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Par: Soumia BELARBI

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FACULTY OF MATHEMATICS

THESIS

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By:Soumia BELARBI

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Contribution To Fractional Differential Equations

Defended, on **17/06/2023** , in front of the evaluating committee composed of:

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Publications

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Abstract

Our main aim in this thesis is to contribute to the theory of fractional differential equations through the study of some fractional differential problems. Our contribution concerns two different problems.

First, we are interested in the investigation of a sequential pantograph fractional differential problem involving the $\Phi-$Caputo fractional derivative, where a new existence and uniqueness criteria of solutions is discussed. The result is based on some standard fixed point theorems combined with the technique of measures of noncompactness.

Secondly, we establish new existence results for a multi-point boundary value problem of nonlinear differential equations using Caputo fractional derivatives with a $\phi-$Laplacian operator. Then, we prove another result for the existence of three positive solutions of the considered problem using Leggett's-Williams fixed point theorem.

Furthermore, examples are presented to illustrate the application of our main results every chapter.

Key words: Caputo and $\Phi-$Caputo derivatives, Pantograph equation, sequential differential equation, fixed point, Leggett's-Williams, Measure of noncompactness, Darbo.

MSC2020-Mathematics Subject Classification: 26A33; 34A08; 34A34; 34B10; 47H08; 47H09; 47H10.

Résumé

Notre objectif principal dans cette thèse est de contribuer à la théorie des équations différentielles fractionnaires à travers l'étude de certains problèmes différentiels fractionnaires. Notre contribution concerne deux problèmes différents.

Tout d'abord, nous nous intéressons à l'étude d'un problème différentiel fractionnaire séquentiel de pantographe impliquant la dérivée fractionnaire Φ-Caputo, où un nouveau critère d'existence et d'unicité des solutions est discuté. Le résultat est basé sur des théorèmes standards de point fixe combinés à la technique des mesures de non-compacité.

Deuxièmement, nous établissons de nouveaux résultats d'existence pour un problème aux limites multipoints d'équations différentielles non linéaires en utilisant les dérivées fractionnaires de Caputo avec un opérateur p- laplacien. Ensuite, nous prouvons un autre résultat pour l'existence de trois solutions positives du problème considéré en utilisant le théorème du point fixe de Leggett's-Williams.

De plus, des exemples sont présentés pour illustrer l'application de nos principaux résultats à chaque chapitre.

Mots clés: Dérivées de Caputo et Φ-Caputo, Équation de pantographe, équation différentielle séquentielle, point fixe, Leggett's-Williams, Mesure de non-compacité, Darbo.

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Contents

Chapter 1

Introduction

When we present the derivative notion, we quickly recognize that we can apply the concept of a derivative to the derivative function itself, and there by introduce the second derivative. Then are successive derivatives of integer order. integration, the inverse operation of differentiation, can possibly be considered as a derivative of order "minus one." However, it is important to note that this interpretation does not hold universally and is limited to specific cases where the necessary conditions are met.

$$..., \int_a^t ds \int_a^s f(\tau)\, d\tau, \int_a^t f(\tau)\, d\tau, f(t), \frac{df(t)}{dt}, \frac{d^2 f(t)}{dt^2}, ...$$

But the curious mind cannot be restrained from asking the question, "What if derivatives of successive orders were not limited to an integer value?" At first glance, one can be a little unwilling, thinking how odd the definition is and how difficult it can be to be manipulated. For instance, based on what we recall from elementary school, exponents serve as a concise representation for what is effectively a repeated multiplication of a numerical number. This concept is simple to grasp and modest in its own right. When taking exponents of non-integer value into account, it is obvious that this physical description might be unclear. In fact, practically everyone can confirm that $x^4 = x \times x \times x \times x$, How would one explain the physical meaning of $x^{1.4}$or, moreover, the transcendental exponent $x^{\sqrt{2}}$? One cannot visualize what it might be like to multiply a number or a quantity by itself 1.4 times or $\sqrt{2}$ times, and yet these expressions have a definite value for any value x, verifiable by an infinite series or, more practically, by a calculator.

Now think about the integral and derivative in the same manner. Despite the fact that they are higher complexity notions by nature, it is still rather simple to physically convey their meaning. As soon as one of these operations, integrations, or differentiations is completed, the concept of doing many more comes naturally. given the fulfillment of a few minor constraints. In this path, we'll examine additional operators, sometimes known as fractional differentiation or integration operators or, more commonly, fractional calculus.

Let's go back to the term "fractional calculus". It does not mean calculating fractions. It also does not mean a fraction of any differential, integral or calculus of variations. The beautiful and mysterious appellation "fractional calculus" is just one of those misnomers that are the essence of mathematics. Just like the terms "natural numbers" and "real numbers," which we use very often, think about these names for a moment. The notion of a "natural number" is a natural abstraction, but is the number itself natural? same thing with the notion of a real number. Real numbers reflect real quantities, but this cannot change the fact that they do not exist. Everything is in order in mathematical analysis, and the notion of a real number facilitates it, but if one wants to calculate something, one immediately realizes that there is no place for real numbers in the true world. So fractional calculus is a term for the theory of arbitrary order integrals and derivatives, which unifies and generalizes the concepts of integer order differentiation and n-times repeated integration.

Fractional calculus has been around for more than 325 years. Both in pure mathematics and in scientific applications, the topic has bloomed and reached its ripeness. However, it would be completely incorrect to categorize fractional calculus as a nascent science. In reality, fractional calculus has roots that are almost as old as classical calculus. In recent years, several books ([74], [79] , [87]) on fractional calculus were published, and in all of them, its history is addressed in some way or another. Additionally, a number of papers ([83], [84]) by Ross deal with various aspects of the history of fractional calculus.

While the birth of classical calculus is associated with Leibniz and Newton [50], one of the earliest remarks on the meaning of non-integer derivatives goes back to an exchange of correspondence between Leibniz, Gauss, Newton, and L'Hôpital in September 1695. (see [74]).

Leibniz's 1697 letter to J. Wallis contains the final mention of fractional calculus during his lifetime [78]. In the discussion of Wallis' infinite product for $\frac{\pi}{2}$, Leibniz points out that this outcome might have been obtained using differential calculus.

The discussion of non-integer order derivatives did not end with Leibniz's passing in 1716 [44] . As a generalization of factorials, the Gamma function was initially created in 1783 while working on number progressions, as was noted in a brief line near the conclusion of his publication.

A bit over half a century after Leibniz's death, the work of J.L. Lagrange indirectly contributed to the field of fractional calculus. In 1772, [62] Lagrange developed the law of exponents for differential operators of integer order.

$$\frac{d^n}{dt^n}\frac{d^m}{dt^m} = \frac{d^{n+m}}{dt^{n+m}}, n, m \in \mathbb{N}.$$

Under certain conditions, this result can be transferred to arbitrary choices of $n, m \in \mathbb{C}$ as demonstrated much later in history [94].

Next, numerous researchers have made additional contributions to the development of this field. We can specifically mention the work of, P.S. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823), J. Liouville (1832), B. Riemann (1847), H.L. Greer (1859), H. Holmgren (1865), A.K. Grunwald (1867), A.V. Letnikov (1868), N.Ya. Sonin (1869), H. Laurent (1884), P.A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892), S. Pincherle (1902), G.H. Hardy and J.E. Littlewood (1917), H. Weyl (1919), P. L evy (1923), A. Marchaud (1927), H.T. Davis (1924), A. Zygmund (1935), E.R. Love (1938), A. Erdelyi (1939), H. Kober (1940), D.V. Widder (1941), M. Riesz (1949) and W. Feller (1952).

Fractional calculus has become of increasing significance due to its applications in many fields of science and engineering ([58], [70]). The first application of fractional calculus was given by Abel $[1-2]$ and includes the solution to the tautochrone problem. This problem is the fact of determining the shape of the curve such that the time of a descent that a frictionless point mass needs to slide down the curve under the work of gravity is independent of the starting point.

There are several additional scientific fields where fractional calculus is used, including biophysics, wave theory, polymers, quantum mechanics, continuum mechanics, field theory, Lie theory, group theory, spectroscopy, artificial neural networks, and other scientific areas ([67] , [72] , [81] , [88] , [93]). Despite having a lengthy history, this calculus has attracted more popularity and attention in recent years due to the intriguing outcomes it has produced when applied to model certain real-world problems ([35], [37] , [89] , [73] , [90] , [95]). What makes fractional calculus special is that there are numerous types of fractional operators, so any scientist modeling real-world phenomena can choose the one that fits their purposes the best. Typically, the definition of any classical fractional derivative is given in terms of a specific integral. The first sincere attempt to provide a logical definition for the fractional derivative is credited to Liouville, who between 1832 and 1837 published nine articles on this subject [94]. Independently, Riemann proposed an approach [82] that turned out to be essentially that of Liouville, and it is since it bears the name "Riemann-Liouville approach" [32]. This last model, due to the power-function kernel in the integral transform definition, can be used to describe processes with power-law behavior, but there are numerous other types of behaviors that occur in nature whose formulations involve single-kernel integrals and which are used to investigate, for example, memory effect problems [20]. The fractional derivative in the Riemann-Liouville sense is noteworthy; however, owing to its improper physical conditions, it has several drawbacks when used to simulate particular physical events. There have been other definitions of fractional calculus offered as well, which confuses many newcomers to the field who anticipate a single definition of fractional derivatives similar to the way there is a single definition of the first-order derivative. The greatest contribution made by Caputo [79] was the creation of a fractional derivative idea that was suitable for physical conditions ([26], [43]).

Additionally, a variety of other fractional operators have been developed, for instance, Hadamard, Erdélyi–Kober, Weyl, Grunwald–Letnikov, Katugampola, Φ-Riemann-Liouville, Φ-Caputo, Φ-Hilfer definitions and many other. Generally, most of these definitions are not alike, with the exception of some; those generalize some of the previously introduced definitions.

In the present thesis, in general, we are concerned with the study of problems involving the Caputo and $\Phi-$Caputo derivatives $[8-9]$. The importance of this $\Phi-$theory is in its applications to real world phenomena like population growth and other models ([8] , [15]). Also, the advantage of the $\Phi-$Caputo fractional derivative is its flexibility to combine all fractional derivatives introduced before (like, for instance, the Caputo, Hadamard, Hadamard-Caputo, and Caputo-Katugampola derivatives). The $\Phi-$Caputo operator is also important since it possesses the semigroup property, which is crucial to obtaining the structure of solutions.

The theory of fractional differential equations is an important branch of differential equation theory, which represents a powerful tool in applied mathematics, and the investigation of fractional differential equations has attracted great interest of researchers. This is due to the fact that fractional differential equations are useful in many fields of engineering and science, including fluid dynamics, fractal theory, diffusion in porous media, fractional biological neurons, traffic flow, polymer rheology, neural network modeling, viscoelastic panels in supersonic gas flow, real systems characterized by power laws, electrodynamics of complex medium, sandwich system identification, and nonlinear oscillation of earthquakes. Also, it is important to state that their nonlocal property is suitable to describe memory phenomena such as nonlocal elasticity, propagation in complex medium, biological, electrochemistry, porous media, viscoelasticity, and electromagnetics; see, for instance, $([74], [79], [83-84])$.These can also be used as the most effective tools for characterizing the hereditary properties of diverse materials and processes. The mathematical models of the word process include memory terms, which ensure that the past and its effects on the present and future are guaranteed [64]. According to research, the majority of physical and biological models that use integer-order derivatives are less realistic than those that use fractional-order derivatives. Books may be used to learn more about the theory and its applications $([13], [16], [39])$ and references therein. Different boundary conditions for fractional-order differential equations have recently been studied. The literature on the subject contains findings about the existence and uniqueness of classical, initial value issue, periodic, anti-periodic, nonlocal, multi-point, integral boundary conditions, and integral fractional boundary conditions; for example, see $[87]$.

Various methods are present in the literature for obtaining the necessary and sufficient conditions for the existence and uniqueness of solutions of fractional differential equations, but in my opinion, the fixed point theory is the main core, and many scientists and mathematicians have tried to apply basic knowledge about the fixed point theory. They proved that it is an interdisciplinary subject that has countless applications in several areas of mathematics and other fields, like data mining, wavelet analysis, game theory, mathematical economics, optimization theory, approximation theory, variational inequality, biology, chemistry, engineering, physics, etc.$([12], [27])$.

Poincare, in 1886, was the first to work in this field. In addition, the Poincaré theorem was also rediscovered by P. Bohl in 1904. Then Brouwer in 1910 [28], proved fixed point theorem for the $f(x) = x$. Which was further extended by Kakutani [54]. In 1922, Banach [19] proved that a contraction mapping whose domain is complete possesses a unique fixed point. The fixed point theory as well as " Banach contraction principle " has been studied and generalized in different spaces for example: Nadler (1969) [76] ,Caristi and Ekeland 1974...

There are diverse extensions of the fixed point theorem in the literature, as Schauder (1930),Tychonoff (1935), Krasnoselskii (1964), and Schaefer (1974) $([10], [60])$...

Furthermore, in 1930 Kuratowski [93], created a new orientation of research by introducing the notion of measure of noncompactness, which gives the degree of noncompactness for bounded sets. This concept is a very useful tool in functional analysis, for example, in metric and topological fixed point theory and operator equation theory in Banach spaces. It also plays an important role in the theory of nonlinear analysis, which has been improved fast recently because of its extensive practical applications in many fields such as engineering, economics, optimal control, and optimization. This notion can be used in the study of single-valued and multivalued mappings. The measure of noncompactness combined with some algebraic arguments is beneficial for studying mathematical formulations, largely explaining the existence of solutions to some nonlinear problems under certain circumstances. Darbo's fixed point theorem [41] is an important application of this measure, in which it generalizes Schauder's fixed point theorem and Banach contraction principle. Thereafter, many articles have been published on the MNC and its application [52].

Outlines and Objectives of the Thesis

The purpose of this thesis is to study the existence and uniqueness of solutions and positives solution to various types of fractional order differential equations.

Chapter 2:

This chapter is devoted to providing a comprehensive overview of fractional calculus, including preliminary, background, and relations results without proofs. It consists of a reminder of the main definitions and properties used in the further work.

We organize this chapter as follows: in Section 2.1, we provide definitions of some elements from fractional calculus theory and special functions; we present some necessary lemmas, theorems, and properties. Section 2.2 is devoted to some notations and definitions of the functional spaces and topological theory, used in this thesis. While the last section related to some fixed point theorems, and other required results.

Chapter 3:

This chapter contains six sections. After the introduction section, in Section 3.2, we give the integral representation of our sequential pantograph fractional differential problem [22]:

$$\begin{cases} {}^cD^{\beta,\Phi}\left({}^cD^{\alpha,\Phi}x(t)+g(t,x(t))\right)=f\left(t,x(t),x(\lambda t),{}^cD^{\alpha,\Phi}x(t)\right),t\in[0,1], \\ x(0)=0, \\ x(1)=\int\limits_0^1 h(s,x(s))\,ds, \end{cases} \tag{1.1}$$

where ${}^cD^{\alpha,\Phi},{}^cD^{\beta,\Phi}$ are the $\Phi-$Caputo derivatives, such that $0<\alpha,\beta\leq 1$, $\lambda\in\mathbb{R}^+$, $0<\lambda<1$, $f:[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $g,h:[0,1]\times\mathbb{R}\to\mathbb{R}$, are given continuous functions

In Section 3.3 and 3.4 we examine the uniqueness of solutions to our $\Phi-$Caputo sequential pantograph fractional differential problem (1.1) with integral boundary condition using Banach contraction principal (BCP for) and Holder inequality.

An existence result is obtained by using Darbo's theorem combined with the measure of noncompactness of Kuratowski in section 3.5, and in the last section, we give two examples to illustrate our results.

Chapter 4: Chapter 4 has the aims to study the existence of solutions and positive solutions of a ϕ-Laplacian fractional multi point boundary value problem [23]:

$$\begin{aligned} D^{\alpha}\left(\rho(t)\,\phi\left(D^{\beta}x(t)\right)\right)+q(t)\,f\left(t,x(t),D^{\beta}x(t)\right) &= 0,0<t<1, \\ \phi\left(D^{\beta}x(0)\right) &= x(0)=0, \\ x'(1)+\sum_{i=1}^{k}\sigma_i x'(\zeta_i) &= 0, \end{aligned} \tag{1.2}$$

where $0<\alpha\leq 1, 1\leq\beta<2, D^{\alpha}, D^{\beta}$ are the standard Caputo fractional derivatives $f\in C\left([0,1]\times\mathbb{R}^2,\mathbb{R}\right)$, $q\in C\left([0,1],\mathbb{R}\right)$, and $\sigma_i\in\mathbb{R}$, $\zeta_i\in(0,1)$ $\sum\limits_{i=1}^{k}\sigma_i x'(\zeta_i)<\infty$. $\phi:(-a,a)\to(-b,b); 0<a,b\leq+\infty$ is an increasing homeomorphism, with $\phi(0)=0$.

Two main results to (1.2) are given. In the Section 4.3, we establish existence of at least one solution using Schauder fixed point theorem. In Section 4.4, we examine the existence of at least three positive solutions using the Leggett-Williams fixed point theorem. An example and numerical simulations are included to show the applicability of our results..

Endly we close our thesis with a conclusion and some perspectives.

Chapter 2

Preliminaries

This chapter is devoted to elementary definitions and basic notions relating to fractional calculus, such as specific functions for fractional integration and fractional derivation and other notions that we will need in the rest of our work.

2.1 Fractional Calculus and Special Functions

2.1.1 The Eulerian Functions

We consider the so-called Eulerian functions, namely the well-known Gamma and Beta functions. These functions play a very important role in the theory of fractional calculus and its applications.

Gamma function

The Euler's Gamma function. Denoted by $\Gamma(z)$, this notation has been introduced by Adrien-Marie Legendre (1752-1833), it has been studied by other eminent mathematicians such as Carl Friedrich Gauss (1777-1855), Christoph Gudermann (1798-1852), Joseph Liouville (1809 -1882), Karl Weierstrass (1815- 1897), Charles Hermite (1822-1901) and many others. The Gamma function belongs to the category of special transcendental functions, and it appears in various fields, such as asymptotic series, definite integration, hypergeometric series, the Riemann zeta function, number theory, etc. For more details on this function, see [79].

Definition 2.1 *The Gamma function* $\Gamma(z)$ *is defined by the integral*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt, \ \Re e(z) > 0. \tag{2.1}$$

This integral representation is the most common for Γ, even if it is valid only in the right half-plane of $\mathbb{C}$. Using integration by parts, (2.1) shows that, at least for $\Re e(\alpha) > 0$, satisfies the simple difference equation

$$\Gamma(z+1) = z\Gamma(z), \tag{2.2}$$

which can be iterated to yield

$$\Gamma(z+n) = z(z+1)\dots(z+n-1)\Gamma(z), n \in \mathbb{N}. \tag{2.3}$$

The recurrence formulas (2.2) and (2.3) can be extended to any $z \in D_\Gamma := \mathbb{C} \setminus \{0, -1, -2, \dots\}$. In particular, since $\Gamma(1) = 1$, we get for non-negative integer values

$$\Gamma(n+1) = n!, n = 0, 1, 2\dots$$

Beta function

Another basic function of fractional calculus is the Beta function. It was discovered by the Swiss mathematical physicist Léonard Euler (1707-1783) and by the French mathematician Adrien - Marie Legendre (1752-1833) more than 200 years ago and was named by the French mathematical physicist and astronomer Jaques Marie Binet (1786-1856). The importance of this function lies in the fact that it is characteristically similar to the fractional integral of many functions, especially polynomials. This function plays an important role when combined with the function Gamma.

Definition 2.2 [79] *The Beta function $B(z,w)$, defined in*

$$D_B := \left\{(z,w) \in \mathbb{C}^2 : \Re e(z) > 0, \Re e(w) > 0\right\},$$

is given by the integral formula

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt. \tag{2.4}$$

The connection between the Beta function and the Gamma function is given by the following relation:

Proposition 2.1 *Let $z, w \in \mathbb{C}$ such that*

$$\Re e(z) > 0, \Re e(w) > 0.$$

Then

$$\Gamma(z)\Gamma(w) = \Gamma(z+w)B(z,w). \tag{2.5}$$

2.1.2 Fractional Integrals and Derivatives

In this section, we discuss the necessary mathematical tools we need in the succeeding chapters. We look at some essential properties of fractional differential operators, limiting our scope to the Riemann-Liouville, Caputo and $\Phi-$Caputo versions.

The easiest access to the idea of the non-integer differential and integral operators studied in the field of fractional calculus is given by Cauchy's well known representation of an n-fold integral as a convolution integral

$$J^n f(t) := \int_a^t \int_a^{\tau_1} \dots \int_a^{\tau_{n-1}} f(\tau)\, d\tau \dots d\tau_2 d\tau_1 = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f(\tau)\, d\tau, n \in \mathbb{N}^*, a, t \in \mathbb{R} \tag{2.6}$$

As seen in the Introduction, different ways to transfer integer-order operations to the non-integer case were developed. Therefore, we start with the most common one, the Riemann-Liouville operators for fractional differentiation and integration [58, 79, 7].

