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Subject

Existence of symmetric positive solutions to the fourth order Lidstone boundary value problem

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To my wife and my daughters.

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Abstract

This work is to show that there are at least three symmetric positive solutions to the non linear equation of fourth order with the boundary conditions Lidstone. This problem is given by:

$$\begin{cases} y^{(4)}(t) = f\left(y(t), -y''(t)\right), \quad 0 \leq t \leq 1, \\ y(0) = y(1) = 0, \\ y''(0) = y''(1) = 0, \end{cases} \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R} \to [0, \infty[$ is continuous.

Key words. Problem of fourth-order for Lidstone, multiple solutions positive, cone, Banach space ordered, Leggett-Williams fixed point theorem.

Contents

Introduction

This work is devoted to the existence of nontrivial positive solutions for a class of fourth order Lindston boundary value problem studied by R.I. Avery, J.M. Davis et J. Henderson [4]

This work presents sufficient conditions for the existence of at least three positive solutions for a nonlinear fourth order equation under Lidstone boundary conditions. The main tool used is the generalized Leggett and Williams fixed point theorem [3]. based on the Krasnozels'kii [15] and Leggett-Williams [17] fixed point theorems in cones.

Growth conditions are imposed on the nonhomogeneous term f and inequalities involving an associated Green's function are used in order to apply the Leggett and Williams fixed point Theorem to cones in ordered Banach space.

Theoretic cone fixed point theorem have been employed commonly in recent year to show the existence of multiple positive solutions.

Over a long period of time, there is much attention focused on questions of positive solutions of boundary value problems for ordinary differential equations, see [1] and the references therein. This is due to the applicability of certain fixed point theorem of Krasnosel'skii[15] or Leggett and Williams [17].

Other attention has been given to the study of positive solutions of boundary value problems.

At least triple positive solutions where ensured by Avery [3] in the second order conjugate boundary value problem, by using the Leggett and Williams fixed point theorem.

In the paper of Henderson and Thompson [12], for certain two-point boundary value problems for n-th order equations, multiple positive solutions have been improved using the same theorem. Avery and J. Henderson [2] improved Avery's results by using a five functional fixed point theorem(which is a generalisation of the Leggett-Williams fixed point theorem).

M. Davis, W. Eloe, J. Henderson [7] make an application of a Leggett-Williams fixed point theorem to yield the existence of at least three positive symmetric concave solutions for the 2m th order Lidstone boundary value problem $y^{2m}(t) = f(y(t), ..., y^{2j}(t), ..., y^{2(m-1)}(t)), 0 \leq$

$t \leq 1, y^{2i}(0) = 0 = y^{2i}(1), 0 \leq i \leq m-1$, where $(-1)^m f > 0$. The emphasis here is that the function f of this equation depends on higher order derivatives. This is the first related study that allows f to depends on more than one order derivative of y.

This paper fellows the work of [2] and [7] by applying the Five Functionals fixed point theorem and allowing to f to depends on the second derivative of y.

This work is divided in three chapters, preceded by an introduction and followed by a conclusion and the bibliographical list of publications.

The first chapter of this paper has the aim to recall some basic fixed point theorems: Banach, Brouwer,Schauder, and Krasnosel'skii fixed point theorem.

The second chapter, has the aim to remind some notions and basic results which are necessary in the presentation of the following chapters of this work and to present the theorem with its proof. This theorem, given by R.I.Avery [3], is a generalisation of the the Leggett-Williams fixed point theorem, which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. The hypotheses of the Leggett-Williams fixed point theorem are based upon the relationships between the operator, the norm and a continuous concave functional. The generalisation theorem allows to choose three convex functionals which are used in place of the norm in part of the hypotheses. In addition a concave functional is introduced with relationships to one of the convex functionals in the same spirit of the Leggett-Williams theorem.

The main results are stated and proved in chapter three. The five functionals fixed point is used to prove the existence of three positive solutions to the integral equations of the form:

$$y(t) = \int_0^1 G(t,s)v(s)ds, \quad 0 \leq t \leq 1,$$

where G is the Green's function which satisfies the Lidstone boundary conditions and v is defined by:

$$v(t) = \int_0^1 G(t,s)f(\int_0^1 G(s,\tau)v(\tau)d\tau, v(s)), \quad 0 \leq t \leq 1.$$

In order to do this, inequalities imposed on the kernel G are used, and on the nonhomogeneous term f.

Chapter 1

Reminders

Summary

1.1 Some Fixed Point Theorems

Definition 1.1.1 *Let X be a set and let $T : X \longrightarrow X$ be a function that maps X into itself. (Such a function is often called an operator, a transformation, or a transform on X, and the notation Tx is often used in place of $T(x)$). A fixed point of T is an element $x \in X$ for which $T(x) = x$.*

Remark 1.1.1 *Note that the definition of a fixed point requires no structure on either the set X or the function T.*

1.1.1 Banach Fixed Point Theorem

Definition 1.1.2 *Let (X, d) be a metric space. A contraction of X (also called a contraction mapping on X) is a function $f : X \longrightarrow X$ that satisfies*

$$\forall x, x' \in X : d(f(x'), f(x)) \leq \beta d\left(x', x\right)$$

for some real number $\beta < 1$. Such a β is called a contraction modulus of f.

Theorem 1.1.1 *Every contraction mapping is continuous.*

Proof : Let $T : X \longrightarrow X$ be a contraction on a metric space(X, d), with modulus β, and let $\overline{x} \in X$. Let $\epsilon > 0$, and let $\delta = \epsilon$. Then $d\left(x, \overline{x}\right) < \delta \Longrightarrow d(Tx; T\overline{x}) \leq \beta\delta < \epsilon$. Therefore T is continuous at $\bar{x}$. Since $\bar{x}$ was arbitrary, T is continuous on X. The above proof actually establishes that a contraction mapping is uniformly continuous.

Definition 1.1.3 *Let (X, d_x) and (Y, d_Y) be metric spaces. A function $f : X \rightarrow Y$ is uniformly continuous if for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\forall x; x' \in X : d_x(x, x') < \delta \Longrightarrow d_Y(f(x), f(x')) < \epsilon.$$

Notice how this definition differs from the definition of continuity: uniform continuity requires that, for a given ϵ , a single will work across the entire domain of f.

Continuity allows that the δ may depend upon the point x at which continuity of f is being evaluated; uniform continuity requires that (for a given ϵ) being within δ of any $x \in X$ guarantees that the image under f is within ϵ of $f(x)$.

Theorem 1.1.2 *Every contraction mapping is uniformly continuous.*

Theorem 1.1.3 *(Banach Fixed Point Theorem)Every contraction mapping on a complete metric space has a unique fixed point. (This is also called the Contraction Mapping Theorem.)*

Proof : *Let $T : X \longrightarrow X$ be a contraction on the complete metric space (X, d), and let be a contraction modulus of T. First we show that T can have at most one fixed point.*

Then we construct a sequence which converges and show that its limit is a fixed point of T.

(a) *Suppose x and $x^{'}$ are fixed points of T. Then $d(x, x^{'}) = d(Tx, Tx^{'}) \leq \beta d(x, x^{'})$; since $\beta < 1$, this implies that $d(x, x^{'}) = 0$, i. e., $x = x^{'}$.*

(b) *Let $x_0 \in X$, and define a sequence $\{x_n\}$ as follows:*

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ \dots \ , x_n = Tx_{n-1} = T^nx_0, \ \dots$$

We first show that adjacent terms of $\{x_n\}$ grow arbitrarily close to one another | speci cally, that $d(x_n, x_{n+1}) \leq \beta^n d(x_0, x_1)$:

$$\begin{aligned} d(x_1, x_2) &\leq \beta d(x_0, x_1) \\ d(x_2, x_3) &\leq \beta d(x_1, x_2) \leq \beta^2 d(x_0, x_1) \\ &\dots \\ d(x_n, x_{n+1}) &\leq \beta d(x_{n-1}, x_n) \leq \beta^n d(x_0, x_1). \end{aligned}$$

Next we show that if $n < m$ then $d(x_n, x_m) < \beta^n \frac{1}{1-\beta} d(x_0, x_1)$:

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leqq \beta^n d(x_0, x_1) \\
d(x_n, x_{n+2}) &\leqq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\
&\leqq \beta^n d(x_0, x_1) + \beta^{n+1} d(x_0, x_1) = \left(\beta^n + \beta^{n+1}\right) d(x_0, x_1) \\
& \quad ... \\
d(x_n, x_m) &\leqq (\beta^n + \beta^{n+1} + ... + \beta^{m-1}) d(x_0, x_1) \\
&= \beta^n (1 + \beta + \beta^2 + ... + \beta^{m-1-n}) d(x_0, x_1) \\
&< \beta^n (1 + \beta + \beta^2 + ...) d(x_0, x_1) \\
&= \beta^n \frac{1}{1-\beta} d(x_0, x_1)
\end{aligned}
$$

Therefore $\{x_n\}$ is Cauchy: for $\epsilon > 0$, let N be large enough that $\beta^N \frac{1}{1-\beta} d(x_0, x_1) < \epsilon$, which ensures that

$$n, m > N \implies d(x_n, x_m) < \epsilon.$$

Since the metric space $(X, d)$is complete, the Cauchy sequence $\{x_n\}$ converges to a point $x^* \in X$. We show that x^* is a fixed point of T: since $x_n \longrightarrow x^*$ and T is continuous, we have $Tx_n \longrightarrow Tx^*$ i.e., $x_{n+1} \longrightarrow Tx^*$. Since $x_{n+1} \longrightarrow x^*$ and $x_{n+1} \longrightarrow Tx^*$, we have $Tx^* = x^*$.

Theorem 1.1.4 *If T is a contraction on a complete metric space (X, d) and β is a contraction modulus of T, then for any $x \in X$,*

$$\forall n \in \mathbb{N} \colon d\left(T^n x, x^*\right) \leq \beta^n d\left(x, x^*\right),$$

where x^* is the unique fixed point of T.

Proof :

$$
\begin{aligned}
d\left(T^n x, x^*\right) &= d\left(TT^{n-1}x, Tx^*\right), \quad \text{because } x^* \text{ is a fixed point of } T \\
&\leq \beta d\left(T^{n-1}x, x^*\right), \text{ because } T \text{ is a contraction} \\
&\leq \beta^2 d\left(T^{n-2}x, x^*\right) \\
&\leq \quad ... \leq \beta^n d\left(T^0 x, x^*\right) \\
&= \beta^n d\left(x, x^*\right).
\end{aligned}
$$

1.1.2 Brouwer Fixed Point Theorem

In the unit-interval example in the preceding section, the Banach Theorem seems somewhat limited. It seems intuitively clear that any continuous function mapping the unit interval into itself will have afixed point, but the Banach Theorem applies only to functions f that satisfy $\left|f'(x)\right| \leq \beta$ for some $\beta < 1$.

An elementary example of this is the function $f(x) = 1 - x$, which has an obvious fixed point at $x = \frac{1}{2}$, but whose derivative satisfies $\left|f'(x)\right| = 1$ everywhere therefore $d(f(x), f(x')) = d(x, x')$ so f is not a contraction and the Banach Fixed Point Theorem doesn't apply to f. The fixed point theorem due to Brouwer covers this case as well as a great many others that the Banach Theorem fails to cover because the relevant functions aren't contractions

Definition 1.1.4 *Let $\mathbb{R}^n$ have its usual inner product $\langle .,. \rangle$ and let $\|.\|$ be the induced norm.*

Let $B^n := \{x \in \mathbb{R}^n : \|x\| < 1\}$ be the open unit ball, $\overline{B}^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ the closed unit ball, and $S^{n-1} := \{x : \|x\| = 1\}$ the unit sphere in $\mathbb{R}^n$.

Theorem 1.1.5 *(Brouwer Fixed Point Theorem). Every continuous map $f : \overline{B}^n \to \overline{B}^n$ has a fixed point. That is there is an $x \in \overline{B}^n$ such that $f(x) = x$.*

In [18] *J. Milnor gave a proof of this result based on elementary multidimensional integral calculus.*

In [19] *C. A. Rogers simplied Milnor's proof.*

Here we give an exposition of the Milnor-Rogers proof.

Lemma 1.1.1 *There is no C^1 map $f : \overline{B}^n \to S^{n-1}$ such that $f(x) = x$ for all $x \in S^{n-1}$.*

Proof : Assume, toward a contradiction, that such an $f : \overline{B}^n \to S^{n-1}$ exists.

For $t \in [0, 1]$ let

$$f_t(x) = (1 - t)x + tf(x) = x + tg(x)$$

where $g(x) = f(x) - x$. Note that for $x \in \overline{B}^n$ that

$$\|f_t(x)\| \leq (1-t)\|x\| + t\|f(x)\| \leq (1-t) + t = 1$$

and therefore $f_t : \overline{B}^n \to \overline{B}^n$.Also note that for all $x \in S^{n-1}$.

$$f_t(x) = (1-t)x + tf(x) = (1-t)x + tx = x$$

and thus f_t fixes all points on S^{n-1}. As f is C^1, the same is true for g and therefore there is a constant C such that for all $x_1, x_2 \in \overline{B}^n$

$$\|g(x_1) - g(x_2)\| \leq C\|x_2 - x_1\|.$$

Now assume that there are distinct points x_1 and x_2 in $\overline{B}^n$ with $f_t(x_1) = f_t(x_2)$.

This implies $x_2 - x_1 = t(g(x_1) - g(x_2))$ and therefore

$$\|x_2 - x_1\| = t\|g(x_1) - g(x_2)\| \leq Ct\|x_2 - x_1\|.$$

As $x_1 \neq x_2$ this implies $Ct \geq 1$. Thus when $t < 1/C$ the function $f_t : \overline{B}^n \to \overline{B}^n$ is injective. Let $G_t = f_t[B^n]$ be the image of the open unit ball under f_t The derivative of f_t, viewed as a linear map $f_t'(x) : \mathbb{R}^n \to \mathbb{R}^n$, is given by

$$f_t'(x) = I + tg'(x)$$

where I is the identity map on $\mathbb{R}^n$. As g is C^1 there is a t_0 such that det $f_t'(x) > 0$ for all $t \in [0, t_0]$. Then by the inverse function theorem G_t is an open set for all $t \in [0, t_0]$.

By possibly making t_0 smaller we also have that f_t is injective for all $t \in [0, t_0]$.

We claim that $G_t = B^n$ for all $t \in [0, t_0]$. Assume that this is not the case. Then the boundary ∂G_t will intersect the open ball B^n at some point y_0. As $y_0 \in \partial G_t$ there is a sequence $x_\ell \in B^n$ such that

$$\lim_{\ell\to\infty} f_t(x_\ell) = y_0.$$

By the compactness of $\overline{B}^n$ we can pass to a subsequence and assume that $\lim\limits_{\ell\to\infty} x_\ell = x_0$ for some $x_0 \in \overline{B}^n$. Then, by the continuity of f , we have

$$f_t(x_0) = y_0.$$

But, as G_t is open and open sets are disjoints from their boundaries, y_0 is not in $G_t = f\left[B^n\right]$ thus

$$x_0 \in \overline{B}^n \setminus B^n = S^{n-1}.$$

But for $x_o \in S^{n-1}$ we have that

$$f_t\left(x_0\right) = x_0,$$

which implies that

$$y_0 = f_t\left(x_0\right) = x_0 \in S^{n-1},$$

which contradicts the assumption that y_0 is in B^n.Therefore for $t \in [0, t_0]$ the map $f_t : \overline{B}^n \to \overline{B}^n$ is a bijection.

