Order No. 265/2025-C/MT

PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

Ministry of Higher Education and Scientific Research

University of Science and Technology Houari Boumediene

Faculty of Mathematics

DOCTORATE THESIS

Presented for obtaining the **degree** of **DOCTOR**

In: MATHEMATICS

Major: Partial differential equations and applications

Submitted by: Ali KAINANE MEZADEK,

Dissertation topic

Rearrangements of functions and their applications in the equations of vortices

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Summary

In my thesis, we prove the existence theory for certain variational problems, which are related to vortex flows of ideal fluids. Such a steady vortex flow of an ideal fluid is governed by a non-linear elliptic equation. The theory that used is similar to the one that was proposed by Benjamin [4] and motivated by Burton [6], in which vortex rings can be obtained as maximisers of a functional that is related to the kinetic energy over the set of rearrangements of a fixed function. In this theory, the function rearranged represents the vorticity in the planar domain, and it represents a quantity related to the vorticity in three dimensions.

The first chapter is started by explaining some preliminary concept of her work (rearrangements of function, the inequalities and theirs applications, the set of rearrangements of a function and it properties and finally the weak closure of the set of rearrangements of a non-negative function defined on a domain having infinite measure).

In the second chapter, we gives briefly the state of equation that governs the steady vortex rings in $\mathbb{R}^3$ followed by Benjamins theories. In fact, Benjamin observed that when an uniform flow of an ideal fluid is in a steady state (possible unsteady), the kinetic energy E and the impulse $\mathcal{I}_2$ in the z-direction of cylindrical coordinates $(r, \theta, z) \in \mathbb{R}^3$ are preserved, and that in the presence of axisymmetry, a quantity related to the vorticity can be seen as rearrangements of a prescribed function. He suggested that the solutions

of one of the two varitional problems

$$\max_{\zeta\in\mathcal{R}(\zeta_0)}(E(\zeta)-\lambda\mathcal{I}_2(\zeta))$$

or

$$\max_{\mathcal{I}_2(\zeta)=I,\zeta\in\mathcal{R}(\zeta_0)}E(\zeta)$$

should give rise to a steady vortex ring of the flow, here is a positive number depending of the speed of the flow at infinity, and $\mathcal{R}(\zeta_0)$ the set of rearrangements of a prescribed function ζ_0 and

$$\mathcal{I}_2(\zeta)=\int\limits_{\mathbb{R}^3}r\zeta.$$

Moreover, if ζ is a solution of one of two variational problem and $\Psi=K\zeta$, satisfies equation

$$\mathcal{L}\Psi=\phi\circ(\Psi-\frac{\lambda}{2}r^2)\text{ in }\Pi \tag{1}$$

with respect to the boundary conditions

$$(BC)\quad:\Psi(0,z)=0,\Psi(r,z)\to 0\text{ and }|\nabla\Psi|\to 0\text{ as }r^2+z^2\to\infty$$

where Π is the half-plane that is defined by $r>0$ and $-\infty<z<\infty$, K is the inverse differential operator of

$$\mathcal{L}=-\left(\frac{1}{r}\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r})+\frac{1}{r^2}\frac{\partial^2}{\partial z^2}\right)$$

with homogeneous Dirichlet boundary conditions on Π and $\phi:\mathbb{R}\to\mathbb{R}$ is an unknown increasing function.

In third chapter, we prove an existence theorem for a steady vortex rings in Poiseuille flow. The flow in question is written in terms of a Stokes stream function $\Psi : \mathbb{R}^3 \to \mathbb{R}$ with respect to cylindrical coordinates $(r, \theta, z) \in \mathbb{R}^3$, which is symmetric about the z-axis direction and approaches at infinity -$\frac{\lambda}{4}r^4$, which represents a flow of velocity field V $= (0, 0, -\lambda r^2)$, where λ is a parameter corresponding to the strength of the background flow at infinity, moreover, the Stokes stream function satisfies the following equation

$$\mathcal{L}\Psi = \phi \circ (\Psi - \frac{\lambda}{4}r^4) \quad \text{in} \quad \Pi \tag{2}$$

with respect to the same boundary condition as uniform flow.

The fourth chapter, we prove also an other existence theorem of steady vortex rings in Poiseuille flow, the flow here is always given by Stokes stream function $\Psi : \Pi \longrightarrow \mathbb{R}$ is symmetric about the z-axis with respect to the cylindrical coordinates (r, θ, z) and approaches $-\lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4)$ at infinity in the negative z-direction, where $\lambda > 0$ and $\sigma \geq 0$ are two numbers corresponding to the strength of the background flow and the velocity in the negative z-direction approaches $\lambda(1 + \sigma r^2)$ at infinity, hence Ψ satisfies the equation

$$\mathcal{L}\Psi = \phi \circ (\Psi - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4)) \quad \text{in} \quad \Pi, \tag{3}$$

where, $\phi : \mathbb{R} \to \mathbb{R}$ is a non-negative increasing function.

In the last chapter in this thesis, we prove an existence existence theory of steady flows described by a variational problem similar to the one governing steady 2-dimensional ideal fluid flows containing symmetric vortex pairs. The flow in question is written in terms of a stream function $\Psi : \Pi \to \mathbb{R}$ with Ψ even in x_1, where Π is the half-plane defined in $\mathbb{R}^2$ by $x_2 > 0$. At infinity the stream function approaches $-\lambda(x_2 + \frac{\sigma}{2} x_2^2)$ which representing a flow of velocity $\lambda(1 + \sigma x_2)$ in the negative x_1-direction, where λ is a positive constant depends the speed of the flow at the infinity and σ is a positive number represents the severity of the shearing. The vorticity is described by $-\Delta\Psi$, where Δ is the Laplacian operator in two dimensions, $-\Delta\Psi$ vanishes outside a bounded region placed symmetrically about the x_2 axis, and avoiding the x_1 axis. The vorticity in the region $x_2 > 0$ is non-negative, and Ψ satisfies the equation

$$-\Delta\Psi = \phi \circ \left(\Psi - \lambda(x_2 + \frac{\sigma}{2} x_2^2)\right), \tag{4}$$

where ϕ is an unknown non-negative function

Acknowledgments

First and foremost, I would like to express my sincere gratitude to Pr. ALIZIANE Tarik professor at the University of Science and Technology Houari Boumediene for giving me the honor of being president of jury and examiner of my work.

A special thank go to my supervisor Pr. REBAH Dirar professor at the University of Science and Technology Houari Boumediene for the continuous support of my Ph.D study and research, for his patience, motivation, enthusiasm, and immense knowledge. his guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my Ph.D study.

Likewise, I am deeply grateful to Dr. BOUTARENE Khaled el ghaouti from the University of Science and Technology Houari Boumediene for accepting to be examiner of my research work. please accept my sincere thanks for your being a member of this jury.

My most respectful thanks are intended to Dr. YOUNSI Abdelhafid at the University of Djelfa who honored me to be examiner of my work. please accept the expression of my deep gratitude.

I would also like to thank my brother (Mohamed) for all discussions which are helped me along the way.

I would like also to thank my parents, brothers and sisters for them sacrifices.

The last word of acknowledgment for all my small family for her encouragement and patience.

Contents

Chapter 1

Introduction

The remainder of the opening chapter will define what rearrangements, or equimeasurable functions, are, and prove basic properties of rearrangements. This definition will be for finite measure spaces. Two functions being rearrangements are shown to be equivalent to six properties: pre-images of upper level sets have equal measures; pre-images of Borel subsets of $\mathbb{R}$ have equal measures; the distribution functions are equal; the decreasing rearrangements are equal; the two functions have equal integrals when composed with non-negative Borel measurable functions; and the positive part of the functions minus arbitrary real values have equal integrals. In colloquial terms, two functions are rearrangements if they take the same values for the same amount. Thirteen properties of rearrangements are shown to hold. The chapter finishes with an characterisation of Douglas for the weak closure of the set of rearrangements. If the reader is familiar with the study of rearrangements, then it should be possible to review this chapter quickly.

1.1 Rearrangements on finite measure space

Definition 1.1 *Let* (Ω, Σ, μ) *and* (Ω', Σ', μ') *be two positive measure space with* $\mu(\Omega) = \mu'(\Omega')$*, where* Σ *and* Σ' *are two* σ*-algebra on* Ω *and* Ω' *respectively. Two real measurable functions* $f \geq 0$ *on* Ω *and* $g \geq 0$ *on* Ω' *are called rearrangement of each other if and only if*

$$\mu(\{x \in \Omega \ | \ f(x) > \alpha\}) = \mu'(\{x \in \Omega' \ | \ g(x) > \alpha\}) \tag{1.1}$$

for each $\alpha > 0$*, provided both sides in (1.1) are finites.*

For a given non-negative mesurable function $f_0 : \Omega \longrightarrow \mathbb{R}$, we use $\mathcal{R}(f_0)$ to denote the set of all rearrangements of f_0 on Ω so

$$\mathcal{R}(f_0) = \{f : \Omega \longrightarrow \mathbb{R}_+, f \ \text{ is a rearrangement of } \ f_0\}. \tag{1.2}$$

In the case $f_0 > 0$ is defined on the half-line $[0, \ \infty)$, Eydeland, Spruck and Turkington [21] characterised the set $\mathcal{R}(f_0)$ as

$$\mathcal{R}(f_0) = \left\{ f \geq 0 \ | \ \int_0^\infty (f - a)_+ dx = \int_0^\infty (f_0 - a)_+ dx \ \ \forall a > 0, \right\}. \tag{1.3}$$

where the subscript + stands for the positive part.

Definition 1.2 *Let* (Ω, Σ, μ) *be a finite measure space, and let* $f : \Omega \to \mathbb{R}$ *be a measurable function. The distribution function of* f *is defined by*

$$F_f : \mathbb{R} \to [0, \ \mu(\Omega)], \quad F_f(t) = \mu\left(\{x \in \Omega : \ \ f(x) > t\}\right).$$

Lemma 1.1 *The distribution function* $F_f : \mathbb{R} \longrightarrow [0,\ \mu(\Omega)]$ *is a decreasing and right-continuous.*

Proof. It is clear that F_f is decreasing function on $\mathbb{R}$ since for $\alpha \leq \beta$, we have

$$\{x \in \Omega \ \mid \ f(x) > \beta\} \subset \{x \in \Omega \ \mid \ f(x) > \alpha\}.$$

Now, let $(\alpha_n)_{n\geq 1}$ be a sequence defined by $\alpha_n = \alpha + \frac{1}{n}$, where $\alpha \in \mathbb{R}$ fixed. Also, we set

$$A(\alpha_n) = \{x \in \Omega \mid f(x) > \alpha_n\}.$$

Then we have $A(\alpha_{n+1}) \subset A(\alpha_n)$, hence by using monotonic convergence of the measure,

$$\{x \in \Omega \ \mid \ f(x) > \alpha\} = \bigcup_{n\geq 1} A(\alpha_n).$$

Thus,

$$\begin{aligned} \lim_{n\to\infty} F_f(\alpha_n) &= \lim_{n\to\infty} \mu(A(\alpha_n)) \\ &= \mu(\{x \in \Omega \mid f(x) > \alpha\}) \\ &= F_f(\alpha). \end{aligned}$$

Therefore F_f is right-continuous function. ■

Now, let $\Omega \subset \mathbb{R}^n$ such that $\mu_n(\Omega) < \infty$, where μ_n is the n-Lebesgue measure in $\mathbb{R}^n$ ($n > 1$) and let $f : \Omega \longrightarrow \mathbb{R}_+$ be a measurable function such that $F_f(\alpha) < \infty$ for each $\alpha > 0$. Here, we list some well known examples of special rearrangements of f.

Definition 1.3 *The symmetric decreasing rearrangement of f, that is denoted by f_Δ is given by*

$$f_\Delta(t) = \begin{cases} \max\{\alpha > 0, F_f(\alpha) \geq 2|t|\} & \text{if } \alpha \text{ exist} \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Example 1.1 *For $x \in [0,\ 1]$ and $y \in [0,\ 2]$, we set $f(x,y) = x + 2y$. Then,*

$$F_f(\alpha) = \frac{(5-\alpha)^2}{4} \quad \text{for } \alpha \in [0,\ 5].$$

Hence

$$f_\Delta(t) = \begin{cases} 5 - 2\sqrt{2|t|} & \text{if } t \in [-\frac{25}{8},\ \frac{25}{8}] \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

Definition 1.4 *The Steiner-symmetrisation of f about the plane $x_n = 0$, that is denoted by f^s is defined by*

$$f^s(x_1, \cdots, x_n) := f_\Delta(x', \cdot), \text{ for all } (x_1, \cdots, x_n) \in \mathbb{R}^n$$

where $x' = (x_1, \cdots, x_{n-1})$.

Note that f^s is a rearrangement of f. Indeed, by using Fubini's Theorem we have

$$\begin{aligned} \mu_n(\{x \mid f^s(x) \geq \alpha\}) &= \int\limits_{\mathbb{R}^{n-1}} \mu_1(\{x_n \mid f^s(x_1, \ldots, x_{n-1}, x_n) \geq \alpha\}) dx_1 \ldots dx_{n-1} \\ &= \int\limits_{\mathbb{R}^{n-1}} \mu_1(\{t \mid f_\Delta(x_1, \ldots, x_{n-1}, t) \geq \alpha\}) dx_1 \ldots dx_{n-1} \\ &= \mu_n(\{x \mid f_(x) \geq \alpha\}). \end{aligned}$$

Definition 1.5 *The spherically decreasing rearrangement f^* of f, also called the Schwarz symmetrisation is defined by*

$$f^*(x) := \max\{\alpha > 0 \mid F(\alpha) \geq |x|^n \omega_n\}$$

where ω_n *denoted the volume of unit ball in* $\mathbb{R}^n$.

Example 1.2 *Let* f *be a function define on* $[0,\ 4]$ *by* $f(x) = \sqrt{x}$*, so* $F_f(\alpha) = 4 - \alpha^2$. *Since* $w_2 = \pi$*, then we find*

$$f^*(x, y) = \sqrt{4 - \pi(x^2 + y^2)} \quad \textit{where} \quad x^2 + y^2 \leq \frac{4}{\pi}.$$

Lemma 1.2 *Let* (Ω, Σ, μ) *be a finite measure space and let* $f : \Omega \longrightarrow \mathbb{R}$ *be a measurable function. We define* $f^\Delta : [0,\ \mu(\Omega)] \longrightarrow \mathbb{R}$ *by*

$$f^\Delta(t) = \max\{\alpha > 0 \ \mid \ F_f(\alpha) \geq t\}. \tag{1.6}$$

Then

1. f^Δ *is a decreasing and right-continuous on* $[0,\ \mu(\Omega)]$,
2. *For all* $t \in [0,\ \mu(\Omega)]$ *and* $\alpha \in \mathbb{R}$,

$$F_f(\alpha) > t \Longleftrightarrow f^\Delta(t) > \alpha.$$

3. f^Δ *is a rearrangement of* f *on* $[0,\ \mu(\Omega)]$.

Proof. (1). For t_1 and t_2 in $[0,\ \mu(\Omega)]$ with $t_1 > t_2$ and for all $\alpha > 0$ we have

$$\{\alpha \in \mathbb{R} \mid F_f(\alpha) \geq t_1\} \subset \{\alpha \in \mathbb{R} \mid F_f(\alpha) \geq t_2\},$$

hence

$$f^\Delta(t_1) = \max\{\alpha > 0 \ \mid \ F_f(\alpha) \geq t_1\} \leq \max\{\alpha > 0 \mid F_f(\alpha) \geq t_2\} = f^\Delta(t_2).$$

Therefore, f^{Δ} is decreasing function on $[0,\ \mu(\Omega)]$.

Now, let $(t_n)_{n\geq 0}$ be a decreasing sequence such that $t_n \to \bar{t} \in [0,\ \mu(\Omega)]$ as $n \to \infty$ with $t_n > \bar{t}$ for all $n \in \mathbb{N}$. We set $x_n = f^{\Delta}(t_n)$ and $\bar{x} = f^{\Delta}(\bar{t})$. Then we have $x_{n+1} \geq x_n$ and $x_n \leq \bar{x}$, which means that the sequence (x_n) is increasing and bounded above by $\bar{x}$, hence there exists $X \in \mathbb{R}$ such that $x_n \to X$ as $n \to \infty$.

By the definition of the decreasing rearrangement, let $\varepsilon > 0$ be arbitrary, then

$$F_f(x_n - \varepsilon) > t_n \geq F_f(x_n + \varepsilon),$$

for all $n \in \mathbb{N}$. If we assume $X < \bar{x}$ and taking $\varepsilon = \frac{\bar{x}-X}{2}$, then, when $n \to \infty$, we obtain

$$\bar{t} \geq F_f(\bar{x} + \varepsilon) \geq F_f(\bar{x} - \varepsilon) > \bar{t}$$

which gives a contradiction, hence, $\bar{x} = X$. Therefore f^{Δ} is right-continuous.

Now, since F_f is decreasing function on $\mathbb{R}$, then for all $t \in [0,\ \mu(\Omega)]$ and $\alpha \in \mathbb{R}_+$,

$$\begin{aligned} F_f(\alpha) > t &\iff \mu(\{x \in \Omega \mid f(x) > \alpha\}) > t \\ &\iff \alpha < \max\{\beta \in \mathbb{R} \mid F_f(\beta) \geq t\} \\ &\iff f^{\Delta}(t) > \alpha. \end{aligned}$$

Thus (2) is verified. It remains just to prove (3). For that, we need to show that for all $\alpha \in \mathbb{R}$ we have $F_f(\alpha) = F_{f^{\Delta}}(\alpha)$. Indeed, according to (2), we have

$$\begin{aligned} F_{f^{\Delta}}(\alpha) &= |\{t \in [0,\ \mu(\Omega)] \mid f^{\Delta}(t) > \alpha\}| \\ &= |\{t \in [0,\ \mu(\Omega)] \mid F_f(\alpha) > t\}| \\ &= |[0,\ F_f(\alpha)]| \\ &= F_f(\alpha). \end{aligned}$$

This completes the proof. ■

Definition 1.6 *Let $\Omega \in \mathbb{R}^n$ be such that $\mu_n(\Omega) < \infty$, and let f be a non negative measurable function defined on Ω. The (essentially) unique decreasing rearrangement f^{Δ} of f is defined by*

$$f^{\Delta}(t) := \max\{\alpha > 0 \mid F_f(\alpha) \geq t\}$$

for all $0 < t < \mu_n(\Omega)$.

The next proposition showed that there are six different properties that are equivalent to two functions being rearrangements of one another. (see Masters [32] Proposition 1.11).

Proposition 1.1 *Let (Ω, Σ, μ) and (Ω', Σ', μ') be two finite measure spaces such that $\mu(\Omega) = \mu'(\Omega')$. Let $f : \Omega \longrightarrow \mathbb{R}$ and $g : \Omega' \longrightarrow \mathbb{R}$ be two measurable functions. Then the following statements are equivalent:*

1. *f and g are rearrangements,*

2. $\mu(f^{-1}(B)) = \mu^{'}(g^{-1}(B))$ for every Borel set $B \subseteq \mathbb{R}$,

3. $F_f = F_g$,

4. $f^{\Delta} = g^{\Delta}$,

5. For every non-negative Borel measurable function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$,

$$\int\limits_{\Omega} \varphi o f d\mu = \int\limits_{\Omega'} \varphi o g d\mu^{'},$$

6. For all $\alpha \in \mathbb{R}$,

$$\int\limits_{\Omega} (f-\alpha)_{+} d\mu = \int\limits_{\Omega'} (g-\alpha)_{+} d\mu^{'},$$

where the subscript + stands for the positive part.

The results of this proposition are more general of Douglas's characterisation proprties for rearrangements functions. In fact, Douglas [18] assumed first that for some non-negative $h \in L^1(\mathbb{R}^n)$, the identity $\mu(A) = \int\limits_{A} h d\mu$ holds for each measurable set $A \subset \Omega$, where μ is denoted the n-Lebesgue measure on $\mathbb{R}^n$ $(n \geq 1)$ and Ω is a bounded set in $\mathbb{R}^n$, then he proved in ([18], section 2.1, theorem 1) some characterisation proprties as in this Propostion 1.1. Furthermore, Burton ([7], lemma 2.1(ii)) proved that if f and g a rearrangements, then $\varphi \circ f$ and $\varphi \circ g$ are rearrangements, for each Borel measurable function $\varphi : \mathbb{R} \to \mathbb{R}$. We end this subsection by setting this result

Lemma 1.3 *Let (Ω, Σ, μ) and $(\Omega^{'}, \Sigma^{'}, \mu^{'})$ be two finite measure spaces such that $\mu(\Omega) = \mu^{'}(\Omega^{'})$. Let $f : \Omega \longrightarrow \mathbb{R}_{+}$ and $g : \Omega^{'} \longrightarrow \mathbb{R}_{+}$ be two measurable functions such that f is a* *rearrangement of g and for every Borel set $B \subseteq \mathbb{R}$*

$$\mu\left(f^{-1}(B)\right) = \mu^{'}\left(g^{-1}(B)\right).$$

Then we have the following propreties:

1. *If $\Phi : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, then $\Phi \circ f$ is a rearrangement of $\Phi \circ g$.*
2. *If $f \in L^p(\mu)$ with $p \in [1, \infty[$, then $g \in L^p(\mu^{'})$ et on a*

$$\int_{\Omega} f^p d\mu = \int_{\Omega'} g^p d\mu^{'}. \tag{1.7}$$

Proof. To prove (1), let $\Phi : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function and let $B \subset \mathbb{R}$ be is Borel subset. Then $\Phi^{-1}(B)$ is a Borel subset of $\mathbb{R}$ and since f is a rearrangement of g, then we have

$$\begin{aligned}
\mu\left((\Phi \circ f)^{-1}(B)\right) &= \mu\left((f^{-1}(\Phi^{-1}(B))\right) \\
&= \mu^{'}\left((g^{-1}(\Phi^{-1}(B))\right) \\
&= \mu^{'}\left((\Phi \circ g)^{-1}(B)\right).
\end{aligned}$$

Thus, $\phi \circ f$ is a rearrangement of $\Phi \circ g$.

Now, for $s > 0$ we set

$$F_f(s) = \{x \in \Omega \mid f(x) \geq s\} \quad \text{and} \quad G_g(s) = \left\{x \in \Omega^{'} \mid g(x) \geq s\right\}.$$

Then, using Fubini'Theorem we get:

$$\begin{aligned}\int\limits_{\Omega} f\, d\mu &= \int\limits_{\Omega}\int\limits_{0}^{f(x)} ds d\mu \\ &= \int\limits_{0}^{\infty}\int\limits_{F_f(s)} d\mu ds \\ &= \int\limits_{0}^{\infty} \mu(F_f(s)) ds \\ &= \int\limits_{0}^{\infty} \mu^{'}(G_g(s)) ds \\ &= \int\limits_{\Omega'} g d\mu^{'}, \qquad (1.8)\end{aligned}$$

hence it follows that if f is a rearrangement of g, then we have (1.8). Now, let $\Phi : \mathbb{R} \to \mathbb{R}$ be a function defined by $\Phi(x) = |x|^p$. Since Φ is a contionuous function then this yields that Φ is a Borel function, hence by using (2), we find that $|f|^p$ is a rearrangement of $|g|^p$. Thus, from (1.8) we find

$$\|f\|_p = \|g\|_p . \qquad (1.9)$$

Thus (2) is proved. This completes the proof. ■

Remark 1.1 *From this lemma, we deduce that if* $\Omega \subset \mathbb{R}^n$ *with* $\mu_n(\Omega) < \infty$ *and if* $f_0 \in L^p(\Omega)$ *is a non-negative function, then the set* $\mathcal{R}(f_0)$ *represents a subspher in* $L^p(\Omega)$.

1.2 Rearrangements inequalities

In this section, we state some inequalities that are needed for our application.

Theorem 1.1 *Let* $\Omega \subset \mathbb{R}^n$ *(*$n \geq 1$*) and* $f, g : \Omega \longrightarrow \mathbb{R}_+$ *be two measurable functions. Then, the following inequalities*

$$\int\limits_{\Omega} f(x)g(x)dx \leq \int\limits_{\mathbb{R}} f_{\Delta}(t)g_{\Delta}(t)dt \tag{1.10}$$

and

$$\int\limits_{\Omega} f(x)g(x)dx \leq \int\limits_{\mathbb{R}^n} f^*(x)g^*(x)dx \tag{1.11}$$

hold.

