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Abdelhamid Bezia

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Opérateurs de Sylvester–Lyapunov pour la résolution de quelques EDP non linéaires en dimension supérieure

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Mr. REZAOUI Med-Salem	Maître de Conférences A, à l'U.S.T.H.B.	Président.
Mr. BETINA Kamel	Professeur, à l'U.S.T.H.B.	Directeur de thèse.
Mr. BENAISSA Abbes	Professeur, à l'U.D.L	Examinateur.
Mr. BERKANE Djamel	Maître de Conférences A, à l'U.S.D.B	Examinateur.
Mr. BEHLOUL Djilali	Professeur, à l'U.S.T.H.B.	Examinateur

University of Sciences and Technology Houari Boumediene

Faculty of Mathematics

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Opérateurs de Sylvester Lyapunov pour la résolution de quelques EDP non linéaires en dimension supérieure

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Abdelhamid Bezia

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Approved by

First Reader

Rezaoui Mohamed Salem
M.C.A, University of Sciences and Technology H. Boumediene

Second Reader

Betina Kamel, Advisor
Professor, University of Sciences and Technology H. Boumediene

Third Reader

Benaissa Abbes
Professor, Bel-Abbes University

Fourth Reader

BERKANE Djamel
M.C.A, Blida University

Fifth Reader

BEHLOUL Djilali
Professor, University of Sciences and Technology H. Boumediene

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Opérateurs de Sylvester Lyapunov pour la résolution de quelques EDP non linéaires en dimension supérieure

(Order No.)

Abdelhamid Bezia

University of Sciences and Technology Houari Boumediene, Faculty of Mathematics, 2017

Major Professor: Kamel Betina, Department of Algebra and Number Theroy, University of Sciences and Technology H. Boumediene

ABSTRACT

In this thesis, we developed a numerical method leading to algebraic systems based on generalized Lyapunov-Sylvester operators to approximate the solution of two-dimensional (2D) time-dependent Boussineq and Kuramoto-Sivashinsky equations. It consists of an order reduction method and a finite difference discretization which is proved to be uniquely solvable, stable and convergent by using Lyapunov criterion and manipulating generalized Lyapunov-Sylvester operators. Some numerical implementations are provided at the end to confirm the good accuracy of the presented scheme

Contents

Chapter 1

Introduction

In this thesis we develop a computational methods for some PDE's using Lyapunov-Sylvester operators. It consists to resolve such problem and prove the invertibility of the algebraic operator yielded in the numerical scheme by appalling topological method Instead of using classical ones such as tri-diagonal transformations. We thus aim to prove that generalized Lyapunov-Sylvester operators can be good candidates for investigating numerical solutions of PDEs in multi-dimensional spaces. The Lyapunov-Sylvester equation have the form

$$\sum_i A_i X_n B_i = C_n \tag{1.1}$$

where A_i and B_i are appropriate matrices depending on the discretization procedure and the problem parameters. X_n represents the numerical solution at time n and C_n is usually depending on the past values X_k, $k \leq n-1$ of the the solution. The equation (1.1) is known as generalized Lyapunov-Sylvester equation. Such equations have their origin in the work of Sylvester on classical matrices equations. In the particular case

$$\sum_i A_i X \overline{A}_i^T = C \tag{1.2}$$

the equation is known restrictively as Lyapunov one. Generally speaking, the equation

$$\sum_i A_i X B_i = C \tag{1.3}$$

is very difficult to be inverted and remains an open problem in algebra. Nevertheless, some works have been developed and proved that under suitable conditions on the coefficient matrices, one may get a unique solution, but it's exact computation remains hard. It necessitates to compute eigenvalues and precisely bounds/estimates of eigenvalues or direct inverses of big matrices which remains a complicated problem and usually inappropriate.

In (Lancaster), a native method to solve (1.3) is investigated based on Kronecker product and equivalent matrix-vector equation. The Sylvester's equation is transformed into a linear equivalent one on the form $Gx = c$, with a matrix G obtained by tensor products issued from the $A_j's$ and the $B_j's$. However, the general case remained already complicated. The authors have been thus restricted to special cases where the matrices A_j and B_j are scalar polynomials based on spacial and fixed matrices A and B. Denote by $\sigma(A)$ the spectrum of A and $\sigma(B)$ the one of B, the spectrum $\sigma(G)$ may then be determined in terms these spectra. Indeed, with the assumptions the $A_j's$ and the $B_j's$, the equation (1.3) can be written as

$$\sum_{j,k} \alpha_{j,k} A^j X B^k = C, \tag{1.4}$$

where α_{jk} are complex numbers. Hence, the tensor matrix G will be written on the form $G = \phi(A, B)$, where ϕ is the 2-variables polynomial $\phi(x, y) = \sum_{j,k} \alpha_{j,k} x^i y^k$. Thus, G is singular if and only if $\phi(\lambda, \mu) = 0$ for some eigenvalues λ and μ of A and B respectively.

However, in numerical studies of PDE's we may be confronted with matrices G where the computation of spectral properties are not easy and necessitate enormous calculus and sometimes, induces slow algorithms and bed convergence rates.

For given matrices $A \in \mathbb{R}^{m\times m}, B \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{m\times n}$, we consider the Sylvester equation given by the form $AX+XB = C$. In the case of square matrices an other criterion of existence of the solution of $AX + XB = C$ was pointed out by Roth (Roth). It was shown that the solution is unique if and only if the matrices

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{1.5}$$

are similar.

A brute force attack to obtain the the solution X is to rewrite the Sylvester equation in standard $mn \times mn$ linear system $G\widetilde{x} = \widetilde{c}$ using the Kronecker Product (Jameson). The Sylvester equation can be solved by Gaussian elimination with $\mathcal{O}(m^3n^3)$ flops. This approach dramatically increases the complexity of the computation, and also cannot preserve the intrinsic properties of the problem in practice (Simoncini). We denote whatever the special structure of the large linear system $G\widetilde{x} = \widetilde{c}$ can be using only rational operation.

In numerical analysis, for solving the Sylvester equation one using the Bratels-Stewart and the Golub-Nash-Van Loan algorithm use $\mathcal{O}(m^3 + n^3)$ floating point operations, if one assume that an $M \times M$ matrix can be reduced to Schur form with $\mathcal{O}(M^3)$ operations. More precise details are given in (Bartels) and (Gloub).

In (Kirrinni) the author describe an algorithm that computes the solution X over an arbitrary field $\mathbb{F}$. The complexity of the algorithm for $A \in \mathbb{F}^{m\times m},\ B \in \mathbb{F}^{n\times n}$ and $m, n \leq N$ is $\mathcal{O}(N^{\beta}.logN)$ arithmetic operations in $\mathbb{F}$, where $\beta > 2$ is such that $M \times M$ matrices can be multiplied with $\mathcal{O}(M^{\beta})$ arithmetic operations. This algorithm is competitive in terms of arithmetic operation with and even faster than the classical algorithms, and by useful for generalizations for other field than $\mathbb{R}$ or $\mathbb{C}$.

Algebraic Lyapunov-Slyvester equations of large dimension arise when using approximations to design controllers for systems modelled by partial differential equations. The present work lies in the whole scope of numerical studies of PDEs essentially Boussinesq and Kuramoto-Sivashinsky equations equations. The method developed in this thesis for the resolution of PDEs consists in replacing time and space partial derivatives by finite-difference approximations in order to transform the continuous problem into linear Lyapunov-Sylvester systems of the form (1.1). An order reduction method is adapted leading to a system of coupled PDEs which is transformed by the next to a discrete algebraic one. The motivation behind the idea of applying Lyapunov operators was already evoked in (Ben Mabrouk 1). We recall in brief that such a method leads to fast convergent and more accurate discrete algebraic systems without going back to the use of tri-diagonal and/or fringe-tridiagonal matrices already used when dealing with multidimensional problems especially in discrete PDEs. For large model order direct solution methods based on eigenvector calculation fail.

To recapitulate, the method developed here is favorable for many reasons

- The first motivation is the fact that it somehow does not change the geometric presentation of the problem as we propose to solve in the same two-dimensional space. We did not project the problem on tri-diagonal representations using the Kronecker product. Relatively to computer architecture, the process of projecting on different spaces and next lifting to the original one may induce degradation of error estimates and slow algorithms.

- The method developed is not just a resolution of a PDE. But, we recall that the resolution itself is not a negligible aim. Further, it proves the efficiency of algebraic operators other than classical tri-diagonal ones.

- We proved here that even when the two systems are equivalent in the sense that they present the same PDE, but with different forms and dimensions, such forms play a major role in the resolution.

- The fact obtaining fast algorithms is very important in computer sciences and makes itself a major aim in computer studies. Recall that the famous method known in mathematical studies of accelerating algorithms in the EM one (expectation-maximisation) which is based on more complicated theories. Here, we proved that we may obtain more rapid algorithms by using just a suitable representation and suitable discerete transformation of the PDE. We got faster algorithms without adding more parameters.

The resulting methods are analysed for local truncation error and stability and we prove that the scheme is uniquely solvable and convergent. Some numerical examples illustrating the method described in our work are given, and proved that the proposed algorithm is more performant and fast convergent compared to the tridiagonal one.

Chapter 2

Lyapunov-Sylvester equations

In the present chapter, we make an attempt to provide a brief overview on the solvability of the Lyapunov-Sylvester equations and the recently results obtained, and do not intend to present a complete list of all the existing results. Lyapunov-Sylvester equations appears in various branches of mathematics. They are very important in control theory and many other branches of engineering. Matrix Sylvester equation have the form

$$AX - XB = C \tag{2.1}$$

where X, and C are $m \times n$-matrices, A is $m \times m$-matrix and B is $n \times n$-matrix over the set of complexes numbers. Matrix equations of the type (2.1) was used for the first time to give a numerical solution of certain boundary value problems in partial differential equations by W.Bickley and J. McNamee (Bickley).

If we denote by A^T is the conjugate transpose, the familiar Lyapunov equation occurs in the theory of stability is given by taking $B = A^T$. Another form is the nonlinear matrix equation

$$AX - XB^T = C + XDX$$

called the algebraic Riccati equation, whose solutions by Newton's method require in each iterative step the solution of a system of two Lyapunov equations, which is equivalent to a Sylvester equation.

Theorem 1 *For any C, a unique solution X of (2.1) exists provided that the eigenvalues of A are distinct from the eigenvalues of B*

$$\mathit{spectrum}\,(A) \cap \mathit{spectrum}\,(B) = \varnothing$$

If u is a eigenvector of A with associated eigenvalue λ, and v is a eigenvector of B^T with associated eigenvalue μ, which means $Au = \lambda u$ and $B^T v = \mu v$. Then $Auv^T = \lambda uv^T$, by transposition we get $v^T B = \mu v^T$. We multiple both side by u, to get $Buv^T = \mu uv^T$, after summation, we obtain

$$Auv^T - uv^T B = (\lambda - \mu)\, uv^T.$$

which means that $\lambda - \mu$ is a eigenvalue of the system (2.1), hence it have a solution if and only if

$$\lambda_i - \mu_j \neq 0 \text{ for all } i, j. \tag{2.2}$$

An other criterion developed by Roth (Roth), the latter stating that the necessary and sufficient condition that the equation (2.1) where A, B, and C are with elements in a field $\mathbb{F}$, have a solution X with elements in $\mathbb{F}$ is that the matrices

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \text{ and } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

be similar.

Roth's theorem has been the subject of a series of papers. It has been extended to matrices over special rings (Guralnick), (Gustafson).

2.1 A naive approach

Given an $m \times n$ matrix A and a $p \times q$ matrix B, their Kronecker product $A \otimes B$, also called their matrix direct product, is an $mp \times nq$ block matrix formed by taking all possible products between the elements of A and the matrix B. More explicitly $A \otimes B$ elements defined by

$$c_{\alpha\beta} = a_{ij} b_{kl},$$

where $\alpha = p\,(i-1) + k$ and $\beta = q(j-1) + l$.

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

If A and B represent linear transformations $V_1 \rightarrow W_1$ and $V_2 \rightarrow W_2$, respectively, then $A \otimes B$ represents the tensor product of the two maps, $V_1 \otimes V_2 \rightarrow W_1 \otimes W_2$.

We denote by $vec\,(C)$ operator as stacking the columns of a matrix C, If C is a $n \times m$, then the elements of $vect\,(C)$ are $c_{11}, c_{21}, ..., c_{12}, ...$ We find that the equation (2.1) written out for the mn ukonwon $x_{11}, x_{21}, ..., x_{12}, ...$ in terms of $c_{11}, c_{21}, ...c_{12}, ...$. In this case we have

$$\left[(I_n \otimes A) - \left(B^T \otimes I_m\right)\right] vect\,(X) = vec\,(C) \tag{2.3}$$

where I_n is the unit matrix of size n.

One may then solve for (2.3) by inverting the matrix $\left[(I_n \otimes A) - \left(B^T \otimes I_m\right)\right]$ or solving the linear equations.

If we proceed with a numerical method to solve a system of N equation with arithmetic complexity of $\mathcal{O}\left(N^3\right)$ (Gaussian elimination for instance). As thee equation (2.3) it is a linear system $N = n^2$, The complexity of this approach will be $\mathcal{O}\left(n^6\right)$. In order to get optimal complexity, the Gaussian elimination can be performed to solve (2.3) and faster algorithm can be used to compute the matrix $\left[(I_n \otimes A) - \left(B^T \otimes I_m\right)\right]$ by exploiting sparsity. Although these improved variants will not be computationally competitive for large n. This approach ignores the structure of the original problem, introducing errors to the solution process unnecessarily.

Remark 2 *When A and B* (2.1) *in can both be reduced to diagonal form by similarity* *transformations*

$$U^{-1}AU = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

and

$$V^{-1}BV = \begin{pmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \ddots & \\ 0 & & & \mu_n \end{pmatrix}$$

Let $\widetilde{X} = (\widetilde{x}_{ij})$ *be the solution of the system*

$$\left(U^{-1}AU\right)\widetilde{X} - \widetilde{X}\left(V^{-1}BV\right) = \left(U^{-1}CV\right)$$

If we put $\widetilde{C} = U^{-1}CV = (\widetilde{c})_{ij}$, *we get then* $\widetilde{x}_{ij} = \frac{1}{\lambda_i - \mu_j}\widetilde{c}_{ij}$, *We multiply the left side of la last equation by* U *and next the right side by* V^{-1}, *we get*

$$A\left(U\widetilde{X}V^{-1}\right) - \left(U\widetilde{X}V^{-1}\right)B = C$$

Finally the solution of system (1.1), is given by $X = U\widetilde{X}V^{-1}$.

We can also see that is this case the system (2.3) can be reduced to diagonal form

$$\left[\left(V^{-1}\right)^T \otimes U\right]\left[I_N \otimes A\right] - \left[\left(B^T\right) \otimes I_M\right]\left[V^T \otimes U^{-1}\right]$$

$$= \begin{pmatrix} \lambda_1 - \mu_1 & & & & & & 0 \\ & \lambda_2 - \mu_1 & & & & & \\ & & \lambda_3 - \mu_1 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_1 - \mu_2 & & \\ & & & & & \ddots & \\ 0 & & & & & & \lambda_M - \mu_N \end{pmatrix}$$

2.2 Explicit solution of $AX - XB = C$

We have seen that the equation $AX - XB = C$ has a unique solution if and only if A and B have no common eigenvalues. So under this condition, the solution matrix X can be explicitly expressed in various forms

Theorem 3 *Let* $P, Q \in \mathbb{C}[t]$, $\deg P = m$ *and* $\deg = n$ *the annihilating polynomial of matrices* A *and* B *respectively are* $P(t) = a_0 + a_1 t + ... + a_m t^m$ *and* $Q(t) = b_0 + b_1 t + ... + b_n t^n$*. If* $X \in \mathbb{C}^{n \times n}$ *is a solution for* (2.1)*, then,*

$$Q(A)X = \sum_{k=1}^{n} a_k \mathcal{C}_k \qquad \text{and} \qquad -XP(B) = \sum_{k=1}^{m} a_k \mathcal{C}_k \tag{2.4}$$

where

$$\begin{cases} \mathcal{C}_1 = C = AX - XB \\ \mathcal{C}_k = \sum_{j=1}^{k} A^{k-j} C B^{j-1} =, \quad k \in \mathbb{N}^* \end{cases}$$

Proof. The idea of the proof is based of the expression

$$A^k X - XB^k = \sum_{j=1}^{k} A^{k-j} (AX - XB) B^{j-1}$$

we have

$$\begin{aligned} \mathcal{C}_1 &= AX - XB = C \\ \mathcal{C}_2 &= A^2 X - XB^2 = A(AX - XB) + (AX - XB)B \\ &= AC_1 + C_1 B \\ \mathcal{C}_3 &= A^3 X - XB^3 \\ &= A\left(A^2 X - XB^2\right) + \left(A^2 X - XB^2\right)B - A(AX - XB)B \\ &= AC_2 + C_2 B - AC_1 B \\ &\quad \cdots \\ \mathcal{C}_k &= A^k X - XB^k = AC_{k-1} + C_{k-1}B - AC_{k-2}B \end{aligned}$$

According the Cayley-Hamilton theorem $P(A) = 0$ and $Q(B) = 0$. Therefore

$$\begin{aligned}\sum_{i=1}^{n} b_i C_i &= \sum_{i=1}^{n} b_i \left(A^i X - X B^i\right) \\ &= Q(A) X - X Q(B) \\ &= Q(A) X\end{aligned}$$

In the same way, we have

$$\begin{aligned}\sum_{i=1}^{m} a_i C_i &= \sum_{i=1}^{m} b_i \left(A^i X - X B^i\right) \\ &= P(A) X - X P(B) \\ &= -X P(B)\end{aligned}$$

■

Corollary 4 *We see that equation (2.1) has a solution if and only if $Q(A)$ is invertible. If in addition $P(\lambda)$ and $Q(\mu)$ are characteristic polynomial of A and B respectively, then*

$$\begin{aligned}P(\lambda) &= \det(\lambda I - A) = a_0 + a_1\lambda + ... + \lambda^m \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_m)\end{aligned}$$

$$\begin{aligned}Q(\mu) &= \det(\mu I - B) = b_0 + b_1\mu + ... + b_n\mu^n \\ &= (\mu - \mu_1)(\mu - \mu_2)...(\mu - \mu_n)\end{aligned}$$

$$-XP(B) = \sum_{k=1}^{m} a_k \mathcal{C}_k$$

$$\begin{aligned}X &= (Q(A))^{-1}\left(\sum_{k=1}^{n} a_k \mathcal{C}_k\right) \\ &= \left(-\sum_{k=1}^{m} a_k \mathcal{C}_k\right)(P(B))^{-1}\end{aligned}$$

where

$$\begin{aligned} Q(A) &= b_0 + b_1 A + ... + b_n A^n \\ &= (A - \mu_1 I)(A - \mu_2 I) + ... + (A - \mu_n I) \end{aligned} \tag{2.5}$$

It is well known that the determinant of a product of matrices is the product of their determinants, as result from 2.5 we see that $Q(A)$ is not invertible if for any i, μ_i is an eigenvalue of A, and similarly $P(B)$ is not invertible if for any i λ_i is an eigenvalue of B. Therefore $P(B)$ and $Q(A)$ are invertible unless (2.2) *hold, (i.e.,$\lambda_i - \mu_j \neq 0$).*

Bocher's identities (Pipes), are useful relations for obtaining the coefficients of the characteristic polynomial, if

$$P(\lambda) = \lambda^m + a_1 \lambda^{m-1} + ... + a_m$$

then coefficients are given by:

$a_1 = -tr(A)$,

$a_2 = \frac{-1}{2}\left[a_1 tr(A) + tr\left(A^2\right)\right]$,

$a_3 = \frac{-1}{3}\left[a_2 tr(A) + a_1 tr\left(A^2\right) + tr\left(A^3\right)\right]$

...

$a_M = \frac{-1}{m}\left[a_{m-1} tr(A) + a_{m-2} tr\left(A^2\right) + ... + tr(A^m)\right] = (-1)^m \det(A)$.