Define a function $F : [0, 1] \to \mathbb{R}$ by

$$F\left(t\right) = \int_{\overline{B}^n} det\ f_t'\left(x\right) dx = \int_{\overline{B}^n} det\ \left(I + tg'\left(x\right)\right) dx$$

where dx is the volume measure on $\mathbb{R}^n$. This is clearly a polynomial in t. And for $t \in [0, t_0]$ the function $f_t : \overline{B}^n \to \overline{B}^n$ is a bijection and so by the change of variable formula for multiple integrals $F\left(t\right)$ is just the volume of the image $f_t\left[\overline{B}^n\right] = \overline{B}^n$. That is

$$F\left(t\right) = \text{Volume }\left(\overline{B}^n\right) \qquad for\ t \in [0, t_0].$$

But a polynomial that is constant on an interval is constant everywhere.

Therefore $F\left(t\right)$ =Volume $\left(\overline{B}^n\right)$ for all $t \in [0, t_0]$ and in particular $F\left(1\right)$ =Volume $\left(\overline{B}^n\right) > 0$.

But

$$f_1\left(x\right) = f\left(x\right) \in S^{n-1}$$

for all x and therefore

$$\left\langle f_1\left(x\right), f_1\left(x\right)\right\rangle = \left\|f_1\left(x\right)\right\|^2 = 1$$

for all x. Thus for any vector $\in \mathbb{R}^n$

$$2\left\langle f_1'\left(x\right)v, f_1\left(x\right)\right\rangle = \frac{d}{dt}\left\langle f_1\left(xt + tv\right), f_1\left(x + tv\right)\right\rangle |_{t=0} = \frac{d}{dt}1 |_{t=0} = 0.$$

This shows that the range of $f_1'(x)$ is contained in $f(x)^{\perp}$, the orthogonal compliment of $f(x)$. But then rank $f_1'(x) < n-1$ for all $x \in \overline{B}^n$ and therefore det $f_1'(x) = 0$ for all $x \in \overline{B}^n$. Whence

$$F(1) = \int_{\overline{B}^n} det\ f_1'(x)\, dx = 0.$$

This contradicts that $F(1) > 0$ and completes the proof.

Proof : of the Brouwer Fixed Point Theorem:

Let $f : \overline{B}^n \to \overline{B}^n$ be a continuous map.

Then by the Stone-Weierstrass theorem there is a sequence of C^1functions $p_\ell : \overline{B}^n \to R^n$ such the$\|f(x) - p_\ell\| < 1/\ell$ for all $x \in \overline{B}^n$.(In fact we can choose the p_ℓ's to be polynomials).

Then

$$\|p_\ell(x)\| \leq \|f(x)\| + \|p_\ell(x) - f(x)\| \leq 1 + 1/\ell.$$

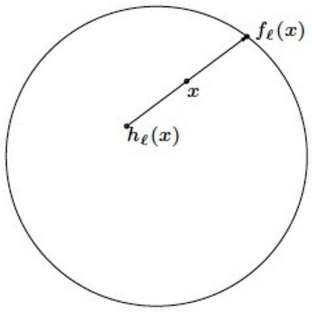
Therefore if

$$h_\ell = (1 + 1/\ell)^{-1} p_\ell$$

we have that $h_\ell : \overline{B}^n \to \overline{B}^n$ and $h_\ell \to f$ uniformly.

We claim that each h_ℓ has a fixed point in $\overline{B}^n$. For if not, let $f_\ell : \overline{B}^n \to S^{n-1}$ be map

$$f_\ell(x) = \text{point where the ray from } h_\ell(x) \text{ to } x \text{ meets } S^{n-1}.$$



If h_ℓ has no fixed point this map is C^1 and has $f_\ell(x) = x$ for all $x \in S^{n-1}$ contradicting Lemma (1.1.1).

Let x_ℓ be a fixed point of h_ℓ, that is $h_\ell(x_\ell) = x_\ell$.

As $\overline{B}^n$ is compact we can pass to a subsquence and assume that $x_\ell \to x_0$ for some x_0 in $\overline{B}^n$. As $h_\ell \to f$ uniformly this implies

$$f(x_0) = \lim_{\ell\to\infty} h_\ell(x_\ell) = \lim_{\ell\to\infty} x_\ell = x_0.$$

That is f has x_0 as a fixed point.

We get as a corollay, important enough to be called a theorem, a version of Lemma (1.1.1) where f is not required to be C^1.

Theorem 1.1.6 *There is no continuous map $f : \overline{B}^n \to S^{n-1}$ with $f(x) = x$ for all $x \in S^{n-1}$.*

Proof : Assume that such an $f : \overline{B}^n \to S^{n-1}$ existed.
Let $g : \overline{B}^n \to \overline{B}^n$ be given by $g(x) = -f(x)$.
Therefore g also maps $\overline{B}^n$ into S^{n-1}.
Therefore if

$$x = g(x) \text{ we have } x \in S^{n-1}.$$

But for

$$x \in S^{n-1}, g(x) = -f(x) = -x \neq x.$$

Thus g has no fixed point, contradicting Theorem (1.1.5).

1.1.3 Schauder Fixed Point Theorem

Lemma 1.1.2 *Every bounded closed subset of a finite dimensional normed space is compact.*

Theorem 1.1.7 *Let K be a nonempty closed convex subset of a normed space.*

Let T be a continuous mapping of K into a cumpact subset of K.

Then T has fixed point in K.

Proof : *Let $\mathcal{E}$ be a Banach space and let $T(K) \subset A$, a compact subset of K. A is contained in a closed convex bounded subset of $\mathcal{E}$.*

$$T(B \cap K) \subset T(K) \subset A \subset B$$

so $T(B \cap K)$ is contained in a compact subset of B, K and there is no loss of generality in supposing that K is bounded.

If A_0 is a countable dense subset of the compact metric space A, then the set of all rational linear combinations of elements of A_0 is a countable dense subset of the closed linear subspace $\mathcal{E}_0$ spanned by A_0 and $A_0 \subset \mathcal{E}_0$.

Then

$$T(K \cap \mathcal{E}_0) \subset T(K) \subset A,$$

a compact subset of $\mathcal{E}_0$, and $K \cap \mathcal{E}_0$ is closed and convex.

Hence without loss of generality we may assume that K is a bounded closed convex subset of a separable normed space $\mathcal{E}$ with a strictly convex norm (Theorem Clarkson).

Given a positive integer n, there exists a $\frac{1}{n}$-net $Tx_1, ..., Tx_m$ say in TK, so that

$$\min_{1 \leq k \leq n} \|Tx - Tx_k\| < \frac{1}{n}\, (x \in K) \tag{1.1.1}$$

Let $\mathcal{E}_n$ denote the linear hull of

$$Tx_1, ..., Tx_m.K_n = K \cap \mathcal{E}_n$$

is a closed bounded subset of $\mathcal{E}_n$ and therefore compact (Lemma 1.1.2). Since the norm is strictly convex, the metric projection P_n of $\mathcal{E}$ onto the convex compact subset K_n exists. $T_n = P_nT$ is a continuous mapping of the non-empty convex compact subset K_n into itself, and therefore by the Brouwer fixed point theorem, it has a fixed point $u_n \varepsilon K_n$,

$$T_n u_n = u_n \tag{1.1.2}$$

By (1.1.1), since

$$Tx_k \in K_n (k = 1, 2, ..., m),$$

we have

$$\|Tx - T_n x\| < \frac{1}{n} \tag{1.1.3}$$

The sequence $\{Tu_n\}$of TK has a subsequence Tu_{n_k} converging to a point $v \in K$. By (1.1.2) and (1.1.3),

$$\|unk - v\| = \|T_{n_k} u_{n_k} - v\| \leq \|T_{n_k} u_{n_k} - Tu_{n_k}\| + \|Tu_{n_k} - v\| < \frac{1}{n} + \|Tu_{n_k} - v\|.$$

Therefore

$$\lim_{k\to\infty} u_{n_k} = u$$

and by continuity of T,

$$\lim_{k\to\infty} Tu_{n_k} = Tv \text{ or } Tv = v.$$

1.1.4 Krasnoselskii's Cone Fixed Point Theorem

Definition 1.1.5 *Let* $\mathcal{E}$ *be a Banach space . A nonempty, closed set* $\mathcal{P} \subset \mathcal{E}$ *is a cone provided*

(*a*) $\alpha u + \beta v \in \mathcal{P}$ *for all* $u, v \in \mathcal{P}$ *and all* $\alpha, \beta \geq 0$ *and*

(*b*) $u, -u \in \mathcal{P}$ *implies* $u = 0$.

Definition 1.1.6 *Let* X *be any space and* f *a map of* X, *or of a subset of* X, *into* X.

A point $x \in X$ *is called a fixed point for* f *if* $x = f(x)$. *The set of all fixed points of* f *is denoted by* $Fix\ f$.

Exemple 1.1.1 *if* $\mathcal{E}$ *is a Banach space and* $K_\varrho = \{x \in \mathcal{E} \mid \|x\| \leq \varrho\}$*is the closed ball in* $\mathcal{E}$ *with center* 0 *and radius* ϱ, *then* $r : \mathcal{E} \to K_\varrho$ *given by*

$$r(y) = \begin{cases} y & \text{for } \|y\| \leq \varrho, \\ \\ \varrho y / \|y\| & \text{for } \|y\| > \varrho, \end{cases}$$

$Fix\ r = K_\varrho$

Let X be a retract normed and U open in X. Recall that by ,$\mathcal{K}\left(\overline{U}, X\right)$ we denote the set of all compact maps from $\overline{U}$ to X, and by $\mathcal{K}_{\partial U}\left(\overline{U}, X\right)$ the set of all maps $f \in \mathcal{K}\left(\overline{U}, X\right)$ that have no fixed points on ∂U. Observing that for $f \in \mathcal{K}_{\partial u}\left(\overline{U}, X\right)$ the map $f_{|U}$ is in $\mathcal{F}\left(\overline{U}, X\right)$.

Definition 1.1.7 *Let* X *be a Banach space,* U *open in* X, *and* $f \in \mathcal{K}_{\partial u}\left(\overline{U}, X\right)$. *The fixed point index* $i(f, U)$ *of* f *is given by*

$$i(f, U) = i\left(f_{|U}, U\right).$$

The properties of the index for compact maps in $\mathcal{F}\left(\overline{U}, X\right)$ immediately translate to properties of the index for maps in $\mathcal{K}_{\partial u}\left(\overline{U}, X\right)$, and we obtain the following basic theorem:

Theorem 1.1.8 *Let X be a Banach space, $U \subset X$ an arbitrary open subset, and $\mathcal{K}_{\partial u}\left(\overline{U}, X\right)$ the set of all compact admissible maps from $\overline{U}$ to X. Then there exists an integer-valued fixed point index function $f \to i\left(f\right)$ for $f \in \mathcal{K}_{\partial u}\left(\overline{U}, X\right)$ with the following properties:*

(I)(Normalization) If $f \in \mathcal{K}_{\partial u}\left(\overline{U}, X\right)$ is a constant map $u \to u_0$. then $i\left(f, U\right) = 1 or\ 0$ depending on whether or not $u_o \in U$.

(II) (Additivity) If $f \in \mathcal{K}_{\partial u}\left(\overline{U}, X\right)$ and Fix $(f) \subset U_l \cup U_2 \subset U$ with U_l, U_2 open and disjoint, then

$$i\left(f, U\right) = i\left(f, U_1\right) + i\left(f, U_2\right).$$

(III) (Homotopy) If $h_t : \overline{U} \to X$ is an admissible compact homotopy in $\mathcal{K}_{\partial u}\left(\overline{U}, X\right)$, then

$$i(h_0, U) = i(h_1, U).$$

(IV) (Existence) If $\ddot{\imath}\left(f, U\right) \neq 0$, then

$$Fix\left(f\right) \neq \emptyset.$$

(V) (Excision) If V is an open subset of U and if $f \in \mathcal{K}_{\partial u}\left(\overline{U}, X\right)$ has no fixed points in $U - V$, then

$$i\left(f, U\right) = i\left(f, V\right)$$

(VI) (Multiplicativity) If $f_1 \in \mathcal{K}_{\partial u}\left(\overline{U_1}, X_1\right)$ and $f_2 \in \mathcal{K}_{\partial u_2}\left(\overline{U_2}, X_2\right)$, then $f_1 \times f_2 \in \mathcal{K}_{\partial u_1}\left(\overline{U_1} \times \overline{U_2}, X_1 \times X_2\right)$ and

$$i\left(f_1 \times f_2, U_1 \times U_2\right) = i\left(f_1, U_1\right) \cdot i\left(f_2, U_2\right)$$

(VII) (Commutativity) Let $X, X^{'}$ be a Banach space, let $U \subset X$, $U^{'} \subset X^{'}$ be open, and let $f : \overline{U} \to X^{'}, g : \overline{U}^{'} \to X$ be continuous maps, at least one of them being compact. Define $V = U \cap f^{-1}\left(U^{'}\right)$ and $V^{'} = U^{'} \cap g^{-1}\left(U\right)$. then:

(i) the maps

$$gf : \overline{V} \to X, \quad fg : \overline{V}^{'} \to X^{'}$$

are compact,

(ii) if $Fix\,(gf) \subset V$ *and* $Fix\,(fg) \subset V'$*, then*

$$i(gf,V) = i(fg,V').$$

We list further frequently used properties of the index:

$(VIII)$ (Contraction) Let (X,A) be a pair of Banach space with A closed in $X, U \subset X$ open, and $f \in \mathcal{K}_{\partial u}\left(\overline{U},X\right)$ with $f\left(\overline{U}\right) \subset A$.

Let $\hat{f} = f_{\overline{u \cap A}} : \overline{U \cap A} \to A$ be the contraction of f. Then

$$\hat{f} \in \mathcal{K}_{\partial(u\cap A)}(\overline{U \cap A}, A) \text{ and } i(f,U) = i(\hat{f}, U \cap A).$$

(IX) (Localization) Let $f \in \mathcal{K}_{\partial u}\left(\overline{U},X\right)$ and $Fix\,(f) \subset U_1 \cup U_2 \subset U$ with U_1, U_2 open and disjoint.

Suppose $i\,(f,U) \neq 0$ and $i\,(f,U_1) = 0$. Then

$$Fix\left(f_{|U_2}\right) \neq \emptyset.$$

(X)(Multiplicity) Let $f \in \mathcal{K}_{\partial u}\left(\overline{U},X\right)$ and $Fix\,(f) \subset U_1 \cup U_2 \subset U$ with U_1, U_2 open and disjoint.