Proof. The proof of these inqualities is the same, so it is suffissant to prove (1.10). Indeed, assume that $f(x) = \mathcal{X}_A(x)$ and $g(x) = \mathcal{X}_B(x)$ where A and B are two measurable subset in Ω. We set $\alpha = \mu_n(A)$ and $\beta = \mu_n(B)$, then, it follows that $f_{\Delta}(t) = \mathcal{X}_{[-\frac{\alpha}{2},\, \frac{\alpha}{2}]}(t)$ and $g_{\Delta}(t) = \mathcal{X}_{[-\frac{\beta}{2},\, \frac{\beta}{2}]}(t)$. Thus, we have

$$\begin{aligned}
\int\limits_{\Omega} f(x)g(x)dx &= \int\limits_{\Omega} \mathcal{X}_A(x)\mathcal{X}_B(x)dx \\
&= \mu_n(A \cap B) \\
&\leq \min\{\mu_n(A),\ \mu_n(B)\} \\
&= \min\{\alpha,\ \beta\} \\
&= \min\{\mu_1([-\frac{\alpha}{2},\ \frac{\alpha}{2}]),\ \mu_1([-\frac{\beta}{2},\ \frac{\beta}{2}])\} \\
&= \mu_1([-\frac{\alpha}{2},\ \frac{\alpha}{2}] \cap [-\frac{\beta}{2},\ \frac{\beta}{2}]) \\
&= \int\limits_{\mathbb{R}} \mathcal{X}_{[-\frac{\alpha}{2},\, \frac{\alpha}{2}]}(t)\mathcal{X}_{[-\frac{\beta}{2},\, \frac{\beta}{2}]}(t)dt \\
&= \int\limits_{\mathbb{R}} f_{\Delta}(t)g_{\Delta}(t)dt.
\end{aligned}$$

Now, for $s > 0$ and $y > 0$, we set $\alpha(s) = \mu_n(A_f(s))$ and $\beta(y) = \mu_n(A_g(y))$ with

$$A_f(s) = \{x \in \Omega \mid f(x) \geq s\} \quad \text{and} \quad A_g(y) = \{x \in \Omega \mid g(x) \geq y\}\,.$$

Since f_Δ et g_Δ are rerrangements of f and g respectively, then we have

$$\alpha(s) = \mu_n(A_f(s)) = \mu_1(A_{f_\Delta}(s)) \;\text{ and }\; \beta(y) = \mu_n(A_g(y)) = \mu_1(A_{g_\Delta}(y))$$

where

$$A_{f_\Delta}(s) = \{t \in \mathbb{R} \mid f_\Delta(t) \geq s\} \quad \text{and} \quad A_{g_\Delta}(y) = \{t \in \mathbb{R} \mid g_\Delta(t) \geq y\}\,.$$

Thus,

$$\begin{aligned}
\int\limits_{\Omega} f(x)g(x)dx &= \int\limits_{\Omega}(\int\limits_{0}^{f(x)} ds)(\int\limits_{0}^{g(x)} dy)dx \\
&= \int\limits_{\Omega}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \mathcal{X}_{A_f(s)}(x)\mathcal{X}_{A_g(y)}(x)dydsdx \\
&= \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \mu_n(A_f(s)\cap A_g(y))dsdy \\
&\leq \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \min\{\alpha(s),\ \beta(y)\}dsdy \\
&= \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \min\{\mu_1([-\frac{\alpha(s)}{2},\ \frac{\alpha(s)}{2}]),\ \mu_1([-\frac{\beta(y)}{2},\ \frac{\beta(y)}{2}]\}dsdy \\
&= \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \mu_1\left([-\frac{\alpha(s)}{2},\ \frac{\alpha(s)}{2}]\cap[-\frac{\beta(y)}{2},\ \frac{\beta(y)}{2}]\right)dsdy \\
&= \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \mu_1\left(A_{f_\Delta}(s)\cap A_{g_\Delta}(y)\right)dsdy \\
&= \int\limits_{\mathbb{R}}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \mathcal{X}_{A_{f_\Delta}(s)}(t)\mathcal{X}_{A_{g_\Delta}(y)}(t)dydsdt \\
&= \int\limits_{\mathbb{R}} f_\Delta(t)g_\Delta(t)dt. \qquad (1.12)
\end{aligned}$$

■

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^n$ such that $\mu_n(\Omega) < \infty$, let f and g are two non-negative functions in $L^p(\Omega)$ and $L^q(\Omega)$ respectively, where $p \in [1,\ \infty)$ and q is the conjugate exponent of p. Then the following inequality*

$$\int\limits_{\Omega} fgd\mu_n \leq \int\limits_{0}^{\mu_n(\Omega)} f^\Delta g^\Delta d\mu_1. \qquad (1.13)$$

holds

Here, f^{Δ} and g^{Δ} are reperesent the (essentially) unique decreasing rearrangement of f and g respectively. Also, this inequality is classical for a proof in this general setting, see Burton [7] H.Hardy & Littelwood [25].

Theorem 1.3 *Let f, g and h be three non-negative measurable functions in $\mathbb{R}^n$ ($n \geq 1$) and let F_f, F_g and F_h be the distributions functions corresponding to f, g and h respectively. Assuming for all $\alpha > 0$, $F_f(\alpha)$, $F_g(\alpha)$ and $F_h(\alpha)$ are finite. Then, the following inequality*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(x-y)h(y)dxdy \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)g^*(x-y)h^*(y)dxdy \tag{1.14}$$

holds. Moreover, If the left-hand side equals ∞, then the right-hand side equals ∞.

This inequality called Riesz'inequality [41], where he proved the case when $n = 1$. Later on, Brascamp, Lieb and Luttinger [5] generalised (1.14) for any $n \geq 1$. Riesz's inequality has particular interest when g is the Newtonian potential, because it gives information about kinetic energy functionals. The reader will find more details about rearrangements inequalities in Lieb and Loss [30].

Lemma 1.4 *Let $p \in [1, \infty)$, let $f \in W^{1,p}(\mathbb{R}^n)$ and let f^* be the spherically decreasing rearrangement. Then*

$$\| \nabla f_0^* \|_p \leq \| \nabla f \|_p \, .$$

In the case of equality, Brothers and Ziemer [14] showed that if $\mu_n(\{f^*(s) > a\})$ is absolutely continuous, then f is almost everywhere equal to a translate of f^*. Burton and McLeod [10] used these results to study maximisation and minimisation of certain weakly continuous when they study the extremisers of the Dirichlet integral over the unit ball in $\mathbb{R}^n$, over all functions whose Laplacians are rearrangements of a given non-negative function, hence they showed that the maximiser and minimiser exist and are unique, furthermore they proved that minimiser and maximiser obtained are radial and monotone. convex functionals.

1.3 Properties of the set of rearrangements

Before we begin the investigation in the properties of the set of rearrangements and its weak closure, we need to the following Definitions

Definition 1.7 *(Atom, Non-atomic Measure Spaces). Let* $(\Omega,\ \Sigma, \mu)$ *be a measure space. An atom of* Ω *is a mesurable set* $A \subset \Sigma$ *such that* $\mu(A) > 0$ *and*

$$\textit{for every } B \in \Sigma, \ \ \textit{eiher } \mu(B) = \mu(A) \ \textit{ or } \ \mu(B) = 0.$$

A meassure space is non-atomic if it has non atom.

Definition 1.8 *(Separable). Let* (Ω, Σ, μ) *be a measure space.* Ω *is seprable if there exists a countable set of measurable set* $\mathcal{D} \subseteq \Sigma$ *such that*

$$\textit{for each } \epsilon > 0 \textit{ and } A \in \Sigma, \ \textit{there exists } B \in \mathcal{D} \textit{ such that } \mu(A \backslash B) + \mu(B \backslash A) < \epsilon.$$

Definition 1.9 *(Uniformly integrable). Let* (Ω, Σ, μ) *be a finite measure space. A family* $\mathcal{G}$ *of measurable functions on* Ω *is called uniformly integrable (or equiintegrable) if for* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for each* $g \in \mathcal{G}$*,*

$$\textit{If } A \in \Sigma \textit{ and } \mu(A) < \delta, \quad \textit{then } \int\limits_A |g| d\mu \le \epsilon.$$

Definition 1.10 *(Measure-Preserving Map, Measure-Preserving Bijection) Let* (Ω, μ) *and* (Ω', μ') *be two finite spaces.* $T : \Omega \to \Omega'$ *is a measure-preserving map if for all* $A \in \Sigma^{'}$*,*

$$T^{-1}(A) \in \Sigma \quad \textit{and} \quad \mu(T^{-1}(A)) = \mu'(A).$$

If $T : \Omega \longrightarrow \Omega^{'}$ *is invertible and both* T*,* T^{-1} *are measure- preserving maps, then* T *is called a measure-preserving bijection.*

Definition 1.11 *(Measure Interval). A finite measure space* (Ω, Σ, μ) *will be called a measure interval if and only if there exists a measure-preserving bijection* $T : \Omega \longrightarrow [0, \mu(\Omega)]$*, where* $[0, \mu(\Omega)]$ *is equipped with the* σ*-algebra of Lebesgue-measurable subsets of* $[0, \mu(\Omega)]$ *and the Lebesgue measure.*

Now with this definition, we have the following result which was proved by Burton [7], Lemma 2.7.

Lemma 1.5 *Let* (Ω, μ)*. Then for* $f, g \in L^p(\Omega, \mu)$*, we have*

$$\|f^{\Delta} - g^{\Delta}\|_p \le \|f - g\|_p,$$

where $p \in [1, \infty)$*.*

This inequality may also be used to show that if $f_0 : \Omega \longrightarrow \mathbb{R}_+$, the set of rearrangements $\mathcal{R}(f_0)$ is closed in $L^p(\Omega, \mu)$. Incidentally, Burton [[7], Lemma 2.11] also showed that $\mathcal{R}(f_0)$ is path-connected.

Since by Remark 1.3, the set of rearrangements of a given function is a subsphere of L^p, then we deduce that the set of rearrangements is closed, it is neither convex, compact nor weakly closed in general. This can make problems involving extremising a functional over a set of rearrangements rather difficult. Accordingly, we now introduce the weak closure $\overline{\mathcal{R}(f_0)}^w$ of the set of rearrangements of a function f_0. Burton [6], Theorem 6 and Lemma 6, proved that on a finite separable non-atomic measure space (Ω, μ), if $f_0 \in L^p(\Omega, \mu)$, then the weak closures of the set $\mathcal{R}(f)$ is convex and weakly sequentially compact in $L^p(\Omega, \mu)$, where $p \in [1, \infty]$.

Lemma 1.6 *Let* $(\Omega,\ \Sigma, \mu)$ *be a finite measure space, let* $1 \leq p \leq \infty$, *let* $f_0 \in L^p(\Omega)$ *and let* $\mathcal{R}(f_0)$ *be the set of rearrangements of* f_0 *on* Ω. *Let* $\overline{\mathcal{R}(f_0)}^w$ *be the closure of* $\mathcal{R}(f_0)$ *in the weak topology on* $L^p(\Omega)$. *If* Ω *is separable and* $1 < p < \infty$, *then* $\overline{\mathcal{R}(f_0)}^w$ *is weakly compact. If* Ω *is separable and* $p = \infty$, *then* $\overline{\mathcal{R}(f_0)}^w$ *is weakly-* sequentially compact. Also,* $1 \leq p < \infty$ *implies* $\overline{\mathcal{R}(f_0)}^w$ *is weakly sequentially compact.*

Proof. Although the proof of this lemma is geven by Burton [6] Lemma 6, we restate the proof Masters [32] Lemma 2.58. Indeed, For all $\mathcal{R}(f_0)$, Lemma 1.3 implies $\|f\|_p = \|f_0\|_p$, hence $\overline{\mathcal{R}(f_0)}^w$ is bounded. When $1 < p < 1$, then $L^p(X)$ is reflexive, according to Royden [40], Proposition 20 (pg. 284). Every closed, bounded, convex subset of reflexive Banach space is weakly compact. Thus, $\overline{\mathcal{R}(f_0)}^w$ is weakly compact. Now, the separability of $(\Omega,\ \Sigma, \mu)$ ensures that $L^1(X)$ is separable, so if $p = 1$, then $(\Omega,\ \Sigma, \mu)$ is a bounded weak-* closed set in the dual of separable Banach space, hence $(\Omega,\ \Sigma, \mu)$ is weak-* sequentially compact. Consider finally the case $p = 1$. We have

$$\lim_{M\to\infty} \int\limits_{|f(x)|>M} |f| d\mu = 0$$

uniformly over $f \in \mathcal{R}(f_0)$. The weak sequential compactness of $\overline{\mathcal{R}(f_0)}^w$ now follows from the Dunford-Pettis criterion for weak compactness in $L^1(X)$. If $1 < p < 1$, then from Royden [40], Theorem 14 (pg. 171), every bounded sequence in $L^p(X)$ has a weakly convergent subsequence, meaning that bounded sets in $L^p(X)$ are weakly sequentially compact. Rearrangements have equal norms by Lemma 1.3, so $\overline{\mathcal{R}(f_0)}^w$ is bounded in $L^p(X)$, meaning the weak closure of the set of rearrangements is weakly sequentially compact. ■

For a special case, if $I = (0,\ 1)$ and $f \in L^1(I)$ is non-negative, Ryff [42] characterised the set $\overline{\mathcal{R}(f)}^w$ as

$$\overline{\mathcal{R}(f)}^w = \left\{ g \in L^1(I) \mid \int\limits_0^x g_\Delta \le \int\limits_0^x f_\Delta \ \text{ for } x \in I, g \ge 0, \text{ and } \ \| g_\Delta \|_1 = \| f_\Delta \|_1 \right\}.$$

Ryff also proved that $\overline{\mathcal{R}(f)}^w$ is the closed convex hull of $\mathcal{R}(f)$, and that the set of extreme points of $\overline{\mathcal{R}(f)}^w$ is simply $\mathcal{R}(f)$. An interesting consequence of Ryff's characterisation is that if f and g are rearrangements, then $\overline{\mathcal{R}(f)}^w = \overline{\mathcal{R}(g)}^w$.

1.4 Douglas's characterisation for the weak closure of the set of rearrangements

Here, we are interested by the characterisation of Douglas [17] for the weak closure of the set of rearrangements. Douglas has given a new characterisation for the weak closure of the set of rearrangements for a given function defined on an unbounded domain having infinite μ-measure. He first characterises the weak closure of the set of rearrangements for non-negative L^p functions defined on the half-line, where $1 < p < \infty$, and then by using the concept of measure preserving transformations, Douglas extended his results to non-negative L^p functions defined on unbounded domains of $\mathbb{R}^n$ having infinite measure with respect to Lebesgue measure μ_n, where $1 < p < \infty$. By using Definition 1.10, we have the following result

Theorem 1.4 *Let Ω be an open unbounded domain in $\mathbb{R}^n$ having infinite μ-measure, where μ is a non-zero a-finite outer regular measure. Further, let $\mu\{x\} = 0$ for each set consisting of a single point $x \in \Omega$. Let $(0, \infty)$ be endowed with Lebesgue measure. Then there exists a measure preserving transformation $T : (0, \infty) \longrightarrow \Omega$.*

Thus, if $A : L^p(\Omega) :\longrightarrow L^p(0,\ \infty)$ a map defined by

$$A(f) = f \circ T,$$

then A is well-defined, bounded and linear, with bounded inverse A^{-1}. Furthermore, for non-negative functions $f, g \in L^P(\Omega)$ we have

$$g \in \mathcal{R}(f) \text{ if and only if } A(g) \in \mathcal{R}(A(f)).$$

Definition 1.12 *Let f and g be two non-negative functions defined on $(0,\infty)$. We say g is a curtailment of f at $l \in [0,\infty]$ if*

$$g = 1_{(0,l)} f^{\Delta}$$

and g is a rearrangement of a curtailment of f at $l \in [0,\infty]$ if g^{Δ} is curtailment of f at some $l \in [0,\infty]$.

This concept together with Definition 1.12 show that if $f_0 \in L^p(\Omega)$ $(1 < p < \infty)$ is a non-negative function, then $g \in L^p(\Omega)$ is a rearrangement of a curtailment of f_0 if and only if $A(g)$ is a rearrangement of a curtailment of $A(f_0)$. We set

$$\mathcal{RC}(f_0) = \left\{ f \geq 0 \mid \exists \alpha \in [0,\infty] \text{ with } f^{\Delta}(s) = f_0^{\Delta}(s) \text{ pour tout } \ s \in [0,\ \alpha] \right\}.$$

and

$$\mathcal{W}(f_0) = \left\{ f \geq 0,\ f \text{ mes on } \Omega,\ \mid \int\limits_U (f-\alpha)_+ d\mu_n \leq \int\limits_U (f_0-\alpha)_+ d\mu_n \ \forall \alpha > 0 \right\},$$

where the subscript + stands for the positive part. The set $\mathcal{RC}(f_0)$ is called the of rearrangements of curtailments of f_0. Douglas proved that $\mathcal{W}(f_0)$ is the weak closure of the set of rearrangements of f_0, furthermore, he proved also that $\mathcal{W}(f_0)$ convex, weakly sequentially compact and the set of extreme points of $\mathcal{W}(f_0)$ is $\mathcal{RC}(f_0)$. All the above stages can be summarised by the following Theorem:

Theorem 1.5 *Let Ω an unbounded domain in $\mathbb{R}^n$ $(n \geq 1)$ of infinite Lebesgue μ_n measure and let $f_0 \in L^p(\Omega)$ $(p \geq 1)$ be a non-negative function. Then we have*

1. $\overline{\mathcal{R}(f_0)}^w = \mathcal{W}(f_0)$,
2. $\mathcal{W}(f_0)$ *is convex and weakly sequentially compact,*
3. *extrem*$(\mathcal{W}(f_0)) = \mathcal{RC}(f_0)$
4. $f \in \mathcal{W}(f_0)$, *then*

$$\|f\|_p \leq \|f_0\|_p, \tag{1.15}$$

moreover

$$\mathcal{R}(f_0) \subset \mathcal{RC}(f_0) \subset \mathcal{W}(f_0). \tag{1.16}$$

Chapter 2

Vortex flows

In this thesis we investigate the motion of the flow of an ideal (incompressible and inviscid) fluid of unit density in various domains in $\mathbb{R}^2$ and $\mathbb{R}^3$. Accordingly we now present the underlying equations describing the fluids motion, and some discussion of their consequences. In a two-dimensional domain Ω with no body forces, the fluid velocity V and the pressure ρ are required to satisfy Eulers equation, together with an incompressibility condition and the condition of tangency of n at the boundary.

2.1 Vortex rings

By a steady vortex ring we mean a figure of revolution Σ that is expected to be homeomorphic to a solid torus in most cases, and is associated with a continuous, axi-symmetrie, solenoidal vector field V (the fluid velocity) defined, in the case of an unbounded fluid, on the real three-dimensional Euclidean space $\mathbb{R}^3$ and having the following properties when we take axes fixed in the ring Σ.

(a) Both Σ and V do not vary with time,

(b) the vorticity field $\omega = \nabla \times V$ has positive magnitude in Σ, vanishes in $\mathbb{R}^3 - \Sigma$, and satisfies a non-linear equation of motion which, among other things, determines the boundary of Σ,

(c) V tends to a constant value at infinity in $\mathbb{R}^3$.

2.1.1 Dynamic's Equation

The motion of an incompressible axisymmetric fluid in $\mathbb{R}^3$ is governed by the following Euler's Equations

$$\frac{DV}{Dt} = \frac{\partial V}{\partial t} + (V.\nabla)V = -\nabla P, \tag{2.1}$$

$$\nabla.V = 0, \tag{2.2}$$

where P is the scalar pressure. Eq (2.1) is called the conservation of momentum equation and Eq (2.2) is called conservation of mass equation (incompressibility equation of the fluid). Since the flow should be axisymmetric without swirl, then with cylindrical coordinates (r, θ, z) in $\mathbb{R}^3$, incompressibility of the fluid (2.2) guarantees the existence of a Stokes stream function $\Psi : \Pi \longrightarrow \mathbb{R}$ for the flow, for which the velocity V field satisfies

$$V = (-\frac{1}{r}\frac{\partial \Psi}{\partial z}, 0, \frac{1}{r}\frac{\partial \Psi}{\partial r}), \tag{2.3}$$

where Π is the half-plane in $\mathbb{R}^3$ defined by $r > 0$, $\theta = 0$ and $z \in \mathbb{R}$. Then the vorticity is given in terms of the velocity by

$$\omega = \nabla \times \ V = (0,\ \bar{\omega},\ 0), \tag{2.4}$$

where

$$\bar{\omega} := \mathcal{L}\Psi \tag{2.5}$$

with

$$\mathcal{L} = -\frac{1}{r}\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}) - \frac{1}{r^2}\frac{\partial^2}{\partial z^2}.$$

Now, for $(r, z) \in \Pi$, we set

$$\zeta(r, z) := \frac{\bar{\omega}(r, z)}{r}. \tag{2.6}$$

The function $\zeta : \Pi \longrightarrow \mathbb{R}$ is called potential vorticity and the region

$$\mathbb{V} := \{(r, z) \in \Pi \ \mid \ \zeta(r, z) \neq 0\}$$

is called the vortex core. Now, by using the well-known identity

$$(V.\nabla)V = \frac{1}{2}\nabla|V|^2 + \omega \times V, \tag{2.7}$$

we get

$$\nabla \times (V.\nabla)V = \nabla \times (\omega \times V) = (V.\nabla)\omega - (\omega.\nabla)V, \tag{2.8}$$

where we have used the fact that $\nabla.V = 0$ and $\nabla.\omega = 0$. In view Eqs (2.3), (2.4) and (2.5), we deduce

$$(V.\nabla)\omega - (\omega.\nabla)V = (0,\ \partial\left(\frac{\bar{\omega}}{r}, \Psi\right),\ 0) \tag{2.9}$$

where $\partial\left(\frac{\bar{\omega}}{r}, \Psi\right)$ is the Jacobien between $\frac{\bar{\omega}}{r}$ and Ψ defined as follows

$$\partial\left(\frac{\bar{\omega}}{r}, \Psi\right) = \frac{\partial(\bar{\omega}/r)}{\partial r} \cdot \frac{\partial \Psi}{\partial z} - \frac{\partial(\bar{\omega}/r)}{\partial z} \cdot \frac{\partial \Psi}{\partial r}.$$

Thus, by using (2.6), (2.8) and (2.9), from (2.1) we obtain

$$\frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + \partial\left(\Psi, \zeta\right) = 0. \tag{2.10}$$

Therefore the total derivative $\frac{D\zeta}{Dt}$ following the motion is zero. In the other hand, in the steady state situation of axisymmetric flow $\frac{\partial \zeta}{\partial t} = 0$, we get

$$\partial\left(\Psi, \zeta\right) = 0,$$

therefore, the Stokes stream function Ψ and the function ζ are functionally dependent. in particular,

$$\zeta = \phi \circ (\Psi), \tag{2.11}$$

where $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is unknown function. Futhermore, if V is an uniform, then V tends a positive constant λ at infinty in the negative the z-direction, hence it follows that the Stokes stearm function Ψ tends to $-\frac{\lambda r^2}{2}$, here λ is the speed of the flow at infinity. The existence problem of steady vortex rings is to show that the non-linear equation

$$\mathcal{L}\Psi = \phi \circ (\Psi - \frac{\lambda}{2} r^2) \quad \text{in} \quad \Pi \tag{2.12}$$

has a solution with respect to the boundary conditions

$$(BC) \quad : \Psi(0, z) = 0, \Psi(r, z) \to 0 \text{ and } |\nabla \Psi| \to 0 \text{ as } r^2 + z^2 \to \infty$$

where Π is the half-plane that is defined by $r > 0$ and $-\infty < z < \infty$, λ is a positive constant corresponding to the speed that the flow approaches at infinity and $\phi : \mathbb{R} \to \mathbb{R}$ is an unknown increasing function. Hence by (2.11), $\Gamma(r, z) = \Psi(r, z) - \frac{\lambda}{2} r^2$ is the Stokes stream function for the flow.

2.1.2 Benjamins's approach

Benjamin [4] claimed that because $\frac{D\zeta}{Dt} = 0$, then the value of ζ for any infinitesimal ring of fluid particles remains constant, and the measure of any set of values assumed by ζ is therefore an invariant of the motion. In other words, the functions ζ of (r, z) evolved during any time interval are rearrangements of the initial function. His approach was concerned with seeking a solution of (2.12) with same boundary conditions (BC), for which ζ is a rearrangement of a prescribed function and for which a value is prescribed for either the speed λ at infinity, or the impulse $I(\zeta)$, which for a fluid of unit density is given by

$$I(\zeta) = \int_{\mathbb{R}^3} r\zeta.$$

The method Benjamin [4] proposed to achieve the existence of a solution is based on solving one of the following variational problems

$$\max_{\zeta \in \mathcal{R}(\zeta_0)} (E(\zeta) - \lambda I(\zeta)) \tag{2.13}$$

or

$$\max_{I(\zeta)=I, \zeta \in \mathcal{R}(\zeta_0)} E(\zeta) \tag{2.14}$$

where $E(\zeta)$ is a functional related to the kinetic energy, $\mathcal{R}(\zeta_0)$ is the set of rearrangements of a given function $\zeta_0 \geq 0$ and I is a positive number. Benjamin advocated as natural formulations on the grounds that $E(\zeta)$, $I(\zeta)$ and $\mathcal{R}(\zeta_0)$ remain constant even in unsteady. One the fact motivate this approach is that $I(\zeta)$ and the volumes of the sets $\{\zeta > \alpha\}$ (for each $\alpha > 0$) are preserved in axisymmetric motions of an ideal fluid in $\mathbb{R}^3$, so the quantities $I(\zeta)$ and the prescribed function ζ_0 are physically meaningful. The first variational problem (2.13) describes the existence of steady vortex rings with a prescribed λ and the second varianal problem (2.14) describes the existence of steady vortex rings with prescribed impluse.

In a bounded axisymmetric domain $\Omega \subset \mathbb{R}^3$, Burton [6] applied his theory to show that for each λ, the variational problem (2.12) has a solution ζ_λ that is satifying equation (2.12) almost every in Ω. Also, he showed that for each λ there exists $\zeta_\lambda \in \mathcal{R}(\zeta_0)$ that maximises the functional $E(\zeta)-\lambda I(\zeta)$ over $\mathcal{R}(\zeta_0)$, and if we set $\Psi := K\zeta_\lambda$, then Ψ satisfies the equation (2.12) almost everywhere in Ω, where $K\zeta_\lambda$ is the weak solution of $\mathcal{L}\Psi = \zeta$ belonging to $\mathcal{H}$, the Hilbert space obtained by completing $\mathcal{D}(\Omega)$ with the scalar product

$$< u, v >_H = \int\limits_{\Omega} \frac{2\pi}{r} \nabla u . \nabla v drdz. \tag{2.15}$$

For the second variational problem (2.14), Burton proved that if I satisfies a certain feasibility condition, then there exists $\zeta \in \mathcal{R}(\zeta_0)$ that maximises the functional $E(\zeta)$ subject to $I(\zeta) = I$ and $\zeta \in \mathcal{R}(\zeta_0)$ (the existence of steady vortex rings with a prescribed impulse), and if $\Psi := K\zeta$, then Ψ satisfies (2.12) almost everywhere in Ω, for some λ and some increasing function ϕ. In each cases $\Psi - \frac{\lambda}{2}r^2$ is the Stokes stream function for the flow.

