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Thème

Problèmes d'évolution en dynamique des populations

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Introduction

L'objet de cette thèse est l'étude de quelques problèmes de mathématiques appliquées. Elle est divisée en trois parties principales.

La première partie est consacrée à l'étude des systèmes de réaction diffusion avec diffusion dégénérée intervenant dans des modèles de dynamique des populations. Cette partie comporte cinq chapitres. Dans le premier chapitre on rappelle les différentes techniques utilisées dans l'étude des systèmes linéaires. En général ces techniques ne s'appliquent pas dans la cas dégénéré. On donne ensuite, quelques résultats relatifs aux systèmes avec diffusion dégénérée. On terminera ce chapitre par une déscription des systèmes étudiés dans cette partie et qui seront étalés dans les chapitres deux à cinq.

La deuxième partie est consacrée à l'implémentation numérique de la contrôlabilité exacte et approchée d'une équation hyperbolique parabolique intervenant dans la modélisation d'une population structurée en âge et en espace. Deux approches différentes sont utilisées. Elles diffèrent par le choix de la fonctionnelle coût à minimiser que l'on fait pour trouver le contrôle qui permet d'atteindre un certain état de la population donnée. Des résultats de simulation sont donnés avec des graphes et des tableaux décrivant la variation de l'erreur en fonction des différents paramètres du problème.

Dans la dernière partie nous présentons un algorithme conçu par A. Khoukhi pour maximiser des fonctionnelles de vraisemblances. La puissance de cet algorithme réside dans le fait qu'il ne nécessite aucune linéarisation des équations d'état ou des équations d'observation, comme c'est trés souvent dans les applications qui utilisent des filtres linéaires comme ceux de Kalmann. On propose deux formes de cet algorithme (l'une hors ligne, l'autre en ligne) pour l'identification des paramètres d'un canal en communication numérique.

Première partie

Systèmes réaction diffusion dégénérés

Chapitre 1

Généralités

1.1 Préambule

La modélisation mathématique de nombreux problèmes physiques et biologiques, s'exprime par des équations aux dérivées partielles de type parabolique non linéaire, qui dégénèrent pour certaines valeurs de fonctions inconnues.

A titre d'exemples, on peut citer l'écoulement d'un gaz ou d'un fluide à travers un milieu poreux, le processus de diffusion-advection modélisée par l'équation dite de Fokker-Planck, la propagation de la chaleur dans un milieu à grande variation de température. En dynamique des populations, on citera le modèle proposé par Mac-Camy et Gurtin [33] pourl'étude de la diffusion des populations biologiques. De nombreux modèles en physique des plasmas, de l'atmosphère et des océans, etc... reléve de ce type d'équations aux dérivées partielles.

On s'intéresse ici au cas où la dégénérescence est dite implicite, car dépendant de la valeur des solutions ou de leurs dérivées .

Dans cette classe de problèmes, il existe une famille d'équations d'évolution qui se présentent sous la forme :

$$\partial_t u_i - div(\varphi_i(u)\nabla u_i) = f_i(t,x,u,\nabla u) \qquad i = 1,\ldots,d. \tag{1.1.1}$$

où : $u = (u_1, u_2, \ldots.., u_d)$ est l'inconnue et les f_i représentent les effets de la réaction et du milieu extérieur sur la diffusion.

1.2 Origine des systèmes de réaction diffusion

1.2.1 Dynamique des populations

Les équations de type (1.1.1) interviennent en dynamique des populations ; reprenons le modèle proposé par Mac-Camy et Gurtin dans la diffusion des populations biologiques (cf [33]). Il est décrit par le système :

$$\begin{aligned} \frac{\partial\rho(t,x,a)}{\partial t} + \frac{\partial\rho(t,x,a)}{\partial a} &+ \mu(a)\rho(t,x,a) - \frac{\partial}{\partial x}(\rho(t,x,a)\frac{\partial u(t,x)}{\partial x}) \\ &= 0, (t,x,a) \in (0,T) \times [0,1] \times (0,A_\dagger) \end{aligned} \tag{1.2.2}$$

avec

$$\text{(1.2.3)}\qquad \begin{cases} \rho(t,0,x) = \int_0^{A_\dagger} \beta(a)\rho(t,a,x)da, \\ u(t,x) = \int_0^{A_\dagger} \rho(t,a,x)da, \\ \rho(t,a,0) = \rho(t,a,1) = 0, \\ \rho(0,a,x) = \rho_0(a,x). \end{cases}$$

où $\rho(t,a,x)$ est la densité de la population d'âge a, au temps t et au point x, $u(t,x)$ est la densité totale en age a, la fonction μ est le taux de décès de la population enfin la fonction β représente le taux de naissance de la population.

En intégrant (1.2.2) par rapport à a et en utilisant (1.2.3), on obtient

$$u_t - (uu_x)_x = \int_0^{A_\dagger} (\beta(a) - \mu(a))\rho(t,a,x)da$$

qui correspond à l'équation (1.1.1) sous la forme suivante :

$$u_t = (u^2)_{xx} + f(t,x)$$

où $\quad f(t,x) = \int_0^{A_\dagger}(\beta(a) - \mu(a))\rho(t,a,x)da$

1.2.2 Systèmes de réaction diffusion

Soit Ω un ouvert borné de $\mathbb{R}^N$ (dans la pratique $N = 2,3$) sensé représenté une surface géographique, une cellule, ... Dans Ω ont lieu des réactions chimiques, diffusion d'une maladie infectieuse ou une rumeur,....

Soient V un volume de contrôle dans Ω, ∂V sa frontière. $\overrightarrow{\nu(x)}$ étant la normale extérieure en x à V.

La vitesse de production de la $i^{ième}$ espèce dans le volume V est égale à la quantité produite par la réaction diminuée de son flux à travers la frontière ∂V, ce qui se traduit par :

$$\frac{\partial}{\partial t}\int_V u_i(t,x)dx = \int_V g_i(t,x,u_1,...,u_d) - \int_{\partial V} j_i d\sigma,$$

g_i représentant le taux de production de la $i^{ième}$ espèce.

Comme V est arbitraire, on obtient la loi de bilan suivante :

$$\partial_t u_i + div\,\overrightarrow{j_i} = g_i.$$

Le vecteur flux $\overrightarrow{j_r}$ est spécifique du phénomène considéré. Il s'écrit en général sous la forme :

$$\text{(1.2.4)}\qquad \overrightarrow{j_r} = -\sum_{i=1,d} a_{ir}(.,u)\nabla u_i + \overrightarrow{\alpha_{ir}}(.,u)u_i.$$

Le premier terme de (1.2.4) représente le terme de diffusion, le deuxième représente le terme de transfert. S'il n'y a pas de transfert de matière alors $\overrightarrow{j_r}$ s'écrit sous la forme simple :

$$\overrightarrow{j_r} = -\sum_{i=1,d} a_{ir}(.,u)\nabla u_r$$

Par ailleur, s'il existe i_0, r_0, tels que $i_0 \neq r_0$ et $a_{i_0 r_0} \neq 0$, c'est à dire si la matrice de diffusion n'est pas diagonale, la diffusion de la $i_0^{ième}$ espèce affecte la production de la $r_0^{ième}$ espèce. Dans le cas général, (voir (1.2.4)), le terme $\overrightarrow{\alpha_{ir}}(.,u)u_r$ détermine le transfert dans la direction du champ vectoriel $\overrightarrow{\alpha_{ir}}(.,u)$, proportionnel à la concentration de la $r^{ième}$ espèce. Ce vecteur de transfert est souvent le gradient du potentiel extérieur. Soit pour simplifier $a_{ii}(x,u) = \phi_i(u_i)$, et $a_{ir}(x,u) = 0$ pour tout $i \neq r$.

1.2.3 Exemples de systèmes de Réaction diffusion

1. On considère la réaction inversible suivante :

$$U + V \leftrightarrows W$$

La modélisation mathématique de ces réactions conduit au système suivant :

$$\begin{cases} \partial_t u = d_1 \Delta u + w - uv & \text{dans} \quad (0,\infty) \times \Omega \\ \partial_t v = d_2 \Delta v + w - uv & \text{dans} \quad (0,\infty) \times \Omega \\ \partial_t w = d_3 \Delta w - w + uv & \text{dans} \quad (0,\infty) \times \Omega \\ B(u,v,w) = 0 & \text{dans} \quad (0,\infty) \times \partial\Omega \\ (u(0,.), v(0,.), w(0,.)) = (u_0, v_0, w_0) & \text{dans} \quad \Omega \end{cases}$$

où u, v, w désignent respectivement les concentrations en U, V, W. d_1, d_2, d_3 sont des constantes strictement positives, représentant les vitesses de réaction, B est un opérateur différentiel définissant les conditions au bord de Ω.

2. Le système qui suit :

$$\begin{cases} \partial_t u_1 - div(a_1(t,x)\nabla u_1) = k_2 u_2 - k_1 u_1 u_5 & \text{dans} \quad (0,\infty) \times \Omega \\ \partial_t u_2 - div(a_2(t,x)\nabla u_2) = -k_2 u_2 + k_1 u_1 u_5 & \text{dans} \quad (0,\infty) \times \Omega \\ \partial_t u_3 - div(a_3(t,x)\nabla u_3) = k_4 u_4 - k_3 u_3 u_5 & \text{dans} \quad (0,\infty) \times \Omega \\ \partial_t u_4 - div(a_4(t,x)\nabla u_4) = -k_4 u_5 + k_3 u_3 u_5 & \text{dans} \quad (0,\infty) \times \Omega \\ \partial_t u_5 - div(a_5(t,x)\nabla u_5) = k_2 u_2 + k_4 u_5 - k_1 u_1 u_5 - k_3 u_3 u_5 & \text{dans} \quad (0,\infty) \times \Omega \\ u_1 = u_2 = u_3 = u_4 = u_5 = 0 & \text{sur } (0,\infty) \times \partial\Omega \\ (u_1(0,.), u_2(0,.), u_3(0,.), u_4(0,.), u_5(0,.)) = (u_{10}, u_{20}, u_{30}, u_{40}, u_{50}) & \text{dans} \quad \Omega \end{cases}$$

modélise le transfert, par l'hémoglobine, de l'oxygène et du dioxyde de carbone dans les poumons. Il est régi par les réaction chimiques suivantes :

$$Hb + O_2 \leftrightarrows HbO_2$$

$$Hb + CO_2 \leftrightarrows HbCO_2$$

u_1, u_2, u_3, u_4, u_5 représentent respectivement les concentrations en hémoglobine, oxygène, oxyhémoglobine, dioxyde de carbone et carboxyhémoglobine.

3. Le système :

$$\begin{cases} S_t - \Delta S^m = -I(\gamma S - \delta) & \text{dans} \quad (0,\infty)\times\Omega \\ I_t - \Delta I^n = I(\gamma S - \delta) & \text{dans} \quad (0,\infty)\times\Omega \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0 & \text{sur} \quad (0,\infty)\times\partial\Omega \\ (S(0,.), I(0,.)) = (S_0, I_0) & \text{dans} \quad \Omega \end{cases}$$

modélise la propagation d'une maladie infectieuse. S représente la densité des individus susceptibles et I représente la densité des individus infectés.

4. L'étude théorique d'une ségrégation raciale entre deux populations (P_1, P_2) conduit à l'étude du système suivant :

$$\begin{cases} \partial_t u_1 + div\,[-\nabla(f_1(u)u_1) - \gamma_1 u_1 \nabla\varphi] = u_1 g_1(u) & \text{dans} \quad (0,\infty)\times\Omega \\ \partial_t u_2 + div\,[-\nabla(f_2(u)u_2) - \gamma_2 u_2 \nabla\varphi] = u_2 g_2(u) & \text{dans} \quad (0,\infty)\times\Omega \\ \left.\begin{array}{c} < -\nabla(f_i(u)u_i - \gamma_i u_i \nabla\varphi), \nu >= 0 \\ \text{ou} \\ u_i = 0 \end{array}\right\} i = 1;2 & \text{sur} \quad (0,\infty)\times\partial\Omega \\ (u_1(0,.), u_2(0,.)) = (u_{10}, u_{20}) & \text{dans} \quad \Omega \end{cases}$$

u_i représente la densité de P_i et φ représente par exemple l'effet des autres pays sur la ségrégation.