Riemann-Liouville approach

Replacing $n \in \mathbb{N}$ with $\alpha \in \mathbb{R}$ in (2.6) and using Euler's Gamma function (2.1) instead of the factorial, we obtain the following definition:

Definition 2.3 [87] *The Riemann-Liouville fractional integral of order $\alpha > 0$, for $f(t) \in L^1(a,b)$, is defined as:*

$$J_{a_+}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, t > a, \alpha > 0 \tag{2.7}$$

$$J_{b_-}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (t-\tau)^{\alpha-1} f(\tau)\, d\tau, t < b, \alpha > 0 \tag{2.8}$$

These integrals are also known as the left-sided and right-sided fractional integrals, or the Riemann-Liouville fractional integrals. The integrals in (2.7) and (2.8) are extensions to a finite interval $[a,b]$ that is half or whole axis.
These can be used on the half axis (a,∞) or $(-\infty,b)$, respectively, depending on the variable integration limit. For the half axis, we write

$$J_{0_+}^{\alpha} f(t) := J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, 0 < t < \infty. \tag{2.9}$$

We have the following properties of the Riemann-Liouville integral operator [83], [84], [87]:

Proposition 2.2 [58]
1∘ *For $\xi, \zeta \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+$*

$$J^{\alpha}(\xi f(t) + \zeta g(t)) = \xi J^{\alpha} f(t) + \zeta J^{\alpha} g(t).$$

2∘ *For $\alpha, \beta \in \mathbb{R}^+$*

$$J^{\alpha}\left(J^{\beta} f(t)\right) = J^{\alpha+\beta} f(t).$$

3∘ *For $\alpha, \beta \in \mathbb{R}^+$*

$$J^{\alpha}\left(J^{\beta} f(t)\right) = J^{\beta}\left(J^{\alpha} f(t)\right).$$

4∘ *For all $\alpha > 0, \beta > -1$, and $t > 0$*

$$J^{\alpha} t^{\beta} = t^{\alpha+\beta} \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.$$

5∘ *For $\alpha \in \mathbb{R}^+, C \in \mathbb{R}$ and $t > 0$*

$$J^{\alpha} C = \frac{C t^{\alpha}}{\Gamma(\alpha+1)}.$$

From the definition of the Riemann-Liouville fractional integral, the fractional derivative is obtained not by replacing α with $-\alpha$ because the integral $\int_a^t (t-\tau)^{\alpha-1} d\tau$ is, in general, divergent. Instead, differentiation of arbitrary order is defined as the composition of ordinary differentiation D^n and fractional integration, i.e., we can define fractional derivative

of order $n-1 < \alpha < n$ by:

Definition 2.4 [79] *Riemann-Liouville fractional derivative of order $\alpha, n-1<\alpha<n$ for $f \in C[a,b], \mathbb{R}$, is defined by:*

$$D_{a_+}^{\alpha} f(t) := \begin{cases} D^n J_{a_+}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau, t>a, a\in\mathbb{R}, n\in\mathbb{N}^* \\ \frac{d^n}{dt^n} f(t), \alpha = n. \end{cases} \tag{2.10}$$

Proposition 2.3 [58]
1○ *For $\alpha \in \mathbb{R}^+$ and $t \in [a,b]$*

$$D_{a_+}^{\alpha} J_a^{\alpha} f(t) = f(t).$$

2○ *For $n-1<\alpha<n$ and $t \in [a,b]$*

$$J_a^{\alpha}\left(D_{a_+}^{\alpha} f(t)\right) = f(t) - \sum_{i=1}^{n} \left[D_{a_+}^{\alpha-i} f(t)\right]_{t=a} \frac{(t-a)^{\alpha-i}}{\Gamma(\alpha-i+1)}. \tag{2.11}$$

3○ *For $n-1<\alpha<n$, $n- <\beta<n$ and $t \in [a,b]$*

$$D_{a_+}^{\alpha}\left(D_{a_+}^{\beta} f(t)\right) = D_{a_+}^{\alpha+\beta} f(t) - \sum_{i=1}^{n-1} \left[D_{a_+}^{\beta-i} f(t)\right]_{t=a} \frac{(t-a)^{-\beta-i}}{\Gamma(1-\beta-i)} \tag{2.12}$$

4○ *For all $\alpha>0, \beta>-1$, and $t \in [a,b]$*

$$D_{a_+}^{\alpha} (t-a)^{\beta} = (t-a)^{\beta-\alpha} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}.$$

5○ *For $\alpha \in \mathbb{R}^+, C \in \mathbb{R}$ and $t \in [a,b]$*

$$D_{a_+}^{\alpha} C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)}.$$

In the development of the theory of fractional integrations and derivations as well as its applications in pure mathematics, the Riemann-Liouville definition played a very important role. However, solving physical problems requires some revision of this well-established approach. Several works have appeared, notably in diffusion and electricity, where fractional derivation is used to better describe certain physical properties.

In general, applied problems require definitions that allow the use of physically interpretable initial conditions. Unfortunately, the definition of the fractional derivative in the sense of Riemann-Liouville leads to initial conditions of fractional types that are difficult to interpret physically. Even though such a value problem or initial condition can be solved well using a diffusive representation, However, in [85], Sabatier et al show that the Riemann-Liouville approach cannot be used to take into account the initial conditions in a physically convenient way. To possibly remedy this situation, Caputo [33] proposes a new definition of the fractional derivative in 1967, which bears his name and incorporates the initial conditions of the function to be processed as well as its integer derivatives, in order to possibly remedy this situation. This approach was adopted by Caputo and Mainardi [32] in their work on viscoelasticity.

Caputo's approach

Definition 2.5 $[32-33]$ *The Caputo fractional derivative of a function* $f \in C^n([a,b],\mathbb{R})$ *is defined by:*

$$^{C}D_{a_+}^{\alpha}f(t) := \left\{ J_{a_+}^{n-\alpha}D^n f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,d\tau, t>a, a\in\mathbb{R}, n\in\mathbb{N}^* \right. \tag{2.13}$$

where $n=[\alpha]+1$ *and* $[\alpha]$ *denotes the integer part of the real number* α.

Proposition 2.4 $[32-33]$*Let* $\alpha,\beta>0,\ t\in[0,b]\ b>0$ *and* $n=[\alpha]+1$ *Then the following relation holds:*

$$^{C}D_{0_+}^{\alpha}t^{\beta} :=^{C} D^{\alpha}t^{\beta} = \begin{cases} t^{\beta-\alpha}\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}, \beta\in\mathbb{N} \text{ and } \beta\geq n \text{ or } \beta\notin\mathbb{N} \text{ and } \beta>n-1, \\ 0, \beta\in\{0,1,...,n-1\}. \end{cases} \tag{2.14}$$

And

$$^{C}D^{\alpha}C=0,$$

for any constant C.

We also have the following Lemmas:

Lemma 2.1 *For* $\alpha>0$ *and* $t\in[0,b]\ b>0$ *the general solution of the equation*

$$^{C}D^{\alpha}f(t)=0$$

is given by

$$f(t)=c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1} \tag{2.15}$$

where $c_i\in\mathbb{R}, i=0,1,2,..,n-1, n=[\alpha]+1$.

Lemma 2.2 *Let* $\alpha>0$. *Then we have*

$$J^{\alpha C}D^{\alpha}f(t)=f(t)+c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1}, \tag{2.16}$$

for some $c_i\in\mathbb{R}, i=0,1,2,..,n-1, n=[\alpha]+1,\ t\in[0,b]\ b>0$.

Lemma 2.3 *Let* $\alpha>\beta>0$, *and* $f\in L^1[0,b]$. *Then we have:*
(i) *The Caputo fractional derivative is linear;*
(ii) $^{C}D^{\alpha}J^{\alpha}f(t)=f(t)$;
(iii) $^{C}D^{\beta}J^{\alpha}f(t)=J^{\alpha-\beta}f(t)$;

Remark 2.1 *We note that if* $f\in C^n[0,b]$.*, then we have*

$$D^{\alpha}f(t) =^{C} D^{\alpha}f(t)+\sum_{i=1}^{n}f^{(i)}(0)\frac{t^{i-\alpha}}{\Gamma(1-\alpha+i)}.$$

Clearly, we see that if $f^{(i)}(0)=0$ *for* $i=0,1,2,..,n-1$ *then we have*

$$D^{\alpha}f(t) =^{C} D^{\alpha}f(t).$$

A novel fractional operator with respect to another function Φ was formulated by Almeida $[8,9]$. This new operator is called the Φ-Caputo fractional operator.

Φ−Caputo's approach

Definition 2.6 [9] *The* $\Phi-$*Riemann–Liouville fractional integral of an integrable function* $f:[a,b]\rightarrow\mathbb{R}$ *of order* $\alpha\in\mathbb{R}_{*}^{+}$ *is defined as*

$$J_{a^{+}}^{\alpha,\Phi}f\left(t\right):=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{t}\Phi'\left(\tau\right)\left(\Phi\left(t\right)-\Phi\left(\tau\right)\right)^{\alpha-1}f\left(\tau\right)d\tau, \tag{2.17}$$

where $\Phi\in C^{n}\left[a,b\right]$ *is an increasing function such that* $\Phi'\left(t\right)\neq0$*, for all* $t\in\left[a,b\right]$ *and* $n=\left[\alpha\right]+1$*.*

Definition 2.7 [9] *The* $\Phi-$*Riemann–Liouville fractional derivative of an integrable function* $f:[a,b]\rightarrow\mathbb{R}$ *of order* $\alpha\in\mathbb{R}_{*}^{+}$ *is defined as*

$$D_{a^{+}}^{\alpha,\Phi}f\left(t\right):=\left(\frac{1}{\Phi'\left(t\right)}\frac{d}{dt}\right)^{n}J_{a^{+}}^{n-\alpha,\Phi}f\left(t\right) \tag{2.18}$$

$$=\frac{1}{\Gamma\left(n-\alpha\right)}\left(\frac{1}{\Phi'\left(t\right)}\frac{d}{dt}\right)^{n}\int_{a}^{t}\Phi'\left(\tau\right)\left(\Phi\left(t\right)-\Phi\left(\tau\right)\right)^{n-\alpha-1}f\left(\tau\right)d\tau, \tag{2.19}$$

where $\Phi\in C^{n}\left[a,b\right]$ *is an increasing function such that* $\Phi'\left(t\right)\neq0$*, for all* $t\in\left[a,b\right]$ *and* $n=\left[\alpha\right]+1$*.*

Proposition 2.5 [9] *The fractional integrals satisfy the semigroup property so for* $\alpha,\beta>0$

$$J_{a^{+}}^{\alpha,\Phi}J_{a^{+}}^{\beta,\Phi}f\left(t\right)=J_{a^{+}}^{\alpha+\beta,\Phi}f\left(t\right)$$

holds.

In the present thesis, we deal with the Φ−Caputo type differential operator.

Definition 2.8 [9] *The* $\Phi-$*Caputo fractional derivative of a given* $f\in C^{n}\left[a,b\right]$ *of order* $\alpha\in\mathbb{R}_{*}^{+}$ *is defined as*

$${}^{C}D_{a^{+}}^{\alpha,\Phi}f\left(t\right):=D_{a^{+}}^{\alpha,\Phi}\left[f\left(t\right)-\sum_{i=1}^{n-1}\frac{f_{\Phi}^{[i]}\left(a\right)}{i!}\left(\Phi\left(t\right)-\Phi\left(a\right)\right)^{i}\right] \tag{2.20}$$

$$=\frac{1}{\Gamma\left(n-\alpha\right)}\int_{a}^{t}\Phi'\left(\tau\right)\left(\Phi\left(t\right)-\Phi\left(\tau\right)\right)^{n-\alpha-1}f_{\Phi}^{[n]}\left(\tau\right)d\tau, \tag{2.21}$$

where $\Phi\in C^{n}\left[a,b\right]$ *is an increasing function such that* $\Phi'\left(t\right)\neq0$*, for all* $t\in\left[a,b\right]$ *and* $n=\left[\alpha\right]+1$ *and*

$$f_{\Phi}^{[n]}\left(t\right):=\left(\frac{1}{\Phi'\left(t\right)}\frac{d}{dt}\right)^{n}f\left(t\right).$$

Proposition 2.6 [9] *Let* $\beta\in\mathbb{R}$ *with* $\beta>0$*, then the* $\Phi-$*Caputo fractional derivative of a power function is given by the formula*

$${}^{C}D_{a^{+}}^{\alpha,\Phi}\left(\Phi\left(t\right)-\Phi\left(a\right)\right)^{\beta}:=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta-\alpha\right)}\left(\Phi\left(t\right)-\Phi\left(a\right)\right)^{\beta-\alpha-1}. \tag{2.22}$$

Theorem 2.1 [9] *Let* $f : [a, b] \to \mathbb{R}$ *and* $\alpha \in \mathbb{R}_*^+$. *Then we have*
(*i*) *For* $f \in C[a, b]$,

$$^C D_{a^+}^{\alpha,\Phi} J_{a^+}^{\alpha,\Phi} f(t) = f(t) \tag{2.23}$$

(*ii*) *For* $f \in C^n[a, b]$

$$J_{a^+}^{\alpha,\Phi C} D_{a^+}^{\alpha,\Phi} f(t) = f(t) - \sum_{i=1}^{n-1} \frac{f_\Phi^{[i]}(a)}{i!} (\Phi(t) - \Phi(a))^i . \tag{2.24}$$

The main advantage of the Φ-Caputo operator is the non-local behavior and the semigroup property, which are clearly preserved. It has been recognized that these types of operators have been successfully used to describe and model many real life phenomena. Other significant features are that the function in its integral kernel can be adapted to accommodate other definitions when replacing it with specific functions, like

1· if $\Phi(t) = t$, then the $\Phi-$Caputo fractional derivative coincide with the Caputo fractional derivative given in Definition 2.5.

2· If $\Phi(t) = \ln t$, then the $\Phi-$Caputo fractional derivative reduce to the Caputo–Hadamard fractional derivative [53].

3· If $\Phi(t) = t^\rho$, then the $\Phi-$Caputo fractional derivative match the Caputo–Erdélyi–Kober fractional derivative [69].

4· If $\Phi(t) = \frac{t^\rho}{\rho}, \rho > 0$, then the $\Phi-$Caputo fractional derivative match the Caputo–Katugampola fractional derivative [55].

2.2 Topics of Functional Analysis

In this part, we present a preliminary in which we recall fundamental notions and results of the theory of functional analysis [42, 59] which represent an indispensable tool in the theory of fractional calculus.

2.2.1 Basic Properties of Banach Space

Definition 2.9 *Given a vector space* X *over a real field. A norm on* X *is a function* $\|\cdot\| : X \to \mathbb{R}$ *such that*

$$\begin{aligned} (i)\, \|x\| &\geq 0, x \in X, \\ (ii)\, \|x\| &= 0, \text{ if and only if } x = 0, \\ (iii)\, \|\alpha x\| &= |\alpha| \|x\|, \alpha \in \mathbb{R}, \\ (iv)\, \|x + y\| &\leq \|x\| + \|y\|, x, y \in X. \end{aligned}$$

Definition 2.10 *A linear space* X *coupled with a norm* $\|\cdot\| : X \to \mathbb{R}$ *is known as a normed linear space. We denote the normed linear space as a pair* $(X, \|\cdot\|)$.

Definition 2.11 *Let* $(X, \|\cdot\|)$ *be a normed linear space. We say that a sequence* $\{x_n\}_{n=1}^\infty$ *in* X *converges to* $x \in X$ *provided that*

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Definition 2.12 *A function $f : X \to \mathbb{R}$ is continuous at $x \in X$ provided that if for each sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that x_n converges to x, then $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x) \in \mathbb{R}$.*

Theorem 2.2 *The norm is a continuous function.*

Example 2.1 *Some examples of norms are:*
1. *The absolute value, $\|x\| = |x|$ for all $x \in \mathbb{R}$.*
2. *The Euclidean norm, $\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ for all $x \in \mathbb{R}^n$.*
3. *The magnitude of a complex number, $\|z\| = \sqrt{\Re e(z)^2 + \Im m(z)^2}$.*

Definition 2.13 *In a normed linear space X , with norm, the open ball with center 0 and radius r is given by*

$$B_n(0, r) = \{x \in X : \|x\|_n < r\}.$$

It is sometimes helpful to codify the idea of "distance" in a normed linear space. The formalization is provided via a function called a metric.

Definition 2.14 *A metric on a non-empty set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:*

$$\begin{aligned} 1.d(x,y) &\geq 0, \\ 2.d(x,y) &= 0 \text{ if and only if } x = y, \\ 3.d(x,y) &= d(y,x), \\ 4.d(x,z) &\leq d(x,y) + d(y,x). \end{aligned}$$

A non-empty set with a metric is called a metric space.

Proposition 2.7 *For a given normed linear space $(X, \|\cdot\|)$. The function $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = \|x - y\|$ is a metric on X.*

Definition 2.15 *A sequence in a normed linear space is called Cauchy if and only if, for all real numbers $\epsilon > 0$ there exists an index $M > 0$ such that*

$$\|x_n - x_m\| < 0,$$

whenever $n, m \geq M$.

Definition 2.16 *A normed linear space $(X, \|\cdot\|)$ is called complete if and only if every Cauchy sequence in X converges to a point in X.*

The idea of a sequence in which two terms of the sequence become arbitrarily close together as one proceeds through the sequence is formally formalized by the definition of a Cauchy sequence. The last element required to define a Banach space is the notion of a complete space.

Definition 2.17 *A Banach space is a complete normed linear space.*

Example 2.2 *The space $C([a,b], \mathbb{R})$ consisting of all continuous functions $f : [a,b] \to \mathbb{R}$ is Banach space with respect to the norm $\|f\| = \sup\{|f(t)|, t \in [a,b]\}$.*

Definition 2.18 *A set M in a Banach space X is compact if for each sequence $\{x_n\} \subset M$ has a subsequence with limit in M.*

Theorem 2.3 *Particularly, a set M in $(\mathbb{R}^n, \|\cdot\|)$ is compact if and only if it is closed and bounded.*

Definition 2.19 *A relatively compact subspace (or relatively compact subset) M of a normed space X is a subset whose closure is compact.*

Every subset of a compact space is relatively compact since the closed subset of compact space is compact.

Definition 2.20 *A subset M of $C([a,b],\mathbb{R})$ is equicontinuous provided for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|x-y\| < \delta$ implies $\|f(x) - f(y)\| < \delta$, for $x, y \in [a,b], f \in M$.*

Definition 2.21 *Let $\{f_n\}$ be a sequence of functions with $f_n : [a,b] \to \mathbb{R}$, then $\{f_n\}$ is uniformly bounded on $[a,b]$ if there exists $c > 0$ such that $|f_n(t)| \leq c$ for all $n \in \mathbb{N}, \in [a,b]$.*

Theorem 2.4 *(**Arzela-Ascoli Theorem**) Let M be an open bounded subset of a Banach spaces X. A subset of $C(M,\mathbb{R})$ is relatively compact if and only if it is bounded and equicontinuous.*

Definition 2.22 *Let X and Y be two Banach spaces. The continuous operator $T : X \to Y$ is completely continuous if it transforms everything bounded in X into a relatively compact part in Y.*

Definition 2.23 *Let X be a real Banach space. A non-empty closed set $P \subset X$ is called a cone of X if it satisfies the following conditions:*

$$\begin{aligned}(1)x &\in P, \mu \geq 0 \text{ implies } \mu x \in P,\\ (2)x &\in P, -x \in P \text{ implies } x = 0.\end{aligned}$$

Every cone $P \subset X$ induces an ordering in X given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 2.24 *A map φ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space X if*

$$\varphi : P \to [0,\infty)$$

is continuous and

$$\varphi(tx + (1-t)y) \geq t\varphi(x) + (1-t)\varphi(y);$$

for all $x, y \in p$ and $t \in [0,1]$.

Definition 2.25 *Similarly, the map ψ is a nonnegative continuous convex functional on a cone P of a real Banach space X if*

$$\psi : P \to [0,\infty)$$

is continuous and

$$\psi(tx + (1-t)y) \leq t\psi(x) + (1-t)\psi(y);$$

for all $x, y \in p$ and $t \in [0,1]$.

Lemma 2.4 *(**Holder's inequality** [71]) Let $1 \leq p \leq \infty$ and let q denote the conjugate exponent defined by:*

$$q = \frac{p}{p-1}; \text{ that is } \frac{1}{p} + \frac{1}{q} = 1.$$

If $u \in L^p(J,\mathbb{R})$ and $v \in L^q(J,\mathbb{R})$, then $uv \in L^1(J,\mathbb{R})$, and

$$\int_J |u(t)v(t)|\,dt \leq \left(\int_J |u(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_J |v(t)|^q\,dt\right)^{\frac{1}{q}}.$$

2.2.2 Measure of Noncompactness

The theory of nonlinear analysis, which has recently been improved quickly due to its extensive practical applications in many fields, including engineering, economics, optimal control, and optimization, heavily relies on the measure of noncompactness, which was first introduced by the main papers of Kuratowski [61] and Darbo [41], developed by Bana's and Goebel [18], and used by many other authors in the literature. The theories of differential and difference equations, integral equations with differential inclusions, and fractional differential equations have all seen success with the use of the measure of noncompactness. [24, 57].

