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Par : **Abdelhakim Dahmani**

## Thème

# Study of the stabilization of certain hyperbolic problems with variable coefficients

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| | | | | |
|---|---|---|---|---|
| **M. M.** | **Medjden** | Professeur | à l'USTHB | Président |
| **M. A.** | **Khemmoudj** | Professeur | à l'USTHB | Directeur de thèse |
| **M. A.** | **Benaissa** | Professeur | à l'Univ. Djillali Liabès, SBA | Examinateur |
| **M. A.** | **Hakem** | Professeur | à l'Univ. Djillali Liabès, SBA | Examinateur |
| **M. A.** | **Touzaline** | Professeur | à l'USTHB | Examinateur |

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University of Sciences and Technology Houari Boumediene

**Faculty of Mathematics**

# Thesis

Submitted to the Department of Analysis in partial fulfillment of the requirements for the degree of **Doctor in Mathematics**

Domain: **Partial Differential Equations**

By: **Abdelhakim Dahmani**

## Topic

# Study of the stabilization of certain hyperbolic problems with variable coefficients

Publicly defended on Wednesday, 07/06/2023, in front of the jury composed by:

| | | | | |
|---|---|---|---|---|
| **M. M.** | **Medjden** | Professor | at USTHB | Chairman |
| **M. A.** | **Khemmoudj** | Professor | at USTHB | Supervisor |
| **M. A.** | **Benaissa** | Professor | at Univ. Djillali Liabès, SBA | Examiner |
| **M. A.** | **Hakem** | Professor | at Univ. Djillali Liabès, SBA | Examiner |
| **M. A.** | **Touzaline** | Professor | at USTHB | Examiner |

# Abstract

This thesis studies the wave equation with dynamic Wentzell boundary conditions with delay feedback in the boundary subjct to a localized Kelvin-Voigt damping.

Using semigroup theory. The problem is shown to be well-posed in a suitable energy space through the use of Lumer-Phillips's theorem and semigroup properties. Then, we show the strong stability of the solution using Arendt-Batty criteria.

For the case where the damping localizing coefficient is smooth, exponential stability is proven by using the Huang-Prüss criteria which is established using a perturbation argument, contradiction argument, and multipliers techniques. On the other hand, if the damping coefficient is discontinuous, polynomial stability is shown using Borichev-Tomilov's criteria, along with a cascade-like technique which allows to merge different stability results and a specific choice of multipliers.

**Key words.** *Delay feedback, Dynamic BC, Exponential stability, Kelvin-Voigt damping, Localized damping, Polynomial stability, Wentzell BC, Wave equation, Well-posedness.*

# Résumé

Dans cette thèse on étudie l'équation des ondes avec des conditions aux bords de type Wentzell dynamique et un retard sur le bord, l'équation est soumise à un amortissement de type Kelvin-Voigt localisé.

En utilisant la théorie des semi-groupes, on montre que le problème est bien-posé dans un espace d'énergie approprié grâce au théorème de Lumer-Phillips et aux propriétés des semi-groupes. Ensuite, on montre la stabilité forte de la solution en utilisant le critère d'Arendt-Batty.

Dans le cas où le coefficient d'amortissement localisé est régulier, la stabilité exponentielle est prouvée en utilisant le critère de Huang-Prüss, établi grâce à une technique de perturbation, un argument de contradiction et des multiplicateurs. En revanche, si le coefficient d'amortissement est discontinu, la stabilité polynomiale est démontrée en utilisant la critère de Borichev-Tomilov, ainsi qu'une technique de cascade qui permet de fusionner plusieurs résultats de stabilité et un choix particulier de multiplicateurs.

**Mots clés.** *Amortissement de Kelvin-Voigt, Amortissement localisé, Condition aux bords de type Wentzell, Condition aux bords dynamique, Équation des ondes, Probleme bien-posé, Retard, Stabilité exponentielle, Stabilité polynomiale.*

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# General Introduction

## 0.1 Introduction

This thesis is devoted to studying the stabilization of the wave equation with dynamical boundary conditions of Wentzell type with a delay feedback on the boundary, where the stabilizing mechanism is done by a localized Kelvin-Voigt damping; Moreover, we consider two different types of regularity for the localizing coefficient. Namely, we will be interested in the following problem

$$
\begin{aligned}
u_{tt}(x,t) - \operatorname{div}\{\nabla u(x,t) + a(x)\nabla u_t(x,t)\} &= 0, && \text{in } \Omega \times \mathbb{R}_+^*, && (1)\\
u(x,t) &= 0, && \text{on } \Gamma_0 \times \mathbb{R}_+^*, && (2)\\
u_{tt}(x,t) + \frac{\partial u}{\partial \nu}(x,t) + a(x)\frac{\partial u_t}{\partial \nu}(x,t) - \Delta_T u(x,t) + k u_t(x,t-\tau) &= 0, && \text{on } \Gamma_1 \times \mathbb{R}_+^*, && (3)\\
u(x,0) = u_0(x), u_t(x,0) &= u_1(x), && \text{in } \Omega, && (4)\\
u_t(x,t-\tau) &= f_0(x,t-\tau), && \text{in } \Omega \times (0,\tau), && (5)
\end{aligned}
$$

where $\Omega$ is an open bounded subset of $\mathbb{R}^n$ with a boundary of class $C^2$ divided into two non empty subsets.

The wave equation with Kelvin-Voigt damping arises in vibration phenomena where the damping originates from the internal friction of the vibrating structure (extension or compression) such as vibration of a body made totally or partially of viscoelastic materials. Damping refers to the process of reducing the energy of a system, resulting from the transformation of one form of energy into another. This process can occur due to external forces such as friction or air resistance, or internal forces such as compression or extension present in viscoelastic materials. One example of energy transformation due to damping is when an object is in motion on a surface, and the mechanical kinetic energy of the object is transformed into heat due to friction. In this context, the term "dissipation" refers to the loss of energy from the system in the form of heat or other forms (sound, light,...etc) that are not useful to the system. Where the dynamical boundary conditions refer to models that don't neglect the velocity on the boundary and the Wentzell boundary conditions are utilized to take into account the potential energy at the boundary.

A lot of work has been done in the context of stabilization of the wave equation and a large number of articles and books treating this topic appeared such that a full review of the literature is beyond the scope and can't fit in our introduction; However, we will review some results related to our problem that we find interesting and highlight our contributions.

The wave equation with different boundary conditions and different types of dampings has been extensively studied in the literature. For instance, it is well-known that the decay of energy is exponential with the classical boundary damping

$$\frac{\partial u}{\partial \nu} = -k u_t, \text{ on } \Gamma_1 \times \mathbb{R}_+^*,$$

where $\Gamma_1$ satisfies the so called Geometric Control Condition (GCC in short) introduced by Bardos, Lebeau and Rauch in [20], as well; a lot of stability results have been shown in many other cases (see for instance [15, 10, 74, 91, 51, 69, 31, 92, 32, 33, 50]). Nevertheless, the situation is more complicated with Wentzell boundary conditions, given that the wave equation with static Wentzel boundary conditions has been studied by Heminna in [46]. The author has shown that without internal damping we cannot guarantee exponential stability based solely on a boundary feedback, even if it is applied within the entire boundary. Still, the strong stability holds in this case.

The wave equation with Wentzell boundary conditions and without damping, was first studied by Lemrabet in [56] then by Lemrabet and Teniou in [57], existence and regularity results have been shown for the linear case. Such a model describes a vibrating body with a thin boundary layer of high rigidity.

Later on, Cavalcanti et al. in [27] showed that in addition to some geometrical assumptions, a linear-like interior frictional damping is sufficient to ensure that the energy decays uniformly at a rate prescribed by the solution to certain ODE, this method was first introduced by Lasiecka and Tataru in [51].

Subsequently, Cavalcanti, Lasiecka and Toundykov in [30] studied the following problem

$$\begin{cases} u_{tt} - \Delta u + a(x) g\left(u_t\right) = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ \dfrac{\partial u}{\partial \nu} - \Delta_T u + u = 0, & \text{on } \Gamma \times \mathbb{R}_+, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in \Omega, \end{cases}$$

where the localized damping doesn't affect the full part of the boundary subject to the Wentzell condition. They proved a perturbed observability inequality from which different decay rates are deduced in accordance with the behavior of the function $g(\cdot)$, while the decay is exponential if $g(\cdot)$ is linearly bounded. Afterwards, this work was extended and generalized in the context of dynamical boundary conditions in [29].

In addition to the fact that dynamical boundary conditions do not neglect the velocity over the boundary, these types of conditions, alongside the interior equation, represent phenomena where two dynamics are coupled, the interior and the boundary ones. It should be noted that adding dynamics to the boundary can significantly affect the stability of the system (see for example [63, 62, 64]). Similarly, in the one dimensional case, it was shown in [59], that the following system

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in (0,1), t \geq 0, \\ u(t,0) = 0, & t \geq 0, \\ m u_{tt}(t,1) + \dfrac{\partial u}{\partial \nu}(t,1) + u_t(t,1) = 0, & t \geq 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in (0,1), \end{cases}$$

is not uniformly stable if $m = 1$ while it is exponentially stable if $m = 0$.

The question of whether we can stabilize the system by acting on one or both dynamics has been addressed by many authors.

Khemmoudj and Medjden in [49] proved the exponential stability for the wave equation with dynamical Wentzell boundary conditions, using a localized linear frictional damping in the interior besides a linear frictional damping on the boundary, while the system was exposed to some non-linear forces in both the interior and at the boundary.

Afterwards, in [28], Cavalcanti, Khemmoudj and Medjden extended the previously mentioned work in a more general context, where they considered the following problem,

$$\begin{cases} u_{tt} + \mathcal{A}u + a(x) g_1(u_t) = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ v_{tt} + \dfrac{\partial u}{\partial v_A} + \mathcal{A}_T v + g_2(v_t) = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+, \\ u = v, & \text{on } \Gamma \times \mathbb{R}_+, \\ u = 0, & \text{on } \Gamma_0 \times \mathbb{R}_+, \\ (u(0), v(0)) = \left(u^0, v^0\right), & \text{in } \Omega \times \Gamma, \\ (u_t(0), v_t(0)) = \left(u^1, v^1\right), & \text{in } \Omega \times \Gamma, \end{cases}$$

where $\mathcal{A}$ and $\mathcal{A}_T$ are linear second order differential operators with variable coefficients that satisfy certain uniform ellipticity conditions. Under some geometrical assumptions the authors proved the uniform decay of the energy, using some energy estimates combined with Riemannian geometric methods due to Lasiecka, Triggiani and Yao (see [52, 53, 89]).

Recently, the following system

$$\begin{cases} u_{tt} - \Delta u - k_\Omega \Delta u_t + c_\Omega u_t = 0, & \text{in } \Omega \times \mathbb{R}_+^*, \\ u = 0, & \text{on } \Gamma_0 \times \mathbb{R}_+^*, \\ u - w = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ w_{tt} - k_\Gamma \Delta_T (\alpha w_t + w) + \partial_v (u + k_\Omega u_t) + c_\Gamma w_t = 0, & \text{in } \Gamma_1 \times \mathbb{R}_+^*, \\ w = 0, & \text{on } \partial\Gamma_1 \times \mathbb{R}_+^*, \\ u(\cdot, \cdot 0) = u_0, \quad u_t(\cdot, \cdot, 0) = u_1, & \text{in } \Omega, \\ w(\cdot, 0) = w_0, \quad w_t(\cdot, 0) = w_1, & \text{in } \Gamma_1, \end{cases}$$

has been studied under the assumption $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ by Lasiecka and Fourrier in [42], they proved that the system is exponentially stable if $k_\Omega > 0$, or $c_\Omega > 0$ and $c_\Gamma > 0$, or $c_\Omega > 0$

and $k_\Gamma \alpha > 0$. Later on, the same problem (without assuming $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$) was treated by Mercier et al. in [72], when $k_\Omega = c_\Omega = \alpha = 0$ and $k_\Gamma = c_\Gamma = 1$. The authors proved the strong stability and the lack of uniform stability when $\Omega$ is the unit disc and $\Gamma_0 = \emptyset$, as well, they could show the polynomial stability under some geometrical assumptions.

Buffe in [23] treated the wave equation with both static and dynamic Wentzell boundary conditions, he proved the logarithmic stability with both localized interior and localized boundary frictional damping, by virtue of a proper Carleman estimate near the boundary established using microlocal analysis, where the Geometric Control Condition is not satisfied. Therefore, there is no hope in obtaining exponential stability, instead, an optimal energy decay was shown (see [55]).

More recently, in [58] the same problem with an additional source term in the boundary has been considered, and the polynomial stability was proven.

On the other hand, the following problem

$$\begin{cases} u_{tt}(x,t) - \operatorname{div}(\nabla u + a(x)\nabla u_t) = 0, & \text{in } \Omega \times \mathbb{R}_+^*, \\ u(x,t) = 0, & \text{on } \Gamma \times \mathbb{R}_+^*, \\ u(x,0) = u_0(x), u'(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{$*$}$$

has been addressed by Liu and Rao in [67]. By assuming that $a \in C^{1,1}(\Omega)$, $\Delta a \in L^\infty$, and $|\nabla a(x)|^2 \leq M_0 a(x)$ almost everywhere in $\Omega$, where $M_0$ is a positive constant, and the damping region $\omega$ is a neighborhood of the whole boundary, they have proven that exponential stability holds using frequency domain approach combined with multipliers technique. Subsequently, Tebou in [84], showed that the exponential stability still holds without need for the boundedness of $\Delta a$ and for a larger class of damping region $\omega$, which in this case should satisfy the so called the Piecewise Multiplier Geometric Condition (PMGC in short) introduced by Liu in [65], that generalizes the Lions Multiplier Geometric Condition. Tebou, in the same paper also proved that if the damping coefficient is only bounded measurable then the energy decays as $t^{-1}$.

In [78], Nicaise and Pignotti studied the following problem

$$\begin{cases} u_{tt}(x,t) - \operatorname{div}(\nabla u + a(x)\nabla u_t) = 0, & \text{in } \Omega \times \mathbb{R}_+^*, \\ u(x,t) = 0, & \text{on } \Gamma_0 \times \mathbb{R}_+^*, \\ \dfrac{\partial u}{\partial \nu} = -a(x)\dfrac{\partial v}{\partial \nu} - k u_t(x, t-\tau) = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ u(x,0) = u_0(x), u'(x,0) = u_1(x), & \text{in } \Omega, \\ u_t(x, t-\tau) = f_0(x, t-\tau), & \text{in } \Omega \times (0,\tau), \end{cases}$$

where the damping coefficient $a(\cdot)$ satisfies the same assumptions imposed by Liu and Rao in [67]. Moreover, assuming some appropriate geometric conditions and relying on an inequality similar to (2.7), they proved the exponential stability by means of frequency domain approach and a perturbation argument. In a previous work [76], the same authors treated a second-order evolution equation with dynamic boundary feedback laws with a delay where the damping was distributed in the whole domain, using the multipliers technique, the exponential stability has been established. Moreover, the authors gave counterexamples where (2.7) doesn't hold, which suggest the optimality of this condition.

It is worth mentioning that the presence of time delay might destabilize systems that are exponentially stable in absence of delay (see [77, 88, 37, 79]).

Burq and Christianson in [25] considered the problem ($*$), by assuming that the damping coefficient $a(\cdot)$ is smooth ($C^\infty$), vanishes nicely, and the damping region $\{a(\cdot) > 0\}$ controls $\Omega$ geometrically, they proved that the energy of this system decays exponentially. Afterwards, Burq and Sun in [26] relaxed the regularity condition on $a(\cdot)$ to become $a \in C^1(\overline{\Omega})$ and $|\nabla a| \leq C\sqrt{a}$ such that the exponential stability always holds.

Note that the smoothness of the coefficient $a(\cdot)$ is crucial for the exponential stability in the case of Kelvin-Voigt damping in contrast to the case of viscous damping as the exponential stability holds for the latter case if the GCC is satisfied regardless of the smoothness of the coefficient. Such a phenomenon can be seen clearly in the work of Liu and Liu [66], where they showed that the locally damped one dimensional wave equation isn't exponentially stable if the coefficient of the Kelvin-Voigt damping is discontinuous; Later on, Alves et al. in [7] proved the polynomial energy decay of type $t^{-2}$ for this case and showed the optimality of this decay rate. Afterwords, the multidimensional case (Namely the problem ($*$))with discontinuous damping coefficient was considered in [87] by Wehbe et al. they proved a polynomial energy decay rate of type $t^{-1}$ if either the damping region $\omega$ satisfies the GCC and meas$(\bar{\omega} \cap \partial\Omega) > 0$ or the damping region $\omega$ contains strictly a subset satisfying the GCC. Furthermore, when the domain $\Omega$ is a square and these geometrical conditions are violated they proved a polynomial energy decay rate of type $t^{-\frac{1}{3}}$ if the damping region $\omega$ is a vertical strip far away from the boundary and of type $t^{-\frac{2}{5}}$ if the damping region $\omega$ is next to the boundary.

Later, in [13] Ammari et al. considered the problem ($*$) without assuming any conditions on the damping region. They established a logarithmic energy decay which is known to be optimal in the general case (see e.g. [24]).

It is noticeable that many significant and important cases remain unaddressed in the literature. Therefore, we propose to investigate the stabilization of the wave equation under boundary dynamics that add energy to the system, primarily from boundary velocity, the Wentzell component, and the boundary delay. Our goal is to comprehensively analyze and understand the long-term behavior of such a system when stabilized by localized Kelvin-Voigt damping. Additionally, we will study the impact of the damping localization coefficient regularity by considering two different types of regularity for this coefficient.

## 0.2 Thesis Overview

This thesis is divided mainly into three chapters, which we present in the following section.

### 0.2.1 Well-Posedness and Strong Stability

In the second chapter, we focus on presenting the problem within the context of semigroup theory. We begin by formulating the problem, and then proceed to establish the existence and uniqueness of a solution. Finally, we demonstrate the strong stability of the solution.

In a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary $\Gamma$ divided into two parts $\Gamma_0$ and $\Gamma_1$, such that $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\text{meas}(\Gamma_i) \neq 0, i = 0, 1$. We consider the following initial boundary value problem

$$
\begin{aligned}
u_{tt}(x,t) - \operatorname{div}(\nabla u(x,t) + a(x)\nabla u_t(x,t)) &= 0, && \text{in } \Omega \times \mathbb{R}_+^*, && (6)\\
u(x,t) &= 0, && \text{on } \Gamma_0 \times \mathbb{R}_+^*, && (7)\\
u_{tt}(x,t) + \frac{\partial u}{\partial \nu}(x,t) + a(x)\frac{\partial u_t}{\partial \nu}(x,t) - \Delta_T u(x,t) + k u_t(x, t-\tau) &= 0, && \text{on } \Gamma_1 \times \mathbb{R}_+^*, && (8)\\
u(x,0) = u_0(x), u_t(x,0) &= u_1(x), && \text{in } \Omega, && (9)\\
u_t(x, t-\tau) &= f_0(x, t-\tau), && \text{in } \Omega \times (0,\tau). && (10)
\end{aligned}
$$

We are interested in the case when the damping is not active everywhere but only on a subset $\omega \subset \Omega$, i.e.

$$a(x) \geq 0 \ a.e \ \text{ in } \Omega, \quad a(x) \geq a_0 > 0 \ a.e. \ \text{ in } \omega,$$

where $\omega$ is an open neighborhood of the part $\Gamma_1$ and $\text{meas}(\bar{\omega} \cap \Gamma_0) > 0$.

A very important ingredient for the results of the second chapter and even the coming chapters is the following inequality

$$a_0 > |k| C_P, \tag{11}$$

where $a_0$ is a lower bound for $a(\cdot)$ in $\omega$ and $C_p$ is a type of Poincaré inequality.

After introducing the following new state (substitution)

$$z(x,\rho,t) = u_t(x, t - \tau\rho), \quad x \in \Gamma_1, \rho \in (0,1), t > 0,$$

and denoting $u_t = v$, we are able to write our problem in the context of semigroup theory

$$
\begin{cases}
\partial_t U = \mathcal{A} U \\
U_0 = (u_0, u_1, u_1|_{\Gamma_1}, f_0(\cdot, - \cdot \tau))^T,
\end{cases} \tag{CB}
$$

where

$$
\mathcal{A}U = \begin{pmatrix}
v \\
\operatorname{div}(\nabla u + a(x)\nabla v) \\
-\frac{\partial u}{\partial \nu} - a(x)\frac{\partial v}{\partial \nu} + \Delta_T u - kz(,1) \\
\frac{1}{\tau} z_\rho
\end{pmatrix}, \forall U \in \mathcal{D}(\mathcal{A}),
$$

with domain

$$\mathcal{D}(\mathcal{A}) := \Big\{ U = (u, v, v|_{\Gamma_1}, z)^T \in \mathcal{H} : \mathcal{A}U \in \mathcal{H} \Big\}$$
$$= \Big\{ U \in V \times V \times L^2(\Gamma_1) \times L^2(\Gamma_1; H^1(0,1)) : \mathrm{div}(\nabla u + a(x)\nabla v) \in L^2(\Omega),$$
$$\frac{\partial u}{\partial \nu} + a(x)\frac{\partial v}{\partial \nu} - \Delta_T u \in L^2(\Gamma_1), v|_{\Gamma_1} = z(\cdot, 0) \Big\},$$

where, $V$ and $\mathcal{H}$ are suitable Hilbert spaces.

Now, we give the following well-posedness result

**Theorem 0.1.** *Assume that the inequality (11) holds. Then for any initial datum $U_0 \in \mathcal{H}$, there exists a unique (weak) solution $U \in C([0, +\infty), \mathcal{H})$ of problem (CB). Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then*

$$U \in C([0, +\infty), \mathcal{D}(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H}).$$

The proof is based on Lumer-Phillips's theorem. Since the domain of $\mathcal{A}$, $\mathcal{D}(\mathcal{A})$, is dense in $\mathcal{H}$, then the proof will be based on the following two points.

- The operator $\mathcal{A}$ is dissipative. i.e. $\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \le 0, \forall U \in \mathcal{D}(\mathcal{A})$.
- The range of $\lambda_0 - \mathcal{A}$, $R(\lambda_0 - \mathcal{A})$, is $\mathcal{H}$. i.e. $\lambda_0 U - \mathcal{A}U = F$ has a solution $U \in \mathcal{D}(\mathcal{A})$ for every $F \in \mathcal{H}$.

Next, we present our first result regarding the stability, namely the strong stability

**Theorem 0.2.** *The semigroup of contractions $(S(t))_{t\ge 0}$ generated by $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is strongly stable on the energy space $\mathcal{H}$ in the sense that*

$$\lim_{t\to\infty} \|S(t)U_0\|_{\mathcal{H}} = 0, \quad \forall U_0 \in \mathcal{H}.$$

This strong stability result is based on Arendt-Batty's criteria (Theorem 1.3) which states that if

1. $\mathcal{A}$ has no eigenvalues in $i\mathbb{R}$,
2. $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable,

then, the semigroup $\left(e^{t\mathcal{A}}\right)_{t\ge 0}$ is strongly stable. Subsequently, the proof of our theorem will be a consequence of the following Lemma and Proposition.

**Lemma 0.3.** *If the inequality (11) is satisfied, then, for all $\beta \in \mathbb{R}$, the operator $(i\beta I - \mathcal{A})$ is injective, i.e.:*

$$\mathrm{Ker}(i\beta I - \mathcal{A}) = \{0\}.$$

**Proposition 0.4.** *If the inequality (11) is satisfied, then for all $\beta \in \mathbb{R}$, we have $(i\beta I - \mathcal{A})$ is surjective, i.e.*

$$R(i\beta I - \mathcal{A}) = \mathcal{H}.$$

Since the operator $\mathcal{A}$ is closed, using the Closed Graph theorem, we infer

$$i\mathbb{R} \subset \rho(\mathcal{A}),$$

which leads to the conclusion of Theorem 0.2.

