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Theme

Degenerate Anisotropic Elliptic Systems with Variable exponents and L^m Data

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Abstract

In a bounded open domain $\Omega \subset \mathbb{R}^N$, where $N \geq 2$, with Lipschitz boundary $\partial\Omega$, we consider the Dirichlet problem for the elliptic systems given by

$$\begin{cases} -\sum_{i=1}^{N} D_i\left(a_i(x, u(x), D_i u(x))\right) + F(x, u) = f(x), & x \quad \in \Omega, \\ u(x) = 0, & x \quad \in \partial\Omega, \end{cases} \tag{0.0.1}$$

where $a_i(x, u, D_i u) = \dfrac{|D_i u|^{p_i(x)-2} D_i u}{(1+|u|)^{\gamma(x)}}$, and the exponents $\gamma(\cdot) > 0$, $p_i(\cdot), i = 1, ..., N$ are continuous functions, here, $u : \Omega \to \mathbb{R}^d$, $d \geq 2$, represents a vector-valued function, $D_i u = \dfrac{\partial u}{\partial x_i}$ denotes the partial derivative of u with respect to x_i, and the vectors fields $a_i : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ are Carathédory functions.

In this thesis, we focus on nonlinear degenerate anisotropic elliptic systems with variable growth and L^m data. Specifically, the differential operator $A(u) = -\sum_{i=1}^{N} D_i\left(a_i(x, u, D_i u)\right)$ of the type pseudo-monotone, which is well defined between $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega; \mathbb{R}^d)$ and its dual space, also it is not coercive if u large. Moreover, we consider the case where the right-hand side term f belongs to $L^m(\Omega; \mathbb{R}^d)$.

On the other hand, to analyze these systems, we work with an appropriate functional setting that involves anisotropic Sobolev spaces with variable exponents $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega; \mathbb{R}^d)$ and weak Lebesgue (Marcinkiewicz) spaces with variable exponents $\mathcal{M}^{p(\cdot)}(\Omega; \mathbb{R}^d)$.

keywords : Elliptic systems, Degenerate coercivity, Anisotropic Sobolev spaces, Weak Lebesgue spaces, Variable exponents, Distributional solutions, Structural conditions, L^m Data.

Résumé

Dans un domaine ouvert borné $\Omega \subset \mathbb{R}^N$, où $N \geq 2$, avec une frontière de Lipschitz $\partial\Omega$, nous considérons le problème de Dirichlet pour les systèmes elliptiques donnés par

$$\begin{cases} -\sum_{i=1}^{N} D_i\left(a_i(x, u(x), D_i u(x))\right) + F(x,u) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{0.0.2}$$

où, $a_i(x,u,D_iu) = \dfrac{|D_iu|^{p_i(x)-2}D_iu}{(1+|u|)^{\gamma(x)}}$, et les exposants $\gamma(.) > 0$, $p_i(\cdot), i = 1, ..., N$ sont des fonctions continues, ici, $u : \Omega \to \mathbb{R}^d$, $d \geq 2$, représente une fonction vectorielle, $D_iu = \dfrac{\partial u}{\partial x_i}$ désigne la dérivée partielle de u par rapport à x_i, et les champs de vecteurs $a_i : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ et $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ sont des fonctions de Carathédory.

Dans cette thèse, nous nous concentrons sur les systèmes elliptiques anisotropes non linéaires dégénérés avec une croissance variable et des données L^m. Plus précisément, l'opérateur différentiel $A(u) = -\sum_{i=1}^{N} D_i\left(a_i(x,u,D_iu)\right)$ du type pseudo-monotone, qui est bien défini entre $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$ et son espace dual, n'est pas coercitif si u est grand. De plus, nous considérons le cas où le terme du côté droit f appartient à $L^m(\Omega;\mathbb{R}^d)$. D'autre part, pour analyser ces systèmes, nous travaillons avec un cadre fonctionnel approprié qui implique des espaces de Sobolev anisotropes avec des exposants variables $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$ et des espaces de Lebesgue faibles (Marcinkiewicz) avec des exposants variables $\mathcal{M}^{p(\cdot)}(\Omega;\mathbb{R}^d)$.

Mots-clés : Systèmes elliptiques, Coercitivité dégénérée, Espaces de Sobolev anisotropes, Espaces de Lebesgue faibles, Exposants variables, Solutions distributionnelles, Conditions structurelles, Données L^m.

Contents

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Notations

Everywhere in the sequel we use the following notations:

- $\mathbb{R}^N$: The N-dimensional Euclidean space with the distance $|x| = \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}}$, where $x = (x_1, x_2, ..., x_N)$ is an element in $\mathbb{R}^N$.
- $\mathbb{M}^{d\times N}$: the real vector space of $d \times N$ matrices.
- Ω: open bounded set in $\mathbb{R}^N$.
- $\overline{\Omega}$: closure of Ω in $\mathbb{R}^N$
- $\partial\Omega$: boundary of Ω.
- $|A|$ or meas(A): Lebesgue measure of the subset A.
- a.e.: abbreviation for almost everywhere (with respect to the Lebesgue measure).
- $V^{'}$: the dual space of V, where V is a Banach space.
- $\langle \cdot, \cdot \rangle$: the duality pairing between V and $V^{'}$.
- $D_i = \dfrac{\partial}{\partial x_i}$: the partial derivative with respect to x_i.
- $\nabla u = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, ..., \dfrac{\partial}{\partial x_N}\right)$: the gradient of u.
- χ_A: the characteristic function of a measurable set A.
- $C(\Omega)$: the space of continuous real-valued functions on Ω.

- $C_0^k(\Omega)$: the space of k times differentiable functions on Ω with continuity 0 on Ω.

- $C_0^\infty(\Omega)$: the space of smooth functions with compact support in Ω.

- $C_+^0(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \min\limits_{x\in\overline{\Omega}} p(x) > 0 \text{ for all } x \in \overline{\Omega} \right\}$.

- $C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \min\limits_{x\in\overline{\Omega}} p(x) > 1 \text{ for all } x \in \overline{\Omega} \right\}$.

- $p^+ = \max\limits_{x\in\overline{\Omega}} p(x)$, and $p^- = \min\limits_{x\in\overline{\Omega}} p(x)$ for $p \in C_+^0(\overline{\Omega})$.

- $p'(\cdot) = \dfrac{p(\cdot)}{p(\cdot)-1}$: the Hölder conjugate exponent of $p \in C_+(\overline{\Omega})$.

- $p^*(\cdot) = \dfrac{Np(\cdot)}{N-p(\cdot)}$ if $1 \leq p(\cdot) < N$, the Sobolev critical exponent of $p \in C(\overline{\Omega})$.

- $L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \,\text{measurable}, \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} < \infty \,\text{for some}\, \lambda \,\text{positive} \right\}$, variable exponent Lebesgue space.

- $\mathcal{M}^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \,\text{measurable}, \sup\limits_{\lambda>0} \lambda \|\chi_{\{|u|>\lambda\}}\|_{L^{p(\cdot)}(\Omega)} < \infty \right\}$, variable exponent weak Lebesgue space.

- $W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega),\ |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$,where $p(\cdot) \in C(\overline{\Omega})$ and $p \geq 1$, variable exponent Sobolev space.

- $D_0^{1,p(\cdot)}(\Omega)$: closure of $C_0^\infty(\Omega)$ with respect to $W^{1,p(\cdot)}(\Omega)$ norm.

- $\left(W^{1,p(\cdot)}(\Omega)\right)'$: the dual space of $W^{1,p(\cdot)}(\Omega)$.

- $\dfrac{1}{\overline{p}(x)} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \dfrac{1}{p_i(x)}$: the harmonic mean of $p_i(x)$.

- For $\overrightarrow{p}(\cdot) = (p_1(\cdot), ..., p_N(\cdot)) \in C\left(\overline{\Omega}\right)$, we set:

  $p_+(x) = \max\{p_1(x), ..., p_N(x)\}$, $p_-(x) = \min\{p_1(x), ..., p_N(x)\}$, $x \in \Omega$.

  $p_+^+ = \max\left\{p_1^+, ..., p_N^+\right\}$, $p_+^- = \max\left\{p_1^-, ..., p_N^-\right\}$, and $p_-^- = \min\left\{p_1^-, ..., p_N^-\right\}$.

- $W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{ u \in L^{p_+(\cdot)}(\Omega) : D_i u \in L^{p_i(\cdot)}(\Omega), i = 1, ..., N \right\}$, the anisotropic variable exponent Sobolev space.

- $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

- $D_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)}$.

- $\left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)\right)'$: the dual space of $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

- $W^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$: the $\mathbb{R}^d$- valued version of $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

- $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$: the $\mathbb{R}^d$- valued version of $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

- $\lambda_g(\cdot) : k \mapsto |\{x \in \Omega : |g| > k\}|\,,\ \forall\, k \geq 0$, the distribution function of g.

General introduction and thesis overview

In recent years, there has been significant development in the field of variable exponent spaces. The emergence of variable exponent Lebesgue spaces dates back to 1931, as documented by W. Orlicz [69]. However, the contemporary evolution commenced with the seminal paper by Kováčik and Ráksoník in 1991 [56]. This work primarily delves into fundamental aspects, including reflexivity, separability, duality, and initial findings related to the embedding and density of smooth functions. Subsequently, these spaces found application in examining functionals in the calculus of variations exhibiting non-standard growth. The comprehensive development of the theory of variable exponent spaces is presented in [28, 31, 38, 41, 44, 79]. Refer to the extensive books [6, 23, 34] for an overview.

One of the reasons for the rapid development of variable exponent function spaces theory can be attributed, in part, to the paradigm of electrorheological fluids proposed by Rajagopal and Růžička [73, 72, 76]. This model necessitates a functional framework incorporating function spaces with varying exponents. To clarify, we present the following model

$$\begin{cases} -\mathrm{div} S + \mathrm{div}\,(v \otimes v) + \nabla \pi = g + [\nabla E]\, P \\ \mathrm{div} v = 0 \end{cases} \tag{0.0.3}$$

where v is the velocity, $\mathrm{div}(v \otimes v)$ is the connective term with $v \otimes v$ denoting the tensor product of the vector v, π the pressure, S the extra stress tensor, g the external body forces, E the electric fields and P the electric polarization. Furthermore, the tensor S satisfies the coercivity, growth condition and monotonicity. In this case, we can use the theory of monotone operator to show the existence of weak solution to problem (0.0.3). For more details about the way, we refer the following papers [77, 78, 33, 32].

Electrorheological fluids change their mechanical properties dramatically when an external electric field is applied. They are one example of smart materials, whose development is currently one of the major task in engineering sciences. Electrorheological fluids have been used in robotics and space technology. The experimental research has been done mainly in the USA, for instance in NASA laboratories. For more information on properties, modelling and the application of variable exponent spaces to these fluids, we refer [63, 30, 80].

In [25], Chen, Levine and Rao proposed a framework for image restoration based on a Laplacian variable exponent, see also [26].

This thesis (see [1]) deals with a class of nonlinear anisotropic elliptic systems with variable exponents and degenerate coercivity. In a bounded open domain $\Omega \subset \mathbb{R}^N$, where $N \geq 2$, with Lipschitz boundary $\partial\Omega$, we consider the Dirichlet problem for the elliptic systems given by

$$\begin{cases} -\sum_{i=1}^{N} D_i \left(a_i(x, u(x), D_i u(x))\right) + F(x, u) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{0.0.4}$$

Our aim is to prove the existence and regularity of distributional solutions for anisotropic nonlinear elliptic systems with variable exponents and degenerate coercivity. The right-hand side of the systems (0.0.4) is in $L^m(\Omega; \mathbb{R}^d)$ where m satisfies the following condition

$$1 < m < \frac{N\overline{p}(x)}{N\overline{p}(x) - N + \overline{p}(x)}, \tag{0.0.5}$$

under additional assumptions on a_i and F.

The primary difficulty of the systems (0.0.4) stems from the fact that, due to the following hypothesis

$$\alpha_1 \frac{|\xi|^{p_i(x)}}{(1+|u|)^{\gamma(x)}} \leq a_i(x, u, \xi).\xi,$$

where α_1 is a positive constant, the differential operator

$$u \longmapsto -\sum_{i=1}^{N} D_i \left(a_i(x, u, D_i u) \right),$$

is not coercive on $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$ despite being well-defined between $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$ and its dual $\left(W^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)\right)'$. Degenerate coercivity implies that as $|u|$ becomes large, $\dfrac{1}{(1+|u|)^{\gamma(\cdot)}}$ tends to zero. This indicates that conventional techniques for elliptic operators are not applicable. To solve this issue, we will address the operator approximation by employing truncations in a_i to derive a coercive differential operator. Subsequently, we will establish anisotropic a priori estimates for the sequence of approximate solutions, ultimately we pass to the limit within the approximate systems. This process will establish the existence of a distributional solution for the systems (0.0.4).

Numerous studies have delved into the context of elliptic problems, and an extensive array of articles and books on this subject has surfaced. A comprehensive overview of the literature exceeds the scope of our introduction and cannot be accommodated. Nonetheless, we will examine select results pertaining to the specific problem that we find noteworthy, highlight our contributions.

Degenerate elliptic equations were initially explored by Boccardo et al. in [16]. Existence and regularity results have been demonstrated for the linear case under various conditions on f. The problem they considered is outlined as follows

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{(1+|u|)^{\gamma}}\right) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{0.0.6}$$

where Ω is a bounded open subset of $\mathbb{R}^N, N > 2$, $0 < \gamma < 1$ and $u : \Omega \to \mathbb{R}$ and $f \in L^m(\Omega)$. Following that, Gao et al. in [48] provided a partial generalization of the anisotropic Laplacian type, where $f \in L^m(\Omega)$. They demonstrated the existence and regularity of weak energy solutions.On the other hand, it is worth pointing out that different ranges have an important impact on the behavior of solutions to the problem (0.0.6), Boccardo et.al in [16], considered the non-existence result of problem (0.0.6) they required that $\gamma > 1$ even f is in $L^\infty(\Omega)$. Several papers have addressed and expanded upon this case; for further details, we recommend consulting the references [75, 50, 53, 81, 71].

The author of [11], has extensively investigated a class of anisotropic elliptic equations with variable exponents. Subsequently, these results were extended to the case where $f \in L^{m(\cdot)}(\Omega)$, as detailed in [67].

Building upon these findings, the authors of [9] expanded the scope of the aforementioned problem to a coercive case. More precisely, they addressed the following problem

$$\begin{cases} -\sum_{i=1}^{N} \left(\dfrac{|D_i u|^{p_i(x)-2} D_i u}{(1+|u|)^{\gamma_i(x)}} \right) + |u|^{s(x)-1} u = f, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

They examined the regularity of these solutions under various assumptions regarding $m(\cdot)$ and $s(\cdot)$, also the regularity result is associated with γ_+^+(defined as $\max_{x\in\overline{\Omega}} \max_{1\le i\le N} \gamma_i(x)$).

The study of nonlinear elliptic equations involving the p-Laplace operator is based on the theory of standard Sobolev spaces $W^{1,p}(\Omega)$ in order to find weak solutions see [36, 46, 21], and their method cannot apply her due to the nonhomogeneous $p_i(\cdot)$-Laplace operators, for this the natural setting for this approach is the use of the variable exponent anisotropic Sobolev spaces $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we refer this paper [41].

Existence of weak solutions u has been profoundly examined in [59, 84, 16], while uniqueness seems to be a delicate matter, see [37, 70]. For the scalar case with lower order term, we refer the reader to [15, 13, 27, 51]. The anisotropic case, in which each component of the gradient $D_i u$ may have a possibly different exponent p_i, is dealt with in [58, 54]. For some papers related to elliptic and parabolic equations with degenerate coercivity, we refer the reader to [8, 11, 45, 66, 51].

At present, to our knowledge, there are only a few results available regarding the regularity of solutions for anisotropic elliptic systems with variable exponents. In [12, 2], the authors explored the existence and regularity of distributional solutions for anisotropic p_i-harmonic systems. In the realm of anisotropic elliptic systems with variable exponents, the author of [10] broadened the scope from p-Laplacian systems to $p(x)$-Laplacian systems, building upon the same structural condition. Subsequently, in [64], the focus shifted towards $p_i(x)$-Laplacian systems characterized by degenerate coercivity and an L^1 right-hand side. The study established that u belongs to the solution space, and the results are as follows u is in $W_0^{1,\overrightarrow{r}(\cdot)}(\Omega;\mathbb{R}^d)$, where $1 \leq r_i(\cdot) < \dfrac{Np_i(\cdot)(\overline{p}(\cdot)-1-\gamma^+)}{\overline{p}(\cdot)(N-1-\gamma(\cdot))-N(\gamma^+-\gamma(\cdot))}$. For some developments on isotropic and anisotropic elliptic systems and recent research, we refer reader to [49, 88, 62, 24, 74].

The systems (0.0.4) with variable exponents is new and has never been studied before when the data f in $L^m(\Omega;\mathbb{R}^d)$. The following inequality holds for m goes to 1:

$$\frac{Nmp_i(\cdot)(\overline{p}(\cdot)-1-\gamma^+)}{Nm(\overline{p}(\cdot)-1-\gamma^+)+(1+\gamma(\cdot))(N-m\overline{p}(\cdot))} > \frac{Np_i(\cdot)(\overline{p}(\cdot)-1-\gamma^+)}{\overline{p}(\cdot)(N-1-\gamma(\cdot))-N(\gamma^+-\gamma(\cdot))}.$$

Thus, the regularity provided by Theorem 3.2.3 improves Theorem 4.2 presented in [64].

The thesis consists of three chapters that are briefly presented below

Chapter 01 : Preliminaries and basic concepts

In the first chapter, we establish the functional framework for our study, which encompasses Lebesgue–Sobolev and weak Lebesgue spaces with variable exponents. Accordingly, we provide an overview of foundational theories related to these spaces. The chapter begins by presenting Lebesuge and Sobolev spaces with variable exponents, then defining the anisotropic Sobolev spaces with variable exponents and presenting fundamental theorems. Subsequently, we delve into the definitions of weak Lebesgue spaces, both in cases of constants and non-constants and we discuss the embedding between these spaces and the Sobolev spaces.

In the following part, our focus is on introducing the truncation function, a crucial element in our analysis. We present the truncation function with values in $\mathbb{R}^d$, using the definition of the tensor product, explicitly establish the derivatives of the truncation function. Finally, we define key concepts and present results related to the theory of monotone/pseudomonotone operators.

Chapter 02: Anisotropic elliptic systems with variable exponents and regular data

In the second chapter, our attention is directed towards the analysis of nonlinear anisotropic elliptic systems characterized by variable exponents and regular data. We then shift our focus to contextualizing the problem within the framework of variational issues and elliptic systems exhibiting non-standard $p_i(x)$-growth conditions. More specifically, we delve into scenarios where the right-hand side is situated in the dual space $\left(W^{1,\overrightarrow{p}(\cdot)}\left(\Omega;\mathbb{R}^d\right)\right)'$.

Our initial step involves formulating the problem under coercive conditions with lower-order terms. Subsequently, we establish the theorem outlining the existence of solutions, formalize the approximation problem, and finally transition to the limit.