Recently, the problem of the existence of steady vortex rings has been studied by Badiani and Burton [2] and Badiani [3], where they showed that the equation (2.12) with the boundary condition (BC) has a solution for which ζ/r belongs to $\mathcal{W}(\zeta_0)$, the weak closure of the set $\mathcal{R}(\zeta_0)$. The two authors found that for each $\lambda > 0$, there exists ζ_λ that maximises the functional $E(\zeta) - \lambda I(\zeta)$ over $\mathcal{W}(\zeta_0)$. Additionally, if we set $\Psi := K\zeta_\lambda$, then Ψ satisfies equation (2.12) almost everywhere in Π for some increasing function ϕ. Furthermore, they showed that if the prescribed function ζ_0 is positive and constant on its support, then ζ_λ is a rearrangement of ζ_0 on Π unless $\zeta_\lambda = 0$. For a prescribed impulse $I(\zeta)$, Burton [8] showed that the functional E can be maximised subject to $I(\zeta) = I > 0$ and $\zeta \in \mathcal{W}(\zeta_0)$. For the cylindrical domain $\Omega \in \mathbb{R}^3$ defined by $-\infty < z < +\infty$ and $0 < r < R$ (R constant), Douglas [17] showed that for all $\lambda > 0$, the functional $E(\zeta) - \lambda I(\zeta)$ attains a maximum value relative to $\mathcal{W}(\zeta_0)$, and every maximiser ζ_λ belongs to $\mathcal{RC}(\zeta_0)$, the set of rearrangements of curtailm ents of prescribed function ζ_0. Additionally, if the prescribed function $\zeta_0 \in L^1 \cap L^p$ where $p > 5/2$, then $\Psi := \widetilde{K}\zeta_\lambda$ satisfies the equation (2.12) almost everywhere in the domain Ω, where $\widetilde{K}\zeta_\lambda$ is the weak solution of $\mathcal{L}\Psi = \zeta$ belonging to $\mathcal{H}$, the Hilbert space which is obtained by completing $\mathcal{D}(\Omega)$ with the scalar product defined in (2.15). In this case also, Burton [13] showed that if λ is small enough and positive, then the functional $E(\zeta) - \lambda I(\zeta)$ attains a maximum value relative to $\mathcal{R}(\zeta_0)$. If ζ_λ is a maximiser and $\Psi := \widetilde{K}\zeta_\lambda$ the weak solution for $\mathcal{L}\Psi = \zeta$, then Ψ satisfies the equation

(2.12) almost everywhere in Ω, for some increasing function ϕ.

For different points of view, Fraenkel and Berger [23] proved the existence of steady vortex rings by proving the existence of a solution for the non-linear equation

$$\mathcal{L}\Psi = k\phi \circ (\Psi - \frac{\lambda}{2}r^2 - \gamma) \text{ in } \Pi \tag{2.16}$$

with respect to (BC), where $k > 0$, $\gamma \geq 0$, $\lambda > 0$ are prescribed and ϕ is a prescribed non-decreasing Hölder continuous function. Fraenkel and Berger achieved this solution by maximising a certain functional on the surface of a sphere in a Sobolev space with energy norm. In the case when $\gamma = 0$ and ϕ is the Heaviside function ($\phi(t) = 0$ if $t < 0$ and $\phi(t) = 1$ if $t > 0$), Fraenkel and Amick [22] showed that any solution in $\mathcal{H}$ for (2.16) is equal, modulo translation in the z-direction, to the explicit "spherical" solution found by Hill [26], where $\mathcal{H}$ is the Hilbert space obtained by completing $\mathcal{D}(\Pi)$ with the scalar product (2.15). By using the contraction mapping theorem, Norbury [34] proved existence of a family of steady vortex rings close to Hills spherical vortex ring. Ambrosetti and Struwe [1] used a minimax method to construct a solution for the equation (2.16) with prescribed k, γ and $\lambda > 0$.

The study of Ni [35] may be regarded as an extension of the work of Fraenkel and Berger [23]; he used a minimax principle to show that such a functional has a critical point which gives rise to an existence theorem for vortex rings. Friedman and Turkington [24] showed that the existence of steady vortex rings is given by studying a variational problem in which the impulse $I(\zeta) = 1$ and the essential supremum of the vorticity is less than or equal to a prescribed constant.

Chapter 3

A constrained variational problem for an existence theorem of steady vortex rings in Poiseuille flow

3.1 Introduction

In this Chapter, we prove an existence theorem for a steady 3-dimensional ideal fluid flow containing axisymmetric steady vortex rings in Poiseuille flow. The flow is written in terms of a Stokes stream function $\Psi : \mathbb{R}^3 \to \mathbb{R}$ with respect to cylindrical coordinates $(r, \theta, z) \in \mathbb{R}^3$, which is symmetric about the z-axis direction and approaches at infinity -$\frac{\lambda}{4}r^4$, which represents a flow of velocity field V $= (0, 0, -\lambda r^2)$, where λ is a parameter corresponding to the strength of the background flow at infinity; hence the velocity is increasing with respect to r along the z-direction. The magnitude of the vorticity ω is given then in terms of the Stokes stream function by $\omega/r = \mathcal{L}\Psi$, where

$$\mathcal{L} := -\left(\frac{1}{r}\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial z^2}\right). \tag{3.1}$$

The function $\zeta := \omega/r$ is called potential vorticity. The vorticity in the region $r > 0$ is non-negative and Ψ satisfies the equation

$$\mathcal{L}\Psi = \phi \circ (\Psi - \frac{\lambda}{4}r^4) \tag{3.2}$$

almost everywhere on $\Pi = \{(r,z) \in \mathbb{R}^2 \mid r > 0\}$, where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing function and λ is a positive constant, hence for Poiseuille flow we obtain $\zeta = \lambda$. Equation (3.2) for arbitrary ϕ represents the relationship that should exist between the vorticity and the Stokes stream function, when the flow is in a steady state, see Lamb [28], page 245. By using Burton's method [8], we prove that the energy E can be maximised subject to the function $\zeta \in \mathcal{R}(\zeta_0)$ and subject to another functional (called the generalised impulse due to vorticity) $\mathcal{I}_4$ is prescribed, moreover, we show that if $\zeta = \mathcal{L}\Psi$ is a maximiser, then there exists a positive number λ and an increasing function $\phi : \mathbb{R} \to \mathbb{R}$ for which (Ψ, ϕ, λ) is a solution for (3.2), where Ψ satisfies the boundary conditions $\Psi(0,z) = 0$ and $\Psi(r,z) \to 0$ as $r^2 + z^2 \to \infty$. Therefore, we prove the existence of a steady vortex ring in Poiseuille flow. This variational method has been proposed by Benjamin [4] in order to prove the existence of steady vortex rings in a uniform flow. In fact, Benjamin [4] observed that when a flow of an ideal fluid is in a steady state, the kinetic energy E and impulse $\mathcal{I}_2$ in the z-direction of cylindrical coordinates $(r,\theta,z) \in \mathbb{R}^3$ are preserved, and that in the presence of axisymmetry, the function ζ can be seen as a rearrangement of a prescribed function ζ_0, Benjamin [4] suggested that the maximiser of the kinetic energy E subject to the function ζ being a rearrangement of a prescribed function ζ_0 and subject to a prescribed impulse $\mathcal{I}_2$ should give rise to a steady vortex ring in the flow. Benjamin [4] also proposed another variational method based on the maximisation of $E - \lambda\mathcal{I}_2$ subject to ζ being a rearrangement, where λ is a positive parameter corresponding to the strength of the flow at infinity. By using this technigue, Rebah [37] proved an existence theory for a variational problem governing a steady vortex ring in Poiseuille flow; he maximised the functional $E - \lambda\mathcal{I}_4$, subject to $\zeta \in \mathcal{R}(\zeta_0)$, where λ is always the parameter corresponding to the strength of the flow at infinity. Rebah [37] showed that if λ is positive and small, then this functional attained its supremum relative to the set where the function ζ is a rearrangement of a prescribed function, furthermore, he proved that if $\zeta = \mathcal{L}\Psi$ is a maximiser, then there exists an increasing function $\phi : \mathbb{R} \to \mathbb{R}$ for which $(\Psi,\ \phi)$ is a solution for (3.2).

3.2 Mathematical formulation

For $X > 0$, let Π_X and $\Pi(X)$ be two subsets of Π defined by

$$\Pi_X = \{(r,z) \in \Pi \mid r < X\} \quad \text{and} \quad \Pi(X) = \{(r,z) \in \Pi \mid r^2 + z^2 < X^2\}.$$

We define a measure ν on Π having density $2\pi r$ with respect to the 2-dimensional Lebesgue measure μ_2, so if A is a 3-dimensional cylindrically set then the volume of A is $\nu(A \cap \Pi)$. The support of any function $\zeta : \Pi \to \mathbb{R}$ is a subset of Π denoted by supp ζ. For some $p \geq 1$, we define the space $L^p(\Pi, \nu)$ of all measurable functions ζ on Π such that $\|\zeta\|_p = (\int\limits_{\Pi} |\zeta(r,z)|^p d\nu)^{\frac{1}{p}} < \infty$. Now, for all (r,z) and (r',z') in Π, we set

$$G(r,r',z,z') = \frac{rr'}{8\pi^2} \int\limits_{-\pi}^{\pi} \frac{\cos\theta d\theta}{(r^2 + r'^2 - 2rr'\cos\theta + (z-z')^2)^{\frac{1}{2}}}; \tag{3.3}$$

then G is the Green's function for the elliptic partial differential operator

$$\mathcal{L} = -\left(\frac{1}{r}\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial z^2} \right)$$

with homogeneous Dirichlet boundary conditions on Π, see Lamb [28], page 237. By setting $a = r^2 + r'^2 + (z-z')^2$ and $b = rr'$, then there exists a number $C \in (\frac{\sqrt{2}}{8}, 2)$, possibly dependent on a and b, for which

$$G(r,r',z,z') = \frac{Cb^2}{\pi^2 a\overline{\rho}} \sinh^{-1} \frac{\overline{\rho}}{\rho}, \tag{3.4}$$

where $\rho^2 = a - 2b$ and $\overline{\rho}^2 = a + 2b$, see Rebah [37], Lemma 4.2. If $\zeta \in L^1(\Pi,\nu) \cap L^p(\Pi,\nu)$ for some $p > 1$, then for all $(r,z) \in \Pi$, we define the function $K\zeta$ by

$$K\zeta(r,z) := \int\limits_{\Pi} G(r,r',z,z')\zeta(r',z')2\pi r'dr'dz'.$$

Moreover, if $p > \frac{5}{2}$ then $K\zeta$ is the weak solution in the distribution sense for the problem $\mathcal{L}\Psi = \zeta$ in Π. Furthermore, if $\Omega \subset \Pi$ is a bounded domain, then $K : L^p(\Omega,\nu) \longrightarrow L^q(\Omega,\nu)$ is a symmetric, positive and compact operator, where q is the conjugate exponent of p, see Burton [6]. The kinetic energy E, the impulse $\mathcal{I}_2$ and the generalised impulse $\mathcal{I}_4$ due to the vorticity are respectively given by

$$E(\zeta) = \frac{1}{2}\int\limits_{\Pi} \zeta(r,z)K\zeta(r,z)d\nu \ , \ \ \mathcal{I}_2(\zeta) = \frac{1}{2}\int\limits_{\Pi} r^2\zeta(r,z)d\nu \quad \text{and} \quad \mathcal{I}_4(\zeta) = \frac{1}{4}\int\limits_{\Pi} r^4\zeta(r,z)d\nu.$$

If ζ_1 and ζ_2 are two non-negative measurable functions and vanishing outside a set of finite volume, we say that ζ_1 is a rearrangement of ζ_2 or ζ_2 is a rearrangement of ζ_1 with respect to ν measure if $\nu(\zeta_1^{-1}[t,\infty)) = \nu(\zeta_2^{-1}[t,\infty))$ for all $t > 0$. In case $\zeta_1 \in L^p(\Pi,\nu)$ where $p \geq 1$, then it follows that $\|\zeta_1\|_p = \|\zeta_2\|_p$; hence $\zeta_2 \in L^p(\Pi,\nu)$. Now, let $\Sigma \subset \Pi$ be a subset of finite volume and let $\zeta : \Sigma \longrightarrow \mathbb{R}_+$ be a measurable function and for $\alpha > 0$, we set

$$\mathbf{F}(\alpha) = \nu(\{(r,z) \in \Sigma \mid \zeta(r,z) > \alpha\}.$$

Then $\mathbf{F}$ is called the distribution function of ζ on Σ. The decreasing rearrangement of ζ on Σ is a decreasing function ζ^{Δ} on $(0,\ \nu(\Sigma))$ that is given by

$$\zeta^{\Delta}(t) := \max\{\alpha > 0 \mid \mathbf{F}(\alpha) \geq t\} \tag{3.5}$$

for all $0 < t < \nu(\Sigma)$. Moreover, if ζ_1 and ζ_2 are two non-negative functions in $L^p(\Sigma,\ \nu)$ and $L^q(\Sigma,\ \nu)$ respectively, where $p > 1$ and q the conjugate exponent of p, then the inequality

$$\int\limits_{\Sigma} \zeta_1(r,z)\zeta_2(r,z)d\nu \leq \int\limits_{0}^{\nu(\Sigma)} \zeta_1^{\Delta}(t)\zeta_2^{\Delta}(t)dt \tag{3.6}$$

is classical for a proof. See for example Burton [6]. Any measurable function ζ on Π is called Steiner-symmetric if ζ satisfies

$$z \geq z' \geq 0 \Rightarrow \zeta(r,z') \geq \zeta(r,z) \geq 0 \quad \text{and} \quad z \in \mathbb{R} \Rightarrow \zeta(r,-z) = \zeta(r,z).$$

We define the Steiner-symmetrisation of ζ_0 as the essentially unique rearrangement ζ_0^s satisfying

$$\mu_1(\{z \mid \zeta_0^s(r,z) \geq t\}) = \mu_1(\{z \mid \zeta_0(r,z) \geq t\})$$

for every $t > 0$ and almost every $r > 0$. For every rearrangement ζ of ζ_0 on Π, the Steiner-symmetrisation ζ^s also lies in $\mathcal{R}(\zeta_0)$, the set of all rearrangements of ζ_0. Also, with the same function ζ_0, we define $\mathcal{W}(\zeta_0)$ and $\mathcal{RC}(\zeta_0)$ as follows:

$$\begin{aligned}\mathcal{RC}(\zeta_0) &= \{\zeta \geq 0 \quad \text{measurable on} \quad \Pi | \zeta^{\Delta} = \zeta_0^{\Delta} 1_{[0,\beta]} \text{ for some } \beta > 0\} \\ \mathcal{W}(\zeta_0) &= \{\zeta \geq 0 \quad \text{measurable on} \quad \Pi | \forall k > 0 \int_{\Pi} (\zeta - k)_+ \leq \int_{\Pi} (\zeta_0 - k)_+ \},\end{aligned}$$

where the subscript + stands for the positive part. Following Burton [8] and Douglas [17, 18] we have $\mathcal{R}(\zeta_0) \subset \mathcal{RC}(\zeta_0) \subset \mathcal{W}(\zeta_0)$. In the case of a non-negative function $\zeta_0 \in L^p(\Pi, \nu)$, where $1 < p < \infty$, Douglas [17, 18] proved that $\mathcal{W}(\zeta_0)$ is convex and weakly sequentially compact, and that, the set of extreme points of $\mathcal{W}(\zeta_0)$ is the set $\mathcal{RC}(\zeta_0)$. The set $\mathcal{W}(\zeta_0)$ is the weak closure of the set $\mathcal{R}(\zeta_0)$ in L^p, and $\mathcal{RC}(\zeta_0)$ will be called the set of all rearrangements of curtailments of ζ_0. Hence, if $\zeta \in \mathcal{W}(\zeta_0)$, then it follows that $\|\zeta\|_p \leq \|\zeta_0\|_p$. Now with all these notations and formulations, we are able to present our main result as follows

Theorem 3.1 *Let* $I > 0$*, let* $p > \frac{5}{2}$ *and let* $\zeta_0 \in L^p(\Pi, \nu)$ *be a non-negative function with compact support. Then*

1. *the functional* E *attains a maximum value subject to* $\zeta \in \mathcal{W}(\zeta_0)$ *and* $\mathcal{I}_4(\zeta) = I$*,*
2. *all maximisers are Steiner-symmetric elements of* $\mathcal{RC}(\zeta_0)$*,*
3. *for any maximiser* ζ*, there exist a positive* λ *and an increasing function* ϕ *such that the function* $\Psi := K\zeta$*,* λ *and* ϕ *satisfy Equation (3.2) almost everywhere in* Π.

4. *There exists a number* $I_* > 0$ *such that if* $I > I_*$ *and* ζ *a maximiser, then* $\zeta \in \mathcal{R}(\zeta_0)$.

In this theorem, the number λ arises as Lagrange multiplier for the constraint $\mathcal{I}_4(\zeta) = I$. Note that in this Chapter and Rebah [37], we construct a solution for the equation (3.2). In Theorem 3.1, the maximiser ζ will be shown to give rise to a solution Ψ of the boundary-value problem for axisymmetric steady vortex rings. Thus, Ψ is just a function of (r, z) only in cylindrical coordinates and satisfies (3.2), $\Psi(r, z) = 0$ when $r = 0$ and $\Psi(r, z) \to 0$ as $r^2 + z^2 \to \infty$. Therefore, the function $\Psi - \frac{\lambda}{4}r^4$ is the Stokes stream function of a steady ideal fluid flow, whose velocity in $\mathbb{R}^3$ is given by

$$\mathrm{V} = \left(-\frac{1}{r}\frac{\partial \Psi}{\partial z}, 0, \frac{1}{r}\frac{\partial \Psi}{\partial r} - \lambda r^2 \right).$$

Hence the vorticity in terms of Ψ is given by

$$\mathrm{curl}\mathrm{V} = (0, r(\mathcal{L}\Psi - \lambda), 0).$$

3.3 Some estimates for the function $K\zeta$

We start this section by stating some estimates for the function $K\zeta$ which has been proved by Rebah [39], where $\zeta \in L^1 \cap L^p(\Pi, \nu)$ is any non-negative function, where $p > 2$.

Lemma 3.1 *There exists a positive number* C *depends on* p *such that for all* $r > 2$ *and for all* $z \in \mathbb{R}$,

$$K\zeta(r, z) \leq C(\|\zeta\|_1 + \|\zeta\|_p) r \log r.$$

Lemma 3.2 *If ζ is Steiner-symmetric, then for any $\alpha \in]0,\ 1[$ and for all $(r, z) \in \Pi$ with $z \neq 0$, there exists a positive number depending on p and α for which the following inequality*

$$K\zeta(r, z) \leq C\mathcal{I}_2(\zeta) r^{-(1-\alpha)}|z|^{-\alpha}.$$

holds.

Lemma 3.3 *If ζ is Steiner-symmetric, then for any $\alpha \in]0,\ 1[$ and for all $(r, z) \in \Pi$ with $|z| \geq 2$, there exists a positive number C depending on p and α for which the following inequality*

$$K\zeta(r, z) \leq C\|\zeta\|_1\left(\frac{r^{1+\alpha}}{|z|} + \frac{r^2}{|z|}\log|z|\right)$$

holds.

Proof. First of all, we observe that if $1 \leq t < \frac{1}{2}|z|$, then $|t+z| \geq |-t+z| \geq \frac{1}{2}z$ if $z > 0$ and $|-t+z| \geq |t+z| \geq -\frac{1}{2}z$ if $z < 0$. Also, if $0 < t < 1$, then $|t+z| \geq |-t+z| \geq (z-1) \geq \frac{1}{2}z$ if $z > 0$ and $|-t+z| \geq |t+z| \leq (-z-1) \geq -\frac{1}{2}z$ if $z < 0$. Thus, if $|z| \geq 2$ and $0 < t < \frac{1}{2}|z|$, then

$$\frac{1}{|t+z|} + \frac{1}{|-t+z|} \leq \frac{4}{|z|} \tag{3.7}$$

We recall now that if $\zeta \in L^1(\Pi, \nu)$ is Steiner-symmetric, then for all $z \neq 0$ we have

$$V(z) := \int_0^\infty \zeta(r, z) 2\pi r dr \leq \frac{\|\zeta\|_1}{|z|}. \tag{3.8}$$

We assume $|z| \geq 2$ and set $z - z' = \pm t$. Then, we can write

$$K\zeta(r, z) = \left(\int\limits_{|t|<1} + \int\limits_{1\leq|t|<\frac{1}{2}|z|} + \int\limits_{|t|\geq\frac{1}{2}|z|} \right) G(r, r', z, z')\zeta(r', z')2\pi r' dr' dz'. \tag{3.9}$$

Now, for $(r,z) \in \Pi$ and $(r',z') \in \Pi$, we set $a = r^2 + r'^2 + (z-z')^2$, $b = rr'$, $\overline{\rho}^2 = a + 2b$ and $\rho^2 = a - 2b$. Since $a > r'^2$ and $\rho > |z - z'|$, then from (3.4) we have

$$G(r,r',z,z') \leq \frac{b^2}{a\rho} \leq \frac{r^2}{|z-z'|}. \tag{3.10}$$

Thus, we get

$$\int\limits_{|t| \geq \frac{1}{2}|z|} G(r,r',z,z')\zeta(r',z')2\pi r' dr' dz' \leq 2\|\zeta\|_1 r^2 |z|^{-1}. \tag{3.11}$$

Now, using (3.8) and (3.7) we have

$$\begin{aligned}
\int\limits_{1 \leq |t| < \frac{1}{2}|z|} \frac{\zeta(r',z')}{|t|} 2\pi r' dr' dz' &= \int\limits_{1}^{\frac{1}{2}|z|} \int\limits_{0}^{\infty} \left(\frac{\zeta(r', t+z)}{|t|} + \frac{\zeta(r', -t+z)}{|t|} \right) 2\pi r' dr' dt \\
&\leq \|\zeta\|_1 \int\limits_{1}^{\frac{1}{2}|z|} \left(\frac{1}{|t+z|} + \frac{1}{|-t+z|}\right) \frac{dt}{|t|} \qquad (3.12) \\
&\leq 4\|\zeta\|_1 \frac{\log|z|}{|z|}. \qquad (3.13)
\end{aligned}$$

Hence, by using (3.10) and (3.12), we obtain

$$\int\limits_{1 \leq |t| \leq \frac{1}{2}|z|} G(r,r',z,z')\zeta(r',z')2\pi r' dr' dz' \leq 4\|\zeta\|_1 r^2 \frac{\log|z|}{|z|}. \tag{3.14}$$

Since $\sinh^{-1} x \leq \frac{2^\alpha}{\alpha} x^\alpha$ for all $x > 1$ and for any $\alpha \in (0,\ 1)$, then by taking $x = \frac{\overline{\rho}}{\rho}$ and using $a > r'^2$, from (3.4) again, we get

$$G(r,r',z,z') \leq \frac{2^\alpha r^{1+\alpha}}{\alpha |z-z'|^\alpha}, \tag{3.15}$$

therefore,

$$\begin{aligned}\int\limits_{|t|<1} \frac{\zeta(r',z')}{|t|^\alpha} 2\pi r' dr' dz' &= \int\limits_0^1 \int\limits_0^\infty \left(\frac{\zeta(r',t+z)}{|t|^\alpha} + \frac{\zeta(r',-t+z)}{|t|^\alpha} \right) 2\pi r' dr' dt \\ &\leq 2\|\zeta\|_1 \int\limits_0^1 \left(\frac{1}{|t+z|} + \frac{1}{|-t+z|} \right) \frac{dt}{|t|^\alpha} \qquad (3.16) \\ &\leq \frac{8\|\zeta\|_1}{(1-\alpha)|z|}, \qquad (3.17)\end{aligned}$$

where we have used (3.8) to obtain the second line and (3.7) to obtain the last line. Thus, from (3.15) and (3.16), we find that

$$\int\limits_{|t|<1} G(r,r',z,z')\zeta(r',z') 2\pi r' dr' dz' \leq \frac{2^{\alpha+3}}{\alpha(1-\alpha)} \|\zeta\|_1 r^{1+\alpha} |z|^{-1}. \qquad (3.18)$$

Therefore, we can choose a positive number C which depends on α, for which the desired estimate follows from (3.9), (3.11), (3.14) and (3.18). ■

Lemma 3.4 *Let* $R = (r,z)$ *and* $R' = (r',z')$ *denote general points of* Π. *Let* ζ *be a non-negative function. Then*

$$K\zeta(r,z) \geq \frac{\sqrt{2}\log 2\, r^2}{32\pi^2(1+|R|^3)} F(\zeta) \quad \textit{where} \quad F(\zeta) = \int\limits_\Pi \frac{r'^2}{1+|R'|^3} \zeta(r',z') 2\pi r' dr' dz'. \qquad (3.19)$$

Proof. We set $a = r^2 + r'^2 + (z-z')^2$ and $b = rr'$. Then from (3.4), it is easy to show

$$G(r,r',z,z') \geq \frac{\sqrt{2} b^2 \log 2}{8\pi^2 a \overline{\rho}} \geq \frac{(\sqrt{2}\log 2) b^2}{32\pi^2(1+|R|^3)(1+|R'|^3)},$$

where we have used the fact that $a\overline{\rho} \leq 4(1+|R|^3)(1+|R'|^3)$. Therefore, the result follows from the above inequality. ■

Lemma 3.5 *Let* λ *be a positive number and* $F(\zeta)$ *defined as in Lemma 3.4. Then for all* $\lambda \in (0, \frac{F(\zeta)}{16\pi^2})$

$$\nu(\{(r,z) \in \Pi \mid K\zeta(r,z) - \frac{\lambda}{4}r^4 > 0\}) \geq \frac{1}{3}(\sqrt{\frac{F(\zeta)}{\lambda}} - 4\pi). \tag{3.20}$$

Proof. By Lemma 3.4, we have

$$K\zeta(r,z) \geq \frac{(\sqrt{2}\log 2)r^2}{32\pi^2(1+|R|^3)}F(\zeta) \geq \frac{r^4}{64\pi^2(1+|R|^3)^2}F(\zeta).$$

Then,

$$\nu(\{(r,z) \in \Pi | K\zeta(r,z) - \frac{\lambda}{4}r^4 > 0\}) \geq \nu(\{|R| \leq (\frac{1}{4\pi}\sqrt{\frac{F(\zeta)}{\lambda}} - 1)^{\frac{1}{3}}\}).$$

This completes the proof. ■

3.4 Some properties for the functional E

In this section, we will use all the previous estimates for the function $K\zeta$ to yield some properties for the functional E, that are needed for the proof of Theorem 3.1. We start this section with a result similar to one proved by [Rebah[38], Lemma 3.7].