### 0.2.2 Exponential Stability: Case of smooth damping coefficient

This chapter is devoted to the study of the exponential stability of our system. To this end, we are going to use the frequency domain approach. Specifically, we shall follow Theorem 1.4 (Huang-Prüss [47, 82]) which states that a $C_0$-semigroup of contractions in a Hilbert space is exponentially stable if and only if

$$i\mathbb{R} \equiv \{i\beta : \beta \in \mathbb{R}\} \subset \rho(\mathcal{A}), \tag{12}$$

and

$$\sup_{\beta\in\mathbb{R}} \left\|(i\beta I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(H)} < \infty. \tag{13}$$

As the first condition was already verified in the previous chapter, we need only to verify the second condition.

The result of the third and also the fourth chapter will be built up on some estimates that are easy to get in $\omega$ but the difficult part is to extend these estimates to the rest of $\Omega$ which needs some regularity assumptions on $a(\cdot)$. Hence, following Nicaise and Pignotti [78] on their inspiration from Liu and Rao [67], we suppose the following assumptions that we illustrate in Figure 1,

**(H)** $meas(\Gamma_1) > 0$,

**(A1)** $\exists\delta > 0,\ a(x) \geq a_0 > 0,\ \forall x \in \mathcal{O}_\delta$, where

$$\mathcal{O}_\delta = \{x \in \Omega,\ \ |x-y| \leq \delta,\ \forall y \in \Gamma_1\},$$

**(A2)** $a \in C^{1,1}\left(\overline{\Omega}\right),\ \Delta a \in L^\infty(\Omega)$.

Also, we assume the following conditions.

There exists a function $q \in C^1(\Omega;\mathbb{R}^n)$ and constants $0 < \alpha < \beta < \delta$ such that

**(D1)** $\partial_j q_k = \partial_k q_j$, $\operatorname{div} q \in C^1(\Omega_\beta)$ and $q \equiv 0$ on $\mathcal{O}_\alpha$, where $\Omega_\beta = \Omega\backslash\mathcal{O}_\beta$,

**(D2)** there exists a constant $\sigma > 0$ such that

$$(\partial_j q_k)_{1\leq k,j\leq n} \geq \sigma I,\ \text{ in } \Omega_\beta,$$

**(D3)** there exists a constant $C > 0$ such that for all $v \in V$ we have

$$|(q\cdot\nabla v)\nabla a - (q\cdot\nabla a)\nabla v| \leq C\sqrt{a}|\nabla v|,\ \text{ in } \Omega_\beta,$$

**(D4)** $q(x)\cdot\nu(x) \leq 0 \quad \forall x \in \Gamma_0$. Then we give a concrete example of $\Omega$ where we can provide an explicit formula for the vector field $q(\cdot)$. Namely, the following remark.

**Remark 0.5.** *If $\Omega$ is a disk in $\mathbb{R}^2$ and $\Gamma_0$ is a suitable connected arc from the boundary, the conditions (**D1**)-(**D4**) hold with $q(x) = m(x)\varrho(x)$ such that $\varrho \in C^1(\overline{\Omega})$ defined by*

$$\varrho(x) = \begin{cases} 1 & \text{if } x \in \Omega_\beta, \\ 0 & \text{if } x \in \mathcal{O}_\alpha, \\ \varrho(x) \in [0,1] & \text{elsewhere }, \end{cases}$$

*and $m(x) = x - x_0$, where $x_0 \in \mathbb{R}^2$ is chosen such that $m(x)\cdot\nu(x) \geq 0$ for every $x$ in $\Gamma_0$ (see Figure 2).*

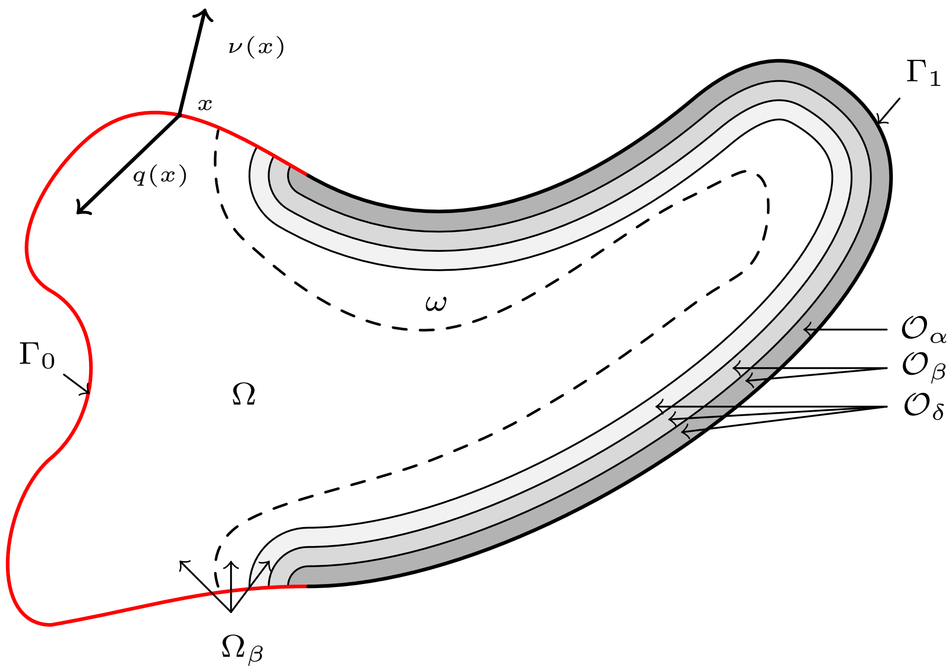


Figure 1: An example of a geometric situation satisfying the assumptions (**D1**)-(**D4**).

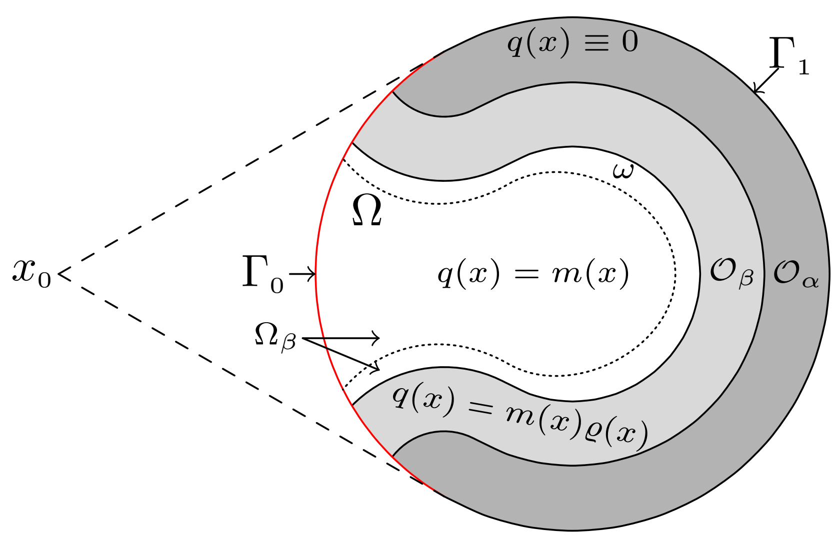


Figure 2: A geometric situation for which there exists a vector field $q$ for which the assumptions (**D1**)-(**D4**) are satisfied.

Now we give the main theorem of the third chapter.

**Theorem 0.6.** *Suppose that the assumptions (**H**),(**A1**),(**A2**),(**D1**)-(**D4**), along with the* *inequality (11) are satisfied, then there are two constants $M > 0$, $\alpha > 0$, such that for all initial data $U_0 \in \mathcal{H}$, the solution $U := (u, u_t, u_t|_{\Gamma_1}, z)$ of the problem (6)-(10) satisfies the following uniform exponential decay estimate*

$$\|U(t)\|_{\mathcal{H}} \leq Me^{-\alpha t}\|U_0\|_{\mathcal{H}}. \tag{14}$$

The proof of this theorem will be divided into several steps, organized in two sections. First of all, in a suitable Hilbest space, $\mathcal{H}_0$, let's define the operator $\mathcal{A}_0$ which corresponds to $\tau = 0$ and $k = 1$, that is

$$\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \to \mathcal{H}_0 : (u, v, v|_{\Gamma_1})^T \to \left(v, \operatorname{div}(\nabla u + a\nabla v), -\frac{\partial u}{\partial \nu} - a(x)\frac{\partial v}{\partial \nu} + \Delta_T u - v\right)^T,$$

with

$$\mathcal{D}(\mathcal{A}_0) := \Big\{U = (u, v, v|_{\Gamma_1})^T \in \mathcal{H}_0 : v \in V, \operatorname{div}(\nabla u + a\nabla v) \in L^2(\Omega), \frac{\partial u}{\partial \nu} + a\frac{\partial v}{\partial \nu} - \Delta_T u \in L^2(\Gamma_1)\Big\}.$$

At this stage, we assume that $\mathcal{A}_0$ generates an exponentially stable semigroup. So, from Theorem 1.4 (Huang-Prüss [47, 82]) we have,

$$\|(i\xi - \mathcal{A}_0)^{-1}\|_{\mathcal{L}(\mathcal{H}_0)} \leq C, \quad \forall \xi \in \mathbb{R}. \tag{15}$$

This resolvent estimate will be used to derive the following estimate

$$\int_{\Gamma_1} |\tilde{v}|^2 d\Gamma \leq C\|F_0\|^2_{\mathcal{H}_0}, \tag{16}$$

which we use in the proof of the stability of the problem with delay (Proposition 4.2). In the next section we study the stability of the delayed problem provided that the operator $\mathcal{A}_0$ generates an exponentially stable $C_0$-Semigroup.

### Stability of the problem with delay

In this section we are going to establish the proof of our main theorem (Theorem 0.6), using the inequality (16).

**Proposition 0.7.** *Under the assumptions (11), **(H)**, **(A1)**,**(A2)**, and **(D1)**-**(D4)**, the operator $\mathcal{A}$ satisfies*

$$\sup_{\beta\in\mathbb{R}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{H}} < +\infty.$$

*Proof.* The proof uses the resolvent estimate (15), the inequality (16), and some classical inequalities. □

At this point the proof of Theorem 0.6 is finished; However, the exponential stability of the problem without delay which was used as an assumption will be proved in the next section.

### Stability of the problem without delay

In this section we prove that the operator $\mathcal{A}_0$ that we introduced in the previous section generates an exponentially stable semigroup. For this matter we use Theorem 1.4 (Huang-Prüss [47, 82]).

**Lemma 0.8.** *For all $\beta \in \mathbb{R}$, one has*

$$\text{Ker}(i\beta I - \mathcal{A}_0) = \{0\}.$$

*Proof.* The proof is based on a standard unique continuation argument. □

**Proposition 0.9.** *For all $\beta \in \mathbb{R}$, one has*

$$R(i\beta I - \mathcal{A}_0) = \mathcal{H}_0.$$

*Proof.* The proof follows the same steps as the damped case which was treated in the strong stability section. □

Next, we present the lemma that's the pillar of our work.

**Lemma 0.10.** *Under the assumptions (H),(**A1**),(**A2**), and (**D1**)-(**D4**), $\mathcal{A}_0$ satisfies*

$$\sup_{\beta \in \mathbb{R}} \|(i\beta I - \mathcal{A}_0)\|_{\mathcal{L}(\mathcal{H}_0)} < \infty. \tag{17}$$

*Proof.* We proceed by contradiction. Suppose that (17) doesn't hold. Then, by the uniform resonance theorem, there exists a sequence $(\beta_n)_{n\in\mathbb{N}} \in \mathbb{R}$ and a sequence $(u_n, v_n, v_n|_{\Gamma_1})_{n\in\mathbb{N}} \in \mathcal{D}(\mathcal{A}_0)$ such that

$$|\beta_n| \to +\infty, \tag{18}$$

$$\|u_n\|_V^2 + \|v_n\|_{L^2(\Omega)} + \|v_n\|_{L^2(\Gamma_1)} = 1, \tag{19}$$

$$i\beta_n \left(u_n, v_n, v_n|_{\Gamma_1}\right) - \mathcal{A}_0 \left(u_n, v_n, v_n|_{\Gamma_1}\right) := (f_n, g_n, h_n) \to 0 \text{ in } \mathcal{H}. \tag{20}$$

Then,

$$i\beta_n u_n - v_n \quad := \quad f_n \to 0 \text{ in } V, \tag{21}$$

$$i\beta_n v_n - \text{div}\left(\nabla u_n + a(x)\nabla v_n\right) \quad := \quad g_n \to 0 \text{ in } L^2(\Omega), \tag{22}$$

$$i\beta_n v_n + \frac{\partial u_n}{\partial \nu} + a(x)\frac{\partial v_n}{\partial \nu} - \Delta_T u_n + v_n \quad := \quad h_n \to 0 \text{ in } L^2(\Gamma_1). \tag{23}$$

We look for a contradiction of the form

$$\|u_n\|_V^2 + \|v_n\|_{L^2(\Omega)} + \|v_n\|_{L^2(\Gamma_1)} = o(1). \tag{24}$$

The proof is divided into several steps.

**Step 1.** A standard analysis gives

$$\|u_n\|_V^2 + \|v_n\|_{L^2(\Omega)} = o(1).$$

**Step 2.** Using simple multipliers we achieve the following estimate

$$\int_\Omega a|\nabla v_n|^2 dx + \int_{\Gamma_1} |v_n|^2 d\Gamma = o(1). \tag{25}$$

and the following equivalence

$$\|u_n\|_V \sim \|v_n\|_{L^2(\Omega)}. \tag{26}$$

So, in order to achieve the contradiction (24), we only need to show

$$\|u_n\|_V = o(1). \tag{27}$$

**Step 3.** From (25), using the assumption (**A1**) and Poincaré's inequality, we find

$$\int_{\mathcal{O}_\delta} |v_n|^2 \, dx = o(1). \tag{28}$$

Then, using the following cut-off function: $\eta \in C^1(\overline{\Omega})$ such that

$$\eta(x) = \begin{cases} 1 & \text{if } x \in \mathcal{O}_{\delta-\epsilon}, \\ 0 & \text{if } x \in \Omega_\delta, \\ \eta(x) \in [0,1] & \text{elsewhere} \end{cases}$$

and suitable multipliers, we deduce

$$\int_{\mathcal{O}_\beta} |\nabla u_n|^2 \, dx + \int_{\Gamma_1} |\nabla_T u_n|^2 \, d\Gamma = o(1). \tag{29}$$

Now, we need only to show

$$\int_{\Omega_\beta} |\nabla u_n|^2 \, dx = o(1).$$

**Step 4.** In this step, using the assumption **(A2)** and (22), we find a uniform bound for

$$\int_\Omega a \, |\beta_n v_n|^2 \, dx + \int_{\Gamma_1} a \, |\beta_n v_n|^2 \, dx \tag{30}$$

which is used in the coming steps.

Until now, we have obtained the estimation of the integral of $\nabla u_n$ on the subdomain $\mathcal{O}_\beta$. Likewise, we will establish a similar estimation on $\Omega_\beta$ which is required to achieve (41). This is the purpose of the following step.

**Step 5.** Using the elliptic regularity, the vector field $q(\cdot)$, and the assumptions (**D1**)-(**D4**), besides well chosen multipliers, we deduce

$$\int_{\Omega_\beta} |\nabla u_n|^2 \, dx = o(1), \tag{31}$$

which leads to the desired contradiction. Hence, the proof is now completed. □

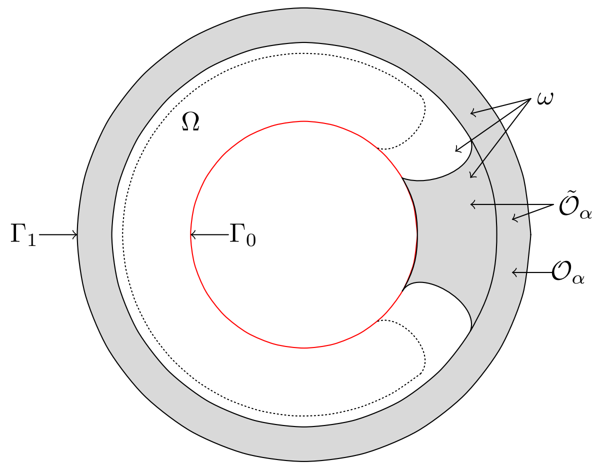


Figure 3: An example where $\Gamma_0$ and $\Gamma_1$ are far away from each other and $\mathcal{O}_\alpha$ doesn't meet $\Gamma_0$.

We finish this chapter by the following remark

**Remark 0.11.** *When $\Omega$ is the crown domain between two circles $\Gamma_0$ and $\Gamma_1$ that constitute the two parts of boundary, the analysis made in this paper still holds except the estimate (28) because of the lack of Poincaré's inequality since $\mathcal{O}_\alpha$ can't meet the part $\Gamma_0$. However, if we consider a set $\tilde{\mathcal{O}}_\alpha$ containing $\mathcal{O}_\alpha$ and such that* $\text{meas}(\overline{\tilde{\mathcal{O}}_\alpha} \cap \Gamma_0) > 0$, *we can use the Poincaré's inequality and (28) still holds (see Figure 3).*

### 0.2.3 Polynomial Stability: Case of discontinuous damping coefficient

In the fourth chapter we study the stability of our problem when the damping coefficient $a(\cdot)$ is discontinuous, namely, when $a(x) = a1_\omega$. We will show that the associated semigroup is polynomially stable with a decay rate of type $t^{-1/2}$. For this matter we will use the perturbation argument from the previous chapter adapted to the discontinuous case and a cascade technique that allows us to merge different stability results for different systems. Particularly, this will be based on an early established result (Cavalcanti et al. [28]), hence, we will put ourselves in the same geometrical situation, that is when $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$ where, $\Gamma_0$ and $\Gamma_1$ are closed and disjoint; besides, $\omega$ is a neighborhood of $\Gamma_1$, where

$$\Gamma_1 = \{x \in \Gamma, m(x) \cdot \nu > 0\},$$

such that $m(x) = x - x_0$ and $x_0$ is an arbitrary point in $\mathbb{R}^n$. Moreover, we make the following two assumptions (see Figure 4).

**(H)** $meas(\Gamma_1) > 0$, **(A)** $\exists \delta > 0$, $\mathcal{O}_\delta \subset \omega$, where

$$\mathcal{O}_\delta = \{x \in \Omega,\ \ |x - y| \leq \delta,\ \forall y \in \Gamma_1\}.$$

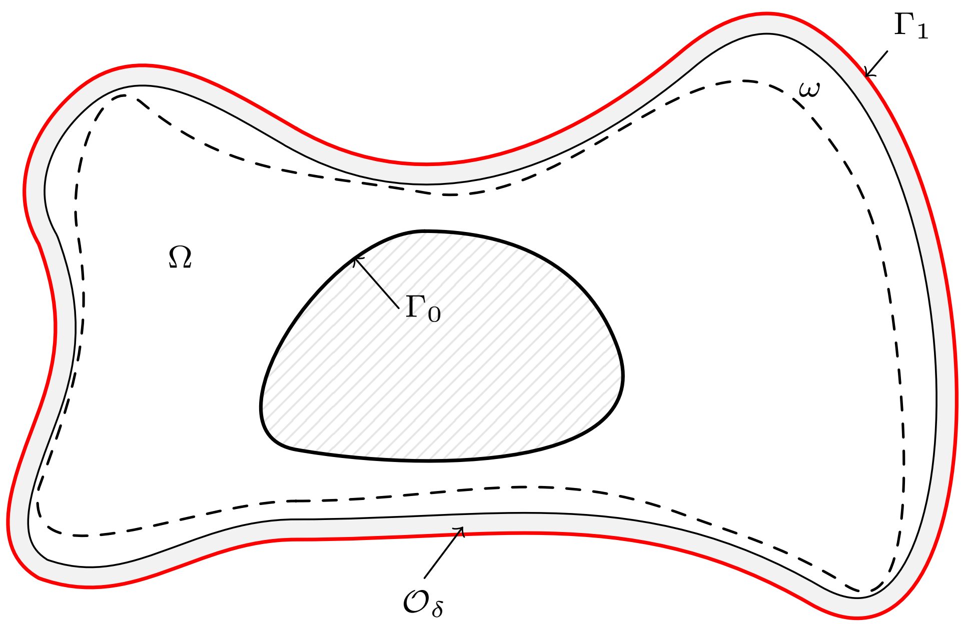


Figure 4: An example of a geometric situation satisfying the assumptions.

After releasing the regularity assumptions on $a(\cdot)$ our system is still stable, however, with a slower decay rate this time. This is the subject of the following theorem.

**Theorem 0.12.** *Suppose that the assumptions (**H**),(**A**) and the inequality (11) are satisfied, then there is a constant $C > 0$, such that for all initial data $U_0 \in \mathcal{D}(\mathcal{A})$ the solution $U := (u, u_t, u_t|_{\Gamma_1}, z)$ of the problem (6)-(10) satisfies the following polynomial decay estimate*

$$\|U\,(t)\,\|_{\mathcal{H}} \leq \frac{C}{\sqrt{t}}\|U_0\|_{\mathcal{D}(\mathcal{A})}. \tag{32}$$

The proof follows the same big lines as the smooth damping coefficient case.

In a suitable Hilbert space $\mathcal{H}_0$ we define the operator $\mathcal{A}_0$, that corresponds to $\tau = 0$ and $k = 0$, that is

$$\mathcal{A}_0 U = \begin{pmatrix} v \\ \operatorname{div}(\nabla u + a\nabla v) \\ -\dfrac{\partial u}{\partial \nu} - a\dfrac{\partial v}{\partial \nu} + \Delta_T u \end{pmatrix}, \forall U \in \mathcal{D}(\mathcal{A}_0),$$

with

$$\mathcal{D}(\mathcal{A}_0) := \Big\{ U = (u, v, w)^T \in \mathcal{H}_0 : v \in V, \operatorname{div}(\nabla u + a\nabla v) \in L^2(\Omega),$$
$$\frac{\partial u}{\partial \nu} + a\frac{\partial v}{\partial \nu} - \Delta_T u \in L^2(\Gamma_1), w = v|_{\Gamma_1} \Big\}.$$

At this point we suppose that $i\mathbb{R} \subset \rho(\mathcal{A}_0)$ and that the operator $\mathcal{A}_0$ generates a polynomially stable semigroup with a decay rate of type $\frac{1}{\sqrt{t}}$, then from Theorem 1.5 (Borichev-Tamilov [22]) we have

$$\| (i\xi - \mathcal{A}_0)^{-1} \|_{\mathcal{L}(\mathcal{H}_0)} \leq C\xi^2, \quad \forall \xi \in \mathbb{R}. \tag{33}$$

This resolvent estimate will be used to derive the estimate

$$\int_{\Gamma_1} |\tilde{v}|^2 d\Gamma \leq C|\xi|^2 \|F_0\|^2_{\mathcal{H}_0}. \tag{34}$$

which we use in the proof of the stability of the problem with delay (Proposition 0.7).

### Stability of the problem with delay

Now we are going to prove the main result using the inequality (34).

**Proposition 0.13.** *Under the assumptions* (***H***),(***A***)*, the inequality (11) and* $|\beta| \geq 1$*, the operator* $\mathcal{A}$ *satisfies*

$$\sup_{\beta\in\mathbb{R}} \frac{1}{\beta^2} \| (i\beta I - \mathcal{A})^{-1} \|_{\mathcal{H}} < +\infty.$$

*Proof.* The proof uses the resolvent estimate (33), the inequality (34) and some classical inequalities. □

Now, we go back and show the polynomial stability of the problem without delay, namely, the problem related to the operator $\mathcal{A}_0$.

