalized properties of binary operations on a lattice.

Example 2.2. *Let* $(D(12), |, gcd, lcm)$ *be the bounded lattice of positive divisors of* 12 *ordered by the divisibility order given by the Hasse diagram in Figure 2.2. Let* $f : D(12) \longrightarrow D(12)$ *be a function and* A, B *two binary operations on* $D(12)$ *defined as follows:*

x	1	2	3	4	6	12
fx	6	2	1	12	3	4

,

$A(x,y)$	1	2	3	4	6	12
1	3	1	3	3	1	1
2	1	2	2	2	1	2
3	3	2	6	6	1	2
4	3	2	6	12	1	4
6	1	1	1	1	1	1
12	1	2	2	4	1	4

and

$B(x,y)$	1	2	3	4	6	12
1	3	6	6	12	3	12
2	6	2	6	12	2	4
3	6	6	6	12	6	12
4	12	12	12	12	12	12
6	3	2	6	12	1	4
12	12	4	12	12	4	4

One easily verifies that A *and* B *are* f*-increasing but they are not increasing. Indeed, Let* $x, y \in D(12)$ *such that* $x \mid y$*. Setting* $x = 3$*,* $y = 6$ *and* $z = 1$*. Then* $x \mid y$*, but* $A(x, z) = A(3, 1) = 3 \nmid 1 = A(6, 1) = A(y, z)$ *and* $B(x, z) = B(3, 1) = 6 \nmid 3 = B(6, 1) = B(y, z)$*. Hence,* A *and* B *are not increasing. Therefore,* A *and* B *are not aggregation operators on* $D(12)$*.*

Furthermore, it is not difficult to check that A *is* f*-conjunctive and* B *is* f*-disjunctive on* $D(12)$*. Notice that* A *(resp.* B*) is not conjunctive (resp. disjunctive) on* $D(12)$*.*

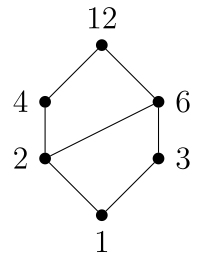


Figure 2.2: The Hasse diagram of the bounded lattice $(D(12), |, gcd, lcm)$.

Definition 2.5. *Let $(L, \leq, \wedge, \vee)$ be a lattice, A a binary operation on L and f a function on L. An element $e \in L$ is called:*

(i) left (resp. right) f-neutral element of A, if $A(e, fx) = x$ (resp. $A(fx, e) = x$), for any $x \in L$;

(ii) f-neutral element of A if it is both a left and a right f-neutral element of A.

Example 2.3. *Let $(L = \{0, a, b, c, 1\}, \leq, \wedge, \vee)$ be the lattice given by the Hasse diagram in Figure 2.3, f a function on L and A a binary operation on L defined as follows:*

x	0	a	b	c	1
fx	1	b	c	a	0

and

$A(x,y)$	0	a	b	c	1
0	1	1	0	0	0
a	1	c	a	b	0
b	0	a	b	c	1
c	0	b	c	a	1
1	0	0	1	1	1

It is not difficult to show that the element $a \in L$ is an f-neutral element of A, but it is not a neutral element of A.

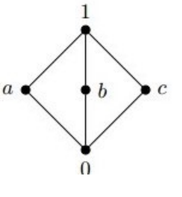


Figure 2.3: The Hasse diagram of the lattice $(L = \{0, a, b, c, 1\}, \leq)$.

The following proposition shows that if any element is both a neutral and an f-neutral element of a binary operation on a lattice, then f is the identity function. The proof is straightforward.

Proposition 2.2. *Let $(L, \leq, \wedge, \vee)$ be a lattice, A a binary operation on L and f a function on L. If $e \in L$ is both a neutral and an f-neutral element of A, then f is the identity function of L.*

Definition 2.6. *Let $(L, \leq, \wedge, \vee)$ be a lattice, A a binary operation on L and f a function on L. A is called f-commut, if $A(fx, y) = A(x, fy)$, for any $x, y \in L$.*

Example 2.4. *Let A a binary operation on L and f a function on L given in Example 2.3. One easily verifies that A is f-commut.*

Definition 2.7. *Let $(L, \leq, \wedge, \vee)$ be a lattice, A a binary operation on L and f a function on L. A is called f-idempotent, if $A(fx, fx) = x$, for any $x \in L$.*

Example 2.5. *Consider the binary operation A and the function f given in Example 2.2. One easily verifies that A is f-idempotent, but not idempotent.*

2.2.2. Binary operations on lattices and their f-extensions

In this subsection, we study the relationships between some interesting properties of binary operations on lattices and their extensions with respect to a given function on that lattice. Moreover, we provide some properties of a binary operation based on an arbitrary function on a lattice in order that it can be represented by the meet and the join operations of that lattice.

Proposition 2.3. *Let $(L, \leq, \wedge, \vee)$ be a lattice and A a binary operation on L. Then it holds that*

(i) If A is increasing, then A is f-increasing, for any isotone function f on L;

(ii) If A is increasing, then A is f-decreasing, for any antitone function f on L.

The following proposition shows that the interaction of the notion of f-increasing with the function composition.

Proposition 2.4. *Let $(L, \leq, \wedge, \vee)$ be a lattice, A a binary operation on L and f, g two functions on L such that g is isotone. If A is f-increasing on L, then A is $f \circ g$-increasing on L.*

Proof. Let $x, y \in L$ such that $x \leq y$. Since g is isotone, it holds that $gx \leq gy$. The fact that A is f-increasing implies that $A(f(g(x)), z) \leq A(f(g(y)), z)$, for any $z \in L$. Thus, $A((f \circ g)(x), z) \leq A((f \circ g)(y), z)$, for any $z \in L$. Therefore, A is $f \circ g$-increasing. □

The above propositions lead to the following corollary.

Corollary 2.1. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and A an f-increasing binary operation on L. The following statements hold:*

(i) If f is isotone, then A is f^n-increasing, for any $n \in \mathbb{N}^*$;

(ii) If f is antitone, then A is f^{2n+1}-increasing, for any $n \in \mathbb{N}$.

Remark 2.1. *Let $(L, \leq, \wedge, \vee)$ be a lattice, A a binary operation on L and f a function on L. The following implications hold:*

(i) If A is conjunctive and $fx \leq x$, for any $x \in L$, then A is f-conjunctive;

(ii) If A is disjunctive and $x \leq fx$, for any $x \in L$, then A is f-disjunctive.

The following proposition shows that a given binary operation on a lattice has at most one f-neutral element.

Proposition 2.5. *Let $(L, \leq, \wedge, \vee)$ be a lattice, A a binary operation on L and f a function on L. If A is f-commut, then A has at most one f-neutral element.*

Proof. Let A be an f-commut binary operation on L having two f-neutral elements $e_1, e_2 \in L$. Then $e_1 = A(fe_1, e_2) = A(e_1, fe_2) = e_2$. Therefore, $e_1 = e_2$. □

In the following, we give an example to explain the result of Proposition 2.5 .

Example 2.6. *Let A be an f-commut binary operation on L and f function on L given in Example 2.3. One easily verifies that $a \in L$ is the only f-neutral element of A.*

In the following theorem, we characterize the f-conjunctive (resp. f-disjunctive) binary operation on a bounded lattice in terms of f-neutral element. This characterization is an extension to that known in (Proposition 5.3, in [54]).

Theorem 2.1. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and A a binary operation on L having an f-neutral element $e \in L$ such that A is f-commut and f-increasing. The following equivalences hold:*

(i) A is f-conjunctive if and only if $e = 1$;

(ii) A is f-disjunctive if and only if $e = 0$.

Proof. (i) The fact that A is f-commut and f-conjunctive imply that $1 = A(f(1), e) = A(1, f(e)) \leq e$. Hence, $e = 1$. Conversely, suppose that $e = 1$ and let $x, y \in L$. Since A is f-commut and f-increasing, it follows that $A(fx, y) = A(x, fy) \leq A(x, f1) = A(fx, 1) = x$. In a similar way, we obtain that $A(x, fy) \leq y$. Hence, A is left and right f-conjunctive. Thus, A is f-conjunctive.

(ii) The proof is dual to that of (i).

□

The following result provides some properties of a binary operation based on an arbitrary function on a lattice in order that it can be represented by the meet and the join operations of that lattice.

Proposition 2.6. *Let* $(L, \leq, \wedge, \vee)$ *be a lattice,* A *a binary operation on* L *and* f *a function on* L*. If* f *is surjective, then the following implications hold:*

(i) if A *is* f*-idempotent,* f*-increasing and* f*-conjunctive, then for any* $a, b \in L$*, there exist* $x, y \in L$ *such that* $A(a, b) = x \wedge y$*;*

(ii) if A *is* f*-idempotent,* f*-increasing and* f*-disjunctive, then for any* $a, b \in L$*, there exist* $x, y \in L$ *such that* $A(a, b) = x \vee y$*.*

Proof. (i) Let $a, b \in L$, then there exist $x, y \in L$ such that $fx = a$ and $fy = b$ by using the fact that f is surjective. Since A is f-increasing and f-idempotent, it follows that $x \wedge y = A(f(x \wedge y), f(x \wedge y)) \leq A(fx, fy) = A(a, b)$. The fact that A is f-conjunctive implies that $A(a, b) = A(fx, fy) \leq x \wedge y$. Thus, $A(a, b) = x \wedge y$.

(ii) The proof is dual to that of (i).

□

Proposition 2.6 leads to the following result.

Proposition 2.7. *Let* $(L, \leq, \wedge, \vee)$ *be a lattice,* A *a binary operation on* L *and* f *a function on* L*. If* f *is surjective, then the following equivalences hold:*

(i) A *is* f*-idempotent,* f*-increasing and* f*-conjunctive if and only if for any* $x, y \in L$ *there is* $A(fx, fy) = x \wedge y$*;*

(ii) A *is* f*-idempotent,* f*-increasing and* f*-disjunctive if and only if for any* $x, y \in L$ *there is* $A(fx, fy) = x \vee y$*.*

Proof. (i) The proof of the direct implication follows from Propositions 2.6. Next, we prove the converse implication. Let $x, y, z \in L$ such that $x \leq y$, it is clear that f is surjective, then there exists $t \in L$ such that $z = ft$. Thus, $A(fx, z) = A(fx, ft) = x \wedge t \leq y \wedge t = A(fy, ft) = A(fy, z)$. Hence, A is left f-increasing. In similar way, we obtain that A is right f-increasing. Now, we prove that A is f-conjunctive. Let $x, y \in L$, since f is surjective, then there exists $s \in L$ such that $y = fs$. Then $A(fx, y) = A(fx, fs) = x \wedge s \leq x$.

In a similar way, we obtain that $A(x, fy) \leq y$. Hence, A is f-conjunctive. It is obvious that A is f-idempotent.

(ii) The proof is dual to that of (i).

□

2.3. f-aggregation operators on bounded lattices

In this section, we extend the notion of aggregation operator on bounded lattices introduced by Mesiar and Komorníková [38] to f-aggregation operator, where f is an arbitrary function on that bounded lattice. Furthermore, various properties of this notion and its links with the notion of an aggregation operator on bounded lattices is discussed.

2.3.1. Definitions and examples

In this subsection, we introduce the notion of the f-aggregation operator on bounded lattices and we give some illustrative examples for clarity. First, we recall the definition of aggregation operator on bounded lattices.

Definition 2.8. *[38] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. An aggregation operator on L is a binary operation A on L which is increasing and it fulfills the boundary conditions $A(0,0) = 0$ and $A(1,1) = 1$.*

Next, we extend this definition by using a given function f on a bounded lattice as follow.

Definition 2.9. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and f a function on L. An aggregation operator with respect to the function f (f-aggregation operator, for short) on L is a binary operation A on L which is f-increasing and it fulfills the f-boundary conditions $A(f(0), f(0)) = 0$ and $A(f(1), f(1)) = 1$.*

Remark 2.2. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and A an f-aggregation operator on L. If f is a constant function on L, then $L = \{0\}$. Indeed, let A an f-aggregation operator on L. Since f is constant, it holds that $fx = fy$, for any $x, y \in L$. Then $0 = A(f(0), f(0)) = A(f(1), f(1)) = 1$. Hence, $0 = 1$. Therefore, $L = \{0\}$.*

For the rest of the work, we will assume that f is not a constant function.

Example 2.7. *Let $f : [0,1] \longrightarrow [0,1]$ be a function and A a binary operation on $[0,1]$ defined by:*

$$fx = \frac{1}{2}x \text{ and } A(x,y) = \begin{cases} 1 & , \text{ if } x = y = \frac{1}{2}; \\ \frac{1}{(1+x)(1+y)}, & \text{if } (x,y) \in]\frac{1}{2},1]\times]\frac{1}{2},1]; \\ xy & , \text{ otherwise.} \end{cases}$$

One easily verifies that A is an f-aggregation operator, but A is not an aggregation operator on $[0,1]$. Indeed, Let $x, y \in [0,1[$ such that $x \leq y$. Setting $x = \frac{2}{3}$, $y = \frac{3}{4}$ and $z = 1$. Then $x \leq y$, but $A(x,z) = \frac{1}{(1+x)(1+z)} = \frac{3}{10} \geq \frac{2}{7} = \frac{1}{(1+y)(1+z)} = A(y,z)$. Hence, A is not increasing. However, A is not an aggregation operator.

Example 2.8. *Let A, B be the binary operations on $D(12)$ and f the function on $D(12)$ defined in Example 2.2. One easily verifies that A and B are f-increasing and fulfills the f-boundary conditions. Thus, A and B are f-aggregation operators on $D(12)$.*

Remark 2.3. *In general, we use the aggregation operators (increasing binary operations) on a given universe to aggregate objects on that universe. While the notion of f-aggregation operators (f-increasing operations) allows the use of non-increasing operations to aggregate objects with respect to specific functions on that universe.*

2.3.2. Properties of f-aggregation operators on bounded lattices

In this subsection, we investigate basic properties of f-aggregation operators on bounded lattices. First, we show that any aggregation operator is an f-aggregation operator, for any isotone function on that lattice and not conversely.

Proposition 2.8. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and A a binary operation on L. If A is an aggregation operator on L, then A is an f-aggregation operator, for any isotone function f on L satisfying $f(0) = 0$ and $f(1) = 1$.*

Proof. Since A is increasing and f is isotone, then Proposition 2.3 guarantees that A is f-increasing. The fact that $f(0) = 0$ and $f(1) = 1$ imply that $A(f(0), f(0)) = 0$ and $A(f(1), f(1)) = 1$. Thus, A is an f-aggregation operator on L.

□

The following counter example shows that the converse implication of Proposition 2.8 does not necessarily hold.

Example 2.9. *Let $(D(30), |, gcd, lcm)$ be the lattice of the positive divisors of 30, f an isotone function on $D(30)$ and A a binary operation on $D(30)$ defined as follows:*

x	1	2	3	5	6	10	15	30
fx	1	5	5	3	10	15	30	30

and

$$A(x,y) = \begin{cases} lcm(x,y), & \text{if } (x,y) \in \{2,6\} \times \{2,6\}; \\ gcd(x,y), & \text{otherwise.} \end{cases} .$$

One easily verifies that $A(f(1), f(1)) = 1$ and $A(f(30), f(30)) = 30$. Next, we prove that A is an f-increasing. Let $x, y \in D(30)$ such that $x \mid y$, since f is isotone, it holds that $fx \mid fy$. The fact that $fx, fy \in D(30) \setminus \{2,6\}$ implies that $A(fx, z) = gcd(fx, z)$ and $A(fy, z) = gcd(fy, z)$. Then $A(fx, z) \mid A(fy, z)$, for any $z \in D(30)$. Hence, A is left f-increasing. Similarly, we prove that A is right f-increasing. Thus, A is an f-aggregation operator on $D(30)$. On the other hand, setting $x = 2$, $y = 10$ and $z = 6$. Then $x \mid y$, but $A(x,z) = A(2,6) = lcm(2,6) = 6$ and $A(y,z) = A(10,6) = gcd(10,6) = 2$. Hence, $6 \nmid 2$, i.e., A is not increasing. Consequently, A is not an aggregation operator.