1.3 Rappel des techniques utilisées dans l'étude de l'existence globale

1.3.1 Cas de la diffusion linéaire

On considère le système de réaction-diffusion suivant

$$(1.3.5) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = f(t,x,u,v) & \text{sur } (0,\infty)\times\Omega \\ \dfrac{\partial v}{\partial t} - d_2 \Delta v = g(t,x,u,v) & \text{sur } (0,\infty)\times\Omega \\ \lambda_1 \dfrac{\partial u}{\partial n} + (1-\lambda_1)(u - \alpha_1) = 0 & \text{sur } (0,\infty)\times\partial\Omega \\ \lambda_2 \dfrac{\partial v}{\partial n} + (1-\lambda_2)(v - \alpha_2) = 0 & \text{sur } (0,\infty)\times\partial\Omega \\ u(0,.) = u_0, v(0,.) = v_0. & \text{sur } \Omega \end{cases}$$

Ω étant un ouvert borné de $\mathbb{R}^N$, de frontière régulière ; d_1, d_2 sont deux constantes positives, f et g sont des fonctions localement lipschitziennes ; on suppose pour $i = 1, 2$ que $\lambda_i \in [0, 1]$ et $\alpha_i \geq 0$. Et nous faisons l'hypothèse que les deux propriétés suivantes sont satisfaites :

(H_1) La positivité des solutions de (1.3.5) est préservée au cours du temps, propriété assurée par l'hypothèse suivante

$$\begin{cases} f(t, x, 0, v) \geq 0, \quad g(t, x, u, 0) \geq 0, \quad \text{pour tout } u, v \geq 0. \\ u_0 \geq 0, \quad v_0 \geq 0. \end{cases}$$

Dans ce cas on dit que f et g sont quasi-positives.

(H_2) La masse totale des composantes u et v est contrôlée au cours du temps, c'est-à-dire que « $f + g$ » est nulle ou raisonnablement majorée.

Ces propriétés sont, comme nous le verrons par la suite, essentielles.

La question est de savoir dans quelle mesure, (H_1) et (H_2) assurent l'existence globale de solutions. Remarquons tout d'abord que les propriétés (H_1) et (H_2) assurent l'existence globale pour le système d'équations différentielles ordinaires associé à (1.3.5) :

$$\begin{cases} u' = f(u, v) \quad \text{sur } (0, \infty) \\ v' = g(u, v) \quad \text{sur } (0, \infty). \end{cases}$$

En effet (H_1) et (H_2) impliquent immédiatement les estimations a priori

$$0 \leq u(t) + v(t) \leq u(0) + v(0)$$

Rappelons que l'existence globale pour une équation différentielle ordinaire implique toujours l'existence globale pour l'équation de réaction-diffusion associée, mais ce résultat est en général faux pour les systèmes. Pour s'en convaincre, des contre-exemples explicites sont évoqués dans [47].

Il est cependant naturel de se poser la question pour les systèmes (1.3.5), d'autant plus que pour ces systèmes, bien souvent dans les situations pratiques qu'ils modélisent, les hypothèses (H_1) et (H_2) sont vérifiées de manière naturelle.

Notons que pour de bonnes conditions au bord ($\alpha_1 = \alpha_2 = 0$, $0 \leq \lambda_1, \lambda_2 \leq 1$), les deux propriétés (H_1) et (H_2) fournissent une estimation L^1 a priori uniforme en temps sur la solution. Ces deux propriétés fournissent ainsi un contrôle de la masse totale du système, ce qui justifie la terminologie de "systèmes avec contrôle de masse". Ces estimations n'entrent cependant pas dans le cadre classique assurant l'existence globale comme c'est le cas des estimations a priori dans L^∞. Rappelons en effet que lorsque $u_0, v_0 \in L^\infty(\Omega)$, l'existence locale de solutions pour (1.3.5) est classique : le système (1.3.5) possède une unique solution classique (u, v) sur un intervalle de temps $[0, T_{\max})$ avec l'alternative suivante,

$$\text{si } T_{\max} < +\infty \text{ alors } \lim_{t \longrightarrow T_{\max}} \|u(t)\|_{\infty,\Omega} + \|v(t)\|_{\infty,\Omega} = +\infty$$

Par conséquent, pour avoir l'existence globale de la solution de (1.3.5), il suffit de montrer que celle-ci reste localement bornée.

L'existence globale nessécite l'adjonction d'hypothèses supplémentaires sur les non-linéarités. Notons aussi que, dans le cas où les coefficients de diffusion sont égaux ($d_1 = d_2$), le système (1.3.5) admet une solution globale. Il suffit pour cela de sommer les deux équations de (1.3.5) et d'utiliser (H_1) et (H_2). Nous obtenons alors par le principe du maximum, une estimation a priori L^∞ sur u et v,

$$\|(u+v)(t)\|_{\infty,\Omega} \leq \|u_0 + v_0\|_{\infty,\Omega}$$

ce qui, d'après ce que nous venons de rappeler ci-dessus, assure l'existence globale de solutions.

Dans le cas où les coefficients de diffusion sont différents, $d_1 \neq d_2$, la situation est beaucoup plus difficile et l'existence globale n'a pu être obtenue qu'avec des des hypothèses supplémentaires.

Pour fixer les idées et situer le problème, mentionnons les techniques les plus souvent utilisées dans ce type de questions : les techniques de monotonie (quasi-monotonie de la non-linéarité), voir [56], les techniques d'effets régularisant basées sur les injections de Sobolev et qui s'appliquent aux systèmes à croissance pôlynomiale de degré petit ou en petites dimensions, (voir [1] et [55]), les techniques d'ensembles invariants, (voir [45] et [56]) qui permettent dans certains cas, d'obtenir l'existence globale pour des systèmes d'équations de réaction diffusion à partir de l'existence globale des systèmes d'équations différentielles ordinaires qui leurs sont associés.

Cependant certains systèmes simples vérifiant les propriétés (H_1) et (H_2) n'entrent dans aucune de ces catégories. C'est le cas du système suivant proposé par R. H. Martin et qui a l'avantage de bien circonscrire la difficulté :

$$(1.3.6) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = -uf(v) & \text{sur} \quad (0,\infty)\times\Omega \\ \dfrac{\partial v}{\partial t} - d_2 \Delta v = uf(v) & \text{sur} \quad (0,\infty)\times\Omega \end{cases}$$

où $f: \ \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ sur $(0,\infty)\times\Omega$

Remarquons tout d'abord que la positivité de la solution et la structure de la première équation fournissent une estimation L^∞ uniforme et a priori sur u. S'il était possible d'établir une estimation à priori uniforme pour $v(t)$, l'existence globale en résulterait. Mais, et c'est là la difficulté, une telle estimation n'est pas du tout évidente (sauf bien sûr dans le cas trivial $d_1 = d_2$). Lorsque $f(v) = v^\beta$ (et $\lambda_1 = \lambda_2 = 1$), avec $\beta < \frac{N+2}{N}$, l'existence globale a été établie par Alikakos [1] avec les méthodes de " bootstrap ", lui permettant d'obtenir une estimation L^∞ sur les solutions à partir d'estimation L^1. L'extension de ce résultat à β quelconque fût d'abord obtenue par Masuda [48] puis par Hollis – Martin – Pierre [38] (voir aussi [46] et [49]), avec une approche différente, dont le caractère général permet de l'appliquer à une large classe de systèmes vérifiant (H_1) et (H_2), dite " classe des systèmes triangulaires ". En effet, cette technique basée sur la théorie de la régularité pour des solutions des équations paraboliques, nécessite, outre (H_1) et (H_2), trois hypothèses supplémentaires :

– Une hypothèse sur la structure des termes non-linéaires (triangularité)

$$f(t,x,u,v) \leq 0 \tag{1.3.7}$$

ce qui donne avec $(H1)$ que $f(t,x,0,v)=0$, pour $v \geq 0$.

– Une hypothèse de croissance polynomiale sur g

$$g(t,x,u,v) \leq K_1(u+v)^\sigma + K_2 \tag{1.3.8}$$

– Une hypothèse sur les conditions au bord qui doivent être de même type i. e.

$$0 < \lambda_1, \lambda_2 \leq 1 \text{ ou } \lambda_1 = \lambda_2 = 0. \tag{1.3.9}$$

Avec cette technique, les estimations L^∞ des solutions sont obtenues en deux étapes :

– Dans une première étape on établit, à l'aide de la théorie de la régularité L^p pour les opérateurs paraboliques et grâce aux hypothèses (H_1) et (H_2) et (1.3.9), une estimation L^p sur v pour tout p fini. En outre, l'hypothèse (1.3.7) assure une estimation L^∞ uniforme et a priori sur u.

– Dans une seconde étape, si g est pôlynomiale, (1.3.8) permet pour p assez grand et par effet régularisant le passage à l'estimation L^∞ sur v à partir de l'estimation L^p.

On peut s'interroger sur les raisons qui nous ont amener à ajouter ces hypothèses, en particulier (1.3.7) qui est beaucoup plus restrictive que (H_1). Il se trouve que Pierre et Schmidt [53] ont mis en évidence des contre-exemples explicites d'explosion en temps fini dans L^∞ en construisant deux fonctions u et v solutions d'un système du type (1.3.5), satisfaisant (H_1),(H_2) et les hypothèses (1.3.8), (1.3.9). Ces exemples justifient a posteriori l'adjonction d'hypothèses supplémentaires comme (1.3.7) pour qu'il y ait existence globale de solutions classiques. Signalons que dans les contre-exemples exhibés, nous avons non seulement (H_2), mais aussi $f + \lambda g \leq 0$ pour λ voisin de 1.

L'étude de l'influence des conditions au bord sur l'existence globale des solutions a permis de démontrer que l'hypothèse (1.3.9) est nécessaire. En effet, Bebernes-Lacey [13, 14] ont obtenu un exemple d'explosion en temps fini à partir du système (1.3.6) avec $\lambda_1 = 0$, $\lambda_2 = 1$ ($u = 1$ et $\dfrac{\partial v}{\partial n} = 0$) et $f(v) = v^p$, $p > 2$. Martin-Pierre [47] ont alors analysé l'existence de solutions dans tous les autres cas.

Pour ce qui est des non-linéarités, des techniques ont été développées pour traiter des croissances légèrement plus fortes que polynomiales par exemple du type $f(v) = e^{v^\beta}$, $\beta < 1$, dans le système (1.3.6), d'abord par Haraux-Youkana [36] à l'aide d'une fonction de Lyapounov, soit . L'extension à $\beta = 1$, fut d'abord obtenue par Barabanova [12] avec une restriction de taille sur les données initiales, puis par Herrero-Lacey-Velàzquez [37] sans aucune restriction de taille. En tout état de cause, une croissance quelconque ne peut être gérée par ces seules méthodes. D'où l'idée d'une autre approche radicalement différente, introduite dans [52] (voir aussi [41] et [46]), dont l'un des principaux avantages est qu'elle ne nécessite aucune hypothèse de croissance sur les termes non linéaires. Elle fournit cependant des solutions faibles sous les propriétés (H_1), (H_2), (1.3.6) et l'hypothèse d'homogénéité suivante sur les conditions au bord,

$$0 \leq \lambda_1, \lambda_2 \leq 1 \text{ ou } \alpha_1 = \alpha_2 = 0.$$

Cette technique repose sur l'exploitation maximale des estimations uniformes a priori dans L^1 (et non plus L^∞) assurées par (H_1) et (H_2), soit

$$\int_\Omega u(t) + v(t) \leq C$$

Contrairement aux estimations a priori dans L^∞, celle-ci n'assure pas immédiatement l'existence globale. Il faut procéder autrement. En tronquant les non-linéarités, un problème approché est obtenu pour les solutions approchées. On obtient d'abord des estimations à priori dans L^1 des termes non linéaires, puis leur compacité grâce aux résultats de compacité classiques pour le noyau de la chaleur. Le passage à la limite découlera d'une technique de compacité faible dans L^1 appliquées aux termes non linéaires qui ne sont a priori que bornés dans L^1. Cette méthode ne s'étend q'au cas de données initiales seulement intégrables.

Cette technique (Technique L^1) a été ultisée par Boudiba–Moulay–Pierre [15], pour étudier le cas où le second membre dépend fortement du gradient, à savoir le système

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f_1(t, x, u, v, \nabla u_1, \nabla v) & \text{sur } (0,\infty) \times \Omega \\ \dfrac{\partial v}{\partial t} - \Delta v = f_2(t, x, u, v, \nabla u_1, \nabla v) & \text{sur } (0,\infty) \times \Omega \\ u = v = 0 & \text{sur } (0,\infty) \times \partial\Omega \\ u(0,.) = u_0, v(0,.) = v_0. & \text{sur } \Omega, \end{cases}$$

avec en faisant les hypothèses suivantes :

1. Il existe une constante positive L_1 telle que

$$f_1(t, x, u_1, v, \nabla u_1, \nabla v) + f_2(t, x, u_1, v, \nabla u_1, \nabla v) \leq L_1(u_1 + v + 1)$$

pour tout $u_1, u_2 \geqslant 0$ et pour presque tout x et tout t.