### Stability of the problem without delay

**Lemma 0.14.** *For all* $\beta \in \mathbb{R}$*, one has*

$$\mathrm{Ker}(i\beta I - \mathcal{A}_0) = \{0\}.$$

*Proof.* Based on a standard unique continuation argument. □

**Proposition 0.15.** *For all* $\beta \in \mathbb{R}$ *one has*

$$R(i\beta I - \mathcal{A}_0) = \mathcal{H}_0. \tag{35}$$

*Proof.* Similar to the delayed case. □

**Lemma 0.16.** *Under the assumptions* (***H***) *and* (***A***)*,* $\mathcal{A}_0$ *satisfies*

$$\sup_{\beta\in\mathbb{R}} \frac{1}{\beta^2} \|(i\beta I - \mathcal{A}_0)^{-1}\|_{\mathcal{L}(\mathcal{H}_0)} < \infty.$$

*Proof.* We will prove the result by contradiction, suppose there exist sequences $\{\beta_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+^*$ and $\{U_n := (u_n, v_n, w_n)\}_{n\in\mathbb{N}}$ in $\mathcal{D}(\mathcal{A}_0)$ such that

$$|\beta_n| \to +\infty, \tag{36}$$

$$\|u_n\|_V^2 + \|v_n\|_{L^2(\Omega)} + \|w_n\|_{L^2(\Gamma_1)} = 1, \tag{37}$$

$$\beta_n^2\{i\beta_n\,(u_n, v_n, w_n) - \mathcal{A}_0\,(u_n, v_n, w_n)\} := (f_n, g_n, h_n) \to 0 \text{ in } \mathcal{H}. \tag{38}$$

Using some simples multipliers, we deduce

$$\int_\Omega a|\nabla v_n|^2 dx = o(\frac{1}{\beta_n^2}). \tag{39}$$

and

$$\|u_n\|_V \sim \|v_n\|_{L^2(\Omega)}. \tag{40}$$

So, all we need to show is

$$\|u_n\|_V = o(1). \tag{41}$$

Again, using some multipliers, we deduce

$$\int_\omega |\beta_n^2 \nabla u_n|^2 dx = o(1). \tag{42}$$

$$\int_{\Gamma_1} |\nabla_T u_n|^2 d\Gamma = o(1). \tag{43}$$

Now we have all the needed estimates on $\omega$ and we need to establish similar estimates on $\Omega\backslash\omega$. To that end, we are going to use a stability result of a similar system with frictional/viscous damping.

We consider the following auxiliary problem,

$$\left\{\begin{array}{rll} \varphi_{tt}(x,t) - \Delta\varphi + a1_\omega\varphi_t = & 0, & \text{in } \Omega\times\mathbb{R}_+^*, \\ \varphi(x,t) = & 0, & \text{on } \Gamma_0\times\mathbb{R}_+^*, \\ \varphi_{tt} + \dfrac{\partial\varphi}{\partial\nu} - \Delta_T\varphi + \varphi_t = & 0, & \text{on } \Gamma_1\times\mathbb{R}_+^*, \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = & \varphi_1(x), & \text{in } \Omega. \end{array}\right. \tag{44}$$

This system is a particular case of the system studied in [28]. From there we can deduce the exponential stability of the system (44), (See Remark 4.1 in [28]).

Now, since the system (44) is exponentially stable in a suitable Hilbert space $\mathcal{H}_{aux}$, henceforth, after Theorem 1.4 (Huang-Prüss [47, 82]) the operator

$$\mathcal{A}_{aux}U = \begin{pmatrix} \psi \\ \Delta\varphi - a1_\omega\psi \\ -\dfrac{\partial\varphi}{\partial\nu} + \Delta_T\varphi - \phi \end{pmatrix}, \forall U \in \mathcal{D}(\mathcal{A}_{aux}),$$

such that,

$$\mathcal{D}(\mathcal{A}_{aux}) := \left\{ U = (\varphi, \psi, \phi)^T \in V^2 \times L^2(\Gamma_1), \Delta\varphi \in L^2(\Omega), \frac{\partial \varphi}{\partial \nu} - \Delta_T \varphi + \phi \in L^2(\Gamma_1) \right\},$$

satisfies the following uniform inequality

$$\|(i\beta_n - \mathcal{A}_{aux})^{-1} F\|_{\mathcal{H}} \leq M \|F\|_{\mathcal{H}}, \quad \forall F \in \mathcal{H}_{aux} \tag{45}$$

for a positive constant $M > 0$. Taking $F = (0, -u_n, 0)$, there exists a unique $(\varphi_n, \psi_n, \phi_n) \in \mathcal{D}(\mathcal{A}_{aux})$ solution of

$$(i\beta_n I - \mathcal{A}_{aux})(\varphi_n, \psi_n, \phi_n) = (0, -u_n, 0).$$

Finally, from (45) we deduce

$$\|\nabla \varphi_n\|^2_{L^2(\Omega)} + \|\nabla_T \varphi_n\|^2_{L^2(\Gamma_1)} + \|\beta_n \varphi_n\|^2_{L^2(\Omega)} + \|\beta_n \varphi_n\|^2_{L^2(\Gamma_1)} \leq C \|u_n\|^2_{L^2(\Omega)}, \tag{46}$$

which we use with some new multipliers to derive the following estimate

$$\int_\Omega |\nabla u_n|^2 dx = o(1) \tag{47}$$

which finishes the proof. $\square$

# Chapter 1

# Preliminaries

This chapter is devoted to presenting the methodology used in this thesis related to the stability of semigroups as well as some well-known results that will be used in the following chapters. Except Theorem 1.4, all other theorems are presented without proofs, however, the relevant references are given.

## 1.1 General notions in stability theory

We are interested in the stability of the following abstract Cauchy problem in a Hilbert space $H$

$$\begin{cases} u_t(t) = \mathcal{A}u(t), t \in (0, +\infty) \\ u(0) = x \in H, \end{cases} \tag{A1}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H \to H$ is a linear unbounded operator.

Two corner stones in the theory of semigroups related to PDEs are the two theorems of Hille-Yosida and Lumer-Phillips.

**Theorem 1.1.** *(Hille-Yosida, [40]) A linear operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ in a Banach space $X$ generates a strongly continuous contraction semigroup $\big(S(t)\big)_{t\geq 0}$ if and only if it is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $\Re\lambda >$ 0 one has $\lambda \in \rho(A)$ and*

$$\|R(\lambda, A)\| \leq \frac{1}{\Re\lambda} \tag{1.1}$$

*where $R(\lambda, A)$ is the resolvent operator.*

**Theorem 1.2.** *(Lumer-Phillips, [40]) For a densely defined, dissipative operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ on a Banach space $X$ the following statements are equivalent.*

- *The closure $\bar{A}$ of $\mathcal{A}$ generates a contraction semigroup.*
- *$R(\lambda - \mathcal{A})$ is dense in $X$ for some (hence all) $\lambda > 0$.*

In practice, the resolvent estimate (1.1) in Theorem 1.1 is difficult to verify, in contrast to the assumptions of Theorem 1.2, which are easier to deal with in most of the rencontred cases in Hilbert spaces, then one gets (1.1) for free.

Assume that $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ generates a $C_0$-semigroup of contraction $\big(S(t)\big)_{t\geq 0}$ in $H$, then the existence of a unique solution to the Cauchy problem (A1) is derived from the properties of $C_0$-semigroups (see e.g. Lemma II.1.3 in [40]), and the solution is given by

$$u(t) = S(t)x, \quad \forall t > 0. \tag{sol}$$

The semigroup $\big(S(t)\big)_{t\in\mathbb{R}^+}$ is called:

- Strongly (asymptotically) stable if for any $x \in H$:

$$\lim_{t\to\infty} \|S(t)x\|_H = 0. \tag{ss}$$

- Exponentially (uniformly) stable if there exist constants $M \geq 0$ and $\alpha > 0$ such that

$$\|S(t)\|_H \leq Me^{-\alpha t}$$

  or equivalently

$$\lim_{t\to\infty} \|S(t)\|_H = 0.$$

- Polynomially stable of type $t^{-\alpha}$if there exists a constant $C > 0$ such that for every $x \in \mathcal{D}(\mathcal{A})$:

$$\|S(t)x\|_H \leq \frac{C}{t^\alpha}\|x\|_{\mathcal{D}(\mathcal{A})}$$

  One must notice that the norm $\|\cdot\|_{\mathcal{D}(\mathcal{A})}$ can't be replaced by $\|\cdot\|_H$ otherwise using the semigroup property polynomial stability will imply exponential stability.

The asymptotic behavior of the solution (sol) or analogously the semigroup $\big(S(t)\big)_{t\geq 0}$ is related to spectral properties of the infinitisimal generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$. From the Hille-Yosida theorem we know that the spectrum of $\mathcal{A}$, $\sigma(\mathcal{A})$ lies in the closed left half of the complex plane. Obviously, if $\mathcal{A}$ has eigenvalues in the imaginary axis then there is no hope for stability; However, if this is not the case, then we have the following theorem

**Theorem 1.3** (Arendt-Batty, [19])**.** *Assume that $\mathcal{A}$ is the generator of a $C_0$-semigroup of contractions $(S(t))_{t\geq 0}$ on a Hilbert space $H$. If*

1. *$\mathcal{A}$ has no pure imaginary eigenvalues,*
2. *$\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable,*

*then the $C_0$-semigroup $(S(t))_{t\geq 0}$ is strongly stable.*

In general, the rate of decay in (ss) can be extremely slow without any further conditions. Nevertheless, in certain specific scenarios such as our case, the rate of decay in (ss) is related to the rate at which the energy of the system (A1) decreases. Therefore it is of interest to investigate if this rate of decay can be quantified.

We know that the semigroup $\left(S(t)\right)_{t\geq 0}$ satisfies the following estimate:

$$\|S(t)\|_H \leq Me^{\alpha t}, \tag{1.2}$$

for some constants $M \geq 1$ and $\alpha$. The smallest possible $\omega$ is called the growth bound and is defined as follows

$$\omega_0 = \omega_0(S) := \inf\{\alpha \in \mathbb{R} : \exists M \geq 1 \text{ such that } \|S(t)\|_H \leq Me^{\alpha t}, \forall t \geq 0\}. \tag{1.3}$$

Another important quantity is the spectral bound, defined as follow

$$s(\mathcal{A}) := \sup\{\Re\lambda, \lambda \in \sigma(\mathcal{A})\}.$$

Obviously, we have

$$\omega_0(S) < 0 \Leftrightarrow \left(S(t)\right)_{t\geq 0} \text{ is exponentially (uniformly) stable} \Rightarrow s(\mathcal{A}) < 0.$$

From Hille-Yosida theorem it follows that

$$s(\mathcal{A}) \leq \omega_0(S).$$

The equality is known to fail in general (see e.g. [81, 40] ) but it holds whenever the semigroup $\left(S(t)\right)_{t\in\mathbb{R}^+}$ is eventually norm continuous which is the case for analytic, eventually differentiable, eventually compact semigroups or semigouprs that are generated by bounded operators (one can see the diagram II.4.26 in [40]).

If we change the spectral bound $s(\mathcal{A})$ by the pseudospectral bound $s_0(\mathcal{A})$

$$s_0(\mathcal{A}) := \inf\{\alpha \in \mathbb{R} : \sup_{\Re\lambda > \alpha} \|R(\lambda, \mathcal{A})\|_H < +\infty\}, \tag{1.4}$$

then, the equality $s_0(\mathcal{A}) = \omega(S)$ always holds in Hilbert spaces. This is the subject of the following well-known and widely used theorem

**Theorem 1.4.** *(Huang-Prüss, [47, 82]) A $C_0$ semigroup $(S(t))_{t\geq 0}$ of contractions on a Hilbert space $H$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{H1}$$

*and*

$$\sup_{\beta\in\mathbb{R}} \left\|(i\beta I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(H)} < \infty, \tag{H2}$$

*where $\mathcal{A}$ is the infinitisimal generator of $(S(t))_{t\geq 0}$ and $\rho(\mathcal{A})$ denotes the resolvent set of the operator $\mathcal{A}$.*

When the condition (H2) is not satisfied, we still hope to understand and quantify the rate of the decay. The following theorem provides a characterization of a slower (polynomial) decay rate in this case.

**Theorem 1.5** (Borichev-Tamilov, [22])**.** *Let $(S(t))_{t\geq 0}$ be a bounded $C_0$-semigroup on a Hilbert space $H$ with generator $\mathcal{A}$ such that $i\mathbb{R} \subset \rho(A)$. Then for a fixed $\ell > 0$ the following conditions are equivalent*

- $\left\|(\mathrm{i}\beta - \mathcal{A})^{-1}\right\| = O\left(|\beta|^{\ell}\right), \beta \to \infty,$
- $\left\|S(t)\mathcal{A}^{-1}\right\| = O\left(t^{-1/\ell}\right), t \to \infty.$

We finish this section by a short and elementary proof of Huang-Prüss theorem (Theorem 1.4) given recently by Filippo Dell'oro and David Seifert in [38].

*Proof.* (of Theorem 1.4) To simplify some expressions in this proof we write $R(\lambda)$ instead of $R(\lambda, \mathcal{A})$ for $\lambda \in \rho(\mathcal{A})$.

The necessary part is standard and well-known. For the sufficiency part we note that, as a consequence of the semigroup property, it suffices to show that $\lim\limits_{t\to\infty} \|S(t)\| = 0$. Suppose then that $i\mathbb{R} \subseteq \rho(\mathcal{A})$ and $\sup\limits_{\beta\in\mathbb{R}} \|R(i\beta)\| < \infty$, and let $K$ be defined as follow

$$K := \sup_{t\geq 0} \|S(t)\| < \infty.$$

Given $\alpha > 0$, we consider the rescaled semigroup $(S_\alpha(t))_{t\geq 0}$ given by $S_\alpha(t) = e^{-\alpha t}S(t)$ for $t \geq 0$. For every $x \in X$ and $t > 0$ we have

$$\|S\alpha(t)x\|^2 = \frac{1}{t}\int_0^t \|S_\alpha(t-\tau)S_\alpha(\tau)x\|^2\, d\tau \leq \frac{K^2}{t}\int_0^\infty \|S_\alpha(\tau)x\|^2\, d\tau. \tag{1.5}$$

Now let $C = \sup\limits_{\beta\in\mathbb{R}} \|R(i\beta)\|$ and $\omega = C^{-1}$. It follows from the resolvent identity that, for $\alpha > 0$ and $\beta \in \mathbb{R}$,

$$\|R(\alpha + i\beta)\| = \|R(i\beta) - \alpha R(\alpha + i\beta)R(i\beta)\| \leq C + C\alpha\|R(\alpha + i\beta)\|,$$

and hence $\|R(\alpha + i\beta)\| \leq (\omega - \alpha)^{-1}$ for all $\alpha \in (0, \omega)$ and $\beta \in \mathbb{R}$. By another application of the resolvent identity we obtain

$$\|R(\alpha + i\beta)x\| = \|R(\omega + i\beta)x + (\omega - \alpha)R(\alpha + i\beta)R(\omega + i\beta)x\| \leq 2\|R(\omega + i\beta)x\|$$

for all $\alpha \in (0, \omega), \beta \in \mathbb{R}$ and $x \in X$. If $\alpha > 0$ and $x \in X$, and if we extend the semigroup $(S_\alpha(t))_{t\geq 0}$ by zero to $\mathbb{R}$,

$$\tilde{S}_\alpha(t) = \begin{cases} S_\alpha(t), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Then, by the integral representation of the resolvent the function $\beta \mapsto R(\alpha + i\beta)x$ is the Fourier transform of the function $t \mapsto S_\alpha(t)x$.

$$R(\alpha + i\beta)x = \int_0^\infty e^{-i\beta t} S_\alpha(t)xdt = \int_{-\infty}^\infty e^{-i\beta t} \tilde{S}_\alpha(t)xdt \tag{1.6}$$

Thus, applying Plancherel's theorem (twice) we deduce that, for $\alpha \in (0, \omega)$,

$$\begin{aligned}\int_0^\infty \|S_\alpha(\tau)x\|^2 d\tau = \int_{-\infty}^\infty \left\|\tilde{S}_\alpha(\tau)x\right\|^2 d\tau = \frac{1}{2\pi}\int_{-\infty}^\infty \|R(\alpha+i\beta)x\|^2 d\beta \le \frac{2}{\pi}\int_{-\infty}^\infty \|R(\omega+i\beta)x\|^2 d\beta \\ = 4\int_0^\infty \|S_\omega(\tau)x\|^2 d\tau \le c^2\|x\|^2\end{aligned} \tag{1.7}$$

where $c = K(2C)^{1/2}$. Combining (1.5) and (1.7) gives $\|S_\alpha(t)\| \le Kct^{-1/2}$ for all $\alpha \in (0, \omega)$ and $t > 0$, and letting $\alpha \to 0^+$we get

$$\|S(t)\| \le \frac{Kc}{t^{1/2}} \to 0, \quad t \to \infty.$$

Thus $(S(t))_{t\ge 0}$ is exponentially stable, as required. □

## 1.2 Functional analysis tools

In this section, for the reader's inconvenience, we provide some functional analysis tools that are employed in the thesis. Throughout this section $X$ and $Y$ are Banach spaces and $\Omega$ is a connected open set in $\mathbb{R}^n$ with a boundary $\partial\Omega$ of class $C^2$.

**Theorem 1.6.** *(Closed graph theorem, [83]) Let $(A, \mathcal{D}(A))$ be a linear operator from $X$ to $Y$. Then if $(A, \mathcal{D}(A))$ is a closed operator and its domain $\mathcal{D}(A)$ is closed in $X$, then the operator is bounded.*

**Theorem 1.7.** *(The uniform resonance/boundedness theorem, [83]) Let $T_n \in \mathcal{L}(X, Y)$, $n \in \mathbb{N}$ be such that $\{T_n\}$ is pointwise bounded in $X$, then it is uniformly bounded, i.e. $\sup\limits_n \|T_n\|_{\mathcal{L}(X,Y)} < \infty$.*

**Theorem 1.8.** *(Fredholm alternative, [83]) Let $A, K \in \mathcal{L}(X, Y)$ and let $K$ be compact. Then $A$ is an isomorphism if and only if $A + K$ is.*

### Sobolev embedding

**Theorem 1.9.** *(Poincaré's inequality, Lemma I.3.1 in [44]) Let $\Gamma_0$ be a portion of $\Gamma = \partial\Omega$ with strictly positive measure. Then for every $u \in H^1_{\Gamma_0}(\Omega)$, there exists a constant $C > 0$ such that*

$$\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}.$$

**Theorem 1.10.** *(Trace operator, Theorem 1.2 in [75]) Let* $\Gamma$ *be at least Lipschitz. Then there exists a uniquely defined, linear and continuous mapping* $T : H^1(\Omega) \to L^2(\Gamma)$ *such that for every* $x \in \Gamma$ *and* $v \in C^\infty(\bar{\Omega})$*, it is defined by* $T(v)(x) = v(x)$*.*

**Theorem 1.11.** *(Theorem 1.4.3.2 in [45]) Let* $s > s' \geq 0$*, then, the injection of* $H^s(\Omega)$ *in* $H^{s'}(\Omega)$ *is compact.*

## Elliptic regularity

**Theorem 1.12.** *(Theorem 8.8 in [43]) Let* $u \in H^1(\Omega)$ *be the weak solution to the equation* $\Delta u = f$ *and* $f \in L^2(\Omega)$*. Then for any subdomain* $\Omega' \Subset \Omega$*, i.e.* $\Omega'$ *has a compact closer in* $\Omega$*, we have* $u \in H^2(\Omega')$*.*

**Theorem 1.13.** *(Theorem 8.12 in [43]) Let* $u \in H_0^1(\Omega)$ *such that* $\Delta u \in L^2(\Omega)$*. Then* $u \in H^2(\Omega)$*.*

## Unique continuation

**Theorem 1.14.** *(Calderón theorem, [54]). Let* $g$ *be such that* $|g(y)| \leq C|y|$*. Let* $\omega \Subset \Omega$*, with* $\omega \neq \emptyset$*. If* $u \in H^2(\Omega)$ *satisfies* $\Delta u = g(u)$ *in* $\Omega$ *and* $u(x) = 0$ *in* $\omega$*, then* $u$ *vanishes in* $\Omega$*.*

# Chapter 2

# Well-Posedness and Strong Stability

## 2.1 Introduction

In this chapter we present our system, then we prove its well-posedness and the strong stability of its solution.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a boundary $\Gamma$ of class $C^2$. We assume that $\Gamma$ is divided into two open parts $\Gamma_0$ and $\Gamma_1$, i.e. $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\operatorname{meas} \Gamma_i \neq 0, i = 0, 1$.
In this domain $\Omega$, we consider the following initial boundary value problem

$$
\begin{aligned}
u_{tt}(x,t) - \operatorname{div}\{\nabla u(x,t) + a(x)\nabla u_t(x,t)\} &= 0, \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (2.1)\\
u(x,t) &= 0, \quad \text{on } \Gamma_0 \times \mathbb{R}_+^*, \quad (2.2)\\
u_{tt}(x,t) + \frac{\partial u}{\partial \nu}(x,t) + a(x)\frac{\partial u_t}{\partial \nu}(x,t) - \Delta_T u(x,t) + k u_t(x,t-\tau) &= 0, \quad \text{on } \Gamma_1 \times \mathbb{R}_+^*, \quad (2.3)\\
u(x,0) = u_0(x), u_t(x,0) &= u_1(x), \quad \text{in } \Omega, \quad (2.4)\\
u_t(x,t-\tau) &= f_0(x,t-\tau), \quad \text{in } \Omega \times (0,\tau), \quad (2.5)
\end{aligned}
$$

where $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$, $\frac{\partial u}{\partial \nu}$ is the normal derivative of $u$, $\Delta_T$ is the Laplace-Beltrami operator defined by

$$\Delta_T u := \operatorname{div}_T \nabla_T u,$$

where $\operatorname{div}_T$ is the tangential divergence such that the following Stokes formula holds

$$\int_{\Gamma_1} \Delta_T u \tilde{u} d\Gamma = -\int_{\Gamma_1} \nabla_T u \nabla_T \tilde{u} d\Gamma, \quad \forall u, \tilde{u} \in H_0^1(\Gamma_1)$$

and $\nabla_T u$ denotes the tangential gradient where

$$\nabla u = \nabla_T u + \frac{\partial u}{\partial \nu} \cdot \nu, \text{ on } \Gamma_1,$$

$\tau > 0$ is the time delay, $k$ is a real number and $a(x) \in L^{\infty}(\Omega)$ satisfies

$$a(x) \geq 0 \ a.e \ \text{ in } \Omega, \quad a(x) \geq a_0 > 0 \ a.e \ \text{ in } \omega.$$

Here, $\omega \subset \Omega$ is an open neighborhood of the part $\Gamma_1$ of the boundary that is supposed to be connected and meas$(\overline{\omega} \cap \Gamma_0) > 0$ (see Figure 2.1). Furthermore, the initial datum $(u_0, u_1, f_0)$ belongs to a suitable space.

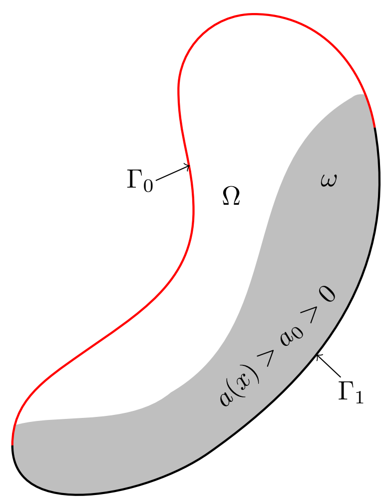


Figure 2.1: Example satisfying the required geometrical assumptions.