In the same line, the following example gives an f-aggregation operator A such that f is not an isotone function and A is not an aggregation operator.

Example 2.10. *Let A be a binary operation on $D(12)$ and f a function on $D(12)$ given in Example 2.2. Then A is an f-aggregation operator, f is not an isotone function and A is not an aggregation operator on $D(12)$.*

Theorem 2.2. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, f a lattice-automorphism on L and A a binary operation on L. Then it holds that A is an f-aggregation operator if and only if A is an f^{-1}-aggregation operator.*

Proof. Since f is a lattice-automorphism, then Proposition 1.2 guarantees that $A(f^{-1}(0), f^{-1}(0)) = A(f(0), f(0)) = 0$ and $A(f^{-1}(1), f^{-1}(1)) = A(f(1), f(1)) = 1$. Assume $x, y \in L$ such that $x \leq y$, then there exist $s, t \in L$ such that $x = f^2(s)$ and $y = f^2(t)$. The fact that f^{-1} is isotone implies that $f^{-2}(x) \leq f^{-2}(y)$, i.e., $s \leq t$. Since A is an f-aggregation operator on L, it follows that $A(fs, z) \leq A(ft, z)$, for any $z \in L$, this equivalent to $A(f^{-1}(f^2(s)), z) \leq A(f^{-1}(f^2(t)), z)$, for any $z \in L$. Hence, $A(f^{-1}(x), z) \leq A(f^{-1}(y), z)$, for any $z \in L$. Thus, A is left f^{-1}-increasing. Similarly, we obtain that A is right f^{-1}-increasing. Therefore, A is an f^{-1}-aggregation operator on L. The proof of the converse implication follows from the fact that $(f^{-1})^{-1} = f$. □

Proposition 2.9. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, f a lattice-*

automorphism on L and A a binary operation on L. The following statements are equivalent:

(i) A is an aggregation operator;

(ii) A is an f-aggregation operator;

(iii) A is an f^{-1}-aggregation operator.

Proof. (i) $\Rightarrow$ (ii): Follows from Propositions 1.2 and 2.8.
(ii) $\Rightarrow$ (iii): The proof is a direct application of Theorem 2.2.
(iii) $\Rightarrow$ (i): Obvious that A satisfies the f-boundary conditions. Next, let $x, y \in L$ such that $x \leq y$, then $fx \leq fy$. Since A is an f^{-1}-aggregation operator, it holds that $A(f^{-1}(fx), z) \leq A(f^{-1}(fy), z)$, for any $z \in L$. Thus, $A(x, z) \leq A(y, z)$, for any $z \in L$. Therefore, A is an aggregation operator on L. □

Proposition 2.10. *Let $(L, \leq, \wedge, \vee)$ be a lattice and $f : L \longrightarrow L$ a lattice-epimorphism. If A is an idempotent f-aggregation operator on L, then A is averaging.*

Proof. Let $x, y \in L$ such that $ft = x$ and $fs = y$. The fact that A is f-increasing implies that $A(x, y) = A(ft, y) \leq A(f(t \vee s), y) = A(f(t \vee s), fs) \leq A(f(t \vee s), f(t \vee s))$. Thus, $A(x, y) \leq A(f(t \vee s), f(t \vee s))$. Since A is idempotent and f is a homomorphism, it follows that $A(x, y) \leq f(t \vee s) = x \vee y$. Analogously, we show that $x \wedge y \leq A(x, y)$. Hence, $x \wedge y \leq A(x, y) \leq x \vee y$. Therefore, A is averaging. □

2.3.3. Composition and transformations of f-aggregation operators on bounded lattices

In this subsection, we study the composition of f-aggregation operators on a bounded lattice. Further, we investigate the transformations of a given f-aggregation operator on a bounded lattice by a lattice-automorphism and a strong negation. First, we show that the aggregation of two f-aggregation operators on a bounded lattice is also an f-aggregation operator.

Proposition 2.11. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and A, F_1, F_2 three binary operations on L. If A is an aggregation operator and F_1, F_2 are two f-aggregation operators on L, then the aggregation of F_1 and F_2 by A defined for any $x, y \in L$ as*

$$A(F_1, F_2)(x, y) = A(F_1(x, y), F_2(x, y)) \,,$$

is also an f-aggregation operator on L.

Proof. Let $x, y \in L$ such that $x \leq y$. Since F_1, F_2 are two f-aggregation operators and A is an aggregation operator on L, it follows that $A(F_1, F_2)(fx, z) = A(F_1(fx, z), F_2(fx, z)) \leq A(F_1(fy, z), F_2(fy, z)) = A(F_1, F_2)(fy, z)$, for any $z \in L$. Thus, $A(F_1, F_2)$ is left f-increasing on L. Similarly, we obtain that $A(F_1, F_2)$ is right f-increasing. Therefore, $A(F_1, F_2)$ is f-increasing. Next, it is obvious that $A(F_1, F_2)$ satisfies the f-boundary conditions. Thus, $A(F_1, F_2)$ is an f-aggregation operator on L. □

Proposition 2.12. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and A, F_1, F_2 three binary operations on L. If A is an f-aggregation operator and F_1, F_2 are two idempotent aggregation operators on L such that $f(F_1(x, y)) = F_1(fx, y) = F_1(x, fy)$ and $f(F_2(x, y)) = F_2(fx, y) = F_2(x, fy)$. Then $A(F_1, F_2)$ is an f-aggregation operator on L.*

Proof. Let $x, y \in L$ such that $x \leq y$. Since F_1, F_2 are two aggregation operators and A is an f-aggregation operator on L, it follows that $A(F_1, F_2)(fx, z) = A(F_1(fx, z), F_2(fx, z)) = A(f(F_1(x, z)), f(F_2(x, z))) \leq A(f(F_1(y, z)), f(F_2(y, z))) = A(F_1, F_2)(fy, z)$, for any $z \in L$. Thus, $A(F_1, F_2)$ is left f-increasing on L. In a similar way, we obtain that $A(F_1, F_2)$ is right f-increasing. Therefore, $A(F_1, F_2)$ is f-increasing. Next, since F_1, F_2 are idempotent, then $A(F_1, F_2)(f(0), f(0)) = A(F_1(f(0), f(0)), F_2(f(0), f(0))) = A(f(0), f(0)) = 0$. Similarly, we obtain that $A(F_1, F_2)(f(1), f(1)) = 1$. Consequently, $A(F_1, F_2)$ is an f-aggregation operator on L. □

Example 2.11. *Let F_1, F_2 be two binary operations on L such that $F_1 = F_2 = \wedge$ and f a meet-translation on L (i.e., $f(x \wedge y) = x \wedge fy$, for any $x, y \in L$). One easily verifies that $f(F_1(x, y)) = F_1(fx, y) = F_1(x, fy)$ and $f(F_2(x, y)) = F_2(fx, y) = F_2(x, fy)$. Since $F_1 = F_2 = \wedge$ are idempotent aggregation operators, then $A(F_1, F_2)$ is an f-aggregation operator on L, for any f-aggregation operator A on L.*

Theorem 2.3. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a lattice-automorphism φ and f a function satisfies $f \circ \varphi = \varphi \circ f$. Then it holds that A is an f-aggregation operator on L if and only if A_φ is an f-aggregation operator on L, where A_φ is a binary operation on L given by:*

$$A_\varphi(x, y) = \varphi^{-1}(A(\varphi x, \varphi y)), \text{ for any } x, y \in L.$$

Proof. Suppose that A is an f-aggregation operator on a bounded lattice L and we show that A_φ is also an f-aggregation operator on L. First, we prove that $A_\varphi(f(0), f(0)) = 0$ and $A_\varphi(f(1), f(1)) = 1$. Since φ is a lattice-automorphism, it follows from Proposition 1.2 that $\varphi(0) = \varphi^{-1}(0) = 0$ and $\varphi(1) = \varphi^{-1}(1) = 1$. From $f \circ \varphi = \varphi \circ f$, it follows that $A_\varphi(f(0), f(0)) = \varphi^{-1}(A(\varphi(f(0)), \varphi(f(0)))) = \varphi^{-1}(A(f(\varphi(0)), f(\varphi(0))))$ and $A_\varphi(f(1), f(1)) = \varphi^{-1}(A(\varphi(f(1)), \varphi(f(1)))) = \varphi^{-1}(A(f(\varphi(1)), f(\varphi(1))))$. Thus, $A_\varphi(f(0), f(0)) = \varphi^{-1}(A(f(0), f(0))) = \varphi^{-1}(0) = 0$ and $A_\varphi(f(1), f(1)) = \varphi^{-1}(A(f(1), f(1))) = \varphi^{-1}(1) = 1$. Next, we prove that A_φ is f-increasing. Let $x, y \in L$ such that $x \leq y$, then $\varphi x \leq \varphi y$. Since A is an f-aggregation operator on L, it follows that $A(f(\varphi x), \varphi z) \leq A(f(\varphi y), \varphi z)$, for any $z \in L$. The fact that φ is a lattice-automorphism guarantees that φ^{-1} is isotone, then $\varphi^{-1}(A(f(\varphi x), \varphi z)) \leq \varphi^{-1}(A(f(\varphi y), \varphi z))$, for any $z \in L$. The equality $f \circ \varphi = \varphi \circ f$ implies that $\varphi^{-1}(A(\varphi(fx), \varphi z)) \leq \varphi^{-1}(A(\varphi(fy), \varphi z))$, for any $z \in L$. Hence, $A_\varphi(fx, z) \leq A_\varphi(fy, z)$, for any $z \in L$. Thus, A_φ is left f-increasing. Similarly, we obtain that A_φ is also right f-increasing. Therefore, A_φ is an f-aggregation operator on L. The proof of the converse implication follows from the fact that $A = (A_\varphi)_{\varphi^{-1}}$. □

In the following, we give an example to explain the result of Theorem 2.3.

Example 2.12. *Let $(D(30), |, gcd, lcm)$ be the lattice of the positive divisors of 30, f, φ two functions on $D(30)$ and A a binary operation on $D(30)$ defined as follows:*

x	1	2	3	5	6	10	15	30
fx	1	15	10	6	5	3	2	30
φx	1	3	5	2	15	6	10	30

and

$A(x, y)$	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	1	30	30	30	6	5	6	30
3	1	30	30	6	6	2	2	30
5	1	30	6	30	2	10	6	30
6	1	30	6	6	3	1	2	30
10	1	10	2	10	2	5	2	10
15	1	6	2	6	2	2	2	6
30	1	30	30	30	30	30	30	30

One easily verifies that φ is a lattice-automorphism and $f \circ \varphi = \varphi \circ f$. Also, it is not difficult to check that A is an f-aggregation operator on $D(30)$ and A_φ defined in the following table is also an f-aggregation operation on $D(30)$.

$A_\varphi(x,y)$	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	1	30	10	30	5	10	5	30
3	1	10	30	30	10	5	15	30
5	1	30	30	30	10	10	3	30
6	1	5	10	10	5	5	5	10
10	1	10	10	30	5	2	1	30
15	1	5	15	15	5	5	3	15
30	1	30	30	30	30	30	30	30

Theorem 2.2, Theorem 2.3 and Proposition 2.4 lead to the following corollary.

Corollary 2.2. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and f, φ two lattice-automorphisms on L such that $f \circ \varphi = \varphi \circ f$. Let A be a binary operation on L. The following statements are equivalent:*

(i) A is an f-aggregation operator;

(ii) A is an f^{-1}-aggregation operator;

(iii) A_φ is an f-aggregation operator;

(iv) A_φ is an f^{-1}-aggregation operator.

For a given binary operation A on a bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$ with a negation N, we denote by A_N its dual, i.e., $A_N(x,y) = N^{-1}(A(Nx, Ny))$, for any $x, y \in L$. One easily observes that if N is a strong negation, then $N^{-1} = N$, $(A_N)_N = A$ and $A_N(x,y) = N(A(Nx, Ny))$, for any $x, y \in L$.

In the same line, the following theorem shows that the transformation of an f-aggregation operator on bounded lattice by a strong negation is also an f-aggregation operator. The proof is analogous to that of Theorem 2.3.

Theorem 2.4. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice with a strong negation N and a function f such that $f \circ N = N \circ f$. Then A is an f-aggregation operator on L if and only if its dual operation A_N is an f-aggregation operator on L, where $A_N(x,y) = N(A(Nx, Ny))$, for any $x, y \in L$.*

In the following, we show an illustrative example of Theorem 2.4.

Example 2.13. *Let f and N be two functions on the bounded lattice $D(30)$ given by:*

x	1	2	3	5	6	10	15	30
fx	15	1	5	2	10	30	6	3
Nx	30	10	15	6	5	2	3	1

Let A be a binary operation on $D(30)$ given by the following table:

$A(x,y)$	1	2	3	5	6	10	15	30
1	1	2	6	2	6	6	1	2
2	2	3	30	1	30	6	1	6
3	30	30	30	30	30	30	6	30
5	2	2	30	5	10	10	1	2
6	6	6	30	5	30	30	1	30
10	6	2	30	10	30	30	2	6
15	1	1	6	1	6	2	1	2
30	2	6	30	2	30	6	2	30

One easily verifies that N is a strong negation, $f \circ N = N \circ f$ and A is an f-aggregation operator. Applying Theorem 2.4 we obtain that A_N defined by the following table:

$A_N(x,y)$	1	2	3	5	6	10	15	30
1	1	5	10	1	10	5	1	10
2	5	1	10	1	2	10	1	5
3	10	10	30	5	30	30	5	30
5	1	1	30	1	6	5	1	5
6	10	2	30	2	6	10	1	10
10	5	5	30	1	30	15	1	10
15	1	1	5	1	1	1	1	1
30	10	5	30	5	10	10	5	30

is also an f-aggregation operator on $D(30)$.

2.4. Smallest and greatest f-aggregation operators on bounded lattices

In this section, we provide some conditions on a given function f to define the smallest and the greatest f-aggregation operators on a bounded lattice.

For a given function f on a bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$, we define the binary operations $A_\perp$ and $A_\top$ as:

$$A_\perp(x,y) = \begin{cases} 1, & if \quad x = y = f(1); \\ 0, & otherwise; \end{cases} \quad \text{and} \quad A_\top(x,y) = \begin{cases} 0, & if \quad x = y = f(0); \\ 1, & otherwise. \end{cases}$$

Remark 2.4. *One can observe that:*

(i) $A_\perp$ *and* $A_\top$ *are not aggregation operators on* L*, in general. However, if* $f(0) = 0$ *and* $f(1) = 1$*, then* $A_\perp$ *(resp.* $A_\top$*) is an aggregation operator on* L*.*

(ii) $A_\perp$ *and* $A_\top$ *are not* f*-aggregation operators on* L*, in general. Indeed, let* f *be a function on* $D(12)$ *defined as follows:*

x	1	2	3	4	6	12
fx	6	2	4	12	6	4

.

It is not difficult to see that $A_\perp$ *(resp.* $A_\top$*) is not* f*-increasing on* $D(12)$ *(*$3 \mid 6$*, but* $A_\perp(f(3), 4) \nmid A_\perp(f(6), 4)$*) (resp.* $2 \mid 6$*, but* $A_\top(f(2), 6) \nmid A_\top(f(6), 6)$*. Thus,* $A_\perp$ *and* $A_\top$ *are not* f*-aggregation operators on* $D(12)$*.*

The following propositions provide some conditions on the function f under which $A_\perp$ and $A_\top$ are f-aggregation operators.