Suppose $i\,(f,U) = 0$ and $i\,(f,U_1) \neq 0$. Then

$$Fix\left(f_{|U_1}\right) \neq \emptyset \text{ and } Fix\left(f_{|U_2}\right) \neq \emptyset.$$

(the proof of theorem can be found in [10]).

Definition 1.1.8 *Let $\mathcal{E}$ be a Banach space. C a convex (not necessarily closed) subset of $\mathcal{E}$, and $U \subset C$ open with $0 \in U$. As before, we denote by $\mathcal{K}_{\partial u}\left(\overline{U},C\right)$ the set of compact maps $f : U \to C$ that are fixed point free on ∂U.*

Definition 1.1.9 *If $\mathcal{P} \subset \mathcal{E}$ is a cone and $\varrho > 0$, we let*

$$B_\varrho = \{x \in \mathcal{P} / \left\|x\right\| < \varrho\}, \qquad S_\varrho = \{x \in \mathcal{P} / \left\|x\right\| = \varrho\}, \qquad K_\varrho = S_\varrho \cup B_\varrho.$$

Lemma 1.1.3 *Let $\mathcal{E}$ be a Banach space and $|\ \ |$ be any norm in $\mathcal{E}$. let $f \in \mathcal{K}_{\partial u}\left(\overline{U},C\right)$. and assurne that one of the following conditions holds for all $x \in \partial U$:*

(i) $\mid f\,(x) \mid \leq \mid x \mid$,

(ii) $| f(x) | \leq | x - f(x) |$,

(iii) $| f(x) |^2 \leq | x |^2 + | x - f(x) |^2$,

(iv) $\langle x, f(x) \rangle \leq (x, x)$. *where* $\langle \cdot \rangle$ *is a scalar product in* E.

Then

$$i(f, U) = 1$$

(the proof of lemma can be found in [10]).

Lemma 1.1.4 *Let* $f \in \mathcal{K}_{S_\varrho}(K_\varrho, P)$ *satisfy* $\|f(x)\| \geq \|x\|$*for all* $x \in S_\varrho$. *Then:*

(a) $i(f, B_\varrho) = 0$,

(b) f *has an eigenvector on* S_ϱ *with characteristic value* $\mu \in (0, 1)$.

(the proof of lemma can be found in [10]).

Theorem 1.1.9 *(Krasnosel'skii). Let* $\mathcal{E}$ *be a Banach space,* $\mathcal{P} \subset \mathcal{E}$ *a proper cone, and assume that* $f : \mathcal{P} \to \mathcal{P}$ *is a completely continuous map such that for some numbers rand* R *with* $0 < r < R$, *one of the following conditions is satisfied:*

(a) $\|f(x)\| \leq \|x\|$ *for* $x \in S_r$, *and* $\|f(x)\| \geq \|x\|$ *for* $x \in S_R$.

(b) $\|f(x)\| \geq \|x\|$ *for* $x \in S_r$, *and* $\|f(x)\| \leq \|x\|$ *for* $x \in S_R$.

Then f *has a fixed point* x *with* $r \leq \|x\| \leq R$.

Proof : We may assume that f has no fixed points on S_r and S_R. We make the following observations:

(i) if *(a)* holds, then $i(f, B_R) = 0$ by (Lemma 1.1.4) and $i(f, B_r) = 1$ by (Lemma 1.1.3),

(ii) if *(b)* holds, then $i(f, B_R) = 1$ by (Lemma 1.1.3) and $i(f, B_r) = 0$ by (Lemma 1.1.4).

1.2 Lebesgue and Arzelà–Ascoli Theorems

1.2.1 The Lebesgue Dominated Convergence Theorem

Definition 1.2.1 *Let*

$$F = \{f_i : X \to Y, i \in I\}$$

be a family of functions indexed by I, where X is an arbitrary set and Y is the metric space with metric d.

We call F uniformly bounded if there exists an element a from Y and a real number M such that

$$d\left(f_i\left(x\right), a\right) \leq M \quad \forall i \in I,\ \forall x \in X.$$

Theorem 1.2.1 *If $\{f_n\}_{n\geq 0}$ is a sequence of measurable functions and $f_n \to f$ a.e, on a set S and there is an integrable*

function g on S such that $|f_n| < g$ for all n, then f is integrable and $\int\limits_S f_n dx \to \int\limits_S f dx$

Proof : Notice that all fn are integrable since $|f_n| < g$ and g is integrable.

Since $f_n \to f$, f is measurable, and f is integrable since $|f| < g$.

Let $\epsilon > 0$ and let T be a finite measure subset of S such that g is bounded on T and $\int\limits_{S-T} g dx < \epsilon$.

Clearly $\{f_n\}$ is uniformly bounded (by a bound for g) on T,

so

$$\int\limits_T f_n dx \to \int\limits_T f dx$$

by the bounded convergence theorem. Since $|f_n| < g$ and $|f| < g$,

$$\left|\int\limits_{S-T} f_n dx\right| < \epsilon \ , \quad \left|\int\limits_{S-T} f dx\right| < \epsilon$$

Therefore, if we choose N so that $\left|\int\limits_T f_n dx - \int\limits_T f dx\right| < \epsilon$ for $n > N$,

then for $n > N$

$$\left|\int\limits_S f_n dx - \int\limits_S f dx\right| \leq \left|\int\limits_T f_n dx - \int\limits_T f dx\right| + \left|\int\limits_{S-T} f_n dx - \int\limits_{S-T} f dx\right| < 3\epsilon$$

The dominated convergence theorem says in essence that pointwise convergence of $\{f_n\}$ implies convergence of the integrals, provided only that the graphs of the f_n all lie in some fixed finite area.

If the f_n are positive and their graphs are not all in some fixed finite area, then the integrals $\int f_n dx$ may get too big, but they are always at

least as big as $\int f dx$ in the limit.

1.2.2 Arzelà–Ascoli Theorem

Definition 1.2.2 *Consider a subset $A \subset C[0,1]$. We call A equicontinuous if,*

$$\forall \epsilon > 0, \quad \exists\, \delta > 0$$

such that,

$$\forall f \in A, \forall x, y \in [0,1], |x-y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

This is very similar to the definition for continuity of a specific function f in A.

The only difference is that, for a given ϵ, we must choose $\delta > 0$ which works for all possible functions f in A. That is, we must choose the before we are allowed to look at the function, making this a property of a set rather than a function, whereas in proving continuity, we choose our δ for a specific function.

Remark 1.2.1 *Since we are trying to show that equi continuity is the additional property which compact sets in $C[0,1]$ contain, our example of a closed, bounded but not compact subset of $C[0,1]$, should fail to be equi continuous.*

We will prove that the unit ball in $C[0,1]$ is not equi continous.

Recall that the unit ball contained the sequence $\{x^n\}_{n\geq 1}$.

Theorem 1.2.2 *For $A \subset C[0,1]$, A is compact if and only if A is closed, bounded, and equicontinuous.*

Proof : A compact $\implies$ A closed, bounded, and equicontinuous.

We have already shown, that A must be closed and bounded. It remains to show that A is equicontinuous.

To do so, we assume first that A is compact but not equicontinuous.

Since it is not equicontinuous, we note that $\exists \epsilon > 0$ such that $\forall \delta \exists x, y \in [0,1]$ and $f \in A$ such that

$$|x-y| < \delta,$$

but

$$|f(x) - f(y)| > \epsilon.$$

In particular, we can create a chain of δ's, which we label by $\delta_n = \frac{1}{n}$, such that $\exists x_n, y_{\ n} \in [0,1]$ and $f_n \in A$

such that

$$|x_n, y_n| < \delta_n = \frac{1}{n},$$

but

$$|f_n(x_n), f_n(y_n)| > \epsilon.$$

This of course defines at least one sequence of functions in A.

We choose one sequence of functions with this property, and we note that by the above this sequence cannot possibly be equicontinuous.

Also, all subsequences$f_{n(k)}$ of this sequence would clearly also have the property that for the $n(k)$'th function in the sequence, $\exists x_{n(k)}, y_{\ n(k)} \in [0,1]$ such that

$$\left|x_{n(k)}, y_{n(k)}\right| < \delta$$

but

$$\left|f_{n(k)}\left(x_{n(k)}\right), f_{n(k)}\left(y_{n(k)}\right)\right| > \epsilon,$$

by the above. And so,it is also true that no subsequence can be equicontinuous.

However, we have shown already that all convergent sequences must in fact be equicontinuous.

And so, under the assumption that A is not equicontinuous, we have demonstrated the existence of a sequence in A with no subsequence that converges.

This is a contradiction with the assumption that A is compact, and so we conclude that A must be equicontinuous.

A closed, bounded, and equicontinuous $\implies$ A compact.

We begin by considering an arbitrary sequence $\{f_n\}_{n\geq 1}$ in A.

We must show that it contains a convergent subsequence. Unfortunately, it is not very clear how to do this.

Intuitively, we may look at the interval $[0,1]$, and find a subsequence which converges pointwise at one point, x_0. We could then find a subsequence of that subsequence which converged at a second point, x_1, and so on.

This would work if $[0, 1]$ had only finitely many points. Unfortunately, the interval has uncountably many points, and so this strategy must be modified.

The first modification is to use a diagonal argument, familiar from previous arguments in analysis, to extend convergence of a subsequence from a finite number of points to a countable set of points. We will then use the equicontinuity property to extend convergence at a well-chosen countable set of points to uniform convergence over the entire interval $[0, 1]$. For now, we continue the proof.

We let $x_1, x_2, .., x_n, ..$ be an enumeration of the rational points of $[0, 1]$.

This is possible since, as we have shown, the rationals are a countable set.

We note that $\{f_n\}_{n\geq 1}$, evaluated at x1, forms an infinite sequence of real numbers.

Since A is closed and bounded, each $\{f_n\}$ must also be bounded, and so our sequence of real numbers, $\{f_n(x_1)\}$, is also bounded.

By the Bolzano-Weierstrass Theorem, then, there exists a subsequence of our sequence of real numbers which converges. This is equivalent to stating that there is a subsequence of $\{f_n\}_{n\geq 1}$ which converges pointwise at x_1.

For notational convenience, we label this sequence $f_{n_1(k)}$, where $n_1(k)$ is a strictly increasing function from the positive integers to the positive integers.

With exactly the same argument, we can create a subsequence of that subsequence which converges at x_2, which we label $f_{n_2(k)}$. Since $f_{n_1(k)}$ converges at x_1 and $f_{n_2(k)}$ is a subsequence of $f_{n_1(k)}$fn1(k), $f_{n_2(k)}$ must also converge at x_1.

We can continue this chain of subsequences, and so obtain a sequence, for each positive integer m, a subsequence $f_{n_m(k)}$ which converges at the rational points $x_1, x_2, .., x_m$, created in such a way that $f_{n_m(k)}$ is a subsequence of $f_{n_{(m-1)}(k)}$.

Thus, for any particular finite number of rational points in $[0, 1]$, we can find a subsequence which converges at those rational points.

As pointed out earlier, this will not be enough to find a subsequence which converges on the entire interval. However, we have not yet used the hypothesis of equicontinuity.

Before doing that, we define a sequence $\{g_n\}_{n\geq 1}$, by making the n'th function in the sequence equal to the n'th function in the sequence $f_{n_m(k)}$.

That is, the n'th function in g is equal to the n'th function of the n'th subsequence of the f_n's.

We note that, for any given rational point x_i, g_n is a subsequence of $f_{n_i(k)}$ for all $n > i$, and so g_n converges at x_i. Thus, this sequence in fact converges at every single rational point on $[0, 1]$. Since the elements of $\{g_n\}$ are all taken from subsequences of $\{f_n\}$, we note that it is also a subsequence of $\{f_n\}$. At this point, it remains to show that g_n converges everywhere on $[0, 1]$, and also that it converges uniformly.

First, we will show that it is a Cauchy sequence. We consider an arbitrary $x \in [0, 1]$. We note immediately that, by the triangle inequality,:

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x) - g_m(x_i)|$$

for any point x_i in $[0, 1]$.

Here, for the first time, we use equicontinuity.

We can choose a such that $|x - x_i| < \delta$ implies both that

$$|g_n(x) - g_n(x_i)| < \frac{\epsilon}{3}$$

and that

$$|g_m(x) - g_m(x_i)| < \frac{\epsilon}{3}.$$

This δ, we recall, is completely independent of m and n, and it is also completely independent of x, x_i.

We note that the rational points are dense in the reals, and so we choose now x_i to be a rational point satisfying $|x - x_i| < \delta$.

As for the middle term, g_n converges at x_i, and so g_n evaluated at x_i forms a Cauchy sequence, after we have already chosen x_i.

Thus, $\exists N > 0$ such that $m, n > N$ forces the middle term to be less than $\frac{\epsilon}{3}$. And so, we have shown that $g_n(x)$ is itself a Cauchy sequence, that is, it converges pointwise everywhere on $[0, 1]$.

We need to show now that this convergence is uniform.

That is, the convergence is essentially independent of x.

The proof above of pointwise convergence depended on x.

Fortunately, it can actually be modified to prove not only pointwise convergence, but uniform convergence. Once again to start, we also $\epsilon > 0$ to be given. Since A is equicontinuous, we can choose a δ independent of n and x such that

$$|x - x_i| < \delta$$

implies that

$$|g_n(x) - g_n(x_i)| < \frac{\epsilon}{3} \text{ for all } n.$$

Now, we partition the interval into intervals of length $\frac{\delta}{2}$. We can now choose exactly one rational point in each such interval. We are now looking at a finite number of rational points, only. Since g converges at each rational point, for each rational point x_j which we are looking at,

there exists an N_j such that $m, n > N_j$ implies that

$$|g_m(x_j) - g_n(x_j)| < \frac{\epsilon}{3}.$$

Now, we define N to be the maximum over all the N_j 's, which exists, since we are taking the maximum over a finite set.

Having done this preparatory work, we will now show that in fact the convergence is uniform.

For ϵ, δ, N and the set of rational points x_j with their associated partition as above, we continue.

Note that for any $x \in [0, 1]$, we can choose one of our special rational points, x_j , that is within δ of x. Choosing this rational point, and forcing m, n to be strictly greater than N, we obtain:

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x) - g_m(x_i)|$$

but

$$|g_n(x) - g_n(x_j)| < \frac{\epsilon}{3}, |g_m(x) - g_m(x_j)| < \frac{\epsilon}{3}$$

since

$$|x - xj| < \delta.$$

We also see that

$$\forall x_j \in [0,1]\,, |g_m(x_j) - g_m(x_j)| < \frac{\epsilon}{3},$$

since we have already imposed the restriction $m, n > N > N_j$.

And so

$$|g_n(x) - g_m(x)| < \epsilon,$$

as we wished to show. Since A is closed and this sequence converges, it must of course converge to a function in A.

Thus, from the assumptions that A is closed, bounded and equicontinuous, we have demonstrated for a general sequence the existence of a convergent subsequence with limit point in A.