Lemma 3.6 *Let* $X > 0$ *and let* $\zeta \in L^p(\Pi_X, \nu)$ *be a non-negative function having support of finite volume, where* $p \geq 2$*. Let* $\lambda > 0$ *and let* $\Psi := K\zeta - \frac{\lambda}{4}r^4$ *be such that* $\zeta = \phi \circ \Psi$ *almost everywhere in* Π_X *for some increasing function* ϕ*. Suppose that* ϕ *has a non-negative indefinite integral* F*. Then*

$$\int_{\Pi_X} F(\Psi)d\nu \geq -\frac{1}{6}\int_{\Pi_X} \zeta(r,z)K\zeta(r,z)d\nu + \frac{\lambda}{3}\int_{\Pi_X} r^4\zeta(r,z)d\nu.$$

If additionally $F(s) = 0$ *for* $s \leq \beta$, *then*

$$\int\limits_{\Pi_X} \zeta(r,z) K\zeta(r,z) d\nu \geq \frac{\lambda}{2} \int\limits_{\Pi_X} r^4 \zeta(r,z) d\nu + \frac{6\beta}{7} \|\zeta\|_1.$$

Proof. We use the same procedure used by Rebah [38] to prove Lemma 3.7. Indeed, for $\zeta \in L^p(\Pi_X, \nu)$, let (ζ_j) be a sequence defined by $\zeta_j = \zeta 1_{(0,X)\times(-j,j)}$, so (ζ_j) converges strongly to ζ in L^p for all $2 \leq p < \infty$. Also by [8], Lemma 4(i), $(K\zeta_j)$ converges strongly to $K\zeta$ in L^∞ and then by using the Monotone Convergence Theorem, we have

$$\int\limits_{\Pi_X} \zeta_j(r,z) K\zeta_j(r,z) d\nu \to \int\limits_{\Pi_X} \zeta(r,z) K\zeta(r,z) d\nu \quad \text{as} \quad j \to \infty \tag{3.21}$$

We set $\Pi(j,X) = (0,X) \times (-j,j)$. Then it follows that

$$\begin{aligned} \int\limits_{\Pi(j,X)} R \cdot \nabla(\Psi + \frac{\lambda}{4} r^4) \zeta(r,z) d\nu &= \int\limits_{\Pi(j,X)} \left(r\frac{\partial \Psi}{\partial r} + z \frac{\partial \Psi}{\partial z} \right) \zeta(r,z) d\nu + \lambda \int\limits_{\Pi(j,X)} r^4 \zeta(r,z) d\nu \\ &= \int\limits_{\Pi(j,X)} R \cdot \nabla(F \circ \Psi)(r,z) d\nu + \lambda \int\limits_{\Pi(j,X)} r^4 \zeta(r,z) d\nu, \end{aligned} \tag{3.22}$$

where, we have used the fact that $\nabla (F \circ \Psi) = (\phi \circ \Psi) \nabla \Psi$. Since $\zeta \in L^p(\Pi, \nu)$ has a support of finite volume, then it follows that $\zeta \in L^1(\Pi, \nu)$, hence $\phi \in L^1$. Therefore , $F \circ \Psi(r,.)$ and $F \circ \Psi(.,z)$ are absolutely continuous for almost every r and z respectively. Thus, integration by parts relative to z yields that

$$\int\limits_{-j}^{j} z \frac{\partial}{\partial z} F(\Psi) dz = j \left[F(\Psi(r,j)) - F(\Psi(r,-j)) \right] - \int\limits_{-j}^{j} F(\Psi) dz$$

for almost all r; hence by using the same calculation for almost all z,

$$\int\limits_{0}^{X} r^2 \frac{\partial}{\partial r} F(\Psi) dr = X^2 F(\Psi(X,z)) - 2 \int\limits_{0}^{X} F(\Psi) r dr.$$

Therefore by substituting into (3.22) and using the fact $\Psi + \frac{\lambda}{4}r^4 = K\zeta$ and $F \geq 0$, we obtain

$$\int\limits_{\Pi(j,X)} \left(R \cdot \nabla K\zeta(r,z)\right)\zeta(r,z)d\nu \geq -3 \int\limits_{\Pi(j,X)} F(\Psi)d\nu + \lambda \int\limits_{\Pi(j,X)} r^4\zeta(r,z)d\nu. \tag{3.23}$$

Since ζ_j has a bounded support, then by using [[24], Lemma 3.1] we have

$$\int\limits_{\Pi_X} \left(R \cdot \nabla K\zeta_j(r,z)\right)\zeta_j(r,z)d\nu = \frac{1}{2}\int\limits_{\Pi_X} \zeta_j(r,z)K\zeta_j(r,z)d\nu.$$

Thus, by using (3.21) and letting $j \to \infty$, the result is

$$\int\limits_{\Pi_X} F(\Psi)d\nu \geq -\frac{1}{6}\int\limits_{\Pi_X} \zeta(r,z)K\zeta(r,z)d\nu + \frac{\lambda}{3}\int\limits_{\Pi_X} r^4\zeta(r,z)d\nu. \tag{3.24}$$

Suppose now that $F(s) = 0$ for $s \leq \beta$, then $\phi(s) = 0$ for $s \leq \beta$ and ϕ is an increasing function; hence

$$F(s) = \int\limits_{\beta}^{s} \phi(t)dt \leq (s-\beta)\phi(s).$$

Therefore, we have

$$\begin{aligned}
\int\limits_{\Pi_X} \zeta(r,z)K\zeta(r,z)d\nu &= \int\limits_{\Pi_X} (\Psi + \frac{\lambda}{4}r^4)\zeta(r,z)d\nu \\
&= \int\limits_{\Pi_X} (\Psi - \beta)F^{'}(\Psi)d\nu + \beta\int\limits_{\Pi_X} \zeta(r,z)d\nu + \frac{\lambda}{4}\int\limits_{\Pi_X} r^4\zeta(r,z)d\nu \\
&\geq \int\limits_{\Pi_X} F(\Psi)d\nu + \beta\|\zeta\|_1 + \frac{\lambda}{4}\int\limits_{\Pi_X} r^4\zeta(r,z)d\nu \\
&\geq \frac{\lambda}{2}\int\limits_{\Pi_X} r^4\zeta(r,z)d\nu + \frac{6\beta}{7}\|\zeta\|_1,
\end{aligned} \tag{3.25}$$

where we have used the inequality (3.24) to obtain the last line. ■

Lemma 3.7 *Let* $\frac{3}{4} < \delta < 1$*, let* $X > 0$ *and let* $O(X)$ *be a subset defined by*

$$O(X) = \left\{(r,z) \in \Pi \mid r^2 + z^2 \geq X^2\right\}.$$

Let $p \geq 1$ *and let* $\zeta \in L^p(\Pi,\nu)$ *be a Steiner-symmetric function having support of finite volume. Then we can choose a positive number* C *depending only on* δ *for which*

$$\int\limits_{O(X)} \zeta(r,z)K\zeta(r,z)d\nu \leq C_\delta \|\zeta_0\|_1^{\frac{2+\delta}{2}} \frac{(\mathcal{I}_4(\zeta))^{\frac{2-\delta}{2}}}{X^{3-2\delta}}. \tag{3.26}$$

Proof. Let $0 < \delta < 1$, let $\alpha_1 = 1 - \delta + \frac{\alpha}{2}$ and let $\alpha_2 = 1 - \alpha$, where $0 < \alpha < \frac{2}{3}\delta$. Then

$$\beta_1 := \int\limits_0^{\frac{\pi}{4}} \frac{d\theta}{(\cos\theta)^{\frac{2(3-2\delta)-3\alpha}{2(3-2\delta)}} (\sin\theta)^{\frac{3(2-2\delta+\alpha)}{2(3-2\delta)}}} < \infty \tag{3.27}$$

and

$$\beta_4 := \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d\theta}{(\cos\theta)^{\frac{3-4\delta+3\alpha}{3-2\delta}} (\sin\theta)^{\frac{3(1-\alpha)}{3-2\delta}}} < \infty. \tag{3.28}$$

Now, we write $O(X) = O_1(X) \cup O_2(X)$, where

$$O_1(X) = \{(r,z) \in O(X) \;\; r \geq |z|\} \text{ and } O_2(X) = \{(r,z) \in O(X) \;\; r < |z|\}.$$

Since ζ is Steiner-symmetric, then by Lemma 3.2 we can have

$$\begin{aligned}
\int\limits_{O(X)} \zeta(r,z)K\zeta(r,z)d\nu &= \sum_{i=1}^{2} \int\limits_{O_i(X)} \zeta(r,z)K\zeta(r,z)d\nu \\
&\leq \mathcal{I}_2(\zeta) \sum_{i=1}^{2} C_i \int\limits_{O_i(X)} \frac{\zeta(r,z)d\nu}{r^{1-\alpha_i}|z|^{\alpha_i}} \\
&= \mathcal{I}_2(\zeta) \sum_{i=1}^{i=2} C_i \int\limits_{O_i(X)} r^{2(1-\delta)}\zeta^{1-\delta}(r,z) \frac{\zeta^\delta(r,z)}{r^{2(1-\delta)+1-\alpha_i}|z|^{\alpha_i}} d\nu \\
&\leq C(\mathcal{I}_2(\zeta))^{2-\delta} \sum_{i=1}^{i=2} \left(\int\limits_{O_i(X)} \frac{\zeta(r,z)d\nu}{r^{\frac{2(1-\delta)+1-\alpha_i}{\delta}}|z|^{\frac{\alpha_i}{\delta}}} \right)^{\delta}
\end{aligned} \tag{3.29}$$

where we have used Hölder's inquality to get the last line, here $C = \max\{C_1, C_2\}$. Now, we set $S = \mathrm{supp}\zeta$, $\Sigma_1 = O_1(X) \cap S$, $\Sigma_2 = O_2(X) \cap S$. Since $\nu(S) < \infty$, then it follows that $\nu(\Sigma_i) < \infty$ for each $i \in \{1, 2\}$. Now, given $\sigma > 0$, we define $F_1(\sigma)$, $F_2(\sigma)$, $\widetilde{F_1}(\sigma)$ and $\widetilde{F_2}(\sigma)$ as

$$F_1(\sigma) = \nu(\{(r,z) \in \Sigma_1 \mid f_1(r,z) > \sigma\}), \quad \widetilde{F_1}(\sigma) = \nu(\{(r,z) \in O_1(X_0) \mid f_1(r,z) > \sigma\}),$$

$$F_2(\sigma) = \nu(\{(r,z) \in \Sigma_2 \mid f_2(r,z) > \sigma\}) \text{ and } \widetilde{F_2}(\sigma) = \nu(\{(r,z) \in O_2(X_0) \mid f_2(r,z) > \sigma\}),$$

where $f_1(r,z) = r^{-\frac{4-2\delta-\alpha}{2\delta}}|z|^{-\frac{2-2\delta+\alpha}{2\delta}}$ and $f_2(r,z) = r^{-\frac{2(1-\delta)+\alpha}{\delta}}|z|^{-\frac{1-\alpha}{\delta}}$. Then F_1, $\widetilde{F_1}$ are the distribution functions of f_1 on Σ_1 and $O_1(X)$ respectively, and F_2, $\widetilde{F_2}$ are the distribution functions of f_2 on Σ_2 and $O_2(X)$ respectively. Since, $\Sigma_i \subset O_i(X)$ for $i \in \{1,\ 2\}$, then it follows that $F_i(\sigma) \leq \widetilde{F_i}(\sigma)$ for each $\sigma > 0$; moreover for each $\xi > 0$ we have

$$\{\sigma > 0 \mid F_i(\sigma) > \xi\} \subset \{\sigma > 0 \mid \widetilde{F_i}(\sigma) > \xi\}. \tag{3.30}$$

Now, we set $r = |R|\cos\theta$ and $z = |R|\sin\theta$. If $(r,z) \in \{(r,z) \in O_1(X) \mid f_1(r,z) > \sigma\}$, then $X < |R| < \sigma^{-\frac{\delta}{3-2\delta}}(\cos\theta)^{-\frac{4-2\delta-\alpha}{2(3-2\delta)}}(\sin\theta)^{-\frac{2-2\delta+\alpha}{2(3-2\delta)}}$ with $\theta \in (-\frac{\pi}{4},\ \frac{\pi}{4})$, hence

$$\begin{aligned} \widetilde{F_1}(\sigma) &= 2\int_0^{\frac{\pi}{4}} \int_X^{\sigma^{-\frac{\delta}{3-2\delta}}(\cos\theta)^{-\frac{4-2\delta-\alpha}{2(3-2\delta)}}(\sin\theta)^{-\frac{\alpha}{3-2\delta}}} 2\pi|R|^2\cos\theta d|R|d\theta \\ &= \frac{4\pi}{3}\left(\frac{\beta_1}{\sigma^{\frac{3\delta}{3-2\delta}}} - \beta_2 X^3\right), \end{aligned} \tag{3.31}$$

where β_1 is defined as in (3.27) and $\beta_2 = \frac{\sqrt{2}}{2}$. Thus, it follows that

$$\{\sigma > 0 \mid \widetilde{F}(\sigma) > t\} \subset \left(0,\ \left(\frac{\beta_1}{\frac{3t}{4\pi} + \beta_2 X^3}\right)^{\frac{3-2\delta}{3\delta}}\right) \subset (0,\ \frac{\beta_3}{X^{\frac{3-2\delta}{\delta}}}), \tag{3.32}$$

where $\beta_3 = (\frac{\beta_1}{\beta_2})^{\frac{3-2\delta}{3\delta}}$. Therefore, from (3.30) and (3.32) we find

$$\{\sigma > 0 \mid F_1(\sigma) > \xi\} \subset (0, \ \frac{\beta_3}{X^{\frac{3-2\delta}{\delta}}}). \tag{3.33}$$

Furthermore, by combining (3.5) with (3.32), for all $0 < \xi < \nu(\Sigma_1)$ we have

$$f_1^*(\xi) \leq \frac{\beta_3}{X^{\frac{3-2\delta}{\delta}}}. \tag{3.34}$$

Now, let ζ^Δ be the decreasing rearrangement of ζ on Σ_1. Then by using (3.6),

$$\begin{aligned}\int\limits_{O_1(X)} \frac{\zeta(r,z)}{r^{\frac{4-2\delta-\alpha}{2\delta}}|z|^{\frac{2-2\delta+\alpha}{2\delta}}}d\nu &= \int\limits_{O_1(X)} f_1(r,z)\zeta(r,z)d\nu \\ &\leq \int\limits_0^{\nu(\Sigma)} \zeta^\Delta(\xi)f_1^*(\xi)d\xi \\ &\leq \frac{\beta_3\|\zeta_0\|_1}{X^{\frac{3-2\delta}{\delta}}}, \end{aligned} \tag{3.35}$$

where we have used (3.34) and the fact $\int\limits_0^{\nu(\Sigma)} \zeta^\Delta(t)dt \leq \|\zeta_0\|_1$ to obtain the last line, since $\zeta \in \mathcal{W}(\zeta_0)$. Now, performing the same calculation we find

$$\int\limits_{O_2(X)} \frac{\zeta(r,z)}{r^{\frac{2(1-\delta)+\alpha}{\delta}}|z|^{\frac{1-\alpha}{\delta}}}d\nu \leq \frac{\beta_6\|\zeta_0\|_1}{X^{\frac{3-2\delta}{\delta}}}, \tag{3.36}$$

where $\beta_6 = (\frac{\beta_4}{\beta_5})^{\frac{3-2\delta}{3\delta}}$ with β_4 is defined as in (3.28) and $\beta_5 = \frac{2-\sqrt{2}}{2}$. Finally, using Hölder's inquality, we have

$$\mathcal{I}_2(\zeta) = \frac{1}{2}\int\limits_\Pi r^2\zeta^{\frac{1}{2}}(r,z)\zeta^{\frac{1}{2}}(r,z)d\nu \leq \frac{1}{2}\|\zeta\|_1^{\frac{1}{2}}\left(\int\limits_\Pi r^4\zeta(r,z)d\nu\right)^{\frac{1}{2}} = \|\zeta\|_1^{\frac{1}{2}}(\mathcal{I}_4(\zeta))^{\frac{1}{2}}. \tag{3.37}$$

Therefore, our inequality follows immediately from (3.29), (3.35), (3.36) and (3.37), where C depends only on β_3 and β_6. ■

Lemma 3.8 *Let* $I > 0$, *let* $p > 2$ *and let* $(\zeta_j)_{j\geq 0}$ *be a sequence of Steiner-symmetric non-negative functions on* Π *that satisfie* $\mathcal{I}_4(\zeta_j) \leq I$. *Suppose that* $\|\zeta_j\|_1$ *is bounded for all* $j \in \mathbb{N}$ *and* $\zeta_j \longrightarrow \zeta$ *weakly in* $L^1 \cap L^p(\Pi, \nu)$. *Then*

$$E(\zeta_j) \longrightarrow E(\zeta) \text{ as } j \longrightarrow \infty \text{ and } \mathcal{I}(\zeta) \leq I.$$

Proof. Since $\mathcal{I}$ is a lower-semi continuous and convex functional over the non-negative functions in $L^1 \cap L^p(\Pi, \nu)$, then $\mathcal{I}_4$ is weakly lower semi-continuous, so $\mathcal{I}_4(\zeta) \leq \liminf_{j\to\infty} \mathcal{I}_4(\zeta_j)$; hence $\mathcal{I}_4(\zeta) \leq I$. Now, for $X > 2$, we set

$$\Pi(X) = \{(r, z) \in \Pi \mid r^2 + z^2 \leq X^2\}.$$

Then, for any Steiner-symmetric function $\zeta \in L^1 \cap L^p(\Pi, \nu)$ that satisfies $\mathcal{I}_4(\zeta) \leq I$, we have

$$\begin{aligned} E(\zeta) - E(\zeta 1_{\Pi(X)}) &= \int\limits_{r^2+z^2>X^2} \zeta(r, z) K\zeta(r, z) d\nu \\ &\leq C\|\zeta\|_1^{\frac{2+\delta}{2}} \frac{I^{\frac{2-\delta}{2}}}{X^{3-2\delta}} \text{ for some } 1 < \delta < 1, \end{aligned}$$

where we have used Lemma 3.7 to obtain the last line. Thus, for a given $\epsilon > 0$, we can choose $\xi_0 > 2$ such that

$$\left|E(\zeta) - E(\zeta 1_{\Pi(X)})\right| \leq \frac{\epsilon}{3} \tag{3.38}$$

for all $X > \xi_0$ and

$$\left|E(\zeta_j) - E(\zeta_j 1_{\Pi(X)})\right| \leq \frac{\epsilon}{3} \tag{3.39}$$

for all $X > \xi_0$ and for all $j \in \mathbb{N}$. In view of the compactness of K as an operator from $L^p(\Pi(X), \nu)$ to $L^q(\Pi(X), \nu)$, then $K\zeta_j \longrightarrow K\zeta$ strongly in $L^q(\Pi(X), \nu)$. Therefore, by using the fact that $\zeta_j \to \zeta$ weakly, for $\epsilon > 0$ we can choose j_0 such that

$$\left|E(\zeta_j 1_{\Pi(X)}) - E(\zeta 1_{\Pi(X)})\right| \leq \frac{\epsilon}{3}, \tag{3.40}$$

for all $j \geq j_0$. Finally, since

$$|E(\zeta_j) - E(\zeta)| \leq |E(\zeta_j) - E(\zeta_j 1_{\Pi(X)})| + |E(\zeta_j 1_{\Pi(X)}) - E(\zeta 1_{\Pi(X)})| + |E(\zeta) - E(\zeta 1_{\Pi(X)})|, \tag{3.41}$$

then from (3.38), (3.39), (3.40) and (3.41) it follows that

$$|E(\zeta_j) - E(\zeta)| \leq \epsilon$$

for all $j \geq \max\{j_0, \xi_0\}$. Therefore $E(\zeta_j) \longrightarrow E(\zeta)$ as $j \longrightarrow \infty$. ■

Lemma 3.9 *Let $0 < I < \infty$, let $p > 2$ and let $\alpha > 0$. Then for any Steiner-symmetric function $\zeta \in L^1 \cap L^p(\Pi, \nu)$ that satisfies $\mathcal{I}(\zeta) = I$, we can choose a positive number X_0 depending only on $\|\zeta\|_1$, $\|\zeta\|_p$ and I such that if $E(\zeta) \geq \alpha$, then $E(\zeta 1_{\Pi(X_0)}) \geq \frac{\alpha}{2}$.*

Proof. For $X_0 > 2$, set $\zeta_1 = \zeta 1_{\Pi(X_0)}$ and $\zeta_2 = \zeta - \zeta_1 = \zeta 1_{\{|R| \geq X_0\}}$. Since K is a positive and symmetric operator, then by using Lemma 3.8 we have

$$E(\zeta) \leq E(\zeta_1) + \int\limits_{\Pi} \zeta_2(r,z) K\zeta(r,z) d\nu \leq E(\zeta_1) + C\|\zeta\|_1^{\frac{2+\delta}{2}\delta} I^{\frac{2-\delta}{2}} X_0^{-(3-2\delta)}$$

for some $\frac{3}{4} < \delta < 1$ where we have used the fact that $\zeta_2 \leq \zeta$ to obtain the right-hand side. Since $E(\zeta) \geq \alpha$, then by taking $X_0 = \left(\frac{2C\|\zeta\|_1^{2+\delta} I^{2-\delta}}{\alpha}\right)^{\frac{1}{2(3-\delta)}}$ we obtain $E(\zeta_1) \geq \frac{\alpha}{2}$. This completes the proof. ■

The last Lemma in this section follows immediately from [8], Lemma 5.

Lemma 3.10 *Let* $\zeta \geq 0$ *be a measurable function on* Π *and let* $F(\zeta)$ *be defined as in Lemma 3.4. For* $\alpha > 0$, *we define* ζ_α *by* $\zeta_\alpha(r,z) = \zeta((1+\alpha)^{-1}r, (1+\alpha)^2\, z)$ *for* $(r,z) \in \Pi$. *Then*

$$E(\zeta_\alpha) > E(\zeta) + \frac{\alpha}{64\pi^2} F^2(\zeta), \tag{3.42}$$

Proof. By using [8], Lemma 5 we have

$$E(\zeta_\alpha) \geq (1+\alpha)E(\zeta) > E(\zeta) + \frac{\alpha}{64\pi^2} F^2(\zeta),$$

since by Lemma 3.4, we have $E(\zeta) > \frac{1}{64\pi^2} F^2(\zeta)$. ■.

3.5 Proof of the main Theorem

To prove our main Theorem, we need to introduce some notation that are needed in the proof. Indeed, for $\zeta_0 \in L^1 \cap L^p(\Pi, \nu)$ $(p > 2)$ a non-negative function having support of finite volume and for $X > 2$, we let $\mathcal{W}_X(\zeta_0)$ to denote the set of all functions ζ in $\mathcal{W}(\zeta_0)$ that are supported in $\Pi_X = (0,\ X) \times \mathbb{R}$. For all $I > 0$ we set

$$\mathbb{T}(I) := \mathcal{W}_X(\zeta_0) \cap \mathcal{I}_4^{-1}(I). \tag{3.43}$$

Also we define the function $\mathbb{F}_X : \mathcal{I}_4(\mathcal{W}_X(\zeta_0)) \to \mathbb{R}_+$ by

$$\mathbb{F}_X(I) = \sup_{\mathbb{T}(I)} E(\zeta). \tag{3.44}$$

We recall the definition of the Hausdorff metric. If A and B are two nonempty closed bounded sets in a metric space $(\mathbb{M}, d)$ we set $h(A,B) := \sup\{d(a,B) | a \in A\}$, where $d(a,B)$ denotes the usual distance from a to B, and we define the Hausdorff metric

$$d_H(A,B) := \max\{h(A,B), h(B,A)\}.$$

Next, we denote by $\partial\mathbb{F}_X(I)$ the generalised gradient of $\mathbb{F}_X(I)$ defined in Clarke [16] at the point I as follows

$$\partial\mathbb{F}_X(I) = \left\{\widetilde{I} \in \mathbb{R} \, | \mathbb{F}^0_X(I; I^*) \geq \widetilde{I} I^* \quad \text{for all} \quad I^* \in \mathbb{R}\right\}, \tag{3.45}$$

where

$$\mathbb{F}^0_X(I; I^*) = \lim \sup_{\bar{I} \to I, t \downarrow 0} \frac{\mathbb{F}_X(\bar{I} + tI^*) - \mathbb{F}_X(\bar{I})}{t}. \tag{3.46}$$

Finally, if we set $Z = \sup \mathcal{I}_4(\mathcal{W}_X(\zeta_0))$, then we obtain $Z \leq \frac{1}{4}\|\zeta_0\|_1 X^4$. Now, with these notations, we have the following results proved by Burton & Emamizadeh [9] and Rebah [38] in the case of planar domains.