Proposition 2.13. *Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and* f *a function on* L*. The following equivalences hold:*

(i) $A_\perp$ *is an* f*-aggregation operator on* L *if and only if for any* $x \leq y$ *it holds* $(fx = f(1) \Rightarrow fy = f(1))$*;*

(ii) $A_\top$ *is an* f*-aggregation operator on* L *if and only if for any* $x \leq y$ *it holds* $(fy = f(0) \Rightarrow fx = f(0))$*.*

Proof. We only give the proof of (i), as (ii) can be proved analogously. Let $x, y \in L$ such that $x \leq y$ and $fx = f(1)$. The fact that $A_\perp$ is f-increasing implies that $A_\perp(fx, z) \leq A_\perp(fy, z)$, for any $z \in L$. For $z = f(1)$, we get that $A_\perp(fy, f(1)) = 1$. Hence, $fy = f(1)$. Next, for the converse implication, let $x, y, z \in L$ such that $x \leq y$. If $A_\perp(fx, z) = 0$, then $A_\perp(fx, z) \leq A_\perp(fy, z)$, for any $z \in L$. If $A_\perp(fx, z) = 1$, then $fx = f(1)$ and $z = f(1)$. Hence, $fy = f(1)$. Thus, $A_\perp(fy, z) = 1$. Then $A_\perp(fx, z) \leq A_\perp(fy, z)$, for any $z \in L$. Therefore, $A_\perp$ is left f-increasing. Similarly, we obtain that $A_\perp$ is right f-increasing. Next, since f is a non-constant function, then $f(0) \neq f(1)$. Thus, $A_\perp(f(0), f(0)) = 0$. Obviously, $A_\perp(f(1), f(1)) = 1$. Hence, $A_\perp$ satisfies the f-boundary conditions. Therefore, $A_\perp$ is an f-aggregation operator on L. □

Proposition 2.13 leads to the following corollaries.

Corollary 2.3. *Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and* f *a function on* L*. If* f *is injective, then* $A_\perp$ *and* $A_\top$ *are* f*-aggregation operators on* L*.*

Corollary 2.4. *Let* $(L, \leq, \wedge, \vee, 0, 1)$ *a bounded lattice and* f *a function on* L*. If* f *satisfies* $fx = f(1)$ *implies* $x = 1$ *(resp.* $fx = f(0)$ *implies* $x = 0$*), then* $A_\perp$

(resp. $A_\top$) is an f-aggregation operator on L.

The following corollary shows that if $A_\perp$ and $A_\top$ are f-aggregation operators on a given bounded lattice, then $A_\perp$ (resp. $A_\top$) is the smallest (resp. the greatest) f-aggregation operator on that lattice.

Corollary 2.5. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and f a function on L. If $A_\perp$ and $A_\top$ are f-aggregation operators on L, then $A_\perp$ (resp. $A_\top$) is the smallest (resp. the greatest) f-aggregation operator on L.*

For a given function f on a bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$, let us denote by $\mathcal{A}_f(L)$ the set of all f-aggregation operators on L. Next, we provide a lattice structure of the set $\mathcal{A}_f(L)$.

Proposition 2.14. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and f a function on L. If $A_\perp$ and $A_\top$ are f-aggregation operators on L, then $(\mathcal{A}_f(L), \sqsubseteq, \sqcap, \sqcup, A_\perp, A_\top)$ is a bounded lattice. Where $A \sqsubseteq B$ if $A(x, y) \leq B(x, y)$, $(A \sqcap B)(x, y) = A(x, y) \wedge B(x, y)$ and $(A \sqcup B)(x, y) = A(x, y) \vee B(x, y)$, for any $A, B \in \mathcal{A}_f(L)$ and $x, y \in L$.*

Proof. Follows from Proposition 2.11 and Corollary 2.5. □

The result of this chapter are published in [34].

3 Class of associative operations on trellises

The different classes of aggregation operations (or associative operations in general) based on additional properties (e.g., commutativity, increasingness, neutral elements) on the unit interval or a bounded lattice are useful in lots of different theoretical and applied areas, for instance, fuzzy logic, fuzzy system modeling, neural networks, expert systems, data sets, aggregation of information, ...etc. Motivated by this usefulness, this chapter is devoted to generalizing some of these classes of associative operations to the trellis setting. Some of these results are either inspired from or discussed in [10, 16, 52, 56], and others are investigated during the preparation of this thesis. For further use, we recall some classes of associative operations on lattices.

3.1. Classes of associative operations on lattices

This section contains basic definitions and properties of specific associative operations on latices. We pay particular attention to the notion of uninorms, triangular norms and triangular conorms on bounded lattices.

3.1.1. Definitions and basic properties

Definition 3.1. *Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and* $e \in L$*. A binary operation* $U : L^2 \to L$ *is called a uninorm if it is commutative, increasing, associative , and has* e *as a neutral element.*

If $e = 1$, then U is called a triangular norm (t-norm, for short). If $e = 0$, then U is called a triangular conorm (t-conorm, for short).

Next, we present some illustrative examples.
Example 3.1. *For a given bounded lattice* $(L, \leq, \wedge, \vee, 0, 1)$*,* $\wedge$ *(resp.* $\vee$*) is a t-norm (resp. a t-conorm) on* L*. Moreover, it well known that* $\wedge$ *(resp.* $\vee$*) is the greatest t-norm (resp. the smallest t-conorm) on* L*.*

Example 3.2. *For a given bounded lattice Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and* T_D *and* S_D *be two binary operations defined on* L *as follows:*

$$T_D(x,y) = \begin{cases} x \wedge y & \textit{if } x = 1 \textit{ or } y = 1; \\ 0 & \textit{otherwise;} \end{cases}$$

and

$$S_D(x,y) = \begin{cases} x \vee y & \textit{if } x = 0 \textit{ or } y = 0; \\ 1 & \textit{otherwise.} \end{cases}$$

One easily verifies that T_D *(resp.* S_D*) is a t-norm (resp. a t-conorm) on* L*. Moreover,* T_D *(resp.* S_D*) is the smallest t-norm (resp. the greatest t-conorm) on* L*.*

Proposition 3.1. *[10] Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice. Then it holds that:*

(i) $T_D \leq T \leq \wedge$*, for any t-norm* T *on* L *;*

(ii) $\vee \leq S \leq S_D$*, or any t-conorm* S *on* L *.*

Let T_1 and T_2 be t-norms on bounded lattices L and M, respectively. A lattice homomorphism $\rho : L \to M$ is a *t-norm morphism* from T_1 into T_2 if there exists a lattice morphism $\psi : M \to L$ such that:

$$T_2(x,y) = \rho\left(T_1(\psi(x), \psi(y))\right), \text{ for any } x, y \in M. \tag{3.1}$$

We will call ψ the *pseudo-inverse* of ρ. Moreover, if ρ is a lattice morphism, then ρ is increasing.

Theorem 3.1. *[4] Let* T_1 *and* T_2 *be t-norms on bounded lattices* L *and* M*, respectively. If* $\rho : L \to M$ *is a t-norm morphism from* T_1 *into* T_2 *with pseudo-inverse* ψ*, then* $\rho \circ \psi = Id_M$*.*

Proof. Let $x \in M$, then it holds that

$$\begin{aligned} x &= T_2\left(x, 1_M\right) \\ &= \rho\left(T_1\left(\psi(x), \psi\left(1_M\right)\right)\right) \text{ by equation (3.1)} \\ &= \rho\left(T_1\left(\psi(x), 1_L\right)\right) \\ &= \rho(\psi(x)). \end{aligned}$$

□

Corollary 3.1. *[4] Let T_1 and T_2 be t-norms on bounded lattices L and M, respectively. If $\rho : L \to M$ is a t-norm morphism from T_1 into T_2 and ψ is a pseudo-inverse, then it holds that ρ is surjective and ψ is injective.*

Remark 3.1. *From the Corollary 3.1, it follows that a t-norm morphism can have several pseudo-inverses.*

Definition 3.2. *[4] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. A function $\rho : L \longrightarrow L$ is an automorphism on L if it is bijective and monotonic with respect to their respective order, i.e. if $x \leq_L y$ then $\rho(x) \leq_L \rho(y)$. Notice that this implies that automorphisms are strictly increasing functions.*

Remark 3.2. *When the bounded lattice is $[0, 1]$, then the notion of automorphism coincides with the usual automorphism notion.*

Proposition 3.2. *[4] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. Then any automorphism on L is a lattice homomorphism.*

In general, the reverse implication of proposition 3.2 is not valid.

Proposition 3.3. *[4] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, T a t-norm on L and ρ an automorphism on L. Then the binary operation T^ρ is defined by:*

$$T^\rho(x, y) = \rho\left(T\left(\rho^{-1}(x), \rho^{-1}(y)\right)\right),$$

is a t-norm on L.

Proof. Analogous to the classical result. □

Corollary 3.2. *Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, T a t-norm on L and ρ an automorphism on L. Then ρ is a t-norm morphism from T into T^ρ.*

Proposition 3.4. *Let $(L, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded modular lattice. Then the binary operation T_Z is defined as follows:*

$$T_Z(x, y) = \begin{cases} x \wedge y & \text{if } x \vee y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

is a t-n m on L.

Proof. One easily verifies that T_Z is commutative, increasing and satisfies the boundary conditions. Now, we only have to show the associativity of T_Z. Let $x, y, z \in L$. On the one hand, we have that

$$T_Z\left(x, T_Z(y, z)\right) = \begin{cases} x \wedge y \wedge z & \text{if } y \vee z = 1 \text{ and } x \vee (y \wedge z) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, it holds that

$$T_Z\left(T_Z(x, y), z\right) = \begin{cases} x \wedge y \wedge z & \text{if } x \vee y = 1 \text{ and } z \vee (x \wedge y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We will show, for instance, that $y \vee z = 1$ and $x \vee (y \wedge z) = 1$ implies $x \vee y = 1$ and $z \vee (x \wedge y) = 1$. Since $x \vee (y \wedge z) = 1$ and $\wedge$ in increasing, it holds that $x \vee y = 1$. The fact that $x \vee (y \wedge z) = 1$ implies $y = y \wedge (x \vee (y \wedge z))$. From $y = z \leq y$ and L is modular, it follows that that $y = (x \wedge y) \vee (y \wedge z)$. Thus, $y \leq (x \wedge y) \vee z$. Since $z \leq (x \wedge y) \vee z$, it holds that $y \vee z \leq (x \wedge y) \vee z$. Then $(x \wedge y) \vee z = 1$. The proof of the converse implication is similar. Hence, T_{Z} is associative. Therefore, T_{Z} is a t-norm on L. □

Remark 3.3. *In general, T_{Z} is not necessarily a t-norm. Indeed, let $(X = \{0, a, b, c, d, e, 1\}, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice given by the Hasse diagram in Figure 3.1.*

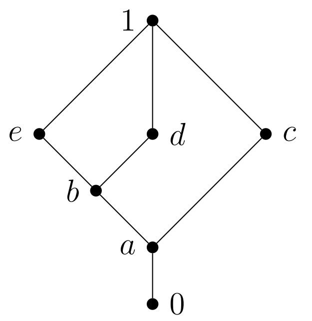


Figure 3.1: The Hasse diagram of the lattice $(X = \{0, a, b, c, d, e, 1\}, \leq)$.

Since $d \vee c = 1, d \wedge c = a$ and $e \vee a = e < 1$, it holds that $T_{\mathrm{Z}}(e, T_{\mathrm{Z}}(d, c)) = T_{\mathrm{Z}}(e, a) = 0$. On the other hand, $e \vee d = 1, e \wedge d = b, b \vee c = 1$ and $b \wedge c = a$ imply that $T_{\mathrm{Z}}(T_{\mathrm{Z}}(e, d), c) = T_{\mathrm{Z}}(b, c) = a > 0$. Hence, T_{Z} is not associative and therefore not a t-norm.

Definition 3.3. *[16] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. A function int: $L \to L$ is called an interior operator on L if, for any $x, y \in L$, it satisfies the following properties:*

(i) $\mathrm{int}(x) \leq x$;

(ii) $\mathrm{int}(x) = \mathrm{int}(\mathrm{int}(x))$;

(iii) $\mathrm{int}(x \wedge y) = \mathrm{int}(x) \wedge int(y)$.

A large class of lattice-valued t-norms can be described using interior operators. The following result proposed a method for generating t-norms on bounded lattices based on interior operators.

Theorem 3.2. *[16] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and int: $L \to L$ an interior operator on L. Then the binary operation $T : L^2 \to L$ is a t-norm on L,* *where*

$$T(x,y) = \begin{cases} x \wedge y & \text{if } 1 \in \{x,y\}; \\ \mathrm{int}(x) \wedge \mathrm{int}(y) & \text{otherwise.} \end{cases}$$

3.1.2. Construction and representations of t-norms on bounded lattices

Consider a bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$, an element $a \in L \backslash \{0,1\}$, a t-norm $V : [a,1]^2 \to [a,1]$. An ordinal sum extension T of V to L is given by (see [46])

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1]^2; \\ x \wedge y & \text{otherwise .} \end{cases} \tag{3.2}$$

However, the above-defined function T need not be a t-norm, in general.

Example 3.3. *Let $(L = \{0,a,b,c,d,1\}, \leq)$ be a bounded lattice given by the Hasse diagram in Figure 3.2 and consider the t-norm $V : [c,1]^2 \to [c,1]$ defined by*

$$V(x,y) = \begin{cases} x \wedge y & 1 \in \{x,y\}; \\ c & \text{otherwise.} \end{cases}$$

Then the operation T is constructed as Table 1 by using the formula (3.2), but T is not a t-norm on L.

T	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	0	b	b
c	0	a	0	c	c	c
d	0	a	b	c	c	d
1	0	a	b	c	d	1

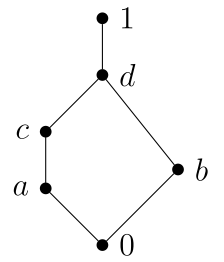


Figure 3.2: The Hasse diagram of the lattice $(L = \{0, a, b, c, d, 1\}, \leq)$.

If we take elements $b, d \in L$, then $b \leq d$. But we have that $T(b, d) = b \| c = T(d, d)$. Hence, the operation T does not satisfy monotonicity. Moreover, $T(T(d, d), b) = T(c, b) = 0$ and $T(d, T(d, b)) = T(d, b) = b$ for elements $b, d \in L$. Hence, the operation T does not satisfy associativity. So, we obtain that T is not a t-norm on L.

Theorem 3.3. *[9] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and $a \in L \backslash \{0, 1\}$. If V is a t-norm on $[a, 1]$, then the binary operation $T : L^2 \to L$ is a t-norm on L, where*

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a, 1[^2; \\ x \wedge y & \text{if } 1 \in \{x,y\}; \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Corollary 3.3. *[9] Let $a \in L \backslash \{0, 1\}$. If we put $V(x,y) = \begin{cases} x \wedge y & 1 \in \{x,y\}, \\ a & \text{otherwise,} \end{cases}$ on [a, 1] in the formula (3.3) in Theorem 3.3. Then the following t-norm is the smallest t-norm on L that extends V.*

$$T(x,y) = \begin{cases} a & \text{if } (x,y) \in [a, 1[^2; \\ x \wedge y & \text{if } 1 \in \{x,y\}; \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.4. *Let $(L = \{0, a, b, c, d, e, 1\}, \leq)$ b a bounded lattice given by the Hasse diagram in Figure 3.3.*

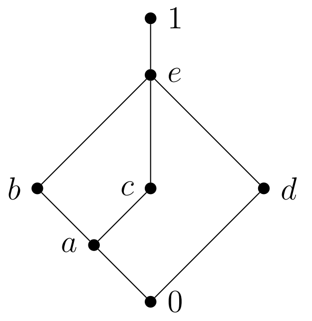


Figure 3.3: The Hasse diagram of the lattice $(L = \{0, a, b, c, d, e, 1\}, \leq)$.

(i) Consider the t-norm $V : [c, 1]^2 \to [c, 1]$ *such that* $V(x, y) = x \wedge y$*. By using Theorem 3.3, it holds that the t-norm* T *on* L *is given by the following table.*

T	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	a
b	0	0	0	0	0	0	b
c	0	0	0	c	0	c	c
d	0	0	0	0	0	0	d
e	0	0	0	c	0	e	e
1	0	a	b	c	d	e	1

(ii) Consider the t-norm $V : [c, 1]^2 \to [c, 1]$ *such that*

$$V(x, y) = \begin{cases} x \wedge y & 1 \in \{x, y\}, \\ c & \text{otherwise.} \end{cases}$$

By using Theorem 3.3, it holds that the t-norm T *on* L *is given by the following table.*

T	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	a
b	0	0	0	0	0	0	b
c	0	0	0	c	0	c	c
d	0	0	0	0	0	0	d
e	0	0	0	c	0	c	e
1	0	a	b	c	d	e	1

Theorem 3.4. *[16] Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and* $a \in L \backslash \{0, 1\}$*. If* V *is a t-norm on* $[a, 1]$*, then the binary operation* $T : L^2 \to L$ *is a t-norm on* L*, where*

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1]^2 \\ x \wedge y & \text{if } 1 \in \{x,y\} \\ x \wedge y \wedge a & \text{otherwise} \end{cases} \tag{3.4}$$

Now, in the following Theorem 3.5. Considering any bounded lattice L, we introduce a construction method for generating t-norms on L by means of a t-norm V acting on $[a, 1]$ for an element $a \in L \backslash \{0, 1\}$. First, we start with the following definition.