First, we recall the definition of the Kuratowski measure of noncompactness and some auxiliary results that are to be used to prove our result. For more details, see ([18], [41], [57], [92]) and the references therein.

Definition 2.26 ([18], [92]) *The Kuratowski measure of noncompactness μ_X over the subset E of a Banach space X is given by*

$$\mu_X(E) = \inf\{\varepsilon > 0 : E \subseteq \cup_{i=1}^{n} E_i \text{ and } diam(E_i) \leq \varepsilon\}, \tag{2.25}$$

where

$$diam(E_i) = \sup\{\|x-y\| : x, y \in E_i\}.$$

The Kuratowski measure of noncompactness satisfies the properties described by the following lemmas ([18], [57], [92]).

Lemma 2.5 *Let X be a Banach space and $E, C \subset X$ be bounded, then the following properties are satisfied:*
1. $\mu_X(E) = 0$ if and only if E is relatively compact;
2. $\mu_X(\overline{E}) = \mu_X(E)$;
3. $\mu_X(C) \leq \mu_X(E)$, when $C \subseteq E$;
4. $\mu_X(C+E) < \mu_X(C) + \mu_X(E)$, where $C + E = \{d/d = c + e; c \in C, e \in E\}$;
5. $\mu_X(\varrho E) = |\varrho|\,\mu_X(E)$ for all $\varrho \in \mathbb{R}$;
6. $\mu_X(ConvE) = \mu_X(E)$;
7. $\mu_X(C \cup E) = \max(\mu_X(C), \mu_X(E))$;
8. $\mu_X(C \cup \{x\}) = \mu_X(C)$ for all $x \in X$.

2.3 Fixed Point Theorems

The theory of fixed point is one of the most powerful tools in modern mathematics. This theory is a beautiful mixture of analysis, topology, and geometry. Fixed point theorems are very useful tools in solving fractional differential equations. Indeed, these theorems not only provide sufficient conditions under which a given function admits a fixed point but also assure us the existence of the solution of a given problem by transforming it into a fixed point problem, and we eventually determine these fixed points, which are the solutions to the given problem. In this section, we collect some very basic fixed point theorems. A more detailed description of the fixed point theory can be found, for instance, in ([4] , [60]).

Definition 2.27 *A point $x \in X$ is called a fixed point of a function $f : X \to X$, if*

$$f(x) = x, x \in X.$$

Definition 2.28 *For a normed space $(X, \|\cdot\|)$, the operator $T : X \to X$ is said to satisfy the Lipschitz condition, if there exists a positive real constant k such that for all X and Y in X,*

$$\|Tx - Ty\| \leq k \|x - y\|.$$

Remark 2.2 *In Definition 2.17, if $0 < k < 1$, the operator T is called a contraction mapping on the normed space $(X, \|\cdot\|)$.*

In several disciplines of science, the Banach contraction principle is important as a fountain of existence and uniqueness theorems. This theorem serves as an example of the analytical use of fixed point theory and the unifying nature of functional analysis techniques. The crucial aspect of the Banach contraction principle is that it demonstrates the existence, singularity, and convergence of consecutive approximations to a problem solution. The primary noteworthy aspect of the outcome is that existence, uniqueness, and determination are all provided by the Banach contraction principle. ([4] , [48]):

Theorem 2.5 ***(Banach fixed point theorem)*** *Let X be a Banach space and T be a contraction mapping with the Lipschitz constant k. Then T has a unique fixed point.*

The second theorem is the following generalization of Banach's fixed point that we take from:

Theorem 2.6 ***(Weissinger's fixed point theorem).*** *Assume $(X, \|\cdot\|)$ to be a normed Banach space and let $\varkappa_j \geq 0$ for every $j \in \mathbb{N}$ such that $\sum_{j=0}^{n-1} \varkappa_j$ converges. Furthermore, let the mapping $T : X \to X$ satisfy the inequality*

$$\left\|T^j x - T^j y\right\| \leq \varkappa_j \|x - y\|,$$

for every $j \in \mathbb{N}$ and every $x, y \in X$. Then, T has a unique fixed point x^*. Moreover, for any $x_0 \in X$, the sequence$\{T^j x_0\}_{j=0}^{\infty}$ converges to this fixed point x^* .

Moreover we also use slightly different results that assert only the existence but not the uniqueness of a fixed point.

Theorem 2.7 ***(Schaefer fixed point theorem)*** *Let X be a Banach space, and let $T : X \to X$ be a completely continuous operator. If the set $\{x \in X : w = \lambda Tx, 0 \leq \lambda \leq 1\}$ is bounded. Then T has at least one fixed point.*

In 1979 a theorem of existence of triple fixed point has been proved by Leggett-Williams [14, 65] :

Theorem 2.8 ***(Leggett-Williams fixed point theorem*** [65]*) Let X be a real Banach space, P be a cone in X. Assume that there exists a nonnegative continuous concave functional on the cone P with $\varphi(x) \leq \|x\|, \forall x \in P$. Furthermore, suppose that $0 < a < b < h \leq c$ are constants and $T : P_c \to P_c$ is a completely continuous operator. If*
(A_1) $\{x \in P(\varphi; b, h) \setminus \varphi(x) > b\} \neq \varnothing$ and $\varphi(Tx) > b$ for every $x \in P(\varphi; b, h)$;
(A_2) $\|Tx\| < a$ for $\|x\| \leq a$
(A_3) $\varphi(Tx) > b$ for $x \in P(\varphi; b, c)$ with $\|Tx\| > h$;
then T has at least three fixed points x_1, x_2 and x_3 in $\overline{P_c}$ such that $\|x_1\| < a, \varphi(x_2) > b, \|x_3\| > a$ with $\varphi(x_3) < b$, where

$$\begin{aligned} P_c &= \{x \in P : \|x\| < c\}, \\ P(\varphi; a, b) &= \{x \in P : \varphi(x) \geq a, \|x\| \leq b\}. \end{aligned}$$

Theorem 2.9 ***(Brouwer fixed point theorem*** [4, 48]*) Let $\overline{B}_1$ be the closed unit ball in the Euclidean space $\mathbb{R}^n$. Then every continuous mapping $f : \overline{B}_1 \to \overline{B}_1$ has a fixed point.*

The first fixed point theorem in an infinite dimensional Banach space was given by Schauder in 1930. The theorem is given below:

Theorem 2.10 ***(Schauder fixed point theorem*** [4, 48]*) Let U be a nonempty, closed, bounded, and convex subset of a Banach space X, and suppose that $T : U \to U$ is a completely continuous operator. Then T has a fixed point U.*

In the fixed point theorems of Brouwer and Schuader, compactness of the space under consideration is sought either completely or partially. Later on, though, the idea of a measure of noncompactness was used to remove the requirement for compactness (in short MNC). Darbo proved the following theorem by utilizing the idea of MNC [41].

Theorem 2.11 ***(Darbo fixed point theorem*** [41]*) Let X be a Banach space and C be a bounded, closed, convex, and nonempty subset of X. Suppose that the continuous mapping $\Upsilon : C \to C$ is a μ_X-contraction. Then Υ has a fixed point in C.*
Where μ_X is Kuratowski measure of noncompactness.

Chapter 3

A Sequential Fractional Differential Problem of Pantograph Type: Existence Uniqueness and Illustrations

3.1 Introduction

In order to extend and developed, In the 1960s the British Railways created a new kind of electric locomotive. The target was to accelerate the trains. The pantograph was a fundamental part of the new high speed electric locomotive. The pantograph's function is to gather current from an overhead wire, which is required for the locomotive to move. There must be no hiccups in the current collection system in order for the electric locomotive to travel smoothly and at high speed. As a result, the pantograph should maintain contact with the overhead catenary wire at all times, especially as it passes the overhead wire's supports, which is a crucial passage.

This impetus led J. R. Ockendon and A. B. Taylor to study the issue of determining the motion of a pantograph head on an electric locomotive in [77] along with other pantograph-related concerns. The goal is to create a mathematical model of the present collecting system and identify the cause of the pantograph head's motion. They came across a special delay differential equation of the form:

$$\{y'(t) = ay(t) + by(\lambda t), t > 0,$$

where a, b are real constants and $0 < \lambda < 1$ for $\lambda \in \mathbb{R}$. When the article was published in 1971.

The pantograph problem is one of the classical models. It is considered as a class of delay differential equations in which the derivative of the function, at any time, depends on the solution at a previous time. Recently, an attention to the pantograph equations has considered ($[5-7], [36]$) due to their applications in modeling numerous processes in real world problems. For example, in [6], a stage-structured model of population growth has been proposed. Then, the proposed model has been employed to study how the electric current is collected by the pantograph of an electric locomotive [77]. In the same sense, in [30], a discretization of the following general pantograph equation has been investigated:

$$\begin{cases} y'(t) = ay(t) + by(\theta(t)) + cy(\phi(t)), t \geq 0, \\ y(0) = y_0, \end{cases}$$

where a, b, c, y_0 are real numbers, θ and ϕ are strictly increasing functions on the nonnegative reals, with $\theta(0) = \phi(0) = 0$, $\theta(t) < t$ and $\phi(t) < t, t > 0$.

We cite also the paper [49], where K. Guan et al. have studied the oscillatory behavior of solutions to the following pantograph problem:

$$\begin{cases} x'(t) = P(t)x(t) - Q(t)x(\alpha t), t \geq t_0, t = t_k, \\ x(t_k) = b_k x(t_k), k = 1, 2, ..., \end{cases}$$

where $0 < \alpha < 1$ and $0 < t_0 < t_1 < ... < t_k < ...$ are fixed points, $P(t), Q(t) \in C([t_0, \infty), [0, \infty))$.

In [47], the authors have addressed and studied the following fractional pantograph equation:

$$\begin{cases} {}^cD^{\alpha}_{0+}(u(s) - P(s)u(\beta s))(t) = f(t, u(t), u(\gamma t)), t \in [0, T], \\ u(0) = u_0, \\ u'(0) = u_1, \end{cases}$$

where $1 < \alpha \leq 2$, ${}^cD^{\alpha}_{0+}$ is the Caputo derivative of order α, $0 < \beta, \gamma < 1$ and f, P are two functions that satisfy some imposed conditions.

Very recently, in [51] using the Ψ-Hilfer derivative, it has been investigated the existence and uniqueness as well as the stability for the following nonlinear neutral pantograph equation:

$$\begin{cases} D^{\alpha,\beta,\Psi}u(t) = g\left(t, u(t), u(\kappa t), D^{\alpha,\beta}u(\kappa t)\right), t \in J = [a, b], \\ I^{1-\gamma,\Psi}u(a) = u_a, \end{cases}$$

where $D^{\alpha,\beta,\Psi}$ is the Ψ-Hilfer derivative of order $\alpha, 0 < \alpha < 1$ and $I^{1-\gamma,\Psi}$ is the Ψ-integral of order $1 - \gamma\,(\gamma = \alpha + \beta - \alpha\beta)$.

In [45], Fazli et al. have discussed the existence and uniqueness result for the fractional problem:

$$\begin{cases} D^{\beta}(D^{\alpha}x(t) + \lambda) = f(t, x(t)), 0 < t \leq 1, \\ x^{(i)}(0) = \mu_i, 0 \leq i < l, \\ x^{(i+\alpha)}(0) = \nu_i, 0 \leq i < n, \end{cases}$$

where $m - 1 < \alpha \leq m$, $n - 1 < \beta \leq n$, $l = \max(n, m)$, $m, n \in \mathbb{N}$, D^{α} is the Caputo derivative, $x(t)$ is the particle displacement, $\lambda \in \mathbb{R}$ is the friction coefficient and f is a noise term.

So, in this work [22], we shall study the following $\Phi-$Caputo sequential pantograph fractional differential problem with integral conditions:

$$\begin{cases} {}^cD^{\beta,\Phi}\left({}^cD^{\alpha,\Phi}x(t) + g(t, x(t))\right) = f\left(t, x(t), x(\lambda t), {}^cD^{\alpha,\Phi}x(t)\right), t \in [0, 1], \\ x(0) = 0, \\ x(1) = \int_0^1 h(s, x(s))\, ds. \end{cases} \tag{3.1}$$

We take into account the conditions that ${}^{c}D^{\alpha,\Phi}, {}^{c}D^{\beta,\Phi}$ are the $\Phi-$Caputo derivatives, such that $0 < \alpha, \beta \leq 1,\ f \in C([0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ g, h \in C([0,1] \times \mathbb{R}, \mathbb{R}),\ \lambda \in \mathbb{R}_{*}^{+}$ and $0 < \lambda < 1$.
It is to note that the problem (1.1) is important since, in one hand, it is more general than the above cited pantograph differential problems, and on the other hand, it is considered as a special type of delay differential problems with integral conditions. Such equations have various applications in chemical engineering, electrodynamics, blood flow models, cellular systems, thermoelectricity, underground water flow, plasma physics, and population dynamics... (see [3, 86]). In this sense, it has been confirmed that models of epidemics that lead to delay equations often have integral conditions that are imposed by the interpretation of these models. The neglect of these integral conditions may lead to solutions that behave in a radically different manner from solutions restricted to obey them [31].

Our investigation to the problem (3.1) is based on the application of Darbo's theorem. This investigation has two motivational reasons: the first one is the fact that Darbo's theorem extends both Schauder and Banach fixed point theorems, so it is better to apply Darbo's theorem instead of Schauder or Banach theorems. The second reason that motivates our application of Darbo theorem is the abundance, by mathematicians, of this important theorem in proving the existence of solutions for a wide class of differential and integral equations. So, we fell motivated to present a contribution in this sense to fill the void and the lack in this field of interest.

3.2 The Pantograph Integral Representation

In this section, we present to the reader the following first result.

Let $E := \left\{x : x \in C([0,1]), {}^{c}D^{\alpha,\Phi}x \in C([0,1])\right\}$ be the Banach space endowed with the norm

$$\|x\|_{E} = \|x\| + \left\|{}^{c}D^{\alpha,\Phi}x\right\|,$$

where $\|x\| = \max_{t\in[0,1]} |x(t)|$ and $\left\|{}^{c}D^{\alpha,\Phi}x\right\| = \max_{t\in[0,1]} \left|{}^{c}D^{\alpha,\Phi}x(t)\right|$.

Using E, we introduce the following hypotheses:

($\mathbf{H}_1$) The function $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there are two positive constants L_f, M_f satisfying

$$\left\|f\left(t, x, \tilde{x}, \tilde{\tilde{x}}\right) - f\left(t, y, \tilde{y}, \tilde{\tilde{y}}\right)\right\| \leq L_f\left(\|x - y\| + \|\tilde{x} - \tilde{y}\|\right) + M_f\left\|\tilde{\tilde{x}} - \tilde{\tilde{y}}\right\|.$$

($\mathbf{H}_2$) The function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a constant $L_g > 0$, such that

$$\|g(t, x) - g(t, y)\| \leq L_g \|x - y\|.$$

($\mathbf{H}_3$) The function $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exist a constant $L_h > 0$, such that

$$\|h(t, x) - h(t, y)\| \leq L_h \|x - y\|.$$

Lemma 3.1 *Let $y \in C([0,1])$. Then the problem*

$$\begin{aligned} {}^{c}D^{\beta,\Phi}\left({}^{c}D^{\alpha,\Phi}x(t)+z(t)\right) &= y(t), t\in[0,1], \\ x(0) &= 0, \\ x(1) &= \int_0^1 h(s,x(s))\,ds, \end{aligned} \tag{3.2}$$

has a unique integral representation, which is given by the expression:

$$\begin{aligned} x(t) &= \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1}y(\tau)}{\Gamma(\alpha+\beta)}d\tau - \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1}z(\tau)}{\Gamma(\alpha)}d\tau \\ &+ \left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha}\Big(\int_0^1 h(s,x(s))\,ds - \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}y(\tau)}{\Gamma(\alpha+\beta)}d\tau \\ &+ \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}z(\tau)}{\Gamma(\alpha)}d\tau\Big). \end{aligned} \tag{3.3}$$

Proof. By Lemma 2.1 and 2.2, we can reduce (3.2) to the equivalent equation:

$${}^{c}D^{\alpha,\Phi}x(t) = J^{\beta,\Phi}y(t) - z(t) + c_0, c_0 \in \mathbb{R}. \tag{3.4}$$

Again, taking the integral operator $J^{\alpha,\Phi}$ on both sides of (3.3), we get

$$x(t) = J^{\alpha+\beta,\Phi}y(t) - J^{\alpha,\Phi}z(t) + c_0\frac{(\Phi(t)-\Phi(0))^{\alpha}}{\Gamma(\alpha+1)} + c_1, c_0, c_1 \in \mathbb{R}.$$

Moreover, for $t \in [0,1]$, using the fact that $x(0)=0$, we find:

$$c_1 = 0.$$

Also, since $x(1) = \int_0^1 h(s,x(s))\,ds$, then one can obtain

$$c_0 = \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right)\left(\int_0^1 h(s,x(s))\,ds - J^{\alpha+\beta,\Phi}y(1) + J^{\alpha,\Phi}z(1)\right).$$

Consequently,

$$x(t) = J^{\alpha+\beta,\Phi}y(t) - J^{\alpha,\Phi}z(t) + \left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha}\left(\int_0^1 h(s,x(s))\,ds - J^{\alpha+\beta,\Phi}y(1) + J^{\alpha,\Phi}z(1)\right),$$

which allows us to obtain the desired result. ■

3.3 One Pantograph Solution Via BCP Principle

We begin this section, by defining the integral operator $\Im : E \rightarrow E$ by the following expression:

$$\begin{aligned}\Im x(t) &= \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1} f\left(\tau, x(\tau), x(\lambda\tau), {}^c D^{\alpha,\Phi} x(\tau)\right)}{\Gamma(\alpha+\beta)} d\tau \\ &- \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1} g(\tau, x(\tau))}{\Gamma(\alpha)} d\tau \\ &+ \left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha} \Big(\int_0^1 h(s, x(s))\, ds \\ &- \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1} f\left(\tau, x(\tau), x(\lambda\tau), {}^c D^{\alpha,\Phi} x(\tau)\right)}{\Gamma(\alpha+\beta)} d\tau \\ &+ \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1} g(\tau, x(\tau))}{\Gamma(\alpha)} d\tau \Big).\end{aligned} \tag{3.5}$$

Then, based on Banach Contraction Principle (BCP for short) [40], we prove the following first main result.

Theorem 3.1 *Let* $(\mathbf{H}_1) - (\mathbf{H}_3)$ *hold. Suppose also that*

$$\Omega := (4L_f + 2M_f)\,\Delta_1 + 2L_g\Delta_2 + L_h\Delta_3 < 1. \tag{3.6}$$

Then, (3.1) *has a unique solution on* $[0,1]$*.*

Proof. Clearly, $\Im$ is itself mapping.