2. Il existe une constante positive L_2 telle que

$$f_1(t, x, u, v, \nabla u_1, \nabla v) \leq L_1(u + v + 1)$$

pour tout $u_1, u_2 \geqslant 0$ et pour presque tout x et tout t.

3. $\sum_{i=1}^{2} |f_i(u, v, \nabla u, \nabla v)| \leq c(u, v)(|\nabla u|^m + |\nabla v|^m + 1)$,

où $1 \leq m < 2$, et $c : [0,\infty)^2 \longrightarrow [0,\infty)$,est une fonction croissante.

4. $u_0, v_0 \in L^1(\Omega)$ et $u_0, v_0 \geqslant 0$.

5. $f_1(t, x, 0, v, 0, q), f_2(t, x, u, 0, p, 0) \geq 0$ pour tout $u, v \geq 0$.

Malheuresement la technique L^1 n'est pas applicable dans le cas des systèmes dégénérés, puisque les résultats de compacité et de continuité qu'on a ne suffisent pas pour utiliser cette dernière.

1.3.2 Cas de la diffusion dégénérée

Les systèmes de réaction diffusion dégénérés n'ont pas eu beaucoup d'engouement de la part des mathématiciens, comme c'était le cas de l'équation des milieux poreux qui était à la mode dans les années soixante-dix et quatre-vingt. Peu de résultats sont connus pour ce type de dégénérescence. Le premier pas dans l'étude des systèmes dégénérés a été franchi par Alikakos [1] qui s'est intérressé à l'équation :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta(|u|^m \, sngu) = \beta(t,x)u & \text{dans }]0,\infty[\times \Omega \\ \left(\dfrac{\partial}{\partial \eta} u\right) u \leq 0 & \text{sur }]0,\infty[\times \partial\Omega \\ u(0,.) = u_0, \; u_0 \in L^\infty & \text{dans } \Omega. \end{cases}$$

En utilisant la technique des itérations de Moser, il arriva à établir le résultat suivant

$$\|u_i\|_{L^\infty(\Omega)} \leq C \|u_i\|_{L^m(\Omega)}.$$

Ce qui veut dire que l'existence globale dépend de l'estimation uniforme de $\|u_i\|_{L^m(\Omega)}$ qui n'est pas toujours facile à établir dans le cas général.

Ce résultat a été généralisé par Le Dung [24], en considérant le système

$$\begin{cases} \dfrac{\partial}{\partial t}(sngu_i \, |u_i|^{m_i}) - d_i(t)\Delta(u_i) = f_i(t,x,u,\nabla u_i) & \text{dans }]0,\infty[\times \Omega \\ \left(\dfrac{\partial}{\partial \eta} u_i\right) u_i \leq 0 & \text{sur }]0,\infty[\times \partial\Omega \\ u(0,.) = u_{i0} \; , u_0 \in L^\infty & \text{dans } \Omega \end{cases}$$

avec $|f_i(t,x,u,\xi)| \leq k_1 \sum\limits_{1 \leq i \leq r} u_i^\alpha + k_2 \|\xi\|^\delta + k_3$, $k_i \geq 0$, $\alpha \in \left[0, \frac{N+2}{N}\right[$, $\delta \in \left[0, \frac{N+2}{N}\right[$ et d_i une fonction continue sur $\mathbb{R}_+$. De plus il existe deux constantes positives d et D telles que : $d \leq d_i \leq D \quad \forall t \geq 0$.

Galaktionov dans [30] a étudié le problème

$$\begin{cases} u_t - \Delta u^{\sigma+1} = u^\beta & \text{dans } (0,\infty) \times \Omega \\ u = 0 & \text{sur } (0,\infty) \times \partial\Omega \\ u(0,.) = u_0 & \text{dans } \Omega \end{cases}$$

et a montré que l'existence ou la non-existence des solutions dépend essentiellement des paramètres β et σ, de la dimension N et de la donnée initiale u_0. Il a étudié également, en collaboration avec Kurdymov et Samarskii, dans [31], l'existence et l'explosion des solutions du système

$$\begin{cases} u_t - \Delta u^{m_1} = v^{p_1} & \text{sur } (0,\infty) \times \Omega \\ v_t - \Delta v^{m_2} = u^{p_2} & \text{sur } (0,\infty) \times \Omega \\ u = v = 0 & \text{sur } (0,\infty) \times \partial\Omega \\ u(0,.) = u_0; v(0,.) = v_0 & \text{sur } \Omega \end{cases}$$

où : $m_1, m_2 > 1$; $p_1, p_2 \geq 1, u_0 \in L^{m_1+1}(\Omega)$, $v_0 \in L^{m_2+1}(\Omega)$. En particulier ils prouvent l'existence de solutions globales sous les hypothèses $p_1 < m_2$ et $p_2 < m_1$, et montrent que dans les cas limites $p_1 = m_2$ ou $p_2 = m_1$, l'existence dépend de la structure géométrique de Ω.

Par la suite, Madallena dans [44], a généralisé l'étude précédente au système plus général

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \Delta u_i^{m_i} = f_i(u_1, u_2, ..., u_d) & \text{sur } (0,\infty)\times\Omega, \\ u_i = 0,\ i = 1,\dots,d, & \text{sur } (0,\infty)\times\partial\Omega, \\ u_i(0,.) = u_{i0},\ i = 1,\dots,d, & \text{sur } \Omega, \end{cases}$$

où : $f_i(0,0,....,0) = 0$, $f_i(u_1, u_2, ..., r_{i-1}, 0, r_{i+1}, ..., r_d) \geqslant 0$ pour tout $r_j \geqslant 0$, $j \neq i$ (quasi-positivité), $u_{i0} \in L^\infty(\Omega)$, $u_{i0} \geqslant 0$, $i = 1,\dots,d$.

Ensuite Laamri, dans [41], a regroupé ces deux derniers travaux. Il a établi, en utilisant la technique L^1, l'existence globale des solutions du système

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \Delta u_i^{m_i} = \displaystyle\sum_{1\le j\le d} c_{ij} u_j^{\alpha_{ij}} + c_i & \text{sur } (0,\infty)\times\Omega \\ u_i = 0,\ i = 1,\dots,d, & \text{sur } (0,\infty)\times\partial\Omega \\ u_i(0,.) = u_{i0}.\ i = 1,\dots,d, & \text{sur } \Omega \end{cases}$$

avec pour $i = 1,\dots,d$ et $j = 1,\dots,d$, $\alpha_{ij} < m_j$, $c_i, c_{ij} \geqslant 0$; $u_{i0} \in L^{m_i+1}(\Omega)$.

1.4 Description des résultats obtenus

Dans cette première partie, nous étudions les systèmes réaction diffusion de la forme

$$\partial_t U_i - \Delta U_i^{m_i} = f_i(x, t, U_1, U_2, ..., U_d), \quad (x,t) \in \Omega\times(0,T) = Q_T.$$

soumis à des conditions initiales et des condition au bord de type Neumann ou Dirichlet homogènes.

Ce système est parabolique dans la région $D = \displaystyle\bigcap_{i=1}^{4}[U_i \neq 0]$ et dégénère en une équation différentielle ordinaire du premier ordre dans la région $Q_T\backslash D$. Pour cette raison, ce système n'admet pas en général de solution classique. Pour cela, une notion de solution faible est requise. Nous adoptons la définition suivante introduite par Oleinik *et al* [51].

Définition 1.1 $(u_1, u_2, ..., u_d)$ *est dite solution faible ci dessus dans* Q_T *si pour tout* $i = 1, 2, ..., d$,

1. $u_i \in C((0,T]; L^2(\Omega))$,
2. $u_i^{m_i} \in L^2(Q_T)$,
3. $\nabla u_i^{m_i}$ *existe au sens des distributions dans* Q_T *et* $\nabla u_i^{m_i} \in (L^2(Q_T))^N$,
4. u_i *satisfait l'identité intégrale suivante*

$$\int_\Omega u_i(x,T)\varphi_i(x,T)dx - \int_{Q_T}\varphi_{it}u_i dxdt + \int_{Q_T}\nabla u_i^{m_i}\nabla\varphi_i dxdt$$
$$= \int_{Q_T} f_i(x,t,U_1,U_2,...,U_d)\varphi_i dxdt + + \int_\Omega u_{i0}(x)\varphi_i(0,x)dx$$

pour tout $\varphi_i \in C^1(\overline{Q_T})$ *telle que* $\varphi_i = 0$ *(pour la condition de Dirichlet et* $\frac{\partial \varphi_i}{\partial \eta} = 0$ *pour celle de Neumann) dans* $(0,T) \times \partial\Omega$.

Les systèmes introduits dans les chapitres qui suivent, interviennent en particulier dans certains modèles de dynamique des populations ; Citons à titre d'exemple la propagation d'une épidémie dans une population spatialement structurée.

Dans ce qui suit nous présentons les un résumés des chapitre 2 à 5 de cette première partie.

1.4.1 Chapitre 2

Dans ce chapitre, nous étudions l'existence globale des solutions faibles du système de réaction diffusion suivant :

$$(1.4.10) \qquad \begin{cases} S_t - \Delta S^m = & -I\left(\gamma S - \delta\right), \\ I_t - \Delta I^n = & I\left(\gamma S - \delta\right) \end{cases} \quad (x,t) \in \Omega \times (0,T) = Q_T,$$

avec les conditions initiales

$$(1.4.11) \qquad S\left(x,0\right) = S_0\left(x\right), \;\; I\left(x,0\right) = I_0\left(x\right); \;\; x \in \Omega,$$

et les conditions au bord de type Neumann homogène

$$(1.4.12) \qquad \frac{\partial S^m}{\partial \eta}(x,t) = \frac{\partial I^n}{\partial \eta}(x,t) = 0 \qquad (x,t) \in \partial\Omega \times (0,T).$$

Ω est un ouvert borné et connexe de $I\!R^N$, de frontière régulière $\partial\Omega$, $m,n > 1$, $I_0,\ S_0 \in C\left(\overline{\Omega}\right)$, $S_0,\ I_0 \geq 0$.
Dans un premier temps, nous démontrons un résultat d'existence et d'unicité de la solution faible en particulier nous montrons le

Théorème 1.1 *Pour toutes données initiales non négatives* (S_0, I_0) *dans* $C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right)$, *il existe une unique solution faible du problème (1.4.10)-(1.4.12) dans* Q_∞ *telle que*
(i) $S \in C(Q_\infty) \bigcap L^\infty(Q_\infty)$;
(ii) $I \in C(Q_\infty)$;
si de plus $S_0 \leq \dfrac{\delta}{\gamma}$ *alors* $I \in C(Q_\infty) \bigcap L^\infty(Q_\infty)$;

Puis nous donnons un résultat sur le comportement asymptotique des solutions, à travers le

Théorème 1.2 *Supposons que* $0 \leq S_0 \leq \dfrac{\delta}{\gamma}$ *et soit* $M = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega (S_0 + I_0)(x)dx$, *alors*

1. *Si* $M \leq \dfrac{\delta}{\gamma}$ *alors*

$$\lim_{t \to +\infty} S(t,.) = M \quad \text{et} \quad \lim_{t \to +\infty} I(t,.) = 0 \;\; \text{dans } C(\bar{\Omega}).$$

2. Si $M > \dfrac{\delta}{\gamma}$ *alors*

$$\lim_{t\to+\infty} S(t,.) = \frac{\delta}{\gamma} \quad et \quad \lim_{t\to+\infty} I(t,.) = M - \frac{\delta}{\gamma} \quad dans\ C(\bar{\Omega}).$$

1.4.2 Chapitre 3

Dans ce chapitre, nous nous intéressons à l'étude de l'existence globale et au comportement asymptotique des solutions faibles du système

$$(1.4.13)\qquad \begin{cases} \partial_t U_1 - \Delta U_1^{m_1} = -\gamma(U_1,U_2,U_3,U_4) - \nu U_1 & = f_1(U_1,U_2,U_3,U_4), \\ \partial_t U_2 - \Delta U_2^{m_2} = \gamma(U_1,U_2,U_3,U_4) - (\lambda+\mu)U_2 & = f_2(U_1,U_2,U_3,U_4), \\ \partial_t U_3 - \Delta U_3^{m_3} = \lambda\pi U_2 - (\alpha+m+\mu)U_3 & = f_3(U_1,U_2,U_3,U_4), \\ \partial_t U_4 - \Delta U_4^{m_4} = (1-\pi)\lambda U_2 + \alpha U_3 + \nu U_1 & = f_4(U_1,U_2,U_3,U_4), \end{cases}$$

dans $\Omega \times (0,+\infty)$, avec les conditions initiales

$$(1.4.14)\qquad U_i(x,0) = U_{i,0}(x) \geq 0, \quad x \in \Omega; \qquad i = 1,..,4.$$

et les conditions au bord de type Neumann homogène

$$(1.4.15)\qquad \frac{\partial U_i^{m_i}}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \qquad i = 1,..,4.$$

Ω est un ouvert borné et connexe de $I\!R^N$, de frontière régulière $\partial\Omega$. On fait les hypothèses suivantes

(H0) $m_i > 1,\ i = 1..4$.