In the case where Sylvester Equation is written on the form $AX + XB = C$ instead of $AX - XB = C$, the explicit solution can be given by

$$\sum_{k=1}^{n} (-1)^{k-1} b_k \mathcal{C}_k = \left(\sum_{k=0}^{n} (-1)^k b_k A^k\right) X$$

and so,

$$X = \widetilde{Q}(A)^{-1}\left[\sum_{k=1}^{n} (-1)^{k-1} b_k \mathcal{C}_k\right]$$

where

$$\widetilde{Q}(A) = \left(\sum_{k=0}^{n} (-1)^k b_k A^k\right) = \prod_{i=1}^{N} (A + \mu_i I)$$

Remark 5 *One consider the Lyapunov equation*

$$AX + XA^T = C,$$

coefficients a_i and b_i of characteristic polynomial coincide. In this case we can also be express $\widetilde{Q}(A)$ in terms of even or odd powers of A only

$$\widetilde{Q}(A) = 2(-1)^N Q(A)$$

For instance, the solution of Lypunov equation $AX + XA^T = C$ for a 2×2 matrices is given by

$$X = \frac{1}{2tr(A)}\left[C + \det(A) A^{-1} C \left(A^{-1}\right)^T\right]. \tag{2.6}$$

and for 3×3 matrices we can express the solution as

$$X = \frac{1}{2(a_1 a_2 - a_3)}\left[A^2 C - ACA^T + CA^{T^2} + \left(a_2 - a_1^2\right) C - a_1 a_3 A^{-1} C \left(A^{-1}\right)^T\right]. \tag{2.7}$$

2.3 Bratels-Stewar Algorithm

The Bartels-Stewart method is one of the most effective schemes for solving the Sylvester equation. It provided the first numerically stable way to systematically solve the symmetric linear matrix equation

$$AX + XB = C \tag{2.8}$$

Bartels-Stewart algorithm (Bartels), make direct use of the matrix-valued structure of the Lyapunov equation, which allows numerical solution in $\mathcal{O}(m^3 + n^3)$. The main idea of the Bartels-Stewart algorithm is to apply the Schur decomposition to transform Sylvester equation into a triangular system which can be solved efficiently by forward or backward substitutions.

The matrix B Hessenberg form by Householder's method, and the upper Hessenberg matrix is in turn reduced to real Schur form by the QR algorithm. The product of the transformations used in the reductions is accumulated to form the matrix V. The reduction of A to lower real Schur form is accomplished by reducing the transpose of A to upper real Schur form and transposing back.

For the Sylvester equation $AX + XB = C$, The matrix A is reduced to a Shur lower matrix A' via an orthogonal transformation U

$$A' = U^T A U = \begin{pmatrix} A'_{11} & & & & 0 \\ A'_{21} & A'_{22} & & & \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & & \ddots & \\ A'_{p1} & A'_{p2} & \cdots & \cdots & A'_{pp} \end{pmatrix}$$

where each block A'_{ii} is at most 2×2 matrix. In the same way B reduced to an upper Schur form with 2×2 submatrices B'_{ii} via a transformation V

$$B' = V^T B V = \begin{pmatrix} B'_{11} & B'_{12} & \cdots & \cdots & B'_{1q} \\ & B'_{22} & \cdots & \cdots & B'_{2q} \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ 0 & & & & B'_{qq} \end{pmatrix}$$

We put now

$$C' = U^T C V = \begin{pmatrix} C'_{11} & \cdots & C'_{1q} \\ \vdots & & \vdots \\ C'_{p1} & \cdots & C'_{pq} \end{pmatrix}$$

and

$$X' = U^T X V = \begin{pmatrix} X'_{11} & \cdots & X'_{1q} \\ \vdots & & \vdots \\ X'_{p1} & \cdots & X'_{pq} \end{pmatrix}$$

We multiply the Lyapunov equation from the right and left with U^T and V respectively,

$$\begin{aligned} AX + XB &= C \\ \left(U^T AU\right)\left(U^T XV\right) + \left(U^T XV\right)\left(V^T BV\right) &= U^T CV \\ A'X' + X'B' &= U^T CV \end{aligned}$$

We obtain the equivalent equation to (2.8) ,

$$A'X' + X'B' = C' \tag{2.9}$$

Since the partitions of C' and X' are conformal with A' , expanding (2.9) to get

$$\begin{aligned} A'_{kk}X'_{kl} + X'_{kl}B'_{ll} &= C'_{kl} - \sum_{j=1}^{k-1} A'_{kj}X'_{jl} - \sum_{i=1}^{l-1} X'_{ki}B'_{il} \\ (k &= 1, 2, ..., p; \quad l = 1, 2, ...q) \end{aligned} \tag{2.10}$$

These equations can be solved sequentially for $X'_{11}, X'_{21}, ..., X'_{p1}, X'_{12}, X'_{22}, ...$ the solution of (2.8) can be given by $X = UX'V^T$.

The solution for X'_{kl} in (2.10) still requires the solution of a matrix equation of the form (2.8). However, in this case the matrices $A\prime_{kk}$ and B'_{ll} are of order at most two. Therefore we can use the native approach (2.3) or explicit method (2.6) to solve the system.

2.3.1 Lyapunov Equation

In principle, the algorithm described for solved Sylvester equation can obviously be used to solve the symmetric problem $AX + XA^T = C$. However, it is possible to take advantage of the symmetry. Recall that all real matrices have a real Schur decomposition: There exists matrices U such that

$$A' = U^T AU$$

where U is an orthogonal matrix and $A' \in \mathbb{R}^{n\times n}$ a block-triangular matrix and $A'_{jj} \in R^{n_j \times n_j}$, $n_j \in \{1,2\}$, $j = 1,..,r$ and $\sum_{j=1}^{r} n_r = n$.

We multiply the Lyapunov equation from the right and left with U^T and U respectively,

$$
\begin{aligned}
AX + XA^T &= C && (2.11a)\\
\left(U^T AU\right)\left(U^T XU\right) + \left(U^T XU\right)\left(U^T A^T U\right) &= U^T CU && (2.11b)\\
A'X' + X'\left(A'\right)^T &= U^T CU && (2.11c)
\end{aligned}
$$

where $Y = U^T XU$. We introduce matrices and corresponding blocks such that,

$$
U^T CU = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \; X' = \begin{pmatrix} X'_{11} & X'_{12} \\ X'_{21} & X'_{22} \end{pmatrix}, \;\; A' = \begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix},
$$

where X'_{22}, C_{22} and $A'_{22}$are at most of order 2 (the size of the last block of A'). This triangularized problem can now be solved with (what we call) backward substitution, similar to backward substitution in Gaussian elimination. By separating the four blocks in the equation $(2.11c)$, we have four equations

$$
\begin{aligned}
C_{11} &= A'_{11}X'_{11} + A'_{12}X'_{21} + X_{11}A'^T_{11} + X'_{12}A'^T_{12} && (2.12)\\
C_{12} &= A'_{11}X'_{12} + R_{12}X'_{22} + X'_{12}A'^T_{22} && (2.13)\\
C_{21} &= A'_{22}X'_{21} + X'_{21}A'^T_{11} + X'_{22}A'^T_{12} && (2.14)\\
C_{22} &= A'_{22}X'_{22} + X'_{22}A'^T_{22} && (2.15)
\end{aligned}
$$

Due to the choice of block sizes, the last equation of size $n_r \times\; n_r$, which can be solved explicitly since n_r it is at most 2. if $n_r = 1$, the equation (2.15) is scalar and we obviously have $X'_{22} = \frac{C_{22}}{2A'_{22}}$ and if $n_r = 2$, we can still solve (2.15) cheaply since it is a 2×2 Lyapunov equation. For instance, we can use the naive approach (5.4), i.e.

$$vec\left(X'_{22}\right) = \left(I \otimes A'_{22} + A'_{22} \otimes I\right)^{-1} vec\left(C_{22}\right).$$

Insert the now known matrix X'_{22} into (2.13) and transposed(2.14) yields

$$\widetilde{C}_{12} \quad : \quad = C_{12} - A'_{12}X'_{22} = A'_{11}X'_{12} + X'_{12}A'^{T}_{22} \tag{2.16}$$

$$\widetilde{C}_{21} \quad : \quad = C^{T}_{21} - A'_{12}X'^{T}_{22} = A'_{11}X'^{T}_{21} + X'^{T}_{21}A'^{T}_{22} \tag{2.17}$$

The equations (2.16) and (2.17) have a particular structure that can be used to directly compute the solutions X'_{12} and X'_{21}. An explicit procedure for the construction of the solution is given by the following result.

Lemma 6 *(Jarlebring) Consider two matrices C, D partitioned in blocks of size $n_1 \times n_p$, $n_2 \times n_p, ..., n_{p-1} \times n_p$ according to*

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_{p-1} \end{bmatrix} \in \mathbb{R}^{N \times n_r}, \quad D = \begin{bmatrix} D_1 \\ \vdots \\ D_{p-1} \end{bmatrix} \in \mathbb{R}^{N \times n_p}$$

Let $C_j, D_j \in \mathbb{R}^{n_j \times n_p}$ and $N = \sum_{j=1}^{p-1} n_j$. Let $W \in \mathbb{R}^{n \times n}$ be a block triangular matrix partitioned as X and D. For $A_{22} \in \mathbb{R}^{n_p \times n_p}$, we that if X satisfies the equation

$$D = WX + XA^{T}_{22}$$

then X_j, $p-1, p-2, ..., 1$ satisfy

$$W_{jj}X_j + X_jA^{T}_{22} = D_j - \sum_{i=j+1}^{p-1} W_{ji}X_i.$$

Similar to the approach we used to compute A_{22}, X_j can expressed and computed explicitly from a small linear system

$$vec\left(X_j\right) = \left(I \otimes T_{jj} + A_{22} \otimes I\right)^{-1} vec\left(\widetilde{C}_j\right)$$

By solving this equation for $j = p-1, .., 1$ for both equations (2.16) and (2.17) we obtain solutions X_{12} and X_{21}. Insertion of obtained X'_{12}, X'_{21} and X'_{22} into (2.12) gives a new Lyapunov equation of size $n - n_p$ and the process can be repeated for the smaller matrix.

2.3.2 Complexity of B-S Algorithm

In numerical analysis, Schur decompositions are usually computed in two steps: first, A is transformed into Hessenberg form with Householder transforms, and then QR iteration is used to get rid of the subdiagonal. Hence, there are intermediate results that are not computed exactly, but approximated by an iterative numerical process. This is appropriate for numerical analysis, but it raises the question whether there is a competitive (say, $\mathcal{O}(m^3 + n^3)$) algorithm that uses only rational operations and can hence be used for arbitrary fields.

The key idea of the B-S algorithm is the orthogonal reduction of A and B to triangular form using QR algorithm for eigenvalues. Since the QR algorithm is an iterative method that, as used, reduces the subdiagonal elements of an upper Hessenberg matrix to zero, some criterion must be adopted for determining when an element is negligible. In (Bartels) an element of H is considered negligible if it is less than or equal to $\varepsilon_H \|H\|_\infty$ where ε_H is a constant supplied by the user. This criterion is appropriate if the elements of H are all of roughly the same size. A different criterion may be required if the elements vary widely and the small elements are significant, as when the elements decrease greatly in size as one passes from the upper left to the lower right corners of A. The QR-algorithm could be tuned to perform well. In practice the total complexity of Bartels-Stewart algorithm is $\mathcal{O}(n^3 + m^3)$ outing point operations, if one assumes that an $M \times M$ matrix can be reduced to Schur form with $\mathcal{O}(M^3)$ operations.

Hessenberg–Schur method proposed by Golub–Nash–Van Loan (Gloub) requires only one of the matrices A or B to be reduced to Schur form. The resulting method is between 30 and 70 faster than B-S Algorithm (Bartels) depending upon the dimension of A and B. Precise bounds are given in (Gloub). The LAPACK (Anderson) algorithms for the Sylvester equation also require both the matrices A and B to be reduced to Schur form.

A detailed study on Bartels-Stewart algorithm for Sylvester equations was done in

(Sorensen) authors give the complexity of different algorithm and revisited the Bartels-Stewart algorithm for Sylvester equations. The schemes constructed are much more efficient compared to the Matlab Sylvester and Lyapunov solver lyap.m

For the backward stability analysis of Bartels-Stewart algorithm, we refer to (Higham 1) and (Higham 2), page, 313. For backward error analysis can refer to (Ghavimi).

2.4 Fast algorithms for the Sylvester equation $AX - XB^T = C$

In (Kirrinni) consider the Sylvester equation oven an arbitrary field $\mathbb{F}$, The autor proposed an efficient algorithm to solve the sylvister equation $AX-XB^T = C$. If $M \times M$ matrices can be multiplied with $\mathcal{O}\left(M^{\beta}\right)$ arithmetic operations then the algorithm computes the solution $X \in \mathbb{F}^{m\times n}$ for $m, n \leq N$, with $\mathcal{O}\left(N^{\beta}.\log N\right)$ arithmetic operations in $\mathbb{F}$. According to the best complexity bound currently known (Coppersmith) we can choose $\beta < 2.367$, for any field $\mathbb{F}$. The idea of the algorithm is to transform the coefficient matrices A and B to a Frobenius form for which the Sylvester equation can be solved easily.

Through this section $\mathbb{F}$ denotes a field, We suppose that $M \times M$ matrices over $\mathbb{F}$ can be multiplied with $\mathcal{O}\left(M^{\beta}\right)$ with $\beta > 2$. If $M \times M$ matrices can be inverted $\mathcal{O}\left(M^{\beta}\right)$ operations, then $\mathcal{O}\left(M^{\beta}\right)$ is also an upper bound for the complexity of $M \times M$ matrix multiplication.

In this section we summarize the results of work done by (Kirrinni). We consider the equation

$$AX - XB^T = C \tag{2.18}$$

$\left(A \in \mathbb{F}^{m\times m},\ B \in \mathbb{F}^{n\times n},\ C, X \in \mathbb{F}^{m\times n}\right)$

Let $U \in GL_m$ and $V \in GL_n$ and define $A' = U^{-1}AU$ and $B' = V^{-1}BV$. Then $Y \in \mathbb{F}^{m\times n}$ solve the equation $A'Y - Y\left(B'\right)^T = U^{-1}C\left(V^{-1}\right)^T$ if and only if $X = UYV^T$ solves $AX - XB^T = C$.

The characteristic polynomial of the matrix $U \in \mathbb{F}^{k\times k}$ is denoted by $\chi_U\left(z\right) = \det\left(z.I_k - U\right)$.

First we show that if $B = \left(b_{i,j}\right)$ is an upper Hessnberg matrix (i.e., $b_{i,j} = 0$ for all $i > j+1$) with all subdiagonal entries $b_{i+1,i}$ equal to 1, then the last column x_n for the solution X fulfills the linear equation $\chi_B(A)\, x_n = d$, where $d \in \mathbb{F}^m$ can be computed from A, B and C. One x_n is known, the other columns of X can be computed form (Kronecker) by backward substitution.

If B is a companion of Frobenius matrix, i.e.

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & \cdots & 0 & -b_1 \\ 0 & 1 & \cdots & 0 & -b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -b_{n-1} \end{pmatrix}$$

then the characteristic polynomial of B is $\chi_B(z) = z^n + b_{n-1}z^{n-1} + ... + b_1 z + b_0$. If A is also a Frobenius matrix, then the equation $\chi_B(A)\, x_n = d$ can be translated into a polynomial equation than can be solved by polynomial gcd computation. Moreover, b can be computed efficiently and the backward substitution for the other columns of X can be performed fast due to the sparsity of A and B

For Frobenius matrices $A \in F^{n\times m}$ and $B \in F^{n\times n}$ with disjoint spectra, the Sylvester equation $AX - XB^T = C$ can be solved with $\mathcal{O}(m{\cdot}n)$ operations

Let $B = (b_{i,j}) \in \mathbb{R}^{n\times n}$. For $1 \le i,\, j \le n$, and let $Y = (y_{i,j})_{1\le i,j\le n}$ be the adjoint matrix of B. the element $y_{i,j} = (-1)^{i+j} . \det B < i, j >$. where the matrix obtained by deleting the ith row and jth column of B is denoted by $B < i, j > \in R^{(n-1)\times(n-1)}$. It well known form linear Algebra that $Y.B = B.Y = \det B.I_n$.

Lemma 7 *If B is a upper Hessenberg with all subdiagonal elements equal to -1. Let $B_k = (b_{i,j})_{1\le i,j\le k}$. Then the entries of the last row the adjoint Y are $y_{n,1} = 1$ and $y_{n,k} = \det B_{k-1}$ for $k > 1$.*

The matrix $B < k, n >$ has the form

$$\begin{pmatrix} B_{k-1} & C \\ 0 & D \end{pmatrix}$$

where D is an upper triangular $(n-k)\times(n-k)$ matrix with all diagonal elements equal to 1 and so $\det D = (-1)^{n-k}$. Hence

$$y_{n,k} = (-1)^{k+n} . \det B < k, n >= (-1)^{k+1} . \det B_{k-1} . \det D = \det B_{k-1}$$

If we apply this result to $R = \mathbb{F}[z]$ and $zI_n - B$ in place of B we get the following corollary

Corollary 8 *Let $B \in F^{n\times n}$ be upper Hessenberg with all subdiagonal elements equal to 1. Then the last row of the adjoint $Y(z)$ of $zI_n - B$ is*

$$\left(1.\chi_{B_1}(z), \chi_{B_2}(z), ..., \chi_{B_n}(z)\right). \tag{2.19}$$

Lemma 9 *Let B be upper Hessenberg with all subdiagonal elements equal to 1. Let $A \in F^{m\times m}$, $C \in F^{m\times n}$ with columns $c_1, c_2, .., c_n$ and let $X \in \mathbb{F}^{m\times n}$ be a solution to the Sylvester equation $AX - XB^T = C$. The the last column x_n of X fulfils*

$$\chi_B(A).x_n = \sum_{k=0}^{n-1} \chi_{B_k}(A).c_{k+1},$$

where $\chi_{B_0}(z) = 1$.

Proof. Let $Y(z) \in F[z]^{n\times n}$ be the adjoint of $zI_n - B$. Then

$$Y(z).(zI_n - B) = \chi_B(z).I_n$$

The last row of $Y(z)$ is given by (2.19) .and

$$\begin{aligned} (Y \otimes I_m).(I_n \otimes A - B \otimes I_m) &= I_n \otimes \chi_B(A) \\ (Y \otimes I_m).vec(c) &= I_n \otimes \chi_B(A).vec(X) \end{aligned}$$

This is an equation in $(\mathbb{F}^m)^n$. On the right hand side, the vector of the last m entries is d because of (2.19), and on the left hand side it $\chi_B(A).x_n$ ■

Note that $\chi_B(A)$ is nonsingular (and hence x_n is uniquely determined) if and only if A and B have no eigenvalues in common. In any case, the other columns $x_1, ..., x_{n-1}$ of X

are determined by x_n and can be computed by backward substitution

Lemma 10 *Let* A, B, C, X *be as in . Then the first* $n-1$ *columns* $x_1, ..., x_{n-1}$ *of* X *are given recursively by*

$$x_{k-1} = Ax_k - b_k x_k - b_{k,k+1}x_{k+1} - ... - b_{k,n}x_n - c_k \text{ for } n \geq k \geq 2.$$

2.4.1 Fast algorithms for Frobenius matrices

If A is a Frobenius matrix, then computing $q(A)$ for a polynomial q can be reduced to polynomial multiplication and division, and linear equations of the form $q(A)x = d$ can be solved by polynomial gcd algorithms.

Let shows in this section how to exploit relations of this type to solve the Sylvester equation for Frobenius matrices A and B efficiently.

Let us first recall some results about polynomial arithmetic. Let $\mu : \mathbb{N} \to \mathbb{R}$ be such that univariate polynomials of degree $\leq n$ with coefficients in $\mathbb{F}$ can be multiplied with $\mathcal{O}(\mu(n))$ arithmetic operations in $\mathbb{F}$. Then we may take $\mu(n) = n \log n$ if $\mathbb{F}$ supports Fast Fourier Transforms and $\mu(n) = n \log n \cdot \log\log n$ for arbitrary $\mathbb{F}$.

Polynomial division is specified as follows: For given $f, g \in \mathbb{F}[z]$ with $\deg f = n \geq m = \deg g$, we want to compute the quotient q and remainder $r \in \mathbb{F}[z]$ determined (uniquely) by $f = qg + r$ and $\deg r < m$. With the usual school algorithm, q and r can be computed with $\mathcal{O}(m\,n)$ operations in $\mathbb{F}$. For asymptotically fast algorithms, the bound $\mathcal{O}(\mu(n-m)+\mu(m))$ is stated in (Burgisser), Corollary (2.26). With the school method for "blocks" of $\mathcal{O}(m)$ coefficients, which are then processed with asymptotically fast techniques, polynomial division can be performed with $\mathcal{O}((n/m)\mu(m))$ operations.

The greatest common divisor d of two polynomials $f, g \in \mathbb{F}[z]$ with $\deg f \leq n$ and $\deg g \leq n$ can be computed with $\mathcal{O}(\mu(n) \cdot\ logn)$ arithmetic operations. Corresponding cofactors, i.e., polynomials $u, v \in \mathbb{F}[z]$ with $u\ f + vg = d$ and $\deg(du) < \deg g$ and $\deg(d \cdot\ v) < \deg f$, can be computed within the same time bound. This follows from (Burgisser) Corollary (3.14).

The algorithm for the Sylvester equation with Frobenius (companion) matrices A and

B is based on the fact that if A is a Frobenius matrix and q is a polynomial, then the equation $q(A)y = d$ can be written in terms of polynomial arithmetic.