Let $t \in [0,1]$. So, we can write

$$\begin{aligned}&|\Im x(t) - \Im y(t)| \\ =& \Bigg| \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\ &\times \left[f\left(\tau, x(\tau), x(\lambda\tau), {}^c D^{\alpha,\Phi} x(\tau)\right) - f\left(\tau, y(\tau), y(\lambda\tau), {}^c D^{\alpha,\Phi} y(\tau)\right)\right] d\tau \\ &- \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1}\left[g(\tau, x(\tau)) - g(\tau, y(\tau))\right]}{\Gamma(\alpha)} d\tau \\ &+ \left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha} \times \Big(\int_0^1 \left[h(s, x(s)) - h(s, y(s))\right] ds \\ &- \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\ &\times \left[f\left(\tau, x(\tau), x(\lambda\tau), {}^c D^{\alpha,\Phi} x(\tau)\right) - f\left(\tau, y(\tau), y(\lambda\tau), {}^c D^{\alpha,\Phi} y(\tau)\right)\right] d\tau \\ &+ \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}\left[g(\tau, x(\tau)) - g(\tau, y(\tau))\right]}{\Gamma(\alpha)} d\tau \Big) \Bigg|.\end{aligned}$$

Using the Lipschitz assumption of f and the two hypotheses $(\mathbf{H}_2)-(\mathbf{H}_3)$, it yields that

$$\begin{aligned}\|\Im x-\Im y\| &\leq \left(2L_f\left(\|x-y\|\right)+M_f\left\|^cD^{\alpha,\Phi}x-^cD^{\alpha,\Phi}y\right\|\right)\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}d\tau\\ &+L_g\|x-y\|\int_0^t\frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1}}{\Gamma(\alpha)}d\tau+L_h\|x-y\|\\ &+\left(2L_f\left(\|x-y\|\right)+M_f\left\|^cD^{\alpha,\Phi}x-^cD^{\alpha,\Phi}y\right\|\right)\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}d\tau,\end{aligned}$$

we obtain

$$\begin{aligned}\|\Im x-\Im y\| &\leq \left(\frac{4L_f(\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{2L_g(\Phi(1)-\Phi(0))^{\alpha}}{\Gamma(\alpha+1)}+L_h\right)\|x-y\|\\ &+\left(\frac{2M_f(\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)\left\|^cD^{\alpha,\Phi}x-^cD^{\alpha,\Phi}y\right\|.\end{aligned}\tag{3.7}$$

On other hand, we have

$$\begin{aligned}{}^cD^{\alpha,\Phi}\Im x(t) &= \int_0^t\frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\beta-1}f\left(\tau,x(\tau),x(\lambda\tau),^cD^{\alpha,\Phi}x(\tau)\right)}{\Gamma(\beta)}d\tau-g(t,x(t))\\ &+\left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right)\times(\int_0^1 h(s,x(s))\,ds\\ &-\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}f\left(\tau,x(\tau),x(\lambda\tau),^cD^{\alpha,\Phi}x(\tau)\right)}{\Gamma(\alpha+\beta)}d\tau\\ &+\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}g(\tau,x(\tau))}{\Gamma(\alpha)}d\tau),\end{aligned}$$

and

$$\begin{aligned}\left|{}^cD^{\alpha,\Phi}\Im x(t)-^cD^{\alpha,\Phi}\Im y(t)\right| &\leq \int_0^t\frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\beta-1}}{\Gamma(\beta)}\\ &\quad\left|f\left(\tau,x(\tau),x(\lambda\tau),^cD^{\alpha,\Phi}x(\tau)\right)-f\left(\tau,y(\tau),y(\lambda\tau),^cD^{\alpha,\Phi}y(\tau)\right)\right|d\tau\\ &\quad+|g(t,x(t))-g(t,y(t))|+\left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right)\\ &\quad\times(\int_0^1|h(s,x(s))-h(s,y(s))|\,ds\\ &\quad+\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\\ &\quad\left|f\left(\tau,x(\tau),x(\lambda\tau),^cD^{\alpha,\Phi}x(\tau)\right)-f\left(\tau,y(\tau),y(\lambda\tau),^cD^{\alpha,\Phi}y(\tau)\right)\right|d\tau\\ &\quad+\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))-g(\tau,y(\tau))|}{\Gamma(\alpha)}d\tau).\end{aligned}$$

Thanks to $(\mathbf{H}_1)-(\mathbf{H}_3)$, we get

$$\begin{aligned}
&\left\|{}^{c}D^{\alpha,\Phi}\Im x-{}^{c}D^{\alpha,\Phi}\Im y\right\| \leq \left(2L_f\left(\|x-y\|\right)+M_f\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\\
&\times\int_0^1\frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\beta-1}}{\Gamma(\beta)}d\tau\\
&+L_g\|x-y\|+\left(\frac{\Gamma(\alpha+1)}{\left(\Phi(1)-\Phi(0)\right)^{\alpha}}\right)\times \qquad (3.8)\\
&\left(L_h\|x-y\|+\left(2L_f\left(\|x-y\|\right)+M_f\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\int_0^1\frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}d\tau\right.\\
&\left.+L_g\|x-y\|\int_0^1\frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha-1}}{\Gamma(\alpha)}d\tau\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left\|{}^{c}D^{\alpha,\Phi}\Im x-{}^{c}D^{\alpha,\Phi}\Im y\right\| \leq{}& \left(\frac{4L_f\left(\Phi(1)-\Phi(0)\right)^{\beta}}{\Gamma(\beta+1)}+2L_g+\left(\frac{\Gamma(\alpha+1)}{\left(\Phi(1)-\Phi(0)\right)^{\alpha}}\right)L_h\right)\\
&\|x-y\| \qquad (3.9)\\
&+\frac{2M_f\left(\Phi(1)-\Phi(0)\right)^{\beta}}{\Gamma(\beta+1)}\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|.
\end{aligned}$$

Finally, from (3.7)and (3.9), it is easy to see that

$$\begin{aligned}
\|\Im x-\Im y\|_E ={}& \|\Im x-\Im y\|+\left\|{}^{c}D^{\alpha,\Phi}\Im x-{}^{c}D^{\alpha,\Phi}\Im y\right\|\\
\leq{}& \left(\frac{4L_f\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{2L_g\left(\Phi(1)-\Phi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}+L_h\right.\\
&\left.+\frac{4L_f\left(\Phi(1)-\Phi(0)\right)^{\beta}}{\Gamma(\beta+1)}+2L_g+\left(\frac{\Gamma(\alpha+1)}{\left(\Phi(1)-\Phi(0)\right)^{\alpha}}\right)L_h\right)\times\|x-y\|\\
&+\left(\frac{2M_f\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{2M_f\left(\Phi(1)-\Phi(0)\right)^{\beta}}{\Gamma(\beta+1)}\right)\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|,\\
\leq{}& \left((4L_f+2M_f)\left(\frac{\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\left(\Phi(1)-\Phi(0)\right)^{\beta}}{\Gamma(\beta+1)}\right)\right.\\
&\left.+(2L_g)\left(\frac{\left(\Phi(1)-\Phi(0)\right)^{\alpha}+\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\right)+L_h\left(\frac{\Gamma(\alpha+1)+\left(\Phi(1)-\Phi(0)\right)^{\alpha}}{\left(\Phi(1)-\Phi(0)\right)^{\alpha}}\right)\right)\\
&\times\|x-y\|_E,\\
\leq{}& \left((4L_f+2M_f)\Delta_1+2L_g\Delta_2+L_h\Delta_3\right)\|x-y\|_E,
\end{aligned}$$

where,

$$\Delta_1=\frac{\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\left(\Phi(1)-\Phi(0)\right)^{\beta}}{\Gamma(\beta+1)},$$

$$\Delta_2 = \frac{(\Phi(1) - \Phi(0))^\alpha + \Gamma(\alpha+1)}{\Gamma(\alpha+1)}$$

and

$$\Delta_3 = \frac{\Gamma(\alpha+1) + (\Phi(1) - \Phi(0))^\alpha}{(\Phi(1) - \Phi(0))^\alpha}.$$

Since $\Omega < 1$, then $\Im$ is contraction mapping. Hence, by the BCP principle, we state that $\Im$ has a unique fixed point, which is the unique solution of (3.1). ■

3.4 One Pantograph Solution Via BCP Principle and Holder Inequality

The following main result deals with the existence of a unique solution of the studied problem by using both the BCP principle and Holder inequality [71]. To prove that result, we need the following hypothesis:

($\mathbf{H}_4$) The function $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists a function φ, such that

$$\left\| f\left(t, x, \tilde{x}, \tilde{\tilde{x}}\right) - f\left(t, y, \tilde{y}, \tilde{\tilde{y}}\right)\right\| \leq \varphi(t)\left(\|x-y\| + \|\tilde{x}-\tilde{y}\| + \left\|\tilde{\tilde{x}} - \tilde{\tilde{y}}\right\|\right),$$

where $t \in [0,1], x, y \in E, \varphi \in L^{\frac{1}{p}}\left([0,1], \mathbb{R}^+\right), p \in (0,1)$ and $\|\varphi\| = \left(\int_0^1 (\varphi(\tau))^{\frac{1}{p}} d\tau\right)^p$.

Theorem 3.2 *If the hypotheses* $(\mathbf{H}_2), (\mathbf{H}_3)$ *and* $(\mathbf{H}_4)$ *are satisfied and*

$$\begin{aligned} F := & \frac{6\|\varphi\|(\Phi(1) - \Phi(0))^{\alpha+\beta-p}}{\Gamma(\alpha+\beta)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p} \\ & + \frac{6\|\varphi\|(\Phi(1) - \Phi(0))^{\beta-p}}{\Gamma(\beta)}\left(\frac{1-p}{\beta-p}\right)^{1-p} \\ & + 2L_g\Delta_2 + L_h\Delta_3 < 1, \end{aligned} \tag{3.10}$$

then, the problem (3.1) *has a unique solution on* $[0,1]$.

Proof. Let $t \in [0,1]$. Then, we have

$$
\begin{aligned}
&\left|\Im x(t)-\Im y(t)\right| \leq \int_0^t \frac{\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
&\times \left|f\left(\tau, x(\tau), x(\lambda\tau), {}^{c}D^{\alpha,\Phi}x(\tau)\right)-f\left(\tau, y(\tau), y(\lambda\tau), {}^{c}D^{\alpha,\Phi}y(\tau)\right)\right| d\tau \\
&+\int_0^t \frac{\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\alpha-1}\left|g(\tau,x(\tau))-g(\tau,y(\tau))\right|}{\Gamma(\alpha)} d\tau \\
&+\left(\frac{\left(\Phi(t)-\Phi(0)\right)}{\left(\Phi(1)-\Phi(0)\right)}\right)^{\alpha}\Big(\int_0^1 \left|h(s,x(s))-h(s,y(s))\right| ds \\
&+\int_0^1 \frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
&\times \left|f\left(\tau, x(\tau), x(\lambda\tau), {}^{c}D^{\alpha,\Phi}x(\tau)\right)-f\left(\tau, y(\tau), y(\lambda\tau), {}^{c}D^{\alpha,\Phi}y(\tau)\right)\right| d\tau \\
&+\int_0^1 \frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha-1}\left|g(\tau,x(\tau))-g(\tau,y(\tau))\right|}{\Gamma(\alpha)} d\tau\Big).
\end{aligned}
$$

Using $(\mathbf{H}_2)-(\mathbf{H}_3)-(\mathbf{H}_4)$, we can write

$$
\begin{aligned}
&\left|\Im x(t)-\Im y(t)\right| \\
\leq\ & \left(2\|x-y\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\int_0^t \frac{\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\alpha+\beta-1}\varphi(\tau)}{\Gamma(\alpha+\beta)} d\tau \\
&+L_g\|x-y\|\int_0^t \frac{\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\alpha-1}}{\Gamma(\alpha)} d\tau + L_g\|x-y\| \\
&+\left(2\|x-y\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\int_0^1 \frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha+\beta-1}\varphi(\tau)}{\Gamma(\alpha+\beta)} d\tau \\
&+L_g\|x-y\|\int_0^1 \frac{\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha-1}}{\Gamma(\alpha)} d\tau.
\end{aligned}
$$

Thanks to Holder inequality, it yields that

$$
\begin{aligned}
\|\Im x-\Im y\| \leq\ & \frac{\left(2\|x-y\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)}{\Gamma(\alpha+\beta)}\left(\int_0^1\left(\Phi'(\tau)\left(\Phi(t)-\Phi(\tau)\right)^{\alpha+\beta-1}\right)^{\frac{1}{1-p}} d\tau\right)^{1-p} \\
&\times\left(\int_0^1 \left(\varphi(\tau)\right)^{\frac{1}{p}} d\tau\right)^{p} \\
&+\frac{2L_g\left(\Phi(1)-\Phi(0)\right)^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|+L_h\|x-y\| \qquad (3.11)\\
&+\frac{\left(2\|x-y\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)}{\Gamma(\alpha+\beta)}\left(\int_0^1\left(\Phi'(\tau)\left(\Phi(1)-\Phi(\tau)\right)^{\alpha+\beta-1}\right)^{\frac{1}{1-p}} d\tau\right)^{1-p} \\
&\times\left(\int_0^1 \left(\varphi(\tau)\right)^{\frac{1}{p}} d\tau\right)^{p}.
\end{aligned}
$$

Therefore,

$$\begin{aligned}\|\Im x-\Im y\| &\leq \frac{\left(2\|x-y\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)(\Phi(1)-\Phi(0))^{\alpha+\beta-p}}{\Gamma(\alpha+\beta)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}\|\varphi\| \\ &+\frac{2L_g(\Phi(1)-\Phi(0))^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|+L_h\|x-y\| \\ &+\frac{\left(2\|x-y\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)(\Phi(1)-\Phi(0))^{\alpha+\beta-p}}{\Gamma(\alpha+\beta)} \\ &\times\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}\|\varphi\|. \end{aligned} \tag{3.12}$$

On other hand, thanks to $(\mathbf{H}_2)-(\mathbf{H}_3)-(\mathbf{H}_4)$ and using the same arguments as in the proof of Theorem 3.1, we can write

$$\begin{aligned}&\left\|{}^{c}D^{\alpha,\Phi}\Im x-{}^{c}D^{\alpha,\Phi}\Im y\right\| \\ \leq\ &\left(2\|x-y\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\int_0^1\frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\beta-1}}{\Gamma(\beta)}\varphi(\tau)\,d\tau \\ &+L_g\|x-y\|+\left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right)(L_h\|x-y\|+\left(2\|x-y\|+^{c}\left\|D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\times \\ &\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\varphi(\tau)\,d\tau+L_g\|x-y\|\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}}{\Gamma(\alpha)}d\tau).\end{aligned}$$

The Holder inequality allows us to obtain

$$\begin{aligned}&\left\|{}^{c}D^{\alpha,\Phi}\Im x-{}^{c}D^{\alpha,\Phi}\Im y\right\|\leq\left(2\|x-y\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right) \\ &\times\frac{1}{\Gamma(\beta)}\left(\int_0^1\left(\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\beta-1}\right)^{\frac{1}{1-p}}d\tau\right)^{1-p}\left(\int_0^1(\varphi(\tau))^{\frac{1}{p}}d\tau\right)^{p} \\ &+L_g\|x-y\|+\left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right)(L_h\|x-y\|+\left(2\|x-y\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right) \\ &\times\frac{1}{\Gamma(\alpha+\beta)}\left(\int_0^1\left(\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}\right)^{\frac{1}{1-p}}d\tau\right)^{1-p}\left(\int_0^1(\varphi(\tau))^{\frac{1}{p}}d\tau\right)^{p} \\ &+L_g\|x-y\|\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}}{\Gamma(\alpha)}d\tau).\end{aligned}$$

Therefore, we can state that

$$\begin{aligned}\left\|{}^{c}D^{\alpha,\Phi}\Im x-{}^{c}D^{\alpha,\Phi}\Im y\right\| &\leq \left(2\|x-y\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\frac{(\Phi(1)-\Phi(0))^{\beta-p}}{\Gamma(\beta)}\left(\frac{1-p}{\beta-p}\right)^{1-p}\|\varphi\| \\ &+L_g\|x-y\|+\left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right)(L_h\|x-y\| \\ &+\left(2\|x-y\|+\left\|{}^{c}D^{\alpha,\Phi}x-{}^{c}D^{\alpha,\Phi}y\right\|\right)\frac{(\Phi(1)-\Phi(0))^{\alpha+\beta-p}}{\Gamma(\alpha+\beta)} \\ &\times\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}\|\varphi\|+\frac{L_g(\Phi(1)-\Phi(0))^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|).\end{aligned}$$

Consequently,

$$\left\| {}^cD^{\alpha,\Phi}\Im x - {}^c D^{\alpha,\Phi}\Im y\right\| \leq \left(2\|x-y\| + \left\|{}^cD^{\alpha,\Phi}x - {}^c D^{\alpha,\Phi}y\right\|\right)\frac{2\|\varphi\|\left(\Phi(1)-\Phi(0)\right)^{\beta-p}}{\Gamma(\beta)}\left(\frac{1-p}{\beta-p}\right)^{1-p}$$
$$+2L_g\|x-y\| + \left(\frac{\Gamma(\alpha+1)}{\left(\Phi(1)-\Phi(0)\right)^{\alpha}}\right)L_h\|x-y\|. \tag{3.13}$$

In view (3.12) and (3.13), we have

$$\begin{aligned}
\|\Im x - \Im y\|_E \leq & \left(\frac{4\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta-p}}{\Gamma(\alpha+\beta)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}\|\varphi\| + \frac{2L_g\left(\Phi(1)-\Phi(0)\right)^{\alpha}}{\Gamma(\alpha+1)} + L_h\right)\|x-y\| \\
& + \frac{2\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta-p}}{\Gamma(\alpha+\beta)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}\|\varphi\|\left\|{}^cD^{\alpha,\Phi}x - {}^c D^{\alpha,\Phi}y\right\| \\
& + \left(\frac{4\|\varphi\|\left(\Phi(1)-\Phi(0)\right)^{\beta-p}}{\Gamma(\beta)}\left(\frac{1-p}{\beta-p}\right)^{1-p} + 2L_g + \left(\frac{\Gamma(\alpha+1)}{\left(\Phi(1)-\Phi(0)\right)^{\alpha}}\right)L_h\right)\|x-y\| \\
& + \frac{2\|\varphi\|\left(\Phi(1)-\Phi(0)\right)^{\beta-p}}{\Gamma(\beta)}\left(\frac{1-p}{\beta-p}\right)^{1-p}\left\|{}^cD^{\alpha,\Phi}x - {}^c D^{\alpha,\Phi}y\right\|, \\
\leq & \left(\frac{6\|\varphi\|\left(\Phi(1)-\Phi(0)\right)^{\alpha+\beta-p}}{\Gamma(\alpha+\beta)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p} + \frac{6\|\varphi\|\left(\Phi(1)-\Phi(0)\right)^{\beta-p}}{\Gamma(\beta)}\left(\frac{1-p}{\beta-p}\right)^{1-p}\right. \\
& \left. + \left(\left(\Phi(1)-\Phi(0)\right)^{\alpha} + \Gamma(\alpha+1)\right)\left(\frac{2L_g}{\Gamma(\alpha+1)} + \frac{L_h}{\left(\Phi(1)-\Phi(0)\right)^{\alpha}}\right)\right)\|x-y\|_E.
\end{aligned}$$

Hence, $\Im$ is a contraction since we have already seen that $\digamma < 1$.
By the BCP principle, we confirm that $\Im$ has a unique fixed point, which is the unique solution of (3.1). ■

3.5 A Solution Via Darbo Theorem

Now, we prove an existence result for the problem (3.1) by Kuratowski MNC and Darbo fixed point theorem. We have:

Theorem 3.3 *Suppose that* $(\mathbf{H}_1)-(\mathbf{H}_3)$ *are valid. Then, the problem* (3.1) *has at least one solution on* $[0,1]$.