(H1) $\mu, \alpha, \nu, m, \lambda, \pi \geq 0,\ \lambda+\mu > 0,\ \alpha+m+\mu > 0$ et $0 \leq \pi \leq 1$.

(H2) $U_{i,0} \in C(\bar{\Omega}),\quad U_{i,0}(x) \geq 0,\ x \in \Omega, \qquad i = 1,..,4.$

(H3) $\gamma : I\!R_+^4 \longrightarrow I\!R_+$ est continue, localement lipschitzienne, de croissance pôlynomiale avec $\gamma(0,U_2,U_3,U_4) = 0$ sur $I\!R_+^3$.

(H4) il existe des constantes positives C_1, C_2 et $0 \leq r < 1$ telles que

$$\gamma(U_1,U_2,U_3,U_4) \leq C_1 + C_2 U_1^r \text{ dans } I\!R_+^4.$$

On montre alors le résultat

Théorème 1.3 *Pour tout quadruplet de conditions initiales continues et non négatives* $(U_{1,0}, U_{2,0}, U_{3,0}, U_{4,0})$*, il existe une unique solution faible* (U_1,U_2,U_3,U_4) *du problème* (1.4.13)–(1.4.15) *dans* Q_∞*, telle que*

(i) Pour tout $i = 1,...,3,\ U_i \in L^1 \cap L^\infty(Q_\infty)$ *et* $\nabla U_i^{m_i},\ \partial_t U_i^{m_i} \in L^2(Q_{\tau,\infty}),\ \tau > 0$ *;*

(ii) $U_4 \in L^1 \cap L^\infty(Q_T)$ *et* $\nabla U_4^{m_4},\ \partial_t U_4^{m_4} \in L^2(Q_{\tau,T}),\ \tau > 0.$

Puis on établit le résultat suivant sur le comportement asymptotique de la solution.

Théorème 1.4 *Il existe deux constantes* $U_1^*, U_4^* \geq 0$ *telles que*

$$\begin{aligned} U_2(.,t), U_3(.,t) &\longrightarrow 0\ ,\ U_1(.,t) \longrightarrow U_1^* \ \ dans\ \ C(\bar{\Omega}) \ \ quand\ \ t \longrightarrow +\infty \\ et\ \overline{U_4}(t) &\longrightarrow U_4^* \ \ dans\ \ L^p(\Omega)\ pour\ tout\ p \geq 1\ \ quand\ \ t \longrightarrow +\infty. \end{aligned}$$

De plus, si $\nu > 0$ *alors* $U_1^* = 0$.

1.4.3 Chapitre 4

Dans ce chapitre, nous étudions le système

(1.4.16)
$$\begin{cases} \partial_t U_1 - \Delta U_1^{m_1} = & -\gamma(U_1,U_2,U_3,U_4) + \sum_{i=1}^4 b_{1i}U_i + \delta U_4 - \nu U_1 - (k_1P+m_1)U_1 \\ & +F_1(x,t) = \ f_1(x,t,U_1,U_2,U_3,U_4), \\ \partial_t U_2 - \Delta U_2^{m_2} = & \gamma(U_1,U_2,U_3,U_4) + b_{22}U_2 - (k_2P+m_2+\lambda+\mu)U_2 \\ & +F_2(x,t) = \ f_2(x,t,U_1,U_2,U_3,U_4), \\ \partial_t U_3 - \Delta U_3^{m_3} = & b_{33}U_3 + \lambda\pi U_2 - (k_3P+\alpha+m_3+m+\mu)U_3 \\ & +F_3(x,t) = \ f_3(x,t,U_1,U_2,U_3,U_4), \\ \partial_t U_4 - \Delta U_4^{m_4} = & b_{44}U_4 + (1-\pi)\lambda U_2 + \alpha U_3 + \nu U_1 - \delta U_4 - (k_4P+m_4)U_4 \\ & +F_4(x,t) = \ f_4(x,t,U_1,U_2,U_3,U_4). \end{cases}$$

dans $\Omega \times (0,+\infty)$, avec les conditions initiales

$$U_i(x,0) = U_{i,0}(x) \geq 0, \quad x \in \Omega; \qquad i = 1..4. \tag{1.4.17}$$

et les conditions au bord de type Neumann homogène

$$\frac{\partial U_i^{m_i}}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \qquad i = 1..4. \tag{1.4.18}$$

Ω est un ouvert borné et connexe de $I\!R^N$, de frontière régulière $\partial\Omega$; $P = \sum_{i=1}^{4} U_i$ la masse totale de la population. On suppose que

(H0) $U_{i,0} \in C(\bar{\Omega})$, $\quad U_{i,0}(x) \geq 0,\ x \in \Omega, \qquad i = 1..4.$
(H1) $m_i > 1,\ i = 1..4.$
(H2) $\mu, \alpha, \nu, m, \lambda, \pi, b_{ii}, b_{1i}, k_i \geq 0,\ i = 1,..,4,\ k_i > 0,\ i = 1,..,4$ et $0 \leq \pi \leq 1$.
(H3) $\gamma : \mathbb{R}_+^4 \longrightarrow \mathbb{R}_+$ est continue, localement lipschitzienne, de croissance pôlynomiale avec $\gamma(0,U_2,U_3,U_4) = 0$ sur $I\!R_+^3$.
(H4) il existe des constantes positives C_1, C_2 et $0 \leq r \leq 1$ telles que

$$\gamma(U_1,U_2,U_3,U_4) \leq (C_1 + C_2 \sum_{i=1}^{4} U_i^r) \text{ sur } \mathbb{R}_+^4.$$

(H5) $F_i,\ i = 1,..,4$ sont des fonctions continues bornées positives sur $\Omega \times (0,+\infty)$.

Dans une première partie on démontre le résultat suivant d'existence, d'unicité et de continuité par rapport aux données.

Théorème 1.5 *Pour tout quadruplet de conditions initiales continues et non négatives* $(U_{1,0}, U_{2,0}, U_{3,0}, U_{4,0})$, *il existe une unique solution faible* (U_1,U_2,U_3,U_4) *du problème* (1.4.16)–(1.4.18) *dans* Q_∞, *telle que*

i) $U_{i,0} \in C((0,+\infty);\bar{\Omega}) \cap L^\infty(Q_\infty)$, *et* $U_i^{m_i} \in H^1(Q_{\tau,T})$ *pour tout* , $0 < \tau < T$, $i = 1..4$.

ii) Il existe une constante $K \geq 0$ telle que
(1.4.19)
$$\int_\Omega \sum_{i=1}^{4} |U_{1,i} - U_{2,i}|(x,t)dx \leq (1 + Kte^{Kt}) \int_\Omega |U_{1,i,0} - U_{2,i,0}|(x)dx, \text{ for all } t > 0.$$

où $U_{j,i}$ est la solution de $(1.4.16) - (1.4.18)$ avec comme condition initiale $U_{j,i,0}$.

On s'intéresse ensuite au problème d'existence d'une solution périodique. On démontre le résultat suivant

Théorème 1.6 *Sous l'hypothèse que les fonctions (ou le second membres) F_i sont T^*-pariodiques, il existe une existe une donnée initiale $(U_{1,0}, U_{2,0}, U_{3,0}, U_{4,0})$ telle que la solution (U_1, U_2, U_3, U_4) du système $(1.4.16) - (1.4.18)$ vérifie pour tout $t \geq 0$, et tout $x \in \Omega$,*

$$U_i(x, t + T^*) = U_i(x, t), \ i = 1, .., 4.$$

Pour terminer on établi que le système dynamique suivant

$$\begin{cases} \partial_t U_i - \Delta\left(|U_i|^{m_i} signU_i\right) = f_i(x, t, U_1, U_2, U_3, U_4), & (x,t) \in \Omega \times (0, +\infty) \\ \frac{\partial(|U_i|^{m_i} signU_i)}{\partial \eta}(x,t) = 0, & x \in \partial\Omega, t > 0, i = 1..4. \\ U_i(x, 0) = U_{i,0}(x), & x \in \Omega; i = 1..4. \end{cases}$$

admet un attracteur global $\mathcal{A} \subset (H^1(\Omega) \cap L^\infty(\Omega))^4$.

1.4.4 Chapitre 5

Dans ce dernier chapitre nous étudions le système suivant
(1.4.20)
$$\begin{cases} \partial_t(u_i) - \Delta(|u_i|^{\sigma_i} u_i) = f_i(u, \nabla u_i) = g_i(u) + \overrightarrow{b_i} \nabla(|u_i|^{m_i - 1} u_i) & \text{dans} \quad (0, \infty) \times \Omega, \\ u_i = 0 & \text{dans} \quad (0, \infty) \times \partial\Omega, \\ u_i(0, .) = u_{i0} & \text{dans} \quad \Omega, \end{cases}$$

où $u = (u_1, ..., u_d), d \in N^*$, $\sigma_i > 0$,, $\overrightarrow{b_i} = \overrightarrow{b_i}(t,x) \in \mathbb{R}^N$ et $m_i > 0$. Nous faisons les hypothèses suivantes, pour tout $i = 1, 2, ..., d$
(H_1) $1 \leq m_i < \sigma_i + 1$,
(H_2) $g_i(0) = 0$,
(H_3) g_i et $\overrightarrow{b_i}$ sont continues, localement lipschitziennes par rapports à leurs arguments,
(H_4) Il existe des constantes $C_i, \alpha_{ij} \geq 0$ avec $\alpha_{ij} < \sigma_j + 1$ telles que

$$\left\|\overrightarrow{b_i}\right\| \leq C_i, \ |g_i(u)| \leq C_i(\sum_{j=1}^{d} |u_j|^{\alpha_{ij}} + 1),$$

(H_5) $u_{i0} \in L^{\sigma_i + 2}(\Omega)$.
Nous commençons par établir le résultat principal de ce chapitre

Théorème 1.7 *Le problème* (1.4.20) *admet une solution globale* $u = (u_1, u_2, ..., u_d)$ *vérifiant :*

$$\|u_i(.,t)\|_{L^\infty(\Omega)} \leq F(\xi) \quad \textit{pour tout } t \geq \xi > 0,$$

où $F(\xi)$ *est une fonction positive et ne dépend que de* ξ*.*

Si, de plus, $u_{i0} \in L^\infty(\Omega)$*,* u *est unique et vérifie pour tout* $t \geq 0$

$$\|u_i(.,t)\|_{L^\infty(\Omega)} \leq C \quad \textit{pour tout } t \geq 0,$$

où C *une constante positive et ne dépend que de* u_0*.*

Enfin, si $u_0 \geq 0$ *et si* g_i *est quasi-positive alors* $u \geq 0$*.*

Ce théorèmé établit l'existence d'un semi-groupe $S(t)$, défini de $(L^\infty(\Omega))^d$ dans $(L^\infty(\Omega))^d$ tel que $S(t)u_0 = u(t,.)$. Son rôle a été crucial dans l'étude du comportement asymptotique des solutions et nous a permis d'obtenir le résultat suivant :

Théorème 1.8 *Le semi-groupe* $S(t)_{t\geq 0}$ *associé au système* (1.4.20) *possède un attracteur global* $\mathcal{A}$ *borné dans* $\mathbb{C}^\alpha$ *et compact dans* $(L^\infty(\Omega))^d$*.*

Nous terminons ce chapitre par l'étude du cas limite, en établissant un résultat d'existence globale, d'extinction ou d'explosion en temps fini.

Théorème 1.9 *Soit*

$$f_i(u, \nabla u_i) = c_{i0} + \sum_{j=1}^{d} c_{ij} u_j^{\sigma_j+1} + \overrightarrow{b_i} \nabla \left(u_i^{\sigma_i+1} \right)$$

1. *Si* $2d \max\limits_{i,j=1,d} c_{ij} + \max\limits_{i=1,d} \left\| \overrightarrow{b_i} \right\| (\lambda+1) < 2\lambda$*(*$\lambda$ *étant la première valeur propre du Laplacien avec condition de Dirichlet homogène sur* $\partial\Omega$*)*
   alors pour tout $u_0 \in (L^\infty(\Omega))^d$*il existe une unique solution faible globalement bornée, qui s'éteint à l'infini si* $c_{i0} = 0$*.*
2. *Si* $\overrightarrow{b_i}$ *est indépendant de* t*,* $\overrightarrow{b_i} \in C^\infty(\overline{\Omega})$ *et si* $c_{ii} > \lambda_i$*, (*λ_i *étant la première valeur propre de* $-\Delta\psi(x) + \overrightarrow{b_i} \nabla\psi(x)$ *avec condition de Dirichlet homogène sur* $\partial\Omega$*)*
   alors toute solution positive non triviale explose en temps fini.