Chapter 04: Anisotropic elliptic system with variable exponents and degenerate coercivity with L^m data

In the third chapter, drawing from the insights of the paper [1], our focus is on nonlinear degenerate anisotropic elliptic systems exhibiting variable growth. Specifically, we explore cases where the right-hand side term f belongs to $L^m(\Omega;\mathbb{R}^d)$. To prove existence and regularity of distributional solutions, we work with an appropriate functional setting that involves anisotropic Sobolev spaces and weak Lebesgue (Marcinkiewicz) spaces with variable exponents. We introduce continuous functions, defined for all $x \in \overline{\Omega}$ and all $i = 1, ..., N$

$$q(x) = \frac{Nm(\overline{p}(x) - 1 - \gamma^+)}{N - m\overline{p}(x)},\ \ q_i(x) = \frac{Nmp_i(x)(\overline{p}(x) - 1 - \gamma^+)}{Nm(\overline{p}(x) - 1 - \gamma^+) + (1 + \gamma(x))(N - m\overline{p}(x))}. \tag{0.0.7}$$

The proof follows the conventional strategy of obtaining uniform estimates for a sequence of suitable approximate solutions $(u_n)_n$ and their weak derivatives $D_i u_n$ in weak Lebesgue spaces with variable exponents $\mathcal{M}^{q(\cdot)}(\Omega;\mathbb{R}^d)$ and $\mathcal{M}^{q_i(\cdot)}(\Omega;\mathbb{R}^d)$, respectively. To establish these estimates, we employ an anisotropic Sobolev inequality and leverage the embedding between Marcinkiewicz and Lebesgue spaces. We demonstrate that u_n belongs to anisotropic Sobolev spaces $W_0^{1,\overrightarrow{r}(\cdot)}(\Omega;\mathbb{R}^d)$ for every $r_i(\cdot) < q_i(\cdot), i = 1, ..., N$ where $q_i(\cdot)$ is as defined in (0.0.7). We then prove almost everywhere convergence of the partial derivatives $D_i u_n$. With this convergence established, we proceed to pass to the limit in the strong L^1 sense in the nonlinear vector fields $a_i(x, u_n, D_i u_n)$ and ultimately conclude that the approximate solutions u_n converge to a solution of (0.0.4).

Chapter 1

Preliminaries and basic concepts

In this chapter, we endeavor to offer a comprehensive examination of pivotal findings arising from functional analyses, laying the groundwork for subsequent utilization. Furthermore, we elucidate essential details pertaining to the requisite function spaces, enhancing the reader's understanding of their fundamental characteristics.

In the context of this chapter, unless specified otherwise, $\Omega \subset \mathbb{R}^N$ is defined as a bounded open set endowed with N-dimensional Lebesgue measure. It is crucial to recognize that the results outlined here are not exhaustively presented. Rather, they will be unfolded as required throughout our study for a more targeted and nuanced exploration.

1.1 Variable exponents Lebesgue / Sobolev spaces

In the following two parts, we present fundamental results concerning Lebesgue-Sobolev spaces $L^{p(.)}(\Omega)$ and $W^{1,p(.)}(\Omega)$. These findings establish the essential groundwork for the exploration of variational problems and elliptic equations featuring non-standard $p(x)$-growth conditions. For a more in-depth understanding, we recommend consulting the works of Musielak [68], Edmunds et al [39, 40], Kováčik and Rákosník [56], Diening [28, 29], and the references provided therein.

Let Ω be a bounded open subset in $\mathbb{R}^N$ with positive measure. Considering the set of all continuous functions $p(.) : \Omega \to (0, +\infty)$, we introduce the set $C_+^0(\overline{\Omega})$ defined as

$$C_+^0(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} p(x) > 0 \text{ for all } x \in \overline{\Omega} \right\}.$$

For any variable exponent $p \in C_+^0(\overline{\Omega})$, We introduce the following notations

$$p^+ = \max_{x \in \overline{\Omega}} p(x) \quad \text{and} \quad p^- = \min_{x \in \overline{\Omega}} p(x).$$

Note that if $p \in C_+^0(\overline{\Omega})$, then, for all $x \in \overline{\Omega}$, $0 < p^- \leq p(x) \leq p^+ < +\infty$. Moreover, if $p^- > 1$, we define the conjugate exponent of $p(.)$ by

$$p^{'}(\cdot) = \frac{p(\cdot)}{p(\cdot) - 1}.$$

Definition 1.1.1. ([23]) Let $p \in C_+^0(\overline{\Omega})$. We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx,$$

is finite. The space $L^{p(\cdot)}(\Omega)$ is equipped with the Luxemburg-Nakano quasi-norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \rho_{p(.)} \left(\frac{u}{\lambda} \right) \leq 1 \right\}, \tag{1.1.1}$$

The space $L^{p(\cdot)}(\Omega)$ is a quasi-Banach space (see [3]). In particular, if $p^- \geq 1$ then the exponent on (1.1.1) defines a norm in $L^{p(\cdot)}(\Omega)$ and the space $\left(L^{p(\cdot)}(\Omega), \|.\|_{L^{p(\cdot)}(\Omega)} \right)$ becomes a separable Banach space.

If $p, q \in C_+^0(\overline{\Omega})$ with $q \leq p$, then the inclusion $L^{p(\cdot)}(\Omega) \subseteq L^{q(\cdot)}(\Omega)$ holds. Moreover, if $1 \leq q^- \leq q \leq p$, the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous, and its norm does not exceed $1 + |\Omega|$. Henceforth, we denote

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} p(x) > 1 \text{ for all } x \in \overline{\Omega} \right\}.$$

Lemma 1.1.2. *([56]) Let $p \in C_+(\overline{\Omega})$, the space $L^{p(\cdot)}(\Omega)$ is reflexive and its dual space can be identified with $L^{p'(\cdot)}(\Omega)$.*

Lemma 1.1.3. *(The Hölder inequality) For all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the following Hölder inequality*

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \leq 2\|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)},$$

holds true, where

- $(p^-)' = \dfrac{p^-}{p^- - 1}$,
- $(p^+)' = \dfrac{p^+}{p^+ - 1}$.

Proposition 1.1.4. *([4]) Let $p \in C_+(\overline{\Omega})$, if $(u_n), u \in L^{p(\cdot)}(\Omega)$, then the following relations holds*

$$(i)\ \|u\|_{L^{p(\cdot)}} < 1 (> 1; = 1) \iff \rho_{p(\cdot)}(u) < 1 (> 1; = 1),$$

$$(ii)\ \min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}; \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \leq \|u\|_{L^{p(\cdot)}(\Omega)} \leq \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}; \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right),$$

$$(iii)\ \min\left(\|u\|^{p^+}_{L^{p(\cdot)}}; \|u\|^{p^-}_{L^{p(\cdot)}}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|^{p^+}_{L^{p(\cdot)}}; \|u\|^{p^-}_{L^{p(\cdot)}}\right),$$

$$(iv)\ \|u_n - u\|_{L^{p(\cdot)}} \to 0 \iff \rho_{p(\cdot)}(u_n - u) \to 0.$$

Throughout this section, we consider $\Omega \subset \mathbb{R}^N$, where $N \geq 2$, as a bounded open domain with a Lipschitz boundary.

Definition 1.1.5. Let $p : \overline{\Omega} \to [1, +\infty)$ be a continuous functions, we define the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that u in $L^{p(\cdot)}(\Omega)$ and ∇u are in $(L^{p(\cdot)}(\Omega))^N$. Subsequently, we express it as

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega),\ |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

where $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_N}\right)$. We make use of the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

Remark 1.1.6. We define $D_0^{1,p(\cdot)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the above norm. Also, the Sobolev space with zero boundary values $W_0^{1,p(\cdot)}(\Omega)$ is the space $W^{1,p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega)$ equipped with the norm of $W^{1,p(\cdot)}(\Omega)$.

As Ω is assumed to be a bounded open Lipschitz domain, we can establish the following definition

Definition 1.1.7. We define $W_0^{1,p(\cdot)}(\Omega)$ the Sobolev space with zero boundary values by

$$W_0^{1,p(\cdot)}(\Omega) = \left\{u \in W^{1,p(\cdot)}(\Omega);\ u = 0 \text{ on } \partial\Omega\right\},$$

endowed with the norm $\|.\|_{W^{1,p(\cdot)}(\Omega)}$.

Lemma 1.1.8. *The function spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are reflexive uniformly convex Banach spaces. Moreover, for any measurable bounded exponent $p(\cdot)$ $\left(1 < p^- \leq p^+ < +\infty\right)$, the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable.*

The following lemma presents the Poincaré inequality.

Lemma 1.1.9. *([6]) For every $u \in W_0^{1,p(\cdot)}(\Omega)$*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq c\|\nabla u\|_{L^{p(\cdot)}(\Omega)}. \tag{1.1.2}$$

For some constant c that depends on Ω and $p(\cdot)$, considering the Poincaré inequality (1.1.2), it becomes possible to define the equivalent norm of the space $W_0^{1,p(\cdot)}(\Omega)$ using the relation

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

We point out that, the above norm is equivalent to the following norm (see [6])

$$\sum_{i=1}^{N} \|D_i u\|_{L^{p(\cdot)}(\Omega)}.$$

Remark 1.1.10. The following inequality in general does not holds

$$\int_{\Omega} |u|^{p(x)} \, dx \leq c \int_{\Omega} |\nabla u|^{p(x)} \, dx$$

but by Proposition 1.1.4 and the inequality (1.1.2), we obtain

$$\int_{\Omega} |u|^{p(x)} \, dx \leq C \max \left(\|Du\|_{L^{p(\cdot)}}^{p^+} ; \|Du\|_{L^{p(\cdot)}}^{p-} \right).$$

The following definition is from [87].

Definition 1.1.11. Let $p : \overline{\Omega} \longrightarrow \mathbb{R}$. If there exist a positive constant C such that

$$|p(x) - p(y)| \leq \frac{-C}{\ln |x-y|}, \ \forall x, y \in \overline{\Omega}, \ |x-y| \leq \frac{1}{2}.$$

Then, $p(\cdot)$ is called log-Hölder continuous on $\overline{\Omega}$.

Theorem 1.1.12. *([6, 52]) Let $p, q \in C_+ \left(\overline{\Omega}\right)$ such that $q(x) < p^*(x)$ in $\overline{\Omega}$, then for every $u \in W_0^{1,p(\cdot)}(\Omega)$*

$$\|u\|_{L^{q(\cdot)}(\Omega)} \leq C \, \|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

with a constant C depending on N, p and Ω. The embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

1.2 Anisotropic Sobolev spaces with variable exponents

Let Ω be a bounded open subset of $\mathbb{R}^N$ and $p_i > 1$, $i = 1, \ldots, N$, $N \geq 2$. We introduce the anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$ which is defined by

$$W_0^{1,\overrightarrow{p}}(\Omega) = \left\{ g \in W_0^{1,1}(\Omega) : D_i g \in L^{p_i}(\Omega), \forall i = 1, ..., N \right\},$$

which is a Banach space under the norm

$$\|g\|_{W_0^{1,\overrightarrow{p}}(\Omega)} = \|g\|_{L^1(\Omega)} + \sum_{i=1}^{N} \|D_i g\|_{L^{p_i}(\Omega)} .$$

We need the anisotropic Sobolev embedding Theorems.

Theorem 1.2.1. *([83]) Suppose $g \in W_0^{1,\overrightarrow{p}}(\Omega)$. Then*

$$\|g\|_{L^q(\Omega)} \leq C \prod_{i=1}^{N} \|D_i g\|_{L^{p_i}(\Omega)}^{\frac{1}{N}} . \tag{1.2.1}$$

where $\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}$ *and:*

$$\begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N - \overline{p}} & if\ \overline{p} < N, \\ q \in [1, \infty) & if\ \overline{p} \geq N, \end{cases}$$

The constant C, depends on $p_1, ..., p_N$, N if $\overline{p} < N$. Furthermore, if $\overline{p} \geq N$, the inequality (1.2.1) is true for all $q \geq 1$ and C depends on q and $|\Omega|$.

Theorem 1.2.2. *([83]) Assume Q is a cube in $\mathbb{R}^N$ with faces parallel to the coordinate planes, and $p_i \geq 1$ for $i = 1, ..., N$. Suppose $u \in W^{1,\overrightarrow{p}}(Q)$, and set*

$$\begin{cases} q = \overline{p}^*, & if\ \overline{p} < N, \\ q \in [1, \infty), & if\ \overline{p} \geq N. \end{cases}$$

Then, there exists a constant C depending on $\overrightarrow{p} = (p_1, ..., p_N)$, N if $\overline{p} < N$. Moreover, if $\overline{p} \geq N$, the inequality (1.2.2) is true for all $q \geq 1$ and C depends on q and $|Q|$.

$$\|u\|_{L^q(Q)} \leq C \prod_{i=1}^{N} \left(\|u\|_{L^{p_i}(Q)} + \|D_i u\|_{L^{p_i}(Q)} \right)^{\frac{1}{N}} . \tag{1.2.2}$$

One has the following Lemma.

Lemma 1.2.3. *([65]) Let $v \in W_0^{1,\overrightarrow{p}}(\Omega)$. Then there exists a positive constant C such that*

$$\|v\|_{L^{p_i}(\Omega)} \leq C\|\partial_i v\|_{L^{p_i}(\Omega)}.$$

We use standard notations for the vector and matrix-valued versions of the space/ norm introduced above. For example, the $\mathbb{R}^d$-valued version of $W^{1,\overrightarrow{p}}(\Omega)$ is denoted by $W^{1,\overrightarrow{p}}(\Omega;\mathbb{R}^d)$.

In this part, we define the anisotropic Lebesgue and Sobolev spaces with variable exponent and give some of their properties. Roughly speaking, anisotropic Lebesgue and Sobolev spaces are functional spaces of Lebesgue's and Sobolev's type in which different space directions have different roles, see [17, 42, 43].

Everywhere in this part, Ω be a bounded open subset of $\mathbb{R}^N, N \geq 2$. Let $p_i : \overline{\Omega} \to [1,+\infty), i = 1, ..., N$, be continuous vectorial functions. We denote by $\overrightarrow{p}(x) = (p_1(x), ..., p_N(x))$, and $p_+(x) = \max_{1\leq i\leq N} p_i(x)$. The anisotropic Sobolev spaces with variable exponents are defined as

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in L^{p_+(\cdot)}(\Omega) : D_i u \in L^{p_i(\cdot)}(\Omega), i = 1, ..., N\right\},$$

which is a Banach space with respect to the norm

$$\|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)} = \|u\|_{L^{p_+(\cdot)}(\Omega)} + \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}. \tag{1.2.3}$$

We define $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as follow

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = W_0^{1,1}(\Omega) \cap W^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

If Ω is a bounded open set with Lipschitz boundary $\partial\Omega$, then

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega) \ ; \ u_{|\partial\Omega} = 0\right\},$$

where, $u_{|\partial\Omega}$ denotes the trace on $\partial\Omega$ of u in $W_0^{1,1}(\Omega)$.

Definition 1.2.4. We define the space $D_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as the intersection of $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $W_0^{1,1}(\Omega)$, thus

$$D_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \overline{C_0^{\infty}(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)}.$$

Remark 1.2.5. It is well-known that in the constant exponent case, that is, when $\overrightarrow{p}(\cdot) = \overrightarrow{p} \in ([1,+\infty))^N$, $D_0^{1,\overrightarrow{p}}(\Omega) = W_0^{1,\overrightarrow{p}}(\Omega)$. However, in the variable exponent case, in general $D_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \subsetneqq W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and the smooth functions are in general not dense in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, but if for all $i = 1, ..., N, p_i$ is log-Hölder continuous, then $C_0^{\infty}(\Omega)$ is dense in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, thus $D_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. The spaces $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$, $D_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ are separable and reflexive Banach spaces when they are supplied with the norm defined in (1.2.3) (see [41]).

Moreover, we proceed to define the function $\overline{p}^*(x)$ for $\overline{p}(x) < N$ as

$$\overline{p}^*(x) = \frac{N\overline{p}(x)}{N - \overline{p}(x)}.$$

Lemma 1.2.6. *([41]) Suppose Ω is a bounded domain in $\mathbb{R}^N$ and $\overrightarrow{p}(x) \in (C_+(\overline{\Omega}))^N$. If $q(\cdot) \in C_+(\overline{\Omega})$ verifies $q(x) < \max(\overline{p}^*(x), p_+(x))$ for all $x \in \overline{\Omega}$, then the embedding*

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega),$$

is continuous and compact.

Lemma 1.2.7. *([41]) Suppose Ω is a bounded domain in $\mathbb{R}^N$ and $\overrightarrow{p}(\cdot) \in (C_+(\overline{\Omega}))^N$. If $p(\cdot) \in C_+(\overline{\Omega})$, satisfies the condition*

$$p_+(x) < \overline{p}^*(x), \forall x \in \overline{\Omega}. \tag{1.2.4}$$

Then, the following Poincaré-type inequality holds

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \leq C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}, \quad \forall u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega),$$

where C is a positive constant independent of u. Thus, $\sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on

$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

In order to facilitate the manipulation of the space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we introduce the following notations

$$\overrightarrow{p}_- = \left\{p_1^-, ..., p_N^-\right\},\ p_+^- = \max\left\{p_1^-, ..., p_N^-\right\},\ p_-^- = \min\left\{p_1^-, ..., p_N^-\right\}.$$

We represent the harmonic mean of $\overrightarrow{p}_-$ as $\overline{p}^-$

$$\frac{1}{\overline{p}^-} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i^-}.$$

If $1 < \overline{p}^- < N$, we define $(\overline{p}^-)^* \in \mathbb{R}^+$ and $p_{-,\infty} \in \mathbb{R}^+$ by

$$(\overline{p}^-)^* = \frac{N}{\sum_{i=1}^{N}\frac{1}{p_i^-} - 1} = \frac{N\overline{p}^-}{N - \overline{p}^-},$$

where $p_{-,\infty} = \max\left\{(\overline{p}^-)^*, p_+^-\right\}$.
In the work of [79], a concise embedding result was established for the space $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$

Theorem 1.2.8. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with Lipschitz boundary. Assume that*

$$\sum_{i=1}^{N}\frac{1}{p_i^-} > 1. \tag{1.2.5}$$

Then, for any $q \in C_+\left(\overline{\Omega}\right)$ satisfying

$$q(x) < p_{-,\infty} \quad \text{for all } x \in \overline{\Omega}, \tag{1.2.6}$$

the embedding

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega),$$

is continuous and compact.

We use standard notations for the vector and matrix-valued versions of the space/ norm introduced above. For example, the $\mathbb{R}^d$-valued version of $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is denoted by $W^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$.

1.3 Weak Lebesgue spaces

Weak Lebesgue spaces are essential for demonstrating uniform and a priori estimates on both the sequence of solutions and their derivatives. In this context, we provide a comprehensive overview, beginning with the constant case. Additionally, we introduce a recent definition of these spaces in the variable case. To initiate, let's revisit the definition of weak Lebesgue spaces, also referred to as Marcinkiewicz spaces

Definition 1.3.1. ([4]) We define the space $\mathcal{M}^q(\Omega)$ for $1 < q < \infty$ as the set of measurable functions $g : \Omega :\to \mathbb{R}$ for which the distribution function

$$\lambda_g(k) = |\{x \in \Omega : |g| > k\}|\,, \;\; \forall\, k \geq 0,$$

satisfies an estimate of the form

$$\lambda_g(k) \leq ck^{-q},$$

for some finite constant c.

Proposition 1.3.2. *The space $\mathcal{M}^q(\Omega)$ is a Banach space under the norm*

$$\|g\|^*_{\mathcal{M}^q(\Omega)} = \sup_{k>0} k^{\frac{1}{q}} \left(\frac{1}{k} \int_0^k g^*(s)\, ds \right),$$

where g^* denotes the non-increasing rearrangement of g

$$g^*(r) = \inf \{k > 0 : \;\lambda_g(k) \leq r\}\,.$$

Remarks 1.3.3.

- We will in what follows use the pseudo norm

$$\|g\|_{\mathcal{M}^q(\Omega)} = \inf \left\{C : \;\lambda_g(k) \leq Ck^{-q}\,, \forall k > 0\right\},$$

which is equivalent to the norm $\|g\|^*_{\mathcal{M}^q(\Omega)}$ i.e., $\left(\|g\|_{\mathcal{M}^q(\Omega)} \leq \|g\|^*_{\mathcal{M}^q(\Omega)} \leq \dfrac{q}{q-1} \|g\|_{\mathcal{M}^q(\Omega)} \right)$.

- It is clear that $L^q(\Omega) \subset \mathcal{M}^q(\Omega)$, let us prove it, if $g \in L^q(\Omega)$, we have

$$\begin{aligned} k^{-q}|\{x \in \Omega : |g(x)| > k\}| &\leq \int_{\{|g|>k\}} |g|^q \, dx \\ &\leq \int_\Omega |g|^q \, dx = \|g\|^q_{L^q(\Omega)}, \end{aligned}$$

then, $\lambda_g(k) \leq \|g\|^q_{L^q(\Omega)}$ and $g \in \mathcal{M}^q(\Omega)$. Moreover, $\|g\|_{\mathcal{M}^q(\Omega)} \leq \|g\|_{L^q(\Omega)}$.