In this chapter, we are going to establish the well-posedness of our system under a suitable relation (Inequality (2.7) in the sequel ) between the function $a(\cdot)$ that characterizes the region where the Kelvin-Voigt damping is active and $k$ the coefficient of the delayed boundary feedback. This relation ensures the dissipativity of our system which is crucial for the forthcoming analysis.

Without loss of generality we can suppose that $\omega$ has a $C^2$ boundary, then from the trace theorem we have

$$\int_{\partial\omega} |v|^2 d\Gamma \leq C\|v\|_{H^1(\omega)}, \forall v \in H^1(\omega),$$

for some positive constant $C$. Hence, the following Poincare-Trace type inequality holds

$$\int_{\Gamma_1} |v|^2 d\Gamma \leq C_P \int_{\omega} |\nabla v|^2 dx, \forall v \in H^1_{\Gamma_0}(\Omega), \tag{2.6}$$

where

$$H^1_{\Gamma_0}(\Omega) = \left\{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\right\},$$

and $C_P$ is the smallest possible positive constant.

We shall show that, when the damping is active in a neighborhood of the part $\Gamma_1$ of the boundary and the following condition is satisfied

$$a_0 > |k| C_P, \tag{2.7}$$

the system (2.1)-(2.5) is well posed and its energy decays to zero. First, we will follow [76] and [77] and transform the boundary delay feedback using an auxiliary state, then using a semigroup approach, Lax-Milgram's lemma and the Lumer-Phillips's theorem we will prove the existence and uniqueness of a solution.

As in [76] and [77], we introduce the following new state,

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \quad x \in \Gamma_1, \rho \in (0,1), t > 0. \tag{2.8}$$

The system (2.1)-(2.5) is then equivalent to:

$$u_{tt}(x,t) - \operatorname{div}(\nabla u + a(x)\nabla u_t) = 0, \qquad \text{in } \Omega \times \mathbb{R}_+^*, \tag{2.9}$$

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \qquad \text{in } \Gamma_1 \times (0,1) \times \mathbb{R}_+^*, \tag{2.10}$$

$$u(x,t) = 0, \qquad \text{on } \Gamma_0 \times \mathbb{R}_+^*, \tag{2.11}$$

$$u_{tt} + \frac{\partial u}{\partial \nu} + a(x)\frac{\partial v}{\partial \nu} - \Delta_T u + kz(x,1,t) = 0, \qquad \text{on } \Gamma_1 \times \mathbb{R}_+^*, \tag{2.12}$$

$$z(x,0,t) = u_t(x,t), \qquad \text{on } \Gamma_1 \times \mathbb{R}_+^*, \tag{2.13}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \qquad \text{in } \Omega, \tag{2.14}$$

$$u_t(x,t-\tau) = f_0(x,t-\tau), \qquad \text{in } \Omega \times (0,\tau). \tag{2.15}$$

We also introduce the following spaces

$$V = \left\{ u \in H^1_{\Gamma_0}(\Omega) : u|_{\Gamma_1} \in H^1_0(\Gamma_1) \right\},$$

and

$$\mathcal{H} = V \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1 \times (0,1)),$$

endowed with the inner product,

$$\left\langle \begin{pmatrix} u \\ v \\ v|_{\Gamma_1} \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{v}|_{\Gamma_1} \\ \tilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}} := \int_\Omega \{\nabla u(x) \cdot \nabla \overline{\tilde{u}}(x) + v(x)\overline{\tilde{v}}(x)\} dx$$

$$+ \int_{\Gamma_1} \{\nabla_T u(x) \cdot \nabla_T \overline{\tilde{u}}(x) + v(x)\overline{\tilde{v}}(x)\} d\Gamma$$

$$+ \xi \int_{\Gamma_1}\int_0^1 z(x,\rho)\overline{\tilde{z}}(x,\rho) d\rho d\Gamma,$$

where $\xi$ is a positive real number such that

$$|k| \leq \frac{\xi}{\tau} \leq \frac{2a_0}{C_p}. \tag{2.16}$$

Now, we introduce our unbounded operator $\mathcal{A}$ on $\mathcal{H}$,

$$\mathcal{A}U = \begin{pmatrix} v \\ \operatorname{div}(\nabla u + a\nabla v) \\ -\dfrac{\partial u}{\partial \nu} - a\dfrac{\partial v}{\partial \nu} + \Delta_T u - kz(\cdot, 1) \\ -\dfrac{1}{\tau} z_\rho \end{pmatrix}, \forall U \in \mathcal{D}(\mathcal{A}),$$

with domain,

$$\begin{aligned} \mathcal{D}(\mathcal{A}) :=& \Big\{ U = (u, v, v|_{\Gamma_1}, z)^T \in \mathcal{H} : \mathcal{A}U \in \mathcal{H} \Big\} \\ =& \Big\{ U \in V \times V \times L^2(\Gamma_1) \times L^2(\Gamma_1; H^1(0,1)) : \operatorname{div}(\nabla u + a\nabla v) \in L^2(\Omega), \\ & \frac{\partial u}{\partial \nu} + a\frac{\partial v}{\partial \nu} - \Delta_T u \in L^2(\Gamma_1), v|_{\Gamma_1} = z(\cdot, 0) \Big\}. \end{aligned}$$

If we denote

$$U = (u, v, v|_{\Gamma_1}, z)^T,$$

where $u_t = c$, we can reformulate our problem in the compact form:

$$\begin{cases} \partial_t U = \mathcal{A}U \\ U_0 = (u_0, u_1, u_1|_{\Gamma_1}, f_0(\cdot, -\cdot\tau))^T. \end{cases} \tag{2.17}$$

Since the vector field $(\nabla u + a\nabla v)$ belongs only to $H(\operatorname{div}, \Omega)$, where

$$H(\operatorname{div}, \Omega) := \{ u \in \big(L^2(\Omega)\big)^n : \operatorname{div} u \in L^2(\Omega) \},$$

we need a valid Green's formula for the related term. However, for a vector field $\Lambda \in H(\operatorname{div}, \Omega)$, $\Lambda \cdot \nu \in H^{-\frac{1}{2}}(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma_1)$. Where $H^{-\frac{1}{2}}(\Gamma_1)$ is the dual space of $\tilde{H}^{\frac{1}{2}}(\Gamma_1)$ the space of functions in $H^{\frac{1}{2}}(\Gamma_1)$ whose extensions outside $\Gamma_1$ by zero lies in $H^{\frac{1}{2}}(\Gamma)$, and we have the following Green's formula (See identity (I.2.17) of [44])

$$\int_\Omega \Lambda \cdot \nabla \varphi dx = -\int_\Omega \operatorname{div} \Lambda \varphi dx + \langle \Lambda \cdot \nu; \varphi \rangle_{\Gamma_1}, \quad \forall \varphi \in H^1_{\Gamma_0}(\Omega), \tag{2.18}$$

where $\langle \cdot; \cdot \rangle_{\Gamma_1}$ is the duality product between $H^{-\frac{1}{2}}(\Gamma_1)$ and $\tilde{H}^{\frac{1}{2}}(\Gamma_1)$.

**Remark 2.1.** *The inequality (2.7) is very important for the stability of the system considered in this thesis. Moreover, it is proved to be necessary in some situations; For example, in the one dimensional setting when $a(\cdot)$ is constant everywhere and the boundary term $u_{tt}$ is multiplied by a sufficiently small parameter $\mu$ as we deduce from the counterexample presented in section (4.3) in [76].*

## 2.2 Well-Posedness

First of all, we would like to draw the reader's attention to the fact that the well-posedness of evolution equations with delay isn't always guaranteed, as one may see in the work of Dreher, Quintilla, and Racke in [39].

Now, we present our first theorem regarding the existence, uniqueness, and regularity of the solution.

**Theorem 2.2.** *Assume that (2.7) holds. Then for any initial datum $U_0 \in \mathcal{H}$, there exists a unique (weak) solution $U \in C([0,+\infty),\mathcal{H})$ of problem (2.17). Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then*

$$U \in C([0,+\infty),\mathcal{D}(\mathcal{A})) \cap C^1([0,+\infty),\mathcal{H}).$$

*Proof.* Take $U = (u, v, v|_{\Gamma_1}, z)^T \in \mathcal{D}(\mathcal{A})$. Then

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =& \left\langle \begin{pmatrix} v \\ \operatorname{div}(\nabla u + a\nabla v) \\ -\dfrac{\partial u}{\partial \nu} - a\dfrac{\partial v}{\partial \nu} + \Delta_T u - kz(\cdot,1) \\ -\dfrac{1}{\tau} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ v|_{\Gamma_1} \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} \\
=& \int_\Omega \nabla v \nabla \overline{u} dx + \int_{\Gamma_1} \nabla_T v \nabla_T \overline{u} d\Gamma + \int_\Omega \operatorname{div}(\nabla u + a\nabla v)\overline{v} dx \\
&+ \int_{\Gamma_1} \big( -\frac{\partial u}{\partial \nu} - a\frac{\partial v}{\partial \nu} + \Delta_T u - kz(\cdot,1)\big)\overline{v} d\Gamma \\
&- \int_\Gamma \int_0^1 \frac{\xi}{\tau} z_\rho \overline{z} d\rho d\Gamma.
\end{aligned}$$

Therefore, by using Green's formula and the definition of $\mathcal{D}(\mathcal{A})$ we get,

$$\begin{aligned}
\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} =& -\int_\Omega a|\nabla v|^2 dx - k\Re \int_{\Gamma_1} z(\cdot,1)\overline{v} d\Gamma \\
&- \frac{\xi}{2\tau}\int_{\Gamma_1} \left(|z(\cdot,1)|^2 - |v|^2\right) d\Gamma,
\end{aligned}$$

where $\Re$ denotes the real part.

By Cauchy–Schwarz and the Young inequalities we find

$$\begin{aligned}
\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \le& -\int_\Omega a|\nabla v|^2 dx + \left(\frac{|k|}{2} + \frac{\xi}{2\tau}\right)\int_{\Gamma_1} |v|^2 d\Gamma \\
&+ \left(\frac{|k|}{2} - \frac{\xi}{2\tau}\right)\int_{\Gamma_1} |z(\cdot,1)|^2 d\Gamma.
\end{aligned}$$

Using (2.7), we deduce that

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq -\left(a_0 - C_P\left(\frac{|k|}{2}+\frac{\xi}{2\tau}\right)\right)\int_\omega |\nabla v|^2 dx + -\left(\frac{\xi}{2\tau}-\frac{|k|}{2}\right)\int_{\Gamma_1}|z(\cdot,1)|^2 d\Gamma \\ - \int_{\Omega\setminus\omega} a|\nabla v|^2 dx. \tag{2.19}$$

Observing that from (2.16) we have,

$$a_0 - C_P\left(\frac{|k|}{2}+\frac{\xi}{2\tau}\right) \geq 0, \quad \frac{\xi}{2\tau}-\frac{|k|}{2} \geq 0, \tag{2.20}$$

hence $\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq 0$, and the operator $\mathcal{A}$ is dissipative.

Now, we will show that the operator $\mathcal{A}$ is surjective. Given $(f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, we seek $(u, v, v|_{\Gamma_1}, z)^T \in \mathcal{D}(\mathcal{A})$ solution of

$$\begin{cases} v = f_1, \\ \operatorname{div}(\nabla u + a(x)\nabla v) = f_2, \\ -\dfrac{\partial u}{\partial \nu} - a(x)\dfrac{\partial v}{\partial \nu} + \Delta_T u - kz(,1) = f_3, \\ -\dfrac{1}{\tau} z_\rho = f_4. \end{cases} \tag{2.21}$$

The first equation in (2.21) gives $v \in V$, whose trace in $\Gamma_1$ is well defined. Then, from the fourth equation we deduce

$$z(x,\rho) = v(x) - \tau\int_0^\rho f_4(x,\sigma)d\sigma \quad \text{ on } \Gamma_1 \times (0,1), \tag{2.22}$$

and in particular

$$z(x,1) = v(x) - \tau\int_0^1 f_4(x,\sigma)d\sigma, \quad \forall x \in \Gamma_1.$$

Since $f_4$ belongs to $L^2(\Gamma_1 \times (0,1))$ then $z \in L^2(\Gamma_1 \times H^1(0,1))$. Moreover, $v, v|_{\Gamma_1}, z$ are uniquely defined, hence, we can define the following functional for every $u, \tilde{u} \in V$,

$$a(u,\tilde{u}) = \int_\Omega \nabla u \nabla\overline{\tilde{u}} dx + \int_{\Gamma_1} \nabla_T u \nabla_T \overline{\tilde{u}} d\Gamma,$$

which is bilinear, continuous, and coercive. Always for every $\tilde{u} \in V$, we define the following linear functional

$$l(\tilde{u}) = -\int_\Omega \left(f_2\overline{\tilde{u}} + a(x)\nabla f_1 \nabla\overline{\tilde{u}}\right)dx - \int_{\Gamma_1}\left(f_3 + kz(\cdot,1)\right)\overline{\tilde{u}} d\Gamma.$$

Using Lax-Milgram theorem we deduce that for every $(f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, there exists a unique $u \in V$ such that

$$a(u,\tilde{u}) = l(\tilde{u}), \quad \forall \tilde{u} \in V. \tag{2.23}$$

Moreover, from (2.23) we have

$$\operatorname{div}(\nabla u + a\nabla v) = f_2, \quad \text{ in } \mathcal{D}'(\Omega).$$

Since $f_2$ is in $L^2(\Omega)$ then, so is $\operatorname{div}(\nabla u + a\nabla v)$. Using this in (2.23) besides the fact that $f_3$ and $z(\cdot, 1)$ belong to $L^2(\Gamma_1)$ we get $\frac{\partial u}{\partial \nu} + a\frac{\partial v}{\partial \nu} - \Delta_T u \in L^2(\Gamma_1)$. Consequently, we conclude the existence of a unique solution $(u, v, v|_{\Gamma_1}, z)^T \in \mathcal{D}(\mathcal{A})$ to (2.21). From (2.21), (2.22), and (2.23) we can deduce

$$\|(u, v, v|_{\Gamma_1}, z)\|_{\mathcal{H}} \leq C\|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}}.$$

So, $0 \in \rho(\mathcal{A})$. Therefore, by contraction principle, we obtain $R(\lambda I - \mathcal{A}) = \mathcal{H}$, for $\lambda > 0$ sufficiently small. Thus, applying the Lumer-Phillips Theorem, we conclude that the operator $\mathcal{A}$ generates a $C_0$ semigroup of contractions on $\mathcal{H}$. □

Let us introduce the energy of the system.

$$E(t) := E(u(t)) = \frac{1}{2}\int_\Omega \left\{u_t^2(x,t) + |\nabla u(x,t)|^2\right\} dx + \frac{1}{2}\int_{\Gamma_1} \left\{u_t^2(x,t) + |\nabla_T u(x,t)|^2\right\} d\Gamma$$
$$+\frac{\xi}{2\tau}\int_{t-\tau}^{t}\int_{\Gamma_1} |u_t(x,\rho)|^2 \, d\Gamma d\rho. \tag{2.24}$$

**Proposition 2.3.** *For any regular solution $U = (u, v, v|_{\Gamma_1}, z)$ of problem (2.1)-(2.5) the energy is decreasing and there exists a positive constant $C$ such that*

$$E'(t) \leq -C\left\{\int_\omega |\nabla u_t(x,t)|^2 \, dx + \int_{\Gamma_1} |u_t(x, t-\tau)|^2 \, d\Gamma\right\} - \int_{\Omega\setminus\omega} a(x) \, |\nabla u_t(x,t)|^2 \, dx$$

*Proof.* It suffices to notice that

$$2E(t) = \|U\|_{\mathcal{H}}^2,$$

hence

$$E'(t) = \Re \, \langle U', U\rangle_{\mathcal{H}} = \Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}}.$$

We then obtain the result owing to (2.19) and the assumption on $\xi$. □

## 2.3 Strong stability

As pointed out in the introduction, the strong stability is a necessary condition for the exponential one. In this section we are going to give a strong stability result for the problem (2.1)-(2.5) under the assumption (2.7). To this end, we shall use Arendt-Batty's criteria [19].

**Theorem 2.4.** *The semigroup of contractions $(S(t))_{t\geq 0}$ generated by $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is strongly stable on the energy space $\mathcal{H}$ in the sense that*

$$\lim_{t\to\infty} \|S(t)U_0\|_{\mathcal{H}} = 0, \quad \forall U_0 \in \mathcal{H}.$$

The criteria of Arendt and Batty [19] ensures that a $C_0$-semigroup of contractions in a reflexive Banach space is strongly stable, if $\sigma(\mathcal{A}) \cap i\mathbb{R}$ contains only a countable number of continuous spectrum of $\mathcal{A}$ and no eigenvalue. Since $0 \in \rho(\mathcal{A})$, we only need to check that (i) $\mathrm{Ker}(i\beta I - \mathcal{A}) = \{0\}$ and (ii) $\mathrm{R}(i\beta I - \mathcal{A}) = \mathcal{H}$, for all real numbers $\beta \neq 0$. Then, by the Hille-Yosida theorem the operator $(i\beta - \mathcal{A})$ is closed and using the closed graph theorem we deduce that its inverse $(i\beta - \mathcal{A})^{-1}$, which is defined everywhere is bounded. Consequently, we deduce that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$.

**Lemma 2.5.** *If (2.7) is satisfied, then, for all $\beta \in \mathbb{R}$, the operator $(i\beta I - \mathcal{A})$ is injective, i.e.,:*

$$\mathrm{Ker}(i\beta I - \mathcal{A}) = \{0\}.$$

*Proof.* Let $U = (u, v, v|_{\Gamma_1}, z)^T \in \mathcal{D}(\mathcal{A})$ be such that

$$\mathcal{A}U = i\beta U. \tag{2.25}$$

Recall (2.19), we have

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq \left(-a_0 + C_P\left(\frac{|k|}{2} + \frac{\xi}{2\tau}\right)\right)\int_\omega |\nabla v|^2 dx + \left(\frac{|k|}{2} - \frac{\xi}{2\tau}\right)\int_{\Gamma_1} |z(x,1)|^2 d\Gamma$$
$$- \int_{\Omega\backslash\omega} a(x)|\nabla v|^2 dx,$$

where $C_p$ is the constant in (2.6). Using (2.20) we infer

$$\int_\omega |\nabla v|^2 dx = 0, \quad \int_{\Gamma_1} |z(x,1)|^2 d\Gamma = 0, \quad \text{and} \quad \int_{\Omega\backslash\omega} a(x)|\nabla v|^2 dx = 0. \tag{2.26}$$

Therefore, from (2.26) we have that

$$a\nabla v = 0, \text{ in } \Omega, \tag{2.27}$$

and

$$\nabla v = 0, \text{ in } \omega.$$

Then for some real constant $c$ we have

$$v = c, \text{ in } \omega. \tag{2.28}$$

In addition, from (2.26), we derive

$$z(x,1) = 0 \text{ on } \Gamma_1. \tag{2.29}$$

Now, note that (2.25) can be rewritten as

$$\begin{cases} i\beta u - v = 0, & \text{in } \Omega, \\ i\beta v - \mathrm{div}(\nabla u + a\nabla v) = 0, & \text{in } \Omega, \\ i\beta v + \dfrac{\partial u}{\partial \nu} + a\dfrac{\partial v}{\partial \nu} - \Delta_T u + kz(\cdot,1) = 0, & \text{in } \Gamma_1, \\ i\beta z + \dfrac{1}{\tau} z_\rho = 0, & \text{in } \Gamma_1 \times (0,1). \end{cases}$$

Inserting (2.27) and (2.29) we have

$$\begin{cases} i\beta u - v = 0, & \text{in } \Omega, \\ \beta^2 u + \Delta u = 0, & \text{in } \Omega, \\ i\beta v + \dfrac{\partial u}{\partial \nu} + a\dfrac{\partial v}{\partial \nu} - \Delta_T u = 0, & \text{in } \Gamma_1, \\ i\beta z + \dfrac{1}{\tau} z_\rho = 0, & \text{in } \Gamma_1 \times (0,1). \end{cases} \tag{2.30}$$

Then, from the fourth equation in (2.30) and (2.29), we obtain

$$z(x,\rho) = 0 \text{ in } \Gamma_1 \times (0,1).$$

Thus

$$v(x) = z(x,0) = 0, \;\; x \in \Gamma_1. \tag{2.31}$$

Therefore, from (2.31) and (2.28), we deduce

$$v = 0, \;\; \text{in } \omega. \tag{2.32}$$

Now, using the first equation of (2.30), we also have

$$u = 0, \;\; \text{in } \omega.$$

Moreover, using the regularity of $u$ and the trace theorem result

$$u = 0, \;\; \text{on } \Gamma_1.$$

Finally, since u satisfies the second equation of (2.30), a unique continuation result (Theorem 1.14 for example) allows us to deduce

$$u = 0, \;\; \text{in } \Omega.$$

Therefore, $(u, v, v|_{\Gamma_1}, z)^T = (0,0,0,0)$. □

**Proposition 2.6.** *If (2.7) is satisfied, then for all $\beta \in \mathbb{R}$, we have that $(i\beta I - \mathcal{A})$ is surjective, i.e.*

$$R(i\beta I - \mathcal{A}) = \mathcal{H}.$$

*Proof.* Since we have already shown in Theorem 2.1 that $R(\mathcal{A}) = \mathcal{H}$, we only need to prove that $R\left(i\beta I - \mathcal{A}\right) = \mathcal{H}$ for all $\beta \in \mathbb{R}$, $\beta \neq 0$.

Given $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$, we need to find $U = (u, v, v|_{\Gamma_1}, z) \in D\left(\mathcal{A}\right)$ such that

$$(i\beta I - \mathcal{A})U = F, \tag{2.33}$$

that is,

$$\begin{cases} i\beta u - v = f_1, & \text{in } \Omega, \\ i\beta v - \operatorname{div}(\nabla u + a\nabla v) = f_2, & \text{in } \Omega, \\ i\beta v + \dfrac{\partial u}{\partial \nu} + a\dfrac{\partial v}{\partial \nu} - \Delta_T u + kz(,1) = f_3, & \text{in } \Gamma_1, \\ i\beta z + \dfrac{1}{\tau} z_\rho = f_4, & \text{in } \Gamma_1 \times (0,1). \end{cases} \tag{2.34}$$

From the fourth identity in (2.34), we have

$$z(x,\rho) = e^{-i\tau\beta\rho}\Big(v + \tau\int_0^\rho e^{i\tau\beta\sigma} f_4(x,\sigma)d\sigma\Big). \tag{2.35}$$

Then,

$$z(x,1) = e^{-i\tau\beta} v + \tau e^{-i\tau\beta}\int_0^1 e^{i\tau\beta\sigma} f_4(x,\sigma)d\sigma. \tag{2.36}$$

Multiplying the second equation in (2.34) by $\varphi \in V$, integrating and using Green's formula (2.18), then using the first and the third equations in (2.34) as well as (2.36) yields,

$$a_\beta(u,\varphi) = l_\beta(\varphi),$$

where

$$a_\beta(u,\varphi) = -\beta^2\int_\Omega u\overline{\varphi}dx + \int_\Omega (1 + i\beta a)\nabla u\nabla\overline{\varphi}dx - \beta^2\int_{\Gamma_1} u\overline{\varphi}d\Gamma + \int_{\Gamma_1}\nabla_T u\nabla_T\overline{\varphi}d\Gamma$$
$$+ i\beta k e^{-i\tau\beta}\int_\Omega u\overline{\varphi}dx,$$

and

$$l_\beta(\varphi) = \int_\Omega f_2\overline{\varphi} + i\beta\int_\Omega f_1\overline{\varphi} + \int_\Omega a\nabla f_1\nabla\overline{\varphi} + \int_{\Gamma_1} f_3\overline{\varphi} + i\beta\int_{\Gamma_1} f_1\overline{\varphi}$$
$$+ k e^{-i\tau\beta}\int_{\Gamma_1} f_1\overline{\varphi} - k\tau e^{-i\tau\beta}\int_{\Gamma_1}\int_0^1 e^{i\tau\beta\sigma} f_4(x,\sigma)\overline{\varphi}d\sigma,$$

which is a linear form over $V$.