Proof. Let ϱ be a positive constant. We consider the set defined by: $B_\varrho = \{x \in E : \|x\|_E \leq \varrho\}$ and let

$$\sup_{t\in[0,1]} |f(t,0,0,0)| := N_f < \infty,$$

$$\sup_{t\in[0,1]} |g(t,0)| := N_g < \infty,$$

and

$$\sup_{t\in[0,1]} |h(t,0)| := N_h < \infty.$$

The set B_ϱ is a closed, bounded and convex of the Banach space E. The proof will be developed as follows:

Claim 1: We prove that, the operator $\Im$ maps the set B_ϱ into itself for any bounded set B_ϱ.
For $x \in B_\varrho$, and $t \in [0,1]$, we have

$$\begin{aligned}
|\Im x(t)| \leq & \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1}\left|f\left(\tau, x(\tau), x(\lambda\tau), {}^{c}D^{\alpha,\Phi}x(\tau)\right)\right|}{\Gamma(\alpha+\beta)}d\tau \\
&+\int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))|}{\Gamma(\alpha)}d\tau \\
&+\left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha}\Big(\int_0^1 |h(s,x(s))|\,ds \\
&+\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}\left|f\left(\tau, x(\tau), x(\lambda\tau), {}^{c}D^{\alpha,\Phi}x(\tau)\right)\right|}{\Gamma(\alpha+\beta)}d\tau \\
&+\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))|}{\Gamma(\alpha)}d\tau\Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
|\Im x(t)| \leq & \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
&\times\left|f\left(\tau, x(\tau), x(\lambda\tau), {}^{c}D^{\alpha,\Phi}x(\tau)\right)-f(\tau,0,0,0)\right|+|f(\tau,0,0,0)|\,d\tau \\
&+\int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))-g(\tau,0)|+|g(\tau,0)|}{\Gamma(\alpha)}d\tau \\
&+\left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha}\Big(\int_0^1 |h(s,x(s))-h(s,0)|+|h(s,0)|\,ds \\
&+\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
&\times\left|f\left(\tau, x(\tau), x(\lambda\tau), {}^{c}D^{\alpha,\Phi}x(\tau)\right)-f(\tau,0,0,0)\right|+|f(\tau,0,0,0)|\,d\tau \\
&+\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))-g(\tau,0)|+|g(\tau,0)|}{\Gamma(\alpha)}d\tau\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\Im x\| \leq & \frac{2(\Phi(1)-\Phi(0))^{\alpha+\beta}\left(2L_f\|x\|+M_f\left\|{}^{c}D^{\alpha,\Phi}x\right\|+N_f\right)}{\Gamma(\alpha+\beta+1)} \\
&+\frac{2(\Phi(1)-\Phi(0))^{\alpha}(L_g\|x\|+N_g)}{\Gamma(\alpha+1)}+L_h\|x\|+N_h.
\end{aligned} \tag{3.14}$$

On other hand, in view of $(\mathbf{H}_1)-(\mathbf{H}_3)$, we can state that

$$\begin{aligned}\left|{}^{c}D^{\alpha,\Phi}\Im x(t)\right| &\leq \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\beta-1}}{\Gamma(\beta)} \\ &\times\left|f\left(\tau,x(\tau),x(\lambda\tau),{}^{c}D^{\alpha,\Phi}x(\tau)\right)-f(\tau,0,0,0)\right|+|f(\tau,0,0,0)|\,d\tau \\ &+|g(\tau,x(\tau))-g(\tau,0)|+|g(\tau,0)| \\ &+\left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right)(\int_0^1 |h(s,x(s))-h(s,0)|+|h(s,0)|\,ds \qquad (3.15) \\ &+\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\ &\times\left|f\left(\tau,x(\tau),x(\lambda\tau),{}^{c}D^{\alpha,\Phi}x(\tau)\right)-f(\tau,0,0,0)\right|+|f(\tau,0,0,0)|\,d\tau \\ &+\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}|g(\tau,x(\tau))-g(\tau,0)|+|g(\tau,0)|}{\Gamma(\alpha)}d\tau).\end{aligned}$$

Then, we obtain

$$\begin{aligned}\left\|{}^{c}D^{\alpha,\Phi}\Im x\right\| &\leq \frac{(\Phi(1)-\Phi(0))^{\beta}\left(2L_f\|x\|+M_f\left\|{}^{c}D^{\alpha,\Phi}x\right\|+N_f\right)}{\Gamma(\beta+1)}+L_g\|x\|+N_g+\left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right) \\ &(L_h\|x\|+N_h+\frac{(\Phi(1)-\Phi(0))^{\alpha+\beta}\left(2L_f\|x\|+M_f\left\|{}^{c}D^{\alpha,\Phi}x\right\|+N_f\right)}{\Gamma(\alpha+\beta+1)} \qquad (3.16) \\ &+\frac{(\Phi(1)-\Phi(0))^{\alpha}(L_g\|x\|+N_g)}{\Gamma(\alpha+1)}).\end{aligned}$$

Thus, (3.14) and (3.16) imply that,

$$\begin{aligned}\|\Im x\|_E &\leq \frac{2(\Phi(1)-\Phi(0))^{\alpha+\beta}\left(2L_f\|x\|+M_f\left\|{}^{c}D^{\alpha,\Phi}x\right\|+N_f\right)}{\Gamma(\alpha+\beta+1)} \\ &+\frac{2(\Phi(1)-\Phi(0))^{\alpha}(L_g\|x\|+N_g)}{\Gamma(\alpha+1)}+L_h\|x\|+N_h. \\ &\frac{(\Phi(1)-\Phi(0))^{\beta}\left(2L_f\|x\|+M_f\left\|{}^{c}D^{\alpha,\Phi}x\right\|+N_f\right)}{\Gamma(\beta+1)}+L_g\|x\| \\ &+N_g+\left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}}\right)(L_h\|x\|+N_h \\ &+\frac{(\Phi(1)-\Phi(0))^{\alpha+\beta}\left(2L_f\|x\|+M_f\left\|{}^{c}D^{\alpha,\Phi}x\right\|+N_f\right)}{\Gamma(\alpha+\beta+1)} \\ &+\frac{(\Phi(1)-\Phi(0))^{\alpha}(L_g\|x\|+N_g)}{\Gamma(\alpha+1)}).\end{aligned}$$

This shows that

$$\begin{aligned}
\|\Im x\|_{E} \leq \Omega \|x\|_{E} &+ \frac{N_f (\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + 2N_g \left(\frac{(\Phi(1)-\Phi(0))^{\alpha}}{\Gamma(\alpha+1)} + 1 \right) \\
&+ N_h \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}} + 1 \right), \\
\leq\ & \Omega\varrho + \frac{N_f (\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + 2N_g \left(\frac{(\Phi(1)-\Phi(0))^{\alpha}}{\Gamma(\alpha+1)} + 1 \right) \\
&+ N_h \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}} + 1 \right), \\
\leq\ & \varrho,
\end{aligned}$$

where,

$$\varrho \geq \frac{\frac{N_f (\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + 2N_g \left(\frac{(\Phi(1)-\Phi(0))^{\alpha}}{\Gamma(\alpha+1)} + 1 \right) + N_h \left(\frac{\Gamma(\alpha+1)}{(\Phi(1)-\Phi(0))^{\alpha}} + 1 \right)}{1-\Omega}.$$

Consequently, this proves that $\Im$ transforms $B\varrho$ into itself.

Claim 2: The application $\Im$ is continuous.

To do this, let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in $B\varrho$, such that $x_n \to x$ when $n \to \infty$. Then, for all $t \in [0,1]$, we have

$$\begin{aligned}
&|\Im x_n(t) - \Im x(t)| \leq \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
&\times \left| f\left(\tau, x_n(\tau), x_n(\lambda\tau), {}^{c}D^{\alpha,\Phi}x_n(\tau)\right) - f\left(\tau, x(\tau), x(\lambda\tau), {}^{c}D^{\alpha,\Phi}x(\tau)\right) \right| d\tau \\
&+ \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1} |g(\tau, x_n(\tau)) - g(\tau, x(\tau))|}{\Gamma(\alpha)} d\tau \\
&+ \left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))} \right)^{\alpha} \\
&\times (\int_0^1 |h(s, x_n(s)) - h(s, x(s))| \, ds \\
&+ \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \\
&\times \left| f\left(\tau, x_n(\tau), x_n(\lambda\tau), {}^{c}D^{\alpha,\Phi}x_n(\tau)\right) - f\left(\tau, x(\tau), x(\lambda\tau), {}^{c}D^{\alpha,\Phi}x(\tau)\right) \right| d\tau \\
&+ \int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}}{\Gamma(\alpha)} |g(\tau, x_n(\tau)) - g(\tau, x(\tau))| \, d\tau),
\end{aligned}$$

Then

$$\|\Im x_n - \Im x\| \leq \left(\frac{4L_f (\Phi(1) - \Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{2L_g (\Phi(1) - \Phi(0))^{\alpha}}{\Gamma(\alpha+1)} + L_h \right) \|x_n - x\| + \left(\frac{2M_f (\Phi(1) - \Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \right) \left\| {}^cD^{\alpha,\Phi} x_n - {}^c D^{\alpha,\Phi} x \right\| . \tag{3.17}$$

Similarly, we have

$$\left\| {}^cD^{\alpha,\Phi} \Im x_n - {}^c D^{\alpha,\Phi} \Im x \right\| \leq \left(\frac{4L_f (\Phi(1) - \Phi(0))^{\beta}}{\Gamma(\beta+1)} + 2L_g + \left(\frac{\Gamma(\alpha+1)}{(\Phi(1) - \Phi(0))^{\alpha}} \right) L_h \right) \|x_n - x\| + \frac{2M_f (\Phi(1) - \Phi(0))^{\beta}}{\Gamma(\beta+1)} \left\| {}^cD^{\alpha,\Phi} x_n - {}^c D^{\alpha,\Phi} x \right\| , \tag{3.18}$$

(3.17) and (3.18) allows us to state that

$$\|\Im x_n - \Im x\|_E \leq \Omega \|x_n - x\|_E .$$

Therefore,

$$\|\Im x_n - \Im x\|_E \to 0, n \to \infty.$$

Hence, $\Im$ is a continuous operator over $B\varrho$.

Claim 3: We shall prove that $\Im(B\varrho)$ is equicontinuous.
Let $t_1, t_2 \in [0,1]; t_1 < t_2$. Since $f(t,x,y,z)$, $g(t,x)$ and $h(t,x)$ are continuous, they are bounded on the compact subset $[0,1] \times [-\varrho, \varrho] \times [-\varrho, \varrho] \times [-\varrho, \varrho]$. Put

$$\vartheta_f = \sup \{ |f(t,x,y,z)| : t \in [0,1], x,y,z \in [-\varrho, \varrho] \} ,$$

$$\vartheta_g = \sup \{ |g(t,x)| : t \in [0,1], x \in [-\varrho, \varrho] \} ,$$

and

$$\vartheta_h = \sup \{ |h(t,x)| : t \in [0,1], x \in [-\varrho, \varrho] \} .$$

Then, we have

$$
\begin{aligned}
& \left|\Im x\left(t_1\right)-\Im x\left(t_2\right)\right| \\
\leq\ & \int_0^{t_1} \frac{\Phi'(\tau)}{\Gamma(\alpha+\beta)} \\
& \times\left[\left(\Phi\left(t_1\right)-\Phi(\tau)\right)^{\alpha+\beta-1}-\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha+\beta-1}\right]\left|f\left(\tau, x(\tau), x(\lambda \tau),{}^c D^{\alpha, \Phi} x(\tau)\right)\right| d\tau \\
& +\int_{t_1}^{t_2} \frac{\Phi'(\tau)\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha+\beta-1}\left|f\left(\tau, x(\tau), x(\lambda \tau),{}^c D^{\alpha, \Phi} x(\tau)\right)\right|}{\Gamma(\alpha+\beta)} d\tau \\
& +\int_0^{t_1} \frac{\Phi'(\tau)\left[\left(\Phi\left(t_1\right)-\Phi(\tau)\right)^{\alpha-1}-\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha-1}\right]|g(\tau, x(\tau))|}{\Gamma(\alpha)} d\tau \\
& +\int_{t_1}^{t_2} \frac{\Phi'(\tau)\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha-1}|g(\tau, x(\tau))|}{\Gamma(\alpha)} d\tau \\
& +\frac{\left(\Phi\left(t_1\right)-\Phi(0)\right)^\alpha-\left(\Phi\left(t_2\right)-\Phi(0)\right)^\alpha}{(\Phi(1)-\Phi(0))^\alpha} \\
& \times\left(\int_0^1|h(s, x(s))| ds+\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}\left|f\left(\tau, x(\tau), x(\lambda \tau),{}^c D^{\alpha, \Phi} x(\tau)\right)\right|}{\Gamma(\alpha+\beta)} d\tau\right. \\
& \left.+\int_0^1 \frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}|g(\tau, x(\tau))|}{\Gamma(\alpha)} d\tau\right).
\end{aligned}
$$

Therefore, we can write

$$
\begin{aligned}
& \left|\Im x\left(t_1\right)-\Im x\left(t_2\right)\right| \leq \frac{\vartheta_f}{\Gamma(\alpha+\beta)} \int_0^{t_1} \Phi'(\tau)\left[\left(\Phi\left(t_1\right)-\Phi(\tau)\right)^{\alpha+\beta-1}-\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha+\beta-1}\right] d\tau \\
& +\frac{\vartheta_f}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} \Phi'(\tau)\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha+\beta-1} d\tau \\
& +\frac{\vartheta g}{\Gamma(\alpha)} \int_0^{t_1} \Phi'(\tau)\left[\left(\Phi\left(t_1\right)-\Phi(\tau)\right)^{\alpha-1}-\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha-1}\right] d\tau \\
& +\frac{\vartheta_g}{\Gamma(\alpha)} \int_{t_1}^{t_2} \Phi'(\tau)\left(\Phi\left(t_2\right)-\Phi(\tau)\right)^{\alpha-1} d\tau \\
& +\left(\frac{\left(\Phi\left(t_1\right)-\Phi(0)\right)^\alpha-\left(\Phi\left(t_2\right)-\Phi(0)\right)^\alpha}{(\Phi(1)-\Phi(0))^\alpha}\right)\left(\vartheta_h+\frac{\vartheta_f(\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\vartheta_g(\Phi(1)-\Phi(0))^\alpha}{\Gamma(\alpha+1)}\right), \\
\leq\ & \frac{2 \vartheta_f}{\Gamma(\alpha+\beta+1)}\left(\Phi\left(t_2\right)-\Phi\left(t_1\right)\right)^{\alpha+\beta}+\frac{2 \vartheta_g}{\Gamma(\alpha+1)}\left(\Phi\left(t_2\right)-\Phi\left(t_1\right)\right)^\alpha \\
& +\left(\frac{\left(\Phi\left(t_1\right)-\Phi(0)\right)^\alpha-\left(\Phi\left(t_2\right)-\Phi(0)\right)^\alpha}{(\Phi(1)-\Phi(0))^\alpha}\right)\left(\vartheta_h+\frac{\vartheta_f(\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\vartheta_g(\Phi(1)-\Phi(0))^\alpha}{\Gamma(\alpha+1)}\right).
\end{aligned}
$$

Hence,

$$
\left|\Im x\left(t_1\right)-\Im x\left(t_2\right)\right| \rightarrow 0 \text{ as } t_1 \rightarrow t_2 .
$$

Also, we can state that

$$\begin{aligned}\left|{}^{c}D^{\alpha,\Phi}\Im x(t_1) -{}^{c} D^{\alpha,\Phi}\Im x(t_2)\right| &\leq \frac{\vartheta_f}{\Gamma(\beta)}\int_0^{t_1}\Phi'(\tau)\left[(\Phi(t_1)-\Phi(\tau))^{\beta-1}-(\Phi(t_2)-\Phi(\tau))^{\beta-1}\right]d\tau \\ &+\frac{\vartheta_f}{\Gamma(\beta)}\int_{t_1}^{t_2}\Phi'(\tau)(\Phi(t_2)-\Phi(\tau))^{\beta-1}d\tau+[g(t_1,x(t_1))-g(t_2,x(t_2) \\ &\leq \frac{\vartheta_f}{\Gamma(\beta+1)}(\Phi(t_2)-\Phi(t_1))^{\beta}+\frac{\vartheta_f}{\Gamma(\beta)}\int_{t_1}^{t_2}\Phi'(\tau)(\Phi(t_2)-\Phi(\tau))^{\beta-1}d\tau \\ &+[g(t_1,x(t_1))-g(t_2,x(t_2))].\end{aligned}$$

By taking t_1 tends to t_2, then, the right-hand side of the last inequality tends to 0.

Consequently, $\Im(B\varrho)$ is equicontinuous.

Claim 4: We show that $\Im$ is a condensing operator.

Let $W \subset B\varrho$ and $t \in [0,1]$. So, we have:

$$\mu_E(\Im W(t)) = \mu_E(\Im x(t), x \in W),$$

where μ_E be the measure of noncompactness introduced on Definition 2.25.

Obviously,

$$\begin{aligned}\mu_E(\Im W(t)) &\leq \int_0^t \frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left\{\mu_E\left(f\left(\tau,x(\tau),x(\lambda\tau),{}^{c}D^{\alpha,\Phi}x(\tau)\right)\right),x\in W\right\}d\tau \\ &+\int_0^t\frac{\Phi'(\tau)(\Phi(t)-\Phi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\{\mu_E(g(\tau,x(\tau))),x\in W\}d\tau \\ &+\left(\frac{(\Phi(t)-\Phi(0))}{(\Phi(1)-\Phi(0))}\right)^{\alpha}(\int_0^1\{\mu_E(h(\tau,x(\tau))),x\in W\}ds \\ &+\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left\{\mu_E\left(f\left(\tau,x(\tau),x(\lambda\tau),{}^{c}D^{\alpha,\Phi}x(\tau)\right)\right),x\in W\right\}d\tau \\ &+\int_0^1\frac{\Phi'(\tau)(\Phi(1)-\Phi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\{\mu_E(g(\tau,x(\tau))),x\in W\}d\tau)), \\ &\leq \Omega\{\mu_E(x(\tau)),x\in W,\tau\in[0,1]\}, \\ &\leq \Omega\{\mu_E(W(\tau)),\tau\in[0,1]\}.\end{aligned}$$

Finally, we can state that

$$\mu_E(\Im W) \leq \Omega\mu_E(W).$$

Therefore, the operator $\Im$ is a contraction.

By Darbo fixed point theorem, the operator has a fixed point, which is a solution of (3.1).

■

3.6 Illustrative Examples

Example 3.1 *Let consider the following pantograph fractional problem:*

$$\begin{cases} {}^{c}D^{\frac{3}{5},\Phi}\left({}^{c}D^{\frac{4}{5},\Phi}x(t)+g(t,x(t))\right)=f\left(t,x(t),x(\lambda t),{}^{c}D^{\frac{4}{5},\Phi}x(t)\right), t\in[0,1], \\ x(0)=0, \\ x(1)=\int\limits_0^1 h(t,x(s))\,ds. \end{cases} \tag{3.19}$$

Let us define $\Phi(t):=2t^2+2t+2$.
In particular, Φ *is an increasing function on* $[0,1]$ *and* $\Phi'(t)$ *is continuous over* $[0,1]$.
By taking

$$f\left(t,x(t),x(\lambda t),{}^{c}D^{\frac{4}{5},\Phi}x(t)\right)=\frac{1}{55\exp(t^2+1)\left[1+\frac{\cos t}{(t^2+1)^2}+x(t)+x\left(\frac{2t}{5}\right)+{}^{c}D^{\frac{4}{5},\Phi}x(t)\right]},$$

$$g(t,x(t))=\frac{t^3-3}{100}x(t),$$

and

$$h(t,x(t))=\frac{1}{200}x(t),$$

we constat that

$$L_f=\frac{1}{101\exp(1)}, M_f=\frac{1}{101\exp(1)}, L_g=\frac{4}{100}, L_h=\frac{1}{200},$$

$$\begin{aligned} \Omega &= (4L_f+2M_f)\left(\frac{(\Phi(1)-\Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{(\Phi(1)-\Phi(0))^{\beta}}{\Gamma(\beta+1)}\right) \\ &\quad +(2L_g)\left(\frac{(\Phi(1)-\Phi(0))^{\alpha}+\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\right)+L_h\left(\frac{\Gamma(\alpha+1)+(\Phi(1)-\Phi(0))^{\alpha}}{(\Phi(1)-\Phi(0))^{\alpha}}\right) \\ &= \left(4\times\frac{1}{101\exp(1)}+2\times\frac{1}{101\exp(1)}\right)\left(\frac{(4)^{\frac{7}{5}}}{\Gamma\left(\frac{12}{5}\right)}+\frac{(4)^{\frac{4}{5}}}{\Gamma\left(\frac{4}{5}+1\right)}\right) \\ &\quad +\left(\frac{8}{100}\right)\left(\frac{(4)^{\frac{3}{5}}+\Gamma\left(\frac{3}{5}+1\right)}{\Gamma\left(\frac{3}{5}+1\right)}\right)+\frac{1}{200}\left(\frac{(4)^{\frac{3}{5}}+\Gamma\left(\frac{3}{5}+1\right)}{(4)^{\frac{3}{5}}}\right) \\ &= 0.486\,30<1. \end{aligned} \tag{3.20}$$

Hence, by Theorem 3.1*, we can state that this example has a unique solution on* $[0,1]$.

Example 3.2 *Let consider the following problem:*

$$\begin{cases} {}^{c}D^{\frac{2}{3},\Phi}\left({}^{c}D^{\frac{3}{4},\Phi}x(t)+g(t,x(t))\right)=f\left(t,x(t),x(\lambda t),{}^{c}D^{\frac{3}{4},\Phi}x(t)\right), t\in[0,1], \\ x(0)=0, \\ x(1)=\int\limits_0^1 h(t,x(s))\,ds. \end{cases} \tag{3.21}$$

Let us also define $\Phi(t) := t^3 + t$.
Therefe, Φ *is an increasing function over* $[0,1]$ *and* $\Phi'(t) := 3t^2 + 1 \neq 0$ *is continuous for all* $t \in [0,1]$*. Moreover, the function* $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ *defined by*

$$f\left(t, x(t), x\left(\frac{3}{5}t\right), {}^{c}D^{\frac{3}{4},\Phi}x(t)\right) = \frac{4 + x(t) + x\left(\frac{3t}{5}\right) + {}^{c}D^{\frac{3}{4},\Phi}x(t)}{\left(98\exp(t) + 2\cos(t^2) + \sin 2t\right)\left(1 + x(t) + x\left(\frac{3t}{5}\right) + {}^{c}D^{\frac{3}{4},\Phi}x(t)\right)},$$

is continuous. In addition, let

$$g(t, x(t)) = \frac{1}{20^2} + \frac{t^2}{4}\left(\frac{1}{10^2}\sin x(t)\right),$$

and

$$h(t, x(t)) = \frac{\cos \pi t}{6(2t+9)} + \frac{\sin x(t)}{36(4t+7)}.$$

Consequently, $(\mathbf{H}_1) - (\mathbf{H}_3)$ *are satisfied with*

$$L_f = \frac{1}{100}, M_f = \frac{1}{100}, L_g = \frac{1}{200}, L_h = 2.2487 \times 10^{-2}$$

and

$$\begin{aligned}
\Omega &= (4L_f + 2M_f)\left(\frac{(\Phi(1) - \Phi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{(\Phi(1) - \Phi(0))^{\beta}}{\Gamma(\beta+1)}\right) \\
&\quad + 2L_g\left(\frac{(\Phi(1) - \Phi(0))^{\alpha} + \Gamma(\alpha+1)}{\Gamma(\alpha+1)}\right) + L_h\left(\frac{\Gamma(\alpha+1) + (\Phi(1) - \Phi(0))^{\alpha}}{(\Phi(1) - \Phi(0))^{\alpha}}\right) \\
&= \left(\frac{4}{100} + \frac{2}{100}\right)\left(\frac{(2)^{\frac{2}{3}+\frac{3}{4}}}{\Gamma\left(\frac{2}{3}+\frac{3}{4}+1\right)} + \frac{(2)^{\frac{2}{3}}}{\Gamma\left(\frac{2}{3}+1\right)}\right) + \frac{1}{100}\left(\frac{(2)^{\frac{3}{4}} + \Gamma\left(\frac{3}{4}+1\right)}{\Gamma\left(\frac{3}{4}+1\right)}\right) \qquad (3.22) \\
&\quad + 2.2487 \times 10^{-2}\left(\frac{\Gamma\left(\frac{3}{4}+1\right) + (2)^{\frac{3}{4}}}{(2)^{\frac{3}{4}}}\right) \\
&= 0.29613 < 1.
\end{aligned}$$

Also, we have

$$N_f = \frac{1}{25}, N_g = \frac{1}{400}, N_h = \frac{1}{54},$$

and

$$\vartheta_f = \frac{1}{100}, \vartheta_g = \frac{1}{200}, \vartheta_h = \frac{1}{54} + \frac{1}{252}.$$

It follows by Theorem 3.3 that the example 3.2 has at least one solution on $[0,1]$.