Proposition 1.3.4. *A useful property of weak Lebesgue spaces is the following version of Hölder's inequality. Let $E \subset \Omega, g \in \mathcal{M}^q(\Omega), r < q$, then*

$$\|g\|_{\mathcal{M}^r(E)} \leq \left(\frac{q}{q-r}\right)^{\frac{1}{r}} |E|^{\frac{1}{r}-\frac{1}{p}} \|g\|_{\mathcal{M}^q(E)},$$

it is then immediate that $\mathcal{M}^q(\Omega) \subset \mathcal{M}^r(\Omega)$ if $r < q$.

Remark 1.3.5. In similarly with the anisotropic Sobolev spaces, we employ conventional notations for the vector/matrix-valued versions of weak Lebesgue spaces.

We recall the definition of weak Lebesgue spaces (Marcinkiewicz spaces) with variable exponents.

Definition 1.3.6. Let $p(\cdot) \in C(\overline{\Omega})$ such that $p^- > 0$. We say that a measurable function $u : \Omega \to \mathbb{R}$ belongs to the Marcinkiewicz space $\mathcal{M}^{p(\cdot)}(\Omega)$ if

$$\|u\|_{\mathcal{M}^{p(\cdot)}(\Omega)} = \sup_{\lambda>0} \lambda \|\chi_{\{|u|>\lambda\}}\|_{L^{p(\cdot)}(\Omega)} < \infty, \tag{1.3.1}$$

where χ_A denotes the characteristic function of a measurable set A.

Proposition 1.1.4 suggests that (1.3.1) is equivalent to asserting the existence of a positive constant M such that

$$\int_{\{|u|>\lambda\}} \lambda^{p(x)} \, dx \leq M, \text{ for all } \lambda > 0. \tag{1.3.2}$$

The following results are form [82].

Proposition 1.3.7. *Let $p, q \in C^0_+(\overline{\Omega})$. If $(p-q)^- > 0$, then*

$$L^{p(\cdot)}(\Omega) \subset \mathcal{M}^{p(\cdot)}(\Omega) \subset \mathcal{M}^{q(\cdot)}(\Omega).$$

Lemma 1.3.8. *Let $\eta(\cdot), r(\cdot)$ in $C(\overline{\Omega})$ such that $r^- > 0$, $(\eta - r)^- > 0$. If $u \in \mathcal{M}^{\eta(\cdot)}(\Omega)$, then $|u|^{r(\cdot)} \in L^1(\Omega)$. In particular, $\mathcal{M}^{\eta(\cdot)}(\Omega) \subset L^{r(\cdot)}(\Omega)$ for all $\eta(\cdot), r(\cdot) \geq 1$ such that $(\eta - r)^- > 0$.*

Moreover, the following property holds.

Lemma 1.3.9. *([61]) If $u \in \mathcal{M}^{p(\cdot)}(\Omega)$, with $p^- > 0$, then*

$$\lambda_u(k) \leq c\frac{1}{k^{p^-}}\left(M + |\Omega|\right), \forall k > 0,$$

where M is the constant appeared in (1.3.2).

Remark 1.3.10. We use standard notations for the vector and matrix-valued versions of the space/ norm introduced above. For example, the $\mathbb{R}^d$-valued version of $\mathcal{M}^{q(\cdot)}(\Omega)$ is denoted by $\mathcal{M}^{q(\cdot)}(\Omega; \mathbb{R}^d)$.

1.4 Truncation function

In the following, our attention is directed towards the introduction of the truncation function, a pivotal component in our analytical framework. We expound on the truncation function, aligning its values within $\mathbb{R}^d$ through the application of the tensor product. Moreover, we explicitly determine the derivatives of the truncation function, shedding light on its mathematical intricacies.

First of all, we begin by the definition of tensor product $a \otimes b$ of two vectors $a, b \in \mathbb{R}^d$, $a = \left((a_i)_{i=1,...,d}\right)^T$ and $b = \left((b_i)_{i=1,...,d}\right)^T$, is defined to be the $d \times d$ matrix of entries $(a_i b_j)_{ij}$ with $i, j = 1, ..., d$. Then

$$\begin{aligned} \otimes : \quad \mathbb{R}^d \times \mathbb{R}^d &\longrightarrow \mathbb{M}^{d\times d} \\ (a, b) \quad &\longrightarrow a \otimes b = \left(\left((a_i b_j)_{j=1,...,d}\right)_{i=1,...,d}\right)^T, \end{aligned}$$

here $\mathbb{M}^{d\times d}$ denotes the space of real $d\times d$ matrices equipped with inner product $a:b=\sum_{i=1}^{d}\sum_{j=1}^{d}a_{ij}b_{ij}$. Moreover, we can write

$$a:b=\begin{pmatrix} a_1b_1 & a_2b_1 & \cdots & a_db_1 \\ a_1b_2 & a_2b_2 & \cdots & a_db_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1b_d & a_2b_d & \cdots & a_db_d \end{pmatrix}.$$

Proposition 1.4.1. *If $u:\Omega\to\mathbb{R}$ is a measurable function such that $T_k(u)\in W^{1,\overrightarrow{p}(.)}(\Omega)$ for all $k>0$, then there exists a unique measurable function $v:\Omega\to\mathbb{R}^d$ such that*

$$\nabla T_k(u)=v1_{\{|u|\leq k\}} \text{ a.e. in } \Omega, \quad T_k(t)=\max\{-k,\min\{k,t\}\} \tag{1.4.1}$$

Furthermore, if $u\in W_0^{1,1}(\Omega)$ then v coincides with standard distributional gradient of u.

The truncation function will be used repeatedly to derive a priori estimates for our approximate solutions. For that reason, we present for any $k>0$, the spherical radially symmetric truncation T_k by

$$\begin{aligned} T_k:\ \mathbb{R}^d &\longrightarrow \mathbb{R}^d \\ s &\longrightarrow T_k(s)=\begin{cases} s, & if \quad |s|\leq k, \\ k\dfrac{s}{|s|}, & if \quad |s|>k. \end{cases} \end{aligned}$$

The mapping $s\mapsto T_k(s)$ as a mapping from $\mathbb{R}^d$ into $\mathbb{R}^d$ is not differentiable at $s\in\partial\Omega$. We clarify the derivative of T_k as follow, we begin by the first case, if $|s|\leq k$ then we have

$$DT_k(s)=\begin{pmatrix} \dfrac{\partial s_1}{\partial s_1} & \dfrac{\partial s_2}{\partial s_1} & \cdots & \dfrac{\partial s_d}{\partial s_1} \\ \dfrac{\partial s_1}{\partial s_2} & \dfrac{\partial s_2}{\partial s_2} & \cdots & \dfrac{\partial s_d}{\partial s_2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial s_1}{\partial s_d} & \dfrac{\partial s_2}{\partial s_d} & \cdots & \dfrac{\partial s_d}{\partial s_d} \end{pmatrix}=I$$

It is worth noting that, the Euclidean norm of a vector $s \in \mathbb{R}^d$ is denoted by $|s| = \left(\sum_{l=1}^{d} |s^l|^2\right)^{\frac{1}{2}}$. Therefore, if $|s| > k : T_k(s) = \left(k\frac{s_1}{|s|}, ..., \frac{s_d}{|s|}\right)$, then we can express our derivative as following

$$DT_k(s) = \begin{pmatrix} \frac{k}{|s|^2}\left(|s| - \frac{s_1^2}{|s|}\right) & -k\frac{s_1 s_2}{|s|^3} & \cdots & -k\frac{s_1 s_d}{|s|^3} \\ -k\frac{s_2 s_1}{|s|^3} & \frac{k}{|s|^2}\left(|s| - \frac{s_2^2}{|s|}\right) & \cdots & -k\frac{s_2 s_d}{|s|^3} \\ \vdots & \vdots & \ddots & \vdots \\ -k\frac{s_d s_1}{|s|^3} & -k\frac{s_d s_2}{|s|^3} & \cdots & \frac{k}{|s|^2}\left(|s| - \frac{s_d^2}{|s|}\right) \end{pmatrix}$$

$$= \frac{k}{|s|}\left(I - \frac{1}{|s|^2}\begin{pmatrix} s_1^2 & s_1 s_2 & \cdots & s_1 s_d \\ s_2 s_1 & s_2^2 & \cdots & s_2 s_d \\ \vdots & \vdots & \ddots & \vdots \\ s_d s_1 & s_d s_2 & \cdots & s_d^2 \end{pmatrix}\right)$$

Thanks to the definition of tensor product, we deduce that

$$\begin{aligned} DT_k : \quad \mathbb{R}^d &\longrightarrow \mathbb{M}^{d\times d} \\ s \quad &\longrightarrow DT_k(s) = \begin{cases} I, & if \quad |s| < k, \\ \frac{k}{|s|}\left(I - \frac{(s \otimes s)}{|s|^2}\right), & if \quad |s| > k, \end{cases} \end{aligned}$$

Remark 1.4.2. We have the following equality for all $s \in \mathbb{R}^d$

$$|T_k(s)| = \begin{cases} |s|, & if \quad |s| \leq k, \\ k, & if \quad |s| > k. \end{cases} \tag{1.4.2}$$

Furthermore, we define the truncation function $T_k^+ : \mathbb{R}^+ \to \mathbb{R}^+$ as follow

$$T_k^+(|s|) = \begin{cases} |s|, & if \quad |s| \leq k, \\ \\ k, & if \quad |s| > k. \end{cases} \tag{1.4.3}$$

From (1.4.2) and (1.4.3), we have the following equality

$$|T_k(s)| = T_k^+(|s|). \tag{1.4.4}$$

It is important to observe that $s \in \mathbb{R}^d$ represents a vector, while $|s| \in \mathbb{R}^+$ represents a norm value. Therefore, $T_k(s)$ on the first side of the equality (1.4.4) is different from $T_k^+(|s|)$ on the right-hand of this equality.

We explore the following cubic truncation function

$$\begin{aligned} \mathcal{T}_k(y) &= (T_k(y_1), ..., T_k(y_d)) \\ &= (\max(-k, \min(k, y_1)), ..., \max(-k, \min(k, y_d)))\,, \end{aligned}$$

which satisfies

$$|\mathcal{T}_k(y)| \leq |y|, \quad |\mathcal{T}_k(y)| \leq dk. \tag{1.4.5}$$

For a comprehensive discussion on T_k, $\mathcal{T}_k$, and other test functions pertinent to elliptic systems, we direct the reader to [57]. This topic is indeed nuanced and requires careful consideration.

Within this section, we intricately explore fundamental outcomes tied to surjectivity, a pivotal factor in establishing the existence of solutions to nonlinear problems involving pseudo-monotone operators. Before unveiling the distinguished theorem concerning pseudo-monotone operators, it is imperative to introduce key definitions referenced in [14].

1.5 The theory of monotone operators with application

The method of monotone operators, initiated by G. Minty in 1962, has played a pivotal role in the development of the theory of pseudo-monotone operators, which has undergone significant advancements and found numerous applications. The extension of this concept to multi-valued operators and its application to variational inequalities have been extensively explored, as discussed in [18, 19, 20].

Throughout the ensuing discussion, we denote V as a real Banach space, with $V^{'}$ representing its topological dual.

Definition 1.5.1. ([85]) An operator

$$\begin{aligned} \mathcal{L}: \quad V \quad &\longrightarrow V^{'} \\ u \quad &\mapsto (v \mapsto \langle \mathcal{L}(u), v\rangle), \end{aligned}$$

is said to be monotone if, for every $u, v \in V$, the inequality $\langle \mathcal{L}u - \mathcal{L}v, u - v\rangle \geq 0$ holds, and it is strictly monotone if this inequality is strict whenever $u \neq v$

Example 1.5.2. We examine the function $h : \mathbb{R} \to \mathbb{R}$ defined as follows

$$h(u) = \begin{cases} |u|^{p-2}u & if \quad |u| \neq 0, \\ 0 & if \quad |u| = 0. \end{cases}$$

if $p > 1$, then h is strictly monotone. This is evident from the inequality

$$\left(|u|^{p-2}u - |v|^{p-2}v\right)(u - v) \geq c\,|u - v|^{p},$$

for all $u, v \in \mathbb{R}$ and fixed $p > 2, c > 0$

Definition 1.5.3. ([85]) An operator $\mathcal{L} : V \to V^{'}$ is said to be a bounded if the image of a bounded subset of V is a bounded open subset of $V^{'}$, i.e.,

$$\forall C > 0, \exists C^{'} > 0 : \|u\|_V \leq C \Longrightarrow \|\mathcal{L}(u)\|_{V'} \leq C^{'}.$$

Example 1.5.4. Let Ω denote a bounded open set in $\mathbb{R}^N$, and let $V = W_0^{1,p}(\Omega)$ equipped with the norm $\|v\|_V = \||\nabla v|\|_{L^p}$. It is known that $V^{'} = W^{-1,p^{'}}(\Omega)$.
Consider the operator $\mathcal{L}(u) = -\text{div}\left(|\nabla u|^{p-2}\nabla u\right)$, where $1 < p < \infty$. For any $\varphi \in W_0^{1,p}(\Omega)$, the definition yields

$$\langle \mathcal{L}(u), \varphi\rangle = \int_\Omega |\nabla u|^{p-2}\, \nabla u.\nabla \varphi \, dx.$$

We assert that the operator $\mathcal{L}$ is bounded on V. Take $\rho > 0$, and for $u \in B_v(0, \rho)$, we can represent

$$\|\mathcal{L}v\|_V = \sup_{\{\varphi \in V, \|\varphi\|_V \leq 1\}} |\langle \mathcal{L}u, \varphi\rangle| = \sup_{\{\varphi \in V, \|\varphi\|_V \leq 1\}} \left| \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla \varphi \, dx\right|.$$

But

$$\begin{aligned} \left|\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi\right| \quad & \leq \int_\Omega |\nabla u|^{p-1}\, |\nabla \varphi| \; dx \\ & \leq \left(\int_\Omega |\nabla u|^p \; dx\right)^{\frac{1}{p'}} \left(\int_\Omega \nabla\varphi \, dx\right)^{\frac{1}{p}} \\ & = \|u\|_V^{p-1}\, \|\varphi\|_{V'} \leq \rho^{p-1}. \end{aligned}$$

Therefore, $\|\mathcal{L}u\|_{V'} \leq \rho^{p-1}$. This establishes that $\mathcal{L}\left(B_V(0, \rho)\right) \subset B_{V'}\left(0, \rho^{p-1}\right)$.

Definition 1.5.5. Consider a reflexive Banach space V. An operator $\mathcal{L} : V \to V^{'}$ is said to be coercive if, for all $v \in V$,

$$\frac{\langle \mathcal{L}v, v\rangle}{\|v\|_V} \to +\infty \quad \text{as} \quad \|v\|_V \to +\infty.$$

Definition 1.5.6. ([7]) Consider a Banach space. An operator $\mathcal{L} : V \to W$ is said to be hemicontinuous at point u_∞ of V if, for any sequence $(u_n)_n$ converging to u_∞ along a line, the sequence $(\mathcal{L}u_n)_n$ converges weakly to $\mathcal{L}(u)$ in W. More explicitly

$$\forall v \in V, \forall (\lambda_n)_n \in \mathbb{R}, \lambda_n \longrightarrow 0, \mathcal{L}\left(u_\infty + \lambda_n v\right) \rightharpoonup \mathcal{L}u_\infty \text{ weakly in } W.$$

If $\mathcal{L}$ is semicontinuous at every point of V, it is described as said to be semicontinuous on V. In reflexive spaces, when $W = V'$, and passing from sequential to continuous, we can define hemicontinuity on V can be defined by ensuring that

$$\forall u, v, w \in V \ \text{ the application } \lambda \longmapsto \langle \mathcal{L}\left(u + \lambda v\right), w \rangle \in \mathbb{R},$$

is continuous for all λ in $\mathbb{R}$.

Example 1.5.7. The operator $\mathcal{L} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by $\mathcal{L}(u) = -\text{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is hemicontinuous. Consider $u, v, w \in W_0^{1,p}(\Omega)$ and $\lambda \in \mathbb{R}$. We aim to demonstrate the continuity of the function from $\mathbb{R}$ to $\mathbb{R}$ given by

$$\begin{aligned} \lambda \mapsto \quad & \langle \mathcal{L}(u + \lambda v), w \rangle \\ & = \int_\Omega \left(|\nabla\left(u + \lambda v\right)|^{p-2}\nabla\left(u + \lambda v\right)\right).\nabla w\, dx \end{aligned}$$

is continuous. Let $\lambda \in \mathbb{R}$ be fixed, and let $(\lambda_n)_n$ be a real sequence converging to λ. Define

$$K_n(x) = \left(|\nabla\left(u + \lambda v\right)|^{p-2}\right).\nabla w.$$

Since a_i are Carathédory functions and $\lambda_n \to \lambda$ in $\mathbb{R}$, we have

$$K_n \to K \ \text{ a.e. in } \ \Omega.$$

Consequently,

$$\begin{aligned} \langle \mathcal{L}(u + \lambda_n v), w \rangle \quad & = \int_\Omega \left|(\nabla u + \lambda_n \nabla v)^{p-2}.\left(\nabla u + \lambda \nabla v\right)\right| |\nabla w| \\ & \leq \int_\Omega |\nabla u + \lambda_n \nabla v|^{p-1} |\nabla w| \; dx \\ & \leq C_p \int_\Omega \left(|\nabla u|^{p-1} + |\lambda_n|^{p-1} |\nabla v|^{p-1}\right) |\nabla w|\, dx \\ & \leq C \int_\Omega \left(|\nabla u|^{p-1} + |\nabla v|^{p-1}\right) |\nabla w|\, dx, \end{aligned}$$

where $C_p = \max\left\{1, 2^{p-2}\right\}$. Since the sequence $(\lambda_n)_n$ is bounded, the Dominated Convergence Theorem implies

$$\lim_{n\mapsto\infty} \langle \mathcal{L}(u+\lambda_n v), w\rangle = \langle \mathcal{L}(u+\lambda v), w\rangle.$$

This establishes the hemicontinuity of $\mathcal{L}$.

Definition 1.5.8. ([14]) Consider a reflexive Banach space V. An operator $\mathcal{L} : V \to V^{'}$ is deemed pseudo-monotone if it satisfies the following conditions

- $\mathcal{L}$ is bounded.

- For any weak convergence $u_m \rightharpoonup u$ in V, and if $\limsup_{m\mapsto+\infty}\langle \mathcal{L}u_m, u_m - v\rangle \leq 0$, then $\liminf_{m\mapsto+\infty}\langle \mathcal{L}u_m, u_m - v\rangle \geq \langle \mathcal{L}u, u - v\rangle$, for every $v \in V$, where $\langle ., .\rangle$ denotes the duality product between V and $V^{'}$.

The fundamental theorem on pseudo-monotone operators is credited to Brézis, as detailed in [22, 85].

Theorem 1.5.9. *Suppose the operator $\mathcal{L} : V \to V^{'}$ is both pseudo-monotone and coercive on the real, separable, and reflexive Banach space V. In this case, for every $f \in V^{'}$, there exists a solution $u \in V$ such that $\mathcal{L}(u) = f$.*

Theorem 1.5.10. *(Browder and Minty) Consider a reflexive Banach space V and an operator $\mathcal{L} : V \to V^{'}$ with the following properties*

- *Boundedness,*

- *Coerciveness,*

- *Hemicontinuity,*

- *Monotonicity.*

Under these conditions, for every $f \in V^{'}$, there exists a solution $u \in V$ such that $\mathcal{L}(u) = f$.