Lemma 11 *Let $A \in \mathbb{F}^{m\times m}$ be a Frobenius matrix. Let $q \in \mathbb{F}[z]$, and let $y = (y_0, y_1, ..., y_{m-1})^T$ and $d = (d_0, d_1, ..., d_{m-1})^T \in F^m$. Let $p_y(z) = y_0 + y_1 z + ... + y_{m-1}z^{m-1}$ and $p_d(z) = d_0 + d_1 z + ... + d_{m-1}z^{m-1}$. Then $q(A).y = d$ if and only if $q.p_y = p_d \bmod \chi_A$.*

Lemma 12 *Let B be a Frobenius matrix. Let $A \in F^{m\times m}$, $C \in F^{m\times n}$ with columns $c_1, c_2, .., c_n$ and let $X \in F^{m\times n}$ be a solution to the Sylvester equation $AX - XB^T = C$. The the last column x_n of X fulfils*

$$\chi_B(A).x_n = \sum_{k=0}^{n-1} A^k.c_{k+1},$$

Proof. If B is a Frobenius matrix, then $\chi_{B_k}(z) = z^k$ for $0 \le k \le n-1$. ■

Lemma 13 *$d = \sum_{k=0}^{n-1} A^k.c_{k+1}$ can be computed with $\mathcal{O}(mn)$ arithmetic operations.*

Multiplication of a vector with A can be done with $2m-1$ operations. The vector $d = \sum_{k=0}^{n-1} A^k.c_{k+1}$ can be computed via Horner's rule with $n-1$ multiplications of $A$with a vector and $n-1$additions in F^m.

Lemma 14 *Let $A \in \mathbb{F}^{m\times m}$ and $B \in \mathbb{F}^{n\times n}$ be Frobenius matrices with disjoint spectra and $C \in \mathbb{F}^{m\times n}$. Then the last column x_n of the solution X of Sylvester Equation can be computed with $\mathcal{O}(mn + \mu(m).\log m)$ arithmetic operations.*

Theorem 15 *(Kirrinni) Let $A \in F^{m\times m}$ and $B \in \mathbb{F}^{n\times n}$ be Frobenius matrices with disjoint spectra and $C \in \mathbb{F}^{m\times n}$. Then the solution X of Sylvester Equation can be computed with $\mathcal{O}(mn)$ arithmetic operations.*

For the case of generic matrices A and B (i.e., the entries of A and B are indeterminates), the problem can be reduced to the Frobenius case by similarity transforms, using Keller-Gehrig's algorithm for computing the characteristic polynomial. The algorithm exploits fast matrix multiplication.

Lemma 16 *Let $A \in \mathbb{F}^{n\times n}$ and $\nu \in \mathbb{F}^n \setminus \{0\}$. If the matrix $U = U(A) = \left(\nu, A\nu, A^2\nu, ..., A^{n-1}\nu\right)$ is nonsingular; then $U^{-1}AU$ is a Frobenius matrix. If the entries of A are algebraically independent over a subfield $\mathbb{F}_0$ of $\mathbb{F}$ and $\nu \in \mathbb{F}_0^n \setminus \{0\}$; then $U(A)$ is nonsingular.*

If the entries of A are algebraically independent; then the matrix U in can be computed A simple algorithm described by Keller-Gehrig (Keller), Section 3 with $\mathcal{O}(n \log n)$ operations.

The inverse of a nonsingular $n \times n$ matrix can be computed with $\mathcal{O}(n^\beta)$ operations, see (Strassen) or (Burgisser), Section 16.4.

matrices $U \in GL_m$ and $V \in GL_n$ such that $A = U^{-1}AU$ and $B = V^{-1}BV$ are Frobenius matrices can be computed with $\mathcal{O}(N^\beta logN)$ operations. The matrix $C = U^{-1}C(V^{-1})^T$ can be computed with $\mathcal{O}(N^\beta)$ operations. The solution Y of the equation $A'Y - Y(B')^T = C'$ can be computed with $\mathcal{O}(mn) \leq \mathcal{O}(N^\beta)$ operations, and the solution $X = UYV^T$ of $AX - XB^T = C$ can then be computed from Y with $\mathcal{O}(N^\beta)$ operations.

2.4.2 The general case

We consider a matrix $A \in \mathbb{F}^{n\times n}$ semicyclic, i.e. upper block triangular, with Frobenius matrices $\mathbb{F}_1, .., \mathbb{F}_l$ on the block diagonal. Any semicyclic matrix $A \in \mathbb{F}^{n\times n}$ can be partitioned as

$$\begin{matrix} n_1 \\ n_0 \\ n_2 \end{matrix} \begin{pmatrix} A_1 & * & * \\ & A_0 & * \\ 0 & & A_2 \end{pmatrix}$$

where $A_1 \in \mathbb{F}^{n_1\times n_1}$ and $A_2 \in \mathbb{F}^{n_2\times n_2}$ are semicyclic, $A_0 \in \mathbb{F}^{n_0\times n_0}$ is a Frobenius matrix, $n = n_1 + n_2 + n_0$, and $n_1, n_2 \leq n/2$.

The crucial step in Keller-Gehrig's algorithm for computing the characteristic polynomial of a matrix A (not necessarily generic) is to transform A into a semicyclic matrix. For $A \in \mathbb{F}^{n\times n}$; a nonsingular matrix U such that $U^{-1}AU$ is semicyclic can be computed with $\mathcal{O}(n^\beta \log n)$ arithmetic operations. An algorithm for this task is described in (Keller), Section 5 and (Burgisser), Section 16.6.

The divide and conquer algorithm for the Sylvester equation uses the following simple complexity result for rectangular matrix multiplication: For $m \leq n$, the product of an $m \times m$ matrix A with an $m \times n$ matrix B can be computed with $\mathcal{O}(m^{\beta-1}n)$ operations.

Lemma 17 *(Kirrinni) Let A, B be semicyclic; one of them a Frobenius matrix; and $m, n \leq N$. Then X can be computed with $\mathcal{O}(N^{\beta})$ arithmetic operations.*

Proof. We prove that if we suppose that A be semicyclic; B be a Frobenius matrix then, X can be computed with $\mathcal{O}(m^{\beta-1}n)$ arithmetic operations in the case where $m \leq n$. and with $\mathcal{O}(m^{\beta})$ arithmetic operations. if $m \geq n$.

First case, If A is not a Frobenius matrix, then we partition A according and partition the rows of X and C accordingly. Then the Sylvester equation reads

$$\begin{pmatrix} A_1 & U_1 & U_2 \\ 0 & A_0 & U_3 \\ 0 & 0 & A_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_0 \\ X_2 \end{pmatrix} - \begin{pmatrix} X_1 \\ X_0 \\ X_2 \end{pmatrix} B^T = \begin{pmatrix} C_1 \\ C_0 \\ C_2 \end{pmatrix}$$

or equivalently

$$A_2X_2 - X_2B^T = C_2, \tag{2.20}$$

$$A_0X_0 - X_0B^T = C_0 - U_3X_2, \tag{2.21}$$

$$A_1X_1 - X_1B^T = C_1 - U_1X_0 - U_2X_2 \tag{2.22}$$

Let $T(M, N)$ denote a time bound for solving the Sylvester equation for the special case specified in the lemma with $m \leq M$ and $n \leq N$. Then X_2 can be computed in time $T(M/2, N)$. The right hand side of (2.21) and (2.22) can be computed in time $\mathcal{O}\left(M^{\beta-1}N\right)$. The solution X_0 can be computed with $\mathcal{O}(M.N)$ operations and X_1 can then computed from (2.22) in time $T(M/2, N)$. This shows that $T(M, N)$ fulfils the recursive estimate.

$$T(M, N) \leq 2.T(M/2, N) + O\left(M^{\beta-1}N\right), \qquad T(1, N) \leq \mathcal{O}(N),$$

which implies the assertion of the lemma.

Secendely, if $m \geq n.$, we reduce to smaller cases by partitioning A as in the first case. The time for computing the r.h.s. of (2.21) and (2.22) is bounded by $\mathcal{O}(M^{\beta})$. (We do not exploit the fact that the matrix multiplications involved here can be computed cheaper if n is small compared with m. The recursive estimate is as follows:

$$T(M,N) \leq 2.T(M/2,N) + O\left(M^{\beta}\right), \qquad T(N,N) \leq \mathcal{O}\left(N^{\beta}\right),$$

This implies that $T(M,N) = \mathcal{O}\left(M^{\beta}\right)$. ■

Lemma 18 *Let A, B be and $m,n \leq N$. Then X can be computed with $\mathcal{O}\left(N^{\beta}\right)$ arithmetic operations.*

Theorem 19 *(Kirrinni) Let $A \in \mathbb{F}^{m\times m}$ and $B \in \mathbb{F}^{n\times n}$ with disjoint spectra; let $C \in \mathbb{F}^{m\times n}$; and let $m,n \leq N$. Then the unique solution X of the Sylvester equation $AX - XB^T = C$ can be computed with $\mathcal{O}\left(N^{\beta}\log N\right)$ arithmetic operations.*

Proof. Compute nonsingular matrices $U \in \mathbb{F}\ m{\times}m$ and $V \in \mathbb{F}\ n{\times}n$ such that $A = U^{-1}AU$ and $B = V^{-1}BV$ are semicyclic. U and V can be computed with $\mathcal{O}(N^{\beta}\log N)$ operations, and A and B can then be computed with $O(N)$ operations. The matrices A and B have disjoint spectra, and the (unique) solution $Y \in \mathbb{F}\ m \times n$ of $AY - YB^T = U^{-1}C(V^{-1})^T$ can be computed according to Lemma 18 with $\mathcal{O}(N^{\beta})$ operations. The solution $X = UYV^T$ of the original equation $AX - XB^T = C$ can now be computed with $\mathcal{O}(N^{\beta})$ operations. ■

Chapter 3

Boussinesq Equation

In this chapter we use Lyapunov-Sylvester algebraic operators to approximate the solutions of Boussinesq equation in higher dimensions without adapting classical developments based on separation of variables, radial solutions, etc. The crucial idea is to prove that simple methods of discretization of PDEs such as finite difference, finite volumes, can be transformed into well adapted algebraic systems such as Lyapunov-Sylvester ones leading to best algorithms when regarded for convergence rates, time execution and error estimates. In this work, fortunately, we are confronted with more complicated but fascinating forms to prove the invertibility of the algebraic operator appearing in the numerical scheme. Instead of using classical methods such as tri-diagonal transformations we applied a topological method to prove the invertibility. This is good as it did not necessitate to compute eigenvalues and precisely bounds/estimates of eigenvalues or direct inverses which remains a complicated problem in general linear algebra and especially for generalized Lyapunov-Sylvester operators. Recall that even though, bounds/estimates of eigenvalues can already be efficient in studying stability. Recall also that block tri-diagonal systems for classical methods can be already used here also and can be solved for example using iterative techniques, or highly structured bandwidth solvers, Kronecker-product techniques, etc. These methods have been subjects of more general discretization. See (ElMikkawy 1; ElMikkawy 2; ElMikkawy 3; ELMikkawy 3; ELMikkawy 5; Jia) for a review on tri-diagonal and block tri-diagonal systems, their advantages as well as their disadvantages.

This work has many folds. One principal aim is to apply non tri-diagonal type algebraic methods to investigates numerical solutions for PDEs in multi-dimensional spaces. We aim to prove that Lyapunov-Sylvester operators can be good candidates for such aim and that they may give best solvers compared to tri-diagonal and/or block tri-diagonal ones. Recall that the later methods are unadvised because of many reasons. First they are costing methods from both the machine memory and time. In higher dimensions, they secondly need to transform the original problem into an external space of projection and thus solve an associated problem in the new space and next to left to the original one. These facts may induce as previously time and accuracy losing.

This chapter is devoted to the development of a numerical method based on two-dimensional finite difference scheme to approximate the solution of the nonlinear Boussinesq equation in $\mathbb{R}^2$ written in the form

$$u_{tt} = \Delta u + qu_{xxxx} + (u^2)_{xx}, \quad ((x,y),t) \in \Omega \times (t_0, +\infty) \tag{3.1}$$

with initial conditions

$$u(x,y,t_0) = u_0(x,y) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,y,t_0) = \varphi(x,y), \quad (x,y) \in \Omega \tag{3.2}$$

and boundary conditions

$$\frac{\partial u}{\partial \eta}(x,y,t) = 0, \quad ((x,y),t) \in \partial\Omega \times (t_0, +\infty). \tag{3.3}$$

To reduce the derivation order, we set

$$v = qu_{xx} + u^2. \tag{3.4}$$

We have to solve the system

$$\begin{aligned}
u_{tt} = \Delta u + v_{xx}, \quad (x,y,t) \in \Omega \times (t_0, +\infty) \\
v = qu_{xx} + u^2, \quad (x,y,t) \in \Omega \times (t_0, +\infty) \\
(u,v)(x,y,t_0) = (u_0,v_0)(x,y), \quad (x,y) \in \overline{\Omega} \\
\frac{\partial u}{\partial t}(x,y,t_0) = \varphi(x,y), \quad (x,y) \in \overline{\Omega} \\
\frac{\partial}{\partial \eta}(u,v)(x,y,t) = 0, \quad (x,y,t) \in \partial\Omega \times (t_0, +\infty)
\end{aligned} \tag{3.5}$$

on a rectangular domain $\Omega =]L_0, L_1[\times]L_0, L_1[$ in $\mathbb{R}^2$. $t_0 \geq 0$ is a real parameter fixed as the initial time, u_{tt} is the second order partial derivative in time, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator in $\mathbb{R}^2$, q is a real constant, u_{xx} and u_{xxxx} are respectively the second order and the fourth order partial derivative according to x. $\frac{\partial}{\partial \eta}$ is the outward normal derivative operator along the boundary $\partial\Omega$. Finally, u, u_0 and φ are real valued functions with u_0 and φ are $\mathcal{C}^2$ on $\overline{\Omega}$. u (and consequently v) is the unknown candidates supposed to be $\mathcal{C}^4$ on $\overline{\Omega}$.

Several papers have been devoted to the study of existence and uniqueness of solutions of problem (3.1) and sometimes exact solutions are developed such as solitary, stationary, time-independent, one-dimensional ones. For example, in the case of a one-direction viscous fluid we may seek solutions of the form $u(x, y, t) = \alpha\psi(x)$. In this case, the problem is transformed into a one variable ordinary differential equation

$$q\psi''^2(x) = ax + b,$$

for some constants a and b depending on the initial-boundary conditions. Therefore, the existence and uniqueness problems are overcame using the well-known theory of ODEs. For more details on these facts, we may refer to (Bao-Dan; Bratsos; Diaz 1; Diaz 2; Jafarizadeh; Kano; Kaya; Liu; Parlange; Song; Varlamov; Wazwaz; Yi1; Yi2).

The Boussinesq equation has a wide reputation in both theoretic and applied fields. It governs the flow of ground water, heat conduction, natural convection in thermodynamics for both volume and fluids in porous media, traveling-waves solutions, self-similar solutions, scattering method, mono and multi dimensional versions, reduction of multi dimensional equations with respect to algebras, etc. In (Dehghan 1), several finite difference schemes such as three fully implicit finite difference schemes, two fully explicit ones, an alternating direction implicit procedure and the Barakat and Clark type explicit formula are discussed and applied to solve a two-dimensional case. In (Dehghan 2), the solution of a generalized Boussinesq equation has been developed by means of the homeotypic perturbation method. It consisted in a technique method that avoids the discretization, linearization, or small perturbations of the equation and thus reduces the numerical computations. Next, (Deghghan 3), a boundary-only meshfree method has been applied to approximate the numerical solution of the classical Boussinesq equation in one dimension. In (Shokri), a collocation and approximation of the numerical solution of the improved Boussinesq equation is obtained based on radial bases. A predictor-corrector scheme is provided and the Not-a-Knot method is used to improve the accuracy in the boundary. For this reason, many studies have been developed discussing the solvability of such equations. In (Diaz 1), a Boussinesq system of hydrodynamics equations arising in a coupling between Navier Stokes equations and thermodynamic ones in the presence of density gradients and where thermodynamical coefficients such as viscosity, specific heat and thermal conductivity are not assumed to be constants and thus leading to a coupled system of quasi-parabolic equations. The authors studied the existence and uniqueness of weak solutions. In this model there are two paradigmatic situations as stated by the authors and related to the fast and the slow heat diffusion. In theoretical mathematical study of such systems, this may correspond to the singular or degenerate character of the heat equation which occur according to the relative behavior of the specific heat of the fluid and its thermal conductivity. By assuming local Hölder approximations the behavior of the solution is studied near the origin. In (Diaz 2), local strong solutions for a parabolic system based on Boussinesq equation are studied for buoyancy-driven flow with viscous heating. A modification of the classical Navier-Stokes-Boussinesq system motivated by unresolved issues regarding the global solvability of the classical system in situations where viscous heating cannot be neglected is developed. The authors applied a simple model to obtain a coupled system of two parabolic equations where a source term involving the square of the gradient of one of the unknowns appears. Local existence and uniqueness in time of strong solutions for the model problem are established. See for instance (Bao-Dan; Ben Mabrouk 1; Bratsos; Clarkson; Diaz 1; Diaz 2; Jafarizadeh; Kano; Kaya; Lai; Liu; Parlange; Song; Varlamov; Wazwaz; Yi1; Yi2) and the references therein for backgrounds on theses facts.

In the organization of this chapter, the next section is concerned with the introduction of the finite difference scheme. Section 3 is devoted to the discretization of the continuous reduced system obtained from (3.1)-(3.3) by the order reduction method. Section 4 deals with the solvability of the discrete Lyapunov equation obtained from the discretization method. In section 5, the consistency of the method is shown and next, the stability and convergence of are proved based on Lyapunov criterion. Finally, some numerical implementation is provided in section 6 leading to the computation of the numerical solution and error estimates.

3.1 Discrete two-dimensional Boussinesq equation

Consider the domain $\Omega =]L_0, L_1[\times]L_0, L_1[\subset \mathbb{R}^2$ and an integer $J \in \mathbb{N}^*$. Denote $h = \frac{L_1 - L_0}{J}$ for the space step, $x_j = L_0 + jh$ and $y_m = L_0 + mh$ for all $(j, m) \in I^2 = \{0, 1, \dots, J\}^2$. Let $l = \Delta t$ be the time step and $t_n = t_0 + nl$, $n \in \mathbb{N}$ for the discrete time grid. For $(j, m) \in I$ and $n \geq 0$, $u^n_{j,m}$ will be the net function $u(x_j, y_m, t_n)$ and $U^n_{j,m}$ the numerical solution. The following discrete approximations will be applied for the different differential operators involved in the problem. For time derivatives, we set as discrete initial condition

$$U^0 = U^{-1} + l\varphi$$

and for $n \geq 1$,

$$u_t \rightsquigarrow \frac{U^{n+1}_{j,m} - U^{n-1}_{j,m}}{2l} \quad \text{and} \quad u_{tt} \rightsquigarrow \frac{U^{n+1}_{j,m} - 2U^n_{j,m} + U^{n-1}_{j,m}}{l^2}$$

and for space derivatives, we shall use

$$u_x \rightsquigarrow (U_x)_{j,m} = \frac{U^n_{j+1,m} - U^n_{j-1,m}}{2h} \quad \text{and} \quad u_y \rightsquigarrow (U_y)_{j,m} = \frac{U^n_{j,m+1} - U^n_{j,m-1}}{2h}$$

for first order derivatives, and

$$u_{xx} \rightsquigarrow (U_{xx})_{j,m} = \frac{U^{n,\alpha}_{j+1,m} - 2U^{n,\alpha}_{j,m} + U^{n,\alpha}_{j-1,m}}{h^2},$$
$$u_{yy} \rightsquigarrow (U_{yy})_{j,m} = \frac{U^{n,\alpha}_{j,m+1} - 2U^{n,\alpha}_{j,m} + U^{n,\alpha}_{j,m-1}}{h^2}$$

for second order ones, where for $n \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}$,

$$u^{n,\alpha} = \alpha U^{n+1} + (1-2\alpha)U^n + \alpha U^{n-1}.$$

Finally, we denote $\sigma = \frac{l^2}{h^2}$ and $\delta = \frac{q}{h^2}$.