Definition 3.4. *Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and* $a, b \in L$*. If* a *and* b *are incomparable, we use the notation* $a \| b$*. We denote the set of all incomparable elements with* a *by* I_a*, i.e.,* $I_a = \{x \in L \mid x \| a\}$*.*

Theorem 3.5. *[16] Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and* $a \in L \backslash \{0, 1\}$*. If* V *is a t-norm on* $[a, 1]$*, then the function* $T : L^2 \to L$ *is a t-norm on* L*, where*

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1[^2, \\ 0 & \text{if } (x,y) \in [0,a[^2 \cup [0,a[\times I_a \cup I_a \times [0,a[\cup I_a \times I_a, \\ x \wedge y & \text{if } 1 \in \{x,y\} \\ x \wedge y \wedge a & \text{otherwise.} \end{cases} \tag{3.5}$$

Remark 3.4. *One easily Observe that the above t-norm* T *can be described as follows:*

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1[^2, \\ 0 & \text{if } (x,y) \in [0,a]^2 \cup [0,a[\times I_a \cup I_a \times [0,a[\cup I_a \times I_a, \\ y \wedge a & \text{if } (x,y) \in [a,1[\times I_a, \\ x \wedge a & \text{if } (x,y) \in I_a \times [a,1[, \\ x & \text{if } (x,y) \in [0,a] \times [a,1[, \\ y & \text{if } (x,y) \in [a,1[\times [0,a] \\ x \wedge y & \text{if } 1 \in \{x,y\}. \end{cases}$$

And represent the t-norm T *on* L *as shown in Figure 3.4.*

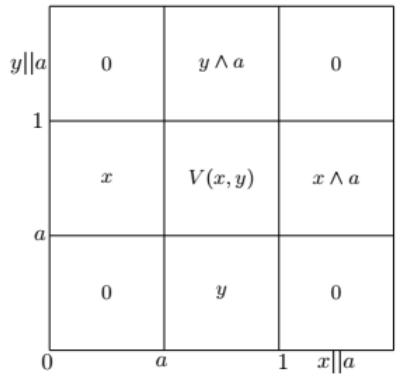


Figure 3.4: The t-norm T on L given in the Theorem 3.5.

3.1.3. Ordinal sum construction for t-norms on bounded lattices

Ordinal sums have been introduced in many different contexts (e.g., for posets, semigroups, t-norms, copulas, aggregation operators, or quite recently for hoops). In [7], Birkhoff provides a definition for building the ordinal sum X and Y of two disjoint posets X, Y. Due to the associativity of this construction we immediately extend this concept to families of pairwise disjoint posets for some linearly ordered index set $(I, \preccurlyeq_I), I \neq \emptyset$. Note that ordinal sums of disjoint posets in the sense of Birkhoff are also referred to as linear sums of posets [12].

Definition 3.5. *[12] Let a linearly ordered index set $(I, \preccurlyeq_I), I \neq \emptyset$ and a family of pairwise disjoint posets $(X_i, \leq_i)_{i \in I}$. The ordinal sum $\bigoplus_{i \in I} X_i$ is defined as the set $\bigcup_{i \in I} X_i$ equipped with the following order $\leq$ such that:*

$$x \leq y \Leftrightarrow (\exists i \in I | x, y \in X_i \text{ and } x \leq_i y) \text{ or } (\exists i, j \in I | x \in X_i,\ y \in X_j \text{ and } i \prec_I j).$$

In this contribution, we focus on ordinal sums of t-norms acting on some bounded lattice that is not necessarily a chain or an ordinal sum of posets. Necessary and sufficient conditions are provided for an ordinal sum operation yielding again a t-norm on some bounded lattice whereas the operation is determined by an arbitrary selection of subintervals as carriers for arbitrary t-norms.

The following definition of an ordinal sum of t-norms defined on subintervals of a bounded lattice $(L, \leq, \wedge, \vee, 0, 1)$, which generalizes the one given in [30, 47] on subintervals of $[0, 1]$.

Definition 3.6. *[46] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, $(I, \preccurlyeq_I)$ a totally* *ordered index set such that* $I \neq \emptyset$, $\{]a_i, b_i[\}_{i\in I}$ *a family of pairwise disjointed subintervals of* L *and* $\left\{T^{[a_i,b_i]}\right\}_{i\in I}$ *a family of t-norms on the corresponding intervals* $\{[a_i, b_i]\}_{i\in I}$. *Then the binary operation* $T = \left\{\left\langle a_i, b_i, T^{[a_i,b_i]}\right\rangle\right\}_{i\in A} : L \times L \to L$ *defined for any* $x, y \in L$, *as*

$$T(x,y) = \begin{cases} T^{[a_i,b_i]}(x,y) & \text{if } x,y \in [a_i,b_i] \text{ and } i \in I; \\ x \wedge y & \text{otherwise.} \end{cases} \tag{3.6}$$

is called ordinal sum of the family $\left\{T^{[a_i,b_i]}\right\}_{i\in I}$ *on the bounded lattice* L.

Theorem 3.6. *[9] Let* $(L, \leq, \wedge, \vee, 0, 1)$ *be a bounded lattice and* $\{a_0, a_1, a_2, \ldots, a_n\}$ *be a finite chain in* L *such that* $a_0 = 1 > a_1 > a_2 > \ldots > a_n = 0$. *Let* $V : [a_1, 1]^2 \to [a_1, 1]$ *be a t-norm on the sublattice* $[a_1, 1]$. *Then the operation* $T = T_n : L^2 \to L$ *defined recursively as follows is a t-norm, where* $V = T_1$ *and the operation* $T_i : [a_i, 1]^2 \to [a_i, 1]$, *for any* $i \in \{2, 3, \ldots, n\}$ *is given by*

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y) & \text{if } (x,y) \in [a_{i-1}, 1\,[^2; \\ x \wedge y & \text{if } 1 \in \{x,y\}; \\ a_i & \text{otherwise.} \end{cases} \tag{3.7}$$

Remark 3.5. *The proof follows easily from Theorem 3.3 by induction and therefore it is omitted. The construction described inductively by formula (3.7) can be considered as an ordinal sum construction for t-norms. Obviously, if* L *in Theorem 3.6 is a chain, then the formula (3.7) reduces to*

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y) & \text{if } (x,y) \in [a_{i-1}, 1\,[^2; \\ x \wedge y & \text{if } 1 \in \{x,y\}; \\ a_i & \text{if } (x,y) \in [a_i, a_{i-1}\,[^2 \cup [a_i, a_{i-1}\,[\times [a_{i-1}, 1\,[\cup [a_{i-1}, 1\,[\times [a_i, a_{i-1}[. \end{cases}$$

Example 3.5. *Let* $(L = \{0, a, b, c, d, 1\}, \leq)$ *be a bounded lattice given by the Hasse diagram in Figure 3.5 and a finite chain* $\{0, a, b, c, 1\}$ *such that* $0 \leq a \leq b \leq c \leq 1$. *Suppose that* $V : [c, 1]^2 \to [c, 1]$ *is a t-norm on the sublattice* $[c, 1]$. *By using Theorem 3.6, where* $V = T_1$, *it holds that the t-norms* $T_2 : [b, 1]^2 \to [b, 1], T_3 : [a, 1]^2 \to [a, 1]$ *and* $T = T_4 : L^2 \to L$ *are defined as follows:*

T_2	b	c	d	1
b	b	b	b	b
c	b	c	b	c
d	b	b	b	d
1	b	c	d	1

,

T_3	a	b	c	d	1
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	c	b	c
d	a	b	b	b	d
1	a	b	c	d	1

and

$T = T_4$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	a	a	a	a
b	0	a	b	b	b	b
c	0	a	b	c	b	c
d	0	a	b	b	b	d
1	0	a	b	c	d	1

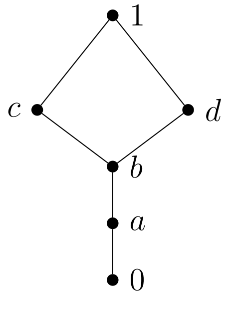


Figure 3.5: The Hasse diagram of the lattice $(L = \{0, a, b, c, d, 1\}, \leq)$.

Remark 3.6. *By using the t-norm T defined by the formula (3.3) in Theorem 3.3, it holds that T on the lattice $(L, \leq, \wedge, \vee, 0, 1)$ given by the Hasse diagram in Figure 3.5 is defined for a given t-norm $V = T_1 : [c, 1]^2 \to [c, 1]$ on the sublattice*

$[c, 1]$ by the following table.

T	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	c	0	c
d	0	0	0	0	0	d
1	0	a	b	c	d	1

3.1.4. A T-partial order obtained from T-norms

A natural partial order for semigroups was defined by H. Mitsch in 1986 (see [39]). In this section, we define a t-partial order obtained from t-norms and investigate its properties (see, e.g. [1, 2]).

Definition 3.7. *[27] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and T a t-norm on L. The following order is called a triangular order (t-order, for short) for a t-norm T.*

$$x \preceq_T y \Leftrightarrow T(\ell, y) = x, \text{ for some } \ell \in L.$$

Proposition 3.5. *[27] The binary relation $\preceq_T$ is a partial order on L.*

Proof. Since $1 \in L$ and $T(1, x) = x, x \preceq Tx$ holds. Thus, the reflexivity is satisfied. Let $x \preceq_T y$ and $y \preceq_T \quad x$. Then there exist ℓ_1, ℓ_2 of L such that $T(\ell_1, y) = x$ and $T(\ell_2, x) = y$. Hence, $x = T(\ell_1, y) \leq T(1, y) = y$; i.e, $x \leq y$. On the other hand, $y = T(\ell_2, x) \leq T(1, x) = x$; i.e, $y \leq x$. So, $x = y$. Thus, the antisymmetry is satisfied. Let $x \preceq_T y$ and $y \preceq_T z$. Then there exist ℓ_1, ℓ_2 of L such that $T(\ell_1, y) = x$ and $T(\ell_2, z) = y$. For $T(\ell_1, \ell_2)$ of $L, T(T(\ell_1, \ell_2), z) = T(\ell_1, T(\ell_2, z)) = T(\ell_1, y) = x$. Thus, $x \preceq_T z$. Hence, $\preceq_T$ satisfies the transitivity. Therefore, $\preceq_T$ is a partial order on L. □

Proposition 3.6. *[27] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and T a t-norm on L. If $(x, y) \in \preceq_T$, then $(x, y) \in \leq$.*

Proof. Let $(x, y) \in \preceq_T$. Then there exists an element ℓ of L such that $x = T(\ell, y) \leq T(1, y) = y$. Thus, $(x, y) \in \leq$. □

Remark 3.7. *If $(x, y) \in \leq$, then $(x, y) \in \preceq_T$ may not be true. Indeed, let $(L = \{0, a, b, c, 1\}, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice given by the Hasse-diagram in Figure 3.6.*

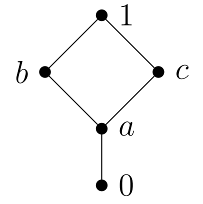


Figure 3.6: The Hasse diagram of the lattice $(L = \{0, a, b, c, 1\}, \leq)$.

Now, suppose that $T = T_D$, we can see $a \leq b$ but $a \not\preceq_{T_D} b$. Indeed, if $a \preceq_{T_D} b$, then there exists an element ℓ of L such that $T_D(\ell, b) = a$. If $\ell = 0$, then $a = 0$, which is a contradiction. If $\ell = a, b$ or c, then $T_D(\ell, b) = 0 = a$. This is a contradiction. If $\ell = 1$, then $T_D(1, b) = b = a$, which is not possible. Therefore, there doesn't exist any element ℓ of L satisfying $T_D(\ell, b) = a$. Thus, $a \not\preceq_{T_D} b$. Hence, the order $\preceq_{T_D}$ on L is given by the following Hasse diagram:

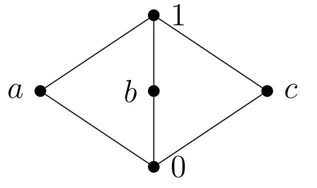


Figure 3.7: The order $\preceq_{T_D}$ on L.

Proposition 3.7. *[27] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and T a t-norm on L. If $T = T_D$, then it holds that $a \wedge_T b = 0$, for any $a \in L \backslash \{0, 1\}$ and $a \vee_T b = 1$, for any $b \in L \backslash \{0, 1, a\}$. Moreover, $(L, \preceq_{T_D})$ is a lattice.*

Proof. Let $a, b \in L \backslash \{0, 1\}$ such that $a \neq b$, since $T_D(a, b) = 0$ and for any $k \in L$, $T_D(a, k) \neq b$ and $T_D(b, k) \neq a$, a and b are not comparable with respect to $\preceq_{T_D}$. We claim that for an arbitrary $a \in L \backslash \{0, 1\}$ it satisfies $a \wedge_{T_D} b = 0$ for any $b \in L \backslash \{0, 1, a\}$. If $a \wedge_{T_D} b = x \neq 0$, then $x \preceq_{T_D} a$ and $x \preceq_{T_D} b$. Thus, there exists $x_1 \in L \backslash \{0\}$ such that $0 \neq x = T_D(a, x_1)$. If $x = a$, then this is a contradiction since a and b aren't comparable with respect to $\preceq_{T_D}$. If $T_D(a, x_1) = 1$ or x_1, then we obtain that $a = 1$. This contradicts the choice of a. Hence, $a \wedge_{T_D} b = 0$. Similarly, let us show that for an arbitrary $a \in L \backslash \{0, 1\}, a \vee_{T_D} b = 1$ for any $b \in L \backslash \{0, 1, a\}$. Let $a \vee_{T_D} b = x$. Then $a \preceq_{T_D} x$ and $b \preceq_{T_D} x$, and so there exist $x_1, x_2 \in L \backslash \{0\}$ such that $T_D(x, x_1) = a$ and $T_D(x, x_2) = b$. If $x = a$, then $T_D(a, x_2) = b$ which is a contradiction since a and b aren't comparable with respect to $\preceq_{T_D}$. Then, $x_1 = a$, so it must be $x = 1$. Therefore, $a \vee_{T_D} b = 1$.

Finally, we have that $(L, \preceq_{T_D})$ is a lattice. Now, we give an example such that $(L, \preceq_T)$ is a lattice and $T \neq T_D$. $\square$

Proposition 3.8. *[27] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and T a t-norm on L. If $a \preceq_T b$, then $T(a, c) \preceq_T T(b, c)$, for any $a, b, c \in L$.*

Proof. Let $a, b \in L$ such that $a \preceq_T b$. Then there exists $x \in L$ such that $T(x, b) = a$. Since $T(a, c) = T(T(x, b), c) = T(x, T(b, c))$, it holds that there exists $x \in L$ such that $T(x, T(b, c)) = T(a, c)$. Thus, $T(a, c) \preceq_T T(b, c)$. $\square$

Corollary 3.4. *[27] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and T a t-norm on L. If $(L, \preceq_T)$ is a lattice, then $T : (L, \preceq_T)^2 \to (L, \preceq_T)$ is a t-norm.*

Let $(L, \leq, \wedge, \vee)$ be a complete lattice, T a t-norm on L and $L_1 \subseteq L$. The notation $T \downarrow L_1$ will be used for the restriction of T to L_1.