Chapitre 2

Global existence and asymptotic behavior for a system of degenerate evolution equations

2.1 Introduction

In this chapter we study global existence and asymptotic behavior for degenerate parabolic system of the form

$$(2.1.1) \qquad \begin{cases} S_t - \Delta S^m = & -I\left(\gamma S - \delta\right), \\ I_t - \Delta I^n = & I\left(\gamma S - \delta\right) \end{cases} \qquad (x,t) \in \Omega \times (0,T) = Q_T,$$

in $\Omega \times (0, +\infty)$, subject to the initial conditions

$$(2.1.2) \qquad S\left(x,0\right) = S_0\left(x\right), \;\; I\left(x,0\right) = I_0\left(x\right); \;\; x \in \Omega,$$

and to the Neumann boundary conditions

$$(2.1.3) \qquad \frac{\partial S^m}{\partial \eta}(x,t) = \frac{\partial I^n}{\partial \eta}(x,t) = 0 \qquad (x,t) \in \partial\Omega \times (0,T).$$

Herein, Ω is an open, bounded and connected domain in $I\!R^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$; Δ is the Laplace operator in $I\!R^N$, $m, n > 1$, I_0, $S_0 \in C\left(\overline{\Omega}\right)$, S_0, $I_0 \geq 0$.
This problem leads to the so-called $(S - I - S)$ model : S, I represent respectively the densities of susceptibles and infectives, γSI is the force of infection or the incidence term, it represents the number of susceptible individuals S infected by contact with infective individuals I per time unit, finally δI is the number of infectives who become susceptibles after recovery.
System (2.1.1)-(2.1.3) is parabolic in the region $D = [S \neq 0] \cap [I \neq 0]$ and degenerate into first order equations on $Q_T \setminus D$. Note that degenerate diffusion is a good approach in modeling slow diffusion of individuals in the spatial spread of an epidemic disease, see Okubo [50].
In the spatially homogeneous case we found one of the models of propagation of an

epidemic disease described in [17, 40]. In fact that model deals with susceptibles, infectives and removed, but if we eliminate the removed ones by adding them to susceptibles we form the model below, without demography (no new borns or deaths) and in that setting it's well known that when $t \longrightarrow +\infty$

$$\begin{cases} (S,I) \longrightarrow (S_0+I_0,0) & \text{if} \quad S_0+I_0 \leq \dfrac{\delta}{\gamma}, \\ (S,I) \longrightarrow (\dfrac{\delta}{\gamma}, S_0+I_0-\dfrac{\delta}{\gamma}) & \text{otherwise.} \end{cases} \tag{2.1.4}$$

This chapter is organized as follows : in section 2 notion of a weak solution is introduced and we state our mean results, in section 3 we will construct our solution as a limit of solutions of quasilinear and nondegenerate problems depending on a parameter ε, derive uniform a priori estimates on these solutions, and prove existence, uniqueness and regularity results in section 4. Finally in the last section we prove the large time behavior result which generalize (2.1.4) .

2.2 Main results

2.2.1 Basic assumptions and notations

Herein, Ω is an open, bounded and connected domain of the N-dimensional Euclidean space $I\!R^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$, a $(N-1)$-dimensional manifold so that locally Ω lies on one side of $\partial\Omega$; $x = (x_1, ..., x_N)$ is the generic element of $I\!R^N$. The gradient with respect to x is ∇ and the Laplace operator in $I\!R^N$ is Δ, $sign_\varepsilon$ is a smooth approximation of the function signum, finally if r is a real number then we set $r^+ = sup(r,0)$, $r^- = sup(-r,0)$.
Then we set $\Omega \times (0,T) = Q_T$ and for $0 \leq \tau < T$, $\Omega \times (\tau,T) = Q_{\tau,T}$. The norm in $L^p(\Omega)$ is $\|.\|_{p,\Omega}$ and the norm in $L^p(Q_{\tau,T})$ is $\| \|_{p,Q_{\tau,T}}$ for $1 \leq p \leq +\infty$ finally $H^1(Q_{\tau,T}) = H^1(\Omega \times (\tau,T))$.

2.2.2 Main results

It's well known that in general problem (2.1.1)-(2.1.3) has no classical solutions. A suitable notion of generalized solution is required : We adopt the notion of weak solution introduced by Oleinik *et al* [51].

Definition 2.1 *A couple of nonnegative functions (S,I) is a solution of system (2.1.1)-(2.1.3) if for each $T > 0$:*

1. $S, I \in C([0,T];\ L^\infty(\Omega))$.
2. ∇S^m , ∇I^n *exist in the sense of distribution and* ∇S^m , $\nabla I^n \in L^2(Q_T)$;
3. *for each functions* $\varphi, \psi \in H^1(Q_T)$

$$\text{(a)} \int_\Omega S(x,T)\,\varphi(x,T)\,dx - \int_0^T \int_\Omega \left[S\varphi_t - \nabla S^m \nabla\varphi - I(\gamma S - \delta)\,\varphi\right] dxdt$$
$$= \int_\Omega S(x,0)\,\varphi(x,0)\,dx$$
$$\text{(b)} \int_\Omega I(x,T)\,\psi(x,T)\,dx - \int_0^T \int_\Omega \left[I\psi_t - \nabla I^n \nabla\psi + I(\gamma S - \delta)\,\psi\right] dxdt$$
$$= \int_\Omega I(x,0)\,\psi(x,0)\,dx$$

We are now ready to state our results.

Theorem 2.1 *For each initial non negative data* (S_0, I_0) *in* $C\left(\overline{\Omega}\right) \times C\left(\overline{\Omega}\right)$ *there exists a unique weak solution* (S, I) *of problem (2.1.1)-(2.1.3) on* Q_∞ *; furthermore*
(i) $S \in C(Q_\infty) \bigcap L^\infty(Q_\infty)$ *;*
(ii) $I \in C(Q_\infty)$ *; and if* $S_0 \le \dfrac{\delta}{\gamma}$ *then* $I \in C(Q_\infty) \bigcap L^\infty(Q_\infty)$ *;*

Remark 2.1 *These results can be extended to the case* $S_0, I_0 \in L^\infty(\Omega)$ *with* $S, I \in C([\tau, T]; L^\infty(\Omega))$ *in the definition of weak solution and use results of Di Benedetto [21] to have :*
i) $S \in C(Q_{\tau,\infty}) \bigcap L^\infty(Q_\infty)$ *for all* $\tau > 0$
ii) $I \in C(Q_{\tau,T})$ *; and if* $S_0 \le \dfrac{\delta}{\gamma}$ *then* $I \in C(Q_{\tau,\infty}) \bigcap L^\infty(Q_\infty)$ *for all* $\tau > 0$.

For the large time behavior of weak solution, we have :

Theorem 2.2 *Assume* $0 \le S_0 \le \dfrac{\delta}{\gamma}$ *and let* $M = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega (S_0 + I_0)(x)dx$*, then*

1. *if* $M \le \dfrac{\delta}{\gamma}$ *then*
$$\lim_{t\to+\infty} S(t,.) = M \quad \textit{and} \quad \lim_{t\to+\infty} I(t,.) = 0 \ \textit{ in } C(\bar{\Omega}).$$
2. *if* $M > \dfrac{\delta}{\gamma}$ *then*
$$\lim_{t\to+\infty} S(t,.) = \frac{\delta}{\gamma} \quad \textit{and} \quad \lim_{t\to+\infty} I(t,.) = M - \frac{\delta}{\gamma} \ \textit{ in } C(\bar{\Omega}).$$

2.3 Auxiliary problem and a priori estimates

In this section we consider in $\Omega \times (0, +\infty)$ the auxiliary quasilinear non degenerate system

$$\text{(2.3.5)} \qquad \begin{cases} S_t - \Delta d_1(S) = & -(I-\varepsilon)\left(\gamma(S-\varepsilon)-\delta\right), \\ I_t - \Delta d_2(I) = & (I-\varepsilon)\left(\gamma(S-\varepsilon)-\delta\right) \end{cases} \qquad (x,t) \in \Omega \times (0,T) = Q_T,$$

subject to the initial and boundary conditions

$$(2.3.6) \qquad \begin{cases} S(x,0) = S_{0,\varepsilon}(x), \;\; I(x,0) = I_{0,\varepsilon}(x)\,; & x \in \Omega, \\ \dfrac{\partial d_1(S)}{\partial n}(x,t) = \dfrac{\partial d_2(I)}{\partial n}(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

$d_1,\, d_2 : I\!R^N \longrightarrow (\frac{\varepsilon}{2}, +\infty)$ are smooth and increasing functions with

$$(2.3.7) \qquad d_1(S) = S^m, \varepsilon \le S; \;\; and \;\; d_2(I) = I^n, \varepsilon \le I.$$

If $U_{0,\varepsilon}$ represents one of the smooth functions $S_{0,\varepsilon}$ or $I_{0,\varepsilon}$ over $\overline{\Omega}$ then we require

$$(2.3.8) \qquad \begin{cases} U_{0,\varepsilon}(x) \ge \varepsilon, \; x \in \Omega, \; 0 < \varepsilon \le 1, \\ \displaystyle\int_\Omega (U_{0,\varepsilon}(x) - \varepsilon)dx = \int_\Omega U_0(x)dx, \\ U_{0,\varepsilon} \longrightarrow U_0 \text{ in } C(\Omega), \text{ as } \varepsilon \longrightarrow 0. \end{cases}$$

We refer to [3, 7] for a construction of such a set of initial data. From standard results, [42], local existence and uniqueness of a classical solution $(S_\varepsilon, I_\varepsilon)$ of $(2.3.5) - (2.3.6)$ in some maximal interval $[0, T_{max,\varepsilon})$ is granted.
It is eay to check that $(\varepsilon, \varepsilon)$ is a subsolution, thus

$$(2.3.9) \qquad 0 < \varepsilon \le S_\varepsilon(x,t), 0 < \varepsilon \le I_\varepsilon(x,t), \; x \in \Omega, \; 0 < t < T_{max,\varepsilon}\,.$$

Then one can apply results in [25] to show global existence, i.e. $T_{max,\varepsilon} = +\infty$, of a classical solution for $(2.3.5) - (2.3.6)$. Using (2.3.7) and (2.3.9) we obtain global existence for

$$(2.3.10) \qquad \begin{cases} S_t - \Delta S^m = & -(I-\varepsilon)\,(\gamma(S-\varepsilon) - \delta)\,, \\ I_t - \Delta I^n = & (I-\varepsilon)\,(\gamma(S-\varepsilon) - \delta)\,, \end{cases} \quad (x,t) \in \Omega \times (0,T) = Q_T,$$

in $\Omega \times (0, +\infty)$, together with (2.3.6).
We derive a priori estimates. First, adding the two equations in (2.3.5) and using a straightforward integration one can derive the conservation of the total mass :

$$(2.3.11) \qquad \int_\Omega S_\varepsilon(x,T)\,dx + \int_\Omega I_\varepsilon(x,T)\,dx = \int_\Omega S_{0,\varepsilon}(x) + I_{0,\varepsilon}(x)\;dx, \quad \forall T \ge 0.$$

In what follows T is a positive number, $M_1, .., M_n$ are positive constants independent of T, $\varepsilon, 0 < \varepsilon \le 1$, and $F_1, .., F_n$ are non decreasing functions of T independent of ε.

Lemma 2.1 *There exist a constant M_1 and a nondecreasing function F_1 independent of ε,
$0 < \varepsilon \le 1$ such that*

$$(2.3.12) \qquad 0 < \varepsilon \le S_\varepsilon(x,t) \le M_1, \qquad x \in \Omega,\; t \ge 0,$$

$$(2.3.13) \qquad 0 < \varepsilon \le I_\varepsilon(x,t) \le F_1(T), \qquad x \in \Omega, \;\; 0 \le t \le T.$$

If $0 \le S_0(x) \le \dfrac{\delta}{\gamma}$ *then F_1 is a constant.*

Proof.