Chapter 2

Leggett-Williams Fixed Point Theorem

Summary

2.1 Preliminaries

Definition 2.1.1 *A Banach space $\mathcal{E}$ is a partially ordered Banach space if there exists a partial ordering $\leq$ on $\mathcal{E}$ satisfying*

(a) $u \leq v$, for $u, v \in \mathcal{E}$ implies $tu \leq tv$, for all $t \geq 0$,

(b) $u_1 \leq v_1$ and $u_2 \leq v_2$, for $u_1, u_2, v_1, v_2 \in \mathcal{E}$, $u_1 + u_2 \leq v_1 + v_2$.

Let $\mathcal{P} \subset \mathcal{E}$ be a cone and define $u \leq v$ if and only if $v - u \in \mathcal{P}$. Then $\leq$ is a partial ordering on $\mathcal{E}$ and we will say that $\leq$ is the partial ordering induced by $\mathcal{P}$. Moreover, $\mathcal{E}$ is a partially ordered Banach space with respect to $\leq$. We also state the following definitions for future reference.

Definition 2.1.2 *The map α is a nonnegative continuous concave functional on $\mathcal{P}$ provided $\alpha : \mathcal{P} \longrightarrow [0, \infty)$ is continuous and*

$$\alpha\left(tx + (1-t)\, y\right) \geq t\alpha\left(x\right) + (1-t)\, \alpha\left(y\right) \text{ for all } x, y \in \mathcal{P} \text{ and } 0 \leq t \leq 1.$$

Similarly we say the map β is a nonnegative continuous convex functional on $\mathcal{P}$ provided $\beta : \mathcal{P} \longrightarrow [0, \infty)$ is continuous and

$$\beta\left(tx + (1-t)\, y\right) \leq t\beta\left(x\right) + (1-t)\, \beta\left(y\right) \text{ for all } x, y \in \mathcal{P} \text{ and } 0 \leq t \leq 1.$$

Definition 2.1.3 *An operator A is completely continuous if A is continuous and compact, i.e. A maps bounded sets into precompact sets.*

Let γ, β, θ be nonnegative continuous convex functionals on $\mathcal{P}$ and α, ψ be nonnegative continuous concave functionals on $\mathcal{P}$.

Then for nonnegative numbers h, a, b, d, and c, we define the following convex sets:

$$\begin{aligned}
P\left(\gamma, c\right) &= \left\{x \in \mathcal{P} \mid \gamma\left(x\right) < c\right\}, \\
P\left(\gamma, \alpha, a, c\right) &= \left\{x \in \mathcal{P} \mid a \leq \alpha\left(x\right), \gamma\left(x\right) \leq c\right\}, \\
Q\left(\gamma, \beta, d, c\right) &= \left\{x \in \mathcal{P} \mid \beta\left(x\right) \leq d, \gamma\left(x\right) \leq c\right\}, \\
P\left(\gamma, \theta, \alpha, a, b, c\right) &= \left\{x \in \mathcal{P} \mid a \leq \alpha\left(x\right), \theta\left(x\right) \leq b, \gamma\left(x\right) \leq c\right\}, \\
Q\left(\gamma, \beta, \psi, h, d, c\right) &= \left\{x \in \mathcal{P} \mid h \leq \psi\left(x\right), \beta\left(x\right) \leq d, \gamma\left(x\right) \leq c\right\}, \\
S\left(\alpha, a, b\right) &= \left\{x \in \mathcal{P} : a \leq \alpha\left(x\right) \quad \text{and} \quad \|x\| \leq b\right\}, \\
P_c &= \left\{x \in \mathcal{P} : \|x\| \leq c\right\}, 0 < c < \infty,
\end{aligned}$$

and

$$P_\infty = \mathcal{P}.$$

2.2 Leggett-Williams Theorem

Let the map $A : P_c \to \mathcal{P}$ satisfying the following property:

$(*)$ A has a continuous extension $A_1 : \mathcal{P} \to \mathcal{P}$ such that range A_1 =range A and A_1 has no fixed points in $\mathcal{P} \setminus P_c$.

Theorem 2.2.1 *suppose $A : P_c \to P_c$ is completely continuous, and suppose there exist a concave positive functional α with $\alpha(x) \leq \|x\|$ $(x \in \mathcal{P})$ and numbers a, b and d, with $0 < d < a < b \leq c$, satisfying the following conditions:*

(1) $\{x \in S(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Ax) > a$ *if* $x \in S(\alpha, a, b)$,

(2) $\|Ax\| < d$ *if* $x \in P_d$,

(3) $\alpha(Ax) > a$ *if* $x \in P_c$ *and* $\|Ax\| > b$.

Then A has at least three fixed points in P_c .

(the proof of theorem can be found in [17]).

Theorem 2.2.2 *Let $A : P_c \to \mathcal{P}$ be a completely continuous operator satisfying property $(*)$ suppose there exist a concave positive functional α with*

$$\alpha(x) \leq \|x\| \, (x \in \mathcal{P})$$

and numbers a, b and d, with

$$0 < d < a < b \leq c,$$

satisfying the following conditions :

(1) $\{x \in S(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ *and* $\alpha(Ax) > a$ *if* $x \in S(\alpha, a, b)$,

(2) $\|Ax\| < d$ *if* $x \in P_d$,

(3) $\alpha(Ax) > a$ *if* $x \in P_c$ *and* $\|Ax\| > b$.

Then A has at least three fixed points in P_c .

Proof : Let A_1 be the extension of A described in property $(*)$ and choose $r \geq c$ such that $A_1(P_r) \subset P_r$.

Note that conditions (1) and (2) of Theorem (2.2.1) hold for A_1.

If

$$x \in S(\alpha, a, b) \text{ and } \|A_1 x\| > b,$$

then

$$A_1 x = Ay$$

for some $y \in P_c$ and $\alpha(Ax) = \alpha(Ay) > a$ (since $\|Ay\| > b$).

Hence condition (3) of Theorem (2.2.1) is satisfied for P_r and A_1 , and A_1 has at least three fixed points in P_r .

Since A_1 has no fixed points in $\mathcal{P} \setminus P_c$, these fixed points lie in P_c and therefore are fixed points of A .

It is possible to obtain two fixed points of A even if A dose not satisfy property $(*)$.

2.3 A generalization of the Leggett-Williams Theorem

Definition 2.3.1 *Let E be a real Banach space, a subset K of E is said to be a retract of E if ther exists a continuous map $Y : E \to K$ such that $Yx = x$ for all $x \in K$.*

Remark 2.3.1 *Any closed convex subset C of E is a retract of E. That is, there is a continuous mapping $R : E \to C$ such that $Rx = x$ for each $x \in C$.*

Theorem 2.3.1 *Let X be a retract of a real Banach space $\mathcal{E}$.*

Then, for every bounded relatively open subset U of X and every completely continuous operator $A : \overline{U} \to X$ which has no fixed points on ∂U (relative to X), there exists an integer $i(A, U, X)$ satisfying the following conditions:

($\mathcal{G}1$) *Normality:*

$$i(A, U, X) = 1 \text{ if } Ax \equiv y_O \in U \text{ for any } x \in \overline{U},$$

($\mathcal{G}2$) *Additivity:*

$$i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$$

whenever U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $\overline{U} - (U_1 \cup U_2)$,

($\mathcal{G}3$) *Homotopy Invariance: $i(H(t, .), U, X)$ is independent of $t \in [0, 1]$ whenever $H : [0, l] \times \overline{U} \to X$ is completely continuous and $H(t, x) \neq x$ for any $(t, x) \in [0, 1] \times \partial U$,*

($\mathcal{G}4$)*Permanence:*

$$i(A, U, X) = i(A, U \cap Y, Y)$$

if Y is a retract of X and $A(\overline{U}) \subset Y$,

($\mathcal{G}5$) *Excision:*

$$i(A, U, X) = i(A, U_0, X)$$

whenever U_0 is an open subset of U such that A has no fixed points in $\overline{U} - U_0$,

($\mathcal{G}6$) *Solution: if*

$$i(A, U, X) \neq 0$$

then A has at least one fixed point in U.

Moreover, $i(A,U,X)$ is uniquely defined.

(the proof of theorem can be found in [8] & [11]).

Theorem 2.3.2 *Let X be a retract of the real Banach space $\mathcal{E}$ and X_1 be a bounded convex retract of X. Let U be a nonempty relatively open subset of X with $U \subset X_1$.*

Suppose that $A : X_1 \to X$ is completely continuous, $A(X_1) \subset X_1$ and A has no fixed points on $X_1 - U$. Then $i(A,U,X) = 1$.

(the proof of theorem can be found in [11]).

corollary 2.3.1 *Let X be a nonempty bounded closed convex set of the real Banach space E.*

Suppose that $A : X \to X$ is completely continuous.

Then

$$i(A,X,X) = 1.$$

The following is a generalization of the Leggett-Williams Fixed Point Theorem.

Theorem 2.3.3 *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{E}$; c, M be positive numbers; α, ψ be nonnegative continuous concave functionals on $\mathcal{P}$ and γ, β, θ be nonnegative continuous convex functionals on P with*

$$\alpha(x) \leq \beta(x) \quad \textit{and} \quad \|x\| \leq M\gamma(x)$$

for aII $x \in \overline{P(\gamma, c)}$. Suppose

$$A : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$$

is completely continuous and there exists nonnegative numbers h, d, a, b with $0 < d < a$ such that:

$$\{P(\gamma,\theta,\alpha,a,b,c) : \alpha(x) > a\} \neq \emptyset \textit{ and } \alpha(Ax) > a \textit{ for } x \in P(\gamma,\theta,\alpha,a,b,c), \tag{$\mathcal{C}1$}$$

$$\{Q(\gamma,\beta,\psi,h,d,c) : \beta(x) < d\} \neq \emptyset \textit{ and } \beta(Ax) < d \textit{ for } x \in Q(\gamma,\beta,\psi,h,d,c), \tag{$\mathcal{C}2$}$$

$$\alpha(Ax) > a \text{ for } x \in P(\gamma, \alpha, a, c) \text{ with } \theta(Ax) > b, \tag{$\mathcal{C}3$}$$

$$\beta(Ax) < d \text{ for } x \in Q(\gamma, \beta, h, d, c) \text{ with } \psi(Ax) < h. \tag{$\mathcal{C}4$}$$

Then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that;

$$\beta(x_1) < d,$$

$$a < \alpha(x_2)$$

and

$$d < \beta(x_3) \text{ with } \alpha(x_3) < a.$$

Proof : Let

$$X = \overline{P(\gamma, c)},$$

$$U = \{x \in P(\gamma, \alpha, a, c) : \alpha(x) > a\}$$

and

$$V = \{x \in Q(\gamma, \beta, d, c) : \beta(x) < d\}.$$

Thus U and V are two nonempty bounded (since $\|x\| \leq M\gamma(x) \leq Mc$ for all $x \in X$) open convex sets relative to X. Also if $x \in V$ then $\beta(x) < d$ hence $\alpha(x) < d$ since $\alpha(x) \leq \beta(x)$ for all $x \in X$.

Therefore U and V are disjoint.

Choose $z_1 \in P(\gamma, \theta, \alpha, a, b, c)$ with $\alpha(z_1) > a$. Let

$$H_1(t, x) = tz_1 + (1-t)Ax,$$

thus $H_1 : [0,1] \times \overline{U} \to X$ is completely cotinuous.

Claim 1: $H_1(t, x) \neq x$ for all $(t, x) \in [0,1] \times \partial U$.

Suppose to the contrary, that is there exists $(t_1, x_1) \in [0,1] \times \partial U$ such that

$$H_1(t_1, x_1) = x_1 .$$

Case 1: $\theta(Ax_1) > b$.

By assumption $\alpha(Ax_1) > a$ hence

$$\alpha(x_1) = \alpha(t_1 z_1 + (1-t_1)Ax_1) \geq t_1\alpha(z_1) + (1-t_1)\alpha(Ax_1) > a$$

which is a contradiction since $x_1 \in \partial U$ implies that $\alpha(x_1) = a$.

Case 2: $\theta(Ax_1) < b$.

We have $x_1 \in P(\gamma, \theta, \alpha, a, b, c)$ by definition since $\alpha(x_1) = a$ and

$$\theta(x_1) = \theta(t_1 z_1 + (1 - t_1) Ax_1) \leq t_1 \theta(z_1) + (1 - t_1) \theta(Ax_1) < b$$

Therefore $\alpha(Ax_1) > a$ by assumption, which is the same contradiction we found in the previous case.

Thus by the homotopy invariance and normality properties of the fixed point index we have

$$i(A, U, X) = i(z_1, U, X) = 1.$$

Choose $z_2 \in Q(\gamma, \beta, \psi, h, d, c)$ with $\beta(z_2) < d$. Let

$$H_2(t, x) = tz_2 + (1 - tAx,)$$

thus $H_2 : [0, 1] \times \overline{V} \to X$ is completely continuous.

Claim 2: $H_2(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial V$

Suppose to the contrary, that is there exists $(t_2, x_2) \in [0, 1] \times \partial V$ such that

$$H_2(t_2, x_2) = x_2 \ .$$

Case 1: $\psi(Ax_2) < h$.

By assumption $\beta(Ax_2) < d$ hence

$$\beta(x_2) = \beta(t_2 z_2 + (1 - t_2) Ax_2) \leq t_2 \beta(z_2) + (1 - t_2) \beta(Ax_2) < d$$

which is a contradiction since $x_2 \in \partial V$ implies that $\beta(Ax_2) < d$.

Case 2: $\psi(Ax_2) \geq h$.

We have $x_2 \in Q(\gamma, \beta, \psi, h, d, c)$ by definition since $\beta(x_2) = d$ and

$$\psi(x_2) = \psi(t_2 z_2 + (1 - t_2) Ax_2) \geq t_2 \psi(z_2) + (1 - t_2) \psi(Ax_2) \geq h$$

Therefore $\beta(x_2) < d$ by assumption, which is the same contradiction we found in the previous case.

Thus by the homotopy invariance and normality properties of the fixed point index we have

$$i\,(A, V, X) = i\,(z_2, V, X) = 1.$$

By Corollary (2.3.1) we have

$$i\,(A, V, X) = 1.$$

Hence by the additivity property

$$i\left(A, X - \left(\overline{V} \cup \overline{U}\right), X\right) = i\,(A, X, X) - i\,(A, V, X) - i\,(A, U, X) = -1.$$

Therefore by the solution property of the fixed point index there exists three fixed points $x_1, x_2, x_3 \in X = \overline{P\,(\gamma, c)}$ for A such that ;

$$\beta\,(x_1) < d,$$

$$a < \alpha\,(x_2)$$

and

$$d < \beta\,(x_3) \;\text{ with }\; \alpha\,(x_3) < a.$$

Chapter 3

Three Symmetric Positive Solutions for The Problem of Lidstone

Summary

3.1 Main results

The propose of this section is to prove the existence positive solutions of the fourth order lidstone boundary value problem (**BVP**):

$$y^{(4)}(t) = f\left(y(t), -y''(t)\right), \quad 0 \leq t \leq 1, \tag{BVP 1}$$

$$y(0) = y(1) = 0, \tag{BVP 2}$$

$$y''(0) = y''(1) = 0, \tag{BVP 3}$$

Where $f : \mathbb{R} \times \mathbb{R} \to [0, +\infty[$ is continuous.