Proposition 3.9. *[27] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and T be a t-norm on L. Then $T \downarrow I(T)$ is a t-norm on $(I(T), \preceq_T)$.*

Proof. Let $a, b \in I(T)$. Then we must show that $T(a, b) \in I(T)$. Since T is associative, it holds that

$$\begin{aligned} T(T(a, b), T(a, b)) &= T(T(T(a, b), a), b) \\ &= T(T(T(b, a), a), b) = T(T(b, T(a, a)), b) \\ &= T(T(b, a), b) = T(T(a, b), b) \\ &= T(a, T(b, b)) = T(a, b) \end{aligned}$$

Then, $T(a, b)$ is an element of $I(T)$. Proposition 3.8 guarantees that T is increasing with respect to $\preceq_T$. Also, T is associative with respect to $\preceq_T$ and. Since $1 \in I(T)$, it holds that $T(x, 1) = x$, for any $x \in I(T)$. Therefore, $T \downarrow I(T)$ is a t-norm on $I(T)$. $\square$

Proposition 3.10. *[27] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and T a t-norm on L such that $K \subseteq L$ be a lattice with respect to the order on L. If $x \wedge_T y = T(x, y)$ for any $x, y \in K$, then $T \downarrow K = \wedge$. Moreover, if $K = L$, then $T = \wedge$.*

Proof. Since $x \wedge_T x = x = T(x, x)$, for any $x \in K$, it holds that $x \wedge y = T(x \wedge y, x \wedge y) \leq T(x, y) \leq x \wedge y$, for any $x, y \in K$. Therefore, $T(x, y) = x \wedge y$. $\square$

Corollary 3.5. *[27] Let $L = [0, 1]$ and T a t-norm on L. Then $(I(T), \preceq_T) = (I(T), \leq)$.*

3.1.5. Representation of a Boolean Algebra by its t-Norms

We will show that a particular subset of τ of all triangular norms on L forms a Boolean algebra that is isomorphic to L, where L is a complete and atomic Boolean algebra.

Definition 3.8. *[44] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice. A subset α of L, which does not contain 0 is a J-set in L, if α satisfies the following condition:*

$$0 < y \leq x \in \alpha \text{ implies } y \in \alpha, \text{ for any } x, y \in L.$$

We note that the empty set is a J-set in L. The importance of a J-set in L can be gauged from the following results, which indicate how to characterize a general t-norm on L.

Theorem 3.7. *[44] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice, T a t-norm on L and $\omega \subseteq L$ such that $\omega = \{x \in L : x \neq 0, T(x, x) = 0\}$. Then it holds that*

(i) ω is a J-set in L;

(ii) If $a \in \omega$ and $b \leq a$, then $T(a, b) = 0$;

(iii) If $a \in \omega, b \leq a$ and $c \leq a$, then $T(b, c) = 0$.

In the following result, we present a new family of t-norms based on J-sets subset in L. This family will be needed to characterize an atomic Boolean algebra.

Theorem 3.8. *[44] Let $(L, \leq, \wedge, \vee, 0, 1)$ be a bounded lattice and α a J-set in L. Then the binary operation T_α on L defined by*

$$T_\alpha(x, y) = \begin{cases} 0 & \text{if} \quad x \neq 1 \neq y \text{ and } x \wedge y \in \alpha; \\ x \wedge y & \text{otherwise.} \end{cases}$$

is a t-norm on L.

Now, let B be an atomic Boolean algebra that is complete (i.e., Boolean algebra which is isomorphic to power sets). If γ is the set of all atoms of B, then B is isomorphic to the Boolean algebra $P(\gamma)$ of all subsets of γ under the set inclusion. Further, γ is a J-set in B such that if $\alpha \subseteq \gamma$, then both α and $\gamma - \alpha$ are J-sets in B.

Proposition 3.11. *[44] Let B be an atomic Boolean algebra and γ is the set of all atoms of B such that $|\gamma| \geq 2$. If $\alpha \subseteq \gamma$ and $\beta \subseteq \gamma$. Then it holds that $\alpha \subseteq \beta$ if and only if $T_\beta \leq T_\alpha$.*

Proof. Let $x, y \in B$, $\alpha \subseteq \gamma$ and $\beta \subseteq \gamma$ such that $\alpha \subseteq \beta$. If $x = 1$, then

$T_\beta(x,y) = y = T_\alpha(x,y)$. If $y = 1$, then $T_\beta(x,y) = x = T_\alpha(x,y)$. If $x \neq 1$ and $y \neq 1$ such that $x \wedge y \in \beta$, then $T_\beta(x,y) = 0 \leq T_\alpha(x,y)$. If $x \neq 1 \neq y$ and $x \wedge y \notin \beta$, then $T_\beta(x,y) = x \wedge y = T_\alpha(x,y)$. Conversely, let $T_\beta \leq T_\alpha$. Let $c \in \alpha$ such that $c \notin \beta$. Then $T_\beta(c,c) = c \wedge c = c > 0 = T_\alpha(c,c)$, hence a contradiction. □

Theorem 3.9. *[44] Let B be an atomic Boolean algebra and γ is the set of all atoms of B such that $|\gamma| \geq 2$. Then $\tau_\gamma = \{T_\alpha \mid \alpha \subseteq \gamma\}$ of t-norms T_α on B defined in Theorem 3.8 forms a Boolean algebra, under the partial ordering $\leq$ for t-norms and meet ($\wedge$), join ($\vee$), and the complement T'_α given by*

$$T_\alpha \wedge T_\beta = T_{\alpha\cup\beta'}$$
$$T_\alpha \vee T_\beta = T_{\alpha\cap\beta'}$$
$$T'_\alpha = T_{\alpha'},$$

where $\alpha' = \gamma - \alpha$. Furthermore, B is isomorphic to τ_γ under the identification map

$$\alpha \longrightarrow T_{\alpha'}, \text{ for any } \alpha \subseteq \gamma.$$

Proof. One easily verifies that $\wedge$ and $\vee$ are commutative and that each operation is distributive over the other. Proposition 3.11 guarantees that $T_\emptyset$ is the greatest member of τ_γ and it is the meet operation $\wedge$ on B. Also, T_γ is the least member of τ_γ and it is the smallest t-norm on B (i.e., T_D). Moreover, since B is a Boolean algebra, it holds that

$$T_\alpha \vee T_{\alpha'} = T_{\alpha\cap\alpha'} = T_\emptyset \quad \text{and} \quad T_\alpha \wedge T_{\alpha'} = T_{\alpha\cup\alpha'} = T_\gamma.$$

According to Huntington [25], it follows that τ_γ is a Boolean algebra. Furthermore, let a map defined on B by:

$$\alpha \longrightarrow T_{\alpha'}, \quad \alpha \subseteq \gamma,$$

y then one easily verifies that $P(\gamma)$ is isomorphic to τ_γ. Therefore, B is isomorphic to τ_γ. □

Example 3.6. *Let $(D(6), |, gcd, lcm, ')$ be a Boolean algebra of the positive divisors of 6. Then $D(6)$ has two atoms are 2 and 3. Thus, Theorem 3.9 guarantees that all t-norms on $D(6)$ are $T_{\{2,3\}} = T_D, T_{\{2\}}, T_{\{3\}}$ and $T_\emptyset = gcd$ defined as follows:*

T_D	1	2	3	6
1	1	1	1	1
2	1	1	1	2
3	1	1	1	3
6	1	2	3	6

gcd	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

$T_{\{2\}}$	1	2	3	6
1	1	1	1	1
2	1	1	1	2
3	1	1	3	3
6	1	2	3	6

$T_{\{3\}}$	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	1	3
6	1	2	3	6

3.2. Special classes of associative operations on bounded trellises

In this section, we study associative operations on a bounded trellis. More precisely, we generalize the class of triangular norms and the class of triangular conorms on bounded lattices to the setting of bounded trellises. We pay particular attention to what happens when we eliminate the property of transitivity.

Notation 3.1. *(i)* $\mathcal{AO}_e(X)$: *the class (or the set) of all binary operations on a bounded trellis* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *that are associative, commutative, increasing, and have an arbitrary element* $e \in X$ *as neutral element. If* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *is a bounded lattices, then* $\mathcal{AO}_e(X)$ *is the class of uninorms on* X*.*

(ii) If $e = 0$, $\mathcal{AO}_0(X)$ *is the class of associative, commutative, increasing, and have* 0 *as a neutral element. If* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *is a bounded lattices, then* $\mathcal{AO}_0(X)$ *is the class of triangular norms (t-norms)on* X*.*

(iii) If $e = 1$, $\mathcal{AO}_1(X)$ *is the class of associative, commutative, increasing, and have* 1 *as a neutral element. This class was called the class of t-norms on a bounded trellis on* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *and it was studied in detail in [56].*

Remark 3.8. *(i) For a given bounded trellis* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$*, the binary operation* $\wedge$ *(resp.* $\vee$*) is not necessarily an element of* $\mathcal{AO}_1(X)$ *(resp. of* $\mathcal{AO}_1(X)$*). Indeed,* $\wedge$ *and* $\vee$ *are not necessarily associative operations.*

(ii) In view of (i), we were more careful to use the same names of the above classes on bounded lattices in the trellis setting. Of course, we generalize these classes to bounded trellises, but they are not extensions (in sense of inclusion) to the same classes on bounded lattices.

Next, we present some elements of the classes $\mathcal{AO}_0(X)$ and $\mathcal{AO}_1(X)$.

Example 3.7. *Let* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *be a bounded trellis.*

(i) The binary operation T_{D} *on* X*, called* drastic t-norm*, defined by*

$$T_{\mathrm{D}}(x,y) = \begin{cases} x & \textit{if } y = 1; \\ y & \textit{if } x = 1; \\ 0 & \textit{otherwise;} \end{cases} \tag{3.8}$$

is the smallest element on $\mathcal{AO}_1(X)$.

(ii) The binary operation S_{D} *on* X*, called* drastic t-conorm*, defined by*

$$S_{\mathrm{D}}(x,y) = \begin{cases} x & \textit{if } y = 0; \\ y & \textit{if } x = 0; \\ 1 & \textit{otherwise;} \end{cases} \tag{3.9}$$

is the greatest element on $\mathcal{AO}_0(X)$.

Example 3.8. *Let* $(X = \{0, a, b, c, d, e, 1\}, \trianglelefteq, \wedge, \vee)$ *be a bounded trellis given by the Hasse diagram in Fig. 3.8.*

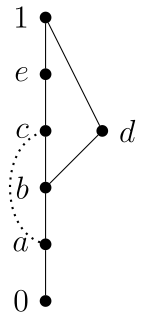


Figure 3.8: The Hasse diagram of the trellis $(X = \{0, a, b, c, d, e, 1\}, \trianglelefteq)$.

The operation S *on* X *defined by the following table is an element of* $\mathcal{AO}_0(L)$.

S	0	a	b	c	d	e	1
0	0	a	b	c	d	e	1
a	a	a	b	e	d	e	1
b	b	b	b	e	d	e	1
c	c	e	e	e	1	e	1
d	d	d	d	1	d	1	1
e	e	e	e	e	1	e	1
1	1	1	1	1	1	1	1

On a bounded modular lattice X [13], the binary operation T_{Z} defined by

$$T_{\mathrm{Z}}(x,y) = \begin{cases} x \wedge y & \text{if } x \vee y = 1; \\ 0 & \text{otherwise;} \end{cases} \tag{3.10}$$

is a t-norm. This example was extended to the setting of bounded modular trellises under a suitable necessary and sufficient condition [56].

Proposition 3.12. [56] *Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded modular trellis. Then it holds that $T_{\mathrm{Z}} \in \mathcal{AO}_1(X)$ if the following condition is satisfied:*

$$(\forall (x,y,z,t) \in X^4)(((x \wedge y \neq 0) \text{ and } (x \vee y = 1)) \text{ implies } (x \vee z) \vee (y \vee t) = 1)\,. \tag{3.11}$$

Dually, we obtain the following example.

On a bounded modular lattice X [13], the binary operation S_{Z} defined by

$$S_{\mathrm{Z}}(x,y) = \begin{cases} x \vee y & \text{if } x \wedge y = 0; \\ 1 & \text{otherwise;} \end{cases} \tag{3.12}$$

is a t-conorm. This example was extended to the setting of bounded modular trellises under a suitable necessary and sufficient condition [56].

Proposition 3.13. *Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded modular trellis. Then it holds that $S_{\mathrm{Z}} \in \mathcal{AO}_0(X)$ if the following condition is satisfied:*

$$(\forall (x,y,z,t) \in X^4)(((x \vee y \neq 1) \text{ and } (x \wedge y = 0)) \text{ implies } (x \wedge z) \wedge (y \wedge t) = 0)\,. \tag{3.13}$$

Remark 3.9. [56] *For a given bounded modular trellis $(X, \trianglelefteq, \wedge, \vee, 0, 1)$, the condition (3.13) does not necessarily hold. Indeed, let $(X = \{0, a, b, c, d, 1\}, \trianglelefteq, \wedge, \vee)$ be the bounded modular trellis given by the Hasse diagram displayed in Figure 1.4. It is clear that X does not satisfy this condition.*

The following example presents a family of elements in $\mathcal{AO}_0(X)$ and $\mathcal{AO}_1(X)$ based on the notions of atom and coatom on a bounded trellis. First, we recall the following definition of atom and coatom on trellises. This definition is a natural generalization of the same notions on lattices (see, e.g. [26]).

Definition 3.9. *Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis. An element $\alpha \in X$ is called:*

(i) atom, if it is a minimal element of the set $X \backslash \{0\}$;

(ii) coatom, if it is a maximal element of the set $X \backslash \{1\}$.

We denote by Atom(X) (resp. Coatom(X)), the set of all atoms (resp. coatoms) of X.

Proposition 3.14. *Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis and $i \in$ Atom(X). The binary operation S_i on X, defined by*

$$S_i(x,y) = \begin{cases} x \vee y & \text{if } x = 0 \text{ or } y = 0; \\ i & \text{if } (x,y) = (i,i); \\ 1 & \text{otherwise.} \end{cases} \tag{3.14}$$

is an element of $\mathcal{AO}_0(X)$.

In a similar way,

Proposition 3.15. *Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis and $i \in$ Coatom(X). The binary operation T_i on X, defined by*

$$T_i(x,y) = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1; \\ i & \text{if } (x,y) = (i,i); \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

is an element of $\mathcal{AO}_1(X)$.

The proof of the above propositions is straightforward.

Next, we present illustrative examples. Before that, we need to mention the following abbreviation. Throughout the rest of the paper, the rows and columns corresponding to 0 and 1 in the tables define elements of $\mathcal{AO}_0(X)$ and $\mathcal{AO}_1(X)$ will not be considered, as they are fixed.

In the following remark, we illustrate that the greatest t-norm on a given bounded trellis does not necessarily exist. This is one of the properties that we lose when considering bounded trellises instead of bounded lattices.

Remark 3.10. *On given bounded lattice L, it is well known that the classes $\mathcal{AO}_0(L)$ and $\mathcal{AO}_1(X)$ have least and greatest elements. In the trellis setting, Ex-*

ample 3.7 mentions that $\mathcal{AO}_0(X)$ has a greatest element and $\mathcal{AO}_1(X)$ has a least element. While $\mathcal{AO}_0(X)$ does not necessarily has a least element, and $\mathcal{AO}_1(X)$ does not necessarily has a greatest element. Indeed, let $(X = \{0, a, b, c, d, e, 1\}, \trianglelefteq, \wedge, \vee)$ be the bounded trellis given in Example 3.8. The binary operations T_1 and T_2 on X defined by the following tables are maximal elements in $\mathcal{AO}_1(X)$ [56].

T_1	a	b	c	d	e
a	0	0	0	a	0
b	0	b	b	b	b
c	0	b	c	b	c
d	a	b	b	d	b
e	0	b	c	b	e

T_2	a	b	c	d	e
a	0	0	0	0	a
b	0	b	b	b	b
c	0	b	c	b	c
d	0	b	b	d	b
e	a	b	c	b	e

3.3. Constructions of elements of $\mathcal{AO}_0(X)$ and $\mathcal{AO}_1(X)$ based on specific subsets on bounded trellises

In this section, we construct several elements of $\mathcal{AO}_0(X)$ and $\mathcal{AO}_1(X)$ based on specific subsets on a given bounded trellis.