Chapter 4

Existence of Solutions and Positive Solutions For a ϕ-Laplacian Fractional Multi Point Boundary Value Problem

4.1 Introduction

In this chapter, we consider the following problem [23] :

$$\begin{aligned} D^{\alpha}\left(\rho\left(t\right)\phi\left(D^{\beta}x\left(t\right)\right)\right)+q\left(t\right)f\left(t,x\left(t\right),D^{\beta}x\left(t\right)\right) &= 0, 0<t<1, \\ \phi\left(D^{\beta}x\left(0\right)\right) &= x\left(0\right)=0, \\ x'\left(1\right)+\sum_{i=1}^{k}\sigma_i x'\left(\zeta_i\right) &= 0, \end{aligned} \tag{4.1}$$

where $0<\alpha\leq 1$, $1\leq\beta<2$, D^{α}, D^{β} are the standard Caputo fractional derivatives $f\in C\left([0,1]\times\mathbb{R}^2,\mathbb{R}\right)$, $q\in C\left([0,1],\mathbb{R}\right)$, and $\sigma_i\in\mathbb{R}$, $\zeta_i\in(0,1)$, $\sum\limits_{i=1}^{k}\sigma_i x'\left(\zeta_i\right)<\infty$. We also suppose that $\phi:(-a,a)\to(-b,b); 0<a,b\leq+\infty$ is an increasing homeomorphism, with $\phi\left(0\right)=0$.

It's been frequently asked about the origin of the p-Laplace operator [25], and the answer was in the 18th century, when constructing a dependable water supply for rapidly expanding urban centers was a major challenge of hydrodynamics engineering. The need for water prompted several new directions in the study of hydrodynamic theory.

Due to the fast advancement of hydrology in the late 18th and early 19th centuries, new theoretical foundations and associated novel measuring techniques were necessary. The book written in 1804 by baron Gaspard Riche de Prony, one of the school's past directors, contained a substantial amount of this study, notably that conducted by French engineers intimately associated with the renowned "École des ponts et chaussées" in Paris. [80]. This book provides a very comprehensive description of French research on the movement of water via pipes and channels. Studies on filtration through soil, sand, and other similarly porous materials were established near the end of the 19th century. Some studies also deal with smaller (thinner) pipelines and hoses. All of the models used in this study are based mathematically on a one-dimensional space.

Civil engineers [29], theoretical engineers and applied mathematicians like Antoine de Chézy and de Prony [38], and mathematicians [63] make up the range of experts active in 18th-century research. The work by Jacob Bear [21] derives and formulates several intriguing mathematical issues in this field. Among these are unsaturated flow and fluid filtration through porous media. [21].

In [66], the author was the first one who introduced a ϕ-Laplacian differential equation when studying the turbulent flow in a porous medium. Hence, the research of fractional differential equations with ϕ-Laplacian operator has already become a focus in recent years, and it has developed very rapidly. It has gained popularity and importance due to its distinguished applications in diverse fields of science and engineering, such as viscoelasticity mechanics, non-Newtonian mechanics, electrochemistry, fluid mechanics, combustion theory, and material science, see ([34] , [56] , [96]).

According to the definition in the literature in ([11] , [17] , [68]), a ϕ-Laplacian operator is said to be singular when the domain of ϕ is finite. Also, we say that ϕ is bounded if its range is finite (i.e. $b < +\infty$) and unbounded if not. Related to this context, we can obtain the following models:

(A) If $a = b = +\infty$(regular unbounded), then we have:

$$\phi_p(x) = |x|^{p-2}x, p > 1,$$

and

$$\phi_p^{-1}(x) = \phi_q(x) = |x|^{q-2}x, \frac{1}{p} + \frac{1}{q} = 1.$$

(B) When $a < +\infty, b = +\infty$ (singular unbounded), then the operator is given by:

$$\phi(x) = \frac{x}{\sqrt{1 - x^2}}.$$

(C) At the end, when $a = +\infty, b < +\infty$ (regular bounded), then:

$$\phi(x) = \frac{x}{\sqrt{1 + x^2}}.$$

Let us now introduce some research papers that have motivated our work. We begin by [34] where the authors studied φ-Laplacian problem:

$$\begin{aligned}
(\varphi (u' (t)))' + f (t, u (t) , u (t)) &= 0, 0 < t < T, \\
u (0) &= A, \\
\phi (u' (0)) &= \tau u (T) + \sum_{i=1}^{k} \tau_i u (\zeta_i) ,
\end{aligned}$$

where $\varphi : (-a, a) \to (-b, b)(0 < a, b \leq +\infty)$ is an increasing homeomorphism such that $\varphi(0) = 0, \tau, \tau_i \in \mathbb{R}, \zeta_i \in (0, T), i = 1, 2, ..., k, A \geq 0$, and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. They proved that even if some of the τ and τ_i are negative, the boundary value problem with singular φ-Laplacian operator is always solvable, and the problem with a bounded φ-Laplacian operator has at least one positive solution.

In [56] H. Khan et al. used fixed-point theorems to study existence of positive solution and Hyers-Ulam stability of the following nonlinear singular fractional differential equation with p-Laplacian:

$$\begin{aligned} D^{\beta}\left(\varphi_p\left(D^{\varepsilon}x(t)\right)\right)+Q(t)f(t,x(t)) &= 0, 0<t<1, \\ \varphi_p\left(D^{\varepsilon}x(t)\right)^{(i)} &\mid_{t=0}=0,\ i=0,1,2...n-1, \\ x^{(j)}(0) &= x''(1)=0, j=0,1,3...n-1, \end{aligned}$$

where $D^{\alpha}, D^{\varepsilon}$ stand for Caputo fractional derivative with $n-1<\varepsilon,\beta\leq n$, n is a positive integer greater than or equal to 3,φ_p is the $p-$Laplacian operator, .f is a continuous function. In [96], the authors considered the following integral BVP of Caputo fractional differential equations with $p-$Laplacian operator and some parameters:

$$\begin{aligned} D^{\beta}\left(\varphi_p\left(D^{\alpha}x(t)\right)\right)+f\left(t,x(t),D^{\beta}x(t)\right) &= 0, t\in(0,1), \\ \varphi_p\left(D^{\alpha}x(0)\right)^{(i)} &= \varphi_p\left(D^{\alpha}x(1)\right)=0, i=1,2...m-1 \\ x(0)+x'(0) &= \int_0^1 g_0(s)x(s)ds+a, \\ x(1)+x'(1) &= \int_0^1 g_1(s)x(s)ds+b, \\ x^{(j)}(0) &= 0, j=2,3...n-1, \end{aligned}$$

where $1<n-1<\alpha<n, 1<m-1<\beta<m$, D^{α} and D^{β} are the Caputo fractional derivatives φ_p is the $p-$Laplacian operator. $g_{0,}g_1\in C\left[(0,1),\mathbb{R}^{+}\right]$, $f\in C\left[(0,1)\times\mathbb{R}^{+}\times\mathbb{R}^{+},\mathbb{R}^{+}\right]$ are given functions. $a,b>0$ are disturbance parameters.

Very recently in [11], G.M. Alqurishi investigated the question of existence of nonnegative solutions for the following fractional BVP involving $p-$Laplacian operator:

$$\begin{aligned} D^{\beta}\left(\varphi_p\left(D^{\alpha}u(t)\right)\right) &= h(t)f(u(t)), 0<t<1, \\ D^{\alpha}u(0) &= u(0)=0, \\ u(1) &= \lambda u(a), \\ D^{\alpha}u(1) &= \eta D^{\alpha}u(b), \end{aligned}$$

where D^{α}, D^{β} are the standard Riemann-Liouville derivatives with $1<\alpha,\beta\leq 2, a,b\in(0,1)$, $\lambda,\eta\geq 0$ and $p>1$, h is a measurable function on $[0,1]$, and the nonlinear term f is a continuous function.

It is to note that the importance of BVPs with multi-point boundary conditions is in their various applications in the study of a variety applied mathematics and physics, see [91] .

To the best of our knowledge, there are few papers devoted to the study of fractional differential equations with a ϕ-Laplacian operator using the Leggett-Williams theorem, which is a generalization of Krasnoselskii's original result and has been applied to study the nonlinear equation modelling certain infectious diseases.[14, 65].

4.2 Preliminaries

To prove the existence of solutions to (4.1), we need the following auxiliary Lemma.

Lemma 4.1 *Let* $y \in C([0,1])$*. Then the BVP*

$$D^{\alpha}\left(\rho(t)\phi\left(D^{\beta}x(t)\right)\right)+y(t) = 0, 0<t<1, \tag{4.2}$$

$$\phi\left(D^{\beta}x(0)\right) = x(0)=0, \tag{4.3}$$

$$x'(1)+\sum_{i=1}^{k}\sigma_i x'(\zeta_i) = 0, \tag{4.4}$$

has a unique solution:

$$x(t)=\int_0^1 G(t,s)\,\phi^{-1}\left(\frac{J^{\alpha}y(s)}{\rho(s)}\right)ds+H(t),$$

where

$$G(t,s)=\begin{cases}\frac{t(1-s)^{\beta-2}}{\Gamma(\beta-1)\left(1+\sum_{i=1}^{k}\sigma_i\right)}-\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}, 0\le s\le t\le 1,\\ \frac{t(1-s)^{\beta-2}}{\Gamma(\beta-1)\left(1+\sum_{i=1}^{k}\sigma_i\right)}, 0\le t\le s\le 1,\end{cases} \tag{4.5}$$

is the Green's function for the problem, and

$$H(t)=\frac{t}{\left(1+\sum_{i=1}^{k}\sigma_i\right)\Gamma(\beta-1)}\sum_{i=1}^{k}\sigma_i\int_0^{\zeta_i}(\zeta_i-s)^{\beta-2}\left(\phi^{-1}\left(\frac{J^{\alpha}y(s)}{\rho(s)}\right)\right)ds.$$

Proof. By the Lemmas 2.1, 2.2, we can reduce the equation of (4.1) to the following equivalent equation

$$\rho(t)\phi\left(D^{\beta}x(t)\right)=-J^{\alpha}y(t)-c_0, c_0\in\mathbb{R},$$

By (4.3), we get

$$c_0=0.$$

Moreover, we have

$$x(t)=-J^{\beta}\left(\phi^{-1}\left(\frac{J^{\alpha}y(t)}{\rho(t)}\right)\right)-d_0-d_1t, d_0,d_1\in\mathbb{R}.$$

So

$$x'(t)=-J^{\beta-1}\left(\phi^{-1}\left(\frac{J^{\alpha}y(t)}{\rho(t)}\right)\right)-d_1, d_1\in\mathbb{R}.$$

From the boundary conditions $(4.3), (4.4)$, we have

$$d_0=0, d_1=-\frac{1}{1+\sum_{i=1}^{k}\sigma_i}\left[J_1^{\beta-1}\left(\phi^{-1}\left(\frac{J^{\alpha}y(t)}{\rho(t)}\right)\right)+\sum_{i=1}^{k}\sigma_i J_{\zeta_i}^{\beta-1}\left(\phi^{-1}\left(\frac{J^{\alpha}y(t)}{\rho(t)}\right)\right)\right]$$

$$x(t) = -J^{\beta}\left(\phi^{-1}\left(\frac{J^{\alpha}y(t)}{\rho(t)}\right)\right) + \frac{t}{1+\sum_{i=1}^{k}\sigma_i}\int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)}\left(\phi^{-1}\left(\frac{J^{\alpha}y(s)}{\rho(s)}\right)\right)ds$$

$$+\frac{t}{1+\sum_{i=1}^{k}\sigma_i}\sum_{i=1}^{k}\sigma_i\int_0^{\zeta_i}\frac{(\zeta_i-s)^{\beta-2}}{\Gamma(\beta-1)}\left(\phi^{-1}\left(\frac{J^{\alpha}y(s)}{\rho(s)}\right)\right)ds \tag{4.6}$$

$$x(t) = \int_0^1 G(t,s)\,\phi^{-1}\left(\frac{J^{\alpha}y(s)}{\rho(s)}\right)ds + \frac{t}{\left(1+\sum_{i=1}^{k}\sigma_i\right)}\sum_{i=1}^{k}\sigma_i\int_0^{\zeta_i}\frac{(\zeta_i-s)^{\beta-2}}{\Gamma(\beta-1)}\left(\phi^{-1}\left(\frac{J^{\alpha}y(s)}{\rho(s)}\right)\right)ds. \tag{4.7}$$

■

In the next lemma, we present some properties of the Green functions G.

Lemma 4.2 *The function $G(t,s)$ defined by (4.5) satisfies the following properties:*
(i) $G(t,s)$ is a continuous function and $G(t,s) \geq 0$ for $(t,s) \in [0,1]\times[0,1]$;
(ii) $G(t,s) \leq \max_{0\leq s\leq 1} G(1,s) = \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)\left(1+\left|\sum_{i=1}^{k}\sigma_i\right|\right)}$;
(iii) $G(t,s) \geq \min_{\frac{1}{2}\leq t\leq 1} G(t,s) = \frac{(1-s)^{\beta-2}}{2\Gamma(\beta-1)\left(1+\left|\sum_{i=1}^{k}\sigma_i\right|\right)} - \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}$.

Proof. The proof is omitted. ■

Let X be the Banach space $X = C^{\beta-1}([0,1]) := \left\{x : x \in C([0,1]), D^{\beta-1}x \in C([0,1])\right\}$, endowed with the norm

$$\|x\|_{\beta-1} = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}\left|D^{\beta-1}x(t)\right|.$$

Define a cone by:

$$P = \left\{x \in X, x(t) \geq 0, D^{\beta-1}x(t) \geq 0\right\}.$$

To be able to investigate the above problem, we use the condition

$$1 + \sum_{i=1}^{k}\sigma_i \neq 0.$$

Also, we assume that ϕ is considered in the case (A).

($\mathbf{H}_1$): The function $f : [0,1]\times\mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

($\mathbf{H}_2$): The function $q : [0,1] \to \mathbb{R}^+$ is continuous and non vanishing on $[0,1]$, and there exits positive constants m, M such that

$$m \leq q(t) \leq M.$$

($\mathbf{H}_3$): The function $\rho \in C^1([0,1])$ not identical zero on any closed subinterval of and there exits positive constants l, L such that

$$l \leq \rho(t) \leq L.$$

4.3 Existence of at least one solution

Firstly, we show that the problem with ϕ-Laplacian operator is always solvable even in the case where some of the σ_i are negative by the use of Schauder fixed point theorem.

Define the integral operator $\Upsilon : X \to X$ by

$$\begin{aligned}
\Upsilon x(t) &= -\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \left(\phi^{-1} \left(\frac{\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} q(\tau) f\left(\tau, x(\tau), D^\beta x(\tau)\right) d\tau}{\rho(s)} \right) \right) ds \\
&+ \frac{t}{1+\sum_{i=1}^k \sigma_i} \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \left(\phi^{-1} \left(\frac{J^\alpha q(s) f\left(s, x(s), D^\beta x(s)\right)}{\rho(s)} \right) \right) ds \qquad (4.8) \\
&+ \frac{t}{1+\sum_{i=1}^k \sigma_i} \sum_{i=1}^k \sigma_i \int_0^{\zeta_i} \frac{(\zeta_i-s)^{\beta-2}}{\Gamma(\beta-1)} \left(\phi^{-1} \left(\frac{J^\alpha q(s) f\left(s, x(s), D^\beta x(s)\right)}{\rho(s)} \right) \right) ds.
\end{aligned}$$

$$\Upsilon x(t) = \int_0^1 G(t,s)\, \phi^{-1} \left(\frac{J^\alpha q(s) f\left(s, x(s), D^\beta x(s)\right)}{\rho(s)} \right) ds + H(t) \qquad (4.9)$$

and the $\beta - 1^{th}$ fractional derivative

$$\begin{aligned}
D^{\beta-1}\Upsilon x(t) &= -\int_0^t \phi^{-1} \left(\frac{\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} q(\tau) f\left(\tau, x(\tau), D^\beta x(\tau)\right) d\tau}{\rho(s)} \right) ds \\
&+ \frac{t^{2-\beta}}{\Gamma(3-\beta)\left(1+\sum_{i=1}^k \sigma_i\right)} \int_0^1 \frac{1}{(1-s)} \left(\phi^{-1} \left(\frac{J^\alpha q(s) f\left(s, x(s), D^\beta x(s)\right)}{\rho(s)} \right) \right) ds \quad (4.10) \\
&+ \frac{t^{2-\beta}}{\Gamma(3-\beta)\left(1+\sum_{i=1}^k \sigma_i\right)} \sum_{i=1}^k \sigma_i \int_0^{\zeta_i} \frac{1}{(\zeta_i - s)} \left(\phi^{-1} \left(\frac{J^\alpha q(s) f\left(s, x(s), D^\beta x(s)\right)}{\rho(s)} \right) \right) ds.
\end{aligned}$$

Lemma 4.3 *Let* $(\mathbf{H}_1) - (\mathbf{H}_3)$ *hold. Then* $\Upsilon : X \to X$ *is completely continuous.*

Proof. First, it is easy to check that $\Upsilon : X \to X$ is well defined in view of the continuity of the functions $G(t,s)$ and f. Let $D \subset X$ be bounded. Then there exists a positive constant $d > 0$ such that $\|x\|_{\beta-1} \leq d$, $x \in D$. Denote

$$R = \sup_{t\in[0,1], x\in D} \left| f\left(t, x(t), D^\beta x(t)\right)\right|. \qquad (4.11)$$

Then, we get

$$\begin{aligned} |\Upsilon x(t)| &\leq \left| \int_0^1 G(t,s)\,\phi^{-1}\left(\frac{\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} q(\tau) f\left(\tau, x(\tau), D^\beta x(\tau)\right) d\tau}{\rho(s)} \right) ds + H(t) \right| \\ &\leq \int_0^1 G(1,s)\,\phi^{-1}\left(\frac{\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left| q(\tau) f\left(\tau, x(\tau), D^\beta x(\tau)\right)\right| d\tau}{|\rho(s)|} \right) ds + |H(t)|. \end{aligned}$$

Thanks to (($\mathbf{H}_2$) and ($\mathbf{H}_3$), we get

$$|\Upsilon x(t)| \leq \phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l} \right) \left| \int_0^1 G(1,s)\,\phi^{-1}(s^\alpha)\,ds \right| + |H(t)|.$$

On other hand, using ($\mathbf{H}_2$) and ($\mathbf{H}_3$), we have

$$\begin{aligned} |H(t)| &= \left| \frac{t}{1+\sum_{i=1}^k \sigma_i} \sum_{i=1}^k \sigma_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{\beta-2}}{\Gamma(\beta-1)} \left(\phi^{-1}\left(\frac{J^\alpha q(s) f\left(s, x(s), D^\beta x(s)\right)}{\rho(t)} \right) \right) \right| \\ &\leq \frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l} \right)}{1+\sum_{i=1}^k \sigma_i} \left| \sum_{i=1}^k \sigma_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{\beta-2}}{\Gamma(\beta-1)} \left(\phi^{-1}(s^\alpha) \right) ds \right| \\ &< \infty \end{aligned}$$

and

$$\begin{aligned} \left| D^{\beta-1}\Upsilon x(t) \right| &\leq \phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l} \right) \int_0^1 \phi^{-1}(s^\alpha)\,ds + \frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l} \right)}{\Gamma(3-\beta)\left(1+\sum_{i=1}^k \sigma_i\right)} \int_0^1 \phi^{-1}(s^\alpha)\,ds \\ &\quad + \frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l} \right)}{\Gamma(3-\beta)\left(1+\sum_{i=1}^k \sigma_i\right)} \sum_{i=1}^k \sigma_i \int_0^{\zeta_i} \frac{\phi^{-1}(s^\alpha)}{(\zeta_i - s)} ds. \end{aligned}$$

Consequently, we get

$$
\begin{aligned}
\left|D^{\beta-1}\Upsilon x\left(t\right)\right| &\leq \frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)}{\alpha\left(q-1\right)+1}\left(\frac{\Gamma\left(3-\beta\right)\left(1+\sum\limits_{i=1}^{k}\sigma_i\right)+1}{\Gamma\left(3-\beta\right)\left(1+\sum\limits_{i=1}^{k}\sigma_i\right)}\right) \\
&\quad +\frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)}{\Gamma\left(3-\beta\right)\left(1+\sum\limits_{i=1}^{k}\sigma_i\right)}\sum_{i=1}^{k}\sigma_i\int\limits_0^{\zeta_i}\frac{\phi^{-1}\left(s^{\alpha}\right)}{\left(\zeta_i-s\right)}ds, \\
&< \infty.
\end{aligned}
$$

Hence $\Upsilon\left(D\right)$ is bounded.