We will show that imposing conditions on f yields at least three positives solutions by using the generalized Leggett-Williams fixed point theorem.

We first give a definition and two lemmas necessary to the main result.

Lemma 3.1.1 *The Green function of the problem* $Lv = -v'' = 0$ *on* $(0, 1)$ *satisfying*

$$Dv = v(0) = v(1) = 0$$

is given by

$$G(t, s) = \begin{cases} (1-s)t & , 0 \leq t \leq s \leq 1, \\ \\ (1-t)s & , 0 \leq s \leq t \leq 1. \end{cases}$$

Proof : Let $G(t, s)$, is the Green function for $-v'' = 0$, satisfying in the boundary conditions **BVP (2).**

The equation

$$-v^{''}=0$$

implies

$$v=at+b$$

we seek the Green function $G(t,s)$ of the problem

$$\begin{aligned} Lv &= 0 \text{ on } (0,1)\,, \\ Dv &= 0. \end{aligned}$$

which satisfies the following conditions:

1- G continues on $t=s$,

2- For $t\neq s$ $\qquad LG\,(t,s)=0$,

3- DG satisfies the boundary conditions,

4-The jump derivative $\lim\limits_{\epsilon\to 0}G^{'}\,(s+\epsilon,t)-G^{'}\,(s-\epsilon,t)=-1$.

The Green function for the linear operator $Lv=0$ is defined as the solution to:

$$G\,(t,s)=\delta\,(t-s)$$

if the $t\neq s$, then the general solution is

$$G(t,s)=at+b=\begin{cases} a_1t+b_1 & ,t<s, \\ \\ a_2t+b_2 & ,t>s. \end{cases}$$

Since G is continuous on $t=s$, the

$$a_1s+b_1=a_2s+b_2,\ \text{so}\ (a_2-a_1)\,s+(b_2-b_1)=0 \tag{3.1.1}$$

and since G satisfies the assumption (4), then

$$(a_2-a_1)=-1. \tag{3.1.2}$$

So from $(3.1.1),(3.1.2)$ we have:

$$\begin{cases} a_2-a_1=-1, \\ \\ b_2-b_1=s. \end{cases} \tag{3.1.3}$$

The boundary conditions $v(0) = v(1) = 0$ implies

$$G(0,s) = G(1,s) \text{ and then}$$

$$b_1 = 0 \quad \text{and} \quad b_2 = -a_2.$$

From (3.1.2) and (3.1.3), we obtain:

$$b_2 = s \quad \text{and} \quad a_1 = 1 - s.$$

Thus we have

$$G(t,s) = \begin{cases} (1-s)t & ,0 \leq t \leq s \leq 1, \\ \\ (1-t)s & ,0 \leq s \leq t \leq 1. \end{cases}$$

We will make use of various properties of G, namely:

Lemma 3.1.2

$$\int_0^1 G(t,s)\,ds = \frac{t(1-t)}{2} \quad ,0 \leq t \leq 1 \tag{3.1.4}$$

$$\int_0^{\frac{1}{r}} G\left(\frac{1}{2}, s\right) ds = \int_{1-\frac{1}{r}}^{1} G\left(\frac{1}{2}, s\right) ds = \frac{1}{4r^2} \quad ,2 < r\ , \tag{3.1.5}$$

$$\int_{\frac{1}{r}}^{\frac{1}{2}} G\left(\frac{1}{2}, s\right) ds = \int_{\frac{1}{2}}^{1-\frac{1}{r}} G\left(\frac{1}{2}, s\right) ds = \frac{r^2-4}{16r^2} \quad ,\forall\ r,\ 2 < r\ , \tag{3.1.6}$$

$$\int_{t_1}^{t_2} G(t_1,s)\,ds + \int_{1-t_2}^{1-t_1} G(t_1,s)\,ds = t_1(t_2 - t_1) \quad ,\forall\ t_1, t_2,\ 0 < t_1 < t_2 < \frac{1}{2}\ , \tag{3.1.7}$$

$$\max_{o \leq r \leq 1} \frac{G\left(\frac{1}{2}, r\right)}{G(t,r)} = \frac{1}{2t} \qquad ,0 < t_1 \leq \frac{1}{2}\ , \tag{3.1.8}$$

$$\min_{o \leq r \leq 1} \frac{G(t_1,r)}{G(t_2,r)} = \frac{t_1}{t_2} \qquad ,0 < t_1 < t_2 \leq \frac{1}{2} \quad . \tag{3.1.9}$$

Proof :

1)
$$\int_0^1 G(t,s)\,ds = \int_0^t s(1-t)\,ds + \int_t^1 t(1-s)\,ds$$
$$= \left[\frac{s^2}{2}(1-t)\right]_0^t + \left[t\left(s-\frac{s^2}{2}\right)\right]_t^1 = \frac{t(1-t)}{2} \qquad , \ \forall\ 0 \le t \le 1.$$

2) Let $r > 2$

when $t = \frac{1}{2}$ and $s < \frac{1}{r} < \frac{1}{2}$ then

$$G\left(\frac{1}{2}, s\right) = s\left(1 - \frac{1}{2}\right) = \frac{1}{2}s$$

and when $t = \frac{1}{2}$ and $r < 1 - \frac{1}{r} < s < 1$ then

$$G\left(\frac{1}{2}, s\right) = \frac{1}{2}(1-s)$$

so that clearly

$$\int_0^{\frac{1}{r}} G\left(\frac{1}{2}, s\right) ds = \int_{1-\frac{1}{r}}^1 G\left(\frac{1}{2}, s\right) ds = \frac{1}{4r^2} \qquad \text{for} \qquad 2 < r\ .$$

3) For the some reasons that for 2, we have (3.1.6) and (3.1.7).

4) We have

$$\max_{o\le r\le 1} \frac{G\left(\frac{1}{2}, r\right)}{G(t,r)} = \max_{o\le r\le 1} \begin{cases} \frac{\frac{r}{2}}{t(1-r)} & ,0 < t \le r \le \frac{1}{2} \\ \frac{\frac{r}{2}}{r(1-t)} & ,0 < r \le t \le \frac{1}{2} \\ \frac{\frac{1}{2}(1-r)}{t(1-r)} & ,0 < t \le \frac{1}{2} \le r \le 1 \end{cases}$$
$$= \max_{o\le r\le 1} \begin{cases} \frac{r}{2t(1-r)} & ,0 < t \le r \le \frac{1}{2} \\ \frac{1}{2(1-t)} & ,0 < r \le t \le \frac{1}{2} \\ \frac{1}{2t} & ,0 < t \le \frac{1}{2} \le r \le 1 \end{cases}$$

first we have $\max\limits_{o\leq r\leq 1}\frac{r}{2(1-r)}=\frac{1}{2}$
so

$$\max_{o\leq r\leq 1}\frac{r}{2t\left(1-r\right)}=\frac{1}{2t}\quad \text{for}\quad 0<t.$$

For $0<t\leq\frac{1}{2}$, $t\leq 1-t$ and then

$$\frac{1}{2\left(1-t\right)}\leq\frac{1}{2t}\quad .$$

Finaly, we find

$$\max_{o\leq r\leq 1}\frac{G\left(\frac{1}{2},r\right)}{G\left(t,r\right)}=\frac{1}{2t}\quad .$$

5)

$$\begin{aligned}\min_{o\leq r\leq 1}\frac{G\left(t_1,r\right)}{G\left(t_2,r\right)} &= \min_{o\leq r\leq 1}\begin{cases}\frac{r(1-t_1)}{r(1-t_2)} & ,0\leq r\leq t_1\leq t_2\leq\frac{1}{2}\\ \\ \frac{t_1(1-r)}{r(1-t_2)} & ,0\leq t_1\leq r\leq t_2\leq\frac{1}{2}\\ \\ \frac{t_1(1-r)}{t_2(1-r)} & ,0\leq t_1\leq t_2\leq r\leq\frac{1}{2}\end{cases}\\ &= \min_{o\leq r\leq 1}\begin{cases}\frac{1-t_1}{1-t_2} & ,0\leq r\leq t_1\leq t_2\leq\frac{1}{2}\\ \\ \frac{t_1(1-r)}{r(1-t_2)} & ,0\leq t_1\leq r\leq t_2\leq\frac{1}{2}\\ \\ \frac{t_1}{t_2} & ,0\leq t_1\leq t_2\leq r\leq\frac{1}{2}\end{cases}\end{aligned}$$

we have

$$\min_{0\leq r\leq 1}\frac{t_1}{1-t_2}\left(\frac{1-r}{r}\right)=\frac{t_1}{1-t_2}$$

and for $t_1\leq t_2\leq\frac{1}{2}$: $\frac{1}{1-t_2}\leq\frac{1}{t_2}$
thus

$$\min_{o\leq r\leq 1}\frac{G\left(t_1,r\right)}{G\left(t_2,r\right)}=\frac{t_1}{t_2}\qquad ,0<t_1<t_2\leq\frac{1}{2}\quad .$$

Let $\mathcal{E} = C[0,1]$ be endowed with the maximum norm,

$$\|v\| = \underset{0 \leq t \leq 1}{max} |v(t)|$$

and for $0 < t_3 < 1/2$ define the set $\mathcal{P} \subset \mathcal{E}$ by

$$\begin{aligned}\mathcal{P} = \{ & v \in \mathcal{E},\, v(t) = v(1-t),\, v(t) \geq 0,\, v(t) \text{ is concave for all } t \in [0,1], \\ & \text{and } \underset{t \in [t_3, 1-t_3]}{min} |v(t)| \geq 2t_3 \|v\| \}.\end{aligned}$$

Proposition 3.1.1 *$\mathcal{P}$ is a closed cone*

Proof : First we prove that $\mathcal{P}$ is a cone

Let u and $v \in \mathcal{P}$

It is obvious that all $\alpha, \beta \geq 0$

- $\alpha u + \beta v \in \mathcal{E}$
- $\alpha u + \beta v \geq 0$
- $(\alpha u + \beta v)(t) = (\alpha u + \beta v)(1-t) \qquad \forall t \in [0,1]$
- $\alpha u + \beta v$ is concave $\qquad \forall t \in [0,1]$

Now we show

$$\min_{t \in [t_3, 1-t_3]} |(\alpha u + \beta v)(t)| \geq 2t_3 \|\alpha u + \beta v\|$$

Since $u, v \in \mathcal{P}$ then

$$\begin{cases} \min\limits_{t \in [t_3, 1-t_3]} |u(t)| \geq 2t_3 \|u\| \\ \\ \min\limits_{t \in [t_3, 1-t_3]} |v(t)| \geq 2t_3 \|v\| \end{cases}$$

As $\alpha u, \beta v \geq 0$ then

$$\begin{cases} \min\limits_{t \in [t_3, 1-t_3]} |\alpha u(t)| \geq 2t_3 \|\alpha u\| & , \ \forall \alpha \geq 0 \\ \\ \min\limits_{t \in [t_3, 1-t_3]} |\beta v(t)| \geq 2t_3 \|\beta v\| & , \ \forall \beta \geq 0 \end{cases}$$

As $\alpha u+\beta v\geq 0$

$$\Longrightarrow \min_{t\in[t_3,1-t_3]}|\alpha u+\beta v|=\min_{t\in[t_3,1-t_3]}(|\alpha u|+|\beta v|)$$

$$\begin{aligned}2t_3\|\alpha u+\beta v\| &\leq 2t_3(\|\alpha u\|+\|\beta v\|)\\ &\leq 2\alpha t_3\|u\|+2t_3\beta\|v\|\\ &\leq \alpha\min_{t\in[t_3,1-t_3]}|u|+\beta\min_{t\in[t_3,1-t_3]}|v|\\ &\leq \min_{t\in[t_3,1-t_3]}(\alpha|u|+\beta|v|)\\ &\leq \min_{t\in[t_3,1-t_3]}|\alpha u+\beta v|\ .\end{aligned}$$

Now we prove that $u,-u\in\mathcal{P}$ imply $u=0$

If $u\in\mathcal{P}$ then

$$\{u\in\mathcal{E}\mid u(t)=u(1-t),u\geq 0\ ,u(t)\ \text{ is concave and } \min_{t\in[t_3,1-t_3]}|u(t)|\geq 2t_3\|u\|\}$$

If $-u\in\mathcal{P}$ then

$$\{-u\in\mathcal{E}\mid -u(t)=-u(1-t),-u\geq 0\ ,-u(t)\ \text{ is concave}$$

$$\text{and} \min_{t\in[t_3,1-t_3]}|-u(t)|\geq 2t_3\|-u\|\}$$

But $u\geq 0$ $\quad ,-u\geq 0$ then $u=0$.

From the above we conclude that $\mathcal{P}$ is a cone.

Now we prove that $\mathcal{P}$ is closed i.e $\forall\ v_n\in\mathcal{P}:\lim_{n\to+\infty}v_n=\overline{v}$ implies $\overline{v}\in\mathcal{P}$

We will prove this in four steps :

Let $v_n\in\mathcal{P}$

Then

$$\{v_n\in\mathcal{E}:v_n(t)=v_n(1-t),v_n\geq 0,v_n \text{ concave } \mathit{and} \min_{t\in[t_3,1-t_3]}|v_n(t)|\geq 2t_3\|v_n\|\}$$

- We first show that

$\overline{v}\in\mathcal{E}$:

Since $v_n(t)=v_n(1-t)$

then

$$\lim_{n\to^+\infty} v_n(t) = \lim_{n\to^+\infty} v_n(1-t)$$

which implies

$$\overline{v}(t) = \overline{v}(1-t) \quad \text{since} \quad v_n \quad \text{is continuous.}$$

Hence $\overline{v} \in \mathcal{E}$.

- $\overline{v} \geq 0$ because $v_n \geq 0 \qquad \forall t \in [0,1]$ and $\lim_{n\to^+\infty} v_n \geq 0$
- $\overline{v}$ is concave :

Indeed since v_n is concave then $\forall \lambda \in [0,1]$, $\forall t_1, t_2 \in [0,1]$

$$v_n(\lambda t_1 + (1-\lambda)t_2) \geq \lambda v_n(t_1) + (1-\lambda)v_n(t_2)$$

then

$$\lim_{n\to^+\infty} [v_n(\lambda t_1 + (1-\lambda)t_2)] \geq \lim_{n\to^+\infty} [\lambda v_n(t_1) + (1-\lambda)v_n(t_2)]$$

which implies

$$\overline{v}(\lambda t_1 + (1-\lambda)t_2) \geq \lambda \overline{v}(t_1) + (1-\lambda)\overline{v}(t_2)$$

hence $\overline{v}$ is concave.

- As v_n concave then

$$\min_{t\in[t_3,1-t_3]} |v_n(t)| = v_n(t_3) \geq 2t_3 \|v_n\|$$

then

$$\lim_{n\to^+\infty} v_n(t_3) \geq \lim_{n\to^+\infty} 2t_3 \|v_n\|$$

$$\overline{v}(t_3) \geq 2t_3 \|\overline{v}\| .$$

So

$$\min_{t\in[t_3,1-t_3]} \overline{v}(t) \geq 2t_3 \|\overline{v}\| .$$

Then $\overline{v} \in \mathcal{P}$

and therefore $\mathcal{P}$ is closed.