Lemma 3.11 *Let* $2 < p < \infty$*, let* $X > 0$ *and let* $\mathbb{B}_X = L^1 \cap L^p(\Pi_X, \nu)$ *be the Banach space associated with the norm* $\|.\|_{\mathbb{B}_X} = \|.\|_1 + \|.\|_p$*. Then* $E \in C^1(\mathbb{B}_X)$*.*

Lemma 3.12 $\mathbb{T}(.)$ *is locally Lipschitz on* $(0, Z)$*.*

Lemma 3.13 *The function* $\mathbb{F}_X$ *is locally Lipschitz on* $(0, Z)$*.*

Lemma 3.14 *Let* $I_0 < I_1$ *be two positive numbers. Suppose that there exists* $X > 0$ *such that for all* $I \in (I_0, I_1)$*, there is a maximiser for* $E(\zeta)$ *subject to* $\zeta \in \mathcal{W}(\zeta_0)$ *and* $\mathcal{I}_4(\zeta) = I$*,* *that is supported in* Π_X. *Then for all* $I \in (I_0,\ I_1)$ *we have*

$$\lambda \in \partial \mathbb{F}_X(I) \Rightarrow \lambda > 0.$$

Proof. Consider $I \in (I_0, I_1)$ and for the given $X > 0$, let ζ be a maximiser for E relative to $\mathcal{W}_X(\zeta_0) \cap \mathcal{I}_4^{-1}(I)$. We have

$$\mathbb{F}_X(I) = E(\zeta).$$

We need first to show that there exists a positive constant $\theta > 0$ such that

$$\mathbb{F}_X(I+h) \geq \mathbb{F}_X(I) + \theta h$$

for all small $h > 0$. To do that, for $t \in (0,1)$, we define the function ζ_t by

$$\zeta_t(r,z) = \zeta((1+t)^{-1}r,\ (1+t)^2)z) \quad \text{for all} \quad (r,z) \in \Pi.$$

Futher, we take $h = \mathcal{I}_4(\zeta_t) - \mathcal{I}_4(\zeta)$. Since $\mathcal{I}_4(\zeta_t) = (1+t)^4 \mathcal{I}_4(\zeta)$ follows immediately from the volume preserving linear change of variables $((1+t)^{-1}r,\ (1+t)^2 z) \hookrightarrow (r,z)$ in the integral defining $\mathcal{I}_4(\zeta)$. Then, for all $t \in (0,\ 1)$ and $I \in (I_0,\ I_1)$ we find $h \leq 15tI_1$. Using now Lemma 3.10 yields

$$E(\zeta_t) \geq E(\zeta) + \frac{t}{64\pi^2} F^2(\zeta), \tag{3.47}$$

where $F(\zeta)$ is defined as in Lemma 3.4. Thus it suffices to find a positive constant C such that $F(\zeta) \geq C$. Indeed, since ζ is a maximiser of E, then we can choose a positive number α depending on ζ_0, I_0 and I_1 only such that $E(\zeta) \geq \alpha$, hence by Lemma 3.9, we can find X_0 depending on $\|\zeta_0\|_1$ and I with $I \in (I_0,\ I_1)$ such that $E(\zeta 1_{\Pi(X_0)}) \geq \frac{\alpha}{2}$, Thus, by using [8], Lemma 4(iii), there exist three positive numbers C such that

$$\int_{\Pi(X_0)} r^2\zeta(r,z)d\nu \geq \frac{\alpha}{C\|\zeta\|_p} := \eta.$$

We can assume that ζ is Steiner symmetric and then it follows that

$$\int_{\Pi(X_0)} \frac{r^2\zeta(r,z)}{1+|R|^3}d\nu \geq \frac{\eta}{(1+X_0^3)}. \tag{3.48}$$

Therefore we have

$$F(\zeta) \geq \frac{\eta^2}{(1+X^3)}.$$

Thus by (3.47) and (3.48), we deduce that there exists a positive constant θ such that for all $I_0 < I < I_1$ and $0 < h < I_1 - I$ we have

$$\mathbb{F}_X(I+h) - \mathbb{F}_X(I) \geq \theta h.$$

It then follows that the function $\mathbb{G}_X(I) = \mathbb{F}_X(I) - \theta I$ is increasing on $[I_0, I_1]$; hence by using the formula (3.46), we find that $\mathbb{G}_X^0(I; I^*) \geq 0$ for all $I^* > 0$ and $I_0 < I < I_1$. Therefore we get $\mathbb{G}_X^0(I, I^*) = C_1 I^*$, where $C_1 = \mathbb{G}_X^0(I, 1) \geq 0$. Now if $I^* < 0$, then we can set $I^* = -I'$, where $I' > 0$; hence by using [9], Proposition 2.1.1, $\mathbb{G}_X^0(I, -I') = (-\mathbb{G}_X)^0(I, I') = C_2 I^{'}$, where $C_2 = (-\mathbb{G}_X)^0(I, 1) \leq 0$ because $-\mathbb{G}_X$ is a decreasing function, so by using (3.45), we find that $\partial\mathbb{G}_X(I) = [-C_2, C_1]$, and therefore we deduce that $\lambda \geq \theta$ for any $\lambda \in \partial\mathbb{F}_X(I)$. This completes the proof. ■

Proof of Theorem 3.1

Now, we are in good position to prove our Theorem. Indeed, for $I > 0$, we set

$$V(I) := \sup\{E(\zeta) \mid \zeta \in \mathcal{W}(\zeta_0),\ \mathcal{I}_4(\zeta) \leq I\}. \tag{3.49}$$

Let $\mathcal{W}_X^s(\zeta_0)$ be the set of all functions in $\mathcal{W}_X(\zeta_0)$ that are Steiner-symmetric relative to the r axis and supported in Π_X, where $\zeta_0 \in L^1(\Pi) \cap L^p(\Pi)$ $(p > 2)$. Let $\{\zeta_j\}_{j=1}^{\infty}$ be a maximising sequence of E relative to $\mathcal{W}(\zeta_0)$ with $\mathcal{I}_4(\zeta_j) \leq I$. By using [29], Theorem 3.7, we have $E(\zeta_j^s) \geq E(\zeta_j)$ and $\mathcal{I}_4(\zeta_j^s) = \mathcal{I}_4(\zeta_j)$. Thus, we can assume $\{\zeta_j\}_{j=1}^{\infty}$ is a maximising sequence lies in $\mathcal{W}_X^s(\zeta_0)$. Since for any $\zeta \in \mathcal{W}_X^s(\zeta_0)$ we have $\|\zeta\|_p \leq \|\zeta_0\|_p$ with $1 \leq p < \infty$, see [2], Lemma 2,9, then it follows that the sequence $\{\zeta_j\}_{j=1}^{\infty}$ is bounded in $\|.\|_1$. We may therefore pass to a subsequence and suppose that $\{\zeta_j\}_{j=1}^{\infty}$ converges weakly in $L^1 \cap L^p(\Pi, \nu)$ to $\bar{\zeta}$, hence by Lemma 3.8, we have $E(\zeta_j) \to E(\bar{\zeta})$ and $\mathcal{I}_4(\bar{\zeta}) \leq I$. Thus, the functional E attains a maximum value relative to $\zeta \in \mathcal{W}(\zeta_0)$ and $\mathcal{I}_4(\zeta) \leq I$ and each maximiser ζ is Steiner-symmetric. Therefore, the problem $V(I)$ has a solution.

To prove (i), we need in the first step to show that every maximiser ζ must satisfy $\mathcal{I}_4(\zeta) = I$. To do this, we consider $\bar{\zeta} \in \mathcal{W}^s(\zeta_0)$ a maximiser for (3.49) and assume that $\mathcal{I}_4(\bar{\zeta}) < I$. For $t \in (0, 1)$, we define $\bar{\zeta}_t$ by

$$\bar{\zeta}_t(r,\ z) = \bar{\zeta}((1+t)^{-1}r,\ (1+t)^2 z) \tag{3.50}$$

for all $(r, z) \in \Pi$. Now, using Lemma 3.10 yields

$$E(\bar{\zeta}_t) \geq E(\bar{\zeta}) + \frac{t}{64\pi^2} F^2(\bar{\zeta}), \tag{3.51}$$

furthermore, by making the change of variables $z' = (1+t)^2 z$ and $r' = (1+t)^{-1} r$ we get

$$\mathcal{I}_4(\bar{\zeta}_t) = (1+t)^4 \mathcal{I}_4(\bar{\zeta}) \leq \mathcal{I}_4(\bar{\zeta}) + 15t\mathcal{I}_4(\bar{\zeta}). \tag{3.52}$$

Since $\mathcal{I}_4(\bar{\zeta}) < I$, then we can choose $t = \frac{I - \mathcal{I}_4(\bar{\zeta})}{30\mathcal{I}_4(\bar{\zeta})}$ to ensure $\mathcal{I}_4(\bar{\zeta}_t) < I$. Thus, we find that

$$E(\bar{\zeta}_t) > E(\bar{\zeta}) \quad \text{and} \quad \mathcal{I}_4(\bar{\zeta}_t) < I;$$

which shows us a contradiction. Therefore if $\bar{\zeta}$ is a maximiser subject to $\zeta \in \mathcal{W}(\zeta_0)$ and $\mathcal{I}_4(\zeta) \leq I$, then $\bar{\zeta}$ must satisfy $\mathcal{I}_4(\bar{\zeta}) = I$.

The second step is to show that there exists a positive number $X > 1$ such that if $\bar{\zeta}$ is a maximiser for E subject to $\zeta \in \mathcal{W}(\zeta_0)$ and $\mathcal{I}_4(\zeta) = I$, then $\bar{\zeta}$ is supported in Π_X. We want to show that if X is chosen large enough, depending on I, $\|\zeta_0\|_1$ and $\|\zeta_0\|_p$, then this leads to a contradiction. To do this, we assume that $\bar{\zeta}$ is a maximiser and for $X > 1$ any positive number depending on I, $\|\zeta_0\|_1$ and $\|\zeta_0\|_p$, we set $w = \bar{\zeta} 1_{\Pi_X}$ and $h = \bar{\zeta} - w$, furthermore, we assume that $h \neq 0$. Indeed, since K is a symmetric and positive operator, then by using Lemma 3.1, the following inequalities

$$\begin{aligned} E(w) = E(\bar{\zeta} - h) &\geq E(\bar{\zeta}) - \int\limits_{r>X} h(r,z) K\bar{\zeta}(r,z) d\nu \\ &\geq E(\bar{\zeta}) - C(\|\zeta_0\|_p + \|\zeta_0\|_1) \int\limits_{r>X} (r \log r) h(r,z) d\nu \\ &\geq E(\bar{\zeta}) - C(\|\zeta_0\|_p + \|\zeta_0\|_1) \mathcal{I}_4(h) \frac{\log X}{X^3} \end{aligned} \tag{3.53}$$

hold, where we have used the fact that $r \log r \leq r^4 \frac{\log X}{X^3}$ for all $r > X > 1$ to obtain the last line. Now, for $t \in (0,\ 1)$ we define w_t as in (3.50), then using (3.51) we get

$$E(w_t) \geq E(w) + \frac{t}{64\pi^2} F^2(w),$$

hence from (3.53) it follows that

$$E(w_t) \geq E(\bar{\zeta}) + \frac{t}{64\pi^2}F^2(w) - C(\|\zeta_0\|_1 + \|\zeta_0\|_p)\frac{\log X}{X^3}\mathcal{I}_4(h). \tag{3.54}$$

Also from (3.52) we have

$$\mathcal{I}_4(w_t) \leq \mathcal{I}_4(w) + 15tI. \tag{3.55}$$

Since $t \in (0,\ 1)$, then we can take

$$t \leq \frac{I - \mathcal{I}_4(w)}{15I} = \frac{\mathcal{I}_4(h)}{15I}$$

to ensure that from (3.55), the result is

$$\mathcal{I}_4(w_t) \leq \mathcal{I}_4(\bar{\zeta}) = I.$$

Now, if we want to have from (3.54) that $E(w_t) > E(\bar{\zeta})$, then it is sufficient to have

$$\frac{t}{64\pi^2}F^2(w) > C(\|\zeta_0\|_1 + \|\zeta_0\|_p)\frac{\log X}{X^3}\mathcal{I}_4(h). \tag{3.56}$$

Thus, we need just find a positive constant e independent of $\bar{\zeta}$ such that $\widetilde{E}(w) \geq e$. Indeed, we recall that

$$\mathbb{F}_X(I) = \sup_{\mathbb{T}(I)} E(\zeta).$$

We note that for $I \geq I_0$ we have $\mathbb{F}_X(I) \geq \mathbb{F}_X(I_0)$. By using Lemma 3.9, we can choose $X_0 > 0$ such that if $\zeta \in \mathcal{W}(\zeta_0)$ and $E(\zeta) \geq m$; then we have $E(\zeta 1_{\Pi(X_0))} \geq \frac{1}{2}m$, where $m = \mathbb{F}_X(I_0)$, hence by using [8], Lemma 4(iii), there exists a positive constant C such that

$$\int\limits_{R \leq X_0} r^2 w(r,z) d\nu \geq \frac{m}{2C\|\zeta_0\|_p}.$$

Thus, it follows that $F(w) \geq e$, where $e = \frac{m}{6C\|\zeta_0\|_p X_0^3}$. Hence from (3.56) we need just to find X such that

$$\frac{e^2}{64\pi^2} - C(\|\zeta_0\|_1 + \|\zeta_0\|_p)\frac{\log X}{X^3}\mathcal{I}_4(h) > 0.$$

This formula allows us to choose first $t = \frac{\mathcal{I}_4(h)}{15I}$ and then, we can choose $X_1 > 0$ such that

$$\frac{\log X_1}{X_1^3} > \frac{e^2}{960C(\|\zeta_0\|_1 + \|\zeta_0\|_p)I}$$

would yield X_1 valid also for all smaller values of I to ensure that if $r \geq X = \max\{1, X_0, X_1\}$, then we find that $E(w_t) > E(\bar{\zeta})$ which shows a contradiction. Hence there exists X large enough depending only on $\|\zeta_0\|_1$, $\|\zeta_0\|_p$ and I such that the maximiser $\bar{\zeta}$ is supported in the region $\Pi_X = (0, X) \times \mathbb{R}$.

Now, since $\bar{\zeta}$ maximises the strictly convex functional E relative to the closed convex set $\mathcal{W}_X(\zeta_0) \cap \mathcal{I}_4^{-1}(I) \cap L^1(\Pi_X)$, then $\bar{\zeta}$ is an extreme point of $\mathcal{W}_X(\zeta_0) \cap \mathcal{I}_4^{-1}(I) \cap L^1(\Pi_X)$. Also since $\mathcal{I}_4$ is linear and bounded on $L^1(\Pi_X)$, then by applying [18], Lemma 2.4, we deduce $\bar{\zeta} \in \mathcal{RC}(\zeta_0)$. Note that although Douglas's result is stated for $1 < p < \infty$, the proof is valid when $p = 1$ also. Thus, (ii) is proved.

To prove (iii), we assume $I \in [I_0, I_1]$ and we recall first the Banach space $\mathbb{B}_X = L^1 \cap L^p(\Pi_X, \nu)$ $(p > 2)$ associated with the norm $\|.\|_{\mathbb{B}_X} = \|.\|_1 + \|.\|_p$, where $X = \max\{1, X_0, X_1\}$. We also assume that $\bar{\zeta}$ is a maximiser for E relative to $\mathbb{T}(I)$. Since we do not know whether $\mathbb{F}_X$ is differentiable everywhere, we are going to use non-smooth analysis to obtain the Lagrange multiplier λ. To do this, we argue as Burton [8] did in his proof. Indeed, by Lemma 3.13, the function $\mathbb{F}_X$ is locally Lipschitz. Also, since $\mathcal{I}_4$ defines a bounded linear functional on X([16], corollary to proposition 2.4.3), then $\bar{\zeta}$ maximises some functional in $\partial(E - \mathbb{F}_X \circ \mathcal{I})(\bar{\zeta}) \subset \mathbb{B}_X^*$ relative to $\mathcal{W}(\zeta_0) \cap \mathbb{B}_X$, where $\mathbb{B}_X^*$ is the dual of $\mathbb{B}_X$ and then $\bar{\zeta}$ maximises $E - \lambda\mathcal{I}_4$ relative to $\mathcal{W}(\zeta_0)$, for some $\lambda \in \partial\mathbb{F}_X(I)$. Hence it follows from Lemma 3.14 that $\lambda > 0$. Therefore $\bar{\zeta}$ maximises

$$\int_{\Pi} \zeta(K\bar{\zeta} - \frac{\lambda}{4}r^4) \tag{3.57}$$

relative to $\mathcal{W}(\zeta_0) \cap \mathbb{B}_X$, where $\lambda > 0$. Since $\mathcal{L}(K\bar{\zeta} - \frac{\lambda}{4}r^4) \geq \bar{\zeta}$ almost everywhere in Π_X, it follows from [7], Lemma 2.15 that there exists an increasing function ϕ such that $\bar{\zeta} = \phi \circ (K\bar{\zeta} - \frac{\lambda}{4}r^4)$ almost everywhere in Π_X. Since $\bar{\zeta}$ is a maximiser, then for every $Y \geq X$ we have $\bar{\zeta} = \phi_Y \circ (K\bar{\zeta} - \frac{\lambda}{4}r^4)$ on Π_Y, where ϕ_Y is increasing function. If $X \leq Y \leq Y_1$, then we can assume that ϕ_{Y_1} is an extension of ϕ_Y, hence we can choose an increasing function ϕ that is extension of ϕ_{Y_1}. Now if we set $\psi := K\bar{\zeta}$, then we have

$$\psi = \phi \circ (\psi - \frac{\lambda}{4}r^4) \tag{3.58}$$

almost everywhere in Π for some increasing function ϕ and positive λ.

It remains just to prove (*iv*). Indeed, let $I > 0$, let $p > \frac{5}{2}$, let q be the conjugate exponent of p, let a be a positive number such that $\nu(\{(r, z) \in \Pi \mid \zeta_0(r, z) > 0\}) = 2\pi^2 a^3$ and we let C denote any positive number depending only on p and a. Let ζ be a maximiser of E subject to $\zeta \in \mathcal{W}(\zeta_0)$ and $\mathcal{I}_4(\zeta) = I$, then this implies that ζ is a Steiner symmetric function and for all $I > 0$, $\zeta \in \mathcal{RC}(\zeta_0)$, hence ζ is supported in the region Π_X, where X is a positive number depending on I. Hence,

$$\zeta^{-1}(0, \infty) \subset \{(r, z) \mid K\zeta(r, z) - \frac{\lambda}{4}r^4 > 0\}$$

apart from a set of zero volume. We set

$$V = \{(r,z) \in \Pi \mid K\zeta(r,z) - \frac{\lambda}{4}r^4 > 0\} \quad \text{and} \quad S = \{(r,z) \mid \zeta(r,z) > 0\}.$$

To prove $\zeta \in \mathcal{R}(\zeta_0)$, we need first to show that if $\nu(V) \geq 2\pi^2 a^3$, then this implies that $\nu(S) \geq 2\pi^2 a^3$. Indeed, let ζ^{Δ} and ζ_0^{Δ} be the decreasing rearrangements of ζ and ζ_0 respectively. Assuming that $\nu(V) \geq 2\pi^2 a^3$ but $\nu(S) < 2\pi^2 a^3$, since $\zeta \in \mathcal{RC}(\zeta_0)$, then there exists a rearrangement w of $\zeta_0^{\Delta} - \zeta^{\Delta}$ supported by the region $V \backslash S$, hence $w + \zeta$ is a rearrangement of ζ_0. Now, since $E(w) > 0$ because w is supported by $V \backslash S$, then this implies that $E(\zeta + w) > E(\zeta)$ which shows a contradiction. Thus, if $\nu(V) \geq 2\pi^2 a^3$, then $\nu(S) \geq 2\pi^2 a^3$. By Lemma 3.5,

$$\nu(V) \geq \frac{1}{3}\Big(\sqrt{\frac{F(\zeta)}{\lambda}} - 4\pi\Big), \tag{3.59}$$

where

$$F(\zeta) = \int\limits_{\Pi_X} \frac{r^2}{1+|R|^3}\zeta(r,z)d\nu.$$

Then, it is sufficient just to show that

$$\frac{1}{\lambda}F(\zeta) \longrightarrow \infty \quad \text{as} \quad I \to \infty.$$

Indeed, since ζ satisfies (3.58), then it follows from Lemma 3.6 that

$$\int\limits_{\Pi_X} \zeta(r,z)K\zeta(r,z)d\nu \geq 2\lambda I + \frac{6\beta}{7}\|\zeta\|_1, \tag{3.60}$$

where we have used the fact that $\mathcal{I}_4(\zeta) = I$. Now, ζ maximises the functional $\left\langle ., K\zeta - \frac{\lambda}{4}r^4 \right\rangle$ on the set $\mathcal{W}(\zeta_0)$, then we can assume that $\phi(t) = 0$ for $t \leq \beta$, where $\beta > 0$ since S is not the whole of V. On the other hand, by setting $Z := E(\zeta_0)$, then we can choose $I_2 > 0$ such that

$$\int\limits_{\Pi_X} \zeta(r,z)K\zeta(r,z)d\nu \geq 2Z \tag{3.61}$$

for all $I \geq \max\{I_0,\ I_1,\ I_2\}$, since ζ is a maximiser. Thus, by combineing (3.60) with (3.61), we find that

$$\begin{aligned} \int\limits_{\Pi_X} \zeta(r,z)K\zeta(r,z)d\nu &\geq \lambda I + \frac{3\beta}{7}\|\zeta\|_1 + Z \\ &\geq \lambda I + Z. \end{aligned} \tag{3.62}$$

Now, for $\delta \in (\frac{3}{4},\ 1)$, we set

$$X_2 = \left(\frac{2C}{Z})\right)^{\frac{1}{3-2\delta}} \|\zeta_0\|_1^{\frac{2+\delta}{2(3-2\delta)}} I^{\frac{2-\delta}{2(3-2\delta)}}, \tag{3.63}$$

then, by using Lemma 3.7 we find

$$\int\limits_{|R|\geq X_2} \zeta(r,z)K\zeta(r,z)d\nu \leq C\|\zeta\|_1^{\frac{2+\delta}{2}} \frac{I^{\frac{2-\delta}{2}}}{X_2^{3-2\delta}} = \frac{Z}{2} \tag{3.64}$$

Next, by using (3.60), we have

$$\begin{aligned} \int\limits_{|R|<X_2} \zeta(r,z)K\zeta(r,z)d\nu &\geq \lambda I + Z - \int\limits_{|R|\geq X_2} \zeta(r,z)K\zeta(r,z)d\nu \\ &\geq \lambda I + \frac{Z}{2}, \end{aligned}$$

where we have used (3.64) to obtain the last line. Finally, we can write

$$\int\limits_{|R|<\log I} \zeta(r,z)K\zeta(r,z)d\nu + \int\limits_{\log I\leq|R|<X_2} \zeta(r,z)K\zeta(r,z)d\nu \geq \lambda I + \frac{Z}{2}. \tag{3.65}$$

Henceforth, we assume that $I \geq I_3 = \max\{e^{16},\ e^{\frac{8\sqrt{2}C\|\zeta_0\|_1^2}{z}}\}$. Form (3.65), we can distinguish two cases. The first case is

$$\int\limits_{|R|<\log I} \zeta(r,z)K\zeta(r,z)d\nu \geq \int\limits_{\log I \leq |R| < X_2} \zeta(r,z)K\zeta(r,z)d\nu.$$

Then from (3.65), we obtain

$$\int\limits_{|R|\leq\log I} \zeta(r,z)K\zeta(r,z)d\nu \geq \frac{\lambda}{2}I. \tag{3.66}$$

We recall that by using [8], Lemma 4(iii) we have

$$K\zeta(r,z) \leq C\|\zeta_0\|_p r^2,$$

hence by using (3.66) we obtain

$$\int\limits_{|R|<\log I} r^2\zeta(r,z)d\nu \geq \frac{\lambda}{2C\|\zeta_0\|_p}I.$$

Thus,

$$\int\limits_{|R|<\log I} \frac{r^2}{1+|R|^3}\zeta(r,z)d\nu \geq \frac{\lambda}{4C\|\zeta_0\|_p}I(\log I)^{-3}.$$

Since $\lambda > 0$, then it follows that

$$\frac{1}{\lambda}F(\zeta) \geq \frac{1}{4C\|\zeta_0\|_p}I(\log I)^{-3}. \tag{3.67}$$

The second case is:

$$\int\limits_{\log I \leq |R| < X_2} \zeta(r,z)K\zeta(r,z)d\nu \geq \int\limits_{|R|<\log I} \zeta(r,z)K\zeta(r,z)d\nu.$$

Indeed, we assume that this inequality holds, then from (3.65) we obtain

$$\int\limits_{\log I\leq|R|<X_2,|z|\geq r}\zeta(r,z)K\zeta(r,z)d\nu+\int\limits_{\log I\leq|R|<X_2,|z|<r}\zeta(r,z)K\zeta(r,z)d\nu\geq\frac{\lambda}{2}I+\frac{Z}{4}. \tag{3.68}$$

Now, for any $t\in(0,1)$ and $r>0$, we have $r^{1+t}\leq 2+r^2$, hence by using Lemma 3.3, we find

$$K\zeta(r,z)\leq C\|\zeta_0\|_1\left(\frac{1}{|z|}+\frac{r^2}{|z|}\log|z|\right) \tag{3.69}$$

provided $|z|\geq 2$. Also, if $\log I\leq|R|\leq X_2$ and $|z|\geq r$, then it follows that

$$|z|\geq\frac{\sqrt{2}}{2}\log I\geq\frac{\sqrt{2}}{2}\log I_3=\max\{16,\frac{8\sqrt{2}C\|\zeta_0\|_1^2}{Z}\}$$

since $I_3=\max\{e^{16},\ e^{\frac{8\sqrt{2}C\|\zeta_0\|_1^2}{Z}}\}$, hence, by using (3.69), we get

$$\begin{aligned}\int\limits_{\log I\leq|R|\leq X_2,|z|\geq r}\zeta(r,z)K\zeta(r,z)d\nu &\leq C\|\zeta_0\|_1\int\limits_{\log I\leq|R|\leq X_2,\ |z|\geq r}\left(\frac{1}{|z|}+\frac{r^2}{|z|}\log|z|\right)\zeta(r,z)d\nu\\ &\leq \frac{Z}{8}+C\|\zeta_0\|_1\int\limits_{\log I\leq|R|\leq X_2,|z|\geq r}\frac{r^2}{|z|}\log|z|\zeta(r,z)d\nu,\end{aligned}$$

moreover, by using Lemma 3.1, from (3.68) the result is

$$C\|\zeta_0\|_p\left(\int\limits_{\log I\leq|R|<X_2,|z|\geq r}\frac{r^2}{|z|}\log|z|\zeta(r,z)d\nu+\int\limits_{\log I\leq|R|<X_2,|z|<r}r\log r\zeta(r,z)d\nu\right)\geq\frac{\lambda}{2}I+\frac{Z}{8},$$

therefore

$$\int\limits_{\log I\leq|R|<X_2,\ |z|\geq r}\frac{r^2}{|z|}\zeta(r,z)d\nu+\int\limits_{\log I\leq|R|<X_2,\ |z|<r}r\zeta(r,z)d\nu\geq\frac{\lambda I}{2C\|\zeta_0\|_p}(\log I)^{-1}. \tag{3.70}$$

Now, $1+|R|^3 \leq 2\sqrt{2}X_2^2|z|$, also, if $2 \leq |R| < X_2$ and $|z| \leq r$, then $1+|R|^3 \leq 2\sqrt{2}X_2^2 r$, hence

$$\begin{aligned}\int\limits_{\log I \leq |R| < X_2} \frac{r^2}{1+|R|^3}\zeta(r,z)d\nu &= \Big(\int\limits_{\log I \leq |R| < X_2, |z| \geq r} + \int\limits_{\log I \leq |R| < X_2, |z| < r}\Big)\frac{r^2}{1+|R|^3}\zeta(r,z)d\nu \\ &\geq \frac{1}{2\sqrt{2}X_2^2}\Big(\int\limits_{\log I \leq |R| < X_2, |z| \geq r} \frac{r^2}{|z|}\zeta(r,z)d\nu + \int\limits_{\log I \leq |R| < X_2, |z| < r} r^2\zeta(r,z)d\nu\Big) \\ &\geq \frac{\lambda I}{2\sqrt{2}C\|\zeta_0\|_p X_2^2}(\log I)^{-1} = \frac{\lambda}{C\|\zeta_0\|_p^{\frac{5-\delta}{3-\delta}}} I^{\frac{1-\delta}{3-2\delta}}(\log I)^{-1},\end{aligned}$$

where we have used the inequality (3.70) and (3.63) to get the last line. Thus,

$$\frac{1}{\lambda}F(\zeta) \geq \frac{1}{C\|\zeta_0\|_p^{\frac{5-\delta}{3-\delta}}} I^{\frac{1-\delta}{3-2\delta}}(\log I)^{-1}. \tag{3.71}$$

Therefore in both cases, from (3.67) and (3.71), we find that

$$\nu(V) = \nu(\{(r,z) \in \Pi \mid K\zeta(r,z) - \frac{\lambda}{4}r^4 > 0\}) \to \infty \quad \text{as} \quad I \to \infty.$$

Thus, we can choose I_4 such that $\nu(V) \geq 2\pi^2 a^3$ for all $I \geq \min\{I_3,\ I_4\}$, therefore $\zeta \in \mathcal{R}(\zeta_0)$. This complets the proof.