Example 1.5.11. (Application of Theorem 1.5.10) Consider a bounded open subset Ω in $\mathbb{R}$, and let the operator $\mathcal{L} : W_0^{1,p}(\Omega) \to W^{-1,p^{'}}(\Omega)$ be defined by

$$\mathcal{L}(u) = -\text{div}\left(|\nabla u|^{p-2}\nabla u\right),\ 1 < p < \infty.$$

The operator $\mathcal{L}$ satisfies all the hypotheses of Theorem 1.5.10, as illustrated in examples 1.5.4, 1.5.2, and 1.5.7, where it is shown to be monotone, bounded, and hemicontinuous. Moreover, $\mathcal{L}$ is coercive: for all $u \in V$, $\langle \mathcal{L}u, u \rangle = \|u\|_V^p$. Consequently, according to Theorem 1.5.10, for every $f \in W^{-1,p'}(\Omega)$, there exists $u \in W_0^{1,p}(\Omega)$ such that

$$-\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = f.$$

Chapter 2

Anisotropic elliptic systems with variable exponents and regular data

In this chapter, we establish the existence of distributional solutions for anisotropic nonlinear elliptic systems with variable exponents and regular data. The system we aim to investigate is presented under the specified conditions outlined below.

2.1 Setting of the problem and assumptions

Let us defining the following anisotropic nonlinear elliptic systems

$$\begin{cases} -\sum_{i=1}^{N} D_i \left(a_i(x, u(x), D_i u(x)) \right) + F(x, u) = f(x), & x \quad \in \Omega, \\ u(x) = 0, & x \quad \in \partial\Omega, \end{cases} \tag{2.1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N, N \geq 2$ with Lipschitz boundary $\partial\Omega$, and the right-hand side f belongs to $\left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega; \mathbb{R}^d)\right)'$. We assume that the variable exponents $p_i(\cdot)$ are in $C_+(\overline{\Omega})$ for all $i = 1, ..., N$. The function $u : \Omega \to \mathbb{R}^d$, where $u = (u_1, ..., u_d)$ for $d \geq 2$, represents a vector-valued function, and $D_i u = \dfrac{\partial u}{\partial x_i}$ denotes the partial derivative of u with respect to x_i.
We make the assumption that the vector fields $a_i : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, where $i = 1, ..., N$ are

Carathédory functions, meet the following conditions: for almost every $x \in \Omega$, and all $u \in \mathbb{R}^d$, and all $\xi, \xi^{'} \in \mathbb{R}^d$, there exist positive constants α_1, α_2. The conditions are given by

$$\alpha_1 |\xi|^{p_i(x)} \leq a_i(x, u, \xi).\xi, \tag{2.1.2}$$

$$|a_i(x, u, \xi)| \leq \alpha_2 \left(|h| + |u|^{\overline{p}^-} + \sum_{j=1}^{N} |\xi|^{p_j(x)} \right)^{1 - \frac{1}{p_i(x)}}, |h| \in L^1(\Omega), \tag{2.1.3}$$

$$\left(a_i(x, u, \xi) - a_i(x, u, \xi^{'}) \right) . \left(\xi - \xi^{'} \right) > 0, \, \xi \neq \xi^{'}, \tag{2.1.4}$$

Additionally, consider the perturbation $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, which depends on the vector-valued function u. This function is Carathédory and satisfies the following conditions for almost every $x \in \Omega$

$$F(x, y). \left(y - y^{'} \right) \geq 0, \;\; \forall y, y^{'} \in \mathbb{R}^d, \;\; |y| = |y^{'}|, \tag{2.1.5}$$

$$F(x, y).y \geq |y|^{s(x)+1}, \;\; \forall y \in \mathbb{R}^d, \tag{2.1.6}$$

$$\sup_{|y| \leq t} |F(x, y)| \in L^1(\Omega), \;\; \forall t \in \mathbb{R}. \tag{2.1.7}$$

2.2 Statement of the result along with its proof

For the systems (2.1.1), the following existence Theorem holds.

Theorem 2.2.1. *Under the hypotheses (2.1.2)-(2.1.7), let* $f \in \left(W_0^{1, \overrightarrow{p}(\cdot)} \left(\Omega; \mathbb{R}^d \right) \right)^{'}$. *Then, there exists a function* $u \in W_0^{1, \overrightarrow{p}(\cdot)}(\Omega; \mathbb{R}^d)$*, which allows to solve (2.1.1) in the sense*

$$\int_{\Omega} \sum_{i=1}^{N} a_i(x, u, D_i u).D_i \varphi \, dx + \int_{\Omega} F(x, u).\varphi \, dx = \langle f.\varphi \rangle, \tag{2.2.1}$$

for all $\varphi \in C_0^{\infty}(\Omega; \mathbb{R}^d)$.

Proof. To establish the aforementioned Theorem, let us examine the sequence of approximate problems

$$\begin{cases} -\sum_{i=1}^{N} D_i\left(a_i(x,u(x),D_iu(x))\right) + F_n(x,u) = f(x), & x \quad \in \Omega, \\ u(x) = 0, & x \quad \in \partial\Omega, \end{cases} \tag{2.2.2}$$

where $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$ and $F_n(x,s) = \dfrac{F(x,s)}{1+F(x,s)/n}, \forall n \in \mathbb{N}$. Note that

$$|F_n(x,s)| \le |F(x,s)| \text{ and } |F_n(x,s)| \le n.$$

We denote by $\mathcal{L}_n : W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d) \to \left(W^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)\right)'$ the operator, for $u, v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$

$$\mathcal{L}_n : u \to \left(v \to \int_\Omega \sum_{i=1}^{N} a_i(x,u,D_iu).D_iv\,dx + \int_\Omega F_n(x,u).v\,dx\right).$$

We consider

$$b(u,v) = \int_\Omega \sum_{i=1}^{N} a_i(x,u,D_iu).D_iv\,dx,$$

and

$$c_n(u,v) = \int_\Omega F_n(x,u).v\,dx,$$

and we seek $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$ such that

$$b(u,v) + c_n(u,v) = \langle f, v\rangle; \forall v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d). \tag{2.2.3}$$

The generalized problem (2.2.4), associated with (2.2.2) is equivalent to

$$\mathcal{L}_n(u)(v) = \langle f, v\rangle, \forall v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d),$$

where $\mathcal{L}_n := \mathcal{B} + \mathcal{C}_n$, with $\mathcal{B}, \mathcal{C}_n : W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d) \to \left(W_0^{1,\overrightarrow{p}(\cdot)}\left(\Omega;\mathbb{R}^d\right)\right)'$ characterized by

$$\langle \mathcal{B}(u)(v)\rangle = b(u,v);\ \langle \mathcal{C}_n(u)(v)\rangle = c_n(u,v).$$

Here $\langle .,. \rangle$ denotes the duality pairing between $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$ and $\left(W_0^{1,\overrightarrow{p}(\cdot)}\left(\Omega;\mathbb{R}^d\right)\right)'$.

- The operator $\mathcal{L}_n$ is bounded on $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$

(i) The boundedness of $\mathcal{B}$. Using Hölder's inequality and the given hypothesis (2.1.3), we obtain

$$\begin{aligned}
&|\langle \mathcal{B}_(u), v\rangle| \\
&\leq \sum_{i=1}^{N} \int_\Omega |a_i(x,u,D_iu)||D_iv|\,dx \\
&\leq \alpha_2 \sum_{i=1}^{N} \int_\Omega \left(|h| + |u|^{\overline{p}^-} + \sum_{j=1}^{N} |D_ju|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} |D_iv|\,dx \\
&\leq 2\alpha_2 \sum_{i=1}^{N} \left\| \left(|h| + |u|^{\overline{p}^-} + \sum_{j=1}^{N} |D_ju|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} \right\|_{L^{p_i'(\cdot)}(\Omega;\mathbb{R}^d)} \|D_iv\|_{L^{p_i(\cdot)}(\Omega;\mathbb{R}^d)} \\
&\leq 2\alpha_2 \sum_{i=1}^{N} \left(1 + \int_\Omega \left(|h| + |u|^{\overline{p}^-} + \sum_{j=1}^{N} |D_ju|^{p_j(x)}\right) dx\right)^{1-\frac{1}{p_-^-}} \sum_{i=1}^{N} \|D_iv\|_{L^{p_i'(\cdot)}(\Omega;\mathbb{R}^d)} \\
&\leq 2\alpha_2 \sum_{i=1}^{N} \left(1 + \int_\Omega |h|\,dx + \int_\Omega |u_n|^{\overline{p}^-}\,dx + \sum_{j=1}^{N} \int_\Omega |D_ju|^{p_j(x)}\,dx\right)^{1-\frac{1}{p_-^-}} \|v\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)} \\
&\leq 2\alpha_2 N \left(1 + C + \int_\Omega |u|^{\overline{p}^-}\,dx + \|u\|^{p_+^+}_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)}\right)^{1-\frac{1}{p_-^-}} \|v\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)}.
\end{aligned} \tag{2.2.4}$$

Therefore, given that the solution $|u|$ belongs to $L^{\overline{p}^-}(\Omega)$, it is due to the existence of $l \in \{1, ..., N\}$ such that $p_l^- \geq \overline{p}^-$. Consequently, $u \in L^{\overline{p}^-}\left(\Omega;\mathbb{R}^d\right)$ by applying Lemma 1.2.3. Based on the final estimate (2.2.4), we can infer that $\mathcal{B}$ is bounded.

(ii) On another note, leveraging Hölder's inequality, we obtain for all $u, v \in W_0^{1,\overrightarrow{p}(\cdot)}\left(\Omega;\mathbb{R}^d\right)$

$$\begin{aligned}
|\langle \mathcal{C}_n(u), v\rangle| &= \left|\int_\Omega F_n(x,u).v\,dx\right| \\
&\leq \int_\Omega |F_n(x,u)||v|\,dx \\
&\leq \left(\frac{1}{p_+^-} + \frac{1}{(p_+')^-}\right) \|F_n(x,u)\|_{L^{p_+'(\cdot)}(\Omega;\mathbb{R}^d)} \|v\|_{L^{p_+(\cdot)}(\Omega;\mathbb{R}^d)} \\
&\leq \left(\frac{1}{p_+^-} + \frac{1}{(p_+')^-}\right) \left(1 + \int_\Omega n^{p_+'(\cdot)}\,dx\right)^{\frac{1}{(p_+')^-}} \|v\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)}.
\end{aligned}$$

This, in turn, implies the boundedness of $\mathcal{C}_n$. Additionally, combining the results from (i) and (ii), we conclude the boundedness of $\mathcal{L}_n$.

- Next, we consider the coerciveness of the operator $\mathcal{L}_n$ on $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$. Indeed, by using the assumptions (2.1.2) and (2.1.6), we have

$$
\begin{aligned}
\frac{\langle \mathcal{L}_n(u), u\rangle}{\|u\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)}} &\geq \alpha_1 \frac{\sum_{i=1}^{N}\int_\Omega a_i\left(x,u,D_i u\right).D_i u}{\|u\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)}} \\
&\geq \alpha_1 \frac{\sum_{i=1}^{N}\int_\Omega |D_i u|^{p_i(x)}\,dx}{\|u\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)}} \\
&\geq \alpha_1 \frac{\sum_{i=1}^{N}\min\left\{\|D_i u\|^{p_i^-}_{L^{p_i(\cdot)}(\Omega;\mathbb{R}^d)};\|D_i u\|^{p_i^+}_{L^{p_i(\cdot)}(\Omega;\mathbb{R}^d)}\right\}}{\|u\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)}} \\
&\geq \alpha_1 \frac{\left(\sum_{i=1}^{N}\|D_i u\|^{p_-^-}_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)} - N\right)}{\|u\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)}} \\
&\geq \frac{\alpha_1\left(\frac{1}{N}\sum_{i=1}^{N}\|D_i u\|_{L^{p_i(\cdot)}(\Omega;\mathbb{R}^d)}\right)^{p_-^-} - \alpha_1 N}{\|u\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)}} \\
&= \frac{\alpha_1}{N^{p_-^-}}\|u\|^{p_-^- -1}_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)} - \frac{\alpha_1 N}{\|u\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)}}.
\end{aligned}
$$

This implies that $\mathcal{L}_n$ is coercive.

- The operator $\mathcal{L}_n$ is hemicontinu.

Let u, v, and $w \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$, and $\lambda \in \mathbb{R}$. We aim to prove the continuity of the function from $\mathbb{R}$ to $\mathbb{R}$ given by

$$
\begin{aligned}
\lambda \quad &\to \langle \mathcal{L}_n\left(u+\lambda v\right), w\rangle \\
&= \int_\Omega \sum_{i=1}^{N} a_i\left(x,u,D_i\left(u+\lambda v\right)\right).D_i w\,dx.
\end{aligned}
$$

Let $\lambda \in \mathbb{R}$ be fixed, and consider the sequence $(\lambda_n)_n$ in $\mathbb{R}$ converging to λ. Define

$$K(x) = a_i\left(x, u, D_i u(u + \lambda_k v)\right).D_i w.$$

Since a_i are Carathédory functions and $\lambda_k \to \lambda$ in $\mathbb{R}$, we have for k tends to $+\infty$

$$a_i\left(x, u, D_i u(u + \lambda_k v)\right) \to a_i\left(x, u, D_i u(u + \lambda v)\right) \text{ a.e. in}\,\Omega. \tag{2.2.5}$$

Furthermore, from the hypotheses (2.1.3) and as the sequence $(\lambda_k)_k$ converges, it is bounded. Therefore, we have

$$\begin{aligned}
&|a_i\left(x, u, D_i u(u + \lambda_k v)\right).D_i w| \leq |a_i\left(x, u, D_i u(u + \lambda_k v)\right)||D_i w| \\
\leq\ & \alpha_2 \left(|h| + |u|^{\overline{p}^-} + \sum_{j=1}^N |D_j u + \lambda_k D_j v|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} |D_i w| \\
\leq\ & \alpha_2 \left(|h| + |u|^{\overline{p}^-} + C_p^{'} \left\{\sum_{j=1}^N |D_j u|^{p_j(x)} + \sum_{j=1}^N |\lambda_k|^{p_j(x)} |D_j v|^{p_j(x)}\right\}\right)^{1-\frac{1}{p_i(x)}} |D_i w| \\
\leq\ & \alpha_2 \left(|h| + |u|^{\overline{p}^-} + C_p^{'} \sum_{j=1}^N |D_j u|^{p_j(x)} + C_p^{'} \left(|\lambda_k|^{p_+^+} + 1\right) \sum_{j=1}^N |D_j v|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} |D_i w| \\
\leq\ & \alpha_2 \left(|h| + |u|^{\overline{p}^-} + C_p^{'} \sum_{j=1}^N |D_j u|^{p_j(x)} + C_p^{'} C \sum_{j=1}^N |D_j v|^{p_j(x)}\right)^{1-\frac{1}{p_-^-}} |D_i w|,
\end{aligned}$$

where $C_p^{'} = 1/\min\left\{1, 2^{1-p_+^+}\right\}$. Put

$$G^k(x) = \left(|h| + |u|^{\overline{p}^-} + C_p^{'} \sum_{j=1}^N |D_j u|^{p_j(x)} + C_p^{'} C \sum_{j=1}^N |D_j v|^{p_j(x)}\right)^{1-\frac{1}{p_-^-}} |D_i w|,$$

Now, let's demonstrate that the function $|G^k|$ is integrable in $L^1(\Omega)$ for all $j = 1, ..., N$. Consequently, we can apply Hölder's inequality

$$
\begin{aligned}
\int_\Omega |G^k(x)|\,dx \quad & \leq \left\| \left(|h| + |u|^{\overline{p}^-} + C_p' \sum_{j=1}^N |D_j u|^{p_j(x)} + C_p' C \sum_{j=1}^N |D_j v|^{p_j(x)} \right)^{1-\frac{1}{p_-^-}} \right\|_{L^{p_i'(\cdot)}(\Omega;\mathbb{R}^d)} \\
& \sum_{i=1}^N \|Dw\|_{L^{p_i(\cdot)}(\Omega;\mathbb{R}^d)} \\
& \leq \left(1 + \int_\Omega \left(|h| + |u|^{\overline{p}^-} + C_p' \sum_{j=1}^N |D_j u|^{p_j(x)} + C_p' C \sum_{j=1}^N |D_j u|^{p_j(x)} \right) dx \right)^{1-\frac{1}{p_-^-}} \\
& \sum_{i=1}^N \|D_i w\|_{L^{p_i(\cdot)}(\Omega;\mathbb{R}^d)} .
\end{aligned}
$$

Given that $|h| \in L^1(\Omega)$, $|u| \in L^{\overline{p}^-}(\Omega)$, and $u, v, w \in W_0^{1,\overrightarrow{p}(\cdot)}\left(\Omega;\mathbb{R}^d\right)$, we can conclude that $|G^k| \in L^1(\Omega)$ using Hölder's inequality. Consequently, by the Dominated Convergence Theorem, we deduce the following

$$
\begin{aligned}
\lim_{k \mapsto +\infty} \int_\Omega K^k(x)\,dx \quad & = \lim_{k \mapsto +\infty} \langle \mathcal{L}_n (u + \lambda_k v), w \rangle\,dx \\
& = \lim_{k \mapsto +\infty} \int_\Omega \sum_{i=1}^N a_i(x, u, D_i(u + \lambda_k v)).D_i w \\
& = \int_\Omega \sum_{i=1}^N a_i(x, u, D_i(u + \lambda v)).D_i w \\
& = \langle \mathcal{L}_n (u + \lambda_k v), w \rangle = \int_\Omega K(x)\,dx,
\end{aligned}
$$

which implies the hemicontinuous of $\mathcal{B}$.

- $\mathcal{B}$ is strictly monotone.

Let $u, v \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$ with $u \neq v$, by the hypothesis (2.1.4), we have

$$
\langle \mathcal{B}(u) - \mathcal{B}(v), u - v \rangle = \int_\Omega \left(a_i(x, u, D_i u) - a_i(x, v, D_i v) \right).D_i (u - v)\ dx > 0.
$$

Consequently, the Theorem applies and ensures existence of least one distributional solution $u_n \in$

$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$ to (2.2.2) in the sense that

$$\int_\Omega \sum_{i=1}^N a_i(x,u_n,D_iu_n).D_i\varphi\,dx + \int_\Omega F_n(x,u_n).\varphi\,dx = \langle f,\varphi\rangle, \tag{2.2.6}$$

for all $\varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$.

Finally, we pass to the limit in the approximate systems to obtain the existence of a distributional solution for problem (2.2.1). Inserting u_n in (2.2.6) and by (2.1.2) and (2.1.6), we have

$$\alpha_1 \sum_{i=1}^N \int_\Omega |D_iu_n|^{p_i(x)}\,dx \le \|f\|_{W_0^{1,\overrightarrow{p}'(\cdot)}(\Omega;\mathbb{R}^d)} \|u_n\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)}.$$

By using Young's inequality and (iii) in the Proposition 1.1.4, we have

$$\frac{\alpha_1}{N^{p_-^-}} \|u_n\|_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)} \le C(\varepsilon^*)\,\|f\|^{(p_-^-)'}_{\left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)\right)'} + \varepsilon^*\,\|u_n\|^{p_-^-}_{W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)} + N\alpha_1, \varepsilon^*>0,$$

it suffice to take $\varepsilon^* = \dfrac{\alpha_1}{2N^{p_-^-}}$ to get the boundedness of the sequence u_n in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$ (still denoted $(u_n)_n$). Then, there exists a function $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d) \text{ and a.e. in } \Omega.$$

We present the following Lemma

Lemma 2.2.2. *Assume* (2.1.2) -(2.1.4)*, and let $(u_n)_n$ be a sequence in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$ such that*

$$u_n \rightharpoonup u \textit{ weakly in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d), \tag{2.2.7}$$

and

$$\int_\Omega \sum_{i=1}^N \left(a_i(x,u_n,D_iu_n) - a_i(x,u_n,D_iu)\right).D_i\left(u_n-u\right)\,dx \longrightarrow 0, \textit{ as } n \to +\infty, \tag{2.2.8}$$

Then

$$u_n \longrightarrow u \textit{ in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d) \textit{ and a.e. in } \Omega, \tag{2.2.9}$$

for a subsequence.

We adopt the techniques outlined in [5, 86] with certain modifications.