For $(j,m) \in \mathring{I}^2$ an interior point of the grid I^2, ($\mathring{I} = \{1, 2, \ldots, J-1\}$), and $n \geq 1$, the following discrete equation is deduced from the first equation in the system (3.5).

$$\begin{aligned}
&U^{n+1}_{j,m} - 2U^n_{j,m} + U^{n-1}_{j,m} \\
&= \sigma\alpha(U^{n+1}_{j-1,m} - 4U^{n+1}_{j,m} + U^{n+1}_{j+1,m} + U^{n+1}_{j,m-1} + U^{n+1}_{j,m+1}) \\
&\quad + \sigma(1-2\alpha)(U^n_{j-1,m} - 4U^n_{j,m} + U^n_{j+1,m} + U^n_{j,m-1} + U^n_{j,m+1}) \\
&\quad + \sigma\alpha(U^{n-1}_{j-1,m} - 4U^{n-1}_{j,m} + U^{n-1}_{j+1,m} + U^{n-1}_{j,m-1} + U^{n-1}_{j,m+1}) \\
&\quad + \sigma\alpha(V^{n+1}_{j-1,m} - 2V^{n+1}_{j,m} + V^{n+1}_{j+1,m}) \\
&\quad + \sigma(1-2\alpha)(V^n_{j-1,m} - 2V^n_{j,m} + V^n_{j+1,m}) \\
&\quad + \sigma\alpha(V^{n-1}_{j-1,m} - 2V^{n-1}_{j,m} + V^{n-1}_{j+1,m}).
\end{aligned} \tag{3.6}$$

Similarly, the following discrete equation is obtained from equation (3.4).

$$\begin{aligned}
V^{n+1}_{j,m} + V^{n-1}_{j,m} &= 2\delta\alpha(U^{n+1}_{j-1,m} - 2U^{n+1}_{j,m} + U^{n+1}_{j+1,m}) \\
&\quad + 2\delta(1-2\alpha)(U^n_{j-1,m} - 2U^n_{j,m} + U^n_{j+1,m}) \\
&\quad + 2\delta\alpha(U^{n-1}_{j-1,m} - 2U^{n-1}_{j,m} + U^{n-1}_{j+1,m}) + 2\widehat{F(U^n_{j,m})}
\end{aligned} \tag{3.7}$$

where

$$F(u) = u^2,\ F^n = F(u^n) \quad \text{and} \quad \widehat{F^n} = \frac{F^{n-1} + F^n}{2}.$$

The discrete boundary conditions are written for $n \geq 0$ as

$$U^n_{1,m} = U^n_{-1,m} \quad \text{and} \quad U^n_{J-1,m} = U^n_{J+1,m}, \tag{3.8}$$

$$U^n_{j,1} = U^n_{j,-1} \quad \text{and} \quad U^n_{j,J-1} = U^n_{j,J+1}. \tag{3.9}$$

The parameter q is related to the equation and has the role of a viscosity-type coefficient and thus it is related to the physical domain of the model. The barycenter parameter α is used to calibrates the position of the approximated solution around the exact one. Of course, these parameters affect surely the numerical solution as well as the error estimates. This fact will be recalled later in the numerical implementations part. In a future work in progress now, we are developing results on numerical solutions of 2D Schrödinger equation on the error estimates as a function on the barycenter calibrations by using variable coefficients α_n instead of constant α. The use of these calibrations permits the use of implicit/explicit schemes by using suitable values. For example for $\alpha = \frac{1}{2}$, the barycenter estimation

$$V^{n,\alpha} = \alpha V^{n+1} + (1-2\alpha)V^n + \alpha V^{n-1} = \frac{V^{n+1} + V^{n-1}}{2}$$

which is an implicit estimation that guarantees an error of order 2 in time.

As motioned in the introduction, the main idea consists in applying Lyapunov-Sylvester operators to approximate the solution of the continuous problem (3.1)-(3.3) or its discrete equivalent system (3.6)-(3.9). Denote

$$a_1 = \frac{1}{2} + 2\alpha\sigma, \quad a_2 = -\alpha\sigma,$$

$$b_1 = 1 - 2(1-2\alpha)\sigma, \quad b_2 = (1-2\alpha)\sigma,$$

$$c_1 = (1-2\alpha)\delta \quad \text{and} \quad c_2 = \alpha\delta.$$

Equation (3.6) becomes

$$\begin{aligned} & a_2U^{n+1}_{j-1,m} + a_1U^{n+1}_{j,m} + a_2U^{n+1}_{j+1,m} + a_2U^{n+1}_{j,m-1} + a_1U^{n+1}_{j,m} + a_2U^{n+1}_{j,m+1} \\ & \quad + a_2(V^{n+1}_{j-1,m} - 2V^{n+1}_{j,m} + V^{n+1}_{j+1,m}) \\ & = b_2U^{n}_{j-1,m} + b_1U^{n}_{j,m} + b_2U^{n}_{j+1,m} + b_2U^{n}_{j,m-1} + b_1U^{n}_{j,m} + b_2U^{n}_{j,m+1} \\ & \quad - a_2U^{n-1}_{j-1,m} - a_1U^{n-1}_{j,m} - a_2U^{n-1}_{j+1,m} - a_2U^{n-1}_{j,m-1} - a_1U^{n-1}_{j,m} - a_2U^{n-1}_{j,m+1} \\ & \quad + b_2(V^{n}_{j-1,m} - 2V^{n}_{j,m} + V^{n}_{j+1,m}) - a_2(V^{n-1}_{j-1,m} - 2V^{n-1}_{j,m} + V^{n-1}_{j+1,m}). \end{aligned} \tag{3.10}$$

Equation (3.7) becomes

$$\begin{aligned} & V^{n+1}_{j,m} - 2c_2(U^{n+1}_{j-1,m} - 2U^{n+1}_{j,m} + U^{n+1}_{j+1,m}) \\ & = 2c_1(U^{n}_{j-1,m} - 2U^{n}_{j,m} + U^{n}_{j+1,m}) \\ & \quad + 2c_2(U^{n-1}_{j-1,m} - 2U^{n-1}_{j,m} + U^{n-1}_{j+1,m}) - V^{n-1}_{j,m} + 2\widehat{F(U^n_{j,m})}. \end{aligned} \tag{3.11}$$

Denote

$$A = \begin{pmatrix} a_1 & 2a_2 & 0 & \dots & \dots & 0 \\ a_2 & a_1 & a_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_2 & a_1 & a_2 \\ 0 & \dots & \dots & 0 & 2a_2 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 2b_2 & 0 & \dots & \dots & 0 \\ b_2 & b_1 & b_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_2 & b_1 & b_2 \\ 0 & \dots & \dots & 0 & 2b_2 & b_1 \end{pmatrix},$$

$$R = \begin{pmatrix} -2 & 2 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 2 & -2 \end{pmatrix}$$

The system (3.10)-(3.11) can be written on the matrix form

$$\begin{aligned} \mathcal{L}_A(U^{n+1}) + a_2 R V^{n+1} &= \mathcal{L}_B(U^n) - \mathcal{L}_A(U^{n-1}) + R(b_2 V^n - a_2 V^{n-1}), \\ V^{n+1} - 2c_2 R U^{n+1} &= 2R(c_1 U^n + c_2 U^{n-1}) - V^{n-1} + 2\widehat{F^n} \end{aligned} \tag{3.12}$$

for all $n \geq 1$ where

$$U^n = (U^n_{j,m})_{0\leq j,m\leq J}, \quad V^n = (V^n_{j,m})_{0\leq j,m\leq J}, \quad F^n = (F(U^n_{j,m}))_{0\leq j,m\leq J}$$

and for a matrix $Q \in \mathcal{M}_{(J+1)^2}(\mathbb{R})$, $\mathcal{L}_Q$ is the Lyapunov operator defined by

$$\mathcal{L}_Q(X) = QX + XQ^T, \quad \forall X \in \mathcal{M}_{(J+1)^2}(\mathbb{R}).$$

Remark that V is obtained from the auxiliary function v that is applied to reduce the order of the original PDEs in u. This reduction yielded the Lyapunov-Sylvester system (3.12) above. A natural question that can be raised here turns around the ordering of U and V. So, we stress the fact that no essential idea is fixed at advance but, this is strongly related to the system obtained. For example, in (3.12) above, it is easy to substitute the second equation into the first to omit the unknown matrix V^{n+1} from the first equation. But in the contrary, it is not easier to do the same for U^{n+1}, due to the difficulty to substitute it from $\mathcal{L}_A(U^{n+1})$. It is also not guaranteed that the part $a_2 R V^{n+1}$ in the first equation is invertible to substitute V^{n+1}. So, it is essentially the final system that shows the ordering in U and V.

3.2 Solvability of the discrete problem

In (Ben Mabrouk 1), the authors have transformed the Lyapunov operator obtained from the discretization method into a standard linear operator acting on one column vector by juxtaposing the columns of the matrix X horizontally which leads to an equivalent linear operator characterized by a fringe-tridiagonal matrix. We used standard computation to prove the invertibility of such an operator. Here. we do not apply the same computations as in (Ben Mabrouk 1), but we develop different arguments.

The first main result is stated as follows.

Theorem 20 *System* (3.12) *is uniquely solvable whenever* U^0 *and* U^1 *are known.*

Proof. It reposes on the inverse of Lyapunov operators. Consider the endomorphism Φ defined on $\mathcal{M}_{(J+1)^2}(\mathbb{R}) \times \mathcal{M}_{(J+1)^2}(\mathbb{R})$ by $\Phi(X,Y) = (AX + XA^T + a_2 RY, \frac{1}{2}Y - c_2 RX)$. To prove Theorem 20, it suffices to show that $ker\Phi$ is reduced to 0. Indeed,

$$\Phi(X,Y) = 0 \Longleftrightarrow (AX + XA^T + a_2 RY, \frac{1}{2}Y - c_2 RX) = (0,0)$$

or equivalently,

$$Y = 2c_2 RX \quad \text{and} \quad (A + 2a_2 c_2 R^2)X + XA^T = 0.$$

So, the problem is transformed to the resolution of a Lyapunov type equation of the form

$$\mathcal{L}_{W,A}(X) = WX + XA^T = 0 \tag{3.13}$$

where W is the matrix given by $W = A + 2a_2 c_2 R^2$. Denoting

$$\omega = 2a_2 c_2, \quad \omega_1 = a_1 + 6\omega, \quad \overline{\omega}_1 = \omega_1 + \omega, \quad \omega_2 = a_2 - 4\omega\,.$$

The matrix W is explicitly given by

$$W = \begin{pmatrix} \omega_1 & 2\omega_2 & 2\omega & 0 & \dots & \dots & \dots & 0 \\ \omega_2 & \overline{\omega}_1 & \omega_2 & \omega & \ddots & \ddots & \ddots & \vdots \\ \omega & \omega_2 & \omega_1 & \omega_2 & \omega & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \omega & \omega_2 & \omega_1 & \omega_2 & \omega \\ \vdots & \ddots & \ddots & \ddots & \omega & \omega_2 & \overline{\omega}_1 & \omega_2 \\ 0 & \dots & \dots & \dots & 0 & 2\omega & 2\omega_2 & \omega_1 \end{pmatrix}$$

■

Next, we use the following preliminary result of differential calculus (See (Cartan) for example).

Lemma 21 *Let E be a finite dimensional ($\mathbb{R}$ or $\mathbb{C}$) vector space and $(\Phi_n)_n$ be a sequence of endomorphisms converging uniformly to an invertible endomorphism Φ. Then, there exists n_0 such that, for any $n \geq n_0$, the endomorphism Φ_n is invertible.*

The proof is simple and can be found in any differential calculus references such as (Cartan). We recall it here for the convenience and clearness of the proof. Recall that the set $Isom(E)$ (the set of isomorphisms on E) is already open in $L(E)$ (the set of endomorphisms of E). Hence, as $\Phi \in \text{Isom}(E)$ there exists a ball $B(\Phi, r) \subset \text{Isom}(E)$. The elements Φ_n are in this ball for large values of n. So these are invertible.

Assume now that $l = o(h^{2+s})$, with $s > 0$ which is always possible. Then, the coefficients appearing in A and W will satisfy as $h \to 0$ the following.

$$A_{i,i} = \frac{1}{2} + \varepsilon\, h^{2+2s} \to \frac{1}{2}.$$

For $1 \leq i \leq J-1$,

$$A_{i,i-1} = A_{i,i+1} = \frac{A_{0,1}}{2} = \frac{A_{J,J-1}}{2} = -\varepsilon h^{2+2s} \to 0.$$

For $2 \leq i \leq J-2$,

$$W_{i,i} = W_{0,0} = W_{J,J} = \frac{1}{2} + 2\alpha\varepsilon\, h^{2+2s} - 12\alpha^2 \varepsilon h^{2s} \to \frac{1}{2}.$$

Similarly,

$$W_{1,1} = W_{J-1,J-1} = \frac{1}{2} + 2\alpha\varepsilon h^{2+2s} - 14\alpha^2 \varepsilon h^{2s} \to \frac{1}{2}$$

and

$$W_{i,i-1} = W_{i,i+1} = \frac{W_{0,1}}{2} = \frac{W_{J,J-1}}{2} = -\alpha\varepsilon h^{2+2s} + 8\alpha^2 \varepsilon h^{2s} \to 0$$

Finally,

$$W_{i,i-2} = W_{i,i+2} = \frac{W_{0,2}}{2} = \frac{W_{J,J-2}}{2} = -2\alpha^2 \varepsilon\, h^{2s} \to 0.$$

Recall that the technique assumption $l = o(h^{2+s})$ is a necessary requirement for the resolution of the present problem and may not be necessary in other PDEs. See for example (Ben Mabrouk 2; Ben Mabrouk 1; Ben Mabrouk 3) for NLS and Heat equations. Next, observing that for all X in the space $\mathcal{M}_{(J+1)^2}(\mathbb{R}) \times \mathcal{M}_{(J+1)^2}(\mathbb{R})$,

$$\begin{aligned}\|(\mathcal{L}_{W,A} - I)(X)\| &= \|(W - \frac{1}{2}I)X + X(A^T - \frac{1}{2}I)\| \\ &\leq [\|W - \frac{1}{2}I\| + \|A^T - \frac{1}{2}I\|]\|X\|,\end{aligned}$$

it results that

$$\|\mathcal{L}_{W,A} - I\| \leq \|W - \frac{1}{2}I\| + \|A^T - \frac{1}{2}I\| \leq C(\alpha)h^{2s}. \tag{3.14}$$

Consequently, the Lyapunov endomorphism $\mathcal{L}_{W,A}$ converges uniformly to the identity I as h goes towards 0 and $l = o(h^{2+s})$ with $s > 0$. Using Lemma 21, the operator $\mathcal{L}_{W,A}$ is invertible for h small enough.

Remark 22 *The strict hypothesis $l = o(h^{2+s})$,$s > 0$ is theoretical and used to prove the invertibility (solvability) of the discrete system, but from the numerical point of view, we will see that even if this assumption is not satisfied, the algorithm converges faster the tri-diagonal classical methods,and with good error estimates.*

3.2.1 Some facts on the convergence of solutions and associated spaces

Usually the problem of convergence depends on different quantities in the model and on the geometry of the domain. Denote

$$\Omega_h = \{(x_j, y_m) \in \mathbb{R}^2 : 0 \leq j, m \leq J\}, \quad \Omega_t = \{t_n : n \in \mathbb{N}\}$$

and define the space of grid functions on Ω_h as

$$\mathcal{V}_h = \{U = (U_{j,m})_{j,m\in\mathbb{Z}} \text{ satisfying (3.8)-(3.9)}\}.$$

On the grid functions space, we usually define some appropriate norms to compute the error estimate between exact solutions of the continuous inhomogenous problem associated to (3.1) and its discrete variant obtained through the discrete scheme.

For $U \in \mathcal{V}_h$ and $V \in \mathcal{V}_h$ define the inner product

$$(U,V)_h = h^2 \sum_{j,m=0}^{J} U_{j,m}.V_{j,m}.$$

This leads to Sobolev norms (or semi-norms) such as

$$\|V\|_h = (V,V)_h^{1/2}, \quad \|V\|_{\infty,h} = \max_{0\leq j,m\leq J} |V_{j,m}|,$$
$$|V|_{1,h} = \Big[h^2 \sum_{j,m=0}^{J} (|(U_x)_{j,m}|^2 + |(U_y)_{j,m}|^2)\Big]^{1/2},$$
$$\|V\|_{2,h} = \Big[h^2 \sum_{j,m=0}^{J} (|\Delta_h U_{j,m}|^2\Big]^{1/2}.$$

Next, as it appears in the continuous problem derivatives of order 4 of the unknown function u, we generally restrict on suitable regularity spaces. It is not sometimes necessary to go to higher derivatives. In the present case for example, we may consider functions that are of class C^4 with respect to x, C^2 with respect to y and class C^2 with respect to t. We get using summation by parts

$$(\Delta_h V, U)_h = (V, \Delta_h U)_h, \quad -(\Delta_h V, V)_h = |V|_{1,h}^2, \quad (\Delta_h^2 V, V)_h = |V|_{2,h}^2.$$

As we work on a finite grid and thus a finite space of grid functions all these norms (semi-norms) are equivalent, and thus there is no essential difference between them. The norms $\|.\|_h$ and $\|\cdot\|_{\infty,h}$ reflects the L^2 convergence, while the semi-norms $|\cdot|_{1,h}$ and $|\cdot|_{1,h}$ reflects somehow the convergence of the discrete derivatives and thus the convergence in the discrete Sobolev space. For more details and backgrounds on these facts we refer to (Ben Mabrouk 1; Ben Mabrouk 2; Ben Mabrouk 3; Bratsos; Dehghan 1; Dehghan 2; **?**; Diaz 1; Diaz 2; Goncalves; Jafarizadeh; Kano; Song; Wazwaz).

3.3 Consistency, stability and convergence of the discrete method

The consistency of the proposed method is done by evaluating the local truncation error arising from the discretization of the system

$$\begin{aligned} u_{tt} - \Delta u - v_{xx} &= 0, \\ v &= q u_{xx} + u^2. \end{aligned} \tag{3.15}$$

The principal part of the first equation is

$$\begin{aligned} \mathcal{L}^1_{u,v}(t,x,y) = \frac{l^2}{12}\frac{\partial^4 u}{\partial t^4} - \frac{h^2}{12}\Big(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\Big) - \alpha l^2 \frac{\partial^2(\Delta u)}{\partial t^2} \\ - \frac{h^2}{12}\frac{\partial^2 v}{\partial x^4} - \alpha l^2 \frac{\partial^4 v}{\partial t^2 \partial x^2} + O(l^2 + h^2). \end{aligned} \tag{3.16}$$

The principal part of the local error truncation due to the second part is

$$\mathcal{L}^2_{u,v}(t,x,y) = \frac{l^2}{2}\frac{\partial^2 v}{\partial t^2} + \frac{l^4}{24}\frac{\partial^4 v}{\partial t^4} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} - \alpha l^2 \frac{\partial^4 u}{\partial t^2 \partial x^2} + O(l^2 + h^2). \tag{3.17}$$

It is clear that the two operators $\mathcal{L}^1_{u,v}$ and $\mathcal{L}^2_{u,v}$ tend toward 0 as l and h tend to 0, which ensures the consistency of the method. Furthermore, the method is consistent with an order 2 in time and space.

We now proceed by proving the stability of the method by applying the Lyapunov criterion. A linear system $\mathcal{L}(x_{n+1}, x_n, x_{n-1}, \dots) = 0$ is stable in the sense of Lyapunov if for any bounded initial solution x_0 the solution x_n remains bounded for all $n \geq 0$. Here, we will precisely prove the following result.

Lemma 23 $\mathcal{P}_n$*: The solution* (U^n, V^n) *is bounded independently of* n *whenever the initial solution* (U^0, V^0) *is bounded.*

We will proceed by recurrence on n. Assume firstly that $\|(U^0, V^0)\| \leq \eta$ for some η positive. Using the system (3.12), we obtain

$$\begin{aligned} \mathcal{L}_{W,A}(U^{n+1}) &= \mathcal{L}_{Z,B}(U^n) + b_2 R V^n - \mathcal{L}_{W,A}(U^{n-1}) - a_2 R(F^{n-1} + F^n), \\ V^{n+1} &= 2c_2 R U^{n+1} + 2R(c_1 U^n + c_2 U^{n-1}) - V^{n-1} + 2\widehat{F^n}. \end{aligned} \tag{3.18}$$

where $Z = B - 2a_2c_1R^2$. Consequently,

$$\begin{aligned} &\|\mathcal{L}_{W,A}(U^{n+1})\| \\ &\leq \|\mathcal{L}_{Z,B}\|\,\|U^n\| + 2|b_2|\,\|V^n\| + \|\mathcal{L}_{W,A}\|\,\|U^{n-1}\| + 2|a_2|(\|F^{n-1}\| + \|F^n\|) \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \|V^{n+1}\| \leq 4|c_2|\,\|U^{n+1}\| + 4(|c_1|\,\|U^n\| + |c_2|\,\|U^{n-1}\|) \\ + \|V^{n-1}\| + \|F^{n-1}\| + \|F^n\|. \end{aligned} \tag{3.20}$$

Next, recall that for $l = o(h^{s+2})$ small enough and $s > 0$, we have

$$\begin{gathered} a_1 = \frac{1}{2} + 2\alpha h^{2s+2} \to \frac{1}{2}, \quad a_2 = -\alpha\, h^{2s+2} \to 0, \\ b_1 = 1 - 2(1-2\alpha)h^{2s+2} \to 1, \quad b_2 = (1-2\alpha)h^{2s+2} \to 0, \\ c_1 = (1-2\alpha)h^{-2} \to \infty, \quad c_2 = \alpha\, h^{2s+2} \to 0, \\ a_2c_1 = -\alpha(1-2\alpha)h^{2s} \to 0. \end{gathered}$$

As a consequence, for h small enough,

$$\|\mathcal{L}_{Z,B}\| \leq 2\|B\| + 2|a_2c_1|\|R\|^2 \leq 2\max(|b_1|, 2|b_2|) + 4|a_2c_1| \leq 2 + 4 = 6, \tag{3.21}$$

and the following lemma is deduced from (3.14).