We define the nonnegative continuous concave functionals $\alpha, \psi$and the nonnegative continuous convex functionals β,θ,γ on $\mathcal{P}$ by

$$\gamma(v) = \max_{t\in[0,t_3]\cup[1-t_3,1]} v\left(t\right), \tag{3.1.10}$$

$$\psi(v) = \min_{t\in\left[\frac{1}{r},1-\frac{1}{r}\right]} v\left(t\right), \tag{3.1.11}$$

$$\beta\left(v\right) = \max_{t\in\left[\frac{1}{r},1-\frac{1}{r}\right]} v\left(t\right), \tag{3.1.12}$$

$$\alpha(v) = \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} v\left(t\right), \tag{3.1.13}$$

$$\theta\left(v\right) = \max_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} v\left(t\right) \tag{3.1.14}$$

where t_1, t_2, and r are nonnegative numbers such that

$$0 < t_1 < t_2 < \frac{1}{2} \text{ and } \frac{1}{r} \leq t_2 \ .$$

Lemma 3.1.3 *The functionals $\gamma, \psi, \beta, \alpha$ and θ are such that*

$$\begin{aligned}
\gamma(v) &= v\left(t_3\right), \\
\psi(v) &= v\left(\tfrac{1}{r}\right), \\
\beta\left(v\right) &= v\left(\tfrac{1}{2}\right), \\
\alpha(v) &= v\left(t_1\right), \\
\theta\left(v\right) &= v\left(t_2\right),
\end{aligned}$$

and for each $v \in \mathcal{P}$

$$\alpha(v) = v\left(t_1\right) \leq v\left(\tfrac{1}{2}\right) = \beta\left(v\right) \tag{3.1.15}$$

$$\|v\| = v\left(\tfrac{1}{2}\right) \leq \frac{v\left(t_3\right)}{2t_3} = \frac{1}{2t_3}\gamma(v) \tag{3.1.16}$$

Before starting the proof of this lemma, we give these two propositions:

Proposition 3.1.2 *There is equivalence between:*

1- f convex on interval $I \subset \mathbb{R}$.

2- $\forall (a,b,c) \in I^3$ such that $a < b < c$, we have

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(c)-f(a)}{c-a} \leq \frac{f(c)-f(b)}{c-b} \quad .$$

Proposition 3.1.3 *If a function f is convex and continue on a segment $I \subset \mathbb{R}$, then*

1- If f has a maximum point $\overline{x}$, then $\overline{x}$ an extreme point of I.

2- If this maximum is also taken in the interior to I, then f is constant.

Proof : of the Lemma (3.1.3)

It is obvious that α, ψ are nonnegative continuous concave functionals and β, θ and γ are nonnegative continuous convex functionals.

Let $v \in \mathcal{P}$, then $-v$ is convex and $v(0) = v(1)$.

So from Proposition 3.1.2 (2) if $0 < t_3 < 1$ then $v(t_3) > v(0)$

and from Proposition 3.1.3 (1) :

$$\begin{aligned} \gamma(v) &= \max_{t \in [0,t_3] \cup [1-t_3,1]} v(t) = \max_{t \in [0,t_3] \cup [1-t_3,1]} \{v(t_3), v(1-t_3)\} \\ &= v(t_3) = v(1-t_3). \end{aligned}$$

For the some reasons, we have:

$$\psi(v) = \min_{t \in \left[\frac{1}{r}, 1-\frac{1}{r}\right]} v(t) = v\left(\tfrac{1}{r}\right).$$

Since $-v$ is convex and $\frac{1}{r} < \frac{1}{2}$ then

from Proposition 3.1.2, for $\frac{1}{r} < \frac{1}{2} < 1 - \frac{1}{r}$ we can show that

$$v\left(\frac{1}{2}\right) > v\left(\frac{1}{r}\right) = v\left(1-\frac{1}{r}\right)$$

and then from Proposition 3.1.3 (2), we have

$$\max_{t \in \left[\frac{1}{r}, 1-\frac{1}{r}\right]} v(t) = v\left(\frac{\frac{1}{r}+1-\frac{1}{r}}{2}\right) = v\left(\tfrac{1}{2}\right).$$

For the equality (3.1.13), we can first see that for $t_1 < t_2 < \frac{1}{2}$, then $t_1 < t_2 < 1 - t_2$

so

$$v(t_1) < v(t_2),$$

and then

$$\begin{aligned}\alpha(v) &= \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} v(t) = \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} \{v(t_1), v(1-t_1)\} \\ &= v(t_1) = v(1-t_1).\end{aligned}$$

And we have also:

$$\theta(v) = \max_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} v(t) = v(1-t_2) = v(t_2).$$

From the above, it is obvious that

$$\alpha(v) = v(t_1) \leq v\left(\tfrac{1}{2}\right).$$

By applying Proposition 3.1.2 (2) to t_3 such that $0 < t_3 < \frac{1}{2}$, we obtain:

$$\frac{-v(t_3) + v(0)}{t_3} \leq \frac{-v\left(\frac{1}{2}\right) + v(0)}{\frac{1}{2}}$$

and so:

$$-\frac{1}{t_3} v(t_3) \leq -2v\left(\tfrac{1}{2}\right) + \left(2 - \tfrac{1}{t_3}\right) v(0) \leq -2v\left(\tfrac{1}{2}\right)$$

since $\left(2 - \frac{1}{t_3}\right) < 0$.

Then

$$\|v\| = v\left(\frac{1}{2}\right) \leq \frac{v(t_3)}{2t_3} = \frac{\gamma(v)}{2t_3} \quad \text{for} \quad o < t_3 < \tfrac{1}{2} \quad .$$

Proposition 3.1.4 *$y \in \mathcal{P}$ is a solution the* ***BVP*** **(1)**,**(2)**,**(3)** *if and only if there exists $v \in \mathcal{P}$ such that*

$$y(t) = \int_0^1 G(t,s)v(s)ds \qquad , 0 \leq t \leq 1,$$

where v is of the form

$$v(t) = \int_0^1 G(t,s) f\left(\int_0^1 G(s,\tau)v(\tau)d\tau, v(s)\right) ds \qquad , 0 \leq t \leq 1.$$

Proof : Let $v \in \mathcal{P}$, we want show that $y \in \mathcal{P}$:

$$y(t) = \int_0^1 G(t,s)v(s)ds \quad ,0 \leq t \leq 1,$$

- It is clear that y is continuous, positive and symmetric.
- $y(t)$ is concave

$$\begin{aligned} y'(t) &= -\int_0^t sv(s)ds + (1-t)tv(t)dt + \int_t^1 (1-s)v(s)ds - t(1-t)v(t)dt \\ &= -\int_0^t sv(s)ds + \int_t^1 (1-s)v(s)ds \ , \end{aligned}$$

$$\begin{aligned} y''(t) &= -tv(t) - (1-t)v(t) \\ &= -v(t), \end{aligned}$$

then

$$y''(t) \leq 0$$

As the second derivative of the $y(t)$ is negative so $y(t)$ is concave.

- $y(t_3) \geq 2t_3 y(1/2)$

$$\min_{t \in [t_3, 1-t_3]} y(t) = y(t_3)$$

G is a positive, continuous and convex function, then

$$\begin{aligned} \min_{t \in [t_3, 1-t_3]} y(t) &= \min_{t \in [t_3, 1-t_3]} \int_0^1 G(t,s) v(s) ds \\ &= \int_0^1 \min_{t \in [t_3, 1-t_3]} G(t,s) v(s) ds \\ &= \int_0^1 G(t_3, s) v(s) ds \\ &= y(t_3) \end{aligned}$$

and

$$\begin{aligned}\max_{t\in[0,1]} y(t) &= \max_{t\in[0,1]} \int_0^1 G(t,s)\, v(s)\, ds \\ &= \int_0^1 \max_{t\in[0,1]} G(t,s)\, v(s)\, ds \\ &= \int_0^1 G\left(\frac{1}{2}, s\right) v(s)\, ds \\ &= y\left(\frac{1}{2}\right).\end{aligned}$$

We have from (3.1.8)

$$\max_{o\leq s\leq 1} \frac{G\left(\frac{1}{2}, s\right)}{G(t,s)} = \frac{1}{2t} \qquad , 0 < t_1 \leq \frac{1}{2}$$

and from that

$$\frac{G\left(\frac{1}{2}, s\right)}{G(t,s)} \leq \frac{1}{2t} \quad , G(t,s) \neq 0 \ \text{ Because } \ t \neq 0.$$

So

$$2t \cdot G\left(\frac{1}{2}, s\right) \leq G(t,s) \quad , 0 < t < \frac{1}{2}$$

By taking $t = t_3 < \frac{1}{2}$

$$\Longrightarrow 2t_3 \cdot G\left(\frac{1}{2}, s\right) \leq G(t_3, s).$$

Now we multiply by the positive function f and from the integration procedure for s, we obtain

$$\int_0^1 G(t_3, s)\, v(s)\, ds \geq 2t_3 \int_0^1 G\left(\frac{1}{2}, s\right) v(s)\, ds$$

which implies

$$\min_{t\in[t_3, 1-t_3]} y(t)\, 2t_3 \geq \max_{t\in[0,1]} y(t)$$

and then

$$y(t_3) \geq 2t_3 y\left(\frac{1}{2}\right) .$$

Thus $y \in \mathcal{P}$.

Now we let $y \in \mathcal{P}$, we want show that $v \in \mathcal{P}$:

Let

$$y(t) = \int_0^t G(t,s)v(s)ds + \int_t^1 G(t,s)v(s)ds$$

then

$$\begin{aligned} y(t) &= \int_0^t s(1-t)v(s)ds + \int_t^1 t(1-s)v(s)ds \\ &= (1-t)\int_0^t sv(s)ds + t\int_t^1 (1-s)v(s)ds \ , \end{aligned}$$

$$\begin{aligned} y'(t) &= -\int_0^t sv(s)ds + (1-t)tv(t)dt + \int_t^1 (1-s)v(s)ds - t(1-t)v(t)dt \\ &= -\int_0^t sv(s)ds + \int_t^1 (1-s)v(s)ds \ , \end{aligned}$$

and

$$\begin{aligned} y''(t) &= -tv(t) - (1-t)v(t) \\ &= -v(t) \ . \end{aligned}$$

$$\begin{aligned} v(t) &= \int_0^t s(1-t)f\left(\int_0^1 G(s,\tau)v(\tau)d\tau, v(s)\right)ds + \int_t^1 t(1-s)f\left(\int_0^1 G(s,\tau)v(\tau)d\tau, v(s)\right)ds \\ &= (1-t)\int_0^t sf\left(\int_0^1 G(s,\tau)v(\tau)d\tau, v(s)\right)ds + t\int_t^1 (1-s)f\left(\int_0^1 G(s,\tau)v(\tau)d\tau, v(s)\right)ds \ , \end{aligned}$$

$$\begin{aligned}
v^{'}(t) &= -\int_0^t sf\left(\int_0^1 G(s,\tau)v(\tau)d\tau, v(s)\right)ds + (1-t)\,tf\left(\int_0^1 G(t,\tau)v(\tau)d\tau, v(t)\right) + \\
&\quad + \int_t^1 (1-s)f\left(\int_0^1 G(s,\tau)v(\tau)d\tau, v(s)\right)ds - t\,(1-t)\,f\left(\int_0^1 G(t,\tau)v(\tau)d\tau, v(t)\right) \\
&= -\int_0^t sf\left(\int_0^1 G(s,\tau)v(\tau)d\tau, v(s)\right)ds + \int_t^1 (1-s)f\left(\int_0^1 G(s,\tau)v(\tau)d\tau, v(s)\right)ds\ ,
\end{aligned}$$

and

$$\begin{aligned}
v^{''}(t) &= -tf\left(\int_0^1 G(t,\tau)v(\tau)d\tau, v(t)\right) - (1-t)\,f\left(\int_0^1 G(t,\tau)v(\tau)d\tau, v(t)\right) \\
&= -f\left(\int_0^1 G(t,\tau)v(\tau)d\tau, v(t)\right).
\end{aligned}$$

So:

$$y^{(4)}(t) = -v^{''}(t) = f\left(y(t), -y^{''}(t)\right).$$

For the boundary conditions:

$$\begin{aligned}
y(0) &= \int_0^1 G(0,s)v(s)ds \\
&= \int_0^1 0(1-s)v(s)ds = 0\ , \\
y(1) &= \int_0^1 s(1-1)v(s)ds = 0\ , \\
y^{''}(0) &= -v^{''}(0) = 0\ ,
\end{aligned}$$

and

$$y^{''}(1) = -v^{''}(1) = 0\ .$$

Thus $v \in \mathcal{P}$.

Now we define the operator A by

$$Av(t) = \int_0^1 G(t,s) f(Bv(s), v(s))\, ds \ ,$$

where

$$Bv(s) = \int_0^1 G(s,\tau)\, v(\tau)\, d\tau.$$

Lemma 3.1.4 *The operator A opers from $\mathcal{P}$ to $\mathcal{P}$.*

Proof : $A : \mathcal{P} \to \mathcal{P}$ Means that $A(\mathcal{P}) \subset \mathcal{P}$

Let $v \in \mathcal{P}$, we want show that $Av \in \mathcal{P}$

- $Av(t) = Av(1-t)$

$$\begin{aligned} Av(1-t) &= \int_0^1 G(1-t,s)\, f(Bv(s), v(s))\, ds \\ &= \int_0^1 G(t,s)\, f(Bv(s), v(s))\, ds \\ &= Av(t) \end{aligned}$$

- $Av(t) = \int_0^1 G(t,s)\, f(Bv(s), v(s))\, ds$

$$\left\{ \begin{array}{c} G(t,s) \geq 0 \\ \\ f(Bv(s), v(s)) \geq 0 \end{array} \right\} \Longrightarrow [G(t,s)\, f(Bv(s), v(s))] \geq 0$$

$$\Longrightarrow Av(t) \geq 0 \qquad , \forall v \in P$$

- $Av(t)$ is concave

$$\begin{aligned}
[Av(t)]' &= \int_0^t [s(1-t)]' f(Bv(s), v(s))\,ds + \int_t^t [t(1-s)]' f(Bv(s), v(s))\,ds \\
&= \int_0^t (-s) f(Bv(s), v(s))\,ds + \int_t^t (1-s) f(Bv(s), v(s))\,ds
\end{aligned}$$

$$\begin{aligned}
[Av(t)]'' &= (-t) f(Bv(s), v(s)) - (1-t) f(Bv(s), v(s)) \\
&= (-t-1+t) f(Bv(s), v(s)) \\
&= -f(Bv(s), v(s))
\end{aligned}$$

As the $f(Bv(s), v(s)) \geq 0$ then

$$-f(Bv(s), v(s)) \leq 0$$

As the second derivative of the $Av(t)$ is negative so $Av(t)$ is concave.