Chapter 4

Steady vortex rings of an ideal fluid in Shear flow and rearrangements of a function

4.1 Introduction

The aim in this Chapter is to prove the existence theorem for certain flows in $\mathbb{R}^3$ called steady axisymmetric vortex-rings in shear flow of an ideal fluid. By steady vortex rings in shear flow, we mean bounded axisymmetric regions of vorticity in an otherwise irrotational flow. We consider axisymmetric flows, without swirl of an incompressible inviscid fluid with unit density whose Stokes stream function $\Psi : \Pi \longrightarrow \mathbb{R}$ is symmetric about the z-axis with respect to the cylindrical coordinates (r, θ, z) and approaches $-\lambda(\frac{1}{2}r^2+\frac{\sigma}{4}r^4)$ at infinity in the negative z-direction, where $\lambda > 0$ and $\sigma \geq 0$ are two numbers corresponding to the strength of the background flow and Π is the half-plane in $\mathbb{R}^3$ defined by $r > 0$, $\theta = 0$ and

-$\infty < z < \infty$. The velocity in the negative z-direction approaches $\lambda(1+\sigma r^2)$ at infinity. If $\sigma = 0$, the Stokes stream function represents a uniform flow having a prescribed velocity λ in the negative z direction at infinity. The vorticity is directed azimuthally and its magnitude ω is given in terms of the Stokes stream function by $\omega/r = \mathcal{L}\Psi$, an expression approaching $\lambda\sigma$ at infinity, where $\mathcal{L}$ is an elliptic differential operator. Hence $r\mathcal{L}$ is the Laplacian operator in three dimensions with respect to the cylindrical coordinates. Steady vortex rings in shear flow of an ideal fluid are governed by the non-linear elliptic partial differential equation

$$\mathcal{L}\Psi = \phi \circ (\Psi - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4)) \quad \text{in} \quad \Pi, \tag{4.1}$$

where, $\phi : \mathbb{R} \to \mathbb{R}$ is a non-negative increasing function. This equation for arbitrary ϕ represents the relationship that should exist between the vorticity and the Stokes stream function for which the flow is in steady state, see Lamb [28]. The main results show that there exists a solution (Ψ, ϕ) for (4.1) with the boundary conditions (BC: $\Psi(0, z) = 0$, $\Psi(r, z) \to 0$ and $|\nabla\Psi| \to 0$ as $r^2 + z^2 \to \infty$), for which the function $\zeta := \omega/r$ is a rearrangement of a prescribed function ζ_0. We use the theory of axisymmetric vortex -rings proposed by Benjamin [4] in the case of the uniform flow ($\sigma = 0$). In this theory, Benjamin advocated that even in unsteady flows, the function $\zeta := \omega/r$ is a rearrangement of the initial state and both the kinetic energy $E(\zeta)$ and the impulse $\mathcal{I}(\zeta)$ are preserved in axisymmetric motions of an ideal fluid. Additionally, he suggested that the steady state flows are characterized as maximisers of the functional $E - \lambda\mathcal{I}$ over the set where the function ζ is a rearrangement of a prescribed function ζ_0. The proof is achieved in two stages. We first show that there exist solutions (Ψ, ϕ) for (4.1) with the same boundary conditions (BC) for which the function ζ belongs to the weak closure in L^p $(p > 2)$ of the set of rearrangements of ζ_0. This is done by studying the maximisers of the functional $\Phi_\lambda^\sigma = E - \lambda \mathcal{I}_\sigma$ over the set of weak closure of the set of rearrangements of ζ_0. Here, $\mathcal{I}_\sigma(\zeta)$ represents a quantity called the generalized impulse depending on σ and defined by

$$\mathcal{I}_\sigma(\zeta) := \int_\Pi (\frac{1}{2}r^2 + \frac{\sigma}{4}r^4)\zeta(r,z)d\nu.$$

This method is similar to the one that was used by Badiani & Burton [2]. In the second stage, we use some estimates, to show that if λ is small and $\sigma \geq \lambda^\epsilon$ for some $\epsilon > 0$, then all the maximisers are in fact rearrangements of the prescribed function ζ_0. Therefore, we prove the existence of steady vortex rings in shear flows of an ideal fluid. We shoud mention that this problem is similar to the problem studied by Elcrati, Fornberg and Miller [19], in the case of flows with swirl and shear.

4.2 The main results

With the notions and mathematical formulations of Chapter 2 and 3, our main results are presented as follows:

Theorem 4.1 *Let $\frac{5}{2} < p < \infty$ and let $\zeta_0 \in L^p(\Pi, \nu)$ be a non-negative function vanishing outside a set of finite positive measure. Then the following hold:*

(i) Φ_λ^σ attains its supremum relative to the set $\mathcal{W}(\zeta_0)$ for all $\lambda > 0$ and $\sigma > 0$.

(ii) If $\lambda > 0$, $\sigma > 0$ and ζ is any maximiser for Φ_λ^σ relative to $\mathcal{W}(\zeta_0)$, then $\zeta \in \mathcal{RC}(\zeta_0)$ and

$$\zeta^{-1}(0,\infty) \subset \{(r,z) \in \Pi | K\zeta(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4) > 0\} \tag{4.2}$$

apart from a set of zero measure; moreover if $\zeta \notin \mathcal{R}(\zeta_0)$, then equality, up to a set of zero measure, holds in (4.2).

(iii) If $\lambda > 0$, $\sigma > 0$ and ζ is any maximiser for Φ_λ^σ relative to $\mathcal{W}(\zeta_0)$, then $\Psi := K\zeta$ satisfies the equation (4.1) almost everywhere in Π, for some increasing function ϕ.

This theorem should be compared with the main result (Theorem 1.2) of Badiani & Burton [2] in the case $\sigma = 0$. In fact, Badiani & Burton proved Theorem 1.2 by showing first the existence of maximising sequences comprising Steiner-symmetric (using Reisz' inequality, see Lieb & Loss [29], Theorem 3.7) supported by the strip $\Pi_X = \{(r,z)|r < X\}$, where X is a positive number provided by the following inequality

$$K\zeta(r,z) - \frac{\lambda}{2}r^2 < 0, \quad \text{for all} \quad r \geq X, \tag{4.3}$$

where $\zeta \in \mathcal{W}(\zeta_0)$. Thus, any maximiser ζ of Φ_λ^0 over $\mathcal{W}(\zeta_0)$ must be a Steiner-symmetric function supported by the strip Π_X. The two authors used this strategy to overcome loss of compactness resulting from the unboundedness of the domain Π.

Theorem 4.2 *Let $0 \leq \epsilon < \frac{1}{5}$. Let the assumptions about p and ζ_0 be the same as in Theorem 4.1 and let ζ be a maximiser of Φ_λ^σ provided by Theorem 4.1. Assuming $\sigma \geq \lambda^\epsilon$. Then there exists a positive number Λ depending only on ζ_0 such that $\zeta \in \mathcal{R}(\zeta_0)$ for all $\lambda \in (0,\Lambda)$.*

In the cylindrical domain $\Omega \subset \mathbb{R}^3$ defined by $|z| < \infty$ and $r < R$ ($R > 0$ constant), Burton [7] used his theory to obtain a similar result in the case of uniform flow; by maximising the functional Φ_λ^0 over the set $\mathcal{R}(\zeta_0)$, where he showed that the maximiser is a rearrangement of a prescribed function ζ_0 for all small positive λ. In the case where the Stokes stream function Ψ approaches at infinity $-\frac{\lambda}{4}r^4$ in the negative z-direction, Rebah [37] proved a similar result and that by using Burton' theory [7]. In a planar domain, Rebah used Burton' theory [7]again to obtain an existence theory of steady vortex pairs in two phase shear flow [38]. This theorem should be compared with the main result (Theorem 1.2) of Badiani & Burton [2] in the case $\sigma = 0$. In fact, Badiani & Burton proved Theorem 1.2 by showing first the existence of maximising sequences comprising Steiner-symmetric (using Reisz' inequality, see Lieb & Loss [29], Theorem 3.7) supported by the strip $\Pi_X = \{(r, z)|r < X\}$, where X is a positive number provided by the following inequality

$$K\zeta(r, z) - \frac{\lambda}{2}r^2 < 0, \quad \text{for all} \quad r \geq X, \tag{4.4}$$

where $\zeta \in \mathcal{W}(\zeta_0)$. Thus, any maximiser ζ of Φ_λ^0 over $\mathcal{W}(\zeta_0)$ must be a Steiner-symmetric function supported by the strip Π_X. The two authors used this strategy to overcome loss of compactness resulting from the unboundedness of the domain Π. Note that in this Chapter, we construct a solution for the equation (4.1). In Theorem 4.2, the maximiser ζ will be shown to give rise to a solution Ψ of the boundary-value problem for axisymmetric steady vortex rings in Poiseuille flow. Thus, Ψ is just a function of (r, z) only in cylindrical coordinates and satisfies (4.1), $\Psi(r,z) = 0$ when $r = 0$ and $\Psi(r,z) \to 0$ as $r^2 + z^2 \to \infty$. Therefore, the function $\Psi - \frac{\lambda}{2}(1 + \frac{\sigma}{2}r^2)r^2$ is the Stokes stream function of a steady ideal fluid flow, whose velocity in $\mathbb{R}^3$ is given by

$$\mathrm{V} = \left(-\frac{1}{r}\frac{\partial \Psi}{\partial z}, 0, \frac{1}{r}\frac{\partial \Psi}{\partial r} - \lambda(1 + \frac{\sigma}{2}r^2) \right).$$

Hence the vorticity in terms of Ψ is given by

$$\mathrm{curl}\mathrm{V} = (0, r(\mathcal{L}\Psi - \lambda), 0).$$

The study of steady vortex rings in an ideal fluid has been the subject of many investigations, specially in the case of uniform flow $\sigma = 0$. For different points of view than Benjamin [4], Fraenkel & Beger [23] proved existence theorems for steady vortex rings by proving the existence of a solution for the non-linear equation

$$\mathcal{L}\Psi = K\phi \circ (\Psi - \frac{\lambda}{2}r^2 - \gamma) \quad \text{in} \quad \Pi \tag{4.5}$$

with respect to (BC), where $K > 0$, $\gamma \geq 0$, $\lambda > 0$ are prescribed and ϕ is a prescribed non-decreasing Hölder continuous function. They used an approach based on variational principle for the Stokes stream function Ψ, where they maximised a certain functional on the surface of a sphere in a Sobolev space with energy norm. In the case when $\gamma = 0$ and ϕ is the Heaviside function ($\phi(t) = 0$ if $t \leq 0$ and $\phi(t) = 1$ if $t > 0$), Fraenkel & Amick [22] showed that any solution in $\mathcal{H}$ for (4.5) is equal, modulo a translation in the z-direction, to the explicit "spherical" solution found by Hill [26], where $\mathcal{H}$ is the Hilbert space obtained by completing $D(\Pi)$ with scalar product defined by

$$\langle u, v\rangle_{\mathcal{H}} = \int\limits_{\Pi} \frac{1}{r^2} \nabla u . \nabla v d\nu.$$

By using the contraction mapping theorem, Norbury [34] proved existence of a family of steady vortex rings close to Hill's spherical vortex rings. Ambrosetti & Struwe [1] used minimax method to construct a solution for (4.5) with prescribed k, γ and λ. The study of Ni [35] may be regarded as an extension of the work of Fraenkel & Berger [9]; he used a minimax principle to show that such functional has a critical point which gives rise to an existence theorem for vortex rings. Friedman & Turkington [24] showed that the existence of steady vortex rings is given by studying a variational problem in which the impulse $\mathcal{I}(\zeta) = I$ and the essential supremum of the vorticity is less than or equal to a prescribed constant.

4.3 Properties of the maximiser

In this section, we will use all estimates for the function $K\zeta$ that were proved in the last Chapter, to find some properties for the functional Φ_λ^σ, where $\zeta \in L^1 \cap L^p(\Pi, \nu)$ $(p > 2)$ is any non-negative function. Throughout in this section, C denotes any positive number depends only on q, where q is the conjugate exponent of p.

Lemma 4.1 *Let $p > \frac{5}{2}$ and let $\zeta_0 \in L^p(\Pi, \nu)$ be a non-negative function having support of finite volume. We assume that λ and σ are small and positive. Then*

(i) There exists a number $X > 2$ (depending on λ, σ and $\|\zeta\|_p$ only) such that, for all

$\zeta \in \mathcal{W}(\zeta_0)$,

$$K\zeta(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4) < 0 \quad \textit{for} \quad r > X.$$

(ii) If $\zeta \in \mathcal{W}(\zeta_0)$ *and* $h = \zeta 1_U$ *where* U *is the set on which* $K\zeta - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4)$ *is nowhere positive, then*

$$\Phi_\lambda^\sigma(\zeta - h) \geq \Phi_\lambda^\sigma(\zeta)$$

with strict inequality unless $h = 0$*, and in particular we can take* $U = (X, \infty) \times \mathbb{R}$*.*

(iii) Any maximiser of Φ_λ^σ *relative to* $\mathcal{W}(\zeta_0)$ *is supported in* $[0,\ X] \times \mathbb{R}$*.*

Proof. Note that if $\zeta \in \mathcal{W}(\zeta_0)$, then $\|\zeta\|_1 \leq \|\zeta_0\|_1$ and $\|\zeta\|_p \leq \|\zeta_0\|_p$, this follows from the definition of $\mathcal{W}(\zeta_0)$, so by using Lemma 3.1, $K\zeta(r,z) \leq\ C\|\zeta_0\|_p\, r\log r$ provided $r > 2$, where C is a positive number depending only on p, since $\zeta_0 \in L^p(\Pi, \nu)$ has support of finite volume. Now, for λ and σ small, we can find $X > 2$ depending on λ and σ for which

$$C\|\zeta_0\|_p \log\ X - \frac{\lambda}{2}X - \frac{\lambda\sigma}{4}X^3 = 0,$$

hence for $\zeta \in \mathcal{W}(\zeta_0)$ and $(r,z) \in \Pi$ with $r \geq X$, we have

$$K\zeta(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4) \leq (C\|\zeta_0\|_p \log\ X - \frac{\lambda}{2}X - \frac{\lambda\sigma}{4}X^3)X = 0.$$

Thus, (i) follows immediately.

Now, let U be the set where $K\zeta - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4)$ is nowhere positive and for $\zeta \in \mathcal{W}(\zeta_0)$ we set $h = \zeta 1_U$. Then we have

$$\begin{aligned}
\Phi^\sigma_\lambda(\zeta - h) &= \Phi^\sigma_\lambda(\zeta) - \int\limits_\Pi (K\zeta - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4))h + E(h) \\
&= \Phi^\sigma_\lambda(\zeta) - \int\limits_U (K\zeta - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4))\zeta + E(h) \\
&\geq \Phi^\sigma_\lambda(\zeta) + E(h).
\end{aligned}$$

Thus (ii) follows since $E(h) > 0$ unless $h = 0$.

Now, since $\zeta \in \mathcal{W}(\zeta_0)$, it follows that $\zeta - h \in \mathcal{W}(\zeta_0)$, hence by using (ii), we get (iii).

Now, in oder to prove the secont part *(ii)* of Theoreme 4.1, we need to the following Lemma proved by Badiani & Burton ■

Lemma 4.2 *Let* $1 < p < \infty$, *let* $\Omega \subset \Pi$ *be a set of infinite* ν*-measure, let* $\zeta_0 \in L^1 \cap L^p(\Pi, \nu)$ *be a non-negative function, let* $\mathcal{W}(\zeta_0, \Omega)$ *be the set of functions in* $\mathcal{W}(\zeta_0)$ *vanishing outside* Ω *and let* $\Psi \in L^\infty(\Omega)$. *Suppose* $\zeta \in \mathcal{W}(\zeta_0, \Omega)$

$$\int\limits_\Omega \zeta' \Psi \leq \int\limits_\Omega \zeta \Psi \quad \forall \zeta' \in \mathcal{W}(\zeta_0, \Omega) \backslash \{\zeta\}.$$

Then

1. $\zeta \in \mathcal{RC}(\zeta_0)$ *and* $\zeta^{-1}(0,\infty) \subset \Psi^{-1}(0,\infty)$, *apart from a set of zero measure;*

2. *If* $\zeta \notin \mathcal{R}(\zeta_0)$, *then* $\zeta^{-1}(0,\infty) = \Psi^{-1}(0,\infty) \cap \Omega$, *apart from a set of zero measure.*

From the last chapter, we deduce that the most important stage in the prove of the Theorem 4.2 is to show that if ζ is any maximiser provided by Theorem 4.1, then the set where $K\zeta(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4) > 0$ has a volume greater than $2\pi^2 a^3$ for some $0 < a < \infty$. To do this , we need to the following lemma

Lemma 4.3 *Let $\lambda > 0$ and let $\sigma \in (0,\ 1)$. Let $F(\zeta)$ be defined as in Lemma 3.4. Then for all $\lambda \in (0, (16\pi^2)^{-1}F(\zeta))$,*

$$\nu\{(r,z) \in \Pi | K\zeta(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4) > 0\} \geq \frac{1}{3}(\sqrt{\frac{F(\zeta)}{\lambda}} - 4\pi). \tag{4.6}$$

Proof. Observe that for all $(r,z) \in \Pi$ and for $\sigma \in (0,\ 1)$ the following inequality

$$\frac{r^2}{1+|R|^3} \geq \frac{2r^2 + \sigma\, r^4}{4(1+|R|^3)^2}$$

holds, so by using Lemma 3.4 it follows that

$$\begin{aligned} K\zeta(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4) &\geq \frac{r^2}{16\pi^2(1+|R|^3)}F(\zeta) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4) \\ &\geq (\frac{(2r^2 + \sigma\, r^4)}{64\pi^2(1+|R|^3)^2})F(\zeta) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4) \\ &\geq (\frac{F(\zeta)}{16\pi^2(1+|R|^3)^2} - \lambda)(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4) \end{aligned}$$

$$\nu\{(r,z) \in \Pi | K\zeta(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4) > 0\} \geq \nu\{|R| \leq (\sqrt{\frac{F(\zeta)}{16\pi^2\lambda}} - 1)^{\frac{1}{3}}\}.$$

Therefore the result follows immediately. This completes the proof. ■

4.4 Proof of our main results

Now, with these all properties, we are in good position to prove our main results.

Proof of the main Theorem 4.1

In order to prove this Theorem, we should use the same procedure as Badiani & Burton [2] used to prove Theorem 1.2. Indeed, let $\lambda > 0$, let $\sigma > 0$ and let $p > \frac{5}{2}$. Note that if

$\zeta \in \mathcal{W}(\zeta_0)$, then it follows that $\|\zeta\|_1 \leq \|\zeta_0\|_1$ and $\|\zeta\|_p \leq \|\zeta_0\|_p$. Now, Lemma 4.1 allows us to fix $X = X(\lambda,\ \sigma) > 0$ such that $K\zeta(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4) < 0$ for all $\zeta \in \mathcal{W}(\zeta_0)$ and all $r \in \Pi$ with $r \geq X$. Moreover, Lemma 4.1 has two consequences for the maximisation for Φ_λ^σ on $\mathcal{W}(\zeta_0)$, firstly, that any maximiser is supported by the strip Π_X and secondly, that any maximising sequence (ζ_n) for Φ_λ^σ can be replaced by one in $\mathcal{W}_X(\zeta_0)$.

Now, to prove (i), consider a maximising sequence (ζ_n) for Φ_λ^σ on $\mathcal{W}_X(\zeta_0)$. From the definitions, the Steiner-symmetrization ζ_n^s of ζ_n lies in $\mathcal{W}_X(\zeta_0)$; moreover $\mathcal{I}(\zeta_n^s) = \mathcal{I}(\zeta_n)$ and by using Reisz' inequality, Lieb & Loss [29], we have $E(\zeta_n^s) \geq E(\zeta_n)$. We may therefore suppose that (ζ_n) lies in $\mathcal{W}_X^s(\zeta_0)$. We may suppose that (ζ_n) converges weakly to $\bar{\zeta}$ in $L^p(\Pi_X)$. Now, since $\mathcal{W}_X^s(\zeta_0)$ is closed and convex, then it follows that $\bar{\zeta} \in \mathcal{W}_X^s(\zeta_0)$. Thus, by Lemma 3.8, $E(\zeta_n) \to E(\bar{\zeta})$ and $\mathcal{I}(\bar{\zeta}) \leq \lim_{n\to\infty} \inf \mathcal{I}(\zeta_n)$ since $\mathcal{I}$ is lower-discontinuous and convex. Hence $\bar{\zeta}$ is the desired maximiser for Φ_λ^σ relative $\mathcal{W}_X(\zeta_0)$.

To prove (ii), we argue as Douglas did for the cylindrical domain. Consider a maximiser $\bar{\zeta} \in \mathcal{W}_X(\zeta_0)$, and define $\Psi(r,z) = K\bar{\zeta}(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4)$, so Ψ is bounded on Π_X. If $\zeta^{'} \in \mathcal{W}_X(\zeta_0)$ and $\zeta^{'} \neq \bar{\zeta}$, then

$$\Phi_\lambda^\sigma(\bar{\zeta}) \geq \Phi_\lambda^\sigma(\zeta^{'}) = \Phi_\lambda^\sigma(\bar{\zeta}) + \int\limits_{\Pi_X} \Psi(\zeta^{'} - \bar{\zeta}) + E(\zeta^{'} - \bar{\zeta}) > \Phi_\lambda^\sigma(\bar{\zeta}) + \int\limits_{\Pi_X} \Psi(\zeta^{'} - \bar{\zeta}),$$

whence

$$\int\limits_{\Pi_X} \Psi\zeta^{'} < \int\limits_{\Pi_X} \Psi\bar{\zeta}.$$

Therefore, we deduce (ii) by applying Lemma 4.2.

Now, for $\bar{\zeta} \in \mathcal{W}_X(\zeta_0)$ a maximiser, we set $\Psi(r,z) := K\bar{\zeta}(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4)$. Then by

Lemma 3.2, $K\zeta(r,z) \to 0$ as $|z| \to \infty$, since $r < X$; hence $\Omega := \Psi^{-1}(0,\ \infty)$ is a bounded open set in Π_X. By (ii) we have $\zeta^{-1}(0,\ \infty) \subset \Omega$, apart from a set of zero measure, hence ζ has bounded support. Arguing as (ii), we further obtain

$$\int_\Omega \Psi(x)\zeta^{'}(x)dx < \int_\Omega \Psi(x)\zeta(x)dx$$

for all $\zeta^{'} \in \mathcal{R}(\zeta_0, \Omega)$, $\zeta^{'} \neq \bar{\zeta}$, and this ensures, by Burton [6], Theorem 2, that $\bar{\zeta} = \phi \circ \Psi$ almost everywhere in Ω, for some increasing function ϕ. Now, $inf(\Psi(\Omega)) = 0$. If we redefine $\phi(t) = 0$ for $t \leq 0$, then ϕ is again an increasing function, and satisfies $\bar{\zeta} = \phi \circ \Psi$ almost everywhere in Π. Thus (iii) is proved.