Now, let's introduce the operator $A_\beta : V \to V'$ by

$$\langle A_\beta u, \tilde{u}\rangle_{V',V} = a_\beta(u,\tilde{u}), \quad \forall \tilde{u} \in V,$$

and define the operators $\mathbb{A}_1$ and $\mathbb{A}_2$ that decompose $A_\beta$ as follows

$$\begin{cases} \mathbb{A}_1 : V \to V' \\ u \to \mathbb{A}_1 u, \end{cases} \quad \begin{cases} \mathbb{A}_2 : V \to V' \\ u \to \mathbb{A}_2 u, \end{cases}$$

such that,

$$\begin{cases} (\mathbb{A}_1 u)(\varphi) & = a_1(u,\varphi), \quad \forall u,\varphi \in V, \\ (\mathbb{A}_2 u)(\varphi) & = a_2(u,\varphi), \quad \forall u,\varphi \in V, \end{cases}$$

where
$$\begin{cases} a_1(u,\varphi) = \left(i\beta k e^{-i\tau\beta} - \beta^2\right) \displaystyle\int_\Omega u\overline{\varphi}dx - \beta^2 \int_{\Gamma_1} u\overline{\varphi}d\Gamma, \\ a_2(u,\varphi) = \displaystyle\int_\Omega \left(1+i\beta a(x)\right)\nabla u\nabla\overline{\varphi}dx + \int_{\Gamma_1} \nabla_T u\nabla_T\overline{\varphi}d\Gamma. \end{cases} \tag{2.37}$$

Our aim here is to prove that $A_\beta = \mathbb{A}_1 + \mathbb{A}_2$ is an isomorphism, for which we are going to prove that $\mathbb{A}_1$ is a compact operator and that $\mathbb{A}_2$ is an isomorphism.

Let $u,\varphi \in V$. We have for suitable constant $C$,

$$\begin{aligned} |a_1(u,\varphi)| &\leq C\|u\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)} + C\|u\|_{L^2\Gamma_1}\|\varphi\|_{L^2(\Gamma_1)} \\ &\leq C\|u\|_V\|\varphi\|_{L^2(\Omega)} + C\|u\|_{L^2\Gamma_1}\|\varphi\|_{L^2(\Gamma_1)}. \end{aligned}$$

Further, for all $s \in (\frac{1}{2},1)$, we have

$$|a_1(u,\varphi)| \leq C\|u\|_V\|\varphi\|_{H^s(\Omega)}.$$

Since the canonical injection of $V$ into $H^1_{\Gamma_0}(\Omega)$ (respectively) $H^1_{\Gamma_0}(\Omega)$ into $H^s_{\Gamma_0}(\Omega)$ is continuous (respectively) compact, then the injection $(H^s(\Omega))' \hookrightarrow V'$ is compact. This proves the compactness of $\mathbb{A}_1$.

On the other hand, we can easily check that $a_2$ is a coercive sesquilinear form on $V$. So, by the Lax-Milgram theorem, the operator $\mathbb{A}_2$ is an isomorphism.

Now, our proof of $A_\beta$ being an isomorphism is reduced to proving that $\operatorname{Ker} A_\beta = \{0\}$.

Let $u \in \operatorname{Ker} A_\beta$, that is

$$a_\beta(u,\varphi) = 0, \forall\varphi \in V.$$

Then from (2.37), we find that

$$-\beta^2 u - \operatorname{div}(\{1+i\beta a\}\nabla u) = 0, \text{ in } \mathcal{D}'(\Omega),$$

$$-\beta^2 u + (1+i\beta a)\frac{\partial u}{\partial\nu} - \Delta_T u + i\beta k e^{-i\tau\beta}u = 0, \text{ on } \Gamma_1.$$

If we set $v = i\beta u$ and $z(x,\rho) = i\beta e^{-i\tau\beta\rho}u(x)$, then $(u,v,v|_{\Gamma_1},z)$ belongs to $\mathcal{D}(\mathcal{A})$ and satisfies

$$(i\beta - \mathcal{A})\begin{pmatrix} u \\ v \\ v|_{\Gamma_1} \\ z \end{pmatrix} = \begin{pmatrix} i\beta u - v \\ i\beta v - \Delta u - \operatorname{div}(a(x)\nabla v) \\ i\beta v + \dfrac{\partial u}{\partial\nu} + a(x)\dfrac{\partial v}{\partial\nu} - \Delta_T u \\ i\beta z + \dfrac{1}{\tau}z_\rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using Lemma 2.5, we easily deduce that $\operatorname{Ker}(i\beta - \mathcal{A}) = \{0\}$. Hence $u = v = v|_{\Gamma_1} = z = 0$ and $\operatorname{Ker} A_\beta = \{0\}$. The proof is now completed. $\square$

# Chapter 3

# Exponential Stability: Case of smooth damping coefficient

In this chapter, we are going to prove the exponential stability of the problem (2.1)-(2.5) when (2.7) is verified and the damping coefficient $a(\cdot)$ satisfies certain regularity assumptions described below.

To this end, we are going to use the frequency domain approach. Specifically, we shall follow Theorem 1.4 (Huang-Prüss [47, 82]).

Since the condition (H1) of Theorem 1.4 (Huang-Prüss [47, 82]) was already verified in the proof of Theorem 2.4, we need only to verify the condition (H2).

The result of this chapter and the next one will be built up on some estimates that are easy to get in $\omega$ but the tricky part is to extend them to the rest of $\Omega$ which needs some regularity assumptions on $a(\cdot)$. Hence, following Nicaise and Pignotti [78] on their inspiration from Liu and Rao [67], we assume the following conditions that we illustrate in Figure 3.1, **(H)** $meas\Gamma_1 > 0$, **(A1)** $\exists \delta > 0,\ a(x) \geq a_0 > 0,\ \forall x \in \mathcal{O}_\delta$, where

$$\mathcal{O}_\delta = \left\{ x \in \Omega,\ \ |x-y| \leq \delta,\ \forall y \in \Gamma_1 \right\},$$

**(A2)** $a \in C^{1,1}\left(\overline{\Omega}\right),\ \Delta a \in L^\infty(\Omega)$.

Also, we assume the following conditions.

There exists a function $q \in C^1(\Omega; \mathbb{R}^n)$ and constants $0 < \alpha < \beta < \delta$ such that **(D1)** $\partial_j q_k = \partial_k q_j$, $\operatorname{div} q \in C^1(\Omega_\beta)$ and $q \equiv 0$ on $\mathcal{O}_\alpha$, where $\Omega_\beta = \Omega \backslash \mathcal{O}_\beta$, **(D2)** there exists a constant $\sigma > 0$ such that

$$(\partial_j q_k)_{1 \leq k,j \leq n} \geq \sigma I, \text{ in } \Omega_\beta,$$

**(D3)** there exists a constant $C > 0$ such that for all $v \in V$ we have

$$|(q \cdot \nabla v) \nabla a - (q \cdot \nabla a) \nabla v| \leq C\sqrt{a}|\nabla v|, \text{ in } \Omega_\beta,$$

**(D4)** $q(x) \cdot \nu(x) \leq 0 \quad \forall x \in \Gamma_0$.

**Remark 3.1.** *If $\Omega$ is a disk in $\mathbb{R}^2$ and $\Gamma_0$ is a suitable connected arc from the boundary, the*

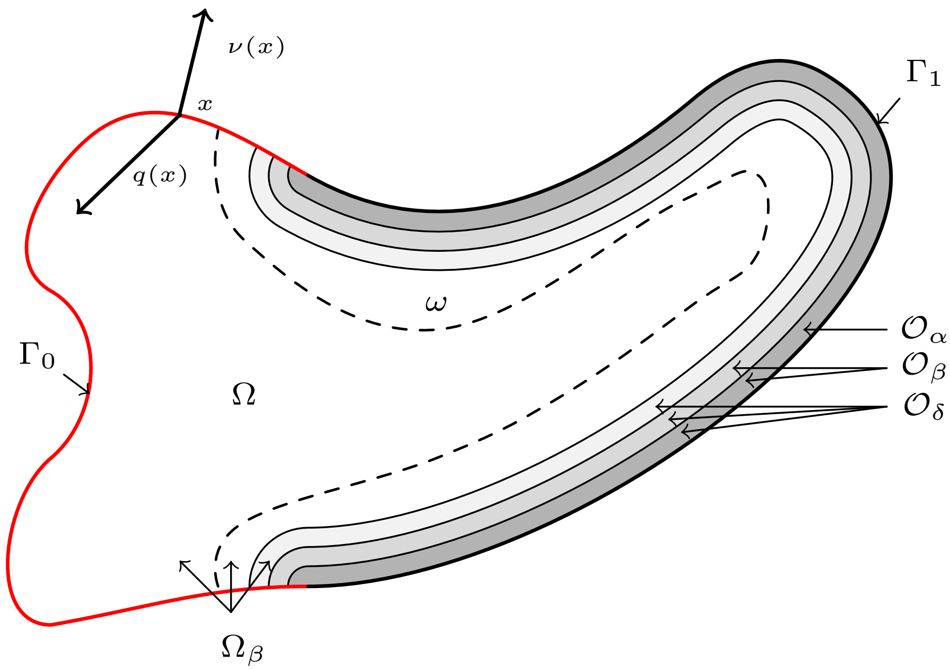


Figure 3.1: An example of a geometric situation satisfying the assumptions (**D1**)-(**D4**).

*conditions (**D1**)-(**D4**) hold with $q(x) = m(x)\varrho(x)$ such that $\varrho \in C^1(\overline{\Omega})$ defined by*

$$\varrho(x) = \begin{cases} 1 & \textit{if } x \in \Omega_\beta, \\ 0 & \textit{if } x \in \mathcal{O}_\alpha, \\ \varrho(x) \in [0,1] & \textit{elsewhere} , \end{cases}$$

*and $m(x) = x - x_0$, while $x_0 \in \mathbb{R}^2$ is chosen such that $m(x) \cdot \nu(x) \geq 0$ for every $x$ in $\Gamma_0$ (see Figure 3.2).*

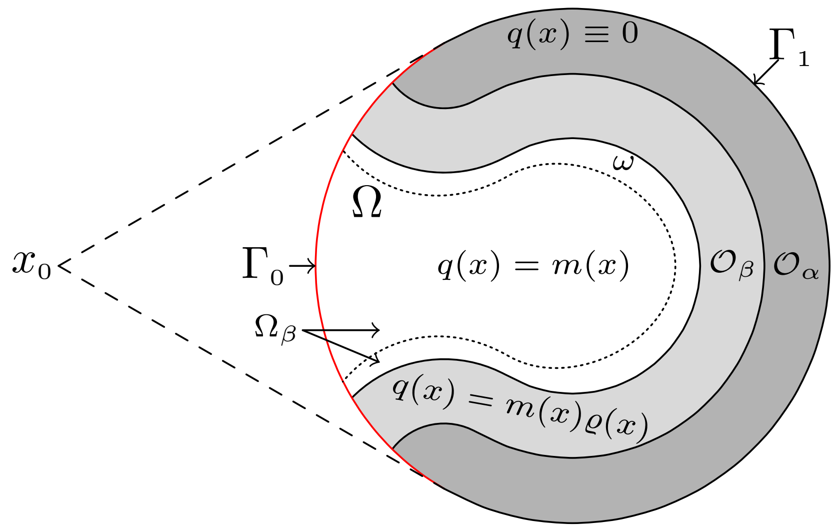


Figure 3.2: A geometric situation for which there exists a vector field $q$ for which the assumptions (**D1**)-(**D4**) are satisfied.

Now we give the main theorem of this chapter.

**Theorem 3.2.** *Suppose that the assumptions (**H**),(**A1**),(**A2**),(**D1**)-(**D4**), along with the inequality (2.7) are satisfied. Then there are two constants $M > 0$, $\alpha > 0$, such that for all initial data $U_0 \in \mathcal{H}$ the solution $U := (u, v, w, z)$ of the problem (2.1)-(2.5) satisfies the following uniform exponential decay estimate*

$$\|U(t)\|_{\mathcal{H}} \leq M e^{-\alpha t} \|U_0\|_{\mathcal{H}}. \tag{3.1}$$

The proof is divided into several parts and is contained in the next two sections. Moreover it is based on a perturbation argument similar to the one used in [78]. We are going to show that our system is exponentially stable by relying on the exponential stability of the same system with no time delay. Let $\mathcal{H}_0$ be the space,

$$\mathcal{H}_0 = V \times L^2(\Omega) \times L^2(\Gamma_1), \tag{3.2}$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ v|_{\Gamma_1} \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{v}|_{\Gamma_1} \end{pmatrix} \right\rangle_{\mathcal{H}_0} := \int_\Omega \left\{ \nabla u(x) \cdot \nabla \overline{\tilde{u}}(x) + v(x)\overline{\tilde{v}}(x) \right\} dx + \int_{\Gamma_1} \left\{ \nabla_T u(x) \cdot \nabla_T \overline{\tilde{u}}(x) + v(x)\overline{\tilde{v}}(x) \right\} d\Gamma,$$

and let $\mathcal{A}_0$ be the operator corresponding to $\tau = 0$ and $k = 1$, that is

$$\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \to \mathcal{H}_0 : (u, v, v|_{\Gamma_1})^T \to \left( v, \operatorname{div}(\nabla u + a\nabla v), -\frac{\partial u}{\partial \nu} - a(x)\frac{\partial v}{\partial \nu} + \Delta_T u - v \right)^T,$$

with

$$\mathcal{D}(\mathcal{A}_0) := \Big\{ U = (u, v, v|_{\Gamma_1})^T \in \mathcal{H}_0 : v \in V, \operatorname{div}(\nabla u + a(x)\nabla v) \in L^2(\Omega),$$
$$\frac{\partial u}{\partial \nu} + a(x)\frac{\partial v}{\partial \nu} - \Delta_T u \in L^2(\Gamma_1) \Big\}.$$

At this stage, we assume that $\mathcal{A}_0$ generates an exponentially stable semigroup. So, from Theorem 1.4 (Huang-Prüss [47, 82]) we have,

$$\begin{cases} (i) & i\mathbb{R} \subset \rho(\mathcal{A}_0) \\ (ii) & \| (i\xi - \mathcal{A}_0)^{-1} \|_{\mathcal{L}(\mathcal{H}_0)} \leq C, \quad \forall \xi \in \mathbb{R}. \end{cases} \tag{3.3}$$

The estimate $(ii)$ in (3.3), will be used to derive the estimate (3.6) which we use to prove the Proposition 4.2. Indeed, for every $F_0 \in \mathcal{H}_0$, the solution $(\tilde{u}, \tilde{v}, \tilde{v}|_{\Gamma_1})^T \in \mathcal{D}(\mathcal{A}_0)$ of

$$(i\xi I - \mathcal{A}_0) \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{v}|_{\Gamma_1} \end{pmatrix} = F_0,$$

satisfies

$$\left\| \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{v}|_{\Gamma_1} \end{pmatrix} \right\|_{\mathcal{H}_0} \leq C \|F_0\|_{\mathcal{H}_0}, \tag{3.4}$$

which is equivalent to

$$\|\tilde{u}\|_{H^1_{\Gamma_0}(\Omega)} + \|\tilde{v}\|_{L^2(\Omega)} + \|\tilde{v}\|_{L^2(\Gamma_1)} \leq C \|F_0\|_{\mathcal{H}_0}. \tag{3.5}$$

Then,

$$\int_{\Gamma_1} |\tilde{v}|^2 d\Gamma \leq C \|F_0\|^2_{\mathcal{H}_0}. \tag{3.6}$$

## 3.1 Stability of the problem with delay

In this section we are going to establish the proof of our main theorem (Theorem 4.1), using the inequality (3.6).

**Proposition 3.3.** *Under the assumptions (2.7),* ***(H)****,* ***(A1)****,* ***(A2)****, and* ***(D1)****-****(D4)****, the operator $\mathcal{A}$ satisfies*

$$\sup_{\beta \in \mathbb{R}} \| (i\beta I - \mathcal{A})^{-1} \|_{\mathcal{H}} < +\infty.$$

*Proof.* For $F \in \mathcal{H}$ and $\beta \in \mathbb{R}$, let $U \in \mathcal{D}(\mathcal{A})$ be a solution of

$$(i\beta I - \mathcal{A}) U = F = (f_1, f_2, f_3, f_4)^T,$$

that is

$$\begin{cases} i\beta u - v &= f_1, \quad \text{in } \Omega, \\ i\beta v - \Delta u - \operatorname{div}(a\nabla v) &= f_2, \quad \text{in } \Omega, \\ i\beta v + \dfrac{\partial u}{\partial \nu} + a\dfrac{\partial v}{\partial \nu} - \Delta_T u + kz(\cdot, 1) &= f_3, \quad \text{on } \Gamma_1, \\ i\beta z + \tau^{-1} z_\rho &= f_4, \quad \text{on } \Gamma_1. \end{cases} \tag{3.7}$$

The first identity of (3.7) gives

$$v = i\beta u - f_1.$$

Recall that, from the identity (2.36), we have

$$v(x) = e^{i\tau\beta} z(x, 1) - \tau \int_0^1 e^{i\tau\beta} f_4(x, \sigma)\, d\sigma,$$

and so,

$$\|v\|_{L^2(\Gamma_1)} \le \|z(., 1)\|_{L^2(\Gamma_1)} + C\|f_4\|_{L^2(\Gamma_1 \times (0,1))}. \tag{3.8}$$

Moreover, from (2.19), we have

$$C \int_{\Gamma_1} |z(x, 1)|^2 d\Gamma \le -\Re \langle \mathcal{A}U, U \rangle_{\mathcal{H}}.$$

Then,

$$C \int_{\Gamma_1} |z(x, 1)|^2 d\Gamma \le \Re \langle F, U \rangle_{\mathcal{H}} \le \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}. \tag{3.9}$$

From (3.9) and (3.8), we deduce

$$||v||^2_{L^2(\Gamma_1)} \le C \left( ||F||^2_{\mathcal{H}} + ||F||_{\mathcal{H}} ||U||_{\mathcal{H}} \right). \tag{3.10}$$

From (2.35), we also have

$$||z||^2_{L^2(\Gamma_1 \times (0,1))} \le C \left( \|v\|_{L^2(\Gamma_1)} + \|f_4\|_{L^2(\Gamma_1 \times (0,1))} \right). \tag{3.11}$$

By using (3.11) in (3.10), we obtain

$$\|v\|^2_{L^2(\Gamma_1)} \le C \Big\{ \|F\|^2_{\mathcal{H}} + \|F\|_{\mathcal{H}} \left( \|(u, v, v|_{\Gamma_1})\|_{\mathcal{H}_0} + \|v\|_{L^2(\Gamma_1)} + \|f_4\|_{L^2(\Gamma_1 \times (0,1))} \right) \Big\}.$$

Therefore,

$$\|v\|^2_{L^2(\Gamma_1)} \le C \left( \|F\|^2_{\mathcal{H}} + \|F\|_{\mathcal{H}} \|(u, v, v|_{\Gamma_1})\|_{\mathcal{H}_0} \right) + C\|F\|_{\mathcal{H}} \|v\|_{L^2(\Gamma_1)},$$

from which follows, by using Young's inequality,

$$\|v\|^2_{L^2(\Gamma_1)} \le C \left( \|F\|^2_{\mathcal{H}} + \|F\|_{\mathcal{H}} \|(u, v, v|_{\Gamma_1})\|_{\mathcal{H}_0} \right). \tag{3.12}$$

Estimates (3.9), (3.11), and (3.12) imply

$$\|z\,(.,1)\,\|^2_{L^2(\Gamma_1)} \leq C\left(\|F\|^2_{\mathcal{H}} + \|F\|_{\mathcal{H}}\|\,(u,v,v|_{\Gamma_1})\,\|_{\mathcal{H}_0}\right). \tag{3.13}$$

We have now to estimate $\|\,(u,v,v|_{\Gamma_1})\,\|_{\mathcal{H}_0}$. For this, let $(\tilde{u},\tilde{v},\tilde{v}|_{\Gamma_1}) \in \mathcal{D}\,(\mathcal{A}_0)$ be the solution of

$$(-i\beta - \mathcal{A}_0)\begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{v}|_{\Gamma_1} \end{pmatrix} = \begin{pmatrix} u \\ -v \\ -v|_{\Gamma_1} \end{pmatrix}, \tag{3.14}$$

which is equivalent to

$$\left\{\begin{array}{rcll} -i\beta\tilde{u} - \tilde{v} &=& u & \text{in } \Omega, \\ -i\beta\tilde{v} - \operatorname{div}\left(\nabla\tilde{u} + a\nabla\tilde{v}\right) &=& -v & \text{in } \Omega, \\ -i\beta\tilde{v} + \dfrac{\partial\tilde{u}}{\partial v} + a\dfrac{\partial\tilde{v}}{\partial v} - \Delta_T\tilde{u} &=& -v & \text{on } \Gamma_1. \end{array}\right. \tag{3.15}$$

From another part we have

$$\left\langle (i\beta - \mathcal{A})\,U, \begin{pmatrix} \tilde{u} \\ -\tilde{v} \\ -\tilde{v}|_{\Gamma_1} \\ 0 \end{pmatrix} \right\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} i\beta u - v \\ i\beta v - \operatorname{div}(\nabla u + a\nabla v) \\ i\beta v + \dfrac{\partial u}{\partial\nu} + a\dfrac{\partial v}{\partial\nu} - \Delta_T u + kz(\cdot,1) \\ i\beta z + \tau^{-1}z_\rho \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ -\tilde{v} \\ -\tilde{v}|_{\Gamma_1} \\ 0 \end{pmatrix} \right\rangle_{\mathcal{H}}$$

$$\begin{aligned} &= \int_\Omega \nabla(i\beta u - v)\cdot\nabla\overline{\tilde{u}}dx + \int_{\Gamma_1}\nabla_T(i\beta u - v)\cdot\nabla_T\overline{\tilde{u}}dx - \int_\Omega(i\beta v - \operatorname{div}(\nabla u + a\nabla v))\cdot\overline{\tilde{v}}dx \\ &\quad - \int_{\Gamma_1}\left(i\beta v + \frac{\partial u}{\partial\nu} + a\frac{\partial v}{\partial\nu} - \Delta_T u + kz(\cdot,1)\right)\overline{\tilde{v}}d\Gamma_1 \\ &= \int_\Omega \nabla u\nabla\overline{(-i\beta\tilde{u} - \tilde{v})}dx + \int_\Omega v\overline{(i\beta\tilde{v} + \operatorname{div}(\nabla\tilde{u} + a\nabla\tilde{v}))}dx + \int_{\Gamma_1}\nabla_T u\nabla_T\overline{(-i\beta\tilde{u} - \tilde{v})}d\Gamma \\ &\quad + \int_{\Gamma_1} v\overline{\left(i\beta\tilde{v} - \frac{\partial\tilde{u}}{\partial\nu} - a\frac{\partial\tilde{v}}{\partial\nu} + \Delta_T\tilde{u}\right)}d\Gamma - \int_\Gamma kz(\cdot,1)\overline{\tilde{v}}d\Gamma. \end{aligned}$$

Then, from (3.15), we have

$$\left\langle (i\beta - \mathcal{A})\,U, \begin{pmatrix} \tilde{u} \\ -\tilde{v} \\ -\tilde{v}|_{\Gamma_1} \\ 0 \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_\Omega\left(|\nabla u|^2 + |v|^2\right)dx + \int_{\Gamma_1}\left(|\nabla_T u|^2 + |v|^2\right)dx - \int_{\Gamma_1} kz\,(\cdot,1)\,\overline{\tilde{v}}d\Gamma,$$

and so

$$\|(u,v,v|_{\Gamma_1})\|_{\mathcal{H}_0} = \left\langle F, \begin{pmatrix} \tilde{u} \\ -\tilde{v} \\ -\tilde{v}|_{\Gamma_1} \\ 0 \end{pmatrix} \right\rangle_{\mathcal{H}} + \int_{\Gamma_1} kz\,(x,1)\,\overline{\tilde{v}}d\Gamma,$$

from which follows, by using Cauchy–Schwarz inequality,

$$\|\,(u,v,v|_{\Gamma_1})\,\|^2_{\mathcal{H}_0} \leq \|F\|_{\mathcal{H}}\|\,(\tilde{u},\tilde{v},\tilde{v}|_{\Gamma_1})\,\|_{\mathcal{H}_0} + |k|\|z\,(.,1)\,\|_{L^2(\Gamma_1)}\|\tilde{v}\|_{L^2(\Gamma_1)}. \tag{3.16}$$

Now, observe that from the estimates (3.4) and (3.6), we have

$$\|\,(\tilde{u},\tilde{v},\tilde{v}|_{\Gamma_1})\,\|_{\mathcal{H}_0} \leq C\|\,(u,v,v|_{\Gamma_1})\,\|_{\mathcal{H}_0}$$

and

$$\|\tilde{v}\|_{L^2(\Gamma_1)} \leq C\|\,(u,v,v|_{\Gamma_1})\,\|_{\mathcal{H}_0}.$$

Using these last two inequalities in (4.26), we obtain

$$\|\,(u,v,v|_{\Gamma_1})\,\|_{\mathcal{H}_0} \leq C\left(\|F\|_{\mathcal{H}} + \|z\,(\cdot,1)\,\|_{L^2(\Gamma_1)}\right).$$

By recalling (3.13), we have

$$\|\,(u,v,v|_{\Gamma_1})\,\|^2_{\mathcal{H}_0} \leq C\left(\|F\|^2_{\mathcal{H}} + \|F\|_{\mathcal{H}}\|\,(u,v,v|_{\Gamma_1})\,\|_{\mathcal{H}_0}\right),$$

and so, using Young's inequality, we obtain

$$\|\,(u,v,v|_{\Gamma_1})\,\|_{\mathcal{H}_0} \leq C\|F\|_{\mathcal{H}_0}. \tag{3.17}$$

Hence, we have

$$\|v\|_{L^2(\Gamma_1)} \leq C\|F\|_{\mathcal{H}},$$

and then, from (3.11),

$$\|z\|_{L^2(\Gamma_1\times(0,1))} \leq C\|F\|_{\mathcal{H}}.$$

This proves that the resolvent of $\mathcal{A}$ is uniformly bounded on the imaginary axis. □

At this point we have finished the proof of Theorem 4.1; However, the exponential stability of the problem without delay which was used as an assumption will be proved in the next section.