1. As $\|S_{0,\varepsilon}\|_{\infty,\Omega} + \dfrac{\delta}{\gamma}$ is a supersolution of equation for S_ε in (2.3.5)-(2.3.6), estimation (2.3.12) follow.
2. Multiplying the equation for I_ε by $p(I_\varepsilon - \varepsilon)^{p-1}$, $p \geq 1$ and integrating over Ω, we obtain
$$\frac{d}{dt}\int_\Omega (I_\varepsilon - \varepsilon)^p(x,t)dx \leq p\gamma \int_\Omega (I_\varepsilon - \varepsilon)^p(S_\varepsilon - \varepsilon)(x,t)dx \qquad t \geq 0.$$
Estimation (2.3.12) and Gronwall's inequality lead to estimate (2.3.13).
3. Now if $0 \leq S_0(x) \leq \dfrac{\delta}{\gamma}$, we can construct $S_{0,\varepsilon}$ such that $0 \leq S_{0,\varepsilon}(x) \leq \dfrac{\delta}{\gamma} + \varepsilon$, then by the maximum principle applied to the equation for S_ε we obtain
$$0 \leq S_\varepsilon(x,t) - \varepsilon \leq \frac{\delta}{\gamma} \qquad x \in \Omega,\ t \geq 0. \tag{2.3.14}$$
A second application of the maximum principle to the equation for I_ε gives
$$\varepsilon \leq I_\varepsilon(x,t) \leq \|I_0\|_{\infty,\Omega} + 1 \qquad x \in \Omega,\ \ 0 \leq t \leq T.$$

□

Lemma 2.2 *There exists a constant M_2 independent of ε, $0 < \varepsilon \leq 1$ such that*

$$\int_0^\infty \int_\Omega (I_\varepsilon - \varepsilon)\left(\delta - \gamma(S_\varepsilon - \varepsilon)\right)^2 dxdt \leq M_2. \tag{2.3.15}$$

Proof. Let $U_\varepsilon = (\gamma(S_\varepsilon - \varepsilon) - \delta)$, then the equation for S_ε can be written as

$$\begin{cases} U_t - m\ div(S_\varepsilon^{m-1}\nabla U) + \gamma(I-\varepsilon)U = 0, & (x,t) \in \Omega \times (0,T), \\ \dfrac{\partial U}{\partial n}(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \\ U(x,0) = \gamma(S_{0,\varepsilon} - \varepsilon) - \delta, & x \in \Omega. \end{cases} \tag{2.3.16}$$

We multiply (2.3.16) by U_ε and integrate over Q_T to find

$$\frac{1}{2}\int_\Omega U_\varepsilon^2(x,T)dx + m\int_0^T\int_\Omega S_\varepsilon^{m-1}\left|\nabla U_\varepsilon\right|^2(x,t)dxdt + \gamma\int_0^T\int_\Omega (I_\varepsilon - \varepsilon)U_\varepsilon^2 dxdt$$
$$= \frac{1}{2}\int_\Omega \left(\gamma(S_{0,\varepsilon} - \varepsilon) - \delta\right)^2(x,0)dx.$$

Then the estimate (2.3.15) follows by (2.3.9). □

Lemma 2.3 *There exists a nondecreasing function F_2 independent of $\varepsilon, 0<\varepsilon\leq 1$ such that*

$$\int_0^T\int_\Omega \left|\nabla S_\varepsilon^m\right|^2(x,t)dxdt+\int_0^T\int_\Omega \left|\nabla I_\varepsilon^n\right|^2(x,t)dxdt\leq F_2(T). \tag{2.3.17}$$

And if $0\leq S_0(x)\leq \frac{\delta}{\gamma}$, $x\in\Omega$ then F_2 is a constant.

Proof. 1. The estimate on ∇I_ε^n is obtained upon multiplying the equation for I_ε by I_ε^n and integrating over $Q_T=\Omega\times(0,T)$:

$$\begin{aligned}\frac{1}{n+1}\int_\Omega I_\varepsilon^{n+1}(x,T)\,dx+\int_0^T\int_\Omega \left|\nabla I_\varepsilon^n\right|^2 dxdt &= \frac{1}{n+1}\int_\Omega I_{0,\varepsilon}^{n+1}(x)\,dx\\ &+\int_0^T\int_\Omega I_\varepsilon^n\left(I_\varepsilon-\varepsilon\right)\left(\gamma(S_\varepsilon-\varepsilon)-\delta\right)(x,t)\,dxdt\end{aligned} \tag{2.3.18}$$

by the Cauchy inequality and (2.3.13) one has
(2.3.19)

$$\begin{aligned}\frac{1}{n+1}\int_\Omega I_\varepsilon^{n+1}(x,T)\,dx+\int_0^T\int_\Omega \left|\nabla I_\varepsilon^n\right|^2 dxdt &\leq \frac{1}{n+1}\int_\Omega I_{0,\varepsilon}^{n+1}(x)\,dx\\ &+(T\ mes(\Omega)\ F_1^{2n+1})^{\frac12}\left(\int_0^T\int_\Omega (I_\varepsilon-\varepsilon)\left(\delta-\gamma(S_\varepsilon-\varepsilon)\right)^2(x,t)dxdt\right)^{\frac12}.\end{aligned}$$

Using Lemma 2.2 one obtains the desired estimate.
If $0\leq S_0(x)\leq \frac{\delta}{\gamma}$, $x\in\Omega$, putting (2.3.14) in (2.3.18) one can obtain

$$\int_0^\infty\int_\Omega \left|\nabla I_\varepsilon^n\right|^2 dxdt\leq \frac{1}{n+1}\int_\Omega I_{0,\varepsilon}^{n+1}(x)\,dx. \tag{2.3.20}$$

2. Now to obtain an estimate on ∇S_ε we multiply the equation for S_ε by S_ε^m, integrate over Q_T and use the Cauchy inequality and Lemma 2.1 we have

$$\begin{aligned}\frac{1}{m+1}\int_\Omega S_\varepsilon^{m+1}(x,T)\,dx+\int_0^T\int_\Omega \left|\nabla S_\varepsilon^m\right|^2 dxdt\\ =\frac{1}{m+1}\int_\Omega S_{0,\varepsilon}^{m+1}(x)\,dx+\int_0^T\int_\Omega S_\varepsilon^m(I_\varepsilon-\varepsilon)\left(\delta-\gamma(S_\varepsilon-\varepsilon)\right)dxdt\end{aligned} \tag{2.3.21}$$

$$\begin{aligned}&\leq\frac{1}{m+1}\int_\Omega S_{0,\varepsilon}^{m+1}(x)\,dx\\ &+M_1^m(T\ mes(\Omega)\ F_1)^{\frac12}\left(\int_0^T\int_\Omega (I_\varepsilon-\varepsilon)\left(\delta-\gamma(S_\varepsilon-\varepsilon)\right)^2(x,t)dxdt\right)^{\frac12}.\end{aligned} \tag{2.3.22}$$

Then the estimate on ∇S_ε^m in (2.3.17) follows from (2.3.22) and (2.3.15) .
If $0\leq S_0(x)\leq \frac{\delta}{\gamma}$, $x\in\Omega$, by integrating equation in I_ε one gets

$$\int_0^\infty\int_\Omega (I_\varepsilon-\varepsilon)\left(\delta-\gamma(S_\varepsilon-\varepsilon)\right)dxdt\leq\int_\Omega I_{0,\varepsilon}(x)dx \tag{2.3.23}$$

then by (2.3.23) and (2.3.21) we have

$$\frac{1}{m+1}\int_0^T\int_\Omega |\nabla S_\varepsilon^m|^2\,dxdt \le \frac{1}{m+1}\int_\Omega S_{0,\varepsilon}^{m+1}(x)\,dx + \left(\frac{\delta}{\gamma}\right)^m \int_\Omega I_{0,\varepsilon}(x)dx \tag{2.3.24}$$

□

Lemma 2.4 *: There exists a nondecreasing functions F_3, F_4 independent of $\varepsilon, 0<\varepsilon\le 1$ such that for all $t>t_0>0$*

$$\int_\Omega |\nabla S_\varepsilon^m|^2(x,t)\,dx + \int_\Omega |\nabla I_\varepsilon^n|^2(x,t)\,dx \le F_3(t),$$

$$\int_{t_0}^t\int_\Omega |(S_\varepsilon^m)_t|^2\,dxds + \int_{t_0}^t\int_\Omega |(I_\varepsilon^n)_t|^2\,dxds \le F_4(t).$$

And if $0\le S_0(x)\le \frac{\delta}{\gamma}$, $x\in\Omega$ then F_3 and F_4 are constants.

Proof.

1. We multiply the equation for S_ε by $(S_\varepsilon^m)_t$ and integrate over $\Omega\times(\tau,t)$, $\frac{t}{2}\le\tau\le t\le T$; to find

(2.3.25)
$$\frac{4m}{(m+1)^2}\int_\tau^t\int_\Omega \left(S_\varepsilon^{\frac{m+1}{2}}\right)_t^2(x,s)dxds + \frac{1}{2}\int_\Omega |\nabla S_\varepsilon^m|^2(x,t)\,dx$$
$$= \int_\tau^t\int_\Omega (S_\varepsilon^m)_t(I_\varepsilon-\varepsilon)(\delta-\gamma(S_\varepsilon-\varepsilon))(x,s)dxds + \frac{1}{2}\int_\Omega |\nabla S_\varepsilon^m|^2(x,\tau)\,dx.$$

Next as $m>1$ we have $(S_\varepsilon^m)_t = \frac{2m}{m+1}\left(S_\varepsilon^{\frac{m-1}{2}}\right)\left(S_\varepsilon^{\frac{m+1}{2}}\right)_t$, so that

$$\begin{aligned}&\int_\tau^t\int_\Omega (S_\varepsilon^m)_t(I_\varepsilon-\varepsilon)(\delta-\gamma(S_\varepsilon-\varepsilon))(x,s)dxds\\ &\le \frac{2m}{(m+1)^2}\int_\tau^t\int_\Omega \left(S_\varepsilon^{\frac{m+1}{2}}\right)_t^2(x,s)dxds\\ &+\frac{m}{2}\|S_\varepsilon\|_{\infty,\Omega}^{m-1}\|I_\varepsilon-\varepsilon\|_{\infty,\Omega}\int_\tau^t\int_\Omega (I_\varepsilon-\varepsilon)(\delta-\gamma(S_\varepsilon-\varepsilon))^2(x,s)dxds.\end{aligned} \tag{2.3.26}$$

Putting this estimate in (2.3.25) one obtains

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla S_\varepsilon^m|^2(x,t)\,dx\\ &\le \frac{1}{2}\int_\Omega |\nabla S_\varepsilon^m|^2(x,\tau)\,dx\\ &+\frac{m}{2}\|S_\varepsilon\|_{\infty,Q_T}^{m-1}\|I_\varepsilon\|_{\infty,Q_T}\int_\tau^t\int_\Omega (I_\varepsilon-\varepsilon)(\delta-\gamma(S_\varepsilon-\varepsilon))^2(x,s)dxds.\end{aligned}$$

Integrating this inequality in τ over $(\frac{t}{2}, t)$ one finds

$$\begin{aligned}
&\int_\Omega |\nabla S_\varepsilon^m|^2 (x,t)\, dx \\
&\leq \frac{2}{t} \int_{\frac{t}{2}}^t \int_\Omega |\nabla S_\varepsilon^m|^2 (x,\tau)\, dx d\tau \\
&+ m \|S_\varepsilon\|_{\infty,Q_T}^{m-1} \|I_\varepsilon\|_{\infty,Q_T} \int_{\frac{t}{2}}^t \int_\Omega (I_\varepsilon - \varepsilon)(\delta - \gamma(S_\varepsilon - \varepsilon))^2(x,\tau) dx d\tau
\end{aligned}$$

The estimate for ∇S_ε follows by Lemmas 2.1, 2.2 and 2.3. In the same way one can obtain the estimate for ∇I_ε.

2. The estimate for $(S_\varepsilon^m)_t$ is immediately deduced from (2.3.25), (2.3.26) and lemma 2.3 keeping in mind that

$$|(S_\varepsilon^m)_t|^2 (x,t) \leq \left(\frac{2m}{m+1}\right)^2 \|S_\varepsilon\|_{\infty,Q_T}^{m-1} \left(S_\varepsilon^{\frac{m+1}{2}}\right)_t^2 (x,t).$$

The estimate for $(I_\varepsilon^n)_t$ follows immediately. □

2.4 Proofs for the existence and uniqueness

In this section we supply a quick proof of theorem 2.1.