Proof. Consider $\triangle_n^i = \left(a_i(x,u_n,D_iu_n) - a_i(x,u_n,D_iu)\right).D_i\left(u_n - u\right)$. According to (2.1.4) we have $\triangle_n^i$ is a positive function. Moreover, from (2.2.8), we have

$$\triangle_n^i \longrightarrow 0 \text{ in } L^1(\Omega;\mathbb{R}^d) \text{ as } n \to \infty.$$

Thanks to (2.2.7), we have $u_n \to u$ a.e. in Ω. Since $\triangle_n^i \longrightarrow \infty$ a.e. in Ω there exists a subset E in Ω with measure zero such that for all $x \in \Omega \backslash E$,

$$|u(x)| < \infty,\ \ |D_iu| < \infty,\ \ h(x) < \infty,\ u_n \to u,\ \triangle_n^i \to 0.$$

Given the hypotheses (2.1.2) and (2.1.3), we have

$$\begin{aligned}
\triangle_n^i(x) \quad &= \left(a_i(x,u_n,D_iu_n) - a_i(x,u_n,D_iu)\right).D_i\left(u_n-u\right)\\
&= a_i(x,u_n,D_iu_n).D_iu_n + a_i(x,u_n,D_iu).D_iu - a_i(x,u_n,D_iu_n).D_iu - a_i(x,u_n,D_iu).D_iu_n\\
&\geq \alpha_1|D_iu_n|^{p_i(x)} + \alpha_1|D_iu|^{p_i(x)} - \alpha_2\left(|h| + |u_n|^{\overline{p}^-} + \sum_{j=1}^N |D_ju_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} |D_iu|\\
&-\alpha_2\left(|h| + |u_n|^{\overline{p}^-} + \sum_{j=1}^N |D_ju|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} |D_iu_n|\\
&\geq \alpha_1|D_iu_n|^{p_i(x)} - C_x\left(1 + \left(\sum_{i=1}^N |D_ju_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} + |D_iu_n|\right)\\
&\geq \alpha_1|D_iu_n|^{p_i(x)} - C_x\left(1 + \frac{1}{C}\left(1 + \sum_{i=1}^N |D_ju_n|^{p_i(x)-1}\right) + |D_iu_n|\right),
\end{aligned}$$

where C_x depending on x, without dependence on n, and $C = \min\left\{1, 2^{1-\frac{1}{p_+^+}}\right\}$.
Since $u_n \to u$ then $(u_n)_n$ is bounded, we obtain

$$\triangle_n^i(x) \geq |D_iu_n|^{p_i(x)}\left(\alpha_1 - \frac{C_x}{|D_iu_n|^{p_i(x)}} - \frac{C_x'}{|D_iu_n|} - \frac{C_x'}{|D_iu_n|^{p_i(x)-1}}\right).$$

Using the standard argument, we conclude that $D_i u_n$ is bounded almost everywhere in Ω. Specifically, if $|D_i u_n| \to \infty$ on a measurable subset $E \in \Omega$, then

$$\lim_{n\to 0}\int_\Omega \triangle_n^i(x)\,dx \geq \lim_{n\to 0}\int_E |D_i u_n|^{p_i(x)}\left(\alpha_1 - \frac{C_x}{|D_i u_\epsilon|^{p_i(x)}} - \frac{C_x^{'}}{|D_i u_n|} - \frac{C_x^{'}}{|D_i u_n|^{p_i(x)-1}}\right)dx = \infty,$$

which is absurd since $\triangle_n^i \longrightarrow 0$ in $L^1(\Omega;\mathbb{R}^d)$.

Let $D_i w$ an accumulation point of $D_i u_n$, we have $|D_i w| < \infty$ and by the continuity of a_i, we obtain

$$\left(a_i(x,u,D_i w) - a_i(x,u,D_i u)\right).\left(D_i w - D_i u\right) = 0.$$

Thanks to (2.1.4), the uniqueness of the accumulation point implies that

$$D_i u_n \longrightarrow D_i u \text{ a.e. in } \Omega, \forall i = 1,...,N.$$

Since $\left(a_i(x,u_n,D_i u_n)\right)_n$ is bounded in $L^{p_i^{'}(\cdot)}(\Omega;\mathbb{R}^d)$ and for all i=1,...,N, we have

$$a_i\left(x,u_n,D_i u_n\right) \longrightarrow a_i\left(x,u,D_i u\right) \text{ a.e. in}\,\Omega.$$

Then we can establish that

$$a_i\left(x,u_n,D_i u_n\right) \rightharpoonup a_i\left(x,u,D_i u\right) \text{ in}\, L^{p_i^{'}(\cdot)}(\Omega;\mathbb{R}^d).$$

Let $\overline{Y}_{n,i} = a_i\left(x,u_n,D_i u_n\right).D_i u_n$ and $\overline{Y}_i = a_i\left(x,u,D_i u\right).D_i u$, then

$$\overline{Y}_{n,i} \longrightarrow \overline{Y}_i \,\text{in}\, L^1\left(\Omega;\mathbb{R}^d\right) \text{ for all}\, i = 1,...,N.$$

According to the condition (2.1.2), we have

$$\alpha_1 |D_i u_n|^{p_i(x)} \leq a_i\left(x,u_n,D_i u_n\right).D_i u_n.$$

Let $Z_{n,i} = D_i u_n$, $Z_i = D_i u$ and $Y_{n,i} = \dfrac{\overline{Y}_{n,i}}{\alpha_1}$, $Y_i = \dfrac{\overline{Y}_i}{\alpha_1}$, in view of the Fatou Lemma, we obtain

$$\int_\Omega 2.Y_i \, dx \leq \liminf_{n\to\infty} \int_\Omega \left(\overline{Y}_{n,i} + \overline{Y}_i - |Z_{n,i} - Z_i|^{p_i(x)} \right) \, dx,$$

then,

$$0 \leq - \limsup_{n\to\infty} \int_\Omega |Z_{n,i} - Z_i|^{p_i(x)} \, dx,$$

and since

$$0 \leq \liminf_{n\to\infty} \int_\Omega |Z_{n,i} - Z_i|^{p_i(x)} \, dx \leq \limsup_{n\to 0} \int_\Omega |Z_{n,i} - Z_i|^{p_i(x)} \, dx \leq 0,$$

it follows that

$$\lim_{n\to\infty} \int_\Omega |D_i u_n - D_i u|^{p_i(x)} \, dx = 0,$$

and we get

$$D_i u_\epsilon \longrightarrow D_i u \,\text{in}\, L^{p_i(.)}(\Omega; \mathbb{R}^d) \,\text{and a.e. in}\, \Omega. \tag{2.2.10}$$

□

In the following, by (2.2.10) this implies for all $i = 1, ..., N$

$$a_i\left(x, u_n, D_i u_n\right) \rightharpoonup a_i\left(x, u, D_i u\right) \quad L^{p_i^{'}(.)(\Omega;\mathbb{R}^d)},$$

from (2.2.7) and Lebesgue dominated convergence Theorem, we obtain, for every $\varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega; \mathbb{R}^d)$ and all $i = 1, ..., N$

$$a_i\left(x, u_n, D_i u_n\right).D_i\varphi \longrightarrow a_i\left(x, u, D_i u\right).D_i\varphi. \tag{2.2.11}$$

□

Chapter 3

Anisotropic elliptic systems with variable exponents and degenerate coercivity with L^m data

Within this chapter, drawing from the insights of the paper [1], our attention is directed towards nonlinear degenerate anisotropic elliptic systems that manifest variable growth. More precisely, we investigate the case in which the right-hand side term f is a member of $L^m(\Omega;\mathbb{R}^d)$. To establish the existence and regularity of distributional solutions, we engage with a suitable functional framework, incorporating anisotropic Sobolev spaces and weak Lebesgue (Marcinkiewicz) spaces characterized by variable exponents.

3.1 Setting of the problem and assumptions

In a bounded open domain $\Omega \subset \mathbb{R}^N$, where $N \geq 2$, with Lipschitz boundary $\partial\Omega$, we consider the Dirichlet problem for the elliptic systems given by

$$\begin{cases} -\sum_{i=1}^{N} D_i \left(a_i(x, u(x), D_i u(x)) \right) + F(x,u) = f(x), & x \quad \in \Omega, \\ u(x) = 0, & x \quad \in \partial\Omega, \end{cases} \tag{3.1.1}$$

here, $u : \Omega \to \mathbb{R}^d$, $d \geq 2$, represents a vector-valued function, $D_i u = \dfrac{\partial u}{\partial x_i}$ denotes the partial derivative of u with respect to x_i, and the vector fields $a_i : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ are Carathédory functions.

We assume that the vector fields $a_i : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, i = 1, ..., N$, satisfying the following conditions: for almost every $x \in \Omega$, and all $u \in \mathbb{R}^d$, and all $\xi, \xi^{'} \in \mathbb{R}^d$, $i = 1, ..., N$, there exist $\alpha_1, \alpha_2 > 0$ and $\gamma(\cdot) \in \mathcal{C}(\overline{\Omega})$ such that $\gamma(\cdot) > 0$ for all $x \in \overline{\Omega}$, and we have

$$\alpha_1 \frac{|\xi|^{p_i(x)}}{(1+|u|)^{\gamma(x)}} \leq a_i(x,u,\xi).\xi, \tag{3.1.2}$$

$$|a_i(x,u,\xi)| \leq \alpha_2 \left(|h| + |u|^{\overline{p}^-} + \sum_{j=1}^{N} |\xi|^{p_j(x)} \right)^{1-\frac{1}{p_i(x)}}, |h| \in L^1(\Omega), \tag{3.1.3}$$

$$\left(a_i(x,u,\xi) - a_i(x,u,\xi^{'}) \right) . \left(\xi - \xi^{'} \right) > 0, \; \xi \neq \xi^{'}. \tag{3.1.4}$$

Moreover, the variable exponents $p_i : \overline{\Omega} \to (1, +\infty)$ and $s : \overline{\Omega} \to (0, +\infty)$ are continuous functions. Let the perturbation $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ depends on the vector-valued function u, which satisfies the following conditions for almost every $x \in \Omega$

$$F(x,y). \left(y - y^{'} \right) \geq 0, \;\; \forall y, y^{'} \in \mathbb{R}^d, \;\; |y| = |y^{'}|, \tag{3.1.5}$$

$$F(x,y).y \geq |y|^{s(x)+1}, \;\; \forall y \in \mathbb{R}^d, \tag{3.1.6}$$

$$\sup_{|y| \leq t} |F(x,y)| \in L^1(\Omega), \;\; \forall t \in \mathbb{R}. \tag{3.1.7}$$

We introduce the following notations

$$\gamma^+ = \max_{x \in \overline{\Omega}} \gamma(x), \;\; \frac{1}{\overline{p}(x)} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i(x)} \;\; \overline{p}^- = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i^-}, \;\; p_i^- = \min_{x \in \overline{\Omega}} p_i(x), \;\; p_i^+ = \max_{x \in \overline{\Omega}} p_i(x) \;\; m^{'} = \frac{m}{m-1}.$$

Assuming

$$0 < \gamma^+ < \overline{p}(x) - 1. \tag{3.1.8}$$

The fundamental problem in extending the results from an equation to a system is to obtain an estimation of the truncation, as the truncation differs for scalar and vector cases. Therefore, an additional structural condition is needed to prove the existence of a solution for the elliptic systems with L^m data. To overcome this obstacle, we have developed a novel technique and we use the following anisotropic version of the so-called (right-) angle condition: for all $x \in \Omega$, $\xi \in \mathbb{R}^d$, and all $s \in \mathbb{R}^d$ with $|s| \leq 1$, we have

$$a_i(x, s, \xi).\left[(I - s \otimes s)\,\xi\right] \geq 0,\ \ i = 1, ..., N, \tag{3.1.9}$$

here $I - s \otimes s$ represents the rank $d - 1$ orthogonal projector onto the space orthogonal to the unit vector $s \in \mathbb{R}^d$. Please refer to the assumed condition in [35]. If $a_{l,i}, l = 1, ..., d$ denotes components of the vector a_i, then the angle condition can be stated more explicitly as

$$\sum_{i,l=1}^{d} a_{l,i}\,(x, s, \xi)\,\xi_l\,(\delta_{l,i} - s_i s_l) \geq 0,$$

here $\delta_{l,l} = 1$ and $\delta_{l,i} = 0$ if $l \neq i$. In the case $d = 1$, the assumption (3.1.9) is void, and the hypotheses (3.1.2)-(3.1.7) are sufficient to prove the results in this paper.
In particular, (3.1.9) implies for all s, $\xi \in \mathbb{R}^d$ the crucial property, for all $i = 1, ..., N$

$$a_i(x, s, \xi).DT_k(s)\xi \geq a_i(x, s, \xi).\xi\chi_{\{|s|\leq k\}}. \tag{3.1.10}$$

As prototype examples, we consider the following models

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D_i\left(\frac{|D_i u(x)|^{p_i(x)-2} D_i u(x)}{(1 + |u(x)|)^{\gamma(x)}}\right) + |u(x)|^{s(x)-1}u(x) = f(x), & x \ \in \Omega, \\ u(x) = 0, & x \ \in \partial\Omega, \end{cases} \tag{3.1.11}$$

and

$$\begin{cases} -\sum_{i=1}^{N} D_i \left(\dfrac{v(u(x))|D_i u(x)|^{\frac{p_i(x)-2}{2}} D_i u(x)}{(1+|u(x)|)^{\gamma(x)}} \right) + |u(x)|^{s(x)-1}u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

here $v : \mathbb{R}^d \longrightarrow (0, +\infty)$ is a bounded continuous function. Moreover, we have $f \in L^m(\Omega; \mathbb{R}^d)$, $s(\cdot)$, and the exponents $p_i(\cdot)$ are restricted as in Theorem 3.2.3.

3.2 Statement of the results

Definition 3.2.1. A function u is a distributional solution to systems (3.1.1) if

$$u \in W_0^{1,1}(\Omega; \mathbb{R}^d) \text{ and } a_i(x, u, D_i u), F(x, u) \in L^1(\Omega; \mathbb{R}^d), \;\; i = 1, ..., N.$$

Additionally

$$\int_\Omega \sum_{i=1}^{N} a_i(x, u, D_i u).D_i\varphi \, dx + \int_\Omega F(x, u).\varphi \, dx = \int_\Omega f.\varphi \, dx, \tag{3.2.1}$$

for all $\varphi \in C_0^\infty(\Omega; \mathbb{R}^d)$.

Our main results are the following

Theorem 3.2.2. *Let $\overrightarrow{p}(\cdot) = (p_1(.), ..., p_N(.)) \in (C_+(\overline{\Omega}))^N$, $\overline{p}(\cdot) < N$ such that (1.2.4) holds. Under our given assumptions (3.1.2)-(3.1.7), let $f \in L^m(\Omega; \mathbb{R}^d)$ with m as follow*

$$m > \frac{N\overline{p}(\cdot)}{N\overline{p}(\cdot) - N + \overline{p}(\cdot)}. \tag{3.2.2}$$

Then the systems (3.1.1) has a distributional solution $u \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega; \mathbb{R}^d)$.

Theorem 3.2.3. *Let $f \in L^m(\Omega;\mathbb{R}^d)$, where m is defined as in (0.0.5) and γ^+ satisfies (3.1.8). We assume that $p_i(\cdot), i = 1, ..., N,$ and $s(.)$ are continuous functions on $\overline{\Omega}$ such that $\overline{p}(\cdot) < N$ for all $i = 1, ..., N$*

$$s(\cdot) \geq p_i(\cdot), \tag{3.2.3}$$

and

$$\begin{aligned} &\frac{Nm(\overline{p}(\cdot) - 1 - \gamma^+) + (1+\gamma(\cdot))(N - m\overline{p}(.))}{Nm(\overline{p}(\cdot) - 1 - \gamma^+)} < p_i(\cdot) \\ &< \frac{Nm(\overline{p}(\cdot) - 1 - \gamma^+) + (1+\gamma(\cdot))(N - m\overline{p}(\cdot))}{(1+\gamma(\cdot))(N - m\overline{p}(\cdot))}. \end{aligned} \tag{3.2.4}$$

Let a_i and F be Carathédory functions, where a_i satisfies (3.1.2)-(3.1.4) and (3.1.9), and F satisfies (3.1.5)-(3.1.7). Then the systems (3.1.1) has a distributional solution $u \in W_0^{1,\overrightarrow{r}(\cdot)}(\Omega;\mathbb{R}^d)$ where $r_i(\cdot), i = 1, ..., N$ are continuous functions on $\overline{\Omega}$ satisfying

$$1 \leq r_i(\cdot) < q_i(\cdot) = \frac{Nmp_i(\cdot)(\overline{p}(\cdot) - 1 - \gamma^+)}{Nm(\overline{p}(.) - 1 - \gamma^+) + (1+\gamma(\cdot))(N - m\overline{p}(\cdot))}, \ \ \forall i = 1, ..., N. \tag{3.2.5}$$

Remark 3.2.4. Let us remark that $1 < \overline{p}(x) < N$ implies

$$\frac{N\overline{p}(\cdot)}{N\overline{p}(\cdot) + \overline{p}(\cdot) - N} < \frac{N}{\overline{p}(\cdot)},$$

which means that the condition $m < \dfrac{N\overline{p}(\cdot)}{N\overline{p}(\cdot) + \overline{p}(\cdot) - N}$ implies $m < \dfrac{N}{\overline{p}(\cdot)}$. Therefore, we have $q_i(\cdot) < p_i(\cdot)$ for all $i = 1, ..., N$. The lower bound of $p_i(\cdot)$ guarantees $\dfrac{r_i(\cdot)}{p_i(\cdot) - 1} > 1$ and the upper bound of $p_i(\cdot)$ guarantees $\dfrac{Nmp_i(\cdot)(\overline{p}(\cdot) - 1 - \gamma^+)}{Nm(\overline{p}(\cdot) - 1 - \gamma^+) + (1+\gamma(\cdot))(N - m\overline{p}(\cdot))} > 1$. However, the condition (3.1.8) ensures that (3.2.4) is well-defined.

Remark 3.2.5. If $m = 1$, then $u \in W_0^{1,\overrightarrow{r}(\cdot)}(\Omega;\mathbb{R}^d)$ with $1 \leq r_i(\cdot) < \dfrac{Np_i(\cdot)(\overline{p}(\cdot) - 1 - \gamma^+)}{\overline{p}(\cdot)(N - 1 - \gamma(\cdot)) - N(\gamma^+ - \gamma(\cdot))}$, which is the same result as in Theorem 3.1 in [64]. Additionally, if $m = 1$ and $d = 1$, and $\gamma(x) = 0$, then $u \in W_0^{1,\overrightarrow{r}(\cdot)}(\Omega)$ with $1 \leq r_i(\cdot) < \dfrac{Np_i(\cdot)(\overline{p}(\cdot) - 1)}{\overline{p}(\cdot)(N - 1)}$, which is the same result as Theorem 3.1 in [11].

In the subsequent section, we introduce a technical lemma, a pivotal outcome of our work, which serves as a key element in proving the estimate of the modulus of u in $\mathcal{M}^{q(\cdot)}(\Omega)$.

3.2.1 Technical Lemma

The following Lemma assumes a central role in substantiating the estimate of the modulus of u within the space $\mathcal{M}^{q(\cdot)}(\Omega)$.