## 3.2 Stability of the problem without delay

In this section we are going to prove that the operator $\mathcal{A}_0$ that we introduced in the previous section generates an exponentially stable semigroup. For this matter we use Theorem 1.4 (Huang-Prüss [47, 82]).

**Lemma 3.4.** *For all $\beta \in \mathbb{R}$, one has*

$$\ker(i\beta I - \mathcal{A}_0) = \{0\}.$$

*Proof.* Let $U = (u, v, v|_{\Gamma_1})^T \in \mathcal{D}(\mathcal{A}_0)$ be such that

$$\mathcal{A}_0(u, v, v|_{\Gamma_1})^T = i\beta(u, v, v|_{\Gamma_1})^T. \tag{3.18}$$

Using Green's formula (2.18) and (3.18), we have

$$0 = \Re\langle \mathcal{A}_0 U, U\rangle_{\mathcal{H}} = -\int_\Omega a|\nabla v|^2 dx - \int_{\Gamma_1} |v|^2 d\Gamma. \tag{3.19}$$

This implies

$$a\nabla v = 0 \text{ in } \Omega \qquad \text{and} \qquad v = 0 \text{ on } \Gamma_1.$$

Arguing like in Lemma 2.5, we find

$$v = 0 \text{ on } \omega,$$

inserting all these on (2.25), we obtain

$$\begin{cases} i\beta u - v &= 0, \quad \text{in } \Omega, \\ \beta^2 u + \Delta u &= 0, \quad \text{in } \Omega, \\ u &= 0, \quad \text{in } \omega. \end{cases}$$

Then using a standard unique continuation argument, we deduce $(u, v, v|_{\Gamma_1}) = 0$. Hence,

$$\ker(i\beta I - \mathcal{A}_0) = \{0\}.$$

□

**Proposition 3.5.** *For all $\beta \in \mathbb{R}$, one has*

$$R(i\beta I - \mathcal{A}_0) = \mathcal{H}_0.$$

*Proof.* The proof follows the same steps as in Proposition 2.6. □

**Lemma 3.6.** *Under the assumptions (H),(**A1**),(**A2**), and (**D1**)-(**D2**), $\mathcal{A}_0$ satisfies*

$$\sup_{\beta\in\mathbb{R}} \|(i\beta I - \mathcal{A}_0)\|_{\mathcal{L}(\mathcal{H}_0)} < \infty. \tag{3.20}$$

*Proof.* We proceed by contradiction. Suppose that (3.20) doesn't hold. Then, by the uniform resonance theorem, there exists a sequence $(\beta_n)_{n\in\mathbb{N}} \in \mathbb{R}$ and a sequence $(u_n, v_n, v_n|_{\Gamma_1})_{n\in\mathbb{N}} \in \mathcal{D}(\mathcal{A}_0)$ such that

$$|\beta_n| \to +\infty, \tag{3.21}$$

$$\|u_n\|_V^2 + \|v_n\|_{L^2(\Omega)}^2 + \|v_n\|_{L^2(\Gamma_1)}^2 = 1, \tag{3.22}$$

$$i\beta_n (u_n, v_n, v_n|_{\Gamma_1}) - \mathcal{A}_0 (u_n, v_n, v_n|_{\Gamma_1}) := (f_n, g_n, h_n) \to 0 \text{ in } \mathcal{H}. \tag{3.23}$$

Then,

$$\begin{aligned} i\beta_n u_n - v_n &:= f_n \to 0 \text{ in } V, & (3.24)\\ i\beta_n v_n - \operatorname{div}(\nabla u_n + a\nabla v_n) &:= g_n \to 0 \text{ in } L^2(\Omega), & (3.25)\\ i\beta_n v_n + \frac{\partial u_n}{\partial \nu} + a(x)\frac{\partial v_n}{\partial \nu} - \Delta_T u_n + v_n &:= h_n \to 0 \text{ in } L^2(\Gamma_1). & (3.26)\end{aligned}$$

We look for a contradiction of the form

$$\|u_n\|_V^2 + \|v_n\|_{L^2(\Omega)}^2 + \|v_n\|_{L^2(\Gamma_1)}^2 = o(1). \tag{3.27}$$

The proof is divided into several steps. **Step 1.** Taking the inner product of (3.24) with $u_n$ in $V$, the product of (3.25) with $v_n$ in $L^2(\Omega)$, and the product of (3.26) with $v_n|_{\Gamma_1}$ in $L^2(\Gamma_1)$, we obtain

$$\begin{aligned} & i\beta_n \|u_n\|_V^2 - \langle u_n, v_n\rangle_V = o(1),\\ & i\beta_n \|v_n\|_{L^2(\Omega)}^2 - \int_\Omega \operatorname{div}(\nabla u_n + a\nabla v_n)\overline{v}_n dx = o(1),\\ & i\beta_n \|v_n\|_{L^2(\Gamma_1)}^2 - \int_{\Gamma_1} (\nabla u_n + a\nabla v_n)\overline{v}_n d\Gamma - \int_{\Gamma_1} \Delta_T u\overline{v}_n + \|v_n\|_{L^2(\Gamma_1)}^2 d\Gamma = o(1).\end{aligned}$$

Integrating by part, summing the above identities then, taking the real part , we deduce

$$\int_\Omega a|\nabla v_n|^2 dx + \int_{\Gamma_1} |v_n|^2 d\Gamma = o(1). \tag{3.28}$$

So,

$$\|v_n\|_{L^2(\Gamma_1)} = o(1)$$

and (3.27) becomes,

$$\|u_n\|_V^2 + \|v_n\|_{L^2(\Omega)}^2 = o(1).$$

**Step 2.** Now, taking the inner product of (3.24) with $v_n$ in $L^2(\Omega)$, (3.25) with $u_n$ in $L^2(\Omega)$ and (3.26) with $u_n|_{\Gamma_1}$ in $L^2(\Gamma_1)$, we have

$$i\beta_n \int_\Omega u_n\overline{v}_n dx - \|v_n\|_{L^2(\Omega)}^2 = o(1), \tag{3.29}$$

$$i\beta_n \int_\Omega v_n\overline{u}_n dx - \int_\Omega \operatorname{div}(\nabla u_n + a\nabla v_n)\overline{u}_n dx = o(1) \tag{3.30}$$

and

$$i\beta_n \int_{\Gamma_1} v_n\overline{u}_n d\Gamma + \int_{\Gamma_1} (\frac{\partial u_n}{\partial \nu} + a\frac{\partial v_n}{\partial \nu})\overline{u}_n d\Gamma - \int_{\Gamma_1} \Delta_T u_n\overline{u}_n d\Gamma + \int_{\Gamma_1} v_n\overline{u}_n d\Gamma = o(1). \tag{3.31}$$

Using Green's formula (2.18) in (3.30), then summing the result with (3.29) and (3.30), we find

$$i\beta_n \int_\Omega u_n \overline{v}_n dx + i\beta_n \int_\Omega v_n \overline{u}_n dx + \int_\Omega |\nabla u_n|^2 dx + \int_{\Gamma_1} |\nabla_T u_n|^2 d\Gamma - \|v_n\|^2_{L^2(\Omega)} + \int_\Omega a \nabla v_n \nabla \overline{u}_n dx + i\beta_n \int_{\Gamma_1} v_n \overline{u}_n d\Gamma + \int_{\Gamma_1} v_n \overline{u}_n dx = o(1). \tag{3.32}$$

From (3.22), (3.24), and the trace theorem, we have

$$\left| i\beta_n \int_{\Gamma_1} v_n \overline{u}_n d\Gamma \right| = \left| \int_{\Gamma_1} v_n (\overline{v}_n + \overline{f}_n) d\Gamma \right| \leq \|v_n\|_{L^2(\Gamma_1)} (\|v_n\|_{L^2(\Gamma_1)} + \|f_n\|_{L^2(\Gamma_1)}) = o(1). \tag{3.33}$$

Using (3.22) and (3.28), we obtain

$$\left| \int_\Omega a \nabla v_n \nabla \overline{u}_n dx \right| \leq \|a\|_\infty^{\frac{1}{2}} \|\nabla u_n\|_{L^2(\Omega)} \Big( \int_\Omega a(x) |\nabla v_n|^2 dx \Big)^{\frac{1}{2}} = o(1), \tag{3.34}$$

and

$$\left| \int_{\Gamma_1} v_n \overline{u}_n dx \right| \leq \|v_n\|_{L^2(\Gamma_1)} \|u_n\|_{L^2(\Gamma_1)} = o(1). \tag{3.35}$$

Now, taking the real part of (3.32) and using (3.33)-(3.35), we deduce

$$\int_\Omega |\nabla u_n|^2 dx + \int_{\Gamma_1} |\nabla_T u_n|^2 d\Gamma - \|v_n\|^2_{L^2(\Omega)} = o(1),$$

hence,

$$\|u_n\|_V \sim \|v_n\|_{L^2(\Omega)}. \tag{3.36}$$

So, in order to achieve the contradiction (3.27), we only need to show

$$\|u_n\|_V = o(1). \tag{3.37}$$

**Step 3.** From (3.28), using the assumption (**A1**) and Poincaré's inequality, we find

$$\int_{\mathcal{O}_\delta} |v_n|^2 \, dx = o(1). \tag{3.38}$$

Now, we introduce a cut-off function $\eta \in C^1(\overline{\Omega})$ such that

$$\eta(x) = \begin{cases} 1 & \text{if } x \in \mathcal{O}_{\delta-\epsilon}, \\ 0 & \text{if } x \in \Omega_\delta, \\ \eta(x) \in [0,1] & \text{elsewhere .} \end{cases}$$

Multiplying (3.25) by $\eta \overline{u}_n$, (3.26) by $\overline{u}_n|_{\Gamma_1}$, integrating by parts and summing, we obtain

$$\int_{\mathcal{O}_\delta} \eta \, |\nabla u_n|^2 \, dx + \int_{\mathcal{O}_\delta} \left( i\beta_n v_n \eta \overline{u}_n + \overline{u}_n \nabla u_n \cdot \nabla \eta + a \nabla v_n \cdot \nabla \left( \eta \overline{u}_n \right) \right) dx + \int_{\Gamma_1} |\nabla_T u_n|^2 \, d\Gamma + i\beta_n \int_{\Gamma_1} v_n \overline{u}_n d\Gamma + \int_{\Gamma_1} v_n \overline{u}_n d\Gamma = o(1).$$

Moreover, from (3.22) and (3.24), we see that $\|\beta_n u_n\|_{L^2(\Omega)} \leq \|v_n\|_{L^2(\Omega)} + \|f_n\|_V$ is uniformly bounded for all $n \geq 1$. Then from (3.28),(3.33), (3.35), and (3.38), we deduce

$$\int_{\mathcal{O}_\beta} |\nabla u_n|^2 \, dx + \int_{\Gamma_1} |\nabla_T u_n|^2 \, d\Gamma = o(1). \tag{3.39}$$

Now, we need only to show

$$\int_{\Omega_\beta} |\nabla u_n|^2 \, dx = o(1).$$

**Step 4.** Multiplying (3.25) by $i\beta_n a v_n$ and integrating by parts we have

$$\begin{aligned} \int_\Omega a |\beta_n v_n|^2 \, dx = &\int_\Omega i\beta_n (\nabla u_n + a\nabla v_n) \nabla (a\overline{v}_n) \, dx - \int_\Omega i\beta_n a\overline{v}_n g_n dx \\ &- i\beta_n \int_{\Gamma_1} \left( \frac{\partial u_n}{\partial \nu} + a\frac{\partial v_n}{\partial \nu} \right) a\overline{v}_n d\Gamma. \end{aligned} \tag{3.40}$$

Multiplying (3.26) by $i\beta_n a v_n|_{\Gamma_1}$ and integrating by parts, we have

$$\begin{aligned} \int_{\Gamma_1} a |\beta_n v_n|^2 \, dx = &\int_{\Gamma_1} i\beta_n \left( \frac{\partial u_n}{\partial \nu} + a\frac{\partial v_n}{\partial \nu} \right) (a\overline{v}_n) \, dx + i\beta_n \int_{\Gamma_1} \nabla_T u_n \nabla_T (a\overline{v}_n) d\Gamma \\ &- \int_{\Gamma_1} i\beta_n a\overline{v}_n h_n dx + i\beta_n \int_{\Gamma_1} a |v_n| \, d\Gamma. \end{aligned} \tag{3.41}$$

Adding (3.40) and (3.41), we get

$$\begin{aligned} \int_\Omega a |\beta_n v_n|^2 \, dx + \int_{\Gamma_1} a |\beta_n v_n|^2 \, dx = &\int_\Omega i\beta_n (\nabla u_n + a\nabla v_n) \nabla (a\overline{v}_n) \, dx \\ &+ i\beta_n \int_{\Gamma_1} \nabla_T u_n \nabla_T (a\overline{v}_n) d\Gamma - \int_\Omega i\beta_n a\overline{v}_n g_n dx - \int_{\Gamma_1} i\beta_n a\overline{v}_n h_n dx \\ &+ i\beta_n \int_{\Gamma_1} a |v_n| \, d\Gamma. \end{aligned} \tag{3.42}$$

Using (3.24) and (3.28), a straightforward computation gives

$$\begin{aligned} \Re \int_\Omega i\beta_n \nabla u_n \cdot \nabla (a\overline{v}_n) \, dx &= \Re \int_\Omega (\nabla v_n + \nabla f_n) \cdot (a\nabla \overline{v}_n + \overline{v}_n \nabla a) \, dx \\ &= \Re \int_\Omega \overline{v}_n \nabla v_n \cdot \nabla a dx + o(1) \\ &\leq C\|\Delta a\|_\infty \int_\Omega |v_n|^2 \, dx + o(1). \end{aligned} \tag{3.43}$$

Similarly, we have

$$\begin{aligned} \Re \int_\Omega i\beta_n a\nabla v_n \cdot \nabla (a\overline{v}_n) \, dx =& \Re \int_\Omega i\beta_n \overline{v}_n a\nabla v_n \cdot \nabla a dx \\ \leq& \frac{1}{3} \int_\Omega a |\beta_n v_n|^2 \, dx + \frac{3\|\nabla a\|_\infty^2}{4} \int_\Omega a |\nabla v_n|^2 \, dx \\ =& \frac{1}{3} \int_\Omega a |\beta_n v_n|^2 \, dx + o(1), \end{aligned} \tag{3.44}$$

$$
\begin{aligned}
\Re \int_\Omega i\beta_n a\overline{v}_n g_n dx \leq & \frac{1}{3}\int_\Omega a\left|\beta_n v_n\right|^2 dx + \frac{3}{4}\int_\Omega a\left|g_n\right|^2 dx \\
= & \frac{1}{3}\int_\Omega a\left|\beta_n v_n\right|^2 dx + o(1),
\end{aligned} \tag{3.45}
$$

$$
\begin{aligned}
\Re \int_{\Gamma_1} i\beta_n \nabla u_n \cdot \nabla\left(a\overline{v}_n\right) dx = & \Re \int_{\Gamma_1} \left(\nabla v_n + \nabla f_n\right)\cdot\left(a\nabla\overline{v}_n + \overline{v}_n\nabla a\right) dx \\
= & \, o(1),
\end{aligned} \tag{3.46}
$$

$$
\begin{aligned}
\Re \int_{\Gamma_1} i\beta_n a\overline{v}_n h_n dx \leq & \frac{1}{3}\int_\Omega a\left|\beta_n v_n\right|^2 dx + \frac{3}{4}\int_{\Gamma_1} a\left|h_n\right|^2 dx \\
= & \frac{1}{3}\int_{\Gamma_1} a\left|\beta_n v_n\right|^2 dx + o(1).
\end{aligned} \tag{3.47}
$$

Inserting (3.43)–(3.47) into (3.42), we get

$$
\int_\Omega a\left|\beta_n v_n\right|^2 dx + \int_{\Gamma_1} a\left|\beta_n v_n\right|^2 dx \leq C\|\Delta a\|_\infty \int_\Omega \left|v_n\right|^2 dx + o(1), \tag{3.48}
$$

which is bounded for all $n \geq 1$ because of the assumption **(A2)** and (3.22). In (3.39), we have obtained the estimation of the integral of $\nabla u_n$ on the subdomain $\mathcal{O}_\beta$. Likewise, we will establish a similar estimation on $\Omega_\beta$ which is required to achieve (3.37). This is the purpose of the following step. **Step 5.** Now, observe that

$$
\operatorname{div}\left(\nabla u_n + a\nabla v_n\right) \in L^2(\Omega)
$$

and

$$
\operatorname{div}\left(\nabla u_n + a\nabla v_n\right) = \operatorname{div}\left(\nabla\left(u_n + av_n\right) - \nabla a v_n\right) \in L^2(\Omega).
$$

Then,

$$
\begin{cases}
\Delta\left(u_n + av_n\right) \in L^2(\Omega), \\
u_n + av_n = 0 \quad \text{ on } \Gamma_0,
\end{cases}
$$

which implies

$$
u_n + av_n \in H^2(\Omega \backslash W),
$$

where $W$ is a neighborhood of $\Gamma_1$. Let

$$
M_n := \nabla u_n + a\nabla v_n = \nabla u_n + o(1). \tag{3.49}
$$

Since $q \equiv 0$ on $\mathcal{O}_\alpha$, $q\cdot M_n \in H^1(\Omega)$ . Now, observe that by (3.25), taking the inner product with $q \cdot M_n$, results in

$$
\Re \int_\Omega i\beta_n v_n q\cdot\overline{M}_n dx - \Re \int_\Omega \operatorname{div} M_n q\cdot\overline{M}_n dx = o(1). \tag{3.50}
$$

Further, using Green's formula, we have

$$
\int_\Omega \operatorname{div} M_n q\cdot\overline{M}_n dx = \int_{\partial\Omega} M_n\cdot\nu q\cdot\overline{M}_n d\Gamma - \int_\Omega M_n \nabla\left(q\cdot\overline{M}_n\right) dx. \tag{3.51}
$$

From (3.51), using (3.49) and the fact that $u = 0$ on $\Gamma_0$, we deduce

$$\begin{aligned}
-\int_\Omega \operatorname{div} M_n q \cdot \overline{M}_n dx &= -\int_{\Gamma_0} q\cdot\nu \left|M_n\cdot\nu\right|^2 d\Gamma + \int_\Omega \left(M_{n,j}\partial_j q_k \overline{M}_{n,k} + M_{n,j} q_k \partial_k \overline{M}_{n,k}\right) dx \\
&= -\int_{\Gamma_0} q\cdot\nu \left|M_n\cdot\nu\right|^2 d\Gamma + \int_\Omega \left(M_{n,j}\partial_j q_k \overline{M}_{n,k} + M_{n,j} q_k \partial_k \overline{M}_{n,j}\right) dx \\
&+ \int_\Omega M_{n,j} q_k \left(\partial_j \overline{M}_{n,k} - \partial_k \overline{M}_{n,j}\right) dx.
\end{aligned} \tag{3.52}$$

Let us denote

$$I = \int_\Omega M_{n,j} q_k \partial_k \overline{M}_{n,j} dx,$$

integrating by parts, we have

$$\begin{aligned}
I &= -\int_\Omega \partial_k \left(M_{n,j} q_k\right) \overline{M}_{n,j} dx + \int_{\Gamma_0} q\cdot\nu \left|M_n\right|^2 d\Gamma \\
&= -\overline{I} - \int_\Omega \operatorname{div} q \left|M_n\right|^2 dx + \int_{\Gamma_0} q\cdot\nu \left|M_n\right|^2 d\Gamma.
\end{aligned}$$

So,

$$\Re I = -\frac{1}{2}\int_\Omega \operatorname{div} q \left|M_n\right|^2 dx + \frac{1}{2}\int_{\Gamma_0} q\cdot\nu \left|M_n\right|^2 d\Gamma. \tag{3.53}$$

Inserting (3.53) in (3.52), we obtain

$$\begin{aligned}
-\Re\int_\Omega \operatorname{div} M_n q \cdot \overline{M}_n dx = &-\frac{1}{2}\int_{\Gamma_0} q\cdot\nu \left|M_n\cdot\nu\right|^2 d\Gamma \\
&+ \Re\int_\Omega \left(M_{n,j}\partial_j q_k \overline{M}_{n,k} - \frac{1}{2}\operatorname{div} q \left|M_n\right|^2\right) dx \\
&+ \Re\int_\Omega M_{n,j} q_k \left(\partial_j \overline{M}_{n,k} - \partial_k \overline{M}_{n,j}\right) dx.
\end{aligned} \tag{3.54}$$