2.4.1 Existence

From estimates established in the previous section one has $(S_\varepsilon - \varepsilon)_{0<\varepsilon\leq 1}$ and $(\nabla S_\varepsilon^m)_{0<\varepsilon\leq 1}$ are respectively bounded in $L^2(Q_T)$ and $(L^2(Q_T))^N$ for a fixed $T > 0$. Then there exists two sequences which are still denoted $(S_\varepsilon - \varepsilon)_{0<\varepsilon\leq 1}$ and $(\nabla S_\varepsilon^m)_{0<\varepsilon\leq 1}$ such that $(S_\varepsilon - \varepsilon)_{0<\varepsilon\leq 1}$ converges weakly to some function S in $L^2(Q_T)$ and $(\nabla S_\varepsilon^m)_{0<\varepsilon\leq 1}$ converges weakly to V in $(L^2(Q_T))^N$. On the other hand $(S_\varepsilon)_{0<\varepsilon\leq 1}$ is bounded in $L^\infty(Q_T)$; using a weak formulation of the equation for S_ε one can invoke the results in Di Benedetto [21] to get $(S_\varepsilon)_{0<\varepsilon\leq 1}$ relatively compact in $C\left(\overline{\Omega} \times [0,T]\right)$. It follows that $(S_\varepsilon - \varepsilon)_{0<\varepsilon\leq 1}$converges to S in $C\left(\overline{\Omega} \times [0,T]\right)$ and $(S_\varepsilon^m)_{0<\varepsilon\leq 1}$converges to S^m in $C\left(\overline{\Omega} \times [0,T]\right)$. As a first consequence of this $V = \nabla S^m$; By the same way one can prove that there is a function I such that $(I_\varepsilon)_{0<\varepsilon\leq 1}$ converges to I in $C\left(\overline{\Omega} \times [0,T]\right)$ and $(\nabla I_\varepsilon^n)_{0<\varepsilon\leq 1}$ converges to ∇I in $(L^2(Q_T))^N$. Now we us multiply equation for S_ε in (2.3.10) by φ, equation for I_ε by ψ, integrate by parts over $\Omega \times (0,T)$ and let ε goes to zero, to conclude that (S, I) is the desired solution. □

2.4.2 Uniqueness

The uniqueness is obtained by choosing an adequate test function in the definition of the weak solution as in Maddalena [44] and Aliziane & Langlais [4].

Let (S_1, I_1) and (S_2, I_2) be two weak solutions of problem (2.1.1) – (2.1.3). They verify the integral identity

$$\begin{aligned}(2.4.27)\qquad &\int_\Omega (S_1 - S_2)(x,T)\varphi_1(x,T)dx - \int_{Q_T} (S_1^m - S_2^m)\Delta\varphi_1(x,t)dxdt \\ &= \int_{Q_T} [\partial_t\varphi_1(S_1 - S_2) - (f(S_1,I_1) - f(S_2,I_2))\varphi_1](x,t)dxdt\end{aligned}$$

and

$$\begin{aligned}(2.4.28)\qquad &\int_\Omega (I_1 - I_2)(x,T)\varphi_2(x,T)dx - \int_{Q_T} (I_1^n - I_2^n)\Delta\varphi_2(x,t)dxdt \\ &= \int_{Q_T} [\partial_t\varphi_2(I_1 - I_2) + (f(S_1,I_1) - f(S_2,I_2))\varphi_2](x,t)dxdt\end{aligned}$$

for every $\varphi_i \in C^1(\bar{Q}_T)$ $i = 1, 2$, such that $\dfrac{\partial\varphi_i}{\partial\eta} = 0$ on $\partial\Omega \times (0,T)$ and $\varphi_i > 0$, where $f(S,I) = I\,(\gamma S - \delta)$. Let us introduce two functions ψ_1, ψ_2 as follows

$$\psi_1(x,t) = \begin{cases} \dfrac{S_1^m - S_2^m}{S_1 - S_2}(x,t), & \text{if } S_1 \neq S_2, \\ 0, & \text{otherwise.} \end{cases},$$

$$\psi_2(x,t) = \begin{cases} \dfrac{I_1^m - I_2^m}{I_1 - I_2}(x,t), & \text{if } I_1 \neq I_2, \\ 0, & \text{otherwise.} \end{cases}$$

Let us consider a sequence of smooth functions $(\psi_{i,\varepsilon})_{\varepsilon\geq 0}$ such that $\psi_{i,\varepsilon} \geq \varepsilon$, $\psi_{i,\varepsilon}$ uniformly bounded in $L^\infty(Q_T)$ and

$$\lim_{\varepsilon\to 0} \|(\psi_{i,\varepsilon} - \psi_i)/\sqrt{\psi_{i,\varepsilon}}\|_{L^2(Q_T)} = 0.$$

For any $0 < \varepsilon \leq 1$, let us introduce the adjoint nondegenerate boundary value problem

$$(2.4.29)\qquad \begin{cases} \partial_t\varphi_i + \psi_{i,\varepsilon}\Delta\varphi_i = 0 & \text{in } \Omega \times (0,T) \\ \dfrac{\partial\varphi_i}{\partial\eta}(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ \varphi_i(x,T) = \chi_i & \text{in } \Omega. \end{cases}$$

For any smooth χ_i with $0 \leq \chi_i(x) \leq 1,\ i = 1, ..., 2$, and any $0 < \varepsilon \leq 1$ this problem has a unique classical solution $\varphi_{i,\varepsilon}$ such that [44]

$$0 \leq \varphi_{i,\varepsilon}(x,t) \leq 1,$$

$$\int_{Q_T} \psi_{i,\varepsilon}(\Delta\varphi_{i,\varepsilon})^2 dxdt \leq K_1,$$

If in (2.4.27) – (2.4.28) we replace φ_i by $\varphi_{i,\varepsilon}$, where $\varphi_{i,\varepsilon}$ is the solution of problem (2.4.29) with

$\chi_1(x) = \chi_{1,\varepsilon}(x) = sign^+_\varepsilon(S_1 - S_2)(x,T)$ and $\chi_2(x) = \chi_{2,\varepsilon}(x) = sign^+_\varepsilon(I_1 - I_2)(x,T)$, then we obtain :

$$\int_\Omega \chi_{1,\varepsilon}(x)(S_1 - S_2)(x,T)dx - \int_{Q_T} (\psi_1 - \psi_{1,\varepsilon})(S_1 - S_2)\Delta\varphi_{1,\varepsilon}(x,t)dxdt$$
$$= - \int_{Q_T} (f(S_1, I_1) - f(S_2, I_2))\varphi_{1,\varepsilon}(x,t)dxdt,$$

and

$$\int_\Omega \chi_{2,\varepsilon}(x)(I_1 - I_2)(x,T)dx - \int_{Q_T} (\psi_2 - \psi_{2,\varepsilon})(I_1 - I_2)\Delta\varphi_{2,\varepsilon}(x,t)dxdt$$
$$= \int_{Q_T} (f(S_2, I_2) - f(S_1, I_1))\varphi_{2,\varepsilon}(x,t)dxdt.$$

Using the local lipschitz continuity of f and the properties of $\psi_{i,\varepsilon}$ and $\varphi_{i,\varepsilon}$ we deduce by letting $\varepsilon \to 0$

$$\int_\Omega \left((S_1 - S_2)^+ + (I_1 - I_2)^+\right)(x,T)dx \le K \int_{Q_T} \left(|S_1 - S_2| + |I_1 - I_2|\right)(x,t)dxdt,$$

where K is the lipschitz constant of f. In a similar fashion we establish an analogous inequality for $(S_1 - S_2)^-$ and $(I_1 - I_2)^-$ and deduce

$$\int_\Omega \left(|S_1 - S_2| + |I_1 - I_2|\right)(x,T)dx \le K \int_{Q_T} \left(|S_1 - S_2| + |I_1 - I_2|\right)(x,t)dxdt.$$

We conclude by using Gronwall's Lemma. □

2.4.3 Regularity results

The regularity results for ∇S^m, ∇I^n, $(S^m)_t$ and $(I^n)_t$ follow from the a priori estimates in Lemmas 2.1, 2.2 and 2.4 :

$$S^m, I^n \in H^1(Q_{\tau,T}) \quad \text{for all} \quad 0 < \tau < T, \tag{2.4.30}$$

and if $S_0 \le \dfrac{\delta}{\gamma}$ then $T = +\infty$. Note that (2.4.30) can be extended to the anisotropic Sobolev space $H^{1,2}_{loc}(Q_T)$ as in [7].

2.5 Large time behavior : proofs.

2.5.1 The ω-limit set

In this section we will assume that $0 \le S_0 \le \dfrac{\delta}{\gamma}$.

By Lemma 2.3 the set $\{(S^m(\cdot,\, t)\,, I^n(\cdot,\, t))\}_{t\ge t_0}$ is bounded in $\left(H^1(\Omega)\right)^2$ hence precompact in $\left(L^2(\Omega)\right)^2$, and we conclude that the ω-limit set :

$$\begin{aligned}\omega\left(S_0, I_0\right) &= \Big\{(U,V) \in \left(H^1(\Omega) \cap L^\infty(\Omega)\right)^2 \text{such that} : \exists t_k \longrightarrow +\infty \\ &\quad \left(S^m, I^n\right)(\,\cdot\,,\; t_k) \longrightarrow (U,V)(\cdot)\,, \text{ on } \left(L^2(\Omega)\right)^2\Big\}\end{aligned}$$

is well defined. Now we will give a characterization of $\omega\left(S_0, I_0\right)$.

Proposition 1 *let $(U,V) \in \omega\left(S_0, I_0\right)$ then (U,V) is solution of the homogeneous Neumann problem*

$$\begin{cases}-\Delta U = -V^{\frac{1}{n}}\left(\gamma U^{\frac{1}{m}} - \delta\right), \\ -\Delta V = V^{\frac{1}{n}}\left(\gamma U^{\frac{1}{m}} - \delta\right),\end{cases} \quad \text{in } \Omega \tag{2.5.31}$$

$$\frac{\partial U}{\partial n} = \frac{\partial V}{\partial n} = 0. \quad \text{in } \partial\Omega \tag{2.5.32}$$

Proof.
let $(U,V) \in \omega\left(S_0, I_0\right)$ then there exists $t_k \longrightarrow +\infty$ such that

$$(U,V)(\cdot) = \lim_{k\to\infty}\left(S^m, I^n\right)(.,\; t_k) \text{ in } \left(L^2(\Omega)\right)^2.$$

Let us consider two sequences U_k, V_k in $L^2\left(\Omega \times (-1,1)\right)$ defined as

$$\begin{matrix} U_k(x,s) = S(x, t_k + s) \\ V_k(x,s) = I(x, t_k + s)\end{matrix} \quad x \in \Omega, \quad -1 < s < 1, \quad k > 0.$$

For each $s \in (-1,1)$

$$\begin{aligned}\int_\Omega \left|U_k^m(x,s) - S^m(x,t_k)\right|^2 dx &= \int_\Omega \left|S^m(x,t_k+s) - S^m(x,t_k)\right| dx \\ &= \int_\Omega \left|\int_{t_k}^{t_k+s}\left(S^m\right)_t dt\right|^2 dx \le \int_\Omega \int_{t_k}^{t_k+s}\left(S^m\right)_t^2 dxdt \\ &\le \int_\Omega \int_{t_k}^{+\infty}\left(S^m\right)_t^2 dxdt.\end{aligned}$$

Hence

$$\left\|U_k^m - S^m(\,.\,,t_k)\right\|_{L^2(\Omega\times(-1,1))} \le \left[2\int_{t_k}^{+\infty}\int_\Omega \left(S^m\right)_t^2 dxdt\right]^{\frac{1}{2}}.$$

by Lemma 2.4 :

$$\lim_{k\to+\infty}\int_{t_k}^{+\infty}\int_\Omega \left(S^m\right)_t^2 dxdt = 0.$$

Then for a subsequence still denoted U_k, V_k : $U_k^m \longrightarrow U$, $V_k^n \longrightarrow V$ in $L^2(\Omega \times (-1,1))$ and almost everywhere in $\Omega \times (-1,1)$ as $k \longrightarrow +\infty$, and then by the Lebesgue dominated convergence theorem we have

$$\begin{array}{c} U_k \longrightarrow U^{\frac{1}{m}}, \qquad V_k \longrightarrow V^{\frac{1}{n}} \\ V_k\left(\gamma U_k - \delta\right) \longrightarrow V^{\frac{1}{n}}\left(\gamma U^{\frac{1}{m}} - \delta\right) \end{array} \text{ in } L^2\left(\Omega \times (-1,1)\,.\right)$$

Next let $\xi \in C^2\left(\bar{\Omega}\right)$ be such that $\dfrac{\partial \xi}{\partial \eta} = 0$ on $\partial\Omega$ and $\rho \in C_0^1\left((-1,1)\right)$; $\rho \geq 0$, $\displaystyle\int_{-1}^{1} \rho(s)\, ds = 1$. We set $\varphi(x,t) = \rho(t - t_k)\,\xi(x)$ and use φ as a test function in the definition of S with $T = t_k + 1$ and $t_k \geq 1$. We get

$$\int_0^T \int_\Omega \left[S\rho_t(t - t_k)\,\xi(x) + S^m \rho(t - t_k)\,\Delta\xi - I\left(\gamma S - \delta\right)\rho(t - t_k)\,\xi\right] dxdt = 0,$$

i.e.