Lemma 3.2.6. *([1]) Let $p_i(\cdot)$ and $s(\cdot)$ be continuous functions, where*

$$s(\cdot) \geq p_i(\cdot), \ i = 1, ..., N, \tag{3.2.6}$$

and m satisfies

$$1 < m < \frac{N\overline{p}(x)}{N\overline{p}(x) - N + \overline{p}(x)}. \tag{3.2.7}$$

Let g be a nonnegative function in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Suppose that there exists a constant c such that

$$\|g\|_{L^{s(\cdot)}(\Omega)} \leq c, \tag{3.2.8}$$

and

$$\sum_{i=1}^{N} \int_{\{g \leq k\}} |D_i g|^{p_i(x)} \, dx \leq c(1+k)^{\gamma^+} \left(\int_{\Omega} |T_k^+(g)|^{m'} \, dx \right)^{\frac{1}{m'}}. \tag{3.2.9}$$

Subsequently, there exists a constant C, depending on c, for all $k > 0$, ensuring that

$$\int_{\{g > k\}} k^{q(x)} \, dx \leq C, \ q(x) = \frac{Nm(\overline{p}(x) - 1 - \gamma^+)}{N - m\overline{p}(x)}, \ \forall x \in \overline{\Omega}.$$

Proof. We initiate our analysis by examining the case, for all x in $\overline{\Omega}$

$$q(x) < \frac{Nm(\overline{p}(x) - 1 - \gamma^+)}{N - m\overline{p}(x)}. \tag{3.2.10}$$

Firstly, let q^+ be a constant satisfying

$$\max_{x \in \overline{\Omega}} q(x) = q^+ < \min_{x \in \overline{\Omega}} \frac{Nm(\overline{p}(x) - 1 - \gamma^+)}{N - m\overline{p}(x)} = \frac{Nm(\overline{p}^- - 1 - \gamma^+)}{N - m\overline{p}^-}.$$

Additionally, given (3.2.7)

$$m < (\overline{p}^{-*})^{'} \Longleftrightarrow m^{'} > \overline{p}^{-*}, \overline{p}^{-*} = \frac{N\overline{p}^{-}}{N-\overline{p}^{-}}. \tag{3.2.11}$$

Thus, we can express

$$\begin{aligned}\left(\int_{\Omega} |T_k^+(g)|^{m^{'}}\, dx\right)^{\frac{1}{m^{'}}} &= \left(\int_{\Omega} |T_k^+(g)|^{\overline{p}^{-*}} |T_k^+(g)|^{m^{'}-\overline{p}^{-*}}\, dx\right)^{\frac{1}{m^{'}}} \\ &\leq k^{1-\frac{\overline{p}^{-*}}{m^{'}}} \left(\int_{\Omega} |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{1}{m^{'}}}.\end{aligned} \tag{3.2.12}$$

Using the hypothesis (3.2.9), we derive

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} |D_i T_k^+(g)|^{p_i(x)}\, dx &= \sum_{i=1}^{N} \int_{\{g\leq k\}} |D_i g|^{p_i(x)}\, dx \\ &\leq c(1+k)^{\gamma^+} \left(\int_{\Omega} |T_k^+(g)|^{m^{'}}\, dx\right)^{\frac{1}{m^{'}}}.\end{aligned}$$

Exploiting the fact that $|D_i T^+ k(g)|^{p_i^-} \leq |D_i T_k^+(g)|^{p_i(x)} + 1$, we can reformulate the preceding inequality for all $i = 1, ..., N$ as follows

$$\begin{aligned}\int_{\Omega} |D_i T_k^+(g)|^{p_i^-}\, dx &\leq \sum_{i=1}^{N} \int_{\Omega} |D_i T_k^+(g)|^{p_i^-}\, dx \\ &\leq c(1+k)^{\gamma^+} \left(\int_{\Omega} |T_k^+(g)|^{m^{'}}\, dx\right)^{\frac{1}{m^{'}}} + |\Omega| \\ &\leq c(1+k)^{\gamma^+} \left(\int_{\Omega} |T_k^+(g)|^{m^{'}}\, dx\right)^{\frac{1}{m^{'}}} + |\Omega|(1+k)^{\gamma^+} \\ &\leq c_1(1+k)^{\gamma^+} \left(\left(\int_{\Omega} |T_k^+(g)|^{m^{'}}\, dx\right)^{\frac{1}{m^{'}}} + 1\right).\end{aligned} \tag{3.2.13}$$

Subsequently, we deduce

$$\prod_{i=1}^{N} \left(\int_{\Omega} |D_i T_k^+(g)|^{p_i^-}\, dx\right)^{\frac{1}{Np_i^-}} \leq c_2(1+k)^{\frac{\gamma^+}{\overline{p}^-}} \left(\left(\int_{\Omega} |T_k^+(g)|^{m^{'}}\, dx\right)^{\frac{1}{m^{'}}} + 1\right)^{\frac{1}{\overline{p}^-}}. \tag{3.2.14}$$

By virtue of the Sobolev inequality, we attain

$$\left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{\overline{p}^-}{\overline{p}^{-*}}} \leq c_3(1+k)^{\gamma^+}\left(\left(\int_\Omega |T_k^+(g)|^{m'}\, dx\right)^{\frac{1}{m'}}+1\right). \tag{3.2.15}$$

By combining (3.2.12) and (3.2.15), we arrive at

$$\left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{\overline{p}^-}{\overline{p}^{-*}}} \leq c_4(1+k)^{\gamma^+}\left(k^{1-\frac{\overline{p}^{-*}}{m'}}\left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{1}{m'}}+1\right). \tag{3.2.16}$$

Drawing on (3.2.11), we can articulate

$$k^{1-\frac{\overline{p}^{-*}}{m'}} \leq (1+k)^{1-\frac{\overline{p}^{-*}}{m'}},$$

This observation, coupled with the estimate (3.2.16), implies

$$\begin{aligned}
&\left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{\overline{p}^-}{\overline{p}^{-*}}} \leq c_4(1+k)^{\gamma^+}\times \\
&\left((1+k)^{1-\frac{\overline{p}^{-*}}{m'}}\left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{1}{m'}}+(1+k)^{1-\frac{\overline{p}^{-*}}{m'}}\right) \\
&\leq c_4\,(1+k)^{1+\gamma^+-\frac{\overline{p}^{-*}}{m'}}\left(\left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{1}{m'}}+1\right).
\end{aligned} \tag{3.2.17}$$

We can distinguish between two cases. First, if $\left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{1}{m'}} \leq 1$, and as $m>1$ then, $\dfrac{\overline{p}^-}{\overline{p}^{-*}}-\dfrac{1}{m'}<\dfrac{\overline{p}^-}{\overline{p}^{-*}}$ and for $k\geq 1$, we have

$$\left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{\overline{p}^-}{\overline{p}^{-*}}-\frac{1}{m'}} \leq \left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{\overline{p}^-}{\overline{p}^{-*}}}+|\Omega| \leq c_5 k^{\gamma^++1-\frac{\overline{p}^{-*}}{m'}}.$$

As a consequence, it follows

$$\left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{\overline{p}^-}{\overline{p}^{-*}}-\frac{1}{m'}} \leq c_5 k^{\gamma^++1-\frac{\overline{p}^{-*}}{m'}}.$$

It is evident that $|T_k^+(g)| = k$ on A_k such that $A_k = \{x \in \Omega : |g| > k\}$, which implies

$$|A_k|^{\frac{\overline{p}^-}{\overline{p}^{-*}} - \frac{1}{m'}} \leq c_6 k^{-(\overline{p}^- - 1 - \gamma^+)}. \tag{3.2.18}$$

Here, $m > 1$ ensures that the exponent $\dfrac{\overline{p}^-}{\overline{p}^{-*}} - \dfrac{1}{m'} > 0$. Direct calculation implies that

$$|A_k| \leq c_7 k^{-\frac{Nm(\overline{p}^- - 1 - \gamma^+)}{N - m\overline{p}^-}}.$$

In the second case, if $\left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{1}{m'}} \geq 1$, and from (3.2.17), we have

$$\left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{\overline{p}^-}{\overline{p}^{-*}}} \leq c_8 (1 + k)^{1 + \gamma^+ - \frac{\overline{p}^{-*}}{\overline{p}^-}} \left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{1}{m'}}.$$

Subsequently,

$$\begin{aligned} k^{\overline{p}^- - \frac{\overline{p}^{-*}}{m'}} |A_k|^{\frac{\overline{p}^-}{\overline{p}^{-*}} - \frac{1}{m'}} &\leq \left(\int_\Omega |T_k^+(g)|^{\overline{p}^{-*}}\, dx\right)^{\frac{\overline{p}^{-*}}{\overline{p}^-} - \frac{1}{m'}} \\ &\leq c_8 (1 + k)^{\gamma^+ + 1 - \frac{\overline{p}^{-*}}{m'}}. \end{aligned}$$

Therefore, for $k \geq 1$, and based on our direct calculation, we have

$$|A_k| \leq c_9 k^{-\frac{Nm(\overline{p}^- - 1 - \gamma^+)}{N - m\overline{p}^-}}.$$

If $k < 1$, it is straightforward to observe that

$$|A_k| \leq |\Omega| \leq |\Omega| k^{-\frac{Nm(\overline{p}^- - 1 - \gamma^+)}{N - m\overline{p}^-}}.$$

This proves that $g \in \mathcal{M}^{\frac{Nm(\overline{p}^- - 1 - \gamma^+)}{N - m\overline{p}^-}}(\Omega)$.

Now, let us consider a continuous variable exponent $q(.)$ on $\overline{\Omega}$ that satisfies the estimate (3.2.10). Additionally, we have

$$q^+ \geq \frac{Nm(\overline{p}^- - 1 - \gamma^+)}{N - m\overline{p}^-}. \tag{3.2.19}$$

Due to the continuity of $q(x), \overline{p}(x)$ on $\overline{\Omega}$, there exists a constant $\delta > 0$ such that

$$\max_{y\in Q(x,\delta)\cap\Omega} q(y) < \min_{y\in Q(x,\delta)\cap\Omega} \frac{Nm(\overline{p}(y)-1-\gamma)}{N-m\overline{p}(y)}, \tag{3.2.20}$$

where $Q(x,\delta)$ is a cube with center x and diameter δ. Observe that $\overline{\Omega}$ is compact and therefore we can cover it with a finite number of cubes $(Q_j)_{j=1,\ldots,l}$ with edges parallel to the coordinate axes. Moreover, there exists a constant $\nu > 0$ such that

$$\delta > |\Omega_j| > \nu,\ \Omega_j := Q_j \cap \Omega,\ \text{for all}\, j = 1, ..., l. \tag{3.2.21}$$

We denote by q_j^+ the local maximum of q on $\overline{\Omega}_j$, respectively ($p_{i,j}^-$ the local minimum of $p_i(.)$ on $\overline{\Omega}_j$), for all $i = 1, ..., N$.

Applying analogous arguments as before, but on a local scale, we observe that the inequality (3.2.13) holds within the region Ω_j. Therefore

$$\int_{\Omega_j} |D_i T_k^+(g)|^{p_{i,j}^-}\, dx \le c_1'(1+k)^{\gamma^+}\left(\left(\int_{\Omega_j} |T_k^+(g)|^{m'}\, dx\right)^{\frac{1}{m'}} + 1\right). \tag{3.2.22}$$

Consequently, we obtain

$$\begin{aligned} 1 + \|D_i T_k^+(g)\|_{L^{p_{i,j}^-}(\Omega_j)} &\le c_2'(1+k)^{\frac{\gamma^+}{p_{i,j}^-}}\left(\left(\int_{\Omega_j} |T_k^+(g)|^{m'}\, dx\right)^{\frac{1}{m'}} + 1\right)^{\frac{1}{p_{i,j}^-}} + 1 \\ &\le c_3'(1+k)^{\frac{\gamma^+}{p_{i,j}^-}}\left(\left(\int_{\Omega_j} |T_k^+(g)|^{m'}\, dx\right)^{\frac{1}{m'}} + 1\right)^{\frac{1}{p_{i,j}^-}} + (1+k)^{\frac{\gamma^+}{p_{i,j}^-}} \\ &\le c_4'(1+k)^{\frac{\gamma^+}{p_{i,j}^-}}\left(\left(\left(\int_{\Omega_j} |T_k^+(g)|^{m'}\, dx\right)^{\frac{1}{m'}} + 1\right)^{\frac{1}{p_{i,j}^-}} + 1\right) \\ &\le c_5'(1+k)^{\frac{\gamma^+}{p_{i,j}^-}}\left(\left(\int_{\Omega_j} |T_k^+(g)|^{m'}\, dx\right)^{\frac{1}{m'}} + 1\right)^{\frac{1}{p_{i,j}^-}}. \end{aligned} \tag{3.2.23}$$

The anisotropic Sobolev inequality 1.2.2 implies that

$$\left(\int_{\Omega_j} |T_k^+(g)|^{\overline{p}_j^{-*}}\,dx\right)^{\frac{1}{\overline{p}_j^{-*}}} \le c_6' \prod_{i=1}^{N}\left(\|g\|_{L^{p_{i,j}^-}(\Omega_j)} + \|D_i T_k^+(g)\|_{L^{p_{i,j}^-}(\Omega_j)}\right)^{\frac{1}{N}}. \tag{3.2.24}$$

Expanding (3.2.6), we get

$$p_{i,j}^- \le s_j^-,\ \ s_j^- = \min_{x\in\overline{\Omega}_j} s(x).$$

By using Proposition 1.1.4 in conjunction with inequality (3.2.8), we can deduce that

$$\begin{aligned}
\|g\|_{L^{p_{i,j}^-}(\Omega_j)} \quad &\le 1 + \int_{\Omega_j} |g|^{p_{i,j}^-}\,dx \\
&\le 1 + |\Omega_j| + \int_{\Omega_j} |g|^{s_j^-}\,dx \\
&\le 1 + 2|\Omega_j| + \int_{\Omega_j} |g|^{s(x)}\,dx \\
&\le 1 + 2|\Omega_j| + \int_{\Omega_j} |g|^{s(x)}\,dx \\
&\le c_7'.
\end{aligned}$$

This implies that

$$\left\{\int_{\Omega_j} |T_k^+(g)|^{\overline{p}_j^{-*}}\,dx\right\}^{\frac{1}{\overline{p}_j^{-*}}} \le c_8' \prod_{i=1}^{N}\left(1 + \|D_i T_k^+(g)\|_{L^{p_{i,j}^-}(\Omega_j)}\right)^{\frac{1}{N}}. \tag{3.2.25}$$

According to (3.2.23) and (3.2.25), we obtain

$$\begin{aligned}
\left(\int_{\Omega_j} |T_k^+(g)|^{\overline{p}_j^{-*}}\,dx\right)^{\frac{1}{\overline{p}_j^{-*}}} \quad &\le c_9' \prod_{i=1}^{N}\left(\left(\int_{\Omega_j} |T_k^+(g)|^{m'}\,dx\right)^{\frac{1}{m'}} + 1\right)^{\frac{1}{N p_{i,j}^-}} \\
&\le c_9' \left(\left(\int_{\Omega_j} |T_k^+(g)|^{m'}\,dx\right)^{\frac{1}{m'}} + 1\right)^{\frac{1}{\overline{p}_j^-}}.
\end{aligned} \tag{3.2.26}$$

However, the inequality (3.2.12) remains valid on Ω_j. Therefore, in light of (3.2.26), we obtain

$$\left(\int_{\Omega_j} |T_k^+(g)|^{\overline{p}_j^{-*}}\, dx\right)^{\frac{\overline{p}_j^-}{\overline{p}_j^{-*}}} \le c_{10}^{'}(1+k)^{1-\frac{\overline{p}_j^{-*}}{m'}} \left(\left(\int_{\Omega_j} |T_k^+(g)|^{\overline{p}_j^{-*}}\, dx\right)^{\frac{1}{m'}} + 1\right). \tag{3.2.27}$$

Put

$$|A_k^j| = |\,\{x \in \Omega_j :\ |g(x)| > k\}\,|, k \ge 0, j = 1, ..., l.$$

Applying a similar approach as before, this time at the local level, we can conclude that for $k \ge 1$

$$|A_k^j| \le c_{11}^{'} k^{-\frac{Nm(\overline{p}_j^- - 1 - \gamma^+)}{N - m\overline{p}_j^-}}.$$

Taking into account that, for all $k < 1$

$$|A_k^j| \le |\Omega_j| k^{-\frac{Nm(\overline{p}_j^- - 1 - \gamma^+)}{N - m\overline{p}_j^-}}.$$

Therefore,

$$g \in \mathcal{M}^{\frac{Nm(\overline{p}_j^- - 1 - \gamma^+)}{N - m\overline{p}_j^-}}(\Omega_j). \tag{3.2.28}$$

Finally, since $q(x) \le q_j^+$, for all $x \in \Omega_j$ and all $j = 1, ..., l$, we have that $g \in \mathcal{M}^{q(.)}(\Omega)$.
Moreover, assuming $q(x) = \dfrac{Nm(\overline{p}(x) - 1 - \gamma^+)}{N - m\overline{p}(x)}, \forall x \in \overline{\Omega}$ and let $\varepsilon \in (0, q^-)$, then

$$\int_{\{g>t\}} t^{q(x)-\varepsilon}\, dx \le C. \tag{3.2.29}$$

By letting ε go to zero, we determine the proof. □

3.2.2 Approximate problems

In order to prove our main results, let us consider the sequence of approximate systems

$$\begin{cases} -\sum_{i=1}^{N} D_i\left(a_i(x, T_n(u_n(x)), D_i u_n(x))\right) + F(x, u_n) = f_n(x), & x \in \Omega, \\ u_n(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.2.30}$$

Let $(f_n)_n$ be a sequence of bounded functions defined in Ω which converge to $f \in L^m(\Omega; \mathbb{R}^d)$ with $m > 1$, and verifies the inequalities

$$|f_n| \leq n \quad \text{and } |f_n| \leq |f|, \quad \forall n \in \mathbb{N}. \tag{3.2.31}$$

and $F(x, u_n) = T_n(F(x, u))$, $n > 0$.

We are going to prove the existence of solution u_n to a systems (3.2.30).

Lemma 3.2.7. *Let $s : \overline{\Omega} \to (0, +\infty)$, $p_i : \overline{\Omega} \to (1, +\infty)$, $i = 1, ..., N$ be continuous function. Assume that (1.2.4) holds. Then, there exists at least one solution $u_n = (u_{1,n}, ..., u_{d,n})$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega; \mathbb{R}^d)$ to systems (3.2.30) in the sense that*

$$\int_\Omega \sum_{i=1}^{N} a_i(x, T_n(u_n), D_i u_n).D_i\varphi \, dx + \int_\Omega F(x, u_n).\varphi \, dx = \int_\Omega f_n.\varphi \, dx \tag{3.2.32}$$

$\forall \varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^d) \cap L^{s(\cdot)}(\Omega, \mathbb{R}^d)$.