Lemma 24 *For h small enough, it holds for all $X \in \mathcal{M}_{(J+1)^2}(\mathbb{R})$ that*

$$\frac{1}{2}\|X\| \leq (1 - C(\alpha)h^{2s})\|X\| \leq \|\mathcal{L}_{W,A}(X)\| \leq (1 + C(\alpha)h^{2s})\|X\| \leq \frac{3}{2}\|X\|.$$

Indeed, recall that equation (3.14) affirms that $\|\mathcal{L}_{W,A} - I\| \leq C(\alpha)h^{2s}$ for some constant $C(\alpha) > 0$. Consequently, for any X we obtain

$$(1 - C(\alpha)h^{2s})\|X\| \leq \|\mathcal{L}_{W,A}(X)\| \leq (1 + C(\alpha)h^{2s})\|X\|.$$

For $h \leq \frac{1}{(2C(\alpha))^{1/2s}}$, we obtain

$$\frac{1}{2} \leq (1 - C(\alpha)h^{2s}) < (1 + C(\alpha)h^{2s}) \leq \frac{3}{2}$$

and thus Lemma 24. As a result, (3.19) yields

$$\frac{1}{2}\|U^{n+1}\| \leq 6\|U^n\| + 2\|V^n\| + \frac{3}{2}\|U^{n-1}\| + 2(\|F^{n-1}\| + \|F^n\|). \tag{3.22}$$

For $n = 0$, this implies

$$\|U^1\| \leq 12\|U^0\| + 4\|V^0\| + 3\|U^{-1}\| + 4(\|F^{-1}\| + \|F^0\|). \tag{3.23}$$

Using the discrete initial condition

$$U^0 = U^{-1} + l\varphi.$$

We identify the function φ to the matrix whose coefficients are $\varphi_{j,m} = \varphi(x_j, y_m)$. We obtain

$$\|U^{-1}\| \leq \|U^0\| + l\|\varphi\|. \tag{3.24}$$

Observing that

$$F^{-1}_{j,m} = F(U^{-1}_{j,m}) = (U^0_{j,m} - l\varphi_{j,m})^2,$$

it follows that

$$|F^{-1}_{j,m}| \leq |U^0_{j,m}|^2 + 2l|\varphi_{j,m}|.|U^0_{j,m}| + l^2|\varphi_{j,m}|^2$$

and consequently,

$$\|F^{-1}\| \leq \|U^0\|^2 + 2l\|\varphi\|\,\|U^0\| + l^2\|\varphi\|^2. \tag{3.25}$$

Hence, equation (3.23) yields

$$\|U^1\| \leq (15 + 8l\|\varphi\|)\|U^0\| + 4\|V^0\| + 8\|F^0\| + 3l\|\varphi\| + 4l^2\|\varphi\|^2. \tag{3.26}$$

Now, the Lyapunov criterion for stability states exactly that for each $\varepsilon > 0$ thee exists $\eta > 0$ such that

$$\|(U^0, V^0)\| \leq \eta \;\Rightarrow\; \|(U^n, V^n)\| \leq \varepsilon, \quad \forall n \geq 0. \tag{3.27}$$

For $n = 1$ and $\|(U^1, V^1)\| \leq \varepsilon$, we seek an $\eta > 0$ for which $\|(U^0, V^0)\| \leq \eta$. Indeed, using

(3.26), this means that, it suffices to find η such that

$$8\eta^2 + (19 + 8l\|\varphi\|)\eta + 3l\|\varphi\| + 4l^2\|\varphi\|^2 - \varepsilon < 0. \tag{3.28}$$

The discriminant of this second order inequality is

$$\Delta(l,h) = (19 + 8l\|\varphi\|)^2 - 32(3l\|\varphi\| + 4l^2\|\varphi\|^2 - \varepsilon). \tag{3.29}$$

For h, l small enough, this is estimated as

$$\Delta(l,h) \sim 361 + 32\varepsilon > 0.$$

Thus there are two zeros of the second order equality above

$$\eta_1 = \frac{\sqrt{\Delta(l,h)} - (19 + 8l\|\varphi\|)}{16} > 0$$

and a second zero $\eta_2 < 0$ rejected. Consequently, choosing $\eta \in]0, \eta_1[$ we obtain (3.28). Finally, (3.26) yields $\|U^1\| \leq \varepsilon$. Next, equation (3.20), for $n = 0$, implies that

$$\|V^1\| \leq A(l,h,\varphi)\|U^0\|^2 + B(l,h,\varphi)\|U^0\| + C(l,h,\varphi) + 16|c_2|\|V^0\|, \tag{3.30}$$

where

$$\begin{aligned} A(l,h,\varphi) &= 3 + 32|c_2|, \\ B(l,h,\varphi) &= 4\Big(|c_1| + 8|c_2|(2 + l\|\varphi\|) + l\|\varphi\| + \frac{1}{h^2}\Big), \\ C(l,h,\varphi) &= 2(1 + 8|c_2|)l^2\|\varphi\|^2 + 4l(4|c_2| + \frac{1}{h^2})\|\varphi\|. \end{aligned}$$

Choosing $\|(U^0, V^0)\| \leq \eta$, it suffices to study the inequality

$$A(l,h,\varphi)\eta^2 + (B(l,h,\varphi) + 16|c_2|)\eta + C(l,h,\varphi) - \varepsilon \leq 0. \tag{3.31}$$

Its discriminant satisfies for h, l small enough,

$$\Delta(l,h) \sim \frac{16}{h^4}(1+20\alpha+|1-2\alpha|)^2 + \frac{128\alpha|q|}{h^2}\varepsilon > 0. \tag{3.32}$$

Here also there are two zeros, $\eta_1' = \frac{\sqrt{\Delta(l,h)}-(B(l,h,\varphi)+16|c_2|)}{2A(l,h,\varphi)} > 0$ and a second one $\eta' < 0$ and thus rejected. As a consequence, for $\eta \in]0, \eta_1'[$ we obtain $\|V^1\| \leq \varepsilon$. Finally, for $\eta \in]0, \eta_0[$ with $\eta_0 = \min(\eta_1, \eta_1')$, we obtain $\|(U^1, V^1)\| \leq \varepsilon$ whenever $\|(U^0, V^0)\| \leq \eta$. Assume now that the (U^k, V^k) is bounded for $k = 1, 2, \ldots, n$ (by ε_1) whenever (U^0, V^0) is bounded by η and let $\varepsilon > 0$. We shall prove that it is possible to choose η satisfying $\|(U^{n+1}, V^{n+1})\| \leq \varepsilon$. Indeed, from ((3.22), we have

$$\|U^{n+1}\| \leq 19\varepsilon_1 + 8\varepsilon_1^2. \tag{3.33}$$

So, one seeks, ε_1 for which $8\varepsilon_1^2 + 19\varepsilon_1 - \varepsilon \leq 0$. Its discriminant $\Delta = 361 + 32\varepsilon$, with one positive zero $\varepsilon_1 = \frac{\sqrt{361+32\varepsilon}-19}{16}$. Then $\|U^{n+1}\| \leq \varepsilon$ whenever $\|(U^k, V^k)\| \leq \varepsilon_1$, $k = 1, 2, \ldots, n$. Next, using (3.20) and (3.33), we have

$$\|V^{n+1}\| \leq (4|c_1| + 80|c_2| + 1)\varepsilon_1 + (32|c_2| + 2)\varepsilon_1^2. \tag{3.34}$$

So, it suffices as previously to choose ε_1 such that

$$(32|c_2| + 2)\varepsilon_1^2 + (4|c_1| + 80|c_2| + 1)\varepsilon_1 - \varepsilon \leq 0.$$

$\Delta = (4|c_1| + 80|c_2| + 1)^2 + 4(32|c_2| + 2)\varepsilon$, with positive zero

$$\varepsilon_1' = \frac{\sqrt{\Delta} - (4|c_1| + 80|c_2| + 1)}{2(32|c_2| + 2)}.$$

Then $\|V^{n+1}\| \leq \varepsilon$ whenever $\|(U^k, V^k)\| \leq \varepsilon_1'$, $k = 1, 2, \ldots, n$. Next, it holds from the recurrence hypothesis for $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_1')$, that there exists $\eta > 0$ for which $\|(U^0, V^0)\| \leq \eta$ implies that $\|(U^k, V^k)\| \leq \varepsilon_0$, for $k = 1, 2, \ldots, n$, which by the next induces that $\|(U^{n+1}, V^{n+1})\| \leq \varepsilon$.

Corollary 25 *As the numerical scheme is consistent and stable, it is then convergent.*

This Corollary is a consequence of the well known Lax-Richtmyer equivalence theorem, which states that for consistent numerical approximations, stability and convergence are equivalent. Recall here that we have already proved in (3.16) and (3.17) that the used scheme is consistent. Next, Lemma 23, Lemma 24 and equation (3.27) yields the stability of the scheme. Consequently, the Lax equivalence Theorem guarantees the convergence. So as Corollary 25.

3.4 Main steps of the algorithm applied

- Compute the matrices of the system
- Initialization: Compute the matrices U^0, U^1, V^0 and V^1
- for $n \geq 2$,

$$U^n = \text{lyap}(W, A, \mathcal{L}_{Z,B}(U^{n-1}) + b_2 R V^{n-1} - \mathcal{L}_{W,A}(U^{n-2}) - a_2 R(F^{n-2} + F^{n-1})),$$

  and

$$V^n = 2c_2 R U^n + 2R(c_1 U^{n-1} + c_2 U^{n-2}) - V^{n-2} + 2\widehat{F^{n-1}}.$$

The tri-diagonal associated system

Consider the lexicographic mesh $k = j(J+1) + m$ for $0 \leq j, m \leq J$, and denote $N = J(J+2)$, and

$$\Lambda_N = \{nJ + n - 1 : n \in \mathbb{N}\}, \quad \widetilde{\Lambda}_N = \{n(J+1) : n \in \mathbb{N}\}, \quad \Theta_N = \Lambda_N \cup \widetilde{\Lambda}_N.$$

Using the Kronecker product we obtain a tri-diagonal block system on the form

$$\begin{aligned} \widetilde{A}U^{n+1} + a_2\widetilde{R}V^{n+1} &= \widetilde{B}U^n - \widetilde{A}U^{n-1} + b_2\widetilde{R}V^n - a_2\widetilde{R}V^{n-1} \\ V^{n+1} - 2c_2\widetilde{R}U^{n+1} &= 2c_1\widetilde{R}U^n + 2c_2\widetilde{R}U^{n-1} - V^{n-1} + 2\widehat{F^n}. \end{aligned} \tag{3.35}$$

The numerical solutions' matrices U^n and V^n are identified here as one-column $(N+1)$-vectors and the matrices $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{R}$ are evaluated as follows.

The matrix $\widetilde{A}$

$$\widetilde{A}_{j,j} = 2a_1 \quad \forall j, 0 \leq j \leq N,$$
$$\widetilde{A}_{j,j+1} = \frac{1}{2}\widetilde{A}_{0,1} = a_2 \quad \forall j, 1 \leq j \leq N,\ j \notin \Theta_N,\ \text{and 0 on } \Lambda_N,$$
$$\widetilde{A}_{j-1,j} = \frac{1}{2}\widetilde{A}_{N,N-1} = a_2 \quad \forall j,\ 1 \leq j \leq N,\ j \notin \Theta_N,\ \text{and 0 on } \widetilde{\Lambda}_N,$$
$$\widetilde{A}_{j,j+J+1} = 2a_2, \quad \forall j,\ 0 \leq j \leq J,$$
$$\widetilde{A}_{j,j+J+1} = a_2, \quad \forall j,\ J+1 \leq j \leq N-J-1,$$
$$\widetilde{A}_{j-J-1,j} = a_2, \quad \forall j,\ J+1 \leq j \leq N-J-1,$$
$$\widetilde{A}_{j-J-1,j} = 2a_2, \quad \forall j,\ N-J \leq j \leq N.$$

The matrix $\widetilde{B}$

$$\widetilde{B}_{j,j} = 2b_1 \quad \forall j,\ 0 \leq j \leq N,$$
$$\widetilde{B}_{j,j+1} = \frac{1}{2}\widetilde{B}_{0,1} = b_2, \quad \forall j,\ 1 \leq j \leq N, ; j \notin \Theta_N,\ \text{and 0 on } \Lambda_N,$$
$$\widetilde{B}_{j-1,j} = \frac{1}{2}\widetilde{B}_{N,N-1} = b_2, \quad \forall j,\ 1 \leq j \leq N,\ j \notin \Theta_N,\ \text{and 0 on } \widetilde{\Lambda}_N,$$
$$\widetilde{B}_{j,j+J+1} = 2b_2 \quad \forall j,\ 0 \leq j \leq J,$$
$$\widetilde{B}_{j,j+J+1} = b_2 \quad \forall j,\ J+1 \leq j \leq N-J-1,$$
$$\widetilde{B}_{j-J-1,j} = b_2 \quad \forall j,\ J+1 \leq j \leq N-J-1,$$
$$\widetilde{B}_{j-J-1,j} = 2b_2 \quad \forall j,\ N-J \leq j \leq N.$$

The matrix $\widetilde{R}$

$$\widetilde{R}_{j,j} = -2 \quad \forall j,\ 0 \leq j \leq N,$$
$$\widetilde{R}_{j,j+J+1} = 2 \quad \forall j,\ 0 \leq j \leq J,$$
$$\widetilde{R}_{j,j-J-1} = 2 \quad \forall j,\ N-J \leq j \leq N,$$
$$\widetilde{R}_{j,j+J+1} = \widetilde{R}_{j-J-1,j} = 1 \quad \forall j,\ J+1 \leq j \leq N-J-1.$$

System (3.35) can be written as a linear standard form

$$\begin{aligned} \widetilde{W}U^{n+1} &= \widetilde{Z}U^n - \widetilde{W}U^{n-1} + b_2\widetilde{R}V^n - 2a_2\widetilde{R}\widehat{F^n} \\ V^{n+1} &= 2\widetilde{R}(c_1 I + c_2\widetilde{Z})U^n + 2(c_2 - c_1)\widetilde{R}U^{n-1} + 2b_2c_2\widetilde{R}^2V^n - V^{n-1} + 2\widehat{F^n}. \end{aligned} \tag{3.36}$$

where $\widetilde{W}$ and $\widetilde{Z}$ are given by $\widetilde{W} = \widetilde{A} + 2c_2a_2\widetilde{R}^2$ and $\widetilde{Z} = \widetilde{B} - 2c_1a_2\widetilde{R}^2$.

3.5 Numerical implementation

We propose in this section to present some numerical examples to validate the theoretical results developed previously. The error between the exact solutions and the numerical ones via an L_2 discrete norm will be estimated. The matrix norm used will be

$$\|X\|_2 = \left(\sum_{i,j=1}^{N} |X_{ij}|^2 \right)^{1/2} \tag{3.37}$$

for a matrix $X = (X_{ij}) \in \mathcal{M}_{N+2}\mathbb{C}$. Denote u^n the net function $u(x, y, t^n)$ and U^n the numerical solution. We propose to compute the discrete error

$$\text{Er} = \max_n \|U^n - u^n\|_2 \tag{3.38}$$

on the grid (x_i, y_j), $0 \leq i, j \leq J + 1$ and the relative error between the exact solution and the numerical one as

$$\text{Relative Er} = \max_n \frac{\|U^n - u^n\|_2}{\|u^n\|_2} \tag{3.39}$$

on the same grid.

3.5.1 A polynomial-exponential example

We develop in this part a classical example based on polynomial function with an exponential envelop. We consider the inhomogeneous problem

$$\begin{aligned}
u_{tt} &= \Delta u + v_{xx} + g(x,y,t), \quad (x,y,t) \in \Omega \times (t_0, T),\\
v &= q u_{xx} + u^2, \quad (x,y,t) \in \Omega \times (t_0, T),\\
(u,v)(x,y,t_0) &= (u_0, v_0)(x,y), \quad (x,y,t) \in \overline{\Omega} \times (t_0, T),\\
\frac{\partial u}{\partial t}(x,y,t_0) &= \varphi(x,y), \quad (x,y) \in \overline{\Omega},\\
\overrightarrow{\nabla}(u,v) &= 0, \quad (x,y,t) \in \partial\Omega \times (t_0, T)
\end{aligned} \tag{3.40}$$

where $\Omega = [-1,1]^2$ and where the right hand term is

$$\begin{aligned}
g(x,y,t) &= [(x^2-1)^2(x^4-58x^2+9) - 48(35x^4-30x^2+3)] + [y^4-14y^2+5]e^{-t}\\
&\quad - 16(x^2-1)^2\big[(x^2-1)^4(15x^2-1) + (y^2-1)^2(7x^2-1)\big]e^{-2t}
\end{aligned} \tag{3.41}$$

The exact solution is

$$u(x,y,t) = [(x^2-1)^4 + (y^2-1)^2]e^{-t}. \tag{3.42}$$

In the following tables, numerical results are provided. We computed for different space and time steps the discrete L_2-error estimates defined by (3.38). The time interval is $[0,1]$ for a choice $t_0 = 0$ and $T = 1$. The following results are obtained for different values of the parameters J (and thus h), l ((and thus N). The parameters q and α are fixed to $q = 0.01$ and $\alpha = 0.25$. We just notice that some variations done on these latter parameters have induced an important variation in the error estimates which explains the effect of the parameter q which has the role of a viscosity-type coefficient and the barycenter parameter α which calibrates the position of the approximated solution around the exact one. Finally, some comparison with the work in (Ben Mabrouk 1) has proved that Lyapunov type operators already result in fast convergent algorithms with a maximum time of execution of 2.014 sd for the present one. The classical tri-diagonal algorithms associated to the same problem with the same discrete scheme and the same parameters yielded a maximum time of 552.012 sd, so a performance of 23.10^{-4} faster algorithm for the present one. We recall that the tests are done on a Pentium Dual Core CPU 2.10 GHz processor and 250 Mo RAM.

Table 3.1:

J	l	Er	Relative Er
10	1/100	$4,0.10^{-3}$	0,1317
16	1/120	$3,3.10^{-3}$	0,1323
20	1/200	$2,0.10^{-3}$	0,1335
24	1/220	$1,8.10^{-3}$	0,1337
30	1/280	$1,4.10^{-3}$	0,1340
40	1/400	$9,8.10^{-4}$	0.1344
50	1/500	$7,8.10^{-4}$	0,1346

3.5.2 A 2-particle interaction example

The example developed hereafter is a model of interaction of two particles or two waves. We consider the inhomogeneous problem

$$\begin{aligned} u_{tt} &= \Delta u + v_{xx} + g(x,y,t), \quad (x,y,t) \in \Omega \times (t_0,T), \\ v &= q u_{xx} + u^2, \quad (x,y,t) \in \Omega \times (t_0,T), \\ (u,v)(x,y,t_0) &= (u_0,v_0)(x,y), \quad (x,y,t) \in \overline{\Omega} \times (t_0,T), \\ \frac{\partial u}{\partial t}(x,y,t_0) &= \varphi(x,y), \quad (x,y) \in \overline{\Omega}, \\ \overrightarrow{\nabla}(u,v) &= 0, \quad (x,y,t) \in \partial\Omega \times (t_0,T) \end{aligned} \tag{3.43}$$

where

$$g(x,y,t) = (4 - 6\psi^2(y))u^2 - \psi^2(x)u.$$

and u is the exact solution given by

$$u(x,y,t) = 2\psi^2(x)\psi^2(y)\theta(t)$$

with

$$\psi(x) = \cos(\frac{x}{2}), \quad \theta(t) = e^{-it}, \varphi(x,y) = -2i\psi^2(x)\psi^2(y)$$

As for the previous example, the following tables shows the numerical computations for different space and time steps the discrete L_2-error estimates defined by (3.38) and the relative error (3.39). The time interval is $[-2\pi, +2\pi]$ for a choice $t_0 = 0$ and $T = 1$. The following results are obtained for different values of the parameters J (and thus h), l ((and thus N). The parameters q and α are fixed here-also the same as previously, $q = 0.01$ and $\alpha = 0.25$. Compared to the tri-diagonal scheme the present one leads a faster convergent algorithms

Table 3.2:

J	l	Er	Relative Er
10	1/100	$4,6.10^{-3}$	0,2311
16	1/120	$4,4.10^{-3}$	0,2372
20	1/200	$2,4.10^{-3}$	0,2506
24	1/220	$2,3.10^{-3}$	0,2671
30	1/280	$2,0.10^{-3}$	0,3074
40	1/400	$1,4.10^{-3}$	0,3592
50	1/500	$7,6.10^{-4}$	0,2355

Remark 26 *For the convenience of this section, we give here some computations of the determinants $\Delta(l,h)$ for different values of the parameters of the discrete scheme Firstly, for both examples above, we can easily see that $\|\varphi\| = 2$ and thus, equation (3.29) yields that*

$$\Delta(l,h) = 361 + 32\varepsilon + 416l - 256l^2.$$

For the different values of l as in the tables 1 and 2, we obtain a positive discriminant leading two zeros with a rejected one. For the discriminant of equation (3.32) we obtain

$$\Delta(l,h) = \frac{676}{h^4} + \frac{8\varepsilon}{h^2}.$$

Hence, the results explained previously hold.