Now, we are in good position to prove Theorem 4.2. Our strategy is based on the showing that if ζ is maximiser of Φ_λ^σ provided by Theorem 4.1, the volume of the set where $K\zeta(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4) > 0$ tends to infinity when $\lambda \to 0$.

Proof of the main Theorem 4.2

Let $p \geq \frac{5}{2}$, let a be a positive number such that $\nu(\{(r,z) \in \Pi | \zeta_0(r,z) > 0\}) = 2\pi^2 a^3$ and let C be any positive number depending only on a and p. Let ζ be the maximiser of Φ_λ^σ over $\mathcal{W}(\zeta_0)$ provided by Theorem 4.1, then this implies that ζ is a Steiner symmetric function and for all $\lambda > 0$ and $\sigma > 0$ we have $\zeta \in \mathcal{RC}(\zeta_0)$. Moreover,

$$\zeta = \phi \circ (K\zeta - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4)), \tag{4.7}$$

almost everywhere in Π for some increasing function ϕ. Hence,

$$\zeta^{-1}(0,\infty) \subset \{(r,z) | K\zeta(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4)) > 0\}$$

apart from a set of zero volume. We now set

$$\mathbf{V} = \{(r,z) \in \Pi | K\zeta(r,z) - \lambda(\frac{1}{2}r^2 + \frac{\sigma}{4}r^4)) > 0\} \quad \text{and} \quad \mathbf{S} = \{(r,z)|\zeta(r,z) > 0\}.$$

To prove $\zeta \in \mathcal{R}(\zeta_0)$, we need first to show that if $\nu(\mathbf{V}) \geq 2\pi^2 a^3$, then this implies that $\nu(\mathbf{S}) \geq 2\pi^2 a^3$. Indeed, let ζ^{Δ} and ζ_0^{Δ} be the decreasing rearrangements of ζ and ζ_0 respectively. Assuming that $\nu(\mathbf{V}) \geq 2\pi^2 a^3$ but $\nu(\mathbf{S}) < 2\pi^2 a^3$. Since by Theorem 4.1, $\zeta \in \mathcal{RC}(\zeta_0)$, then there exists a rearrangement w of $\zeta_0^{\Delta} - \zeta^{\Delta}$ supported by the region $\mathbf{V}\backslash\mathbf{S}$, hence $w + \zeta$ is a rearrangement of ζ_0. Now, since $\Phi_\lambda^\sigma(w) > 0$ because w is supported by $\mathbf{V}\backslash\mathbf{S}$, then this implies that $\Phi_\lambda^\sigma(\zeta + w) > \Phi_\lambda^\sigma(\zeta)$ which shows a contradiction. Thus, if $\nu(\mathbf{V}) \geq 2\pi^2 a^3$, then $\nu(\mathbf{S}) \geq 2\pi^2 a^3$. By Lemma 4.3

$$\nu(\mathbf{V}) \geq \frac{4\pi}{3}(\sqrt{\frac{F(\zeta)}{16\pi^2\lambda}} - 1). \tag{4.8}$$

provided $\lambda \in (0,\ (16\pi^2)^{-1}F(\zeta))$ and $\sigma \in (0,\ 1)$ where $F(\zeta)$ is defined as in Lemma 3.4. If we want to prove that $\nu(\mathbf{V}) \geq 2\pi^2 a^3$, then it is sufficient for us to find a lower bound depending on λ for $F(\zeta)$, for which the right side in (4.8) tends to ∞ as $\lambda \to 0$. Henceforth, we assume that $\sigma \in (\lambda^\epsilon,\ 1)$, with $0 \leq \epsilon < \frac{1-3\tau}{17-3\tau}$ and $\tau = (\frac{1}{3})^9$. Also, let X be the solution of equation

$$C\|\zeta_0\|_p \log t - \lambda(\frac{1}{2}t + \frac{\sigma}{4}t^3) = 0, \tag{4.9}$$

so for $\sigma \in (\lambda^\epsilon,\ 1)$ we have $X \to \infty$ as $\lambda \to 0$, which means that we can choose $\lambda_0 > 0$ to ensure that $X > 2$ for all $\lambda \in (0, \lambda_0)$, moreover

$$\frac{1}{\lambda} \geq \left(\frac{X^3}{4C\|\zeta_0\|_p \log X}\right)^{\frac{1}{1+\epsilon}}. \tag{4.10}$$

Therefore, by Lemma 4.1 and (4.9) we can take Π_X as the strip that supported the maximiser ζ. Also, we can write $\Pi_X = \bigcup_{j=1}^{j=3} S_j(X)$, where

$$S_1(X) = \{(r,z) \in \Pi_X | r \leq t_2(\frac{X^3}{\log X})^{\frac{1}{4}} \text{ and } |z| \geq t_1(X^3 \log X)^{\frac{1}{2}}\},$$

$$S_2(X) = \{(r,z) \in \Pi_X | t_2(\frac{X^3}{\log X})^{\frac{1}{4}} \leq r < X \text{ and } |z| \geq t_1(X^3 \log X)^{\frac{1}{2}}\},$$

$$S_3(X) = (0,\ X) \times (-t_1(X^3 \log X)^{\frac{1}{2}},\ t_1(X^3 \log X)^{\frac{1}{2}}),$$

and t_1 and t_2 are two positive numbers satisfying $\frac{t_2^2}{t_1} = \frac{\alpha}{4C\|\zeta_0\|_1\|\zeta_0\|_p}$ and $t_2^2 t_1 = 4$. Now, let λ_1 be the positive numbers chosen so that $\alpha = \Phi^1_{\lambda_1}(\zeta_0) > 0$, then for $0 < \lambda < \min\{\lambda_0, \lambda_1\}$ we have $\Phi^\sigma_\lambda(\zeta) \geq \alpha$; hence it follows that

$$\begin{aligned}\int\limits_{\Pi_X} \zeta(r,z)K\zeta(r,z)d\nu &\geq 2\alpha + \lambda \int\limits_{\Pi_X} r^2\zeta(r,z)d\nu + \frac{\lambda\sigma}{2}\int\limits_{\Pi_X} r^4\zeta(r,z)d\nu \\ &\geq 2\alpha + (\frac{\lambda}{X^2} + \frac{\lambda\sigma}{2})\int\limits_{\Pi_X} r^4\zeta(r,z)d\nu, \qquad (4.11)\end{aligned}$$

where the last line has been obtained by using the fact that

$$\int\limits_{\Pi_X} r^2\zeta(r,z)d\nu \geq X^{-2}\int\limits_{\Pi_X} r^4\zeta(r,z)d\nu.$$

By Lemma 3.2, for all $r \geq 2$ and $|z| \geq 2$, we can choose a positive number C for which

$$K\zeta(r,z) \leq C\|\zeta_0\|_p r^2 \frac{\log|z|}{|z|}. \qquad (4.12)$$

Then from (4.11) it follows that

$$\begin{aligned}\int\limits_{S_3(X)} \zeta(r,z)K\zeta(r,z)d\nu &\geq 2\alpha - \int\limits_{S_1(X)} \zeta(r,z)K\zeta(r,z)d\nu + (\frac{\lambda}{X^2} + \frac{\lambda\sigma}{2})\int\limits_{\Pi_X} r^4\zeta(r,z)d\nu \\ &\quad - \int\limits_{S_2(X)} \zeta(r,z)K\zeta(r,z)d\nu. \qquad (4.13)\end{aligned}$$

Now, using (4.12) yields

$$\int\limits_{S_1(X)} \zeta(r,z)K\zeta(r,z)d\nu \leq C\|\zeta_0\|_p\|\zeta_0\|_1(t_2^2(\frac{X^3}{\log X})^{\frac{1}{2}}\frac{4\log X}{t_1(X^3\log X)^{\frac{1}{2}}} = \alpha. \tag{4.14}$$

In the other hand, by (4.12) we have

$$\begin{aligned}\int\limits_{S_2(X)} \zeta(r,z)K\zeta(r,z)d\nu &\leq C\|\zeta_0\|_p \int\limits_{S_2(X)} \frac{r^4\log|z|}{r^2|z|}\zeta(r,z)d\nu \\ &\leq \frac{4C\|\zeta_0\|_p(\log X)}{t_2^2 t_1 X^3}\int\limits_{\Pi_X} r^4\zeta(r,z)d\nu \\ &= (\frac{\lambda}{2X^2}+\frac{\lambda\sigma}{4})\int\limits_{\Pi_X} r^4\zeta(r,z)d\nu, \end{aligned} \tag{4.15}$$

where we have used (4.9) to obtain the last line. Thus, by combining (4.13) with (4.14) into (4.15) we find

$$\int\limits_{S_3(X)} \zeta(r,z)K\zeta(r,z)d\nu \geq \alpha \tag{4.16}$$

Now, using the fact that ζ is Steiner-symmetric and the above inequality, it follows that

$$\begin{aligned}\int\limits_{r<X,|z|\leq X} \zeta(r,z)K\zeta(r,z)d\nu &\geq \frac{X}{t_1(X^3\log X)^{\frac{1}{2}}}\int\limits_{S_3(X)} \zeta(r,z)K\zeta(r,z)d\nu \\ &\geq \frac{\alpha}{t_1}X^{-\frac{1}{2}}(\log X)^{-\frac{1}{2}}.\end{aligned}$$

Thus, by Lemma 3.1 and Lemma (3.2) we obtain

$$\int\limits_{r<X,|z|<X,r\geq|z|} r\zeta(r,z)d\nu + \int\limits_{r<X,|z|<X,r\leq|z|} \frac{r^2}{|z|}\zeta(r,z)d\nu \geq \frac{\alpha}{Ct_1\|\zeta_0\|_p}X^{-\frac{1}{2}}(\log X)^{-\frac{3}{2}} \tag{4.17}$$

We recall that

$$F(\zeta) = \int\limits_{\Pi} \frac{r^2}{1+|R|^3}\zeta(r,z)d\nu,$$

where $R^2 = r^2 + z^2$. Now, if $r < X$, $|z| < X$ and $r \geq |z|$, then it follows that $R^3 \leq 4X^2 r$, also if if $r < X$, $|z| < X$ and $r \leq |z|$, then $R^3 \leq 4X^2|z|$, hence by using (4.17) we have

$$\begin{aligned} F(\zeta) &\geq \int\limits_{r<X,|z|<X} \frac{r^2}{1+|R|^3}\zeta(r,z)d\nu \\ &\geq \frac{1}{4X^2}\left(\int\limits_{r<X,|z|<X,r\geq|z|} r\zeta(r,z)d\nu + \int\limits_{r<X,|z|<X,r\leq|z|} \frac{r^2}{|z|}\zeta(r,z)d\nu\right) \\ &\geq \frac{\alpha}{Ct_1\|\zeta_0\|_p}X^{-\frac{5}{2}}(\log X)^{-\frac{3}{2}}. \end{aligned} \tag{4.18}$$

Therefore

$$\frac{1}{\lambda}F(\zeta) \geq C\|\zeta_0\|_p^{-\frac{\epsilon}{1+\epsilon}} X^{\frac{1-5\epsilon}{2(1+\epsilon)}}(\log X)^{-\frac{5}{2}}. \tag{4.19}$$

Since $\epsilon \in (0, \frac{1}{5})$ and $X \to \infty$ as $\lambda \to 0$, then this yields that the right-side in (4.19) tends to ∞ as $\lambda \to 0$, which means that we can choose $\lambda_2 > 0$ such that if $\lambda \in (0, \Lambda)$, where $\Lambda = \min\{\lambda_0, \lambda_1, \lambda_2\}$, then $\nu(\mathbf{V}) \geq 2\pi a^3$. Therefore, there exists $\Lambda > 0$ such that if $\lambda \in (0, \Lambda)$, then $\zeta \in \mathcal{R}(\zeta_0)$. This completes the proof of Theorem 4.2.

Chapter 5

Steady vortex pairs in two-phase shear flow

5.1 Introduction

In this Chapter, we study the existence theory of steady flows described by a variational problem similar to the one governing steady 2-dimensional ideal fluid flows containing symmetric vortex pairs. The flow in question is written in terms of a stream function $\Psi : \Pi \to \mathbb{R}$ with Ψ even in x_1, where Π is the half-plane defined in $\mathbb{R}^2$ by $x_2 > 0$. At infinity the stream function approaches $-\lambda(x_2 + \frac{\sigma}{2}x_2^2)$ which representing a flow of velocity $\lambda(1+\sigma x_2)$ in the negative x_1-direction, where λ is a positive constant depends the speed of the flow at the infinity and σ is a positive number represents the severity of the shearing. The vorticity is described by $-\Delta\Psi$, where Δ is the Laplacian operator in two dimensions, $-\Delta\Psi$ vanishes outside a bounded region placed symmetrically about the x_2 axis, and avoiding the x_1 axis. The vorticity in the region $x_2 > 0$ is non-negative, and Ψ satisfies the equation

$$-\Delta\Psi = \phi \circ \Big(\Psi - \lambda(x_2 + \frac{\sigma}{2}x_2^2)\Big)\,, \tag{5.1}$$

where ϕ is an unknown non-negative function. This equation represents the relationship should exist between vorticity and the stream function when the flow is in a steady state, see Lamb [28], page 244. The solution of this problem can be extended by reflection form Π to $\mathbb{R}^2$ by making the stream function odd in x_2 to yield a symmetric vortex pairs in a two phase shear flow. The results of this study prove that for $\lambda > 0$ small, a solution (Ψ, ϕ) of this problem exists, for which the vorticity field is a rearrangement of a prescribed non-negative function $\zeta_0 \in L^p$ $(p > 2)$ which has support of finite measure. This approach is an application of a theory proposed by Benjamin [4] for vortex rings in three dimensions

In [38], Rebah showed the same result by maximising a functional related to the kinetic energy over the set of rearrangements of ζ_0 to obtain the vorticity ζ. The variational principle for this problem has two mathematical difficulties, the first is in the nature of the set of rearrangements (as a subset of L^p). The second in loss of compactness which arises from unbounded domain Π. In order to overcome these difficulties, Rebah [38] solved this problem in bounded domain in Π by using Burton' theory [6]; passing to the unbounded domain is accomplished by using some estimates to show that a solution in a sufficiently large bounded domain is in fact valid throughout the half-plane. In this chapter, we use another strategy to obtain the same result that found by Rebah [38], we first show that the functional related to kinetic energy attains its supremum relative to the closed convex hull (in suitable L^p-space) of rearrangements of ζ_0. A convexity argument shows that any maximiser must be an extreme points of this set, where Douglas's [17, 18] characterization of the extreme points are rearrangements of curtailments.

Burton [7] was concerned with this problem, specifically in the case when the stream function approaches $-\lambda x_2$ at infinity. The relationship between the vorticity and the stream function is given

$$-\Delta\Psi = \phi \circ (\Psi - \lambda x_2),$$

almost everywhere in Π for some increasing function ϕ. For small positive λ, Burton proved the existence of a solution for this equation for which the vorticity is a rearrangement of a given non-negative function. Also, in this case, other authors were concerned with similar problems. In particular, Badiani [3] proved the existence of steady planar flow of an ideal fluid past an obstacle, and Turkington [21, 24] showed the existence of vortex pairs in flows occupying the whole space $\mathbb{R}^2$ or $\mathbb{R}^2 \backslash D$, where D is a bounded, simply connected region, symmetric in the x_1-direction, containing the origin in its interior and having smooth boundary.

5.2 Mathematical formulation

We denote points of $\mathbb{R}^2$ by $x = (x_1, x_2)$, and we consider always the half-plane Π defined by $x_2 > 0$. For X, we define Π_X by the set of all points in Π such that $x_2 < X$, so Π_X is a strip of Π. We use $|A|$ to denote the two-dimensional Lebesgue measure of a measurable set $A \subset \mathbb{R}^2$. For $p > 2$ and $\zeta \in L^1 \cap L^p(\Pi)$, we define the function $K\zeta$ by

$$K\zeta(x) = \frac{1}{2\pi} \int\limits_{\Pi} \log\left(\frac{|x-\bar{y}|}{|x-y|}\right) \zeta(y)dy;$$

hence, by Burton [11], $K\zeta$ is the solution for the problem $-\Delta\Psi = \zeta$, $\Psi(x_1, 0) = 0$ and $\Psi(x) \to 0$ as $|x| \to \infty$. Here the function G defined by

$$G(x,y) = \frac{1}{2\pi} \log\left(\frac{|x-\bar{y}|}{|x-y|}\right)$$

is the Green function for $-\Delta$ with homogeneous Dirichlet boundary condition on Π, where $\bar{y} = (y_1, -y_2)$ is called the reflection of the point $y \in \Pi$. The kinetic energy and the generalised impulse are given respectively in terms of ζ

$$E(\zeta) = \frac{1}{2} \int\limits_{\Pi} \zeta(x) K\zeta(x) dx,$$

$$\mathcal{I}_\sigma(\zeta) = \int\limits_{\Pi} (x_2 + \frac{\sigma}{2} x_2^2) \zeta(x) dx.$$

Also, for $\lambda > 0$, we define $\Phi_\lambda^\sigma(\zeta) \in [-\infty\ , \infty[$ by

$$\Phi_\lambda^\sigma(\zeta) = \frac{1}{2} \int\limits_{\Pi} \left(K\zeta(x) - 2\lambda x_2 - \lambda\sigma x_2^2\right) \zeta(x) dx,$$

so that if $\mathcal{I}_\sigma(\zeta)$ is finite, then $\Phi_\lambda^\sigma(\zeta) = E(\zeta) - \lambda \mathcal{I}_\sigma(\zeta)$. The functional Φ_λ^σ is called the total energy.

A non-negative measurable function ζ defined on Π will be called Steiner-symmetric if

$$0 \le x_1 \le x_1' \Longrightarrow \zeta(-x_1, x_2) = \zeta(x_1, x_2) \ge \zeta(x_1', x_2).$$

Now, if ζ_0 is a non-negative measurable function on Π that satisfies $|\zeta_0^{-1}(k,\infty)| < \infty$ for all $k > 0$, the decreasing rearrangement ζ_0^{Δ} of ζ_0 can be defined as the essentially unique non-negative decreasing function on $(0,\infty)$ that satisfies also $|\zeta_0^{-1}(k,\infty)| = \mu_1((\zeta_0^{\Delta})^{-1}(k,\infty))$ for all $k > 0$, where μ_1 is the one-dimensional Lebesgue measure. We use $\mathcal{F}(\zeta_0)$ to denote the set of all rearrangements of ζ_0 on Π; hence $\mathcal{F}(\zeta_0)$ can have some functions, denoted by ζ_s , that are Steiner-symmetric. With the same function ζ_0 we define the sets $\mathcal{W}(\zeta_0)$ and $\mathcal{RC}(\zeta_0)$ as follows:

$$\mathcal{RC}(\zeta_0) = \{\zeta \geq 0 \text{ measurable on } \Pi \,|\, \zeta^s = \zeta_0^{\Delta} 1_{[0,\beta]} \text{ for some } \beta \geq 0\},$$

$$\mathcal{W}(\zeta_0) = \{\zeta \geq 0 \text{ measurable on } \Pi \mid \forall k > 0 \int_{\Pi} (\zeta - k)_+ \leq \int_{\Pi} (\zeta_0 - k)_+\},$$

where the sign + represents the positive part. If $\Omega \subset \Pi$ is measurable, then $\mathcal{R}(\zeta_0, \Omega)$, $\mathcal{RC}(\zeta_0, \Omega)$ and $\mathcal{W}(\zeta_0, \Omega)$ denote the sets of functions in $\mathcal{F}(\zeta_0)$, $\mathcal{RC}(\zeta_0)$ and $\mathcal{W}(\zeta_0)$ vanishing outside Ω. Moreover, when $|\Omega| = \infty$ and $\zeta_0 \in L^p(\Omega)$ is a non-negative, Douglas [17] proved the following results

Proposition 5.1 *1. $\mathcal{W}(\zeta_0, \Omega)$ is weakly compact convex set in $L^p(\Omega)$;*

2. $\mathcal{RC}(\zeta_0, \Omega)$ is the set of extreme points of $\mathcal{W}(\zeta_0, \Omega)$;

3. $\mathcal{W}(\zeta_0, \Omega)$ is the closed convex hull of $\mathcal{F}(\zeta_0, \Omega)$;

4. $\mathcal{R}(\zeta_0, \Omega)$ is weakly dense in $\mathcal{W}(\zeta_0, \Omega)$

The elements of $\mathcal{RC}(\zeta_0, \Omega)$ are called the rearrangements of curtailment of ζ_0. According to these properties, we have

$$\mathcal{F}(\zeta_0) \subset \mathcal{RC}(\zeta_0) \subset \mathcal{W}(\zeta_0),$$

moreover, if $\zeta \mathcal{W}(\zeta_0)$, then it follows that $\|\zeta\|_p \leq \|\zeta_0\|_p$, where ≥ 1. Now with all this notation and definitions our main results are given as follows:

Theorem 5.1 *Let $2 < p < \infty$ and let $\zeta_0 \in L^p(\Pi)$ be a non-negative function having support of finite measure. Then, the following hold*

- *(i) The functional Φ_λ^σ attains its supremum relative to $\mathcal{W}(\zeta_0)$ for every λ and σ*
- *(ii) If $\lambda > 0, \quad \sigma > 0$ and ζ is any maximiser for Φ_λ^σ relative to $\mathcal{W}(\zeta_0)$, then $\zeta \in \mathcal{RC}(\zeta_0)$ and*

  $$\zeta^{-1}(0, \infty) \subset \{x \in \Pi \ | \ K\zeta(x) - \lambda(x_2 + \frac{\sigma}{2}x_2^2) > 0\}$$

  .
- *(iii) If $\lambda > 0, \quad \sigma > 0$ and ζ is a maximiser, then there exists an increasing function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\Psi := K\zeta$ and ϕ satisfies equation (5.1) almost everywhere in Π.*
- *(iv) If $\sigma \geq \lambda^\epsilon$, There exists $\lambda_0 > 0$ depending on ζ_0 such if ζ is a maximiser for Φ_λ^σ relative $\mathcal{W}(\zeta_0)$ and $0 < \lambda < \lambda_0$, then $\zeta \in \mathcal{R}(\zeta_0)$.*

In order to prove this resul, we will use the same strategy that Badian and Burton used in [2] to prove the existence of steady of vortex ring.

5.3 Estimate for the function $K\zeta$ and properties of E

Let us start with some lemmas presenting some estimates for the function $K\zeta$ and some properties for the functional E which are proved by Rebah in [38] and Burton [7].

Lemma 5.1 *For any non-negative function $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ $(p > 1)$, and for all $x \in \Pi$ we have*

$$K\zeta(x) \geq \frac{x_2}{2\pi(1+|x|^2)} \int\limits_{\Pi} \frac{y_2}{1+|y|^2}\zeta(y)dy.$$

Furthermore

$$E(\zeta) \geq \frac{1}{4\pi}\left(\int\limits_{\Pi} \frac{y_2}{1+|y|^2}\zeta(y)dy\right)^2.$$

Lemma 5.2 *Let $\sigma \in (0,\ 1)$, $\lambda > 0$, let $p > 1$ and let $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ be a non-negative function independent of λ. Then for all $\lambda \in (0, \frac{C(\zeta)}{2\pi})$, we have*

$$|\{x \in \Pi | K\zeta(x) - \lambda(x_2 + \frac{\sigma}{2}x_2^2) > 0\}| \geq \pi((\frac{C(\zeta)}{2\pi\lambda})^{1/2} - 1),$$

where

$$C(\zeta) = \int\limits_{\Pi} \frac{y_2}{1+|y|^2}\zeta(y)dy.$$

Proof. By using Lemma 5.1, for all $x \in \Pi$ we have

$$K\zeta(x) \geq \frac{x_2 C(\zeta)}{2\pi(1+|x|^2)}.$$

Since $x_2 < 1 + |x|^2$ and $\sigma \in (0,\ 1)$, then it follows that

$$K\zeta(x) \geq \left(\frac{x_2 + \frac{\sigma}{2}x_2^2}{2\pi(1+|x|^2)^2}\right) C(\zeta);$$

hence we find that

$$K\zeta(x) - \lambda(x_2 + \frac{\sigma}{2}x_2^2) \geq \left(\frac{C(\zeta)}{2\pi(1+|x|^2)^2} - \lambda\right).$$

Therefore we have

$$\begin{aligned} |\{x \in \Pi | K\zeta(x) - \frac{\lambda}{2}x_2^2 > 0\}| &\geq |\{x \in \Pi | |x|^2 < (\frac{C(\zeta)}{2\pi\lambda})^{1/2} - 1\}| \\ &= \frac{\pi}{2}((\frac{C(\zeta)}{2\pi\lambda})^{1/2} - 1). \end{aligned}$$

This completes the proof. ■

Lemma 5.3 *Let $\zeta \in L^p(\Pi)$ $(p \geq 2)$ be a non-negative function having support of finite measure. Then for all $x \in \Pi$, there exists a positive number C depends only on q for which the following inequality*

$$K\zeta(x) \leq C\|\zeta\|_p \log(1 + 2x_2)$$

holds, where q the conjugate exponent of p.