Proof. Consider the following systems

$$\begin{cases} -\sum_{i=1}^{N} D_i\left(a_i(x, T_n(u_{n_m}(x)), D_i u_{n_m}(x))\right) + F(x, u_{n_m}) = f_{n_m}(x), & x \in \Omega, \\ u_{n_m}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.2.33}$$

In a similar way to the results obtained in Application 2.1, we deduce that there exists a solution

$u_{n_m} \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^d)$ to systems (3.2.33), which satisfies

$$\int_\Omega \sum_{i=1}^N a_i(x, T_n(u_{n_m}), D_i u_{n_m}).D_i\varphi \, dx + \int_\Omega F(x, u_{n_m}).\varphi \, dx = \int_\Omega f_{n_m}.\varphi \, dx \tag{3.2.34}$$

for all $\varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^d)$. Using (3.1.2) and (3.1.6), Hölder inequality, Young's inequality and inserting $\varphi = u_{n_m}$ as test function in (3.2.35), we obtain

$$\begin{aligned}
\sum_{i=1}^N \int_\Omega |D_i u_{n_m}|^{p_i(x)} \, dx \qquad & \leq C_1(n) \left(1 + \int_\Omega |u_{n_m}| \, dx\right) \\
& \leq C_1(n)\varepsilon^* \left(1 + \int_\Omega |u_{n_m}|^{p_-^-} \, dx\right) \\
& \leq C_2(n)\varepsilon^* \left(1 + \int_\Omega |D_i u_{n_m}|^{p_-^-} \, dx\right) \\
& \leq C_3(n)\varepsilon^* \left(1 + \sum_{i=1}^N \int_\Omega |D_i u_{n_m}|^{p_i(x)} \, dx\right).
\end{aligned}$$

We put $\varepsilon^* = \dfrac{1}{2C_3(n)}$, we get

$$\sum_{i=1}^N \int_\Omega |D_i u_{n_m}|^{p_i(x)} \, dx \leq C_4(n), i = 1, ..., N,$$

where $C_i(n), i = 1, ..., 4$ are positive constants depending on n. Consequently, there exists a sequence $u_n \subset W_0^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^d)$ such that

$$u_{n_m} \rightharpoonup u_n \text{ weakly in } W_0^{1,\overrightarrow{p}(\cdot)}(\Omega, \mathbb{R}^d) \text{ and a.e. in } \Omega.$$

Using (3.1.4) and arguing as in the proof of (3.2.36), we obtain

$$D_i u_{n_m} \longrightarrow D_i u_n \text{ strongly in } L^{p_i(\cdot)}(\Omega; \mathbb{R}^d) \text{ and a.e. in } \Omega. \tag{3.2.35}$$

So,

$$a_i(x, T_n(u_{n_m}), D_i u_{n_m}) \rightharpoonup a_i(x, T_n(u_n), D_i u_n) \text{ weakly in } \left(L^{p_i'(\cdot)}(\Omega; \mathbb{R}^d)\right)'. \tag{3.2.36}$$

Taking $T_k(u_{n_m})$ as a test function in (3.2.35), by (3.1.2) and (3.1.10), we obtain

$$\int_\Omega \sum_{i=1}^{N} a_i(x, T_n(u_{n_m}), D_i u_{n_m}).DT_k(u_{n_m}) D_i u_{n_m}\, dx \geq \alpha_1 \sum_{i=1}^{N} \int_{\{|u_{n_m}|\leq k\}} \frac{|D_i u_{n_m}|^{p_i(x)}}{(1+n)^{\gamma(x)}}\, dx$$

By (3.1.6), (3.1.7) and the fact that

$$\frac{u_{n_m}}{|u_{n_m}|}.T_m\left(F(x,u_{n_m})\right) \geq \left|T_m\left(F(x,u_{n_m})\right)\right|, \ |u_{n_m}| > 0,$$

this implies

$$\int_\Omega T_m\left(F(x,u_{n_m})\right).T_k\left(u_{n_m}\right)\, dx \geq k \int_{\{|u_{n_m}|>k\}} \left|T_m\left(F(x,u_{n_m})\right)\right|\, dx.$$

However, given that

$$|T_k(s)| \leq M + k1_{\{|s|>M\}}, \forall s \in \mathbb{R}, M > 0.$$

On the other hand, we have

$$\int_\Omega |f_n|\, |T_k(u_{n_m})|\, dx \leq M \, \|f_n\|_{L^1\left(\Omega;\mathbb{R}^d\right)} + k \int_{\{|u_{n_m}|>M\}} |f_n|\, dx.$$

Through the above mentioned results, we obtain for all $M > 0$

$$\int_{\{|u_{n_m}|>K\}} \left|T_k\left(F(x,u_{n_m})\right)\right|\, dx \leq \frac{M\, \|f_n\|_{L^1(\Omega;\mathbb{R}^d)}}{k} + \int_{\{|u_{n_m}|>M\}} |f_n|\, dx. \tag{3.2.37}$$

Let $E \subset \Omega$ be any measurable set, we write

$$\int_E \left|T_m\left(F(x,u_{n_m})\right)\right|\, dx = \int_{\{E\cap\{|u_{n_m}|\leq k\}\}} \left|T_m\left(F(x,u_{n_m})\right)\right|\, dx + \int_{\{E\cap\{|u_{n_m}|>k\}\}} \left|T_m\left(F(x,u_{n_m})\right)\right|\, dx$$

then, by (3.1.7) and (3.2.37), we deduce the sequence $(T_m(F(x,u_{n_m})))_m$ is equi-integrable in $L^1(\Omega;\mathbb{R}^d)$. Therefore, we can obtain (3.2.32) by passing to the limit in (3.2.35). □

3.2.3 Uniform estimates

In this part, we establish uniform estimates for both the approximate solutions, denoted as u_n, and their partial derivatives. Consistently throughout the chapter, the various constants introduced in this section will represent positive values dependent solely on the problem's data, remaining unaffected by the variable n.

Lemma 3.2.8. *Let $f_n \in L^m(\Omega; \mathbb{R}^d)$ and m be restricted as in Theorem 3.2.2, and let the sequence (u_n) be a solution satisfying (3.2.30). Then, there exists a constant c depending on $\|f_n\|_{L^m(\Omega;\mathbb{R}^d)}$ but not on u_n, such that*

$$\|u_n\|_{W_0^{1,\vec{p}(\cdot)}(\Omega;\mathbb{R}^d)} \leq c.$$

Proof. Inserting u_n in (3.2.30) yields

$$\int_\Omega \sum_{i=1}^{N} a_i\left(x, T_n(u_n), D_i u_n\right).D_i u_n\, dx + \int_\Omega F(x, u_n).u_n\, dx = \int_\Omega f_n.u_n\, dx.$$

The assumption (3.1.6) implies that $F(x, u_n).u_n \geq 0$, then

$$\int_\Omega \sum_{i=1}^{N} a_i\left(x, T_n(u_n), D_i u_n\right).D_i u_n\, dx \leq \int_\Omega f_n.u_n\, dx. \tag{3.2.38}$$

By the degenerate coercivity (3.1.2) and (3.2.38), we find that

$$\begin{aligned} \alpha_1(1+n)^{-\gamma^+} \sum_{i=1}^{N} \int_\Omega |D_i u_n|^{p_i(x)}\, dx \qquad &\leq \int_\Omega \sum_{i=1}^{N} a_i(x, T_k(u_n), D_i u_n).D_i u_n\, dx \\ &\leq \int_\Omega f_n.u_n\, dx. \end{aligned}$$

Using Hölder's inequality, we get

$$\alpha_1(1+n)^{-\gamma^+} \sum_{i=1}^{N} \int_\Omega |D_i u_n|^{p_i(x)}\, dx \leq \|f_n\|_{L^m(\Omega;\mathbb{R}^d)} \|u_n\|_{L^{m'}(\Omega;\mathbb{R}^d)}.$$

Since $m > (\overline{p}^*(x))'$ then $m' < \overline{p}^*(x)$. Therefore, thanks to Lemma 1.2.6 and assumption (1.2.4)

$$m' < \max\left(\overline{p}^*(x), p_+(x)\right) = \overline{p}^*(x), \text{ for all}\, x \in \overline{\Omega}.$$

Consequently, we have the following embedding continuous

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d) \hookrightarrow L^{m'}(\Omega;\mathbb{R}^d),$$

then, there exists a positive constant C_1', such that

$$\sum_{i=1}^{N}\int_{\Omega}|D_i u_n|^{p_i(x)}\,dx \le \frac{C_1'\,(1+n)^{\gamma^+}\,\|f_n\|_{L^m}}{\alpha_1}\sum_{i=1}^{N}\|D_i u_n\|_{L^{p_i(\cdot)}(\Omega;\mathbb{R}^d)}.$$

Put $C_2' = \dfrac{C_1'\,(1+n)^{\gamma^+}\,\|f_n\|_{L^m}}{\alpha_1}$, we recall the following well-know inequalities that holds for any $\alpha_j \ge 0$ $j \in \{1,...,N\}$ and a real $p > 0$

$$\sum_{j=1}^{N}(\alpha_j)^p \le N\left(\sum_{j=1}^{N}\alpha_j\right)^p. \tag{3.2.39}$$

Using Proposition 1.1.4 and (3.2.39), we get

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}|D_i u_n|^{p_i(x)}\,dx \qquad &\le C_3'\sum_{i=1}^{N}\left(\int_{\Omega}|D_i u_n|^{p_i(x)}\,dx\right)^{\frac{1}{\beta_i}}\\ &\le C_4'\left(\sum_{i=1}^{N}\int_{\Omega}|D_i u_n|^{p_i(x)}\,dx\right)^{\frac{1}{\beta_i}},\end{aligned}$$

where

$$\beta_i = \begin{cases} p_i^-, & \text{if } \|D_i u_n\|_{L^{p_i(.)}} \ge 1,\\ p_i^+, & \text{if } \|D_i u_n\|_{L^{p_i(.)}} \le 1.\end{cases}$$

Then

$$\left(\sum_{i=1}^{N}\int_{\Omega}|D_i u_n|^{p_i(x)}\,dx\right)^{1-\frac{1}{\beta_i}} \le C_4'.$$

Hence, the sequence u_n is bounded in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$. □

Lemma 3.2.9. *There exists a positive constant c such that*

$$\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}|D_i u_n|^{p_i(x)}\,dx \le c(1+k)^{\gamma^+}\left(\int_\Omega |T_k(u_n)|^{m'}\,dx\right)^{\frac{1}{m'}}. \tag{3.2.40}$$

Moreover

$$\int_\Omega |F(x,u_n)|\,dx \le C, \tag{3.2.41}$$

also

$$\int_\Omega |u_n|^{s(x)}\,dx \le C. \tag{3.2.42}$$

The constant C is positive and depends solely on the problem's data, but not on n, the $|\cdot|$ represents the vector modulus.

Proof. Taking $T_k(u_n)$ as a test function in (3.2.30)

$$\int_{\{|u_n|\le k\}}\sum_{i=1}^{N} a_i(x,T_n(u_n),D_i u_n).D_i u_n\,dx + \int_\Omega F(x,u_n).T_k(u_n)\,dx = \int_\Omega f_n.T_k(u_n)\,dx. \tag{3.2.43}$$

The assumption (3.1.6) implies that $F(x,u_n).T_k(u_n)\ge 0$, then

$$\int_{\{|u_n|\le k\}}\sum_{i=1}^{N} a_i(x,T_n(u_n),D_i u_n).D_i u_n\,dx \le \int_\Omega f_n.T_k(u_n)\,dx. \tag{3.2.44}$$

By the degenerate coercivity together with the angle condition (3.1.10) and (3.2.44), we deduce that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}\frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma(x)}}\,dx\\
\le \frac{1}{\alpha_1}\quad &\int_{\{|u_n|\le k\}}\sum_{i=1}^{N} a_i(x,T_n(u_n),D_i u_n).D_i u_n\,dx\\
\le \frac{1}{\alpha_1}\quad &\int_\Omega f_n.T_k(u_n)\,dx.
\end{aligned}$$

Applying Hölder's inequality for $m > 1$, which implies

$$\sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma(x)}}\, dx \leq \frac{1}{\alpha_1} \|f_n\|_{L^m(\Omega;\mathbb{R}^d)} \left(\int_\Omega |T_k(u_n)|^{m'}\, dx \right)^{\frac{1}{m'}}. \tag{3.2.45}$$

For $k > 0$,

$$(1+k)^{-\gamma^+} \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} |D_i u_n|^{p_i(x)}\, dx \leq \frac{1}{\alpha_1} \|f_n\|_{L^m(\Omega;\mathbb{R}^d)} \left(\int_\Omega |T_k(u_n)|^{m'}\, dx \right)^{\frac{1}{m'}},$$

and thus we obtain the result (3.2.40), where $c = \dfrac{1}{\alpha_1} \|f_n\|_{L^m(\Omega;\mathbb{R}^d)}$. Therefore, from (3.2.45)

$$\begin{aligned} \int_{\{|u_n| \leq k\}} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma(x)}}\, dx &\leq \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma(x)}}\, dx \\ &\leq c|\Omega|^{\frac{1}{m'}} k \\ &\leq c' (1+k). \end{aligned} \tag{3.2.46}$$

By (3.2.46) and for $k \geq 1$

$$\begin{aligned} \int_{\{|u_n| \leq k\}} k^{-1-\gamma(x)} |D_i u_n|^{p_i(x)}\, dx &\leq \int_{\{|u_n| \leq k\}} 2^{1+\gamma^+} (1+k)^{-1-\gamma(x)} |D_i u_n|^{p_i(x)}\, dx \\ &\leq 2^{1+\gamma^+} (1+k)^{-1} \int_{\{|u_n| \leq k\}} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma(x)}}\, dx \\ &\leq c^*. \end{aligned} \tag{3.2.47}$$

For the proof of (3.2.41), by (3.1.6) and (3.2.43), we obtain

$$\int_\Omega \sum_{i=1}^{N} a_i(x, T_n(u_n), D_i u_n).DT_k(u_n) D_i u_n\, dx + k \int_{\{|u_n| > k\}} \frac{u_n}{|u_n|}.F(x, u_n)\, dx \leq \int_\Omega f_n.T_k(u_n)\, dx,$$

using (3.1.5), we have

$$\frac{u_n}{|u_n|}.F(x, u_n) \geq |F(x, u_n)|, \ |u_n| > 0. \tag{3.2.48}$$

Then

$$\alpha_1 \sum_{i=1}^{N} \int_{\{|u_n|\le k\}} \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{\gamma(x)}} + k \int_{\{|u_n|>k\}} \frac{u_n}{|u_n|}.F(x,u_n)\,dx \le k\,\|f_n\|_{L^m(\Omega;\mathbb{R}^d)} + c$$

Thus

$$\int_{\{|u_n|>k\}} |F(x,u_n)|\;dx \le C, \tag{3.2.49}$$

Consequently, by (3.1.7) and (3.2.49), we derive (3.2.41). Finally, we combine (3.1.6) and (3.2.41) to obtain (3.2.42). This ends the proof of Lemma. □

Lemma 3.2.10. *Assuming p_i is defined as in (3.2.4), $s(\cdot) > 0$, and u_n is a solution of (3.2.30) in the sense of (3.2.32). Then there exists a constant C such that*

$$\|u_n\|_{\mathcal{M}^{q(\cdot)}(\Omega;\mathbb{R}^d)} \le C,\;\; q(x) = \frac{Nm(\overline{p}(x)-1-\gamma^+)}{N-m\overline{p}(x)}. \tag{3.2.50}$$

Additionally

$$\|D_i u_n\|_{\mathcal{M}^{q_i(\cdot)}(\Omega;\mathbb{R}^d)} \le C,\;\;\; q_i(x) = \frac{Nmp_i(x)(\overline{p}(x)-1-\gamma^+)}{Nm(\overline{p}(x)-1-\gamma^+)+(1+\gamma(x))(N-m\overline{p}(x))}. \tag{3.2.51}$$

Proof. By Lemma 3.2.9 and this fact $|D_i|u_n|| \le |D_i u_n|$, and thanks to remark 1.4.2, we get

$$\begin{aligned}\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} |D_i|u_n||^{p_i(x)}\,dx \quad &\le c(1+k)^{\gamma^+}\left(\int_\Omega |T_k(u_n)|^{m'}\,dx\right)^{\frac{1}{m'}}\\ &\le c(1+k)^{\gamma^+}\left(\int_\Omega |T_k^+(|u_n|)|^{m'}\,dx\right)^{\frac{1}{m'}}.\end{aligned}$$

By applying Lemma 3.2.6 to $|u_n|$ gives $\||u_n|\|_{\mathcal{M}^{q(\cdot)}(\Omega)} \le C$. This proves (3.2.50).

We will now focus on proving the derivatives estimate. Putting $\theta_i(\cdot) = \dfrac{p_i(\cdot)}{q(\cdot)+\gamma(\cdot)+1}$ for $i =$

$1, ..., N$, then, for $k \geq 1$ and from the estimates (3.2.47) and (3.2.50) we conclude

$$\begin{aligned}\int_{\{|D_i u_n|^{\theta_i(x)}>k\}} k^{q(x)}\,dx &\leq \int_{\{|D_i u_n|^{\theta_i(x)}>k\}\cap\{|u_n|\leq k\}} k^{q(x)}\,dx + \int_{\{|u_n|>k\}} k^{q(x)}\,dx \\ &\leq \int_{\{|u_n|\leq k\}} k^{q(x)} \left(\frac{|D_i u_n|^{\theta_i(x)}}{k}\right)^{\frac{p_i(x)}{\theta_i(x)}} dx + C \\ &\leq \int_{\{|u|\leq k\}} k^{-1-\gamma(x)} |D_i u_n|^{p_i(x)}\,dx + C.\end{aligned}$$

Invoking (3.2.47), we get for all $k \geq 1$

$$\int_{\{|D_i u_n|^{\theta_i(x)}>k\}} k^{q(x)}\,dx \leq C. \tag{3.2.52}$$

If $k \in (0,1)$, we have

$$\int_{\{|D_i u_n|^{\theta_i(x)}>k\}} k^{q(x)}\,dx \leq \int_\Omega k^{q(x)}\,dx \leq |\Omega|.$$

Consequently, for all $k > 0$, we obtain

$$\int_{\{|D_i u_n|^{\theta_i(x)}>k\}} k^{q(x)}\,dx \leq C.$$

This proves that for all $i = 1, ..., N$, $|D_i u_n|$ is bounded in $\mathcal{M}^{q_i(\cdot)}(\Omega)$, where

$$\begin{aligned} q_i(x) &= \theta_i(x) q(x) = \frac{p_i(x) q(x)}{q(x) + \gamma(x) + 1} \\ &= \frac{N m p_i(x)(\overline{p}(x) - 1 - \gamma^+)}{N m(\overline{p}(x) - 1 - \gamma^+) + (1 + \gamma(x)(N - m\overline{p}(x)))}.\end{aligned}$$

This ends the proof of Lemma. □

Thanks to the Lemma 1.3.8, we conclude that $D_i u_n$ is bounded in $L^{r_i(\cdot)}(\Omega; \mathbb{R}^d)$ for all $r_i(\cdot)$ in $C_+(\overline{\Omega})$ satisfying (3.2.5), so we have for all $i = 1, ..., N$

$$\|D_i u_n\|_{L^{r_i(\cdot)}(\Omega;\mathbb{R}^d)} \leq C. \tag{3.2.53}$$

3.2.4 Proof of the main results

The proof of Theorem 3.2.2 is similar to that of Theorem 3.2.3. Therefore, here we will only provide the proof of Theorem 3.2.3.

Let $(T_n(f))_n := (f_n)_n \in L^\infty(\Omega;\mathbb{R}^d)$, be a sequence of functions, such that

$$f_n \to f \quad \text{in} \quad L^m(\Omega;\mathbb{R}^d), \quad \text{as} \quad n \to +\infty, \tag{3.2.54}$$

and

$$\|f_n\|_{L^m(\Omega;\mathbb{R}^d)} \leq \|f\|_{L^m(\Omega;\mathbb{R}^d)}. \tag{3.2.55}$$

Let u_n be a solution of the problem (3.2.30) that satisfies the weak formulation (3.2.32).

From Lemma 1.3.8, the sequence $(u_n)_n$ is bounded in $W_0^{1,\overrightarrow{r}(\cdot)}(\Omega;\mathbb{R}^d)$, where $r_i(x)$ is defined such as (3.2.5). Therefore, $(u_n)_n$ is bounded in the Sobolev space

$$W_0^{1,r_-^-}(\Omega;\mathbb{R}^d), \;\; r_-^- = \min_{1\leq i\leq N}\min_{x\in\overline{\Omega}} r_i(x). \tag{3.2.56}$$

Thanks to the Rellich embedding Theorem, we can extract a subsequence denoted again as $(u_n)_n$ such that

$$u_n \to u \;\; \text{strongly in} \;\; L^{r_-^-}(\Omega;\mathbb{R}^d) \text{ and } a.e. \text{ in}\, \Omega. \tag{3.2.57}$$

In order to prove the convergence almost every where of derivatives $D_i u_n$ for all $i = 1, ..., N$, we need to present and prove the fundamental Lemma 3.2.13 .Furthermore, we use the analogous ways in [86] with some modifications.

Firstly, we introduce the following notation

$$p_+^+ = \max_{1\leq i\leq N}\max_{x\in\overline{\Omega}} p_i(x), u_n^k = \mathcal{T}_k(u_n), u^k = \mathcal{T}_k(u)$$

.