3.6 Conclusion

This chapter investigated the solution of the well-known Boussinesq equation in two-dimensional case by applying a two-dimensional finite difference discretization. The Boussinesq equation in its original form is a 4-th order partial differential equation. Thus, in a first step it was recasted into a system of second order partial differential equations using a reduction order idea. Next, the system has been transformed into an algebraic discrete system involving Lyapunov-Syslvester matrix terms by using a full time-space discretization. Solvability, consistency, stability and convergence are then established by applying well-known methods such as Lax-Richtmyer equivalence theorem and Lyapunov Stability and by examining the Lyapunov-Sylvester operators. The method was finally improved by developing numerical examples. It was shown to be efficient by means of error estimates as well as time execution algorithms compared to classical ones.

Chapter 4

Kuramoto-Sivashinsky equation

The Kuramoto-Sivashinsky equation (KS) is one of the most famous equations in math-physics for many decades. It has its origin in the work of Kuramoto since the 70-th decade of the 20-th century in his study of reaction-diffusion equation. The equation was then considered by Sivashinsky in modeling small thermal diffusion instabilities for laminar flames and modeling the reference flux of a film layer on an inclined plane. Since then the KS equation has experienced a growing development in theoretical mathematics, numerical as well as physical mechanics, nonlinear physics, hydrodynamics, chemistry, plasma physics, particle distributions advection, surface morphology, ...etc. See (Benachour), (Hase), (Hong), (Jaya), (Kuramoto), (Nadjafikhah), (Procaccia), (Rost), (Sivashinsky 1), (Sivashinsky 2). In (Nadjafikhah), the symmetry problem of the model was studied based on the theory of Lie algebras. In (Benachour), an anisotropic version of the KS equation has been proposed leading to global resolutions of the equation on rectangular domaines. Sufficient conditions were given for the existence of global solution.

From the dimensional point of view, KS equation has been widely studied in one dimension, (Giacomelli), (Goodman), (Ilyashenko), (Nicolaenko), (Otto), (Temam). However, this equation since its appearance is related to the modeling of flame spread which is a two-dimensional problem. In this context, an important result was developed in (Sell) where the authors showed by adapting the method developed in (Raugel) the existence of a set of bounded solutions on a rectangular domain. This major importance of the two-dimensional model was a main motivation behind this work.

The present chapter is devoted to the development of a computational method based on two-dimensional finite difference scheme to approximate the solution of the nonlinear

Kuramoto-Sivashinsky equation

$$\frac{\partial u}{\partial t} = q\Delta u - \kappa\Delta^2 u + \lambda\left|\nabla u\right|^2, \; ((x,y),t) \in \Omega \times (t_0, +\infty), \tag{4.1}$$

with initial conditions

$$u(x,y,t_0) = \varphi(x,y) \; ; \quad (x,y) \in \Omega \tag{4.2}$$

and boundary conditions

$$\frac{\partial u}{\partial \eta}(x,y,t) = 0 \; ; \quad ((x,y),t) \in \partial\Omega \times (t_0, +\infty), \tag{4.3}$$

on a rectangular domain $\Omega = [L_0, L_1] \times [L_0, L_1]$ in $\mathbb{R}^2$, $t_0 \geq 0$ is a real parameter fixed as the initial time. $\frac{\partial}{\partial t}$ is the time derivative, ∇ is the space gradient operator and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator in $\mathbb{R}^2$, q, κ, λ are real parameters. φ and ψ are twice differentiable real valued functions on $\overline{\Omega}$.

We propose to apply an order reduction of the derivation and thus to solve a coupled system of equation involving second order differential operators. We set $v = qu - \kappa\Delta u$ and thus we have to solve the system

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta v + \lambda\left|\nabla u\right|^2, \quad (x,y,t) \in \Omega \times (t_0, +\infty) \\ v = qu - \kappa\Delta u, \quad (x,y,t) \in \Omega \times (t_0, +\infty) \\ (u,v)(x,y,t_0) = (\varphi,\psi)(x,y), \quad (x,y) \in \overline{\Omega} \\ \overrightarrow{\nabla}(u,v)(x,y,t) = 0, \quad (x,y,t) \in \partial\Omega \times (t_0, +\infty) \end{cases} \tag{4.4}$$

A model representing a nonlinear dynamical system defined in a two-dimensional space is considered, where the solution $u(x,y,t)$ satisfies a fourth order partial differential equation of the form (4.1), where u is the height of the interface, q is a pre-factor proportional to the coefficient of surface tension expressed by $q\nabla^2 u$. The quantity $\kappa\nabla^4 u$ represents the result of the diffusion surface due to the chemical potential gradient induced curvature. The pre-factor κ represents the surface diffusion term. The quantity $\lambda|\nabla u|^2$ is due to the existence of overhangs and vacancies during the deposition process. Finally, the quantity

$q\nabla^2 u + \lambda \left|\nabla u\right|^2$ is referred to as modeling the effect of deposited atoms. cf. (Hong).

In this chapter we propose to serve of algebraic operators to approximate the solutions of the Kuramoto-Sivashinsky(K-S) equation in the two spatial and one temporal dimension without adapting classical developments based on separation of variables, radial solutions, tri-diagonal operators, ... etc.

Thus, one motivation here and issued from (Bezia) consists to resolve such problem and prove the invertibility of the algebraic operator yielded in the numerical scheme by appalling topological method Instead of using classical ones such as tri-diagonal transformations. We thus aim to prove that generalized Lyapunov-Sylvester operators can be good candidates for investigating numerical solutions of PDEs in multi-dimensional spaces.

The chapter is organized as follows. In next section the discretization of (4.4) is developed, the solvability of the scheme is analyzed. In section 4, the consistency of the method is shown and next, the stability and convergence are proved based on Lyapunov method and Lax-Richtmyer theorem. Finally, a numerical implementation is provided leading to the computation of the numerical solution and error estimates.

4.1 The numerical scheme

Let $J \in \mathbb{N}^*$ and $h = \frac{L_1 - L_0}{J}$ be the space step. Denote for $(j,m) \in I^2 = \{0,1,...,J\}^2$, $x_j = L_0 + jh$ and $y_m = L_0 + mh$. Next, let $l = \Delta t$ be the time step and $t_n = t_0 + nl$, $n \in \mathbb{N}$ be the time grid. We denote also $\widetilde{\Omega} = \left\{(x_j, y_m, t_n)\,;(x_j, y_m, t_n) \in I^2 \times \mathbb{N}\right\}$ the associated discrete space. Finally, for $(j,m) \in I^2$ and $n \geq 0$, $u^n_{j,m}$ denotes the net function $u(x_j, y_m, t_n)$ and $U^n_{j,m}$ is the numerical solution.

The following discrete approximations will be applied for the different differential operators involved in the problem. For time derivatives, we set

$$\frac{\partial u}{\partial t} \rightsquigarrow \frac{U^{n+1}_{j,m} - U^{n-1}_{j,m}}{2l}$$

and for space derivatives, we shall use for the one order

$$\frac{\partial u}{\partial x} \rightsquigarrow \frac{U^n_{j+1,m} - U^n_{j-1,m}}{2h}, \quad \text{and} \quad \frac{\partial u}{\partial y} \rightsquigarrow \frac{U^n_{j,m+1} - U^n_{j,m-1}}{2h}$$

and for the second order ones, we set

$$\frac{\partial^2 u}{\partial x^2} \rightsquigarrow \frac{U^{n,\alpha}_{j+1,m} - 2U^{n,\alpha}_{j,m} + U^{n,\alpha}_{j-1,m}}{h^2}, \text{ and } \quad \frac{\partial^2 u}{\partial y^2} \rightsquigarrow \frac{U^{n,\alpha}_{j,m+1} - 2U^{n,\alpha}_{j,m} + U^{n,\alpha}_{j,m-1}}{h^2}.$$

Finally, for $n \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}$, we denote

$$U^{n,\alpha} = \alpha U^{n-1} + (1 - 2\alpha)\, U^n + \alpha U^{n+1}$$

to designate the estimation of $U^n_{j,m}$ with an α-extrapolation/interpolation barycenter method. By applying these discrete approximations, we obtain

$$\begin{aligned} U^{n+1}_{j,m} - U^{n-1}_{j,m} &= \frac{2l}{h^2}\left[V^{n,\beta}_{j-1,m} - 2V^{n,\beta}_{j,m} + V^{n,\beta}_{j+1,m} + V^{n,\beta}_{j,m-1} - 2V^{n,\beta}_{j,m} + V^{n,\beta}_{j,m+1}\right] \\ &\quad + \frac{2l}{h^2}\lambda\left[\frac{1}{4}\left(U^n_{j+1,m} - U^n_{j-1,m}\right)^2 + \frac{1}{4}\left(U^n_{j,m+1} - U^n_{j,m-1}\right)^2\right]. \end{aligned}$$

In what follows, we set

$$F^n_{j,m} = \frac{1}{4}\left[\left(U^n_{j+1,m} - U^n_{j-1,m}\right)^2 + \left(U^n_{j,m+1} - U^n_{j,m-1}\right)^2\right]$$

We thus get

$$\begin{aligned} U^{n+1}_{j,m} - U^{n-1}_{j,m} &= \sigma[\beta V^{n+1}_{j-1,m} + (1-2\beta)\, V^n_{j-1,m} + \beta V^{n-1}_{j-1,m} \\ &\quad -2\beta V^{n+1}_{j,m} - 2\,(1-2\beta)\, V^n_{j,m} - 2\beta V^{n-1}_{j,m} \\ &\quad +\beta V^{n+1}_{j+1,m} + (1-2\beta)\, V^n_{j+1,m} + \beta V^{n-1}_{j+1,m} \\ &\quad +\beta V^{n+1}_{j,m-1} + (1-2\beta)\, V^n_{j,m-1} + \beta V^{n-1}_{j,m-1} \\ &\quad -2\beta V^{n+1}_{j,m} - 2\,(1-2\beta)\, V^n_{j,m} - 2\beta V^{n-1}_{j,m} \\ &\quad +\beta V^{n+1}_{j,m+1} + (1-2\beta)\, V^n_{j,m+1} + \beta V^{n-1}_{j,m+1}] + \sigma\lambda F^n_{j,m} \end{aligned}$$

where $\sigma = \frac{2l}{h^2}$. Otherwise, this can be written as

$$\begin{aligned} & U_{j,m}^{n+1} - \sigma\beta\left[V_{j-1,m}^{n+1} - 2V_{j,m}^{n+1} + V_{j+1,m}^{n+1} + V_{j,m-1}^{n+1} - 2V_{j,m}^{n+1} + V_{j,m+1}^{n+1}\right] \\ = \; & U_{j,m}^{n-1} + \sigma(1-2\beta)\left[V_{j-1,m}^{n} - 2V_{j,m}^{n} + V_{j+1,m}^{n} + V_{j,m-1}^{n} - 2V_{j,m}^{n} + V_{j,m+1}^{n}\right] \\ & +\sigma\beta\left[V_{j-1,m}^{n-1} - 2V_{j,m}^{n-1} + V_{j+1,m}^{n-1} + V_{j,m-1}^{n-1} - 2V_{j,m}^{n-1} + V_{j,m+1}^{n-1}\right] + \sigma\lambda F_{j,m}^{n}. \end{aligned}$$

Taking into account the boundary conditions, we obtain the full matrix expression

$$\begin{aligned} U^{n+1} - \sigma\beta\left(AV^{n+1} + V^{n+1}A^T\right) &= U^{n-1} + \sigma(1-2\beta)\left(AV^n + V^nA^T\right) \\ &\quad +\sigma\beta\left(AV^{n-1} + V^{n-1}A^T\right) \\ &\quad +\sigma\lambda F^n \end{aligned} \tag{4.5}$$

where

$$A = \begin{pmatrix} -2 & 2 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 2 & -2 \end{pmatrix}$$

$U^n = \left(U_{j,m}^n\right)_{0\le j,m\le J}$ and $V = \left(V_{j,m}^n\right)_{0\le j,m\le J}$.

Next, for $Q \in M_{(J+1)^2}(\mathbb{R})$, we denote $\mathcal{L}_Q$ the linear operator which associates to $X \in M_{(J+1)^2}(\mathbb{R})$ its image $\mathcal{L}_Q(X) = QX + XQ^T$. The last equation will be written as

$$U^{n+1} - \sigma\beta\mathcal{L}_A\left(V^{n+1}\right) = U^{n-1} + \sigma(1-2\beta)\mathcal{L}_A(V^n) + \sigma\beta\mathcal{L}_A\left(V^{n-1}\right) + \sigma\lambda F^n. \tag{4.6}$$

Now, applying similar techniques as previously we obtain

$$V_{j,m}^{n,\beta} = qU_{j,m}^{n,\alpha} - \frac{\kappa}{h^2}\left(U_{j-1,m}^{n,\alpha} - 2U_{j,m}^{n,\alpha} + U_{j+1,m}^{n,\alpha} + U_{j,m-1}^{n,\alpha} - 2U_{j,m}^{n,\alpha} + U_{j,m+1}^{n,\alpha}\right).$$

Let $\delta = \frac{1}{h^2}$. This results in the following equation

$$\begin{aligned}
\beta V_{j,m}^{n+1} + (1-2\beta)\, V_{j,m}^{n} + \beta V_{j,m}^{n-1} \quad = \quad & q\left[\alpha U_{j,m}^{n+1} + (1-2\alpha)\, U_{j,m}^{n} + \alpha U_{j,m}^{n-1}\right] \\
& -\delta\kappa[\alpha U_{j-1,m}^{n+1} + (1-2\alpha)\, U_{j-1,m}^{n} + \alpha U_{j-1,m}^{n-1} \\
& -2\alpha U_{j,m}^{n+1} - 2\,(1-2\alpha)\, U_{j,m}^{n} - 2\alpha U_{j,m}^{n-1} \\
& +\alpha U_{j+1,m}^{n+1} + (1-2\alpha)\, U_{j+1,m}^{n} + \alpha U_{j+1,m}^{n-1} \\
& +\alpha U_{j,m-1}^{n+1} + (1-2\alpha)\, U_{j,m-1}^{n} + \alpha U_{j,m-1}^{n-1} \\
& -2\alpha U_{j,m}^{n+1} - 2\,(1-2\alpha)\, U_{j,m}^{n} - 2\alpha U_{j,m}^{n-1} \\
& +\alpha U_{j,m+1}^{n+1} + (1-2\alpha)\, U_{j,m+1}^{n} + \alpha U_{j,m+1}^{n-1}],
\end{aligned}$$

or equivalently,

$$\begin{aligned}
& \beta V_{j,m}^{n+1} - q\alpha U_{j,m}^{n+1} \\
& +\delta\kappa\alpha\left[U_{j-1,m}^{n+1} - 2U_{j,m}^{n+1} + U_{j+1,m}^{n+1} + U_{j,m-1}^{n+1} - 2U_{j,m}^{n+1} + U_{j,m+1}^{n+1}\right] \\
= \quad & -(1-2\beta)\, V_{j,m}^{n} - \beta V_{j,m}^{n-1} + q\left[(1-2\alpha)\, U_{j,m}^{n} + \alpha U_{j,m}^{n-1}\right] \\
& -\delta\kappa[(1-2\alpha)\left[U_{j-1,m}^{n} - 2U_{j,m}^{n} + U_{j+1,m}^{n} + U_{j,m-1}^{n} - 2U_{j,m}^{n} + U_{j,m+1}^{n}\right] \\
& +\left[U_{j-1,m}^{n-1} - 2U_{j,m}^{n-1} + U_{j+1,m}^{n-1} + U_{j,m-1}^{n-1} - 2U_{j,m}^{n-1} + U_{j,m+1}^{n-1}\right]].
\end{aligned}$$

Now, applying the boundary conditions, we obtain

$$\begin{aligned}
& \beta V^{n+1} - \alpha\left[qU^{n+1} - \delta\kappa\mathcal{L}_A\left(U^{n+1}\right)\right] \\
= \quad & (1-2\alpha)\left(qU^{n} - \delta\kappa\mathcal{L}_A\left(U^{n}\right)\right) + \alpha\left(qU^{n-1} - \delta\kappa\mathcal{L}_A\left(U^{n-1}\right)\right) \\
& -(1-2\beta)\, V^{n} - \beta V^{n-1}. \qquad (4.7)
\end{aligned}$$

As a result, we obtain finally the following discrete coupled system.

$$\begin{cases} U^{n+1}-\sigma\beta\mathcal{L}_A\left(V^{n+1}\right)=U^{n-1}+\sigma\left(1-2\beta\right)\mathcal{L}_A\left(V^n\right)+\sigma\beta\mathcal{L}_A\left(V^{n-1}\right)+\sigma\lambda F^n \\ \beta V^{n+1}-\alpha\left[qU^{n+1}-\delta\kappa\mathcal{L}_A\left(U^{n+1}\right)\right]=\left(1-2\alpha\right)\left(qU^n-\delta\kappa\mathcal{L}_A\left(U^n\right)\right) \\ \qquad +\alpha\left(qU^{n-1}-\delta\kappa\mathcal{L}_A\left(U^{n-1}\right)\right) \\ \qquad -\left(1-2\beta\right)V^n-\beta V^{n-1} \end{cases} \tag{4.8}$$

4.2 Solvability of the discrete method

In this section we will examine the solvability of the discrete scheme. the main idea is by transforming the system (4.8) to an equality of the form $\left(U^{n+1},V^{n+1}\right)=\phi\left(U^n,V^n,U^{n-1},V^{n-1}\right)$ with an appropriate function ϕ. We prove precisely that ϕ can be expressed with general Lyapunov-Sylvester form. Next using general properties of such operators we prove that ϕ is an isomorphism. In most studies, even recent ones such as (Ben Mabrouk 2) the authors proved that ϕ or some modified versions may be contractive by inserting translation-dilation parameters leading to fixed point theory. In the present context it seams that such transformation is not possible as $\|\phi\|$ may be greater than 1 and thus no contraction may occur. To overcome this problem we come back to differential calculus and topological properties.

Theorem 27 *The system* (4.8) *is uniquely solvable whenever* U^0 *and* U^1 *are known.*

The proof of this result is based on the lemma

Proof Consider the endomorphism ϕ on $M_{(J+1)^2}(\mathbb{R})\times M_{(J+1)^2}(\mathbb{R})$ defined by

$$\phi\left(X,Y\right)=\left(X-\sigma\beta\mathcal{L}_A\left(Y\right),\beta Y-\alpha\Gamma\left(X\right)\right) \tag{4.9}$$

where $\Gamma\left(X\right)=qX-\delta\kappa\mathcal{L}_A\left(X\right)$. To prove Theorem 2, we show that ϕ is a one to one, and so $\ker\phi$ is reduced to 0. Indeed,

$$\phi\left(X,Y\right)=0\Leftrightarrow\left(X-\sigma\beta\mathcal{L}_A\left(Y\right),\beta Y-\alpha\left[qX-\delta\kappa\mathcal{L}_A\left(X\right)\right]\right)=\left(0,0\right).$$

Which is equivalent to

$$\begin{cases} X = \sigma\beta\mathcal{L}_A(Y) \\ \beta Y = \alpha\Gamma(X) \end{cases} \tag{4.10}$$

for $\beta \neq 0$ we get

$$Y = \sigma\alpha\Gamma\mathcal{L}_A(Y) \tag{4.11}$$

Choosing $l = o\left(h^{s+4}\right)$ (which is always possible), the operator $\mathcal{K} = I - \sigma\alpha\Gamma\mathcal{L}_A$ tends uniformly to I whenever h tends to zero. Indeed, denote

$$W = \sigma\alpha\left(qA - \delta\kappa A^2\right) = \frac{2l}{h^2}\alpha\left(qA - \frac{1}{h^2}\kappa A^2\right).$$

We have

$$\mathcal{K}(X) = X - \sigma\alpha\Gamma\mathcal{L}_A(X) = X - WX - XW^T + 2\sigma\alpha\delta\kappa AXA^T.$$

thus,

$$\begin{aligned} \|(\mathcal{K} - I)X\| &= \left\|WX + XW^T + 2(\sigma\alpha\delta\kappa)AXA^T\right\| \\ &\leq 2\|W\|\|X\| + 32\sigma\alpha\delta|\kappa|\|X\| \end{aligned}$$

Since we have $\|W\| \leq 4\sigma|\alpha|[|q| + 4\delta|\kappa|]$, we obtain,

$$\|(\mathcal{K} - I)X\| \leq 8\sigma\alpha[|q| + 8\delta|\kappa|]\|X\|$$

For $l = o\left(h^{4+s}\right)$ this implies that,

$$\|(\mathcal{K}(X) - I(X))\| \leq 16\alpha\left[|q|h^{2+s} + 8h^s|\kappa|\right]\|X\|$$

Consequently the operator $\mathcal{K}$ converge uniformly to the identity whenever h tends towered 0 and $l = o\left(h^{4+s}\right)$, with $s > 0$. Thus, using Lemma 1 ϕ is invertible for l,h small enough with $l = o\left(h^{4+s}\right)$.