For any $0\leq t_1\leq t_2\leq 1$ and $x\in D$, we have

$$
\begin{aligned}
\left|\Upsilon x\left(t_2\right)-\Upsilon x\left(t_1\right)\right| = \Bigg| &\int\limits_0^1\left(G\left(t_2,s\right)-G\left(t_1,s\right)\right)\phi^{-1}\left(\frac{\int\limits_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}q\left(\tau\right)f\left(\tau,x\left(\tau\right),D^{\beta}x\left(\tau\right)\right)d\tau}{\rho\left(s\right)}\right)ds\Bigg| \\
&+\frac{\left(t_2-t_1\right)}{1+\sum\limits_{i=1}^{k}\sigma_i}\sum_{i=1}^{k}\sigma_i\int\limits_0^{\zeta_i}\frac{\left(\zeta_i-s\right)^{\beta-2}}{\Gamma\left(\beta-1\right)}\left(\phi^{-1}\left(\frac{J^{\alpha}q\left(s\right)f\left(s,x\left(s\right),D^{\beta}x\left(s\right)\right)}{\rho\left(s\right)}\right)\right)ds|,
\end{aligned}
$$

so

$$
\begin{aligned}
\left|\Upsilon x\left(t_2\right)-\Upsilon x\left(t_1\right)\right| &\leq \phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)\int\limits_0^1\left|G\left(t_2,s\right)-G\left(t_1,s\right)\right|\phi^{-1}\left(s^{\alpha}\right)ds \\
&\quad +\frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)\left(t_2-t_1\right)}{1+\sum\limits_{i=1}^{k}\sigma_i}\left|\sum_{i=1}^{k}\sigma_i\int\limits_0^{\zeta_i}\frac{\left(\zeta_i-s\right)^{\beta-2}}{\Gamma\left(\beta-1\right)}\left(\phi^{-1}\left(s^{\alpha}\right)\right)ds\right|.
\end{aligned}
$$

Thus,

$$
\left|\Upsilon x\left(t_2\right)-\Upsilon x\left(t_1\right)\right|\to 0.
$$

for $x\in D, t_1\to t_2$.

Similarly,

$$\begin{aligned}\left|D^{\beta-1}\Upsilon x\left(t_{2}\right)-D^{\beta-1}\Upsilon x\left(t_{1}\right)\right| &\leq \phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)\left|\int_{0}^{t_{2}}\phi^{-1}\left(s^{\alpha}\right)ds-\int_{0}^{t_{1}}\phi^{-1}\left(s^{\alpha}\right)ds\right| \\ &+\frac{\left|t_{2}^{2-\beta}-t_{1}^{2-\beta}\right|}{\Gamma\left(3-\beta\right)\left(1+\sum_{i=1}^{k}\sigma_{i}\right)}\int_{0}^{1}\frac{\left(\phi^{-1}\left(\frac{J^{\alpha}q(s)f\left(s,x(s),D^{\beta}x(s)\right)}{\rho(s)}\right)\right)}{(1-s)}ds \\ &+\frac{\left|t_{2}^{2-\beta}-t_{1}^{2-\beta}\right|}{\Gamma\left(3-\beta\right)\left(1+\sum_{i=1}^{k}\sigma_{i}\right)}\sum_{i=1}^{k}\sigma_{i}\int_{0}^{\zeta_{i}}\frac{\left(\phi^{-1}\left(\frac{J^{\alpha}q(s)f\left(s,x(s),D^{\beta}x(s)\right)}{\rho(s)}\right)\right)}{\left(\zeta_{i}-s\right)}ds, \\ &\leq \frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)}{\alpha\left(q-1\right)+1}\left|t_{2}^{\alpha(q-1)+1}-t_{1}^{\alpha(q-1)+1}\right| \\ &+\frac{\left|t_{2}^{2-\beta}-t_{1}^{2-\beta}\right|\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)}{\Gamma\left(3-\beta\right)\left(1+\sum_{i=1}^{k}\sigma_{i}\right)}\int_{0}^{1}\frac{\phi^{-1}\left(s^{\alpha}\right)}{(1-s)}ds \\ &+\frac{\left|t_{2}^{2-\beta}-t_{1}^{2-\beta}\right|}{\Gamma\left(3-\beta\right)\left(1+\sum_{i=1}^{k}\sigma_{i}\right)}\sum_{i=1}^{k}\sigma_{i}\int_{0}^{\zeta_{i}}\frac{\phi^{-1}\left(s^{\alpha}\right)}{\left(\zeta_{i}-s\right)}ds, \\ &\to 0,\end{aligned}$$

for $x\in D, t_1\to t_2$.

That is to say, $\Upsilon\left(D\right)$ is equicontinuous. By the Arzela-Ascoli theorem [46], we state that $\Upsilon : X\to X$ is completely continuous. The proof is completed. ■

Theorem 4.1 *The problem* (4.1) *has at least one solution provided that* $((\mathbf{H}_1)-(\mathbf{H}_3))$ *hold and* $d>0$, *such that*

$$\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)\left(\frac{\Gamma\left(\alpha q-\alpha+1\right)}{\Gamma\left(\beta+\alpha q-\alpha\right)}+\frac{\left(\Gamma\left(3-\beta\right)+\left(\alpha\left(q-1\right)+1\right)\right)}{\left(\alpha\left(q-1\right)+1\right)\Gamma\left(3-\beta\right)}\right)=:d.$$

Proof. Since Υ is completely continuous; we have just to prove that $\Upsilon\left(D\right)\subset D$.

Let $x\in D$. We will show that $\Upsilon x\in D$. For each $t\in[0,1]$, we have

$$\left|\Upsilon x\left(t\right)\right|=\left|\int_{0}^{1}G\left(t,s\right)\phi^{-1}\left(\frac{\int_{0}^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}q\left(\tau\right)f\left(\tau,x\left(\tau\right),D^{\beta}x\left(\tau\right)\right)d\tau}{\rho\left(s\right)}\right)ds\right.$$

$$\left.+\frac{t}{1+\sum_{i=1}^{k}\sigma_{i}}\sum_{i=1}^{k}\sigma_{i}\int_{0}^{\zeta_{i}}\frac{\left(\zeta_{i}-s\right)^{\beta-2}}{\Gamma\left(\beta-1\right)}\left(\phi^{-1}\left(\frac{J^{\alpha}q\left(s\right)f\left(s,x\left(s\right),D^{\beta}x\left(s\right)\right)}{\rho\left(s\right)}\right)\right)ds\right|$$

$$\begin{aligned}
\left|\Upsilon x\left(t\right)\right| &\leq \frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)}{\Gamma\left(\beta-1\right)\left(1+\left|\sum\limits_{i=1}^{k}\sigma_i\right|\right)}\left(\int\limits_0^1\left(1-s\right)^{\beta-2}\phi^{-1}\left(s^{\alpha}\right)ds+\left|\sum_{i=1}^{k}\sigma_i\int\limits_0^{\zeta_i}\left(\zeta_i-s\right)^{\beta-2}\phi^{-1}\left(s^{\alpha}\right)ds\right|\right)\\
&\leq \frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)\Gamma\left(\alpha q-\alpha+1\right)}{\Gamma\left(\beta+\alpha q-\alpha\right)}.
\end{aligned}$$

On other hand, we have

$$\begin{aligned}
\left|D^{\beta-1}\Upsilon x\left(t\right)\right| &\leq \phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)\left(\int\limits_0^1\phi^{-1}\left(s^{\alpha}\right)ds+\frac{1}{\Gamma\left(3-\beta\right)\left(1+\sum\limits_{i=1}^{k}\sigma_i\right)}\int\limits_0^1\frac{\phi^{-1}\left(s^{\alpha}\right)}{\left(1-s\right)}ds\right)\\
&\quad+\frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)}{\Gamma\left(3-\beta\right)\left(1+\sum\limits_{i=1}^{k}\sigma_i\right)}\sum_{i=1}^{k}\sigma_i\int\limits_0^{\zeta_i}\frac{\phi^{-1}\left(s^{\alpha}\right)}{\left(\zeta_i-s\right)}ds. \qquad (4.12)
\end{aligned}$$

In particular,

$$\left|D^{\beta-1}\Upsilon x\left(t\right)\right| \leq \frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)\left(\Gamma\left(3-\beta\right)+\left(\alpha\left(q-1\right)+1\right)\right)}{\left(\alpha\left(q-1\right)+1\right)\Gamma\left(3-\beta\right)}.$$

Therefore,

$$\begin{aligned}
\left\|\Upsilon x\right\| &\leq \frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)\Gamma\left(\alpha q-\alpha+1\right)}{\Gamma\left(\beta+\alpha q-\alpha\right)}+\frac{\phi^{-1}\left(\frac{\frac{MR}{\Gamma(\alpha+1)}}{l}\right)\left(\Gamma\left(3-\beta\right)+\left(\alpha\left(q-1\right)+1\right)\right)}{\left(\alpha\left(q-1\right)+1\right)\Gamma\left(3-\beta\right)}\\
&\leq d.
\end{aligned}$$

Finally, we deduce that Υ has a fixed point x which is a solution of (4.1). ■

4.4 Existence of at least three positive solutions

In this section we use the Leggett-Williams fixed point theorem to prove the existence of at least three positive solutions.

So, for convenience, the following notation is introduced:

$$\Im_1=\frac{\left(\alpha q-\alpha+1\right)\Gamma\left(3-\beta\right)\Gamma\left(\beta+\alpha q-\alpha\right)}{\phi^{-1}\left(\frac{M}{l\Gamma(\alpha+1)}\right)\left(\left(\alpha q-\alpha+1\right)\Gamma\left(3-\beta\right)+\left(\Gamma\left(3-\beta\right)\Gamma\left(\beta+\alpha q-\alpha\right)+\Gamma\left(\beta+\alpha q-\alpha\right)\right)\right)}.$$

$$\Im_2=\frac{2^{\beta}\Gamma\left(\beta+1\right)}{\phi^{-1}\left(\frac{m}{2^{\alpha}L\Gamma(\alpha+1)}\right)\left(\beta-1\right)}.$$

$$\Im_3=\frac{\Gamma\left(\beta\right)\Gamma\left(3-\beta\right)}{\phi^{-1}\left(\frac{M}{l\Gamma(\alpha+1)}\right)\left(\Gamma\left(\beta\right)\Gamma\left(3-\beta\right)+\Gamma\left(\beta\right)+\Gamma\left(3-\beta\right)\right)}.$$

Theorem 4.2 *Let $f : [0,1] \times \mathbb{R}_+^2 \to \mathbb{R}^+$, $q : [0,1] \to \mathbb{R}^+$ and $\rho \in C^1([0,1], \mathbb{R}_*^+)$ be continuous functions. Suppose that there exist constants $0 < a < b < c$ such that the following assumptions hold:*
$(\mathbf{H}_4)$ $\sum\limits_{i=1}^{k} \sigma_i \geq 0$;
$(\mathbf{H}_5)$ $f(t,x,y) \leq \phi(c\Im_1)$ *for* $(t,x,y) \in [0,1] \times [0,c] \times [0,c]$;
$(\mathbf{H}_6)$ $f(t,x,y) > \phi(b\Im_2)$ *for* $(t,x,y) \in [\frac{1}{2},1] \times [b,c] \times [0,c]$;
$(\mathbf{H}_7)$ $f(t,x,y) \leq \phi(a\Im_3)$ *for* $(t,x,y) \in [0,1] \times [0,a] \times [0,c]$;
So, the BVP (4.1) *has at least three positive solutions* x_1, x_2 *and* x_3.

Proof. We will show that all the conditions of the Leggett-Williams fixed point theorem are satisfied for the operator Υ defined by (4.9).

Since $G(t,s) \geq 0$, for $s,t \in [0,1]$, we have $\Upsilon x(t) \geq 0$ for all $x \in P$. Hence, $\Upsilon : P \to P$.

For $x \in P_c$, we have $\|x\|_{\beta-1} < c$. By condition $(\mathbf{H}_5)$, we have

$$\begin{aligned}
\|\Upsilon x(t)\|_{\beta-1} &\leq \max_{0\leq t\leq 1} \Upsilon x(t) + \max_{0\leq t\leq 1} D^{\beta-1}\Upsilon x(t) \\
&\leq \frac{\phi^{-1}\left(\frac{\frac{M}{\Gamma(\alpha+1)}}{l}\right)}{\Gamma(\beta-1)\left(1+\sum\limits_{i=1}^{k}\sigma_i\right)} \int_0^1 (1-s)^{\beta-2}\phi^{-1}\left(s^\alpha f\left(s,x(s),D^\beta x(s)\right)\right)ds \\
&\quad + \frac{\phi^{-1}\left(\frac{\frac{M}{\Gamma(\alpha+1)}}{l}\right)}{\Gamma(\beta-1)\left(1+\sum\limits_{i=1}^{k}\sigma_i\right)} \sum_{i=1}^{k}\sigma_i \int_0^{\zeta_i} (\zeta_i-s)^{\beta-2}\phi^{-1}\left(s^\alpha f\left(s,x(s),D^\beta x(s)\right)\right)ds \\
&\quad + \phi^{-1}\left(\frac{\frac{M}{\Gamma(\alpha+1)}}{l}\right)\left(\int_0^1 \phi^{-1}\left(s^\alpha f\left(s,x(s),D^\beta x(s)\right)\right)ds + \frac{\int_0^1 \frac{\phi^{-1}\left(s^\alpha f\left(s,x(s),D^\beta x(s)\right)\right)}{(1-s)}ds}{\Gamma(3-\beta)\left(1+\sum\limits_{i=1}^{k}\sigma_i\right)}\right) \\
&\quad + \frac{\phi^{-1}\left(\frac{\frac{M}{\Gamma(\alpha+1)}}{l}\right)}{\Gamma(3-\beta)\left(1+\sum\limits_{i=1}^{k}\sigma_i\right)} \sum_{i=1}^{k}\sigma_i \int_0^{\zeta_i} \frac{\phi^{-1}\left(s^\alpha f\left(s,x(s),D^\beta x(s)\right)\right)}{(\zeta_i-s)}ds. \\
&\leq \frac{\phi^{-1}\left(\frac{\frac{M}{\Gamma(\alpha+1)}}{l}\right)}{\Gamma(\beta-1)} \int_0^1 (1-s)^{\beta-2}\phi^{-1}\left(s^\alpha f\left(s,x(s),D^\beta x(s)\right)\right)ds \\
&\quad + \phi^{-1}\left(\frac{\frac{M}{\Gamma(\alpha+1)}}{l}\right)\left(\int_0^1 \phi^{-1}\left(s^\alpha f\left(s,x(s),D^\beta x(s)\right)\right)ds + \frac{\int_0^1 \frac{\phi^{-1}\left(s^\alpha f\left(s,x(s),D^\beta x(s)\right)\right)}{(1-s)}ds}{\Gamma(3-\beta)}\right) \\
&\leq c.
\end{aligned}$$

Which implies that $\Upsilon : \overline{P_c} \to \overline{P_c}$.

If $x \in \overline{P_a}$, then by condition ($\mathbf{H}_7$), it yields that

$$\begin{aligned}
\|\Upsilon x\|_{\beta-1} &= \max_{0\le t\le 1} \Upsilon x(t) + \max_{0\le t\le 1} D^{\beta-1}\Upsilon x(t) \\
&\le \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)\left(1+\sum_{i=1}^k \sigma_i\right)} \phi^{-1}\left(\frac{\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} q(\tau) f\left(\tau, x(\tau), D^\beta x(\tau)\right) d\tau}{\rho(s)}\right) ds \\
&\quad + \frac{1}{1+\sum_{i=1}^k \sigma_i} \sum_{i=1}^k \sigma_i \int_0^1 \frac{(1-s)^{\beta-2}}{\Gamma(\beta-1)} \left(\phi^{-1}\left(\frac{J^\alpha q(s) f\left(s, x(s), D^\beta x(s)\right)}{\rho(s)}\right)\right) ds \\
&\quad + \int_0^1 \phi^{-1}\left(\frac{\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} q(\tau) f\left(\tau, x(\tau), D^\beta x(\tau)\right) d\tau}{\rho(s)}\right) ds + \frac{\int_0^1 \frac{\left(\phi^{-1}\left(\frac{J^\alpha q(s) f\left(s,x(s),D^\beta x(s)\right)}{\rho(s)}\right)\right)}{(1-s)} ds}{\Gamma(3-\beta)\left(1+\sum_{i=1}^k \sigma_i\right)} \\
&\quad + \frac{\sum_{i=1}^k \sigma_i \int_0^{\zeta_i} \frac{\left(\phi^{-1}\left(\frac{J^\alpha q(s) f\left(s,x(s),D^\beta x(s)\right)}{\rho(s)}\right)\right)}{(\zeta_i - s)} ds}{\Gamma(3-\beta)\left(1+\sum_{i=1}^k \sigma_i\right)}, \\
&< \frac{\phi^{-1}\left(\frac{M}{l\Gamma(\alpha+1)}\right)}{\Gamma(\beta-1)} \int_0^1 (1-s)^{\beta-2} \phi^{-1}(s^\alpha) \phi^{-1}\left(f\left(s, x(s), D^\beta x(s)\right)\right) ds \\
&\quad + \phi^{-1}\left(\frac{M}{l\Gamma(\alpha+1)}\right) \int_0^1 \phi^{-1}(s^\alpha) \phi^{-1}\left(f\left(s, x(s), D^\beta x(s)\right)\right) ds \\
&\quad + \frac{\phi^{-1}\left(\frac{M}{l\Gamma(\alpha+1)}\right)}{\Gamma(3-\beta)} \int_0^1 \phi^{-1}(s^\alpha) \phi^{-1}\left(f\left(s, x(s), D^\beta x(s)\right)\right) ds \\
&< \phi^{-1}\left(\frac{M}{l\Gamma(\alpha+1)}\right) \left(\frac{\Gamma(\beta)\Gamma(3-\beta) + \Gamma(\beta) + \Gamma(3-\beta)}{\Gamma(\beta)\Gamma(3-\beta)}\right) \\
&\quad \times \max_{(s,x)\in[0,1]\times[0,a]} \left\{\phi^{-1}\left(f\left(s, x(s), D^\beta x(s)\right)\right)\right\} \\
&\le a.
\end{aligned}$$

This shows that $\Upsilon : \overline{P_a} \to \overline{P_a}$, that is, $\|\Upsilon x\|_{\beta-1} \le a$ for $x \in \overline{P_a}$.

Finally, we show that (A_3) of Theorem 2.9 also holds. Assume that $x \in \overline{P_b}$, define a concave nonnegative continuous functional φ on P by $\varphi(x) = \min_{t\in[\frac{1}{2},1]} x(t)$, we choose $x(t) = \frac{b+c}{2}$ for $t \in [0,1]$. It is easy to see that $x \in P(\varphi; b, c)$ and $\varphi(x) = \varphi((b+c)/2) > b$; consequently, the set $\{x \in P(\varphi; b, c) \setminus \varphi(x) > b\} \neq \varnothing$.