Lemma 3.2.11. *Let us define the vector $\Delta_n^i(u_n,u)$, $\forall i=1,...,N$ such that*

$$\Delta_n^i(u_n,u) := \left(a_i\left(x,T_n(u_n),D_iu_n\right) - a_i\left(x,T_n(u_n),D_iu\right)\right).D_i(u_n-u). \tag{3.2.58}$$

Then

$$\left\{\Delta_n^i(u_n,u)\right\}^{\frac{1}{p_+^+}} \leq C\left(1+\sum_{i=1}^N |D_iu_n| + \sum_{i=1}^N |D_iu|\right) \tag{3.2.59}$$

Proof. Using assumption (3.1.3) with the fact that $|T_n(u_n)| \leq \epsilon$, we find

$$\begin{aligned}
\Delta_n^i(u_n,u) \quad & \leq \left|a_i\left(x,T_n(u_n),D_iu_n\right) - a_i\left(x,T_n(u_n),D_iu\right)\right| |D_iu_n - D_iu| \\
& \leq \left(\left|a_i\left(x,T_n(u_n),D_iu_n\right)\right| + \left|a_i\left(x,T_n(u_n),D_iu\right)\right|\right) \times \left(|D_iu_n| + |D_iu|\right) \\
& \leq \left\{C_1\left(n^{\overline{p}^-} + |h| + \sum_{j=1}^N |D_ju_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} + C_2\left(n^{\overline{p}^-} + |h| + \sum_{j=1}^N |D_ju|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}\right\} \\
& \times \left\{|D_iu_n| + |D_iu|\right\} \\
& \leq \left\{C_1\left(n^{\overline{p}^-} + |h| + \sum_{j=1}^N |D_ju_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} + C_2\left(n^{\overline{p}^-} + |h| + \sum_{j=1}^N |D_ju|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}\right\} \\
& \times \left\{\sum_{i=1}^N |D_iu_n| + \sum_{i=1}^N |D_iu|\right\},
\end{aligned}$$

we recall that $(a+b)^\alpha \leq \max\left(1,2^{\alpha-1}\right)(a^\alpha+b^\alpha)$. Obtaining

$$\begin{aligned}
\Delta_n^i(u_n,u) \quad & \leq C'\left(1+\sum_{i=1}^N |D_iu_n|^{p_i(x)-1} + \sum_{i=1}^N |D_iu|^{p_i(x)-1}\right)\left(\sum_{i=1}^N |D_iu_\epsilon| + \sum_{i=1}^N |D_iu|\right) \\
& \leq C^*\left(1+\sum_{i=1}^N |D_iu_n|^{p_+^+-1} + \sum_{i=1}^N |D_iu|^{p_+^+-1}\right)\left(\sum_{i=1}^N |D_iu_n| + \sum_{i=1}^N |D_iu|\right)
\end{aligned} \tag{3.2.60}$$

By using the inequality

$$\left(\sum_{i=1}^N |\beta_i|^{p-1}\right)\left(\sum_{i=1}^N |\beta_i|\right) \leq N\sum_{i=1}^N |\beta_i|^p,$$

then (3.2.60) get as follow

$$\Delta_\epsilon^i(u_n,u) \leq C_1^* \left(1 + \sum_{i=1}^N |D_i u_n|^{p_+^+} + \sum_{i=1}^N |D_i u|^{p_+^+}\right).$$

This ends the proof of Lemma. □

Lemma 3.2.12. *Our aim is to prove that, For all $\varepsilon^* > 0$, we have*

$$\limsup_{\varepsilon^*} \sum_{l=1}^d \int_{\{|u_{l,n}-u_l^k| \leq \varepsilon^*\}} a_{l,i}(x, T_n(u_n), D_i u_n) D_i(u_{l,n} - u_l^k)\, dx \leq \theta(\varepsilon^*), \tag{3.2.61}$$

with $\lim_{\varepsilon^* \to 0} \theta(\varepsilon^*) = 0$.

Proof. Inserting $\varphi = \mathcal{T}_{\varepsilon^*}(u_n - u^k)$ into (3.2.32), we get

$$\sum_{i=1}^N \sum_{l=1}^d \int_{\{|u_{l,n}-u_l^k| \leq \varepsilon^*\}} a_{l,i}(x, T_n(u_n), D_i u_n) D_{l,i}(u_n - u^k)\, dx \quad + \quad \int_\Omega F(x, u_n).\mathcal{T}_{\varepsilon^*}(u_{l,n} - u_l^k)\, dx$$
$$= \int_\Omega f_n.\mathcal{T}_{\varepsilon^*}(u_n - u^k)\, dx.$$

From and (3.2.59), we have the following

$$\sum_{l=1}^d \int_{\{|u_{l,n}-u_l^k| \leq \varepsilon^*\}} a_{l,i}(x, T_n(u_n), D_i u_n) D_{l,i}(u_{l,n} - u_l^k)\, dx \leq C\varepsilon^*,$$

where $C = 2d\left(\|F(x,u_n)\|_{L^1(\Omega;\mathbb{R}^d)} + \|f\|_{L^m(\Omega;\mathbb{R}^d)}\right)$. □

Lemma 3.2.13. *for all $i = 1, ..., N$, we have*

$$\lim_{\epsilon \to 0} \sum_{i=1}^N \sum_{l=1}^d \int_\Omega \left(a_{l,i}(x, T_n(u_n), D_i u_n) - a_{l,i}(x, T_n(u_n), D_i u)\right) D_i\left(u_{l,n} - u_l\right)\, dx = 0. \tag{3.2.62}$$

Proof. We define the integral S_n as follows

$$S_n = \int_\Omega \left\{\Delta_i(u_n, u)\right\}^{\frac{1}{p_+^+}}\, dx, \tag{3.2.63}$$

such that $\Delta_n^i(u_n,u)$ is defined as (3.2.58). Please note that $0 \le S_n < \infty$. Let us write $S_n = S_n^1 + S_n^2$, where

$$S_n^1 = \int_{\{|u|>k\}} \left\{\Delta_n^i(u_n,u)\right\}^{\frac{1}{p_+^+}} \quad \text{and} \quad S_n^2 = \int_{\{|u|\le k\}} \left\{\Delta_n^i(u_n,u)\right\}^{\frac{1}{p_+^+}}. \tag{3.2.64}$$

By invoking (3.2.59) and using Hölder's inequality

$$S_n^1 \le C_1' \left|\{|u|>K\}\right|^{1-\frac{1}{p_+^+}} + C_3 \left\{ \|D_i u_\epsilon\|_{L^{r_-^-}(\Omega;\mathbb{R}^d)} + \|D_i u\|_{L^{r_-^-}(\Omega;\mathbb{R}^d)} \right\} \left|\{|u|>K\}\right|^{1-\frac{1}{r_-^-}}.$$

Consequently

$$S_n^1 \le C \left|\{|u|>K\}\right|^{1-\frac{1}{p_+^+}} + C \left|\{|u|>K\}\right|^{1-\frac{1}{r_-^-}}.$$

Then

$$S_n^1 \le \frac{1}{k^{r_-^-(1-\frac{1}{p_+})}} + \frac{1}{k^{r_-^- - 1}}.$$

Letting k, n tends to infinity and zero,respectively, obtaining

$$\lim_{k\to+\infty} \limsup_{n\to\infty} S_n^1 = 0. \tag{3.2.65}$$

Now, we divide the integral S_n^2 on the sets $\left\{|u_n - u^k| > \varepsilon^*\right\}$ and $\left\{|u_n - u^k| \le \varepsilon^*\right\}$, for all $\varepsilon^* > 0$, getting

$$S_{n,\varepsilon^*}^3 = \int_{\left\{|u|\le k, |u_n - u^k| > \varepsilon^*\right\}} \left\{\Delta_n^i(u_n,u^k)\right\}^{\frac{1}{p_+^+}} \, dx.$$

and

$$S_{n,\varepsilon^*}^4 = \int_{\left\{|u|\le k, |u_n - u^k| \le \varepsilon^*\right\}} \left\{\Delta_n^i(u_n,u^k)\right\}^{\frac{1}{p_+^+}} \, dx.$$

with similar arguments as in the proof of (3.2.65)

$$\begin{aligned} S_{n,\varepsilon^*}^3 \le C \left|\left\{|u_n - u^k| > n\right\}\right|^{1-\frac{1}{p_+^+}} \quad + \quad C \left|\left\{|u_n - u^k| > n\right\}\right|^{1-\frac{1}{q_0}} \\ = o(1) \, (as \;\; n \to \infty). \end{aligned} \tag{3.2.66}$$

We define

$$S_{n,\varepsilon^*}^4 = M_{n,\varepsilon^*}^1 - M_{n,\varepsilon^*}^2, \tag{3.2.67}$$

where

$$
\begin{aligned}
M^1_{n,\varepsilon^*} &= \sum_{l=1}^{d}\int_{\{|u_{l,n}-u_l^k|\le\varepsilon^*\}}\Big\{a_{l,i}(x,T_n(u_n),D_iu_n)D_i(u_{l,n}-u_l^k)\Big\}^{\frac{1}{p_+^+}}\,dx,\\
M^2_{n,\varepsilon^*} &= \sum_{l=1}^{d}\int_{\{|u_{l,n}-u_l^k|\le\varepsilon^*\}}\Big\{a_{l,i}(x,T_n(u_n),D_iu^k)D_i(u_{l,n}-u_l^k)\Big\}^{\frac{1}{p_+^+}}\,dx.
\end{aligned}
$$

Thanks to Lemma 3.2.12, we have

$$
\lim M^1_{n,\varepsilon^*} = \lim_{\varepsilon^*\to 0}\limsup_n \sum_{i=1}^{N}\sum_{l=1}^{d}\int_{\{|u_{l,n}-u_l^k|\le\varepsilon^*\}} a_{l,i}(x,T_n(u_n),D_iu_n)D_{l,i}(u_{l,n}-u_l^k) = 0. \quad (3.2.68)
$$

For $|u_{l,n}-u_l|\le\varepsilon^*\le 1$, we have $|u_{l,n}| = |u_{l,n}-u_l^k+u_l^k|\le 1+k$. Since $u_{l,n}^k = \mathcal{T}_k(u_{l,n}) \Rightarrow u_{l,n}^{k+1} = \mathcal{T}_{k+1}(u_{l,n}), |u_{l,n}^{k+1}|\le 1+k$, then $u_{l,n} = u_{l,n}^{k+1}$.

Its easy to verify that $\mathcal{T}_k(u_n)\rightharpoonup\mathcal{T}_k(u)$ in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$, which is implies

$$
u_{l,n}^{k+1}\rightharpoonup u_l^{k+1}\ in\ W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d).
$$

Consequently

$$
D_i(u_{l,n}^{k+1}-u_l^k)\rightharpoonup D_i(u_l^{k+1}-u_l^k)\text{ in }L^{p_i(\cdot)}(\Omega;\mathbb{R}^d).
$$

The dominated convergence Theorem implies that

$$
a_{l,i}(x,T_n(u_n),D_iu^k)\to a_{l,i}(x,u,D_iu^k)\text{ strongly in }L^{p_i'(\cdot)}(\Omega;\mathbb{R}^d);\forall l=1,...,d.
$$

Then

$$
\begin{aligned}
\lim_{n\to 0} M^2_{n,\varepsilon^*} &= \sum_{l=1}^{d}\int_{\{|u_l-u_l^k|\le\varepsilon^*\}} a_{l,i}(x,u,D_iu^k)D_i(u_l^{k+1}-u_l^k)\,dx\\
&= \sum_{l=1}^{d}\int_{\Omega} a_{l,i}(x,u,D_iu^k)D_i\mathcal{T}_{\varepsilon^*}(u_l-u_l^k)\,dx.
\end{aligned}
\quad (3.2.69)
$$

It is straightforward to confirm that $D_i\mathcal{T}_{\varepsilon^*}(u_l-u_l^k)\rightharpoonup 0$ in $L^{p_i(\cdot)}(\Omega;\mathbb{R}^d)$ $\forall i=1,...,N, l=1,...,d$.

As a result

$$
\lim_{\varepsilon^*\to 0}\lim_{n\to\infty} M^2_{\varepsilon^*,n} = 0. \quad (3.2.70)
$$

We combine (3.2.65),(3.2.66),(3.2.68) and (3.2.70), we conclude

$$\limsup_{n\to\infty} S_n = o(1)(as\, n \to \infty) + o(1)(as\, n \to +\infty, \varepsilon^* \to 0).$$

This implies (3.2.62). □

Following the same argument as in Application 2.1, we deduce

$$D_i u_n \to D_i u \text{ a.e. in } \Omega. \tag{3.2.71}$$

Lemma 3.2.14. *For all i=1,...,N, as $n \to \infty$, we have*

$$F(x, u_n) \to F(x, u) \;\; \textit{strongly in} \;\; L^1(\Omega; \mathbb{R}^d), \tag{3.2.72}$$

and

$$a_i(x, T_n(u_n), D_i u_n) \to a_i(x, u, D_i u) \;\; \textit{strongly in} \;\; L^1(\Omega; \mathbb{R}^d). \tag{3.2.73}$$

Proof. As F is a Caratheödory function, and based on equation (3.2.57)

$$F(x, u_n) \to F(x, u) \;\; \text{a.e. in} \;\; \Omega. \tag{3.2.74}$$

Under the assumptions (3.1.7), (3.2.41), and (3.2.74), and employing methodologies similar to those presented in [10], we deduce (3.2.72). Consequently, by combining (3.2.57) and (3.2.71), we obtain

$$a_i(x, T_n(u_n), D_i u_n) \to a_i(x, u, D_i u) \;\; \text{a.e. in} \;\; \Omega. \tag{3.2.75}$$

Next, we aim to demonstrate that a_i is bounded in $L^{\frac{r_i(.)}{p_i(\cdot)-1}}(\Omega; \mathbb{R}^d)$, where $r_i(x)$ is a continuous function on $\overline{\Omega}$ satisfying (3.2.5).
The choice of $\dfrac{r_i(\cdot)}{p_i(\cdot) - 1} > 1$ is possible, given the condition (3.2.4). Thus, for all $i = 1, ..., N$, we have

$$1 < \frac{r_i(\cdot)}{p_i(\cdot) - 1} < \frac{N m p_i(\cdot)\left(\overline{p}(\cdot) - 1 - \gamma^+\right)}{\left(p_i(\cdot) - 1\right)\left(N m(\overline{p}(\cdot) - 1 - \gamma^+) + (1 + \gamma(\cdot))(N - m\overline{p}(\cdot))\right)}. \tag{3.2.76}$$

By using (3.2.5), we can choose ϕ as a continuous function on $\overline{\Omega}$ in a manner that

$$\frac{(p_i(\cdot)-1)\, r_i(\cdot)}{p_i(\cdot)} < \phi(.) < \frac{Nm\,(\overline{p}(\cdot)-1-\gamma^+)}{Nm\,(\overline{p}(\cdot)-1-\gamma^+)+(1+\gamma(\cdot))\,(N-m\overline{p}(\cdot))} < 1,$$

and

$$\frac{1}{p_i(\cdot)} < \phi(\cdot) < \frac{Nm\,(\overline{p}(\cdot)-1-\gamma^+)}{Nm\,(\overline{p}(\cdot)-1-\gamma^+)+(1+\gamma(\cdot))\,(N-m\overline{p}(.))}.$$

Then, we have

$$1 \leq p_i(\cdot)\phi(\cdot) < \frac{Nmp_i(\cdot)\,(\overline{p}(\cdot)-1-\gamma^+)}{Nm\,(\overline{p}(\cdot)-1-\gamma^+)+(1+\gamma(\cdot))\,(N-m\overline{p}(\cdot))}, \tag{3.2.77}$$

and

$$\frac{(p_i(\cdot)-1)\, r_i(\cdot)}{p_i(\cdot)\phi(\cdot)} < 1. \tag{3.2.78}$$

Using the assumption (3.1.3) and by (3.2.78), we obtain for all $i = 1, ..., N$

$$\begin{aligned} |a_i(x, T_n(u_n), D_i u_n)|^{r_i(x)} &\leq \left\{ |h|^{\phi(x)} + |u_n|^{\phi(x)\overline{p}^-} + \sum_{j=1}^{N} |D_j u_n|^{\phi(x)p_j(x)} \right\}^{\frac{(p_i(x)-1)r_i(x)}{p_i(x)\phi(x)}} \\ &\leq \left\{ |h| + |u_n|^{\overline{p}^-} + \sum_{j=1}^{N} |D_j u_n|^{\phi(x)p_j(x)} \right\}. \end{aligned} \tag{3.2.79}$$

Therefore, since the solution $|u_n|$ is in $L^{\overline{p}^-}(\Omega)$, it is because, there exists $j \in \{1, ..., N\}$ such that $p_j^- \geq \overline{p}^-$, then $|u_n| \in L^{\overline{p}^-}(\Omega)$ by using Lemma 1.2.3, based on the last estimate, Lemma 3.2.10, and (3.2.77). From this, we can conclude that $a_i(x, T_n(u_n), D_i u_n)$ is bounded in $L^{r_i(.)}(\Omega; \mathbb{R}^d)$. Since we have the almost everywhere convergence (3.2.75), we can apply the Vitali Theorem, we obtain (3.2.73) for all $i = 1, ..., N$. By (3.2.73), and (3.2.72), so that

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega \sum_{i=1}^{N} a_i(x, T_n(u_n), D_i u_n).D_i\varphi\, dx + \lim_{n\to\infty} \int_\Omega F(x, u_n).\varphi\, dx \\ &= \int_\Omega \sum_{i=1}^{N} a_i\,(x, u, D_i u)\,.D_i\varphi\, dx + \int_\Omega F(x, u).\varphi\, dx,\ \ \forall \varphi \in C^\infty(\Omega; \mathbb{R}^d). \end{aligned}$$

□

Conclusion and Perspectives

In the study conducted in this thesis, we focus on studying a nonlinear anisotropic elliptic systems more specifically a generalization of $p_i(x)$-Laplacian with degenerate coercivity and L^m data.

In conclusion, when dealing with the theory of nonlinear systems of partial differential equations, particularly those with nonlinear terms dependent on the gradient's natural growth, it is imperative to introduce an essential condition to establish the existence of distributional solutions.

We have been able to prove the following results

The first result considers the case where f has a high summability, more specifically $m > (\overline{p}^*(x))'$, then we prove the solution u is bounded in $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$.

In the second result, if we decrease the summability of f, i.e., f in $L^m\left(\Omega;\mathbb{R}^d\right)$ where m satisfies $1 < m < (\overline{p}^*(x))'$, we find the solutions which do not in general belong any more to $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega;\mathbb{R}^d)$, more precisely, we found the solution u is in $W_0^{1,\overrightarrow{r}(\cdot)}(\Omega;\mathbb{R}^d)$ where $r_i(x)$ are continuous functions on $\overline{\Omega}$ which satisfy

$$1 \leq r_i(\cdot) < \frac{Nmp_i(\cdot)(\overline{p}(\cdot) - 1 - \gamma^+)}{Nm(\overline{p}(\cdot) - 1 - \gamma^+) + (1 + \gamma(\cdot))(N - m\overline{p}(\cdot))}, \ \forall i = 1, ..., N.$$

In our result, we get an optimal solution because the regularity given in the case f in $L^m(\Omega;\mathbb{R}^d)$ where $m > 1$ is better than when $m = 1$.

Future research topics can include cover the following points

- In the context of anisotropic elliptic problems with variable exponents, the presence of lower-order terms, under suitable conditions, has allowed us to derive uniform estimates for solutions to relevant problems approximating (3.1.1). It would be of great interest to further investigate problem (3.1.1) in the absence of the lower-order term, i.e., when $F = 0$, and strive to prove the existence and regularity of results for problem (3.1.1) based on the summability of the data $f \in L^m(\Omega; \mathbb{R}^d)$.

- In conclusion, research establishes the non-existence of a p-Laplacian with degenerate coercivity, as stipulated in the defined problem $-div\left(\frac{\nabla u}{(1+|u|)^{\theta}}\right)$, when $\theta > 1$. The pivotal role of the parameter θ in shaping the problem's behaviour is underscored, even when the function f demonstrates high regularity. Looking ahead, a crucial question arises: Can we determine the non-existence of the $-\sum_{i=1}^{N}\left(\frac{|D_i u|^{p_i(x)-2} D_i u}{(1+|u|)^{\theta(x)}}\right) + |u|^{s(x)-1}u$ when $\theta > \overline{p}(x) - 1$, whether in the presence of a lower-order term or in its absence?

- A challenging and interesting problem is to consider nonlinear parabolic systems with the principal part without or having a degenerate coercivity

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{N}\left(D_i\left(|D_i u|^{p_i(x)-2} D_i u\right)\right) + |u|^{s(x)-1}u = f, & x \in (0,T)\times\Omega \\ u(x,0) = u_0, & x \in \Omega \end{cases}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{N}\left(D_i\left(\frac{|D_i u|^{p_i(x)-2} D_i u}{(1+|u|)^{\gamma_i(x)}}\right)\right) + |u|^{s(x)-1}u = f, & x \in (0,T)\times\Omega \\ u(x,0) = u_0, & x \in \Omega \end{cases}$$

where $u : \Omega \to \mathbb{R}^d$ whatever the data f in L^m, L^1 or the data is a radon measure, we have a big obstacle related essentially to dealing with, there were not a lot of studies before about systems, what are the structural conditions that we can use it to prove the existence and regularity of the solution?

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