For $\beta = 0$ we obtain the system

$$\begin{cases} U^{n+1} = U^{n-1} + \sigma \mathcal{L}_A\left(V^n\right) + \sigma\lambda F^n \\ V^n = \alpha\Gamma\left(U^{n+1}\right) + (1-2\alpha)\,\Gamma\left(U^n\right) + \alpha\Gamma\left(U^{n-1}\right) \end{cases} \tag{4.12}$$

and thus,

$$U^{n+1} - \sigma\alpha\Gamma\mathcal{L}_A\left(U^{n+1}\right) = \sigma[(1-2\alpha)\,\Gamma\mathcal{L}_A\left(U^n\right) + U^{n-1} + \alpha\Gamma\mathcal{L}_A\left(U^{n-1}\right) + \lambda F^n] \tag{4.13}$$

For the same assumption on l and h as above the same operator $\mathcal{K}(X) = X - \sigma\alpha\Gamma\mathcal{L}_A(X)$ tends toward the identity as h tends to 0.

4.3 Consistency

The consistency of the proposed method will be proved by evaluating the local truncation error arising from the scheme introduced for the discretization of the system (4.4)

Applying Taylor Taylor's expansion, and assuming that u and v to be sufficiently differentiable, we get

$$U_{j,m}^{n+1} = u + l\frac{\partial u}{\partial t} + \frac{l^2}{2}\frac{\partial^2 u}{\partial t^2} + \frac{l^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{l^4}{24}\frac{\partial^4 u}{\partial t^4} + ...$$

Similarly,

$$U_{j,m}^{n-1} = u - l\frac{\partial u}{\partial t} + \frac{l^2}{2}\frac{\partial^2 u}{\partial t^2} - \frac{l^3}{6}\frac{\partial^3 u}{\partial t^3} + \frac{l^4}{24}\frac{\partial^4 u}{\partial t^4} + ...$$

Hence,

$$\frac{1}{2l}\left[U_{j,m}^{n+1} - U_{j,m}^{n-1}\right] = \frac{\partial u}{\partial t} + \frac{l^2}{6}\frac{\partial^3 u}{\partial t^3} + ...$$

Next, we get also

$$\begin{aligned}
V_{j-1,m}^{n-1} \quad = \quad & \left[v - l\frac{\partial v}{\partial t} + \frac{l^2}{2}\frac{\partial^2 v}{\partial t^2} - \frac{l^3}{6}\frac{\partial^3 v}{\partial t^3} + \frac{l^4}{24}\frac{\partial^4 v}{\partial t^4}\right] \\
& -h\left[\frac{\partial v}{\partial x} - l\frac{\partial^2 v}{\partial t\partial x} + \frac{l^2}{2}\frac{\partial^3 v}{\partial t^2\partial x} - \frac{l^3}{6}\frac{\partial^4 v}{\partial t^3\partial x} + \frac{l^4}{24}\frac{\partial^5 v}{\partial t^4\partial x}\right] \\
& +\frac{h^2}{2}\left[\frac{\partial^2 v}{\partial x^2} - l\frac{\partial^3 v}{\partial t\partial x^2} + \frac{l^2}{2}\frac{\partial^4 v}{\partial t^2\partial x^2} - \frac{l^3}{6}\frac{\partial^5 v}{\partial t^3\partial x^2} + \frac{l^4}{24}\frac{\partial^6 v}{\partial t^4\partial x^2}\right] \\
& -\frac{h^3}{6}\left[\frac{\partial^3 v}{\partial x^3} - l\frac{\partial^4 v}{\partial t\partial x^3} + \frac{l^2}{2}\frac{\partial^5 v}{\partial t^2\partial x^3} - \frac{l^3}{6}\frac{\partial^6 v}{\partial t^3\partial x^3} + \frac{l^4}{24}\frac{\partial^7 v}{\partial t^4\partial x^3}\right] \\
& +\frac{h^4}{24}\left[\frac{\partial^4 v}{\partial x^4} - l\frac{\partial^5 v}{\partial t\partial x^4} + \frac{l^2}{2}\frac{\partial^6 v}{\partial t^2\partial x^4} - \frac{l^3}{6}\frac{\partial^7 v}{\partial t^3\partial x^4} + \frac{l^4}{24}\frac{\partial^8 v}{\partial t^4\partial x^4}\right] + ...
\end{aligned}$$

and

$$V_{j-1,m}^{n} = v - h\frac{\partial v}{\partial x} + \frac{h^2}{2}\frac{\partial^2 v}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 v}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 v}{\partial x^4} + ...$$

and finally,

$$\begin{aligned}
V_{j-1,m}^{n+1} \quad = \quad & \left[v + l\frac{\partial v}{\partial t} + \frac{l^2}{2}\frac{\partial^2 v}{\partial t^2} + \frac{l^3}{6}\frac{\partial^3 v}{\partial t^3} + \frac{l^4}{24}\frac{\partial^4 v}{\partial t^4}\right] \\
& -h\left[\frac{\partial v}{\partial x} + l\frac{\partial^2 v}{\partial t\partial x} + \frac{l^2}{2}\frac{\partial^3 v}{\partial t^2\partial x} + \frac{l^3}{6}\frac{\partial^4 v}{\partial t^3\partial x} + \frac{l^4}{24}\frac{\partial^5 v}{\partial t^4\partial x}\right] \\
& +\frac{h^2}{2}\left[\frac{\partial^2 v}{\partial x^2} + l\frac{\partial^3 v}{\partial t\partial x^2} + \frac{l^2}{2}\frac{\partial^4 v}{\partial t^2\partial x^2} + \frac{l^3}{6}\frac{\partial^5 v}{\partial t^3\partial x^2} + \frac{l^4}{24}\frac{\partial^6 v}{\partial t^4\partial x^2}\right] \\
& -\frac{h^3}{6}\left[\frac{\partial^3 v}{\partial x^3} + l\frac{\partial^4 v}{\partial t\partial x^3} + \frac{l^2}{2}\frac{\partial^5 v}{\partial t^2\partial x^3} + \frac{l^3}{6}\frac{\partial^6 v}{\partial t^3\partial x^3} + \frac{l^4}{24}\frac{\partial^7 v}{\partial t^4\partial x^3}\right] \\
& +\frac{h^4}{24}\left[\frac{\partial^4 v}{\partial x^4} + l\frac{\partial^5 v}{\partial t\partial x^4} + \frac{l^2}{2}\frac{\partial^6 v}{\partial t^2\partial x^4} + \frac{l^3}{6}\frac{\partial^7 v}{\partial t^3\partial x^4} + \frac{l^4}{24}\frac{\partial^8 v}{\partial t^4\partial x^4}\right] + ...
\end{aligned}$$

Thus,

$$
\begin{aligned}
V_{j-1,m}^{n,\beta} &= v + \beta l^2 \frac{\partial^2 v}{\partial t^2} + \beta \frac{l^4}{12} \frac{\partial^4 v}{\partial t^4} \\
&+h \left[-\frac{\partial v}{\partial x} - \beta l^2 \frac{\partial^3 v}{\partial t^2 \partial x} - \beta \frac{l^4}{12} \frac{\partial^5 v}{\partial t^4 \partial x} \right] \\
&+\frac{h^2}{2} \left[\frac{\partial^2 v}{\partial x^2} + \beta l^2 \frac{\partial^4 v}{\partial t^2 \partial x^2} + \beta \frac{l^4}{12} \frac{\partial^6 v}{\partial t^4 \partial x^2} \right] \\
&+\frac{h^3}{6} \left[-\frac{\partial^3 v}{\partial x^3} - \beta l^2 \frac{\partial^5 v}{\partial t^2 \partial x^3} - \beta \frac{l^4}{12} \frac{\partial^7 v}{\partial t^4 \partial x^3} \right] \\
&+\frac{h^4}{24} \left[\frac{\partial^4 v}{\partial x^4} + \beta l^2 \frac{\partial^6 v}{\partial t^2 \partial x^4} + \beta \frac{l^4}{12} \frac{\partial^8 v}{\partial t^4 \partial x^4} \right] + ...
\end{aligned}
$$

Similarly,

$$
V_{j,m}^{n,\beta} = v + \beta l^2 \frac{\partial^2 v}{\partial t^2} + \beta \frac{l^4}{12} \frac{\partial^4 v}{\partial t^4} + ...
$$

We have also

$$
\begin{aligned}
V_{j+1,m}^{n,\beta} &= v + \beta l^2 \frac{\partial^2 v}{\partial t^2} + \beta \frac{l^4}{12} \frac{\partial^4 v}{\partial t^4} \\
&+h \left[\frac{\partial v}{\partial x} + \beta l^2 \frac{\partial^3 v}{\partial t^2 \partial x} + \beta \frac{l^4}{12} \frac{\partial^5 v}{\partial t^4 \partial x} \right] \\
&+\frac{h^2}{2} \left[\frac{\partial^2 v}{\partial x^2} + \beta l^2 \frac{\partial^4 v}{\partial t^2 \partial x^2} + \beta \frac{l^4}{12} \frac{\partial^6 v}{\partial t^4 \partial x^2} \right] \\
&+\frac{h^3}{6} \left[\frac{\partial^3 v}{\partial x^3} + \beta l^2 \frac{\partial^5 v}{\partial t^2 \partial x^3} + \beta \frac{l^4}{12} \frac{\partial^7 v}{\partial t^4 \partial x^3} \right] \\
&+\frac{h^4}{24} \left[\frac{\partial^4 v}{\partial x^4} + \beta l^2 \frac{\partial^6 v}{\partial t^2 \partial x^4} + \beta \frac{l^4}{12} \frac{\partial^8 v}{\partial t^4 \partial x^4} \right] + ...
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\frac{1}{h^2} \left[V_{j,m-1}^{n,\beta} - 2V_{j,m}^{n,\beta} + V_{j,m+1}^{n,\beta} \right] &= \left[\frac{\partial^2 v}{\partial y^2} + \beta l^2 \frac{\partial^4 v}{\partial t^2 \partial y^2} + \beta \frac{l^4}{12} \frac{\partial^6 v}{\partial t^4 \partial y^2} \right] \\
&+\frac{h^2}{12} \left[\frac{\partial^4 v}{\partial y^4} + \beta l^2 \frac{\partial^6 v}{\partial t^2 \partial y^4} + \beta \frac{l^4}{12} \frac{\partial^8 v}{\partial t^4 \partial y^4} \right] + ...
\end{aligned}
$$

Now,

$$\begin{aligned}
& \frac{1}{2l}\left[U_{j,m}^{n+1}-U_{j,m}^{n-1}\right] \\
& -\frac{1}{h^2}\left[V_{j-1,m}^{n,\beta}-2V_{j,m}^{n,\beta}+V_{j+1,m}^{n,\beta}+V_{j,m-1}^{n,\beta}-2V_{j,m}^{n,\beta}+V_{j,m+1}^{n,\beta}\right] \\
= \; & \frac{\partial u}{\partial t}-\Delta v+\frac{l^2}{6}\frac{\partial^3 u}{\partial t^3}-\beta l^2\frac{\partial^2}{\partial t^2}(\Delta v)-\frac{h^2}{12}\left(\frac{\partial^4 v}{\partial x^4}+\frac{\partial^4 v}{\partial y^4}\right) \\
& +\beta l^2\frac{\partial^2}{\partial t^2}\left(\frac{\partial^4 v}{\partial x^4}+\frac{\partial^4 v}{\partial y^4}\right)-\beta\frac{l^4}{12}\left[\frac{\partial^6 v}{\partial t^4\partial x^2}+\frac{\partial^6 v}{\partial t^4\partial y^2}\right] \\
& -\beta\frac{l^4}{12}\frac{h^2}{12}\left[\frac{\partial^8 v}{\partial t^4\partial x^4}+\frac{\partial^8 v}{\partial t^4\partial y^4}\right]+...
\end{aligned}$$

We now examine the second equation in (4.4). Applying the same calculus as above, we get

$$v=(qv-\kappa(\Delta u))-\beta l^2\frac{\partial^2 v}{\partial t^2}+q\alpha l^2\frac{\partial^2 u}{\partial t^2}-\kappa\alpha l^2\frac{\partial(\Delta u)}{\partial t^2}-\kappa\frac{h^2}{12}\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)+o\left(l^2+h^2\right).$$

It results from above that the principal part of the first equation in system 4.4 is

$$\begin{aligned}
\mathcal{L}_{u,v}^1(t,x,y) &= \beta l^2\frac{\partial^2 v}{\partial t^2}-\alpha q l^2\frac{\partial^2 u}{\partial t^2}-\alpha\kappa l^2\frac{\partial^2(\Delta v)}{\partial t^2} \\
& \quad -\kappa\frac{l^2}{12}\left(\frac{\partial^4 v}{\partial x^4}+\frac{\partial^4 v}{\partial y^4}\right)+o\left(l^2+h^2\right).
\end{aligned} \tag{4.14}$$

The principal part of the local error truncation due to the second equation is

$$\begin{aligned}
\mathcal{L}_{u,v}^2(t,x,y) &= \beta\frac{l^2}{2}\frac{\partial^2}{\partial t^2}(v-qu-\kappa\alpha\Delta u) \\
& \quad -\kappa\frac{h^2}{12}\left(\frac{\partial^4 v}{\partial x^4}+\frac{\partial^4 v}{\partial y^4}\right)+o\left(l^2+h^2\right).
\end{aligned} \tag{4.15}$$

As a result, we get the following lemma.

Lemma 28 *The numerical method is consistent with an order* 2 *in space and time.*

Proof It is clear that the two operators $\mathcal{L}_{u,v}^1$ and $\mathcal{L}_{u,v}^2$ tend towards 0 as l and h tend to 0, which ensures the consistency of the method. Furthermore, the method is consistent with an order 2 in time and space.

4.4 Stability and convergence

The stability of the discrete scheme will be evaluated using Lyapunov criterion which states that a linear system $\mathcal{L}(x_{n+1}, x_n, x_{n-1}, ...) = 0$ is stable in the sense of Lyapunov if for any bounded initial solution x_0, the solution x_n remains bounded for all $n \geq 0$. In this section we prove precisely the following result.

Lemma 29 *The solution (U^n, V^n) is bounded independently of n whenever the initial solution (U^0, V^0) is.*

Proof We proceed by recurrence on n. Assume $\|(U_0, V_0)\| \leq \eta$ for some positive η. The system (4.8) can be written on the form

$$\begin{cases} U^{n+1} - \sigma\beta\mathcal{L}_A\left(V^{n+1}\right) = U^{n-1} + \sigma(1-2\beta)\mathcal{L}_A(V^n) + \sigma\beta\mathcal{L}_A\left(V^{n-1}\right) + \sigma\lambda F^n. \\ \beta V^{n+1} = \alpha\Gamma\left(U^{n+1}\right) + (1-2\alpha)\Gamma(U^n) + \alpha\Gamma\left(U^{n-1}\right) - (1-2\beta)V^n - \beta V^{n-1}. \end{cases} \tag{4.16}$$

The last equation gives,

$$\begin{aligned} \beta\mathcal{L}_A\left(V^{n+1}\right) &= \alpha\mathcal{L}_A\Gamma\left(U^{n+1}\right) + (1-2\alpha)\mathcal{L}_A\Gamma(U^n) \\ &\quad +\alpha\mathcal{L}_A\Gamma\left(U^{n-1}\right) - (1-2\beta)\mathcal{L}_A(V^n) - \beta\mathcal{L}_A\left(V^{n-1}\right). \end{aligned}$$

Substituting in the first one, we obtain

$$\mathcal{K}\left(U^{n+1}\right) = \sigma(1-2\alpha)\mathcal{L}_A\Gamma(U^n) + \sigma\alpha\mathcal{L}_A\Gamma\left(U^{n-1}\right) + U^{n-1} + \sigma\lambda F^n. \tag{4.17}$$

Next, recall that

$$\left|F_{j,m}^n\right| = \frac{1}{4}\left|\left(U_{j+1,m}^n - U_{j-1,m}^n\right)^2 + \left(U_{j,m+1}^n - U_{j,m-1}^n\right)^2\right|.$$

Thus,

$$\left|F_{j,m}^n\right| \leq \frac{1}{2}\left[\left(U_{j+1,m}^n\right)^2 + \left(U_{j-1,m}^n\right)^2 + \left(U_{j,m+1}^n\right)^2 + \left(U_{j,m-1}^n\right)^2\right]$$

and consequently,

$$\|F^n\|_2 \leq 2\,\|U^n\|_2^2 \,. \tag{4.18}$$

Finally, (4.17) yields that

$$\begin{aligned}\left\|\mathcal{K}\left(U^{n+1}\right)\right\| &\leq \sigma\,|1-2\alpha|\,\|\mathcal{L}_A\Gamma\left(U^n\right)\| + \sigma\,|\alpha|\,\left\|\mathcal{L}_A\Gamma\left(U^{n-1}\right)\right\| + \left\|U^{n-1}\right\| \\ &\quad +\sigma\,|\lambda|\,\|F^n\|\,.\end{aligned}$$

Setting $\omega = |q| + 8\delta\,|\kappa|$, we obtain

$$\left\|\mathcal{K}\left(U^{n+1}\right)\right\| \leq 8\omega\sigma\,|1-2\alpha|\,\|U^n\| + [1+8\omega\sigma\,|\alpha|]\,\left\|U^{n-1}\right\| + 2\sigma\,|\lambda|\,\|U^n\|^2\,. \tag{4.19}$$

We now evaluate $\left\|V^{n+1}\right\|$. Applying Γ for the first equation in the system (4.16), we get

$$\begin{aligned}\Gamma\left(U^{n+1}\right) &= \sigma\beta\Gamma\mathcal{L}_A\left(V^{n+1}\right) + \Gamma\left(U^{n-1}\right) + \sigma\left(1-2\beta\right)\Gamma\left(\mathcal{L}_A\left(V^n\right)\right) \\ &\quad +\sigma\beta\Gamma\left(\mathcal{L}_A\left(V^{n-1}\right)\right) + \sigma\lambda\Gamma\left(F^n\right).\end{aligned}$$

By replacing in the second equation of (4.10) we obtain

$$\begin{aligned}\beta\mathcal{K}\left(V^{n+1}\right) &= \left(1-2\beta\right)\left[\sigma\alpha\Gamma\left(\mathcal{L}_A\left(V^n\right)\right) - V^n\right] \\ &\quad +\beta\left[\sigma\alpha\Gamma\left(\mathcal{L}_A\left(V^{n-1}\right)\right) - V^{n-1}\right] \\ &\quad +2\alpha\Gamma\left(U^{n-1}\right) + \left(1-2\alpha\right)\Gamma\left(U^n\right) + \sigma\alpha\lambda\Gamma\left(F^n\right),\end{aligned} \tag{4.20}$$

We get from (4.20) and (4.18),

$$\begin{aligned}\left\|\beta\mathcal{K}\left(V^{n+1}\right)\right\| &\leq |1-2\beta|\left[8\sigma\,|\alpha|\,\omega+1\right]\|V^n\| \\ &\quad +|\beta|\left[\left[8\sigma\,|\alpha|\,\omega+1\right]\left\|V^{n-1}\right\|\right] + 2\,|\alpha|\,\omega\left\|U^{n-1}\right\| \\ &\quad +\left(1-2\alpha\right)\omega\,\|U^n\| + 2\sigma\alpha\lambda\omega\,\|U^n\|^2\,.\end{aligned} \tag{4.21}$$

Now coming back to (4.4) and applying boundary conditions, we get

$$U^{-1} = U^0 + l\widetilde{\varphi} \quad \text{and} \quad V^{-1} = qU^0 + \widetilde{\psi} \tag{4.22}$$

where,

$$\widetilde{\varphi} = -q\Delta\varphi + \kappa\Delta^2\varphi - \lambda\left|\nabla\varphi\right|^2$$

and

$$\widetilde{\psi} = -\left(lq^2+\kappa\right)\Delta\varphi + 2l\kappa q\Delta^2\varphi - l\kappa^2\Delta^3\varphi - \lambda lq\left|\nabla\varphi\right|^2 + \lambda l\kappa\Delta\left(\left|\nabla\varphi\right|^2\right).$$

Hence,

$$\left\|U^{-1}\right\| \leq \left\|U^0\right\| + l\left\|\widetilde{\varphi}\right\| \quad \text{and} \quad \left\|V^{-1}\right\| \leq |q|\left\|U^0\right\| + \left\|\widetilde{\psi}\right\|. \tag{4.23}$$

Now, the Lyapunov criterion for stability states exactly that

$$\forall\varepsilon > 0, \exists\eta > 0: \ \left\|\left(U^0, V^0\right)\right\| \leq \eta \Rightarrow \left\|\left(U^n, V^n\right)\right\| \leq \varepsilon, \qquad \forall n \geq 0.$$

For $n = 1$, and any ε given such that $\left\|\left(U^1, V^1\right)\right\| \leq \varepsilon$, we seek an $\eta > 0$ for which $\left\|\left(U^0, V^0\right)\right\| < \eta$.