Moreover, for $x \in P(\varphi; b, c)$, put $\theta = \frac{2^{\beta}\Gamma(\beta)}{\phi^{-1}\left(\frac{m}{2^{\alpha}L\Gamma(\alpha+1)}\right)\left(1-\frac{1}{\beta}\right)}$ then, we have

$$
\begin{aligned}
\varphi(\Upsilon x) &= \min_{\frac{1}{2}\leq t\leq 1} \Upsilon x(t) \\
&\geq \int_0^1 \min_{\frac{1}{2}\leq t\leq 1} G(t,s)\,\phi^{-1}\left(\frac{\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} q(\tau) f\left(\tau, x(\tau), D^{\beta}x(\tau)\right) d\tau}{\rho(s)}\right) ds \\
&\quad + \min_{\frac{1}{2}\leq t\leq 1} H(t) \\
&\geq \int_0^{\frac{1}{2}} \left(\frac{(1-s)^{\beta-2}}{2\Gamma(\beta-1)\left(1+\sum_{i=1}^{k}\sigma_i\right)} - \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\right)\phi^{-1}\left(\frac{\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} q(\tau) f\left(\tau, x(\tau), D^{\beta}x(\tau)\right) d\tau}{\rho(s)}\right) \\
&\quad + \frac{\frac{1}{2}}{1+\sum_{i=1}^{k}\sigma_i}\sum_{i=1}^{k}\sigma_i \int_0^{\frac{1}{2}} \frac{\left(\frac{1}{2}-s\right)^{\beta-2}}{\Gamma(\beta-1)}\left(\phi^{-1}\left(\frac{J^{\alpha}q(s) f\left(s, x(s), D^{\beta}x(s)\right)}{\rho(s)}\right)\right) ds \\
&\geq \frac{1}{2\Gamma(\beta-1)}\int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right)^{\beta-2}\left(\phi^{-1}\left(\frac{J^{\alpha}q(s) f\left(s, x(s), D^{\beta}x(s)\right)}{\rho(s)}\right)\right) ds \\
&\quad - \int_0^{\frac{1}{2}} \frac{\left(\frac{1}{2}-s\right)^{\beta-1}}{\Gamma(\beta)}\left(\phi^{-1}\left(\frac{J^{\alpha}q(s) f\left(s, x(s), D^{\beta}x(s)\right)}{\rho(s)}\right)\right) ds \\
&> \frac{b\phi^{-1}\left(\frac{m}{2^{\alpha}L\Gamma(\alpha+1)}\right)}{2\Gamma(\beta-1)}\int_0^{\frac{1}{2}}\left(\frac{1}{2}-s\right)^{\beta-2} ds - \frac{b\phi^{-1}\left(\frac{m}{2^{\alpha}L\Gamma(\alpha+1)}\right)}{\Gamma(\beta)}\int_0^{\frac{1}{2}}\left(\frac{1}{2}-s\right)^{\beta-1} ds \\
&= \frac{b\phi^{-1}\left(\frac{m}{2^{\alpha}L\Gamma(\alpha+1)}\right)}{2^{\beta}\Gamma(\beta)} - \frac{b\phi^{-1}\left(\frac{m}{2^{\alpha}L\Gamma(\alpha+1)}\right)}{2^{\beta}\Gamma(\beta+1)} \\
&= b\theta \times \frac{\phi^{-1}\left(\frac{m}{2^{\alpha}L\Gamma(\alpha+1)}\right)\left(1-\frac{1}{\beta}\right)}{2^{\beta}\Gamma(\beta)} \\
&= b.
\end{aligned}
$$

So all the conditions of theorem 2.8 are satisfied. Thus, Υ has at least three fixed points. So, the BVP (4.1) has at least three positive solutions x_1, x_2 and x_3 in $\overline{P_c}$, such that $\|x_1\|_{\beta-1} < a, \min_{\frac{1}{2}\leq t\leq 1}|x_2| > b, \|x_3\| > a$ with $\min_{\frac{1}{2}\leq t\leq 1}|x_3| < b$.

The proof is completed. ■

4.4.1 An example

Example 4.1 *Consider the following problem*

$$D^{\frac{2}{3}}\left(\rho(t)\,\phi\left(D^{\frac{3}{2}}x(t)\right)\right)+q(t)\,f\left(t,x(t),D^{\frac{3}{2}}x(t)\right) = 0, 0<t<1,$$
$$\phi\left(D^{\frac{3}{2}}x(0)\right) = x(0)=0, \qquad (4.13)$$
$$x'(1)+\sum_{i=1}^{k}\sigma_i x'(\zeta_i) = 0,$$

where

$$\rho(t)=\frac{20\sin^2 t}{t^2+2t+5}+\frac{2}{11},$$
$$q(t)=\frac{(7\pi+2)\cos^2 t}{2\sqrt{1+t^2}}+\frac{2}{9},$$

$$f : [0,1]\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+,$$
$$(t,x,y) \to \frac{\cos t}{10^5\exp\left(\frac{1}{1+x^2}\right)}+\frac{50}{\exp\left(\frac{200}{15+150x^2}\right)}$$

We can see clearly that the function f is continuous.

Let

$$p=\frac{5}{4}, q=5, k=3, \zeta_1=\frac{1}{3}, \zeta_2=\frac{2}{3}, \zeta_3=\frac{3}{4}, \sigma_1=\frac{1}{2}, \sigma_2=\frac{5}{2}, \sigma_3=3.$$

and

$$a=\frac{1}{7}, b=1, c=7$$

We can easily show that

$$\frac{2}{11}\leq\rho(t)\leq\frac{46}{11},$$
$$\frac{2}{9}\leq q(t)\leq\frac{9(7\pi+2)+4}{18},$$
$$\sum_{i=1}^{3}\sigma_i=\frac{1}{2}+\frac{5}{2}+3\geq 0;$$

$$\Im_1 = \frac{\left(\frac{8}{3}+1\right)\Gamma\left(3-\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+\frac{8}{3}\right)}{\left(\frac{\frac{9(7\pi+2)+4}{18}}{\frac{2}{11}\times\Gamma\left(\frac{2}{3}+1\right)}\right)^4\left(\left(\frac{8}{3}+1\right)\Gamma\left(3-\frac{3}{2}\right)+\Gamma\left(3-\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+\frac{8}{3}\right)+\Gamma\left(\frac{3}{2}+\frac{8}{3}\right)\right)} = 4.5546\times 10^{-8},$$

$$\Im_2=\frac{2^{\frac{3}{2}}\times\Gamma\left(\frac{3}{2}+1\right)}{\left(\frac{\frac{2}{9}}{2^{\frac{2}{3}}\times\frac{46}{11}\times\Gamma\left(\frac{2}{3}+1\right)}\right)^3\left(\frac{\frac{2}{9}}{2^{\frac{2}{3}}\times\frac{46}{11}\times\Gamma\left(\frac{2}{3}+1\right)}\right)\left(\frac{3}{2}-1\right)}=3.9768\times 10^6,$$

$$\Im_3=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(3-\frac{3}{2}\right)}{\left(\frac{\frac{9(7\pi+2)+4}{18}}{\frac{2}{11}\times\Gamma\left(\frac{2}{3}+1\right)}\right)^3\left(\frac{\frac{9(7\pi+2)+4}{18}}{\frac{2}{11}\times\Gamma\left(\frac{2}{3}+1\right)}\right)\left(\Gamma\left(\frac{3}{2}\right)\Gamma\left(3-\frac{3}{2}\right)+\Gamma\left(\frac{3}{2}\right)+\Gamma\left(3-\frac{3}{2}\right)\right)}=1.0001\times 10^{-8}.$$

So, one can check that the function f satisfies

$(\mathbf{H}_5)$ $f(t,x,y) = \frac{\cos t}{10^5 \exp\left(\frac{1}{1+x^2}\right)} + \frac{50}{\exp\left(\frac{200}{15+150x^2}\right)} \leq 8.296\,8\times 10^{-5} \leq \phi\left(c\Im_1\right) = \frac{\left(7\times 4.\,554\,6\times 10^{-8}\right)}{\left(7\times 4.\,554\,6\times 10^{-8}\right)^{\frac{3}{4}}} = 2.3762\times 10^{-2}$ for $(t,x,y)\in[0,1]\times[0,7]\times[0,7]$;

$(\mathbf{H}_6)$ $f(t,x,y) > 48.\,661 > \phi\left(b\Im_2\right) = \frac{3.976\,8\times 10^6}{\left(3.976\,8\times 10^6\right)^{\frac{3}{4}}} = 44.656$, for $(t,x,y)\in[\frac{1}{2},1]\times[1,7]\times[0,7]$;

$(\mathbf{H}_7)$ $f(t,x,y) \leq 7.779\,8\times 10^{-4} \leq \phi\left(a\Im_3\right) = \frac{\left(\frac{1.000\,1\times 10^{-8}}{7}\right)}{\left(\frac{1.000\,1\times 10^{-8}}{7}\right)^{\frac{3}{4}}} = 6.148\times 10^{-3}$ for $(t,x,y)\in[0,1]\times[0,\frac{1}{7}]\times[0,7]$.

That is to say that all the conditions of Theorem 4.2 hold. Thus, Theorem 4.2 implies that the example has at least three positive solutions x_1, x_2 and x_3 in $\overline{P_c}$, such that $\|x_1\|_{\beta-1} < \frac{1}{7}$, $\min_{\frac{1}{2}\leq t\leq 1}|x_2| > 1$, $\|x_3\| > \frac{1}{7}$ with $\min_{\frac{1}{2}\leq t\leq 1}|x_3| < 1$.

Numerical simulations

Now, we review a numerical method for the Caputo derivative. The behavior of the aforementioned fractional $\phi-$Laplacian multi point boundary value problem is next investigated for a set of parameters.

To do this, we must first acquire a simplified fractional differential system similar to the problem under study. We use a fourth-order Runge-Kutta integrator.

As we observe, the problem (4.1) can be reduced as follow:

$$\begin{aligned} D^{\alpha}\left(\rho\left(t\right)\phi\left(D^{\beta}x\left(t\right)\right)\right) &= -q\left(t\right)f\left(t,x\left(t\right),D^{\beta}x\left(t\right)\right), 0<t<1 \\ 0 &< \alpha\leq 1, 1\leq\beta<2. \end{aligned}$$

That is,

$$\begin{cases} D^{\beta}x\left(t\right) = \phi^{-1}\left(\frac{z(t)}{\rho(t)}\right), \\ D^{\alpha}z\left(t\right) = -q\left(t\right)f\left(t,x\left(t\right),D^{\beta}x\left(t\right)\right). \end{cases}$$

Consequently,

$$\begin{cases} D^{2}x\left(t\right) = D^{2-\beta}\left(\phi^{-1}\left(\frac{z(t)}{\rho(t)}\right)\right), \\ D^{1}z\left(t\right) = -D^{1-\alpha}q\left(t\right)f\left(t,x\left(t\right),D^{\beta}x\left(t\right)\right). \end{cases}$$

Then,

$$\begin{cases} D^{1}y\left(t\right) = x\left(t\right), \\ D^{1}x\left(t\right) = D^{2-\beta}\left(\phi^{-1}\left(\frac{z(t)}{\rho(t)}\right)\right), \\ D^{1}z\left(t\right) = -D^{1-\alpha}q\left(t\right)f\left(t,x\left(t\right),D^{\beta}x\left(t\right)\right). \end{cases}$$

Taking into account the condition described in (4.13) above, the fourth-order Runge-Kutta method and the Caputo approach are combined to create numerical simulations. Figures show behaviors plotted:

Remark 4.1 *In summarizing our results, we find that the cases are consistent..*

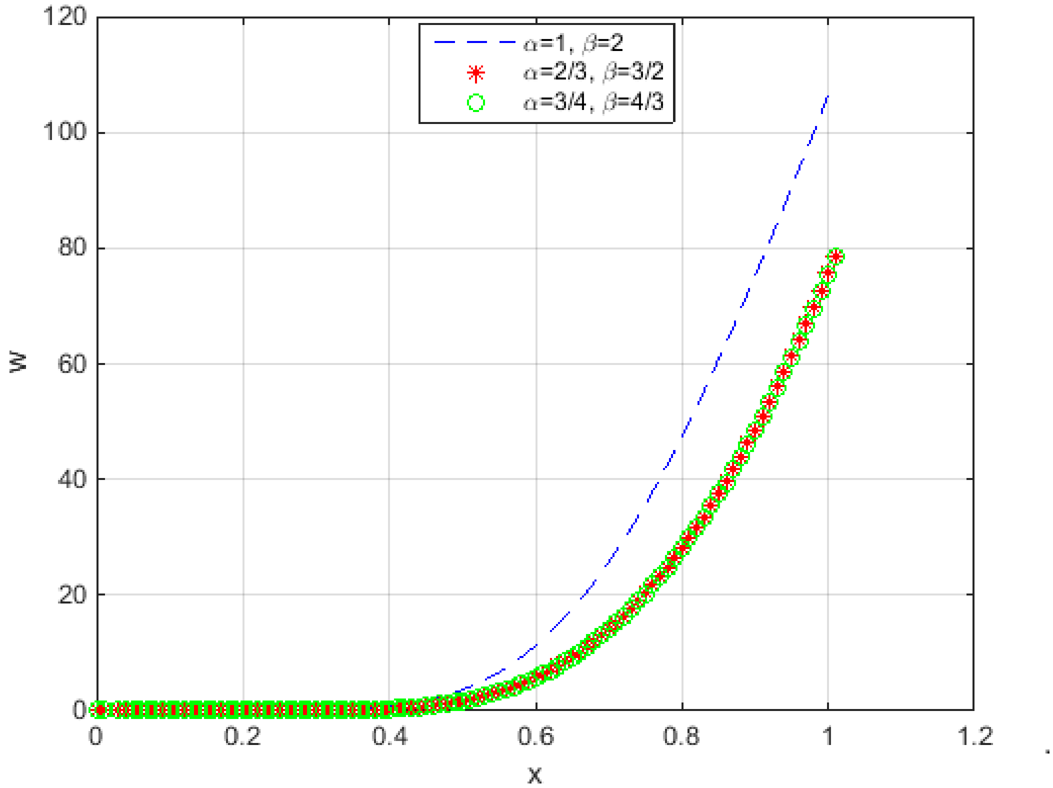

Figure 4.1: Numerical simulations for the example(4.1) for different values of α and β.

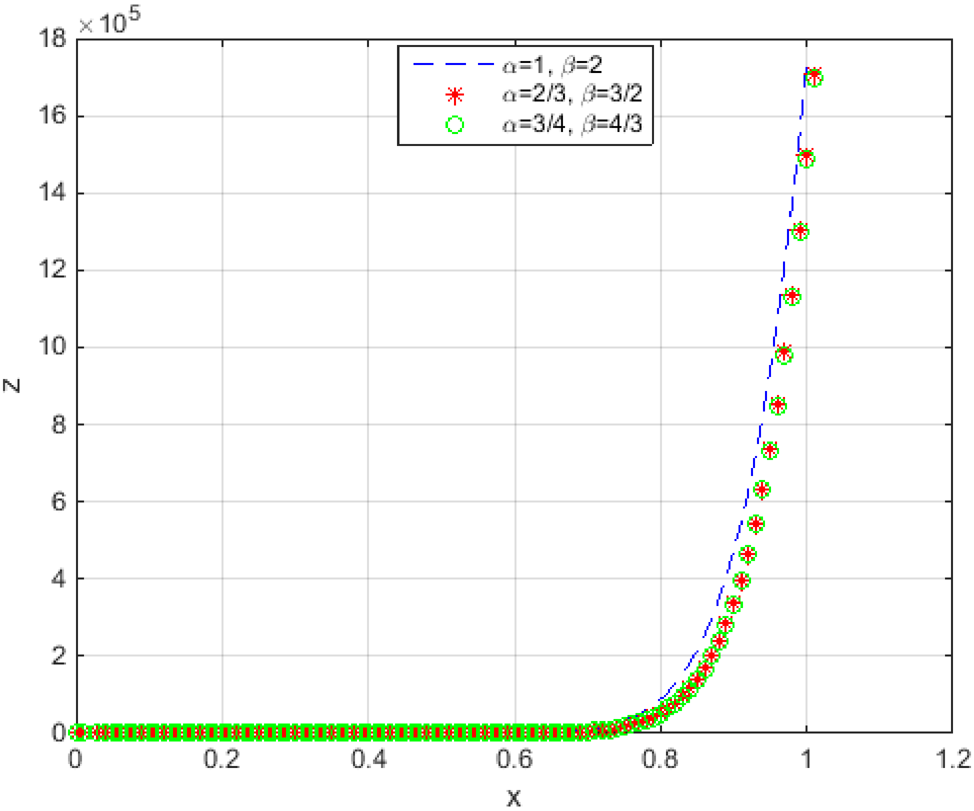

Figure 4.2: Numerical simulations for the example(4.1) for different values of α and β.

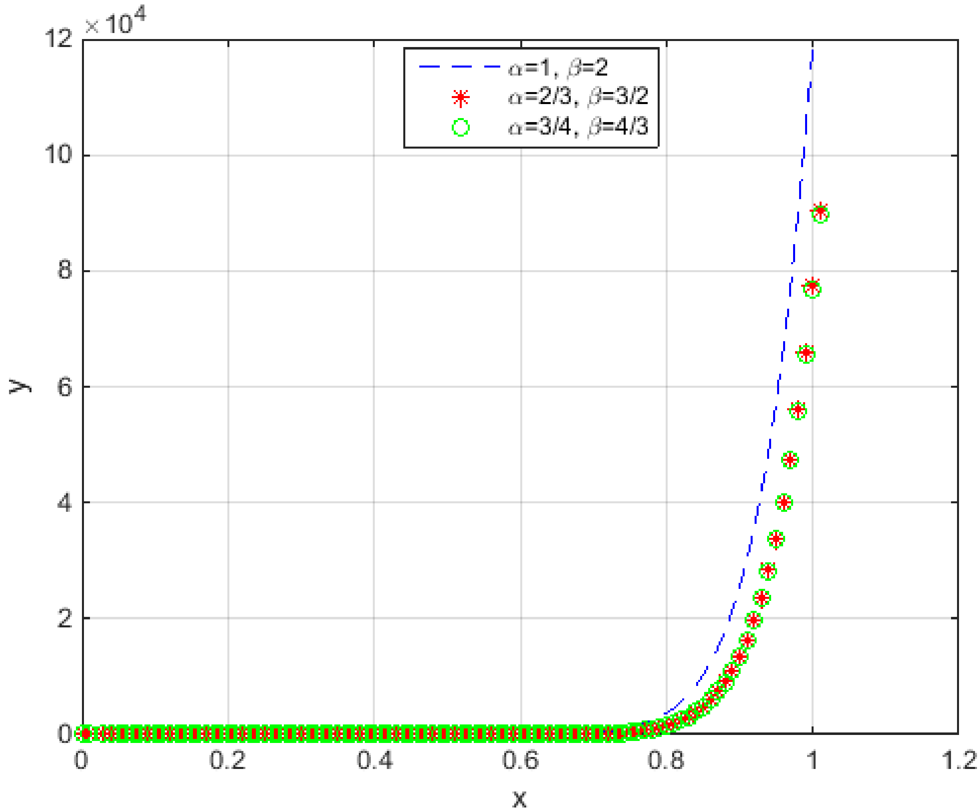

Figure 4.3: Numerical simulations for the example(4.1) for different values of α and β.

Chapter 5

Conclusion and future study

The work accomplished in the frame of this thesis has been a significant enrichment of our knowledge about some concerns the spirit of the fractional calculus and fixed point theory. The treated issues have given us the opportunity to take part, even modestly, in the development of this popular and useful field of mathematics.

Our objective contributions in this thesis research, focused on the existence and uniqueness of solutions for two different kind of nonlinear fractional differential equations involving different types of fractional derivatives and integrals such Caputo and $\Phi-$Caputo derivative.

After presenting the necessary preliminary concepts useful to well understanding the present work, we have developed in the third chapter the study of a sequential Pantograph fractional differential equation using the $\Phi-$Caputo derivative.

In this chapter, we defined the Kuratowski measure of noncompactness on a bounded subset of a Banach space to obtain the existence result by employing the Darbo fixed point theorem.

Further, we investigated sufficient conditions for existence and uniqueness of solutions for the considered problems by the help of classical fixed point theory, such as Banach contraction type and Holder inequality.

in the fourth chapter, we have exposed our contribution by settled some satisfactory conditions for existence of triple solutions and properties of Green's function are provided for a $\varphi-$Laplacian fractional multi point boundary value problem using the Leggett-Williams theorem and Schauder fixed point theorem. The obtained results have been verified by constructed an appropriate examples and numerical simulations.

Beyond the themes that we have approached in this thesis, and motivated by the huge success of the application of fractional calculus in general and fractional differential equations in particularly in many areas and by the unsolved problems in this theory, our attention can be directed to many of perspectives and possible generalizations, it would be interesting to extend the results of the second chapter using the $\Phi-$generalized Caputo $K-$fractional derivatives [75] with another technique, other fixed point theorem, new measures of noncompactness why not other inequalities and determine the conditions that befit closer to obtain the better results.

We consider also a more generalized $p-$Laplacian fractional order differential equation on the whole line coupled with integral type boundary conditions.

To the best of our knowledge, there are no papers that consider the fractional boundary value problems with the $p-$Laplacian operator and integral conditions on the whole real line.

Finding an interpretation of this research subject in real life seems to be very difficult, which is why we are looking forward to working with these aspects in our future investigations.

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