3.3.1. Notations and auxiliary results

In this subsection, we introduce the following functions based on specific subsets on a given bounded trellis $(X, \trianglelefteq, \wedge, \vee)$ in order to define elements of $\mathcal{AO}_0(X)$ and $\mathcal{AO}_1(X)$. First, we recall the following notations and auxiliary results. For more detail, we refer to [56].

For a given trellis $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ and $A \subseteq X$, we recall the definition of the following mapping $\lambda_A \colon X \to X$:

$$\lambda_A(x) = \bigvee\{a \in A \mid a \trianglelefteq x\} = \bigvee(A \cap \downarrow x)\,.$$

Remark 3.11. [56] *In general, the mapping λ_A is not well defined since the supremum $\bigvee(A \cap \downarrow x)$ does not necessarily exist. However, if A is a finite subset of X^{rtr}, then Proposition 1.5 guarantees that it is well-defined.*

For further use, we recall the following properties of λ_A.

Proposition 3.16. [56] *Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ be a trellis and A a finite subset of X^{rtr}. For any $x \in X$, it holds that*

(i) $\lambda_A(x) \trianglelefteq x$;

(ii) if $x \in A$, *then* $\lambda_A(x) = x$.

The following proposition lists additional properties of λ_A in case A is a $\vee$-sub-trellis of $\mathbb{T}$. Since λ_A behaves as the identity on A and maps to A, we refer to this mapping as an *embedding*.

Proposition 3.17. [56] *Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee)$ be a trellis and A a finite subset of X^{rtr}. If A is a $\vee$-sub-trellis of $\mathbb{T}$, then it holds that*

(i) $\lambda_A(X) \subseteq A$;

(ii) λ_A *is idempotent, i.e.,* $\lambda_A(\lambda_A(x)) = \lambda_A(x)$, *for any* $x \in X$;

(iii) λ_A *is increasing, i.e., if* $x \trianglelefteq y$, *then* $\lambda_A(x) \trianglelefteq \lambda_A(y)$, *for any* $x, y \in X$.

The following theorem provides an element of $\mathcal{AO}_1(X)$ based on the above mapping λ_A, where when A is a sub-trellis of $(X, \trianglelefteq, \wedge, \vee, 0, 1)$.

Theorem 3.10. [56] *Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis and A a finite subset of X^{rtr}. If A is a sub-trellis of $\mathbb{T}$, then the binary operation $T^{[A]}$ defined by:*

$$T^{[A]}(x, y) = \begin{cases} y & , \text{ if } x = 1 \\ x & , \text{ if } y = 1 \\ \lambda_A(x \wedge y) & , \text{ otherwise.} \end{cases}$$

is an element of $\mathcal{AO}_1(X)$.

In the same line, the following result provides an element of $\mathcal{AO}_0(X)$. The proof is dual to that of Theorem 3.10.

Theorem 3.11. *Let $\mathbb{T} = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, A a finite subset of X^{ltr} and $\gamma \colon X \to X$ defined as:*

$$\gamma_A(x) = \bigwedge\{a \in A \mid x \trianglelefteq a\} = \bigwedge(A \cap \uparrow x) \,.$$

If A is a sub-trellis of $\mathbb{T}$, then the binary operation $S^{[A]}$ defined by:

$$S^{[A]}(x, y) = \begin{cases} y & , \text{ if } x = 0 \\ x & , \text{ if } y = 0 \\ \gamma(x \wedge y) & , \text{ otherwise.} \end{cases}$$

is an element of $\mathcal{AO}_0(X)$.

Remark 3.12. *In general, the mapping γ_A is not well defined since the infmum $\bigwedge(A\cap\uparrow x)$ does not necessarily exist. However, if A is a finite subset of X^{ltr}, then it is well-defined.*

Next, we will use the following notations:

(i) $\gamma^\alpha(x) = \bigwedge\{a \in X^\alpha \mid x \trianglelefteq a\} = \bigwedge(X^\alpha \cap \uparrow x)$;

(ii) $\lambda^\alpha(x) = \bigvee\{a \in X^\alpha \mid a \trianglelefteq x\} = \bigvee(X^\alpha \cap \downarrow x)$.

The following propositions are immediate.

Proposition 3.18. *Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis and $\alpha \in \{\mathrm{dis}, \mathrm{ass}, \wedge\text{-}\mathrm{ass}, \vee\text{-}\mathrm{ass}, \mathrm{tr}, \mathrm{rtr}\}$. If X^α is finite subtrellis of X, then it holds that*

(i) $x \trianglelefteq \gamma^\alpha(x)$, for any $x \in X$;

(ii) if $x \in X^\alpha$, then $\gamma^\alpha(x) = x$.

(iii) γ^α is idempotent;

(iv) γ^α is isotone;

(v) γ_A is a join-homomorphism, i.e., $\gamma_A(x\vee y) = \gamma_A(x)\vee\gamma_A(y)$, for any $x, y \in X$.

Proposition 3.19. *Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis and $\alpha \in \{\mathrm{dis}, \mathrm{ass}, \wedge\text{-}\mathrm{ass}, \vee\text{-}\mathrm{ass}, \mathrm{tr}, \mathrm{rtr}\}$. If X^α is finite subtrellis of X, then it holds that*

(i) $\lambda^\alpha(x) \trianglelefteq x$, for any $x \in X$;

(ii) if $x \in X^\alpha$, then $\lambda^\alpha(x) = x$.

(iii) λ^α is idempotent;

(iv) λ^α is isotone;

(v) λ_A is a meet-homomorphism, i.e., $\lambda_A(x \wedge y) = \lambda_A(x) \wedge \lambda_A(y)$, for any $x, y \in X$.

3.3.2. Elements of $\mathcal{AO}_0(X)$ based on specific subsets on bounded trellises

In this subsection, we give some examples of elements of $\mathcal{AO}_0(X)$ on a bounded trellis $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ based on its specific subsets of left-transitive elements.

In view of Theorem 3.11 and Proposition 3.18, we derive the following propositions that define elements of $\mathcal{AO}_0(X)$.

Proposition 3.20. *Let* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *be a bounded trellis and* $\alpha \in \{\text{ass}, \wedge\text{-ass}, \vee\text{-ass}, \text{tr}, \text{ltr}\}$. *If* $(X^\alpha, \trianglelefteq, \wedge, \vee)$ *is finite subtrellis, then the binary operation* T_α *defined by*

$$S_\alpha(x, y) = \begin{cases} x \vee y & \text{if } x, y \in X^\alpha \text{ or } x = 0 \text{ or } y = 0; \\ \gamma^\alpha(x) \vee \gamma^\alpha(y) & \text{otherwise;} \end{cases}$$

is an element of $\mathcal{AO}_0(X)$.

The fact that X^{dis} is a sublattice of any trellis leads to the following particular case.

Proposition 3.21. *Let* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *be a bounded trellis. If* X^{dis} *is finite, then the binary operation* S_{dis} *defined by*

$$S_{\text{dis}}(x, y) = \begin{cases} x \vee y & \text{if } x, y \in X^{\text{dis}} \text{ or } x = 0 \text{ or } y = 0; \\ \gamma^{\text{dis}}(x) \vee \gamma^{\text{dis}}(y) & \text{otherwise;} \end{cases}$$

is an element of $\mathcal{AO}_0(X)$.

Under the same condition of Propositions 3.20 and 3.21, we obtain the following result. The proof is achieved by Propositions 3.18.

Proposition 3.22. *Let* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *be a bounded trellis and* $\alpha \in \{\text{dis}, \text{ass}, \wedge\text{-ass}, \vee\text{-ass}, \text{tr}, \text{ltr}\}$. *It holds that*

$$S_\alpha(x, y) \in X^\alpha, \text{ for any } x, y \in X\,.$$

Remark 3.13. ***(coincidence)*** *If* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *is a bounded lattice, then* S_α *coincides with the join* ($\vee$)*, for any* $\alpha \in \{\text{dis}, \text{ass}, \wedge\text{-ass}, \vee\text{-ass}, \text{tr}, \text{ltr}\}$.

Example 3.9. *Let* $(X = \{0, a, b, c, d, 1\}, \trianglelefteq)$ *be a bounded trellis given by the Hasse diagram in Figure 3.9.*

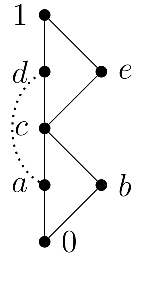


Figure 3.9: The Hasse diagram of the trellis $(X = \{0, a, b, c, d, e, 1\}, \trianglelefteq)$.

One easily verifies that

$X^{\mathrm{rtr}} = \{0, b, c, d, e, 1\}$;
$X^{\mathrm{ltr}} = \{0, a, b, c, e, 1\}$;
$X^{\mathrm{tr}} = \{0, b, e, 1\}$;
$X^{\wedge\text{-ass}} = \{0, b, 1\}$;
$X^{\vee\text{-ass}} = \{0, e, 1\}$;
$X^{\mathrm{dis}} = X^{\mathrm{ass}} = \{0, 1\}$.

The above subsets are subtrellises of X. *Then the binary operations* S_α, *where* $\alpha \in \{\mathrm{dis}, \mathrm{ass}, \wedge\text{-ass}, \vee\text{-ass}, \mathrm{tr}, \mathrm{ltr}\}$ *defined in Propositions 3.20 and 3.21 are are elements of* $\mathcal{AO}_0(X)$.

Before providing the operations S_α, *we present the functions* γ^α, *for any* $\alpha \in \{\mathrm{dis}, \mathrm{ass}, \wedge\text{-ass}, \vee\text{-ass}, \mathrm{tr}, \mathrm{ltr}\}$.

γ^α	0	a	b	c	d	e	1
$\gamma^{\mathrm{dis}}(x)$	0	1	1	1	1	1	1
$\gamma^{\mathrm{ass}}(x)$	0	1	1	1	1	1	1
$\gamma^{\wedge\text{-ass}}(x)$	0	1	b	1	1	1	1
$\gamma^{\vee\text{-ass}}(x)$	0	e	e	e	1	e	1
$\gamma^{\mathrm{tr}}(x)$	0	e	b	e	1	e	1
$\gamma^{\mathrm{ltr}}(x)$	0	a	b	c	1	e	1

It is clear that $S_{\mathrm{ass}} = S_{\mathrm{dis}} = S_{\mathrm{D}}$.

$S_{\wedge\text{-ass}}$	a	b	c	d	e
a	1	1	1	1	1
b	1	b	1	1	1
c	1	1	1	1	1
d	1	1	1	1	1
e	1	1	1	1	1

$S_{\vee\text{-ass}}$	a	b	c	d	e
a	e	e	e	1	e
b	e	e	e	1	e
c	e	e	e	1	e
d	1	1	1	1	1
e	e	e	e	1	e

S_{tr}	a	b	c	d	e
a	e	e	e	1	e
b	e	b	e	1	e
c	e	e	e	1	e
d	1	1	1	1	1
e	e	e	e	1	e

S_{ltr}	a	b	c	d	e
a	a	b	c	1	e
b	c	b	c	1	e
c	c	c	c	1	e
d	1	1	1	1	1
e	e	e	e	1	e

In this example, $\mathcal{AO}_0(X)$ has a least element and it is given by:

S	a	b	c	d	e
a	a	c	c	1	e
b	c	b	c	d	e
c	c	c	c	1	e
d	1	d	1	1	1
e	e	e	e	1	e

3.3.3. Elements of $\mathcal{AO}_1(X)$ based on specific subsets on bounded trellises

In this subsection, we give some examples of elements of $\mathcal{AO}_1(X)$ on a bounded trellis $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ based on its specific subsets of right-transitive elements.

In view of Theorem 3.10 and Proposition 3.19, we derive the following propositions that define elements of $\mathcal{AO}_1(X)$.

Proposition 3.23. *Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis and $\alpha \in \{\text{ass}, \wedge\text{-ass}, \vee\text{-ass}, \text{tr}, \text{rtr}\}$. If $(X^\alpha, \trianglelefteq, \wedge, \vee)$ is finite subtrellis, then the binary operation T_α defined by*

$$T_\alpha(x, y) = \begin{cases} x \wedge y & \text{if } x, y \in X^\alpha \text{ or } x = 1 \text{ or } y = 1; \\ \lambda^\alpha(x) \wedge \lambda^\alpha(y) & \text{otherwise;} \end{cases}$$

is an element of $\mathcal{AO}_1(X)$.

The fact that T_{dis} is a sublattice of any trellis leads to the following particular case.

Proposition 3.24. *Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis. If X^{dis} is finite, then the binary operation T_{dis} defined by*

$$T_{\text{dis}}(x, y) = \begin{cases} x \wedge y & \text{if } x, y \in X^{\text{dis}} \text{ or } x = 1 \text{ or } y = 1; \\ \lambda^{\text{dis}}(x) \wedge \lambda^{\text{dis}}(y) & \text{otherwise;} \end{cases}$$

is an element of $\mathcal{AO}_1(X)$.

Under the same condition of Propositions 3.23 and 3.24, we obtain the following result. The proof is achieved by Propositions 3.19.

Proposition 3.25. *Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis and $\alpha \in \{\text{dis}, \text{ass}, \wedge\text{-ass}, \vee\text{-ass}, \text{tr}, \text{rtr}\}$. It holds that*

$$T_\alpha(x, y) \in X^\alpha, \text{ for any } x, y \in X\,.$$

Remark 3.14. *(**coincidence**) If $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ is a bounded lattice, then T_α coincides with the meet $(\wedge)$, for any $\alpha \in \{\text{dis}, \text{ass}, \wedge\text{-ass}, \vee\text{-ass}, \text{tr}, \text{rtr}\}$.*

Example 3.10. *Let $(X = \{0, a, b, c, d, 1\}, \trianglelefteq)$ be a bounded trellis given in Example 3.9. The binary operations T_α, where $\alpha \in \{\text{dis}, \text{ass}, \wedge\text{-ass}, \vee\text{-ass}, \text{tr}, \text{rtr}\}$ defined in Propositions 3.23 and 3.24 are elements of $\mathcal{AO}_1(X)$. Before providing the operations T_α, we present the functions λ^α, for any $\alpha \in \{\text{dis}, \text{ass}, \wedge\text{-ass}, \vee\text{-ass}, \text{tr}, \text{rtr}\}$.*

λ^α	0	a	b	c	d	e	1
$\lambda^{\text{dis}}(x)$	0	0	0	0	0	0	1
$\lambda^{\text{ass}}(x)$	0	0	0	0	0	0	1
$\lambda^{\wedge\text{-ass}}(x)$	0	0	b	b	b	b	1
$\lambda^{\vee\text{-ass}}(x)$	0	0	0	0	0	e	1
$\lambda^{\text{tr}}(x)$	0	0	0	0	d	e	1
$\lambda^{\text{rtr}}(x)$	0	0	b	c	d	d	1

It is clear that $T_{\text{ass}} = T_{\text{dis}} = T_{\text{D}}$.

$T_{\wedge\text{-ass}}$	a	b	c	d	e
a	0	0	0	0	0
b	0	b	b	b	b
c	0	b	b	b	b
d	0	b	b	b	b
e	0	b	b	b	b

$T_{\vee\text{-ass}}$	a	b	c	d	e
a	0	0	0	0	0
b	0	0	0	0	0
c	0	0	0	0	0
d	0	0	0	0	0
e	0	0	0	0	e

T_{tr}	a	b	c	d	e
a	0	0	0	0	0
b	0	b	b	b	b
c	0	b	b	b	b
d	0	b	b	b	b
e	0	b	b	b	e

T_{rtr}	a	b	c	d	e
a	0	0	0	0	0
b	0	b	b	b	b
c	0	b	c	c	c
d	0	b	c	d	c
e	0	b	c	c	e

In this example, $\mathcal{AO}_1(X)$ has a greatest element and it is given by:

T	a	b	c	d	e
a	0	0	0	0	a
b	0	b	b	b	b
c	0	b	c	c	c
d	0	b	c	d	c
e	a	b	c	c	e

4 Class of weakly associative operations on trellises

In the previous chapter, we have studied two classes of associative operations (associative aggregation operations) on a bounded trellis $(X, \trianglelefteq, \wedge, \vee, 0, 1)$, the class $\mathcal{AO}_1(X)$ and the class $\mathcal{AO}_0(X)$. As the meet and the join operations of a given trellis are not in these classes, we aim in this chapter to study a more general classes that contain them.