- $Av(t_3) \geq 2t_3 Av(1/2)$

$$\min_{t\in[t_3,1-t_3]} Av(t) = Av(t_3)$$

G is a positive, continuous and convex function, then

$$\begin{aligned}
\min_{t\in[t_3,1-t_3]} Av(t) &= \min_{t\in[t_3,1-t_3]} \int_0^1 G(t,s) f(Bv(s), v(s))\,ds \\
&= \int_0^1 \min_{t\in[t_3,1-t_3]} G(t,s) f(Bv(s), v(s))\,ds \\
&= \int_0^1 G(t_3,s) f(Bv(s), v(s))\,ds \\
&= Av(t_3)
\end{aligned}$$

and

$$\begin{aligned}
\max_{t\in[0,1]} Av(t) &= \max_{t\in[0,1]} \int_0^1 G(t,s)\, f(Bv(s), v(s))\, ds \\
&= \int_0^1 \max_{t\in[0,1]} G(t,s)\, f(Bv(s), v(s))\, ds \\
&= \int_0^1 G\left(\frac{1}{2}, s\right) f(Bv(s), v(s))\, ds \\
&= Av\left(\frac{1}{2}\right).
\end{aligned}$$

We have from (3.1.8)

$$\max_{o\leq s\leq 1} \frac{G\left(\frac{1}{2}, s\right)}{G(t,s)} = \frac{1}{2t} \qquad , 0 < t_1 \leq \frac{1}{2}$$

and from that

$$\frac{G\left(\frac{1}{2}, s\right)}{G(t,s)} \leq \frac{1}{2t} \quad , G(t,s) \neq 0 \quad \text{Because} \quad t \neq 0$$

so

$$2t \cdot G\left(\frac{1}{2}, s\right) \leq G(t,s) \quad , 0 < t < \frac{1}{2}$$

by taking $t = t_3 < \frac{1}{2}$

$$\Longrightarrow 2t_3 \cdot G\left(\frac{1}{2}, s\right) \leq G(t_3, s).$$

Now we multiply by the positive function f and from the integration procedure for s, we obtain

$$\int_0^1 G(t_3, s)\, f(Bv(s), v(s))\, ds \geq 2t_3 \int_0^1 G\left(\frac{1}{2}, s\right) f(Bv(s), v(s))\, ds$$

which implies that

$$\min_{t\in[t_3,1-t_3]} Av\left(t\right)2t_3 \geq \max_{t\in[0,1]} Av\left(t\right)$$

and then

$$Av\left(t_3\right) \geq 2t_3 Av\left(\frac{1}{2}\right) .$$

From the above

$$A\left(\mathcal{P}\right) \subset \mathcal{P}$$

and therefore

$$A : \mathcal{P} \to \mathcal{P} .$$

Proposition 3.1.5 *The operator A is completely continuous.*

Proof : A is completely continuous, that is A is continuous and compact

- A continuous :

Let $v_n \in \mathcal{P}$ such that $\lim\limits_{n\to^+\infty} v_n = \overline{v}$, $v_n \in \mathcal{P}$

since v_n and $\overline{v} \in \mathcal{P} \subset C\left[0,1\right]$

then there exists a constant $M_0 > 0$

such than

$$\|v_n\| \leq M_0 \quad \text{and} \quad \|\overline{v}\| \leq M_0 .$$

If $\|v\| \leq M_0$, then

$$\begin{aligned}
\left|Bv\left(s\right)\right| &= \left|\int_0^1 G\left(s,\tau\right)v\left(\tau\right)d\tau\right| \leq \int_0^1 \left|G\left(s,\tau\right)v\left(\tau\right)d\tau\right| \\
&\leq \|v\| \int_0^1 G\left(s,\tau\right)d\tau \\
&\leq M_0 \int_0^1 G\left(s,\tau\right)d\tau \\
&\leq M_0 \frac{s\left(1-s\right)}{2} \qquad ,\forall s \in [0,1] \\
&\leq \frac{M_0}{2}
\end{aligned}$$

and so Bv is bounded.

We can also see that the operator B is continuous on v .

In fact

$$\begin{aligned} |Bv_n - B\overline{v}| &= \left| \int_0^1 G(s,\tau)(v_n(\tau) - \overline{v}(\tau))\, d\tau \right| \\ &\leq \int_0^1 G(\tau,\tau)|v_n(\tau) - \overline{v}(\tau)|\, d\tau \end{aligned}$$

and so if $v_n \underset{n\to+\infty}{\to} \overline{v}$ then $Bv_n \underset{n\to+\infty}{\to} B\overline{v}$.

Let

$$M_1 = \max\{f(Bv,v),\ v \in [0,M_0]\}$$

and let

$$\rho_n = G(s,s)\, f(Bv_n(s), v_n(s)) \ ,$$

then

$$|\rho_n| \leq M_1 G(s,s) \ .$$

Since f is continuous then for each $\epsilon > 0$, there exists $\delta > 0$ such that if $|v_1 - v_2| < \delta$ then

$$|f(Bv_2(s), v_2(s)) - f(Bv_1(s), v_1(s))| < \frac{\epsilon}{G(s,s)} \ .$$

In the view of

$$v_n(s) \to \overline{v}_n(s) \quad \text{as} \quad n \to +\infty \ ,$$

there exists $N \in \mathbb{N}_+^*$ such that for $n > N$ with

$$|v_n(s) - \overline{v}_n(s)| < \delta \ ,$$

we have

$$|f(Bv_n(s), v_n(s)) - f(B\overline{v}(s), \overline{v}(s))| < \frac{\epsilon}{G(s,s)} \ .$$

Thus, for $\epsilon > 0$, there exists $N > 0$, such that when $n > N$,

$$G(s,s)\,|f(Bv_n(s), v_n(s)) - f(B\overline{v}(s), \overline{v}(s))| < \epsilon \ \text{ a . e } [0,1] \ .$$

Then for $t \in [0,1]$, we have

$$\begin{aligned} |Av_n(t) - A\overline{v}(t)| &\leq \left| \int_0^1 G(t,s) f(Bv_n(s), v_n(s)) - f(B\overline{v}(s), \overline{v}(s))\, ds \right| \\ &\leq \int_0^1 G(t,s) \left| f(Bv_n(s), v_n(s)) - f(B\overline{v}(s), \overline{v}(s)) \right| ds \end{aligned}$$

and an application of Lebesgue's dominated convergence theorem implies:

$$\left|\overline{A}v_n(t) - A\overline{v}(t)\right| \to 0 \text{ as } n \to \infty , \quad t \in [0,1]$$

so the operator $A : \mathcal{P} \to \mathcal{P}$ is continuous.

- A compact

Let $\Omega \subset C[0,1]$, Ω bounded

$$\exists\ R > 0 : \Omega \subset \{v \in C[0,1], \|v\| \leq R\}$$

Let M such that

$$M = \max\{f(Bv, v), v \in \Omega\}$$

For any $v \in \Omega$, we have

$$\begin{aligned} |Av(t)| &= \left| \int_0^1 G(t,s) f(Bv(s), v(s))\, ds \right| \\ &\leq M \int_0^1 G(t,s)\, ds \\ &\leq \frac{M}{2} \ . \end{aligned}$$

which implies that $A(\Omega)$ is uniformly bounded.

Furthermore, for any $v \in \Omega$ and $t \in [0,1]$, we have

$$\begin{aligned}
\left|(Av)'(t)\right| &= \left|\left(\int_0^1 G(t,s)\, f(Bv(s), v(s))\, ds\right)'\right| \\
&= \left|\int_0^t [s(1-t)]'\, f(Bv(s), v(s))\, ds + \int_t^1 [t(1-s)]'\, f(Bv(s), v(s))\, ds\right| \\
&= \left|\int_0^t (-s)\, f(Bv(s), v(s))\, ds + \int_t^1 (1-s)\, f(Bv(s), v(s))\, ds\right| \\
&\leq M\left|\int_0^t -s ds + \int_t^1 (1-s)\, ds\right| \\
&\leq M\left|\frac{1-2t}{2}\right| \leq \frac{3}{2}M \quad .
\end{aligned}$$

Hence

$$\left\|(Av)'\right\| \leq \frac{3}{2}(M)$$

and then $A(\Omega)$ is equicontinuous.

So, according to the theory of Arzela - Ascoli, $A(\Omega)$ is relatively compact and therefore A is compact.

In light of these preliminaries, we are now ready to present the main result of this section.

Theorem 3.1.1 *Suppose there exist* $0 < a < b < \frac{t_2}{t_1}b \leq c$ *such that* f *satisfies each of the following growth conditions:*

(G1) $f(z,w) < \left(\frac{8r^2}{r^2-4}\right)\left(a - \frac{c}{r^2 t_3(1-t_3)}\right)$ *for all* $(z,w) \in \left[\frac{a(r-2)}{r^3}, \frac{a}{8}\right] \times \left[\frac{2a}{r}, a\right]$,

(G2) $f(z,w) \geq \frac{b}{t_1(t_2-t_1)}$ *for* $(z,w) \in \left[bt_1(t_2-t_1), \frac{c\left(t_1^2+t_2(1-2t_2)\right)}{4t_3} + \frac{bt_2\left(t_2^2-t_1^2\right)}{2t_1}\right] \times \left[b, \frac{t_2}{t_1}b\right]$,

(G3) $f(z,w) \leq \frac{2c}{t_3(1-t_3)}$ *for* $(z,w) \in \left[0, \frac{c}{16t_3}\right] \times \left[0, \frac{c}{2t_3}\right]$

Then the Lidstone ***BVP*** **(1)**,**(2)**,**(3)** *has at least three symmetric positive solutions* y_1, y_2, y_3, *such that*

$$\begin{aligned}
\max_{t\in[0,t_3]\cup[1-t_3,1]} -y_i''(t) &\leq c, \quad \textit{for} \;\; i=1,2,3 \\
\min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} -y_1''(t) &> b, \\
\max_{t\in\left[\frac{1}{r},1-\frac{1}{r}\right]} -y_2''(t) &< a,
\end{aligned}$$

and

$$\min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} -y_3''(t) < b \quad \textit{with} \max_{t\in\left[\frac{1}{r},1-\frac{1}{r}\right]} -y_3''(t) > a,$$

for some $v_1,\ v_2,\ v_3 \in \mathcal{P}$ *satisfying*

$$y_i(t) = \int_0^1 G(t,s)\, v_i(s)\, ds, \quad i=1,2,3.$$

Proposition 3.1.6 $A : \overline{P(\gamma,c)} \to \overline{P(\gamma,c)}$

Proof : We have identified previously the completely continuous operator A in lemma (3.1.5) by

$$Av(t) = \int_0^1 G(t,s) f\left(Bv(s), v(s)\right) ds$$

Where

$$Bv(s) = \int_0^1 G(s,\tau)\, v(\tau)\, d\tau.$$

Also, for all $v \in \mathcal{P}$, by (3.1.15) we have $\alpha(v) \leq \beta(v)$ and by $(3.1.16)$, $\|v\| \leq \frac{\gamma(v)}{2t_3}$.
If

$$v \in \overline{P(\gamma,c)} = \{v \in \mathcal{P}/\gamma(v) \leq c\}$$

then

$$\|v\| \leq \frac{\gamma(v)}{2t_3} \leq \frac{c}{2t_3}$$

which implies that, for $s \in [0,1]$,

$$v(s) \in \left[0, \frac{c}{2t_3}\right].$$

We have $Bv(s) = \int\limits_0^1 G(s,\tau)\, v(\tau)\, d\tau$

$$\begin{aligned} |Bv(s)| &= \left| \int_0^1 G(s,\tau)\, v(\tau)\, d\tau \right| \\ &\leq \|v\| \int_0^1 G(s,\tau)\, d\tau \\ &\leq \|v\| \frac{s(1-s)}{2} \quad , \qquad \text{for } s = \frac{1}{2} \\ &\leq \|v\| \frac{1}{8} \\ &\leq \frac{c}{2t_3} \times \frac{1}{8} = \frac{c}{16t_3} \end{aligned}$$

then

$$Bv(s) \in \left[0, \frac{c}{16t_3}\right].$$

By condition (G3) we have

$$\begin{aligned} \gamma(Av) &= \max_{t \in [0,t_3] \cup [1-t_3,1]} \int_0^1 G(t,s) f(Bv(s), v(s))\, ds \\ &= \int_0^1 G(t_3,s) f(Bv(s), v(s))\, ds \\ &\leq \frac{2c}{t_3(1-t_3)} \int_0^1 G(t_3,s) ds \\ &\leq \frac{2c}{t_3(1-t_3)} \times \frac{t_3(1-t_3)}{2} \\ &= c \end{aligned}$$

Therefore, $A : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$.

Proof : (of Theoreme 3.1.1)

We seek three fixed points $v_1, v_2, v_3 \in \mathcal{P}$ of A which satisfy the conclusion of the theorem.

For this, using the assumptions G1,G2 and G3, we will verify the hypothesis of theorem (2.2.3) :

We recall that

$$P\left(\gamma,\theta,\alpha,b,\frac{t_2}{t_1}b,c\right)=\left\{v\in\mathcal{P}\mid b\leq\alpha\left(v\right),\theta\left(v\right)\leq\frac{t_2}{t_1}b,\gamma\left(v\right)\leq c\right\}$$

and

$$Q\left(\gamma,\beta,\psi,\frac{2a}{r},a,c\right)=\left\{v\in\mathcal{P}\mid\frac{2a}{r}\leq\psi\left(v\right),\beta\left(v\right)\leq a,\gamma\left(v\right)\leq c\right\}$$

so, it is immediate that the sets

$$S_1=\left\{v\in P\left(\gamma,\theta,\alpha,b,\frac{t_2}{t_1}b,c\right)\mid\alpha\left(v\right)>b\right\}\neq\emptyset,$$

$$S_2=\left\{v\in Q\left(\gamma,\beta,\psi,\frac{2a}{r},a,c\right)\mid\beta\left(v\right)<a\right\}\neq\emptyset$$

And thus the first parts of $(\mathcal{C}1)$ and $(\mathcal{C}2)$ are satisfied.

In order to show the second part of $(\mathcal{C}1)$ holds, let $v\in P\left(\gamma,\theta,\alpha,b,\frac{t_2}{t_1}b,c\right)$.