Note that

$$\begin{aligned}
\partial_j M_{n,k} - \partial_k M_{n,j} &= \partial_j\left(\partial_k u_n + a\partial_k v_n\right) - \partial_k\left(\partial_j u_n + a\partial_j v_n\right) \\
&= \partial_j a \partial_k v_n - \partial_k a \partial_j v_n.
\end{aligned}$$

Then,

$$\begin{aligned}
\Re\int_\Omega M_{n,j} q_k \left(\partial_j \overline{M}_{n,k} - \partial_k \overline{M}_{n,j}\right) dx &= \Re\int_\Omega M_{n,j} q_k \left(\partial_j a \partial_k \overline{v}_n - \partial_k a \partial_j \overline{v}_n\right) dx \\
&= \Re\int_\Omega \left(M_n\cdot\nabla a q\cdot\nabla\overline{v} - M_n\cdot\nabla\overline{v} q\cdot\nabla a\right) dx.
\end{aligned} \tag{3.55}$$

Using (3.55) in (3.54), we have

$$\begin{aligned}
-\Re\int_\Omega \operatorname{div} M_n q\cdot \overline{M}_n dx &= -\frac{1}{2}\int_{\Gamma_0} q\cdot\nu\left|M_n\cdot\nu\right|^2 d\Gamma \\
&+\Re\int_\Omega\left(M_{n,j}\partial_j q_k\overline{M}_{n,k}-\frac{1}{2}\operatorname{div} q\left|M_n\right|^2\right)dx \\
&+\Re\int_\Omega\left(M_n\cdot\nabla a q\cdot\nabla\overline{v}_n - M_n\cdot\nabla\overline{v}_n q\cdot\nabla a\right)dx.
\end{aligned} \tag{3.56}$$

Now, recalling (3.48), we observe that

$$\begin{aligned}
\Re\int_\Omega i\beta_n v_n q\cdot\overline{M}_n dx &= \Re\int_\Omega i\beta_n v_n q\cdot\nabla\overline{u}_n dx + o(1) \\
&= -\Re\int_\Omega v_n q\cdot\nabla\left(\overline{v}_n+\overline{f}_n\right)dx + o(1) \\
&= -\Re\int_\Omega v_n q\cdot\nabla\overline{v}_n dx + o(1).
\end{aligned} \tag{3.57}$$

We have

$$\int_\Omega v_n q\cdot\nabla\overline{v}_n dx = -\int_\Omega\left(\left|v_n\right|^2\operatorname{div} q + q\cdot\nabla v_n\overline{v}_n\right)dx,$$

then,

$$\Re\int_\Omega v_n q\cdot\nabla\overline{v}_n dx = -\frac{1}{2}\int_\Omega\left|v_n\right|^2\operatorname{div} q dx. \tag{3.58}$$

From (3.57) and (3.58), we obtain

$$\Re\int_\Omega i\beta_n v_n q\cdot\overline{M}_n dx = \frac{1}{2}\int_\Omega\left|v_n\right|^2\operatorname{div} q dx + o(1). \tag{3.59}$$

Now, let $h\in C^{0,1}(\overline{\Omega})$. Multiplying (3.24) by $h\overline{v}_n$, (3.25) and (3.26) by $h\overline{u}_n$, we get

$$\int_\Omega i\beta_n h u_n\overline{v}_n dx - \int_\Omega h\left|v_n\right|^2 dx = o(1), \tag{3.60}$$

$$\begin{aligned}
\int_\Omega i\beta_n h v_n\overline{u}_n dx &+ \int_\Omega\left(\nabla u_n + a\nabla v_n\right)\cdot\nabla\left(h\overline{u}_n\right)dx \\
&- \int_{\Gamma_1}\left(\frac{\partial u_n}{\partial\nu}+a\frac{\partial v_n}{\partial\nu}\right)h\overline{u}_n d\Gamma = o(1),
\end{aligned} \tag{3.61}$$

$$\begin{aligned}
-\int_{\Gamma_1} ih\beta_n v_n\overline{u}_n d\Gamma &+ \int_{\Gamma_1}\left(\frac{\partial u_n}{\partial\nu}+a\frac{\partial v_n}{\partial\nu}\right)h\overline{u}_n d\Gamma \\
&- \int_{\Gamma_1}\Delta_T u_n h\overline{u}_n d\Gamma + \int_{\Gamma_1} h v_n\overline{u}_n d\Gamma = o(1).
\end{aligned} \tag{3.62}$$

Using (3.28), (3.35), and (3.39), we deduce from (3.61) and (3.62)

$$\int_\Omega i\beta_n h v_n\overline{u}_n dx + \int_\Omega\left(\nabla u_n + a\nabla v_n\right)\cdot\nabla\left(h\overline{u}_n\right)dx = o(1). \tag{3.63}$$

Recalling (3.22) and (3.28), we deduce from (3.63)

$$\int_{\Omega} i\beta_n h v_n \overline{u}_n dx + \int_{\Omega} \left(h\left|\nabla u_n\right|^2 + \nabla u_n \cdot \nabla h \overline{u}_n\right) dx = o(1). \tag{3.64}$$

Taking the real part of the sum of (3.60) and (3.64) we have

$$\int_{\Omega} h\left|\nabla u_n\right|^2 dx = \int_{\Omega} h\left|v_n\right|^2 dx - \int_{\Omega} \overline{u}_n \nabla u_n \cdot \nabla h dx + o(1). \tag{3.65}$$

Moreover, using (3.24),

$$\int_{\Omega} \overline{u}_n \nabla u_n \cdot \nabla h dx = -\frac{i}{\beta_n} \int_{\Omega} \left(\overline{v}_n + \overline{f}_n\right) \nabla u_n \cdot \nabla h dx = o(1). \tag{3.66}$$

From (3.65) and (3.66) we get

$$\int_{\Omega} h\left|v_n\right|^2 dx = \int_{\Omega} h\left|\nabla u_n\right|^2 dx + o(1). \tag{3.67}$$

Choosing $h = \operatorname{div} q$ in (3.67) and using it in (3.59), we deduce

$$\Re \int_{\Omega} i\beta_n v_n q \cdot \overline{M}_n dx = \int_{\Omega} \frac{1}{2} \operatorname{div} q\left|\nabla u_n\right|^2 dx + o(1). \tag{3.68}$$

By using the definition (3.49) of $M_n$ , we can estimate the second integral in the right–hand side of (3.56) as

$$\int_{\Omega}\left(M_{n,j}\partial_j q_k \overline{M}_{n,k} - \frac{1}{2} \operatorname{div} q\left|M_n\right|^2\right) dx = \int_{\Omega}\left(\partial_j u_n \partial_j q_k \partial_k \overline{u}_n - \frac{1}{2} \operatorname{div} q\left|\nabla u_n\right|^2\right) dx + o(1). \tag{3.69}$$

From (**D3**), (3.28), and the fact that $M_n$ is bounded in $L^2(\Omega)$, we can estimate the third integral in the right–hand side of (3.56) as

$$\int_{\Omega}\left|M_n \cdot \left\{(q \cdot \nabla \overline{v}_n)\nabla a - (q \cdot \nabla a)\nabla \overline{v}_n\right\}\right| dx \leq C \int_{\Omega}\left|M_n\right|^2 dx \int_{\Omega} a\left|\nabla v_n\right|^2 dx = o(1). \tag{3.70}$$

Coming back to (3.50), by substituting (3.56) (taking into account (3.69) and (3.70)) and (3.68), we deduce

$$\int_{\Omega} \partial_j q_k \partial_j u_n \partial_k \overline{u}_n dx - \frac{1}{2} \int_{\Gamma_0} q \cdot \nu\left|M_n \cdot \nu\right|^2 d\Gamma = o(1)$$

and so, recalling (**D2**) and (3.39), we have

$$C \int_{\Omega_\beta}\left|\nabla u_n\right|^2 dx - \frac{1}{2} \int_{\Gamma_0} q \cdot \nu\left|M_n \cdot \nu\right|^2 d\Gamma = o(1),$$

for some $C > 0$. By the assumption (**D4**) we then get

$$\int_{\Omega_\beta} |\nabla u_n|^2 \, dx = o(1). \tag{3.71}$$

The estimations (3.71) and (3.39) together with (3.36) and (3.28) show the targeted estimation (3.27), which leads to the desired contradiction. Hence, the proof is now completed. □

**Remark 3.7.** *When $\Omega$ is the crown domain between two circles $\Gamma_0$ and $\Gamma_1$ that constitute the two parts of boundary, the analysis made in this paper still holds except for the estimate (3.38) because of the lack of Poincaré's inequality since $\mathcal{O}_\alpha$ can't meet the part $\Gamma_0$. However, if we consider a set $\tilde{\mathcal{O}}_\alpha$ containing $\mathcal{O}_\alpha$ and such that* $\text{meas}(\overline{\tilde{\mathcal{O}}_\alpha} \cap \Gamma_0) > 0$, *we can use the Poincaré's inequality and (3.38) still holds (see Figure 3.3).*

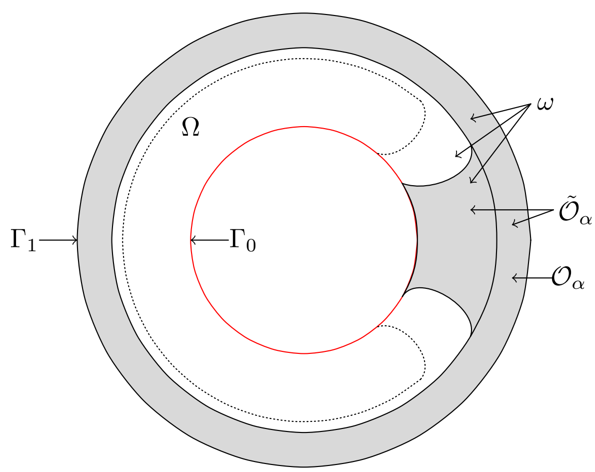


Figure 3.3: An example where $\Gamma_0$ and $\Gamma_1$ are far away from each other and $\mathcal{O}_\alpha$ doesn't meet $\Gamma_0$.

# Chapter 4

# Polynomial Stability: Case of discontinuous damping coefficient

In this chapter we are interested in the stability of the discontinuous damping version of the system (2.1)-(2.5), namely, when $a(x) = a1_\omega$. We are going to prove the polynomial stability of the associated semigroup with a decay rate of type $t^{-1/2}$, for which we will use the perturbation argument from the previous chapter adapted to the discontinuous case and a cascade technique that allows us to merge different stability results for different systems. Particularly, this will be based on a previously established result (Cavalcanti, Khemmoudj & Medjden [28]), hence, we adopt the same geometrical situation, that is when $\partial\Omega = \Gamma = \Gamma_0 \cup \Gamma_1$, such that, $\Gamma_0$ and $\Gamma_1$ are closed and disjoint. Besides, $\omega$ is a neighborhood of $\Gamma_1$, where

$$\Gamma_1 = \{x \in \Gamma, m(x) \cdot \nu > 0\},$$

such that $m(x) = x - x_0$ and $x_0$ is an arbitrary point from $\mathbb{R}^n$. Moreover, we suppose the following two assumptions (see Figure 4.1).

**(H)** $meas\Gamma_1 > 0$, **(A)** $\exists \delta > 0$, $\mathcal{O}_\delta \subset \omega$, where

$$\mathcal{O}_\delta = \{x \in \Omega,\ \ |x - y| \leq \delta,\ \forall y \in \Gamma_1\}.$$

After releasing the regularity assumptions on $a(\cdot)$, our system is still stable, however, with a slower decay rate this time. This is the subject of the following theorem.

**Theorem 4.1.** *Suppose that the assumptions **(H)**,**(A)** and the inequality (2.7) are satisfied, then there is a constant $C > 0$, such that for all initial data $U_0 \in \mathcal{D}(\mathcal{A})$ the solution $U := (u, u_t, u_t|_{\Gamma_1}, z)$ of the problem (2.1)-(2.5) satisfies the following polynomial decay estimate*

$$\|U(t)\|_{\mathcal{H}} \leq \frac{C}{\sqrt{t}} \|U_0\|_{\mathcal{D}(\mathcal{A})}. \tag{4.1}$$

The proof follows the same general lines as the smooth damping coefficient case.

Let's introduce our undelayed system, given the space,

$$\mathcal{H}_0 = V \times L^2(\Omega) \times L^2(\Gamma_1), \tag{4.2}$$

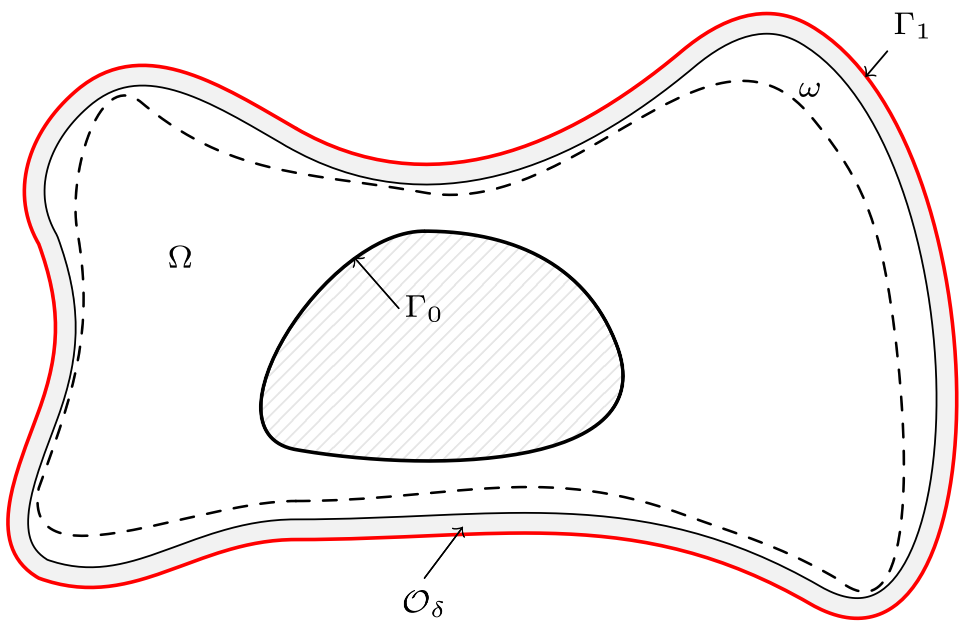


Figure 4.1: An example of a geometric situation satisfying the assumptions.

equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \right\rangle_{\mathcal{H}_0} := \int_\Omega \left\{ \nabla u(x) \cdot \nabla \overline{\tilde{u}}(x) + v(x) \overline{\tilde{v}}(x) \right\} dx + \int_{\Gamma_1} \left\{ \nabla_T u(x) \cdot \nabla_T \overline{\tilde{u}}(x) + w(x) \overline{\tilde{w}}(x) \right\} d\Gamma,$$

where $V$ is defined in the second chapter. In this space $\mathcal{H}_0$ let $\mathcal{A}_0$ be the operator corresponding to $\tau = 0$ and $k = 0$, that is

$$\mathcal{A}_0 U = \begin{pmatrix} v \\ \operatorname{div}(\nabla u + a(x) \nabla v) \\ -\dfrac{\partial u}{\partial \nu} - a(x) \dfrac{\partial v}{\partial \nu} + \Delta_T u \end{pmatrix}, \forall U \in \mathcal{D}(\mathcal{A}_0),$$

with

$$\mathcal{D}(\mathcal{A}_0) := \Big\{ U = (u, v, w)^T \in \mathcal{H}_0 : v \in V, \operatorname{div}(\nabla u + a(x) \nabla v) \in L^2(\Omega),$$
$$\frac{\partial u}{\partial \nu} + a(x) \frac{\partial v}{\partial \nu} - \Delta_T u \in L^2(\Gamma_1), w = v|_{\Gamma_1} \Big\}.$$

After some computation we find

$$\Re \langle \mathcal{A}_0 U, U \rangle_{\mathcal{H}} = - \int_\Omega a(x) |\nabla v|^2 dx. \tag{4.3}$$

Now we suppose that $i\mathbb{R} \subset \rho(\mathcal{A}_0)$ and that the operator $\mathcal{A}_0$ generates a polynomially stable semigroup with a decay rate of type $\dfrac{1}{\sqrt{t}}$. Then from Theorem 1.5 (Borichev-Tamilov [22]) we have,

$$\| \left(i\xi - \mathcal{A}_0\right)^{-1} \|_{\mathcal{L}(\mathcal{H}_0)} \leq C\xi^2, \quad \forall \xi \in \mathbb{R}, \tag{4.4}$$

for some positive constant $C$.

The estimate (4.4) will be used to derive the estimate (4.8) which we use to prove the Proposition (4.2).

Indeed, from (4.4), for every $F_0 \in \mathcal{H}_0$, the solution $(\tilde{u}, \tilde{v}, \tilde{w})^T \in \mathcal{D}\left(\mathcal{A}_0\right)$ of

$$(i\xi I - \mathcal{A}_0) \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} = F_0, \tag{4.5}$$

satisfies

$$\left\| \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \right\|_{\mathcal{H}_0} \leq C|\xi|^2 ||F_0||_{\mathcal{H}_0}. \tag{4.6}$$

Moreover,

$$\Re \left\langle \mathcal{A}_0 \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \right\rangle_{\mathcal{H}_0} = -\int_\Omega a|\nabla \tilde{v}|^2 dx.$$

Then,

$$\begin{aligned} \int_\Omega a|\nabla \tilde{v}|^2 dx &= \Re \left\langle (i\xi - \mathcal{A}_0) \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \right\rangle_{\mathcal{H}_0} \\ &= \Re \left\langle F_0, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \right\rangle_{\mathcal{H}_0} \leq \|F_0\|_{\mathcal{H}_0} \left\| \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} \right\|_{\mathcal{H}_0} \leq C|\xi|^2 \|F_0\|^2_{\mathcal{H}_0}, \end{aligned} \tag{4.7}$$

Using the trace theorem, we obtain

$$\int_{\Gamma_1} |\tilde{v}|^2 d\Gamma \leq C|\xi|^2 \|F_0\|^2_{\mathcal{H}_0}. \tag{4.8}$$

## 4.1 Stability of the problem with delay

Now we prove the main result using the inequality (4.8).

**Proposition 4.2.** *Under the assumptions (**H**),(**A**), the inequality (2.7) and* $|\beta| \geq 1$, *the operator* $\mathcal{A}$ *satisfies*

$$\sup_{\beta\in\mathbb{R}} \frac{1}{\beta^2} \| \left(i\beta I - \mathcal{A}\right)^{-1} \|_{\mathcal{H}} < \infty.$$

*Proof.* For $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$ and $\beta \in \mathbb{R}$, let $U \in \mathcal{D}(\mathcal{A})$ be a solution of

$$(i\beta I - \mathcal{A})\, U = F, \tag{4.9}$$

that is

$$\begin{cases} i\beta u - v &= f_1, \quad \text{in } \Omega, \\ i\beta v - \Delta u - \operatorname{div}(a\nabla v) &= f_2, \quad \text{in } \Omega, \\ i\beta w + \dfrac{\partial u}{\partial \nu} + a(x)\dfrac{\partial v}{\partial \nu} - \Delta_T u + kz(\cdot, 1) &= f_3, \quad \text{on } \Gamma_1, \\ i\beta z + \tau^{-1} z_\rho &= f_4, \quad \text{on } \Gamma_1. \end{cases} \tag{4.10}$$

The first identity of (4.10) gives

$$v = i\beta u - f_1. \tag{4.11}$$

Recall that from the identity (2.36) we have (on $\Gamma_1$)

$$v\,(x) = e^{i\tau\beta} z\,(x, 1) - \tau \int_0^1 e^{i\tau\beta} f_4\,(x, \sigma)\, d\sigma, \tag{4.12}$$

and so,

$$\|v\|_{L^2(\Gamma_1)} \leq \|z\,(., 1)\,\|_{L^2(\Gamma_1)} + C\|f_4\|_{L^2(\Gamma_1\times(0,1))}. \tag{4.13}$$

Moreover, from Proposition 2.3, we have

$$C \int_{\Gamma_1} |z\,(x, 1)\,|^2 d\Gamma \leq -\Re \left\langle \mathcal{A}U, U\right\rangle_{\mathcal{H}}. \tag{4.14}$$

Then,

$$C \int_{\Gamma_1} |z\,(x, 1)\,|^2 d\Gamma \leq \Re \left\langle F, U\right\rangle_{\mathcal{H}} \leq \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{4.15}$$

From (4.15) and (4.13) we deduce

$$||v||^2_{L^2(\Gamma_1)} \leq C\left(||F||^2_{\mathcal{H}} + ||F||_{\mathcal{H}}||U||_{\mathcal{H}}\right). \tag{4.16}$$

From (2.35) also we have

$$||z||_{L^2(\Gamma_1\times(0,1))} \leq C\left(\|v\|_{L^2(\Gamma_1)} + \|f_4\|_{L^2(\Gamma_1\times(0,1))}\right). \tag{4.17}$$

By using (4.17) in (4.16) we obtain

$$\|v\|_{L^2(\Gamma_1)}^2 \le C\Big\{\|F\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}} \left(\| (u, v, v|_{\Gamma_1}) \|_{\mathcal{H}_0} + \|v\|_{L^2(\Gamma_1)} + \|f_4\|_{L^2(\Gamma_1\times(0,1))}\right)\Big\}, \tag{4.18}$$

and therefore

$$\|v\|_{L^2(\Gamma_1)}^2 \le C\left(\|F\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}}\| (u, v, v|_{\Gamma_1}) \|_{\mathcal{H}_0}\right) + C\|F\|_{\mathcal{H}}\|w\|_{L^2(\Gamma_1)}, \tag{4.19}$$

from which follows, by using Young's inequality,

$$\|v\|_{L^2(\Gamma_1)}^2 \le C\left(\|F\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}}\| (u, v, v|_{\Gamma_1}) \|_{\mathcal{H}_0}\right). \tag{4.20}$$

Estimates (4.15), (4.17) and (4.20) imply

$$\|z(., 1)\|_{L^2(\Gamma_1)}^2 \le C\left(\|F\|_{\mathcal{H}}^2 + \|F\|_{\mathcal{H}}\| (u, v, v|_{\Gamma_1}) \|_{\mathcal{H}_0}\right). \tag{4.21}$$

We have now to estimate $\| (u, v, v|_{\Gamma_1}) \|_{\mathcal{H}_0}$. For this, let $(\tilde{u}, \tilde{v}, \tilde{v}|_{\Gamma_1}) \in \mathcal{D}(\mathcal{A}_0)$ be the solution of

$$(-i\beta - \mathcal{A}_0)\begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{v}|_{\Gamma_1}\end{pmatrix} = \begin{pmatrix} u \\ -v \\ -v|_{\Gamma_1}\end{pmatrix}, \tag{4.22}$$

which is equivalent to

$$\begin{cases} -i\beta\tilde{u} - \tilde{v} &= u \quad \text{in } \Omega, \\ -i\beta\tilde{v} - \operatorname{div}(\nabla\tilde{u} + a\nabla\tilde{v}) &= -v \quad \text{in } \Omega, \\ -i\beta\tilde{v} + \dfrac{\partial\tilde{u}}{\partial v} + a\dfrac{\partial\tilde{v}}{\partial v} - \Delta_T\tilde{u} &= -v \quad \text{on } \Gamma_1. \end{cases} \tag{4.23}$$