4.1. Definitions, examples and basic properties

In this section, we study two classes of weakly associative operations and investigate its various properties. A class of weakly associative operations with neutral element 1 and a class of weakly associative operations with neutral element 0. Several elements of these classes are given, and others are constructed. These classes are extensions of the classes $\mathcal{AO}_1(X)$ and $\mathcal{AO}_0(X)$. First, we introduce the definitions of weakly-associative and weakly-increasing operations on a bounded trellis.

4.1.1. Definitions and examples

Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis and $x_1, x_2, \cdots, x_n \in X$. For further use, we recall that if $\{x_1, x_2, \cdots, x_n\} \cap X^{\wedge\text{-ass}} \neq \emptyset$ (resp. $\{x_1, x_2, \cdots, x_n\} \cap X^{\vee\text{-ass}} \neq \emptyset$), then we said that $[x_1, x_2, \cdots, x_n] \in X^{\wedge\text{-ass}}$ (resp. $[x_1, x_2, \cdots, x_n] \in X^{\vee\text{-ass}}$).

Definition 4.1. *Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis and F a binary operation on X.*

(i) F is called weakly-increasing if it satisfies:

$$x \trianglelefteq y \Rightarrow F(x, z) \trianglelefteq F(y, z), \text{ for any } ([x, y] \in X^{\text{tr}} \text{ and } z \in X);$$

(ii) F is weakly-associative if it satisfies:

$$F(x, F(y, z)) = F(F(x, y), z), \text{ for any } ([x, y, z] \in X^{\wedge-ass} \text{ or } [x, y, z] \in X^{\vee-ass}) .$$

Next, we illustrate the previous definition weakly-increasing and weakly-associative operations on a bounded trellis.

Example 4.1. *Let* $(X = \{0, a, b, c, 1\}, \trianglelefteq, \wedge, \vee)$ *be a trellis given by the Hasse diagram in Figure 4.1 and* F, G *two binary operations defined by the following tables:*

$F(x,y)$	0	a	b	c	1
0	a	a	b	c	1
a	b	b	c	c	1
b	b	b	c	c	1
c	c	1	1	1	1
1	1	1	1	1	1

$G(x,y)$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	b	c	1
b	0	a	c	c	1
c	0	0	1	b	c
1	0	a	c	c	1

One easily verifies that F *is weakly-increasing and* G *is weakly-associative.*

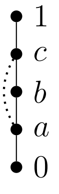


Figure 4.1: Hasse diagram of the trellis $(X = \{0, a, b, c, 1\}, \trianglelefteq)$.

Notation 4.1. *Let* $\mathcal{WAO}_e(X)$ *denotes the class (or the set) of all binary operations on a bounded trellis* $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ *that are commutative, weakly-increasing, weakly-associative and have* e *as a neutral element.*

Remark 4.1. *(i)* $\mathcal{WAO}_1(X)$ *(resp.* $\mathcal{WAO}_0(X)$*) extends the class of all t-norms (resp. the class of t-conorms) on the bounded trellis* X *studied in [56].*

(ii) In general, one can easily observe that $\mathcal{AO}_e(X) \subseteq \mathcal{WAO}_e(X)$*, for any* $e \in \{0, 1\}$.

Next, we give some examples of these classes.

Example 4.2. *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a trellis. It holds that*

(i) $\wedge \in \mathcal{WAO}_1(X)$;

(ii) $\vee \in \mathcal{WAO}_0(X)$;

(iii) The binary operations T_D *(resp.* S_D*) defined in Example 3.2 is an element of* $\mathcal{WAO}_1(X)$ *(resp.* $\mathcal{WAO}_0(X)$*).*

4.1.2. Basic properties of $\mathcal{WAO}_0(X)$ and $\mathcal{WAO}_1(X)$

In this subsection, we investigate some properties of $\mathcal{WAO}_1(X)$ and $\mathcal{WAO}_0(X)$.

The following Proposition shows the duality between the two classes $\mathcal{WAO}_1(X)$ and $\mathcal{WAO}_0(X)$. We recall that for a given bounded trellis $(X, \trianglelefteq, \wedge, \vee, 0, 1)$, its dual bounded trellis is defined as $(X^*, \trianglelefteq^*, \wedge^*, \vee^*, 0^*, 1^*)$, where $X^* = X$, $x \trianglelefteq^* y$ if and only if $y \trianglelefteq x$, $0^* = 1$ and $1^* = 0$.

Proposition 4.1. *Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis and F a binary operation on X. Then the following implications hold:*

(i) If $F \in \mathcal{WAO}_1(X)$, then $F \in \mathcal{WAO}_0(X^*)$;

(ii) If $F \in \mathcal{WAO}_0(X)$, then $F \in \mathcal{WAO}_1(X^*)$.

Proof. The proof is straightforward. □

Proposition 4.2. *Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis. The following implications hold:*

(i) Any element of $\mathcal{WAO}_1(X)$ is conjunctive;

(ii) Any element of $\mathcal{WAO}_0(X)$ is disjunctive.

Proof. (i) Let $T \in \mathcal{WAO}_1(X)$ and $x, y \in X$. Since $1 \in X^{tr}$ and T is weakly-increasing and commutative, it follows that $T(x, y) \trianglelefteq T(1, y)$ and $T(x, y) \trianglelefteq T(x, 1)$. The fact that 1 is the neutral element of T implies that $T(x, y) \trianglelefteq y$ and $T(x, y) \trianglelefteq x$. Thus, $T(x, y) \trianglelefteq x \wedge y$. Therefore, T is conjunctive.

(ii) The proof is dual to that of (i).

□

Proposition 4.3. *Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis and F a binary operation on X. Then the following implications hold:*

(i) If $F \in \mathcal{WAO}_1(X)$, then $F(x, 0) = 0$, for any $x \in X$;

(ii) If $F \in \mathcal{WAO}_0(X)$, then $F(x, 1) = 1$, for any $x \in X$.

Proof.

(i) Suppose that $F \in \mathcal{WAO}_1(X)$ and $x \in X$. Since $1 \in X^{tr}$, $x \trianglelefteq 1$ and F is weakly-increasing, it follows that $F(x, 0) \trianglelefteq F(1, 0) = 0$. Thus, $F(x, 0) = 0$, for any $x \in X$.

(ii) The proof is dual to that of (i).

□

Remark 4.2. *If the cardinal of X is greater than 1 (i.e., $|X| > 1$), then*

$$\mathcal{WAO}_0(X) \cap \mathcal{WAO}_1(X) = \emptyset.$$

4.1.3. Psoset structures of $\mathcal{WAO}_0(X)$ and $\mathcal{WAO}_1(X)$

In this subsection, we discuss the bounded psoset structures of $\mathcal{WAO}_1(X)$ and $\mathcal{WAO}_0(X)$.

For any $F_1, F_2 \in \mathcal{WAO}_e(X)$, we define:

$$F_1 \trianglelefteq_{\mathcal{WAO}} F_2 \text{ if and only if } F_1(x, y) \trianglelefteq F_2(x, y), \text{ for any } x, y \in X.$$

The following result is a natural generalization to that of triangular norms in the trellis setting[56].

Proposition 4.4. *Let $(X, \trianglelefteq_X, \wedge_X, \vee_X, 0, 1)$ be a bounded trellis. Then it holds that:*

(i) $T_D \trianglelefteq_{\mathcal{WAO}} F \trianglelefteq_{\mathcal{WAO}} \wedge$, for any $F \in \mathcal{WAO}_1(X)$;

(ii) $\vee \trianglelefteq_{\mathcal{WAO}} F \trianglelefteq_{\mathcal{WAO}} S_D$, for any $F \in \mathcal{WAO}_0(X)$.

Proof. (i) On the one hand, Proposition 4.2 guarantees that $F \trianglelefteq_{\mathcal{WAO}} \wedge$, for any $F \in \mathcal{WAO}_1(X)$. On the other hand, $T_D(x, y) = 0 \trianglelefteq T(x, y)$, for any $(x, y) \in (X \setminus \{1\})^2$. If $x = 1$ (resp. $y = 1$), it holds that $T_D(1, y) = y = F(1, y)$ (resp. $T_D(x, 1) = x = F(x, 1)$). Hence, $T_D(x, y) \trianglelefteq F(x, y)$, for any $x, y \in X$. Thus, $T_D \trianglelefteq_{\mathcal{WAO}} F \trianglelefteq_{\mathcal{WAO}} \wedge$, for any $F \in \mathcal{WAO}_1(X)$.

(ii) The proof is dual to that of (i).

□

In a bounded Trellis $(X, \trianglelefteq_X, \wedge_X, \vee_X, 0, 1)$, the structures $(\mathcal{WAO}_1(X), \trianglelefteq_{\mathcal{WAO}}, T_D, \wedge)$ and $(\mathcal{WAO}_0(X), \trianglelefteq_{\mathcal{WAO}}, \vee, S_D)$ are bounded psosets.

Remark 4.3. *The bounded psosets* $(\mathcal{WAO}_1(X), \trianglelefteq_{\mathcal{WAO}}, T_D, \wedge)$ *and* $(\mathcal{WAO}_0(X), \trianglelefteq_{\mathcal{WAO}}, \vee, S_D)$ *are not necessary bounded trellises, since the meet (resp. the join) of any two elements is not necessary an element of* $\mathcal{WAO}_1(X)$ *or* $\mathcal{WAO}_0(X)$*.*

The following proposition shows a case when an element of $\mathcal{WAO}_1(X)$ (resp. an element of $\mathcal{WAO}_0(X)$) coincides with the meet (resp. the join) operation. It is particular case of the weaker types of increasing binary operations on a bounded trellis that coincide with the meet (resp. the join) operation [55].

Proposition 4.5. *Let* $(X, \trianglelefteq, \wedge, \vee)$ *be a bounded trellis and* F *a binary operation on* X*. The following statements hold:*

(i) If $F \in \mathcal{WAO}_1(X)$*, idempotent and satisfying* $F(x \wedge y, x \wedge y) \trianglelefteq F(x, y)$*, for any* $x, y \in X$*, then* F *is the meet* $(\wedge)$ *operation of* X*;*

(ii) If $F \in \mathcal{WAO}_0(X)$*, idempotent and satisfying* $F(x, y) \trianglelefteq F(x \vee y, x \vee y)$*, for any* $x, y \in X$*, then* F *is the join* $(\vee)$ *operation of* X*.*

Proof. (i) On the one hand, since $F \in \mathcal{WAO}_1(X)$ which means that F is conjunctive, it holds that $F(x, y) \trianglelefteq x \wedge y$, for any $x, y \in X$. On the other hand, the fact that F is idempotent and satisfying $F(x \wedge y, x \wedge y) \trianglelefteq F(x, y)$, for any $x, y \in X$ implies that $x \wedge y = F(x \wedge y, x \wedge y) \trianglelefteq F(x, y)$. Thus, F is the meet operation $(\wedge)$ of X.

(ii) The proof is dual to that of (i).

□

Remark 4.4. *The converse of the above Proposition 4.5 is immediate.*

4.2. Constructions of some elements of $\mathcal{WAO}_0(X)$ and $\mathcal{WAO}_1(X)$

In this section, we construct some elements of $\mathcal{WAO}_1(X)$ and $\mathcal{WAO}_0(X)$. For the increasingness and associativity properties, we use similar techniques as in [35].

Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis and $e \in X$. Let T_e and S_e two binary operations on X defined as follows:

$$T_e(x,y) = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1; \\ (x \wedge y) \wedge e & \text{otherwise;} \end{cases}$$

and

$$S_e(x,y) = \begin{cases} x \vee y & \text{if } x = 0 \text{ or } y = 0; \\ (x \vee y) \vee e & \text{otherwise.} \end{cases}$$

Remark 4.5. [35] *In general, T_e (resp. S_e) is not necessarily an element of $\mathcal{WAO}_1(X)$ (resp. $\mathcal{WAO}_0(X)$). Indeed, let $(X = \{0, a, b, c, d, e, f, 1\}, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis given by the Hasse diagram in Figure 4.2.*

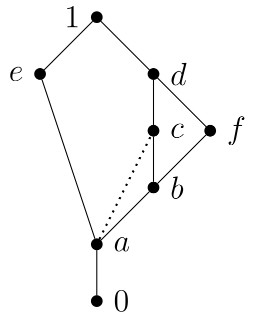


Figure 4.2: Hasse diagram of the trellis $(X = \{0, a, b, c, d, e, f, 1\}, \trianglelefteq)$.

Sitting $x = f$ and $y = d$, then $x \trianglelefteq y$ and $(x, y) \in (X^{tr})^2$. Since $T_e(f, c) = (f \wedge c) \wedge e = a \not\trianglelefteq T_e(d, c) = (d \wedge c) \wedge e = 0$, it follows that T_e is not weakly-increasing. Therefore, $T_e \notin \mathcal{WAO}_1(X)$.

In view of remark 4.5, we give sufficient conditions under which the binary operation T_e is an element of $\mathcal{WAO}_1(X)$.

Proposition 4.6. [35] *Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis. The following implications hold:*

(i) If $e \in X^{\wedge-ass}$, then $T_e \in \mathcal{WAO}_1(X)$;

(ii) If $e \in X^{\vee-ass}$, then $S_e \in \mathcal{WAO}_0(X)$.

Proof. We only give the proof of (i), as the proof of (ii) is similar. One easily verifies that T_e is commutative and satisfies the boundary condition. Now, let $(x, y) \in X \times X^{tr}$ such that $x \trianglelefteq y$ and $z \in X$. Then we discuss the following two possible cases:

(i) If $z = 1$, then $T_e(x, z) = x \trianglelefteq y = T_e(y, z)$.

(ii) If $z \neq 1$, then we have three possible cases:

(i) If $x = 1$, then $y = 1$ and $T_e(x, z) = z \trianglelefteq z = T_e(y, z)$.

(ii) If $x \neq 1$ and $y = 1$, then the fact that $e \in X^{\wedge-ass}$ implies that $T_e(x, z) = (x \wedge z) \wedge e = (x \wedge e) \wedge z \trianglelefteq z = T_e(y, z)$. Thus, $T_e(x, z) \trianglelefteq T_e(y, z)$.

(iii) If $y \neq 1$, then $T_e(x, z) = (x \wedge z) \wedge e$ and $T_e(y, z) = (y \wedge z) \wedge e$. Since $x \trianglelefteq y$ and $y \in X^{tr}$, it follows that $x \wedge z \trianglelefteq y \wedge z$. The fact that $e \in X^{\wedge-ass}$ implies $(x \wedge z) \wedge e \trianglelefteq (y \wedge z) \wedge e$. Thus, $T_e(x, z) \trianglelefteq T_e(y, z)$.

Therefore, T_e is weakly-increasing.

Now, we prove that T_e is weakly-associative. Let $x, y, z \in X$ such that $[x, y, z] \in X^{\wedge-ass}$. Since $e \in X^{\wedge-ass}$, it holds that

$$
\begin{aligned}
T_e(x, T_e(y, z)) &= (x \wedge ((y \wedge z) \wedge e)) \wedge e \\
&= ((x \wedge (y \wedge z)) \wedge e) \wedge e \\
&= (((x \wedge y) \wedge z) \wedge e) \wedge e \\
&= (((x \wedge y) \wedge e) \wedge z) \wedge e \\
&= (T_e(x, y) \wedge z) \wedge e \\
&= T_e(T_e(x, y), z)\,.
\end{aligned}
$$

Hence, T_e is weakly-associative. Therefore, $T_e \in \mathcal{WAO}_1(X)$. □

Remark 4.6. *Particular cases: since $0, 1 \in X^{\wedge-ass}$, we recognize that*

(i) $T_0 = T_D$ *and* $T_1 = \wedge$;

(ii) $S_0 = \vee$ *and* $S_1 = S_D$.