For each $s\in[t_1,t_2]\cup[1-t_2,1-t_1]$ and from (3.1.14)

$$v\left(s\right)\leq\theta\left(v\right)\leq\frac{t_2}{t_1}b$$

and from (3.1.13)

$$b\leq\alpha\left(v\right)\leq v\left(s\right).$$

We will show now that

$$bt_1\left(t_2-t_1\right)\leq Bv\left(s\right)\leq\frac{c\left(t_1^2+t_2\left(1-t_2\right)\right)}{4t_3}+\frac{bt_2\left(t_2^2-t_1^2\right)}{2t_1}$$

We have

$$\begin{aligned}
Bv\left(s\right) &= \int_0^1 G\left(s,\tau\right)v\left(\tau\right)d\tau\\
&\geq \left(\int_{t_1}^{t_2} G\left(s,\tau\right)v\left(\tau\right)d\tau+\int_{1-t_2}^{1-t_1} G\left(s,\tau\right)v\left(\tau\right)d\tau\right)\\
&\geq b\left(\int_{t_1}^{t_2} G\left(s,\tau\right)d\tau+\int_{1-t_2}^{1-t_1} G\left(s,\tau\right)d\tau\right)\\
&\geq bs\left(t_2-t_1\right)
\end{aligned}$$

by using (3.1.7)

$$\geq bt_1 \left(t_2 - t_1\right).$$

For the second part of inequality

$$\begin{aligned}
Bv\left(s\right) &= \int\limits_0^1 G\left(s,\tau\right) v\left(\tau\right) d\tau \\
&\leq \left(\int\limits_0^{t_1} G\left(s,\tau\right) v\left(\tau\right) d\tau + \int\limits_{t1}^{t_2} G\left(s,\tau\right) v\left(\tau\right) + d\tau \int\limits_{t_2}^{1-t_2} G\left(s,\tau\right) v\left(\tau\right) d\tau + \right. \\
&\quad \left. + \int\limits_{1-t_2}^{1-t_1} G\left(s,\tau\right) v\left(\tau\right) + \int\limits_{1-t_1}^{1} G\left(s,\tau\right) v\left(\tau\right) d\tau \right)
\end{aligned}$$

we know that from (3.1.7)

$$\int\limits_{t_1}^{t_2} G\left(s,\tau\right) v\left(\tau\right) d\tau + \int\limits_{1-t_2}^{1-t_1} G\left(s,\tau\right) v\left(\tau\right) d\tau \leq \frac{t_2}{t_1} b \left(\int\limits_{t_1}^{t_2} G\left(t_2,\tau\right) d\tau + \int\limits_{1-t_2}^{1-t_1} G\left(t_2,\tau\right) d\tau \right)$$

as the

$$\begin{aligned}
\int\limits_{t_1}^{t_2} G\left(t_2,\tau\right) d\tau &= \int\limits_{t_1}^{t_2} \tau \left(1 - t_2\right) d\tau \\
&= \left(1 - t_2\right) \frac{\left(t_2^2 - t_1^2\right)}{2}
\end{aligned}$$

and

$$\begin{aligned}
\int\limits_{1-t_2}^{1-t_1} G\left(t_2,\tau\right) &= \int\limits_{1-t_2}^{1-t_1} t_2 \left(1 - \tau\right) d\tau \\
&= \frac{t_2 \left(t_2^2 - t_1^2\right)}{2}
\end{aligned}$$

then

$$\int\limits_{t_1}^{t_2} G\left(s,\tau\right) v\left(\tau\right) d\tau + \int\limits_{1-t_2}^{1-t_1} G\left(s,\tau\right) v\left(\tau\right) d\tau \leq \frac{bt_2 \left(t_2^2 - t_1^2\right)}{2t_1}$$

As well

$$\int\limits_0^{t_1} G\left(s,\tau\right) v\left(\tau\right) d\tau + \int\limits_{t_2}^{1-t_2} G\left(s,\tau\right) v\left(\tau\right) d\tau + \int\limits_{1-t_1}^{1} G\left(s,\tau\right) v\left(\tau\right) d\tau$$

Here we will take the largest value for $s \in [t_1, t_2] \cup [1-t_2, 1-t_1]$, $\tau \in [0,1]$ and $v(s) \in \left[0, \frac{c}{2t_3}\right]$

Then

$$\begin{aligned}
&\int_0^{t_1} G(s,\tau)\,v(\tau)\,d\tau + \int_{t_2}^{1-t_2} G(s,\tau)\,v(\tau)\,d\tau + \int_{1-t_1}^{1} G(s,\tau)\,v(\tau)\,d\tau \\
&\leq \frac{c}{2t_3}\left(\int_0^{t_1} G(t_2,\tau)\,d\tau + \int_{t_2}^{1-t_2} G(t_2,\tau)\,d\tau + \int_{1-t_1}^{1} G(t_2,\tau)\,d\tau\right)
\end{aligned}$$

as

$$\begin{aligned}
\int_0^{t_1} G(t_2,\tau)\,d\tau &= \int_0^{t_1} \tau(1-t_2)\,d\tau \\
&= (1-t_2)\frac{t_1^2}{2},
\end{aligned}$$

$$\begin{aligned}
\int_{t_2}^{1-t_2} G(t_2,\tau)\,d\tau &= \int_{t_2}^{1-t_2} t_2(1-\tau)\,d\tau \\
&= \frac{t_2}{2} - \frac{2t_2^2}{2}
\end{aligned}$$

and

$$\begin{aligned}
\int_{1-t_1}^{1} G(t_2,\tau)\,d\tau &= \int_{1-t_1}^{1} t_2(1-\tau)\,d\tau \\
&= \frac{t_1^2 t_2}{2}
\end{aligned}$$

then

$$\int_0^{t_1} G(s,\tau)\,v(\tau)\,d\tau + \int_{t_2}^{1-t_2} G(s,\tau)\,v(\tau)\,d\tau + \int_{1-t_1}^{1} G(s,\tau)\,v(\tau)\,d\tau \leq \frac{c\left(t_1^2 + t_2(1-2t_2)\right)}{4t_3},$$

from the above

$$bt_1(t_2-t_1) \leq Bv(s) \leq \frac{c\left(t_1^2 + t_2(1-t_2)\right)}{4t_3} + \frac{bt_2\left(t_2^2 - t_1^2\right)}{2t_1}$$

and by condition (G2) and (3.1.4) we see

$$\int_{t_1}^{t_2} G(t_1,s)\,ds + \int_{1-t2}^{1-t_1} G(t_1,s)\,ds = t_1(t_2-t_1) \quad ,\forall\ 0<t_1<t_2<\frac{1}{2}$$

$$\begin{aligned}
\alpha(Av) &= \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} \int_0^1 G(t,s)\,f(Bv(s),v(s))\,ds \\
&= \int_0^1 G(t_1,s)\,f(Bv(s),v(s))\,ds \\
&> \int_{t_1}^{t_2} G(t_1,s)\,f(Bv(s),v(s))\,ds + \int_{1-t_2}^{1-t_1} G(t_1,s)\,f(Bv(s),v(s))\,ds \\
&\geq \left(\frac{b}{t_1(t_2-t_1)}\right)\int_{t_1}^{t_2} G(t_1,s)\,ds + \left(\frac{b}{t_1(t_2-t_1)}\right)\int_{1-t_2}^{1-t_1} G(t_1,s)\,ds \\
&= \left(\frac{b}{t_1(t_2-t_1)}\right)\left(\int_{t_1}^{t_2} G(t_1,s)\,ds + \int_{1-t_2}^{1-t_1} G(t_1,s)\,ds\right) \\
&= \left(\frac{b}{t_1(t_2-t_1)}\right)(t_1(t_2-t_1)) \\
&= b.
\end{aligned}$$

To verify the second part of $(\mathcal{C}2)$, let

$$v\in Q\left(\gamma,\beta,\psi,\frac{2a}{r},a,c\right),$$

this implies that for each

$$s\in\left[\frac{1}{r},1-\frac{1}{r}\right],$$

we have

$$v(s)\leq\beta(v)\leq a$$

and

$$\frac{2a}{r}\leq\psi(v)\leq v(s)$$

so that

$$\frac{2a}{r}\leq v(s)\leq a\ .$$

So, for the operator B, we have :

$$\begin{aligned}
Bv(s) &= \int_0^1 G(s,\tau)\, v(\tau)\, d\tau \\
&\leq a \int_0^1 G\left(\tfrac{1}{2},\tau\right) d\tau \\
&= a\left(\frac{\frac{1}{2}\left(1-\frac{1}{2}\right)}{2}\right) \\
&= \frac{a}{8}\ .
\end{aligned}$$

Since $v(s) \geq \frac{2a}{r}$ then

$$\begin{aligned}
Bv(s) &= \int_0^1 G(s,\tau)\, v(\tau)\, d\tau \geq \frac{2a}{r}\int_0^1 G(s,\tau)\, d\tau \\
&\geq \frac{2a}{r}\int_0^1 G\left(\tfrac{1}{r},\tau\right) d\tau \\
&= \frac{2a}{r}\left(\frac{\frac{1}{r}\left(1-\frac{1}{r}\right)}{2}\right) \\
&= \frac{a(r-1)}{r^3} \\
&\geq \frac{a(r-2)}{r^3}\ .
\end{aligned}$$

Then

$$Bv(s) \in \left[\frac{a(r-2)}{r^3}, \frac{a}{8}\right]$$

Frist, we notice that

$$\left[\frac{a(r-2)}{r^3}, \frac{a}{8}\right] \subset \left[0, \frac{c}{16t_3}\right] \quad \text{and} \quad \left[\frac{2a}{r}, a\right] \subset \left[0, \frac{c}{2t_3}\right].$$

Thus, by condition (G1) and the calculations in (3.1.2) and (3.1.3), we have

$$\begin{aligned}
\beta(Av) &= \max_{t\in\left[\frac{1}{r},1-\frac{1}{r}\right]}\int_0^1 G(t,s)\,f(Bv(s),v(s))\,ds \\
&= \int_0^1 G\left(\tfrac{1}{2},s\right) f(Bv(s),v(s))\,ds \\
&= \left(\int_0^{\frac{1}{r}} G\left(\tfrac{1}{2},s\right) f(Bv(s),v(s))\,ds + \int_{\frac{1}{r}}^{\frac{1}{2}} G\left(\tfrac{1}{2},s\right) f(Bv(s),v(s))\,ds + \right. \\
&\quad \left. + \int_{\frac{1}{2}}^{1-\frac{1}{r}} G\left(\tfrac{1}{2},s\right) f(Bv(s),v(s))\,ds + \int_{1-\frac{1}{r}}^{1} G\left(\tfrac{1}{2},s\right) f(Bv(s),v(s))\,ds\right) \\
&= 2\int_0^{\frac{1}{r}} G\left(\tfrac{1}{2},s\right) f(Bv(s),v(s))\,ds + 2\int_{\frac{1}{r}}^{\frac{1}{2}} G\left(\tfrac{1}{2},s\right) f(Bv(s),v(s))\,ds
\end{aligned}$$

by condition (G3) and the calculation in (3.1.2)

$$\begin{aligned}
2\int_0^{\frac{1}{r}} G\left(\tfrac{1}{2},s\right) f(Bv(s),v(s))\,ds &< \frac{4c}{t_3(1-t_3)}\int_0^{\frac{1}{r}} G\left(\tfrac{1}{2},s\right) ds \\
&= \frac{4c}{t_3(1-t_3)} \times \frac{1}{4r^2} \\
&= \frac{c}{r^2 t_3(1-t_3)}
\end{aligned}$$

Also by condition (G1) and the calculation in (3.1.3)

$$\begin{aligned}
2\int_{\frac{1}{r}}^{\frac{1}{2}} G\left(\tfrac{1}{2},s\right) f(Bv(s),v(s))\,ds &< 2\left(\frac{8r^2}{r^2-4}\right)\left(a-\frac{c}{r^2t_3(1-t_3)}\right)\int_{\frac{1}{r}}^{\frac{1}{2}} G\left(\tfrac{1}{2},s\right) ds \\
&= 2\left(\frac{8r^2}{r^2-4}\right)\left(a-\frac{c}{r^2t_3(1-t_3)}\right)\left(\frac{r^2-4}{16r^2}\right) \\
&= \left(\frac{8r^2}{r^2-4}\right)\left(a-\frac{c}{r^2t_3(1-t_3)}\right)\left(\frac{r^2-4}{8r^2}\right) \\
&= \left(a-\frac{c}{r^2t_3(1-t_3)}\right) \qquad \text{for} \quad 2<r
\end{aligned}$$

and then

$$\beta\left(Av\right) < \left(\frac{c}{r^2 t_3\left(1-t_3\right)}\right) + \left(a - \frac{c}{r^2 t_3\left(1-t_3\right)}\right) = a\ .$$

To show $(\mathcal{C}3)$ holds, suppose

$$v \in P\left(\gamma, \alpha, b, c\right)$$

with

$$\theta\left(Av\right) > \frac{t_2}{t_1} b\ .$$

Using (3.1.9), we get

$$\begin{aligned}
\alpha\left(Av\right) &= \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} \int_0^1 G(t,s) f\left(Bv\left(s\right), v(s)\right) ds \\
&= \int_0^1 G(t_1,s) f\left(Bv\left(s\right), v(s)\right) ds \\
&= \int_0^1 \frac{G(t_1,s)}{G(t_2,s)} G(t_2,s) f\left(Bv\left(s\right), v(s)\right) ds \\
&\geq \frac{t_1}{t_2} \int_0^1 G(t_2,s) f\left(Bv\left(s\right), v(s)\right) ds \\
&= \frac{t_1}{t_2} \theta\left(Av\right) \\
&> b
\end{aligned}$$

Finally, to show $(\mathcal{C}4)$, we take

$$v \in Q\left(\gamma, \beta, a, c\right)$$

with

$$\psi(Av) < \frac{2a}{r}$$

Using (3.1.8), we get

$$
\begin{aligned}
\beta\left(Av\right) &= \max_{t\in\left[\frac{1}{r},1-\frac{1}{r}\right]} \int_0^1 G(t,s)f\left(Bv\left(s\right),v(s)\right)ds \\
&= \int_0^1 G(\tfrac{1}{2},s)f\left(Bv\left(s\right),v(s)\right)ds \\
&= \int_0^1 \frac{G(\frac{1}{2},s)}{G(\frac{1}{r},s)}G(\tfrac{1}{r},s)f\left(Bv\left(s\right),v(s)\right)ds \\
&\leq \frac{r}{2}\int_0^1 G(\tfrac{1}{r},s)f\left(Bv\left(s\right),v(s)\right)ds \\
&= \frac{r}{2}\psi(Av) \\
&< a
\end{aligned}
$$

Therefore, the hypotheses of a generalisation Leggett-Williams theorem are satisfied and as a result there exist three positive solutions y_1, y_2, and y_3 for the Lidstone **BVP** (**1**),(**2**),(**3**).

Moreover, these solutions are of the form

$$
y_i\left(t\right) = \int_0^1 G\left(t,s\right)v_i\left(s\right)ds, \quad i=1,2,3,
$$

for some

$$
v_1, v_2, v_3 \in \mathcal{P}.
$$

Conclusion

In this study, we discussed the existence of solutions to the fourth order Lidstone boundary value problem.

After rewriting the initial value problem into an equivalent fixed point problem defined by an integral operator, the general Leggett-Williams fixed point was used to prove the existence of three fixed points of this operator.

Sufficient conditions on the nonhomogeneous term of the fourth order Lidstone boundary value problem, and inequalities involving on the kernel of the integral operator are given to show the existence of three positive solutions of this problem.

The generalized Leggett-Williams fixed point is based upon the construction of three convex functionals which are used in place of the norm used in the case of the Leggett-Williams fixed point. In addition a concave functional is introduced with relationships to the one the convex functional. This hypotheses allow to give the main tool dealing with a fixed point theorem of a completely continuous operator defined on a cone of an ordered Banach space.

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