By direct substitution in (4.19), for $n = 0$, we obtain

$$\left\|\mathcal{K}\left(U^1\right)\right\| \leq 8\omega\sigma\left|1-2\alpha\right|\left\|U^0\right\| + \left[1+8\omega\sigma\left|\alpha\right|\right]\left\|U^{-1}\right\| + 2\sigma\left|\lambda\right|\left\|U^0\right\|^2.$$

From (4.23), we obtain

$$\begin{aligned}\left\|\mathcal{K}\left(U^1\right)\right\| \leq\ & 2\sigma\left|\lambda\right|\left\|U^0\right\|^2 + \left(1+8\omega\sigma\left[\left|1-2\alpha\right|+\left|\alpha\right|\right]\right)\left\|U^0\right\| \\ & + l\left(1+8\sigma\left|\alpha\right|\omega\right)\left\|\widetilde{\varphi}\right\|.\end{aligned}$$

Observing that,

$$\left|1-2\alpha\right| + \left|\alpha\right| \leq \left(1+3\left|\alpha\right|\right),$$

we get

$$\left\|\mathcal{K}\left(U^1\right)\right\| \leq 2\sigma\left|\lambda\right|\left\|U^0\right\|^2 + 8\omega\sigma\left(1+3\left|\alpha\right|\right)\left\|U^0\right\| + l\left(1+8\sigma\left|\alpha\right|\omega\right)\left\|\widetilde{\varphi}\right\|.$$

Next choosing $l = o\left(h^{4+s}\right)$ small enough, we obtain

$$\left\|U^1\right\| \leq 4\sigma\left|\lambda\right|\left\|U^0\right\|^2 + 16\omega\sigma\left(1+3\left|\alpha\right|\right)\left\|U^0\right\| + 2l\left(1+8\sigma\left|\alpha\right|\omega\right)\left\|\widetilde{\varphi}\right\|.$$

Now, for $\varepsilon > 0$, we seek $\eta > 0$ such that

$$4\sigma\left|\lambda\right|\eta^2 + 16\omega\sigma\left(1+3\left|\alpha\right|\right)\eta + 2l\left(1+8\sigma\left|\alpha\right|\omega\right)\left\|\widetilde{\varphi}\right\| < \varepsilon \tag{4.24}$$

or otherwise,

$$4\sigma\left|\lambda\right|\eta^2 + 16\omega\sigma\left(1+3\left|\alpha\right|\right)\eta + 2l\left(1+8\sigma\left|\alpha\right|\omega\right)\left\|\widetilde{\varphi}\right\| - \varepsilon < 0.$$

The discriminant of the last inequality is

$$\Delta' = 64\left(\omega\sigma\left(1+3\left|\alpha\right|\right)\right)^2 - 4\sigma\left|\lambda\right|\left(2l\left(1+8\sigma\left|\alpha\right|\omega\right)\left\|\widetilde{\varphi}\right\| - \varepsilon\right).$$

For the same assumption on l and h as above,

$$\Delta' \sim 64\left(\omega\sigma\left(1+3\left|\alpha\right|\right)\right)^2 + 4\sigma\left|\lambda\right|\varepsilon > 0.$$

Consequently, there are two zeros $\eta_1 < \eta_2$ of the inequality above. Furthermore, replacing η with 0 we get a negative quantity, thus $\eta_1 < 0 < \eta_2$. As a result, η_2 is the good candidate. Now, choosing $\left\|(U^0, V^0)\right\| \leq \eta_2$, we get immediately $\left\|U^1\right\| < \varepsilon$.

Next, already with $n = 0$, we get similarly to the previous case

$$\begin{aligned}\left\|\beta\mathcal{K}\left(V^1\right)\right\| \leq\ & \left|1-2\beta\right|\left[8\sigma\left|\alpha\right|\omega+1\right]\left\|V^0\right\| + \left|\beta\right|\left[\left[8\sigma\left|\alpha\right|\omega+1\right]\left\|V^{-1}\right\|\right] \\ & +2\left|\alpha\right|\omega\left\|U^{-1}\right\| + \left|1-2\alpha\right|\omega\left\|U^0\right\| + 2\sigma\alpha\lambda\omega\left\|U^0\right\|^2\end{aligned}$$

Choosing $l = o\left(h^{4+s}\right)$ small enough as above, and $\mu = 8\sigma\left|\alpha\right|\omega + 1$, we obtain

$$\begin{aligned} \frac{|\beta|}{2}\left\|V^1\right\| &\le \left\|\beta\mathcal{K}\left(V^1\right)\right\| \le \mu\left|1-2\beta\right|\left\|V^0\right\| + \mu\left|\beta\right|\left[\left\|V^{-1}\right\|\right] \\ &\quad +2\left|\alpha\right|\omega\left\|U^{-1}\right\| + \left|1-2\alpha\right|\omega\left\|U^0\right\| + 2\sigma\alpha\lambda\omega\left\|U^0\right\|^2 \end{aligned}$$

Next, recall that

$$\left\|U^{-1}\right\| \le \left\|U^0\right\| + \left\|\widetilde{\varphi}\right\| \text{ and } \left\|V^{-1}\right\| \le |q|\left\|U^0\right\| + l\left\|\widetilde{\psi}\right\|,$$

we get

$$\begin{aligned} |\beta|\left\|V^1\right\| &\le 2\mu\left|1-2\beta\right|\left\|V^0\right\| + 2\mu\left|\beta\right|\left(|q|\left\|U^0\right\| + \left\|\widetilde{\psi}\right\|\right) \\ &\quad +4\left|\alpha\right|\omega\left(\left\|U^0\right\| + l\left\|\widetilde{\varphi}\right\|\right) + 2\left|1-2\alpha\right|\omega\left\|U^0\right\| + 4\sigma\alpha\lambda\omega\left\|U^0\right\|^2. \end{aligned}$$

Henceforth,

$$\begin{aligned} |\beta|\left\|V^1\right\| &\le 4\sigma\alpha\lambda\omega\left\|U^0\right\|^2 + 2\mu\left|1-2\beta\right|\left\|V^0\right\| \\ &\quad +2\left(\mu\left|q\right|\left|\beta\right| + \omega\left(2\left|\alpha\right| + \left|1-2\alpha\right|\right)\right)\left\|U^0\right\| \\ &\quad +2\mu\left|\beta\right|\left\|\widetilde{\psi}\right\| + 4\left|\alpha\right|\omega l\left\|\widetilde{\varphi}\right\|. \end{aligned}$$

Now, proceeding as for U^1, we prove that for all $\varepsilon > 0$, there is an $\eta_2' > 0$ satisfying $\left\|V^1\right\| < \varepsilon$ whenever $\left\|\left(U^0, V^0\right)\right\| \le \eta_2'$. Finally $\eta = \min\left(\eta_2, \eta_2'\right)$ answers the question.
Assume now that $\left(U^k, V^k\right)$ is bounded for $k = 1, 2, ..., n$ by ε_1 whenever $\left(U^0, V^0\right)$ is bounded by η and let $\varepsilon > 0$. We shall prove that it is possible to choose η satisfying $\left\|\left(U^{n+1}, V^{n+1}\right)\right\| \le \varepsilon$.

Lemma 30 *(Banach) Let E, F be two Banach spaces and $\phi : E \to F$ be linear. If ϕ is continue and bijective then ϕ is a homeomorphism.*

Consider as above the endomorphism ϕ on $M_{(J+1)^2}\left(\mathbb{R}\right) \times M_{(J+1)^2}\left(\mathbb{R}\right)$ defined by

$$\phi\left(X, Y\right) = \left(X - \sigma\beta\mathcal{L}_A\left(Y\right), \beta Y - \alpha\left[qX - \delta\kappa\mathcal{L}_A\left(X\right)\right]\right).$$

Consider also

$$f_1(X,Y,Z,W) = X + \alpha\beta\mathcal{L}_A(Y) + \sigma(1-2\beta) + \sigma\beta\mathcal{L}_A(Z) + \sigma\lambda W$$

and

$$f_2(X,Y,Z,T) = \alpha\Gamma(X) - \beta Y + (1-2\alpha)\Gamma(Z) - (1-2\beta)T.$$

We obtain thus

$$\begin{aligned}\phi\left(U^{n+1},V^{n+1}\right) &= \left(U^{n+1} - \sigma\beta\mathcal{L}_A\left(V^{n+1}\right), \beta V^{n+1} - \alpha\Gamma\left(U^{n+1}\right)\right) \\ &= \left(f_1\left(U^{n-1},V^{n-1},V^n,F^n\right), f_2\left(U^{n-1},V^{n-1},U^n,V^n\right)\right).\end{aligned}$$

We have already proved that ϕ is one to one for l and h small enough. Since ϕ is linear and continuous, then ϕ has a continuous inverse function. So, ϕ is a homeomorphism on $M_{(J+1)^2}(\mathbb{R}) \times M_{(J+1)^2}(\mathbb{R})$. Furthermore

$$\left(U^{n+1},V^{n+1}\right) = \phi^{-1}\left(f_1\left(U^{n-1},V^{n-1},V^n,F^n\right), f_2\left(U^{n-1},V^{n-1},U^n,V^n\right)\right).$$

As ϕ^{-1} is continuous, there exists $C>0$ such that

$$\begin{aligned}\left\|\left(U^{n+1},V^{n+1}\right)\right\| &= \left\|\phi^{-1}\left(f_1\left(U^{n-1},V^{n-1},V^n,F^n\right), f_2\left(U^{n-1},V^{n-1},U^n,V^n\right)\right)\right\| \\ &\leq C\left\|\left(f_1\left(U^{n-1},V^{n-1},V^n,F^n\right), f_2\left(U^{n-1},V^{n-1},U^n,V^n\right)\right)\right\|.\end{aligned}$$

By choosing $\left\|\left(U^k,V^k\right)\right\| \leq \eta$, for all $k = 0,1,...n$, we get

$$\left\|f_1\left(U^{n-1},V^{n-1},V^n,F^n\right)\right\| \leq 2\sigma|\lambda|\eta^2 + (1+8\sigma(1+3|\beta|))\eta.$$

Now, it suffices to prove that there exists $\eta>0$ for which

$$(1+8\sigma(1+3|\beta|))\eta + 2\sigma|\lambda|\eta^2 \leq \varepsilon \quad \Leftrightarrow \quad 2\sigma|\lambda|\eta^2 + (1+8\sigma(1+3|\beta|))\eta - \varepsilon \leq 0.$$

The discernments is

$$\Delta = (1 + 8\sigma (1 + 3|\beta|))^2 + 8\sigma |\lambda| \varepsilon > 0.$$

Hence, there are two zeros of the last equality

$$\eta_1 = \frac{-(1 + 8\sigma(1 + 3|\beta|)) - \sqrt{\Delta}}{4\sigma|\lambda|} \text{ and } \eta_2 = \frac{-(1 + 8\sigma(1 + 3|\beta|)) + \sqrt{\Delta}}{4\sigma|\lambda|}.$$

It is straightforward that $0 \in]\eta_1, \eta_2[$. Hence $\eta_2 > 0$. Now for f_2 we get

$$\left\| f_2\left(U^{n-1}, V^{n-1}, U^n, V^n\right)\right\| \leq [\omega(1 + 3|\alpha|) + (1 + 3|\beta|)]\,\eta.$$

For

$$\eta \leq \eta_3 = \frac{\varepsilon}{\omega(1 + 3|\alpha|) + (1 + 3|\beta|)},$$

we obtain $\left\| V^{n+1}\right\| \leq \varepsilon$. Finally, choosing η the minimum between η_2 and η_3 the criterion is proved.

Lemma 31 *As the numerical scheme is consistent and stable, it is then convergent.*

4.5 Numerical Implementations

We propose in this section to present some numerical examples to validate the theoretical results developed previously. As in the previous chapter, the error between the exact solutions and the numerical ones via an L_2 discrete norm (3.37) will be estimated. Denote u^n the net function $u(x, y, t^n)$ and U^n the numerical solution. We propose to compute the discrete error (3.38) on the grid (x_i, y_j), $0 \leq i, j \leq J + 1$ and to validate the convergence rate of the proposed schemes we propose to compute the proportion

$$C = \frac{Er}{l^2 + h^2}. \qquad (4.25)$$

We consider the inhomogeneous problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta v + \lambda \left|\nabla u\right|^2 + g(x,y,t), \quad (x,y,t) \in \Omega \times (t_0, +\infty) \\ v = qu - \kappa \Delta u, \quad (x,y,t) \in \Omega \times (t_0, +\infty) \\ (u,v)\,(x,y,t_0) = (\varphi,\psi)\,(x,y), \quad (x,y) \in \overline{\Omega} \\ \overrightarrow{\nabla}(u,v)(x,y,t) = 0, \quad (x,y,t) \in \partial\Omega \times (t_0, +\infty) \end{cases} \tag{4.26}$$

on the rectangular domain $\Omega = [-1,1] \times [-1,1]$, where

$$g(x,y,t) = Ke(t)[e(t)[C^4(x)S^2(x)C^6(y) + C^6(x)C^4(y)S^2(y)] \tag{4.27}$$

$$- \left[C(x)S^2(x)C(y)S^2(y)\right]] \tag{4.28}$$

and the exact solution

$$(u,v)(x,y,t) = (e(t)C^3(x)C^3(y), ...),$$

with

$$C(x) = \cos(\frac{\pi x}{2}) \;\; , \;\; S(x) = \sin(\frac{\pi x}{2}) \;\; , \;\; e(t) = e^{-9\pi^4 t/2} \quad \text{and} \quad K = \frac{9\pi^4}{2}.$$

In the following tables, numerical results are provided. We computed for different space and time steps the discrete L_2-error estimates defined by (3.38). The time interval is $[0,1]$ for a choice $t_0 = 0$ and $T = 1$. The following results are obtained for different values of the parameters J (and thus h), l ((and thus N). The parameters α and β are fixed to $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{5}$. We just notice that some variations done on these latter parameters have induced an important variation in the error estimates which explains their effect as they calibrates the position of the approximated solution around the exact one. The parameters q, λ and κ have the role of viscosity-type coefficients and fixed to the values $\kappa = 1$, $\lambda = -2\pi^2$ and $q = \frac{11\pi^2}{2}$. The following tables outputs the error estimates relatively to the discrete L^2-norm defined above for different values of the space and time steps. First, we provide estimates when the optimal condition $l = o(h^{4+s})$, $s > 0$ is fulfilled. s is fixed to 0.01 for the first table. Next, to control more the effect of such assumption which is due to the presence of a second order Laplacian in the original problem, we tested the convergence of the scheme at some orders less than the optimal power fixed to 4. The second table provides the estimates with a slightly sub-critical power $4-s$, $s>0$ small enough. Here also s is fixed to 0.01, and finally in the third table, we tested the discrete scheme for a strong sub-critical power.

$l=o(h^{4.01})$			$l=o(h^{3.99})$			$l=o(h^{3.01})$		
J	N	$Er2$	J	N	$Er2$	J	N	$Er2$
10	640	$1,25.10^{-5}$	10	616	$1,35.10^{-5}$	10	128	$3,10.10^{-4}$
12	1320	$2,46.10^{-6}$	12	1273	$2,55.10^{-6}$	12	220	$8,82.10^{-5}$
14	2450	$6,14.10^{-7}$	14	2355	$6,65.10^{-7}$	14	350	$2,99.10^{-5}$
16	4183	$1,84.10^{-7}$	16	4012	$2,00.10^{-7}$	16	523	$1,17.10^{-5}$
18	6707	$6,38.10^{-8}$	18	6419	$6,96.10^{-8}$	18	746	$9,59.10^{-6}$
20	10233	$2,46.10^{-8}$	20	7973	$4,06.10^{-8}$	20	1024	$2,45.10^{-6}$
22	14997	$1,04.10^{-8}$	22	14295	$1,14.10^{-8}$	22	1364	$1,26.10^{-6}$
24	21258	$4,76.10^{-9}$	24	20228	$5,26.10^{-9}$	24	1772	$6,84.10^{-7}$
25	25039	$3,29.10^{-9}$	25	23806	$3,64.10^{-9}$	25	2004	$5,14.10^{-7}$
30	52015	$6,36.10^{-10}$	30	49273	$7,09.10^{-10}$	30	3468	$1,34.10^{-7}$

4.6 Conclusion

This chapter investigated the solution of the well-known Kuramoto-Sivashinsky equation in two-dimensional case by applying a two-dimensional finite difference discretization. The original equation is a 4-th order partial differential equation. Thus, in a first step it was recasted into a system of second order partial differential equations by applying a reduction order. Next, the continuous system of simultaneous coupled PDEs has been transformed into an algebraic discrete system involving a generalized Lyapunov-Syslvester type operators. Solvability, consistency, stability and convergence are then established by applying well-known methods such as Lax-Richtmyer equivalence theorem and Lyapunov Stability and by examining the topological properties of the obtained Lyapunov-Sylvester type operators. The method was finally improved by developing a numerical example.

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Z. Yi, Y. Ling-Ya; *Rational and Periodic Wave Solutions of Two-Dimensional Boussinesq Equation*. Commun. Theor. Phys. 49(4) (2008), 815-824.

CURRICULUM VITAE

Abdelhamid Bezia

First name: Abdelhamid
Family name: BEZIA
Birth day: April, 13th, 1989
Address: Laboratory of Algebra and Number Theory,
Department of Algebra and number theory,
Faculty of mathematics,
University of Science and technology, Hourai Boumediene, Algeria
E-mail:, abdelhamid.bezia@gmail.com or abezia@usthb.dz
Web Site laboratory: http://www.latn.usthb.dz

EDUCATION

University of Science and technology, Hourai Boumediene, Algeria/ Faculty of mathematics

- October 2011- Present: Ph.D student.
  Topic: Sylvester-Lyapunov operator for solving some nonlinear PDEs in higher dimension. Supervisors: Betina Kamel.
- October 2009- June 2011: Master Degree in mathematics and applications, option Algebra and geometry.
- October 2007- June 2009: Diploma degree in mathematics LMD.

University Dr. Yahia Fares of Médéa.

- October 2006- June 2007: Common core curriculums in mathematics and computer science.

Mohamed Bouguera Scendry School

- June 2006: Baccalaureate Certificate, option: exact sciences.

PUBLICATIONS AND PROJECTS

- Bezia, Abdelhamid, Anouar Ben Mabrouk and Kamel Betina. "*Lyapunov-Sylvesters operators for (2+1)-Boussinesq equation.*" Electronic Journal of Differential Equations 2016.268 (2016): 1-19.
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- Bezia Abdelhamid. "*Enseignement des structures algébriques en première année universitaire, Espace mathématique francophone*", Geneva 2012.

SCIENTIFIC ACTIVITIES

- Research internship, Laboratoire de mathématiques de Besançon, from Spetember 15, 2016 to October 15, 2016.
- Study week on Maths-Info Entreprises (SEMIE) Campus de Saint Martin d'Hères (Grenoble), 24 - 28 October 2016.
- The fifth conference of GGTM, Canférence à la mémoire de Mahamed Salah Daouendi, Tunis, Cite des Sciences- 24-27 Mars 2014
- The special project "Youth Speak French teachers - Training and entry into the profession", the conference (Francophone Mathematic Space) held in Fribourg (Switzerland) from 30 January to 2 February 2012 and in Geneva 3 to 7 February 2012.

TEACHING EXPERIENCE
Academic Teaching Assistant, (University Dr. Yahia Fares of Médéa and U.S.T.H.B)
Course responsibilities: Algebra, Analysis, Complex analysis.

REFERENCES
Dr. Kamel Bentina, Mathematics Full Professor
Faculty of mathematics, Department of Algebra and number theory
University of Science and technology, Hourai Boumediene, Algeria
kamelbetina@gmail.com

Dr. Anouar Ben Mabrouk
Departement de Mathématiques,
Institut Supérieur de Mathéamatiques Appliquées et Informatique de Kairouan, Tunisia.
anouar.benmabrouk@fsm.rnu.tn

Pr. Djilali Behloul, Mathematics Full Professor
Faculty of Computer Science and electronic, Department of Computer Science
University of Science and technology, Hourai Boumediene, Algeria
dbehloul@usthb.dz, dbehloul@yahoo.fr