The following result is slight modification of Propositions 4.6. The proof follows the same method.

Proposition 4.7. *Let $\mathbb{X} = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded modular trellis and the binary operations Z and Z^* defined as follows:*

$$
Z(x, y) = \begin{cases} x \wedge y & \text{if } x \vee y = 1; \\ 0 & \text{otherwise;} \end{cases} \quad \text{and } Z^*(x, y) = \begin{cases} x \vee y & \text{if } x \wedge y = 0; \\ 1 & \text{otherwise;} \end{cases}
$$

Then, $Z \in \mathcal{WAO}_1(X)$ and $Z^* \in \mathcal{WAO}_0(X)$.

In the following result, we propose a new ordinal sum construction of $\mathcal{WAO}_1(X)$ and $\mathcal{WAO}_0(X)$ on bounded trellises according to [16]. We start by the following immediate proposition.

Proposition 4.8. [35] *Let $(X, \trianglelefteq, \wedge, \vee)$ be a trellis and $a, b \in X^{ass}$ such that $a \trianglelefteq b$. The following subintervals of X defined as:*

$$[a, b] = \{x \in X | a \trianglelefteq x \trianglelefteq b\},$$

$$(a, b] = \{x \in X | a \triangleleft x \trianglelefteq b\},$$

$$[a, b) = \{x \in X | a \trianglelefteq x \triangleleft b\},$$

$$(a, b) = \{x \in X | a \triangleleft x \triangleleft b\},$$

are subtrellises of X.

Theorem 4.1. [35] *Let $(X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis and $a \in X^{ass} \backslash \{0, 1\}$. If $V : [a, 1]^2 \to [a, 1]$ an element of $\mathcal{WAO}_1([a, 1])$ and $W : [0, a]^2 \to [0, a]$ an element of $\mathcal{WAO}_0([0, a])$, then the binary operations $T : X^2 \to X$ and $S : X^2 \to X$ defined as follows:*

$$T(x, y) = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1; \\ V(x, y) & \text{if } x, y \in [a, 1); \\ x \wedge y \wedge a & \text{otherwise;} \end{cases}$$

and

$$S(x, y) = \begin{cases} x \vee y & \text{if } x = 0 \text{ or } y = 0; \\ W(x, y) & \text{if } x, y \in (0, a]; \\ x \vee y \vee a & \text{otherwise;} \end{cases}$$

are elements of $\mathcal{WAO}_1(X)$ and $\mathcal{WAO}_0(X)$, respectively.

Proof. The proof is similar to that of S. One easily verifies that T is commutative and satisfies the boundary conditions. Now, let $x, y \in X \times X^{tr}$ such that $x \trianglelefteq y$. Then we discuss the following possible cases:

(i) If $x = 1$ or $z = 1$, then $T(x, z) = T(y, z)$.

(ii) If $x, z \in [a, 1)$, then, also $a \trianglelefteq y$ and $T(x, z) = V(x, z) \trianglelefteq V(y, z) = T(y, z)$.

(iii) If $x \notin [a, 1)$ and $z \in [a, 1)$, it holds that $T(x, z) = x \wedge a$ and we have three possible cases:

  (i) If $y = 1$, then $T(y, z) = z$. Since $a \in X^{ass}$, then $T(x, z) = x \wedge a \trianglelefteq z = T(y, z)$.

  (ii) If $y \in [a, 1)$, then $T(y, z) = V(y, z) \in [a, 1)$. Since, $a \in X^{ass}$, it follows that $T(x, z) = x \wedge a \trianglelefteq V(y, z) = T(y, z)$.

(iii) If $y \notin [a, 1]$, then $T(y, z) = y \wedge a$. Since $a \in X^{ass}$, then it follows that $T(x, z) = x \wedge a \trianglelefteq y \wedge a = T(y, z)$.

(iv) If $x \notin [a, 1]$ and $z \notin [a, 1]$, then $T(x, z) = x \wedge z \wedge a$ and $T(y, z) = y \wedge z \wedge a$. Thus, Proposition 4.5 guarantees that $T(x, z) = x \wedge z \wedge a \trianglelefteq y \wedge z \wedge a = T(y, z)$.

Hence, T is weakly-increasing. Next, we prove that T is weakly-associative. Let $x, y, z \in X$ such that $[x, y, z] \in X^\wedge$. The proof is split into all possible cases.

(i) If $x, y \in [a, 1)$, then we have two cases:

(a) If $z \in [a, 1)$, then:

$$\begin{aligned} T(x, T(y, z)) &= T(x, V(y, z)) \\ &= V(x, V(y, z)) \\ &= V(V(x, y), z) \\ &= T(V(x, y), z) \\ &= T(T(x, y), z). \end{aligned}$$

(b) If $z \in X \setminus [a, 1)$, then:

$$\begin{aligned} T(x, T(y, z)) &= T(x, y \wedge z \wedge a) \\ &= x \wedge (y \wedge z \wedge a) \wedge a \\ &= z \wedge a \quad (\text{car, } a \in X^{ass}) \\ &= V(x, y) \wedge z \wedge a \\ &= T(V(x, y), z) \\ &= T(T(x, y), z). \end{aligned}$$

(ii) If $x \in [a, 1)$ and $y \in X \setminus [a, 1)$, then we have two cases:

(a) If $z \in [a, 1)$, then this case have been studied in $(i.b)$.

(b) If $z \in X \setminus [a, 1)$, then:

$$\begin{aligned} T(x, T(y, z)) &= T(x, y \wedge z \wedge a) \\ &= x \wedge (y \wedge z \wedge a) \wedge a \\ &= (x \wedge y \wedge a) \wedge z \wedge a \quad (\text{car, } a \in X^{ass}) \\ &= T(x \wedge y \wedge a, z) \\ &= T(T(x, y), z). \end{aligned}$$

(iii) If $x, y \in X \setminus [a, 1)$, then we have two cases:

(a) If $z \in [a, 1)$, then this case have been studied in $(ii.b)$.

(b) If $z \in X \setminus [a, 1)$, then:

$$\begin{aligned} T(x, T(y,z)) &= T(x, y \wedge z \wedge a) \\ &= x \wedge y \wedge z \wedge a \\ &= T(x \wedge y \wedge a, z) \\ &= T(T(x,y), z). \end{aligned}$$

Hence, T is weakly-associative on X. Therefore, $T \in \mathcal{WAO}_1(X)$. □

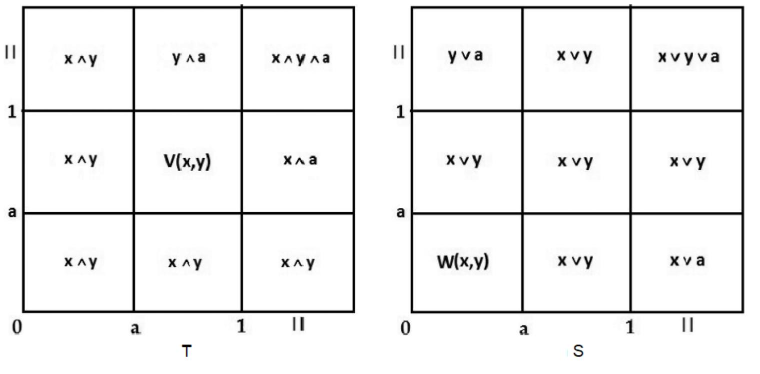
One easily Observes that T and S on a bounded trellis considered in Theorem 4.1 can be described as follows:

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1)^2; \\ y \wedge a & \text{if } x \in [a,1), y \| a; \\ x \wedge a & \text{if } y \in [a,1), x \| a; \\ x \wedge y \wedge a & \text{if } x \| a, y \| a; \\ x \wedge y & \text{otherwise;} \end{cases}$$

and

$$S(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in (0,a]^2; \\ y \vee a & \text{if } x \in (0,a], y \| a; \\ x \vee a & \text{if } y \in (0,a], x \| a; \\ x \vee y \vee a & \text{if } x \| a, y \| a; \\ x \vee y & \text{otherwise.} \end{cases}$$

Thus, we get T and S by the next figures:



4.3. Relationship among $\mathcal{WAO}_e(X)$ and isomorphisms on bounded trellises

In this section, we conjugate elements of $\mathcal{WAO}_1(X)$ (resp. elements of $\mathcal{WAO}_0(X)$) and an isomorphism map on a bounded trellis X. First, we start by the following proposition.

Proposition 4.9. *Let* $(X_1, \trianglelefteq_1, \wedge_1, \vee_1)$, $(X_2, \trianglelefteq_2, \wedge_2, \vee_2)$ *be two trellises and* $\rho : X_1 \longrightarrow X_2$ *an isomorphism. Then* $\rho(X_1^{tr}) \subseteq X_2^{tr}$ *.*

Proof. Let $x, y \in X_2$ and $z \in \rho(X_1^{tr})$ such that $x \trianglelefteq_2 y \trianglelefteq_2 z$. Then there exist $x', y' \in X_1$ and $z' \in X_1^{tr}$ such that $\rho(x') \trianglelefteq_2 \rho(y') \trianglelefteq_2 \rho(z')$. From the increasingness of ρ^{-1}, it holds that $x' \trianglelefteq_1 y' \trianglelefteq_1 z'$. The fact that $z' \in X_1^{tr}$ implies that $x' \trianglelefteq_1 z'$, i.e., $x' \wedge_1 z' = x'$. Since ρ is homomorphism; it follows that $\rho(x') \wedge_2 \rho(z') = \rho(x' \wedge_1 z') = \rho(x')$. Hence, $\rho(x') \trianglelefteq_2 \rho(z')$, i.e., $x \trianglelefteq_2 z$. Thus, $z \in X^{tr}$. Therefore, $\rho(X_1^{tr}) \subseteq X_2^{tr}$. □

Proposition 4.10. *Let* $(X_1, \trianglelefteq_1, \wedge_1, \vee_1)$, $(X_2, \trianglelefteq_2, \wedge_2, \vee_2)$ *be two trellises and* $\rho : X_1 \longrightarrow X_2$ *an isomorphism. Then* $[x, y, z] \in X_1^{\wedge_1}$ *(resp.* $[x, y, z] \in X_1^{\vee_1}$*) if and only if* $[\rho(x), \rho(y), \rho(z)] \in X_2^{\wedge_2}$ *(resp.* $[\rho(x), \rho(y), \rho(z)] \in X_2^{\vee_2}$*).*

Proof. Let $x, y, z \in X_1$ such that $[x, y, z] \in X_1^{\wedge_1}$. Since ρ is an isomorphism, then

$$\begin{aligned}
\rho(x) \wedge_2 (\rho(y) \wedge_2 \rho(z)) &= \rho(x) \wedge_2 \rho(y \wedge_1 z)) \\
&= \rho(x \wedge_1 (y \wedge_1 z)) \\
&= \rho((x \wedge_1 y) \wedge_1 z) \\
&= \rho(x \wedge_1 y) \wedge_2 \rho(z)) \\
&= (\rho(x) \wedge_2 \rho(y)) \wedge_2 \rho(z)\,.
\end{aligned}$$

Therefore, $[\rho(x), \rho(y), \rho(z)] \in X_2^{\wedge_2}$. In a similar way, we prove that $[x, y, z] \in X_1^{\vee_1}$ if and only if $[\rho(x), \rho(y), \rho(z)] \in X_2^{\vee_2}$. □

Proposition 4.11. $(X_1, \trianglelefteq_1, \wedge_1, \vee_1, 0_1, 1_1)$, $(X_2, \trianglelefteq_2, \wedge_2, \vee_2, 0_2, 1_2)$ *be two bounded trellises,* $T \in \mathcal{WAO}_1(X_2)$ *and* $\rho : X_1 \longrightarrow X_2$ *an isomorphism. Then the binary operation* T^ρ *defined by:*

$$T^\rho(x, y) = \rho^{-1}(T(\rho(x), \rho(y))), \textit{ for any } x, y \in X_1\,,$$

is an element of $\mathcal{WAO}_1(X_1)$*.*

Proof. One easily verifies that T^ρ is commutative and satisfies the boundary condition. Now, let $(x,y) \in X_1 \times X_1^{tr}$ such that $x \trianglelefteq_1 y$. Proposition 4.9 assures that $\rho(y) \in X_2^{tr}$. Since T is weakly-increasing, it holds that $T(\rho(x),\rho(z)) \trianglelefteq_2 T(\rho(y),\rho(z))$, for any $z \in X_1$. The fact that ρ^{-1} is increasing on X_2 implies that $\rho^{-1}(T(\rho(x),\rho(z))) \trianglelefteq_1 \rho^{-1}(T(\rho(y),\rho(z)))$, for any $z \in X_1$, i.e., $T^\rho(x,z) \trianglelefteq_1 T^\rho(y,z)$, for any $z \in X_1$. Hence, T^ρ is weakly-increasing on X_1. Next, we prove that T^ρ is weakly-associative. Let $x,y,z \in X_1$ such that $[x,y,z] \in X_1^{\wedge_1}$. Proposition 4.10 assures that $(\rho(x),\rho(y),\rho(z)) \in X_2^{\wedge_2}$. Thus

$$\begin{aligned}
T^\rho(T^\rho(x,y),z) &= \rho^{-1}(T(\rho(T^\rho(x,y)),\rho(z))) \\
&= \rho^{-1}(T(\rho(\rho^{-1}(T(\rho(x),\rho(y)))),\rho(z))) \\
&= \rho^{-1}(T(T(\rho(x),\rho(y)),\rho(z))) \\
&= \rho^{-1}(T(\rho(x),T(\rho(y),\rho(z)))) \\
&= \rho^{-1}(T(\rho(x),\rho(\rho^{-1}(T(\rho(y),\rho(z)))))) \\
&= \rho^{-1}(T(\rho(x),\rho(T^\rho(y,z)))) \\
&= T^\rho(x,T^\rho(y,z))\,.
\end{aligned}$$

Hence, T^ρ is weakly-associative on X_1. Therefore, $T^\rho \in \mathcal{WAO}_1(X_1)$. □

Notice that in a bounded trellis $(X,\trianglelefteq,\wedge,\vee,0,1)$, the identity map Id_X of X (i.e., $Id_X(x) = x$, for any $x \in X$) is an isomorphism (automorphism). Then $T^{Id_X} = T$, for any $T \in \mathcal{WAO}_1(X)$.

Dually, we have the following result for the elements of $\mathcal{WAO}_0(X)$.

Proposition 4.12. $(X_1,\trianglelefteq_1,\wedge_1,\vee_1,0_1,1_1)$, $(X_2,\trianglelefteq_2,\wedge_2,\vee_2,0_2,1_2)$ *be two bounded trellises,* $S \in \mathcal{WAO}_0(X_2)$ *and* $\rho : X_1 \longrightarrow X_2$ *an isomorphism. Then the binary operation* S^ρ *defined by:*

$$S^\rho(x,y) = \rho^{-1}(S(\rho(x),\rho(y))),\ \textit{for any } x,y \in X_1\,,$$

is an element of $\mathcal{WAO}_0(X_1)$.

General conclusions and future research

In this thesis, we have studied specific algebraic operations on lattices and trellises. First, we have generalized the notion of aggregation operators to f-aggregation operators on a bounded lattice and investigated their properties. This generalization is based on an arbitrary function f on that lattice. To that end, a lot of preparatory work was required. In particular, several properties of binary operations in terms of a given on a lattice have been investigated. We have ended this part by finding the smallest and greatest f-aggregation operators on a bounded lattice.

In the second part, we have studied particular classes of associative operations on trellises in chapter 3. More precisely, on a given trellis $(X, \trianglelefteq, \wedge, \vee, 0, 1)$, we have studied $\mathcal{AO}_1(X)$ (resp. $\mathcal{AO}_0(X)$) the class of associative, commutative, increasing, and have 1 (resp. 0) as a neutral element. These classes generalize the classes of t-norms and t-conorms on bounded lattices. In chapter 4, we have extended the same classes on bounded trellises by considering a weakest associativity property.

Future efforts will be directed to the study other important classes of associative operations on bounded trellis. We anticipate that it will be interesting to study other classes of algebraic operations on trellises with suitable weaker types of associativity.

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