Proof. First of all, for x and y in Π we have

$$\begin{aligned} |x - \bar{y}|^2 &= |x-y|^2 + 4x_2y_2 \\ &\leq |x-y|^2 + 4x_2(y_2 - x_2) + 4x_2^2 \\ &\leq |x-y|^2 + 4x_2|x-y| + 4x_2^2. \\ &= (|x-y| + 2x_2)^2, \end{aligned}$$

hence

$$G(x,y) = \frac{1}{2\pi}\log\left(\frac{|x-\bar{y}|}{|x-y|}\right) \leq \frac{1}{2\pi}\log\left(1 + \frac{2x_2}{|x-y|}\right). \tag{5.2}$$

Thus, it follows that

$$\int\limits_{|x-y|\geq 1} G(x,y)\zeta(y)dy \leq \frac{\|\zeta\|_1}{2\pi}\log(1+2x_2). \tag{5.3}$$

Since $t \longmapsto t^{-1}\log(1+2x_2t)$ is a decreasing function on $]1,\ \infty[$ for any $x_2 > 0$, then for all $t > 1$ we get

$$\frac{\log(1+2x_2t)}{t} \leq \log(1+2x_2), \tag{5.4}$$

hence, it follows

$$\begin{aligned}\int\limits_{|x-y|<1} \log\left(1+\frac{2x_2}{|x-y|}\right)\zeta(y)dy &= \int\limits_{|x-y|<1} |x-y|\log\left(1+\frac{2x_2}{|x-y|}\right)\frac{\zeta(y)}{|x-y|}dy \\ &\leq \log\left(1+2x_2\right)\int\limits_{|x-y|<1} \frac{\zeta(y)}{|x-y|}dy \\ &\leq (\frac{2\pi}{2-q})^{\frac{1}{q}} \parallel \zeta \parallel_p \log\left(1+2x_2\right), \end{aligned} \tag{5.5}$$

where we used holer's inequality to obtain the last line. Therefore, from (5.3), (5.4) and (5.5) we can choose a positive number C depends only on q for which our inequality holds. This completes the proof. ■

Lemma 5.4 *If $\zeta \in L^1(\Pi)$ is a Steiner-symmetric, then for any $\alpha \in (0,1)$ and for all $x = (x_1, x_2) \in \Pi$ with $x_1 \neq 0$, the following inequality*

$$K\zeta(x) \leq C\|\zeta\|_1\left(\frac{x_2}{|x_1|}\right)^{\alpha}$$

holds, where C is a positive numbber depends on α.

Proof. We adapt the same method used by Rebah [37] to prove Lemma 4.4. Indeed, for all $(x_1, x_2) \in \Pi$ with $x_1 \neq 0$ we set $(x_1 - y_1) = \pm t$ and write

$$K\zeta(x) = \left(\int\limits_{|t| \leq \frac{1}{2}|x_1|} + \int\limits_{|t| > \frac{1}{2}|x_1|} \right) G(x,y)\zeta(y)dy \tag{5.6}$$

Now, for any $\alpha \in (0,1)$, from (5.2) we can have

$$\begin{aligned} G(x,y) &\leq \frac{1}{2\alpha\pi} \log\left(1 + \frac{2x_2}{|x-y|}\right)^{\alpha} \\ &\leq \frac{2^{\alpha-1}}{\alpha\pi}\left(\frac{x_2}{|t|}\right)^{\alpha}, \end{aligned} \tag{5.7}$$

where we have used that the fact that $|x-y| \geq |t|$. Thus, it follows that

$$\int\limits_{|t| > \frac{1}{2}|x_1|} G(x,y)\zeta(y)dy \leq \frac{2^{2\alpha-1}}{\alpha\pi}\|\zeta\|_1 (\frac{x_2}{|x_1|})^{\alpha}. \tag{5.8}$$

We recall that if $\zeta \in L^1(\Pi)$ is a Steiner-symmetric, then

$$V(x_1) := \int\limits_0^{\infty} \zeta(x_1,\ x_2)dx \leq \frac{\|\zeta\|_1}{|x_1|},$$

hence

$$\begin{aligned} \int\limits_{|t| \leq \frac{1}{2}|x_1|} (\frac{x_2}{|x_1|})^{\alpha}\zeta(y)dy &= x_2^{\alpha} \int\limits_0^{\frac{1}{2}|x_1|} \int\limits_0^{\infty} \left(\frac{\zeta(x_1+t, y_2)}{|t|^{\alpha}} + \frac{\zeta(x_1-t, y_2)}{|t|^{\alpha}} \right) dy_2 dt \\ &\leq 2x_2^{\alpha}\|\zeta\|_1 \int\limits_0^{\frac{1}{2}|x_1|} \left(\frac{1}{|t+x_1|} + \frac{}{|t-x_1|} \right) \frac{dt}{|t|^{\alpha}}. \end{aligned} \tag{5.9}$$

Since $0 < t \leq \frac{1}{2}|x_1|$ then it follows that $|t + x_1| > |-t + x_1| > \frac{1}{2}|x_1|$ if $x_1 > 0$ and $|-t + x_1| > |t + x_1| > \frac{1}{2}|x_1|$ if $x_1 < 0$. Thus, by using (5.7) and (5.9) we get

$$\int\limits_{|t| \leq \frac{1}{2}|x_1|} G(x,y)\zeta(y)dy \leq \frac{2^{2\alpha}\|\zeta\|_1}{\pi\alpha(1-\alpha)}\left(\frac{x_2}{|x_1|}\right)^{\alpha}. \tag{5.10}$$

Therefore, our inequality follows immediately from (5.6), (5.7) and (5.10), where C is a positive number depends on α. ∎

Lemma 5.5 *Let $2 \leq p < \infty$, let q be the conjugate exponent of p and let Ω be a bounded open subset of Π. Then $K : L^p(\Omega) \to L^q(\Omega)$ is compact in the sense that if $\{\zeta_n\}_{n=1}^{\infty}$ is a sequence of functions bounded in $L^p(\Pi)$ and vanishing outside Ω, then the restriction to Ω of $K\zeta_n$ has a subsequence converging in the q-norm .*

Proof. The case when $p > 2$ has been proved in Burton [9], Lemma 8, so we need just to prove the case where $p = 2$. Now, it is easy to show

$$G(x, y) \leq \frac{2^{1/k} k (x_2 y_2)^{1/2k}}{2\pi |x - y|^{1/k}}$$

for all $k \geq 1$ and all x and y in Π, so

$$\int\limits_{\Omega}\int\limits_{\Omega} |G(x, y)|^2 dx dy \leq \frac{2^{2/k} k^2}{4\pi^2} \int\limits_{\Omega}\int\limits_{\Omega} \frac{(x_2 y_2)^{1/k}}{|x - y|^{2/k}} dx dy < \infty,$$

hence $G \in L^2(\Omega \times \Omega)$. Since $L^2(\Omega)$ is a Hilbert space, then it follows from [7, Theorem vi] that K is Hilbert-Schmidt operator, so K is compact. ∎

Lemma 5.6 *Let $2 < p < \infty$ and let $\zeta \in L^p(\Pi)$ have bounded support. Then $\nabla K\zeta(x) = O(|x|^{-2})$ and $K\zeta(x) = O(|x|^{-1})$ as $|x| \to \infty$, and*

$$\int\limits_{\Pi} |\nabla K\zeta|^2 dx = \int\limits_{\Pi} \zeta K\zeta dx < \infty.$$

This lemma shows that the operator K is strictly positive on $L^p(\Omega)$ for any open subset Ω in Π and for $p > 2$, see [Burton [11] Lemma 7].

5.4 Properties of the maximiser

In this section, we use all estimates found in Section 3 to give some properties for the maximiser. We start this suction by stating some results proved by Burton [11].

Lemma 5.7 *Let $\zeta \in L^1(\Pi)$ be a non-negative function and have bounded support. Then*

(i) $\int\limits_{\Pi} \zeta(x)K\zeta(x)dx \leq \int\limits_{\Pi} \zeta^s(x)K\zeta^s(x)dx,$

(ii) if ζ is Steiner-symmetric then $K\zeta$ is Steiner -symmetric.

This lemma shows that any maximiser for the functional Φ_λ^σ is Steiner-symmetric.

Lemma 5.8 *Let $2 < p < \infty$ and let $\zeta_0 \in L^1(\Pi) \cap L^p(\Pi)$ be a non-negative function. Let $X > 0$ and let $\mathcal{W}_X^s(\zeta_0)$ denote the set of all functions in $\mathcal{W}(\zeta_0)$ that are Steiner-symmetric about the x_2 axis and supported by Π_X. Let $\{\zeta_j\}_{j=1}^\infty \subset \mathcal{W}_X^s(\zeta_0)$ be a sequence converging to ζ in $L^p(\Pi)$ weakly. Then we have*

$$E(\zeta_j) \longrightarrow E(\zeta) \quad \textit{as} \quad j \to \infty.$$

Proof. If $\zeta \in \mathcal{W}_X^s(\zeta_0)$, then by using Lemma 5.4 we have

$$K\zeta(x) \leq C\|\zeta_0\|_1 (\frac{X}{|x_1|})^\alpha$$

for all $(x_1,\ x_2) \in \Pi_X$ with $x_1 \neq 0$, where $\alpha \in (0,\ 1)$. Thus, for any $\xi > 2$ we have

$$\left| \int\limits_{x_2 < X, |x_1| \geq \xi} (\zeta_j(x)K\zeta_j(x) - \zeta(x)K\zeta(x))dx \right| \leq C\|\zeta_0\|_1^2 (\frac{X}{\xi})^\alpha,$$

hence that for given $\varepsilon > 0$, we can find $\xi_0 > 2$ such that for all $\xi > \xi_0$ and all $j \in \mathbb{N}$

$$\left| \int_{x_2<X,|x_1|>\xi} (\zeta_j(x)K\zeta_j(x) - \zeta(x)K\zeta(x))dx \right| < \frac{\varepsilon}{3}. \tag{5.11}$$

Now, setting $\Pi(X,\xi_0) = (-\xi,\ \xi) \times (0,\ X)$ and we write

$$\begin{aligned} \int_{\Pi(X,\xi_0)} (\zeta_j(x)K\zeta_j(x) - \zeta(x)K\zeta(x))dx \;&=\; \int_{\Pi(X,\xi_0)} (\zeta_j(x) - \zeta(x))K\zeta(x)dx \\ &\quad + \int_{\Pi(X,\xi_0)} \zeta_j(x)(K\zeta_j(x) - K\zeta(x))dx. \end{aligned}$$

By using Hölder's inequality we get

$$\left| \int_{\Pi(X,\xi_0)} \zeta_j(x)(K\zeta_j(x) - K\zeta(x))dx \right| \le \|\zeta_0\|_p \|K\zeta_j - K\zeta\|_{L^q(\Pi(X,\xi_0))}.$$

From Lemma 5.5

$$K : L^p(\Pi(X,\xi_0)) \to L^q(\Pi(X,\xi_0))$$

is a compact operator, where q is the conjugate exponent of p. Thus by using [4, Remark 2, page 91] $K\zeta_j \to K\zeta$ strongly in $L^q(\Pi(X,\xi_0))$, so we can choose j_0 such that for all $j > j_0$

$$\left| \int_{\Pi(X,\xi_0)} \zeta_j(x)(K\zeta_j(x) - K\zeta(x))dx \right| < \frac{\varepsilon}{3}. \tag{5.12}$$

Since $\{\zeta_j\}_{j\ge1}$ converges weakly to ζ, then for given $\varepsilon > 0$ we can find j_1 such that

$$\left| \int_{\Pi(X,\xi_0)} (\zeta_j(x) - \zeta(x))K\zeta(x)dx \right| < \frac{\varepsilon}{3} \tag{5.13}$$

for all $j > j_1$. Therefore it follows from (5.11), (5.12), (5.12) and (5.13) that

$$\left| \int_{\Pi} (\zeta_j(x)K\zeta_j(x) - \zeta(x)K\zeta(x))dx \right| < \varepsilon$$

for all $j \geq \max\{j_0, j_1\}$. Hence we have

$$E(\zeta_j) \to E(\zeta) \quad \text{as} \quad j \to \infty.$$

This completes the proof. ■

Lemma 5.9 *Let $\Omega \subset \Pi$ be a set of infinite measure, let $\zeta_0 \in L^1 \cap L^p(\Pi)$ be a non-negative function and let $\Psi \in L^\infty(\Omega)$. Let $\zeta \in \mathcal{W}(\zeta_0, \Omega)$ be such that*

$$\int\limits_\Omega \zeta^{'}(x)\Psi(x) \leq \int\limits_\Omega \zeta(x)\Psi(x)dx \quad \forall \zeta^{'} \in \mathcal{W}(\zeta_0, \Omega) \ \text{ with } \ \zeta^{'} \neq \zeta,$$

where $\mathcal{W}(\zeta_0, \Omega)$ is the set of all functions in $\mathcal{W}(\zeta_0)$ vanishing outside Ω. Then

1. *$\zeta \in \mathcal{RC}(\zeta_0)$ and $\zeta^{-1}(0,\infty) \subset \Psi^{-1}(0,\infty)$, apart from a set of zero measure;*
2. *If $\zeta \notin \mathcal{F}(\zeta_0)$, then $\zeta^{-1}(0,\infty) = \Psi^{-1}(0,\infty) \cap \Omega$, apart from a set of zero measure.*

Proof. For (1), suppose that the second assertion fails. Then there exists a measurable set $A \subset \zeta^{-1}(0,\infty)$ having positive measure such that $\Psi \leq 0$ on A. Define $\bar{\zeta} = \zeta 1_{\Omega \ A}$, then it follows that

$$\int\limits_\Omega \bar{\zeta}(x)\Psi(x)dx \geq \int\limits_\Omega \zeta(x)\Psi(x)dx,$$

which is a contradiction.

To prove (2), we assume on the contrary that $A^{'} = \{x \in \Omega \mid \Psi(x) > 0, \zeta(x) = 0\}$ has positive measure. Since $\zeta \in \mathcal{RC}(\zeta_0, \Omega) \backslash \mathcal{F}(\zeta_0, \Omega)$, there exist $\alpha > \beta > 0$ and $\eta > 0$ such that, $\beta < t < \alpha$, then $\zeta_0^\Delta(t) > \eta$ and $\zeta^\Delta(t) = 0$. Choose $B \subset A^{'}$ having $0 < |B| < \alpha - \beta$. Then defining $\zeta^{'} = \eta 1_B + \zeta 1_{\Omega \backslash B}$ yields a contradiction as above hence (2).

5.5 Proofs of our main results

We are going to prove Theorem 5.1 by following the same argument that used in the last chapter. Indeed, we note that if $\zeta \in \mathcal{W}(\zeta_0)$, then it follows that $\|\zeta\|_1 \leq \|\zeta_0\|_1$ and $\|\zeta\|_p \leq \|\zeta_0\|_p$, hence Lemma 5.3 ensures that $K\zeta(x) - \lambda(x_2 + \frac{\sigma}{2}x_2^2)$ is bounded above on Π, for $\lambda > 0$ and $\sigma > 0$ which means that $\Phi_\lambda^\sigma(\zeta) < \infty$. Moreover, Lemma 5.3 allows us to fix $X = X(\lambda,\ \sigma) > 0$ such that $K\zeta(x) - \lambda(x_2 + \frac{\sigma}{2}x_2^2) < 0$ for all $\zeta \in \mathcal{W}(\zeta_0)$ and all $x \in \Pi$ with $x_2 \geq X$. We perform a reduction to functions vanishing outside Π_X. Consider $\zeta \in \mathcal{W}(\zeta_0)$ with $\Phi_\lambda^\sigma(\zeta) > -\infty$, let $A = \mathrm{supp}(\zeta) \cap \Pi_X$ and let $B = \mathrm{supp}(\zeta) \backslash \Pi_X$. Suppose $|B| > 0$ and set $\zeta 1_A$, $\zeta 1_B$. Then it follows that $\zeta 1_A \in \mathcal{W}(\zeta_0)$ and

$$\begin{aligned}
\Phi_\lambda^\sigma(\zeta 1_A) &= \Phi_\lambda^\sigma(\zeta - \zeta 1_B) \\
&= \Phi_\lambda^\sigma(\zeta) - \int_\Pi (K\zeta - \lambda(x_2 + \frac{\sigma}{2}x_2^2))\zeta 1_B + E(\zeta 1_B) \\
&> \Phi_\lambda^\sigma(\zeta),
\end{aligned}$$

since $K\zeta(x) - \lambda(x_2 + \frac{\sigma}{2}x_2^2) < 0$ in B and $E(\zeta 1_B) > 0$. This calculation has two consequences for the maximisation for Φ_λ^σ on $\mathcal{W}(\zeta_0)$, firstly, that any maximiser is supported by the strip Π_X and secondly, that any maximising sequence (ζ_n) for Φ_λ on $\mathcal{W}(\zeta_0)$ is supported by Π_X, where Π_X is the region where $x \in \Pi$ with $x_2 < X$.

Now, to prove (i), consider a maximising sequence (ζ_n) for Φ_λ^σ on $\mathcal{W}_X(\zeta_0)$. From the definition 1.4, the Steiner-symmetrisation ζ_n^s of ζ_n lies in $\mathcal{W}_X(\zeta_0)$; moreover $\mathcal{I}_\sigma(\zeta_n^s) = \mathcal{I}_\sigma(\zeta_n)$ and by Lemma 5.7, we have $E(\zeta_n^s) \geq E(\zeta_n)$. We may therefore suppose that (ζ_n) lies in $\mathcal{W}_X^s(\zeta_0)$. We may suppose that (ζ_n) converges weakly in $L^p(\Pi_X)$, say to $\bar{\zeta}$. Now,

$\bar{\zeta} \in \mathcal{W}^s_X(\zeta_0)$, since this is closed and convex. Thus, by Lemma 5.8, $E(\zeta_n) \to E(\bar{\zeta})$ and $\mathcal{I}_\sigma(\bar{\zeta}) \leq \liminf\limits_{n\to\infty} \mathcal{I}(\zeta_n)$ since $\mathcal{I}_\sigma$ is lower-semicontinuous and convex. Hence $\bar{\zeta}$ is the desire maximiser for Φ^σ_λ relative $\mathcal{W}_X(\zeta_0)$.

To prove (ii), we consider a maximiser $\zeta\mathcal{W}_X(\zeta_0)$, and define $\Psi(x) := K\zeta(x) - \lambda(x_2 + \frac{\sigma}{2}x_2^2)$, so Ψ is bounded on Π_X. If $\zeta' \in \mathcal{W}_X(\zeta_0)$ and $\zeta' \neq \zeta$, then

$$\begin{aligned} \Phi^\sigma_\lambda(\zeta) &\geq \Phi_\lambda(\zeta') \\ &= \Phi^\sigma_\lambda(\zeta) + \int\limits_{\Pi_X} \Psi(\zeta' - \zeta) + E(\zeta' - \zeta) \\ &> \Phi^\sigma_\lambda(\zeta) + \int\limits_{\Pi_X} \Psi(\zeta' - \zeta), \end{aligned}$$

whence

$$\int\limits_{\Pi_X} \Psi\zeta' < \int\limits_{\Pi_X} \Psi\zeta.$$

Therefore, we deduce (ii) by applying Lemma 5.9.

To prove (iii), consider a maximiser $\zeta \in \mathcal{W}_X(\zeta_0)$ and we recall that

$$\Psi(x) := K\zeta(x) - \lambda(x_2 + \frac{\sigma}{2}x_2^2).$$

By Lemma 5.6, we have $K\zeta(x) \to 0$ as $|x| \to \infty$, and hence

$$\Omega := \Psi^{-1}(0,\ \infty) = \{x \in \Pi_X \mid K\zeta(x) - \lambda(x_2 + \frac{\sigma}{2}x_2^2) > 0\}$$

is bounded open set in Π_X since by (ii) from above, $\zeta^{-1}(0,\ \infty) \subset \Omega$, apart from a set of zero measure, hence ζ has bounded support. Arguing as (ii), we further obtain

$$\int\limits_{\Omega} \Psi(x)\zeta'(x)dx < \int\limits_{\Omega} \Psi(x)\zeta(x)dx$$

for all $\zeta^{'} \in \mathcal{R}(\zeta, \Omega)$, $\zeta^{'} \neq \zeta$, and this ensures, by Burton [8], Theorem 2, that

$$\zeta = \phi \circ \Psi$$

almost everywhere in Ω, for some increasing function ϕ. Now, $\inf(\Psi(\Omega)) = 0$. If we redefine $\phi(t) = 0$ for $t \leq 0$, then ϕ is again increasing function, and satisfies $\zeta = \phi \circ \Psi$ almost everywhere in Π.

It remains now just to prove (iv). Indeed, let ζ be is a maximiser for Φ_{λ}^{λ} in $\mathcal{W}(\zeta_0)$ and let a be such that $|\mathrm{supp}\zeta_0| = \pi a^2$. To prove $\zeta \in \mathcal{R}(\zeta_0)$, we need firts to show that if $|\Omega| \geq 2\pi^2$, then this gives $|\mathrm{supp}\zeta| \geq \pi a^2$. For that, we assume that $|\Omega| \geq 2\pi^2$ but $|\mathrm{supp}\zeta| \leq \pi a^2$. Since by (ii), $\zeta \in \mathcal{RC}(\zeta_0)$; then there exists a rearrangement w of $\zeta_0^{\Delta} - \zeta^{\Delta}$ supported by the region $\Omega \backslash \mathrm{supp}\zeta$, where ζ_0^{Δ} and ζ^{Δ} are the decreasing rearrangements of ζ_0 and ζ respectively. Then it follows that $\zeta + w$ is a rearrangement of ζ_0, hence we find that $\Phi_{\lambda}^{\sigma}(\zeta + w) > \Phi_{\lambda}^{\sigma}(\zeta)$ which shows a contradiction with the fact that ζ is maximiser. Now, to prove $|\Omega| \geq 2\pi^2$, we need just to show that $|\Omega| \to \infty$ as $\lambda \to 0$. we recall that

$$\Omega := \{x \in \Pi \mid K\zeta(x) - \lambda(x_2 + \frac{\sigma}{2}x_2^2) > 0\}.$$

Also, by Lemma 5.3, we can choose a positive number C_1 such that

$$K\zeta(x) \leq C_1 \|\zeta_0\|_1 \log x_2 \tag{5.14}$$

for all $x_2 \geq 1$ and also since ζ is Steiner-symmetric, then we can choose a positive number C_2 such that from Lemma 5.4, we have

$$K\zeta(x) \leq C_2 \|\zeta_0\|_1 \left(\frac{x_2}{|x_1|}\right)^{\frac{1}{2}} \tag{5.15}$$

provided $x_1 \neq 0$. Now, for $\lambda > 0$ and $\sigma > 0$,let $X := X(\sigma,\ \lambda)$ be chosen so that

$$C\|\zeta_0\|_1 \log X - \lambda(x_2 + \frac{\sigma}{2}x_2^2) = 0, \tag{5.16}$$

where $C = \max\{C_1, C_2\}$. Then for $\epsilon \in (0,\ 1)$ and $\sigma \geq \lambda^\epsilon$ we have $X \to \infty$ as $\lambda \to 0$, moreover

$$\frac{1}{\lambda} \geq \left(\frac{X^2}{2C\|\zeta_0\|_1 \log X}\right)^{\dfrac{1}{1+\epsilon}}. \tag{5.17}$$

Now, let $\lambda_0 > 0$ be chosen so that $\alpha = \Phi^\sigma_{\lambda_0}(\zeta_0) > 0$ and such that if $\sigma \geq \lambda^\epsilon$ $\lambda \in (0,\ \lambda_0)$ then

$$\Phi_\lambda(\zeta)^\sigma \geq \alpha;$$

hence it follows that

$$\int\limits_{x_2<X} \zeta(x)K\zeta(x)dx \geq 2\alpha. \tag{5.18}$$

We set $t = \dfrac{\alpha^2}{C\|\zeta_0\|_1}$, since that the maximiser ζ is Steiner-symmetric function, then by applying Lemma 5.4 we find

$$\int\limits_{x_2<X,\ x_2\leq t|x_1|} \zeta(x)K\zeta(x)dx \leq \alpha.$$

Thus, from (5.18), it follows that

$$\int\limits_{x_2<X, |x_1|\leq \epsilon X} \zeta(x)K\zeta(x)dx \geq \alpha. \tag{5.19}$$

provided $\lambda \in (0,\ \lambda_0)$ and $\sigma \geq \lambda^\epsilon$. Therefore we have

$$\int\limits_{x_2<2, |x_1|\leq \epsilon X} \zeta(x)K\zeta(x)dx + \int\limits_{2\leq x_2<X, |x_1|\leq \epsilon X} \zeta(x)K\zeta(x)dx \geq \alpha. \tag{5.20}$$

We have two cases, the first case is

$$\int_{x_2<2,|x_1|\le\epsilon X} \zeta(x)K\zeta(x)dx \ge \int_{2\le x_2<X,|x_1|\le\epsilon X} \zeta(x)K\zeta(x)dx. \tag{5.21}$$

Then from (5.21) we get

$$\int_{x_2<2,|x_1|\le\epsilon X} \zeta(x)K\zeta(x)dx \ge \frac{\alpha}{2}, \tag{5.22}$$

thus by Lemma 5.3

$$\int_{x_2<2,|x_1|\le\epsilon X} \zeta(x)dx \ge \frac{\alpha}{2C\|\zeta_0\|_1}\log X, \tag{5.23}$$

hence, we can find a positive number β depends on α and C for which

$$C(\zeta) \ge \int_{x_2<2,|x_1|\le\epsilon X} \frac{x_2}{1+\mid x\mid^2}\zeta(x)dx \ge \frac{\beta}{\|\zeta\|_1 \log X} \tag{5.24}$$

and therefore by using (5.17) and Lemma 5.2

$$\mid \{x \in \Pi_X \mid K\zeta(x) - \lambda(x_2 + \frac{\sigma}{2}x_2^2) > 0\} \mid \ge \gamma \frac{X^2}{\| \zeta_0 \|}(\log X)^{-\frac{2+\epsilon}{1+\epsilon}}$$

Now we write Then it follows that $|\Omega| \to \infty$ as $\lambda \to 0$. Therefore, we can choose $\lambda_2 > 0$ such that $|\Omega| \ge \pi a^2$ for all $\lambda \in (0, \Lambda)$, where $\Lambda = \min\{\lambda_0,\ \lambda_1,\ \lambda_2\}$. This completes the proof. ■

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