From another part we have

$$\left\langle (i\beta - \mathcal{A})U, \begin{pmatrix} \tilde{u} \\ -\tilde{v} \\ -\tilde{v}|_{\Gamma_1} \\ 0 \end{pmatrix}\right\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} i\beta u - v \\ i\beta v - \operatorname{div}(\nabla u + a\nabla v) \\ i\beta v + \dfrac{\partial u}{\partial\nu} + a(x)\dfrac{\partial v}{\partial\nu} - \Delta_T u + kz(\cdot,1) \\ i\beta z + \tau^{-1}z_\rho \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ -\tilde{v} \\ -\tilde{v}|_{\Gamma_1} \\ 0 \end{pmatrix}\right\rangle_{\mathcal{H}}$$

$$= \int_\Omega \nabla(i\beta u - v)\cdot\nabla\overline{\tilde{u}}dx + \int_{\Gamma_1}\nabla_T(i\beta u - v)\cdot\nabla_T\overline{\tilde{u}}d\Gamma$$

$$- \int_\Omega (i\beta v - \operatorname{div}(\nabla u + a\nabla v))\cdot\overline{\tilde{v}}dx - \int_{\Gamma_1}\left(i\beta v + \frac{\partial u}{\partial\nu} + a(x)\frac{\partial v}{\partial\nu} - \Delta_T u + kz(\cdot,1)\right)\overline{\tilde{v}}d\Gamma_1$$

$$= \int_\Omega \nabla u\nabla\overline{(-i\beta\tilde{u} - \tilde{v})}dx + \int_{\Gamma_1}\nabla_T u\nabla_T\overline{(-i\beta\tilde{u} - \tilde{v})}d\Gamma + \int_\Omega v\overline{(i\beta\tilde{v} + \operatorname{div}(\nabla\tilde{u} + a\nabla\tilde{v}))}dx$$

$$+ \int_{\Gamma_1} v\overline{\left(i\beta\tilde{v} - \frac{\partial\tilde{u}}{\partial\nu} - a\frac{\partial\tilde{v}}{\partial\nu} + \Delta_T\tilde{u}\right)}d\Gamma - \int_\Gamma kz(\cdot,1)\overline{\tilde{v}}d\Gamma.$$

Then, from (4.23), we have

$$\left\langle (i\beta - \mathcal{A})\, U, \begin{pmatrix} \tilde{u} \\ -\tilde{v} \\ -\tilde{w} \\ 0 \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_{\Omega} \left(|\nabla u|^2 + |v|^2\right) dx + \int_{\Gamma_1} \left(|\nabla_T u|^2 + |v|^2\right) dx - \int_{\Gamma_1} kz(x,1)\overline{\tilde{v}} d\Gamma, \tag{4.24}$$

and so

$$\|(u,v,v)\|^2_{\mathcal{H}_0} = \left\langle F, \begin{pmatrix} \tilde{u} \\ -\tilde{v} \\ -\tilde{w} \\ 0 \end{pmatrix} \right\rangle_{\mathcal{H}} + \int_{\Gamma_1} kz\left(\cdot,1\right) \overline{\tilde{v}} d\Gamma \tag{4.25}$$

from which follows, by using Cauchy–Schwarz inequality,

$$\|\left(u,v,v\right)\|^2_{\mathcal{H}_0} \leq \|F\|_{\mathcal{H}} \|\left(\tilde{u},\tilde{v},\tilde{v}\right)\|_{\mathcal{H}_0} + |k| \|z\left(.,1\right)\|_{L^2(\Gamma_1)} \|\tilde{w}\|_{L^2(\Gamma_1)}. \tag{4.26}$$

Now, observe that from the estimates (4.6) and (4.8) we have

$$\|\left(\tilde{u},\tilde{v},\tilde{w}\right)\|_{\mathcal{H}_0} \leq C|\beta|^2 \|\left(u,v,w\right)\|_{\mathcal{H}_0}, \tag{4.27}$$

and

$$\|\tilde{w}\|_{L^2(\Gamma_1)} \leq C|\beta| \|\left(u,v,w\right)\|_{\mathcal{H}_0}. \tag{4.28}$$

Using these last inequalities and (4.21) in (4.26), we obtain

$$\|\left(u,v,w\right)\|^2_{\mathcal{H}_0} \leq C|\beta|^2 \|F\|_{\mathcal{H}} \|\left(u,v,w\right)\|_{\mathcal{H}_0} + C|\beta| \|F\|^{\frac{1}{2}}_{\mathcal{H}} \|\left(u,v,w\right)\|^{\frac{3}{2}}_{\mathcal{H}_0}. \tag{4.29}$$

Then by Young's inequality, we have

$$\|\left(u,v,w\right)\|_{\mathcal{H}_0} \leq C|\beta|^2 \|F\|_{\mathcal{H}_0}, \tag{4.30}$$

and from (4.17), we get

$$\|z\|_{L^2(\Gamma_1\times(0,1))} \leq C|\beta|^2 \|F\|_{\mathcal{H}}. \tag{4.31}$$

This proves the estimate on the resolvent of $\mathcal{A}$. □

Now, we go back and show the polynomial stability of the problem without delay, namely, the problem related to the operator $\mathcal{A}_0$.

## 4.2 Stability of the problem without delay

**Lemma 4.3.** *For all $\beta \in \mathbb{R}$, one has*

$$\operatorname{Ker}(i\beta I - \mathcal{A}_0) = \{0\}.$$

*Proof.* Let $U = (u, v, w)^T \in \mathcal{D}(\mathcal{A}_0)$ be such that

$$\mathcal{A}_0(u, v, w)^T = i\beta(u, v, w)^T. \tag{4.32}$$

The inequality

$$\Re\langle \mathcal{A}_0 U, U\rangle_{\mathcal{H}} \leq -\int_\Omega a(x)|\nabla v|^2 dx, \tag{4.33}$$

implies that

$$a(x)\nabla v = 0 \text{ in } \Omega \qquad \text{and} \qquad v = c \text{ in } \omega, \tag{4.34}$$

By an argument similar to the one in Lemma 2.5 we find

$$v = 0 \text{ on } \omega. \tag{4.35}$$

Inserting all these in (4.32) we obtain

$$\begin{cases} \beta^2 u + \Delta u &= 0, \quad \text{in } \Omega, \\ u &= 0, \quad \text{in } \omega. \end{cases} \tag{4.36}$$

Then using a standard unique continuation argument we deduce $(u, v, w) = 0$. Hence $\operatorname{Ker}(i\beta I - \mathcal{A}_0) = \{0\}$. □

**Proposition 4.4.** *For all $\beta \in \mathbb{R}$, one has*

$$R(i\beta I - \mathcal{A}_0) = \mathcal{H}_0. \tag{4.37}$$

*Proof.* Similar to the delayed case. □

**Lemma 4.5.** *Under the assumptions (**H**) and (**A**), $\mathcal{A}_0$ satisfies*

$$\sup_{\beta\in\mathbb{R}} \frac{1}{\beta^2}\|(i\beta I - \mathcal{A}_0)^{-1}\|_{\mathcal{L}(\mathcal{H}_0)} < \infty.$$

*Proof.* We will prove this lemma by contradiction. Suppose there exist sequences $\{\beta_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+^*$ and $\{U_n := (u_n, v_n, w_n)\}_{n\in\mathbb{N}}$ in $\mathcal{D}(\mathcal{A}_0)$ such that

$$|\beta_n| \to +\infty, \tag{4.38}$$

$$\|u_n\|_V^2 + \|v_n\|_{L^2(\Omega)}^2 + \|w_n\|_{L^2(\Gamma_1)}^2 = 1, \tag{4.39}$$

$$\beta_n^2\{i\beta_n (u_n, v_n, w_n) - \mathcal{A}_0 (u_n, v_n, w_n)\} := (f_n, g_n, h_n) \to 0 \text{ in } \mathcal{H}. \tag{4.40}$$

Then,

$$i\beta_n u_n - v_n := \frac{f_n}{\beta_n^2} \to 0 \text{ in } V, \tag{4.41}$$

$$i\beta_n v_n - \operatorname{div}(\nabla u_n + a\nabla v_n) := \frac{g_n}{\beta_n^2} \to 0 \text{ in } L^2(\Omega), \tag{4.42}$$

$$i\beta_n w_n + \frac{\partial u_n}{\partial \nu} + a(x)\frac{\partial v_n}{\partial \nu} - \Delta_T u_n = \frac{h_n}{\beta_n^2} \to 0 \text{ in } L^2(\Gamma_1). \tag{4.43}$$

Taking the inner product of (4.41) with $u_n$ in $V$, the product of (4.42) with $v_n$ in $L^2(\Omega)$,and the product of (4.43) with $w_n$ in $L^2(\Gamma_1)$, we obtain

$$\begin{aligned}
&i\beta_n \|u_n\|_V^2 - \langle v_n, u_n\rangle_V = o(\frac{1}{\beta_n^2}),\\
&i\beta_n \|v_n\|_{L^2(\Omega)}^2 - \int_\Omega \operatorname{div}(\nabla u_n + a\nabla v_n)\overline{v}_n dx = o(\frac{1}{\beta_n^2}),\\
&i\beta_n \|w_n\|_{L^2(\Gamma_1)}^2 + \int_{\Gamma_1} (\partial_\nu u_n + a\partial_\nu v_n)\overline{w}_n d\Gamma - \int_{\Gamma_1} \Delta_T u\overline{w}_n = o(\frac{1}{\beta_n^2}).
\end{aligned} \tag{4.44}$$

Integrating by part, summing the above identities, and taking to consideration that $w_n = v_n|_{\Gamma_1}$, then, taking the real part , we deduce

$$\int_\Omega a|\nabla v_n|^2 dx = o(\frac{1}{\beta_n^2}). \tag{4.45}$$

Now, taking the inner product of (4.41) with $v_n$ in $L^2(\Omega)$, (4.42) with $u_n$ in $L^2(\Omega)$ and (4.43) with $u_n$ in $L^2(\Gamma_1)$, we have

$$i\beta_n \langle u_n, v_n\rangle_{L^2(\Omega)} - \|v_n\|_{L^2(\Omega)}^2 = o(\frac{1}{\beta_n^2}), \tag{4.46}$$

$$i\beta_n \langle v_n, u_n\rangle_{L^2(\Omega)} - \int_\Omega \operatorname{div}(\nabla u_n + a\nabla v_n)\overline{u}_n dx = o(\frac{1}{\beta_n^2}), \tag{4.47}$$

and

$$i\beta_n \langle w_n, u_n\rangle_{L^2(\Gamma_1)} + \int_{\Gamma_1} (\partial_\nu u_n + a\partial_\nu v_n)\overline{u}_n d\Gamma - \int_{\Gamma_1} \Delta_T u_n\overline{u}_n d\Gamma = o(\frac{1}{\beta_n^2}). \tag{4.48}$$

Using Green's formula in (4.47), summing the result with (4.46) and (4.48), taking the real part, then taking to consideration (4.45) and

$$\int_{\Gamma_1} w_n\overline{u}_n d\Gamma \le C\|w_n\|_{L^2(\Gamma_1)}^2 \int_\Omega a|\nabla u_n|^2 dx, \tag{4.49}$$

we deduce

$$\|u_n\|_V \sim \|v_n\|_{L^2(\Omega)}. \tag{4.50}$$

Now, we want to prove that

$$\|u_n\|_V = o(1). \tag{4.51}$$

Using (4.38), (4.39) and (4.41) we deduce

$$\|\beta_n u_n\|_{L^2(\Omega)} = O(1). \tag{4.52}$$

From (4.45), the assumption on $a$ and Poincaré's inequality we obtain

$$\int_\omega |\beta_n v_n|^2 dx = o(1). \tag{4.53}$$

Now, multiplying (4.41) by $i\beta_n\overline{u}_n$, integrating over $\omega$ and using estimations (4.53), (4.52) and the fact that $f_n$ converges to 0 in $L^2(\Omega)$ we get

$$\int_\omega |\beta_n u_n|^2 dx = o(1). \tag{4.54}$$

Taking the gradient of (4.41) then multiplying by $-i\beta_n^3\nabla\overline{u}_n$, integrating over $\omega$ and using Cauchy-Schwarz's and Young's inequality, estimate (4.45), and the fact that $f_n$ converges to 0 in $V$ we get

$$\int_\omega |\beta_n^2 \nabla u_n|^2 dx = o(1). \tag{4.55}$$

Let's define the following function, $\eta \in C^1(\overline{\Omega})$

$$\eta(x) = \begin{cases} 1 & \text{if } x \in \mathcal{O}_{\delta-\epsilon} \\ 0 & \text{if } x \in \Omega_\delta \\ [0,1] & \text{elsewhere.} \end{cases} \tag{4.56}$$

Multiplying (4.42) by $\eta\overline{u}_n$, (4.43) by $\overline{u}_n|_{\Gamma_1}$, integrating by parts and summing, we obtain

$$\int_{\Gamma_1} |\nabla_T u_n|^2 d\Gamma = o(1). \tag{4.57}$$

Now we have all the needed estimates on $\omega$ and we need to establish similar estimates on $\Omega \setminus \omega$. To that end, we use a stability result of a similar system with frictional damping.

We consider the following auxiliary problem

$$\begin{cases} \varphi_{tt}(x,t) - \Delta\varphi + a1_\omega\varphi_t = & 0, & \text{in } \Omega \times \mathbb{R}_+^*, \\ \varphi(x,t) = & 0, & \text{on } \Gamma_0 \times \mathbb{R}_+^*, \\ \varphi_{tt} + \dfrac{\partial\varphi}{\partial\nu} - \Delta_T\varphi + \varphi_t = & 0, & \text{on } \Gamma_1 \times \mathbb{R}_+^*, \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = & \varphi_1(x), & \text{in } \Omega, \end{cases} \tag{4.58}$$

This system is a particular case of the system studied in [28] from where we can deduce its exponential stability, (See Remark 4.1 in [28]).

Since the system (4.58) is exponentially stable in the associated energy space $\mathcal{H}_{aux} = V \times L^2(\Omega) \times L^2(\Gamma_1)$, by Theorem 1.4 (Huang-Prüss [47, 82]) we conclude that the following operator

$$\mathcal{A}_{aux}U = \begin{pmatrix} \psi \\ \Delta\varphi - a1_\omega\psi \\ -\dfrac{\partial\varphi}{\partial\nu} + \Delta_T\varphi - \phi \end{pmatrix}, \forall U \in \mathcal{D}(\mathcal{A}_{aux}),$$

with domain,

$$\mathcal{D}(\mathcal{A}_{aux}) := \Big\{ U = (\varphi,\psi,\phi)^T \in V^2 \times L^2(\Gamma_1), \Delta\varphi \in L^2(\Omega), \frac{\partial\varphi}{\partial\nu} - \Delta_T\varphi + \phi \in L^2(\Gamma_1) \Big\},$$

satisfies the following uniform inequality

$$\|(i\beta_n - \mathcal{A}_{aux})^{-1}F\|_{\mathcal{H}} \le M\|F\|_{\mathcal{H}}, \quad \forall F \in \mathcal{H}_{aux}, \tag{4.59}$$

for a positive constant $M > 0$.

Taking $F = (0, -u_n, 0)$, there exists a unique solution $(\varphi_n, \psi_n, \phi_n) \in \mathcal{D}(\mathcal{A}_{aux})$ solution of

$$(i\beta_n I - \mathcal{A}_{aux})(\varphi_n, \psi_n, \phi_n) = (0, -u_n, 0),$$

equivalently

$$\begin{aligned} i\beta_n\varphi_n - \psi_n &:= 0 \text{ in } V, \\ i\beta_n\psi_n - \Delta\varphi_n + a(x)\psi_n &:= -u_n \text{ in } L^2(\Omega), \\ i\beta_n\phi_n + \frac{\partial\varphi_n}{\partial\nu} - \Delta_T\varphi + \phi_n &= 0 \text{ in } L^2(\Gamma_1), \end{aligned}$$

which yields

$$\begin{cases} \beta_n^2\varphi_n + \Delta\varphi_n - ia\beta_n 1_\omega\varphi_n &= u_n, \quad \text{in } \Omega, \\ \varphi &= 0, \quad \text{on } \Gamma_0, \\ -\beta_n^2\varphi_n + \dfrac{\partial\varphi_n}{\partial\nu} - \Delta_T\varphi_n + i\beta_n\varphi &= 0, \quad \text{in } \Gamma_1. \end{cases} \tag{4.60}$$

Moreover, from (4.59) we deduce

$$\|\nabla\varphi_n\|^2_{L^2(\Omega)} + \|\nabla_T\varphi_n\|^2_{L^2(\Gamma_1)} + \|\beta_n\varphi_n\|^2_{L^2(\Omega)} + \|\beta_n\varphi_n\|^2_{L^2(\Gamma_1)} \le C\|u_n\|^2_{L^2(\Omega)}. \tag{4.61}$$

Now we use (4.60), (4.61) and a special choice of multipliers to deduce the needed estimate.

Hence, multiplying (4.41) by $i\beta_n^3\overline{\varphi}_n$, (4.42) by $\beta_n^2\overline{\varphi}_n$ and (4.43) by $\beta_n^2\overline{\varphi}_n$, using Green's formula and additing the resulting equations we get

$$\begin{aligned} -\int_\Omega \beta_n^4 u_n\overline{\varphi}_n + \beta_n^2\int_\Omega \nabla u_n\nabla\overline{\varphi}_n + \beta_n^2\int_\Omega a\nabla v_n\nabla\overline{\varphi}_n + i\beta_n^3\int_{\Gamma_1} w_n\overline{\varphi}_n \\ + \beta_n^2\int_{\Gamma_1}\nabla_T u_n\nabla_T\overline{\varphi}_n = i\beta_n\int_\Omega f_n\overline{\varphi}_n + \int_\Omega g_n\overline{\varphi}_n + \int_{\Gamma_1} h_n\overline{\varphi}_n. \end{aligned} \tag{4.62}$$

Using (4.52), (4.61), and the fact that $(f_n, g_n, h_n) \to 0$ in $\mathcal{H}_0$, we get

$$-\int_\Omega \beta_n^4 u_n\overline{\varphi}_n + \beta_n^2\int_\Omega \nabla u_n\nabla\overline{\varphi}_n + i\beta_n^3\int_{\Gamma_1} w_n\overline{\varphi}_n \quad +\beta_n^2\int_{\Gamma_1}\nabla_T u_n\nabla_T\overline{\varphi}_n = o(1). \tag{4.63}$$

Integrating by parts and using (4.60) we find

$$-a\beta_n^3\int_\Omega 1_\omega u_n\overline{\varphi}_n - \beta_n^2\int_\Omega |u_n|^2dx + \beta_n^4\int_{\Gamma_1} u_n\overline{\varphi}_n + i\beta_n^3\int_{\Gamma_1} u_n\overline{\varphi}_n d\Gamma + i\beta_n^3\int_{\Gamma_1} w_n\overline{\varphi}_n = o(1). \tag{4.64}$$

We estimate the last term in the left hand side, as

$$\begin{aligned} |i\beta_n^3 \int_{\Gamma_1} w_n \overline{\varphi}_n| &\leq \|\beta_n v_n\|_{L^2(\Gamma_1)} \|\beta_n^2 \varphi\|_{L^2(\Gamma_1)} \\ &\leq C\|\beta_n \nabla v_n\|_{L^2(O_\delta)} \|\beta_n^2 \varphi\|_{L^2(\Gamma_1)} \\ &\leq C\|\beta_n \nabla v_n\|_{L^2(O_\delta)} \|\beta_n u_n\|_{L^2(\Omega)} = o(1). \end{aligned} \tag{4.65}$$

Using (4.55), (4.61) and the trace inequality we deduce

$$\begin{aligned} |\beta_n^4 \int_{\Gamma_1} u_n \overline{\varphi}_n d\Gamma + i\beta_n^3 \int_{\Gamma_1} u_n \overline{\varphi}_n d\Gamma| &\leq C|\beta_n^4 \int_{\Gamma_1} u_n \overline{\varphi}_n| \\ &\leq C\|\beta_n^2 u_n\|_{L^2(\Gamma_1)} \|\beta_n^2 \varphi\|_{L^2(\Gamma_1)} \\ &\leq C\|\beta_n^2 \nabla_n\|_{L^2(O_\delta)} \|\beta_n u_n\|_{L^2(\Omega)} = o(1). \end{aligned} \tag{4.66}$$

After (4.65) and (4.66), (4.64) becomes

$$a\beta_n^3 \int_\Omega 1_\omega u_n \overline{\varphi}_n + \beta_n^2 \int_\Omega |u_n|^2 dx = o(1). \tag{4.67}$$

Now, using (4.55) and (4.61) we deduce

$$\begin{aligned} |a\beta_n^3 \int_\Omega 1_\omega u_n \overline{\varphi}_n| &\leq C\|\beta_n u_n\|_{L^2(\omega)} \|\beta_n^2 \varphi_n\|_{L^2(\omega)} \\ &\leq C\|\beta_n u_n\|_{L^2(\omega)} \|\beta_n^2 \varphi_n\|_{L^2(\Omega)} \\ &\leq C\|\beta_n u_n\|_{L^2(\omega)} \|\beta_n u_n\|_{L^2(\Omega)} = o(1). \end{aligned} \tag{4.68}$$

Hence, we get

$$\beta_n^2 \int_\Omega |u_n|^2 dx = o(1). \tag{4.69}$$

Next, multiplying (4.42) and (4.43) by $\overline{u}_n$, taking to consideration (4.45), (4.49) and (4.49) we deduce

$$\int_\Omega |\nabla u_n|^2 dx = o(1) \tag{4.70}$$

which gives the desired estimation. □

# Conclusion and Perspectives

In the study conducted in this thesis, we explored the effectiveness of localized Kelvin-Voigt damping in stabilizing the wave equation with dynamic Wentzell boundary conditions and a delay boundary feedback. Our findings indicate that when the damping is smoothly localized, it stabilizes the system exponentially. However, when the damping is discontinuously localized, we could show a polynomial decay rate of the form $t^{-1/2}$ only. Our results demonstrate that the Kelvin-Voigt damping is robust enough to stabilize the usual energy in addition to the boundary energy generated by the Wentzell term, the velocity over the boundary and the time delay feedback. Additionally, the decay rate is heavily dependent on the regularity of the damping. The slow of the decay observed in the discontinuous case is due to the reflected waves at the damping region because only a portion of waves proportional to the regularity of the damping coefficient gets absorbed and damped. Future research topics can include can cover the following points:

- ✓ The Poincaré inequality is crucial for the stability results in both the third and fourth chapters. In practice we can face problems where the damping region doesn't meet the Dirichlet boundary, hence, the Poincaré inequality fails. It is of interest to consider such cases.

- ✓ According to the literature, specifically the work of Alves and al. in [7], the decay rate presented in Chapter 4 may be not optimal. A logical next step would be addressing this and seeking out an optimal or improved decay rate.

- ✓ Tebou in [84] relaxed the conditions on the damping coefficient and region considered by Liu and Rao in [67] that we adopted in Chapter 3. Namely, he assumed $a \in W^{1,\infty}(\Omega)$, $|\nabla a(x)|^2 \leq Ca(x)$ and $\omega$ satisfy the PMGC. One may consider the relaxed conditions and see whether we still get the same exponential stability results.

- ✓ Relaxing the geometrical conditions is of practical interest since one always seeks to limit the intervention to the model. Hence, considering more general cases for the damping region such as in [13] is interesting.

- ✓ A challenging and interesting problem is to consider time dependent damping coefficients.

- ✓ Considering a boundary damping with interior delay is interesting from the mathematical point view and is very challenging even in simpler cases such as static boundary conditions without the Wentzell term.

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