$$\int_{t_k-1}^{t_k+1} \int_\Omega \left[S\rho_t(t - t_k)\,\xi(x) + S^m \rho(t - t_k)\,\Delta\xi - I\left(\gamma S - \delta\right)\rho(t - t_k)\,\xi\right] dxdt = 0.$$

Setting $s = t - t_k$ we get

$$\int_{-1}^{1} \int_\Omega U_k \rho_t(s)\,\xi(x) + U_k^m \rho(s)\,\Delta\xi - I\left(\gamma U_k - \delta\right)\rho(s)\,\xi dxds = 0.$$

Passing to the limit as $k \longrightarrow +\infty$, we obtain :

$$\left(\int_{-1}^{1} \rho'(s)\, ds\right)\left(\int_\Omega U^{\frac{1}{m}} \xi(x)\, dx\right) + \left(\int_{-1}^{1} \rho(s)\, ds\right)\left(\int_\Omega U(x)\,\Delta\xi(x) - V^{\frac{1}{n}}\left(\gamma U^{\frac{1}{m}} - \delta\right)\xi dx\right) = 0$$

Since $\int_{-1}^{1} \rho'(s)\, ds = 0$ we conclude that for any $\xi \in C^2\left(\bar{\Omega}\right)$, $\dfrac{\partial \xi}{\partial \eta} = 0$ on $\partial\Omega$, we have

$$\int_\Omega U(x)\,\Delta\xi - V^{\frac{1}{n}}\left(\gamma U^{\frac{1}{m}} - \delta\right)\xi dx = 0.$$

Thus U is a weak solution of

$$\begin{cases} -\Delta U = -V^{\frac{1}{n}}\left(\gamma U^{\frac{1}{m}} - \delta\right), & \text{in } \Omega \\ \dfrac{\partial U}{\partial n} = 0. & \text{on } \partial\Omega \end{cases}$$

In the same way, we prove that V is a weak solution of

$$\begin{cases} -\Delta V = V^{\frac{1}{n}}\left(\gamma U^{\frac{1}{m}} - \delta\right), & \text{in } \Omega \\ \dfrac{\partial V}{\partial n} = 0. & \text{on } \partial\Omega \end{cases}$$

□

2.5.2 Proof of Theorem 2.2

1. The semi-orbits $\{(S(.,t), I(.,t)), t \geq 0\}$ are relatively compact in $(C(\overline{\Omega}))^2$ because they are bounded in $(L^{\infty}(Q_{\infty}))^2$ by Lemma 2.1 and one may use results of Di Benedetto [21].
Let $(U,V) \in \omega\,(S_0, I_0)$ then (U,V) is a solution of $(2.5.31)-(2.5.32)$. The function $U+V$ is a solution of

$$\begin{cases} -\Delta\,(U+V) = 0, & \text{in } \Omega \\ \dfrac{\partial}{\partial n}\,(U+V) = 0, & \text{. on } \partial\Omega \end{cases}$$

Then $U+V$ is constant.
Now multiply equation for V by V and integrate over Ω, to obtain

$$\int_{\Omega} |\nabla V|^2\, dx + \int_{\Omega} V^{1+\frac{1}{n}}(\delta - \gamma U^{\frac{1}{m}})dx = 0. \tag{2.5.33}$$

Because $(\delta - \gamma U^{\frac{1}{m}}) \geq 0$, we conclude that V is constant, and then also U is constant. Putting these conclusions in (2.5.33) one has

$$V^{1+\frac{1}{n}}\left(\gamma U^{\frac{1}{m}} - \delta\right) = 0 \Longrightarrow V = 0 \text{ or } U = \left(\frac{\delta}{\gamma}\right)^m.$$

Now using estimate (2.3.11), we get

$$U^{\frac{1}{m}} + V^{\frac{1}{n}} = \frac{1}{|\Omega|}\int_{\Omega} S_0\,(x) + I_0\,(x)\, dx.$$

Keeping in mind $S \leq \dfrac{\delta}{\gamma}$ i.e. $U \leq \left(\dfrac{\delta}{\gamma}\right)^m$, then
either $V = 0 \implies U = \left(\dfrac{1}{|\Omega|}\displaystyle\int_{\Omega} S_0\,(x) + I_0\,(x)\, dx\right)^m \leq \left(\dfrac{\delta}{\gamma}\right)^m$
and this is possible only if $\dfrac{1}{|\Omega|}\displaystyle\int_{\Omega} S_0\,(x) + I_0\,(x)\, dx \leq \dfrac{\delta}{\gamma}$,
or $U = \left(\dfrac{\delta}{\gamma}\right)^m \implies V^{\frac{1}{n}} = \dfrac{1}{|\Omega|}\displaystyle\int_{\Omega} S_0\,(x) + I_0\,(x)\, dx - \dfrac{\delta}{\gamma} \geq 0$ and this is possible only if $\dfrac{1}{|\Omega|}\displaystyle\int_{\Omega} S_0\,(x) + I_0\,(x)\, dx \geq \dfrac{\delta}{\gamma}$. □

Chapitre 3

Global existence and asymptotic behavior for a degenerate diffusive seir model

3.1 Introduction

In this chapter we shall be concerned with a degenerate parabolic system of the form

$$(3.1.1) \qquad \begin{cases} \partial_t U_1 - \Delta U_1^{m_1} = -\gamma(U_1,U_2,U_3,U_4) - \nu U_1 & = f_1(U_1,U_2,U_3,U_4), \\ \partial_t U_2 - \Delta U_2^{m_2} = \gamma(U_1,U_2,U_3,U_4) - (\lambda+\mu)U_2 & = f_2(U_1,U_2,U_3,U_4), \\ \partial_t U_3 - \Delta U_3^{m_3} = \lambda\pi U_2 - (\alpha+m+\mu)U_3 & = f_3(U_1,U_2,U_3,U_4), \\ \partial_t U_4 - \Delta U_4^{m_4} = (1-\pi)\lambda U_2 + \alpha U_3 + \nu U_1 & = f_4(U_1,U_2,U_3,U_4). \end{cases}$$

in $\Omega \times (0,+\infty)$, subject to the initial conditions

$$(3.1.2) \qquad U_i(x,0) = U_{i,0}(x) \geq 0, \quad x \in \Omega; \qquad i = 1..4.$$

and to the Neumann boundary conditions

$$(3.1.3) \qquad \frac{\partial U_i^{m_i}}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \qquad i = 1..4.$$

Herein, Ω is an open, bounded and connected domain in $I\!R^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$; Δ is the Laplace operator in $I\!R^N$. Powers m_i verify $m_i > 1$, $i = 1..4$.

In the spatially homogeneous case and for $\nu = \mu = \alpha = m = 0$ and $\pi = 1$ this problem reduces to one of the models of propagation of an epidemic disease devised in Kermack and McKendricks [40], namely

$$\begin{cases} S' & = -\gamma S I, \\ I' & = +\gamma S I - \lambda I, \\ R' & = +\lambda I. \end{cases}$$

In that setting it is known, *loc. cit.*, that $I(t) \to 0$ as $t \to +\infty$, while the large time behavior of $S(t)$ and $R(t)$ depends on the initial state (S_0, I_0, R_0) ; note that for $t > 0, S(t) + I(t) + R(t) = S_0 + I_0 + R_0$.

This basic model served as a starting point for many further developments, both from epidemiological or mathematical point of vue : see the books of Busenberg and Cooke [16] or Capasso [18] and their references. These lead to so-called $(S-E-I-R)$ models : S is the distribution of susceptible individuals in a given population, $\gamma(S,E,I,R)$ is the incidence term or number of susceptible individuals infected by contact with an infective individual I per time unit and becoming exposed E, while R is the density of removed or resistant (immune) individuals. Then λ (resp. α) is the inverse of the duration of the exposed stage (resp. infective stage) or rate at which exposed individuals enter the infective class (resp. infective individuals who do not die from the disease recover), m is the death-rate induced by the disease. The last two parameters are control parameters : first ν is a vaccination rate ; next, for a population of animals, as it is considered here as in Anderson *et al* [10], Fromont *et al* [28], Courchamp *et al* [20] or Langlais and Suppo [43], μ is an elimination rate of exposed and infective individuals. Lastly, as it is suggested by the FeLV, a retrovirus of domestic cats (*Felis catus*) see [28], one also introduces a parameter π measuring the proportion of exposed individuals which actually develop the disease after the exposed stage, the remaining proportion $1-\pi$ becoming resistant.

The nonlinear incidence term γ takes various forms as it can be found from the literature ; at least two of them are widely used in applications

$$\gamma(S,E,I,R) = \begin{cases} \gamma SI, & [10,\ 18,\ 40], \quad \text{mass action in [16, 18] , or pseudo-mass action in [39, 22] .} \\ \gamma \dfrac{SI}{S+E+I+R}, & [20,\ 28,\ 43], \quad \text{proportionate mixing in [16] or true mass action in [39, 22] .} \end{cases}$$

We refer to De Jong *et al*, [39] and Diekmann *et al* [22] for a discussion supporting the second one in populations of varying size and Fromont *et al* [29] for a specific discussion in the case of a cat population. See Capasso and Serio [17] and Capasso [18] for more general incidence terms. Note that no demographical effect is considered in our model.

A mathematical analysis of the model of Kermack and McKendricks for spatially structured populations with linear diffusion, i.e. $m_i = 1,\ i = 1..4$, is performed in Webb [58]. Nonlinear but nondegenerate diffusion terms are introduced in Fitzgibbon *et al* [25]. Global existence and large time behavior results are derived therein. Homogeneous Neumann boundary conditions correspond to isolated populations.

A comprehensive analysis of generic $(S-E-I-R)$ models with linear diffusion is initiated in Fitzgibbon and Langlais [26] and Fitzgibbon *et al* [27]. These models include a logistic effect on the demography, yielding $L^1(\Omega)$ a priori estimates on solutions independent of the initial data for large time ; this allows to use a bootstrapping argument to show global existence and exhibit a global attractor in $(C(\overline{\Omega}))^4$.

For degenerate reaction-diffusion equations, a similar approach is followed in Le Dung [24]. In our case, $L^1(\Omega)$ a priori estimates can be established for nonegative solutions upon integrating over $\Omega \times (0,t)$

$$\sum_{i=1}^{4} \int_\Omega U_i(x,t)dx \le \sum_{i=1}^{4} \int_\Omega U_{i,0}(x)dx \quad \text{for all } t > 0,$$

but they cannot be found to be independent of the initial data. Moreover, generally speaking, the large time behavior of solutions depends on these initial datas, as it can be already seen for spatially homogeneous problems see §§3.5.3. This can also be checked on the disease free model : assuming $U_{i,0}(x) \equiv 0$ in Ω $i = 2..4$, the uniqueness result given in Theorem 3.1 implies $U_i(x,t) \equiv 0$ in $\Omega \times (0,+\infty)$ $i = 2..4$. Then, it should be clear that $\gamma(U_1, 0, 0, 0) = 0$ for any reasonable incidence term so that the equation for U_1 reads

$$\partial_t U_1 - \Delta U_1^{m_1} + \nu U_1 = 0 \text{ in } \Omega \times (0,+\infty); \tag{3.1.4}$$

this is the so-called porous medium equation. Now U_1 verifies homogeneous Neumann boundary conditions and it is well-known (see Alikakos [2]) that as $t \longrightarrow +\infty$

$$\begin{cases} U_1(.,t) \longrightarrow 0 & \text{if } \nu > 0, \\ U_1(.,t) \longrightarrow \dfrac{1}{mes(\Omega)} \displaystyle\int_\Omega U_{1,0}(x)dx & \text{if } \nu = 0. \end{cases}$$

The case of mass action incidence was studied by Aliziane and Moulay [7] and they established the long time behavior of the solution of the SIS model, Aliziane and Langlais [6] study the case of models include a logistic effect on the demography and they established global existence result of the solution and existence of periodic solution We also obtain the existence of the global attractor. Finally Hadjadj *et al* [35] study the case where the source term depends on gradient of solution, they study existence of globally bounded weak solutions or blow-up, depending on the relations between the parameters that appear in the problem.

3.2 Main results

3.2.1 Basic assumptions and notations

Herein, Ω is an open, bounded and connected domain of the N-dimensional Euclidian space $I\!R^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$, a $(N-1)$-dimensional manifold so that locally Ω lies on one side of $\partial\Omega$; $x = (x_1, ..., x_N)$ is the generic element of $I\!R^N$. Next we shall denote the gradient with respect to x by ∇ and the Laplace operator in $I\!R^N$ by Δ.

Then we set $\Omega \times (0,T) = Q_T$ and for $0 \leq \tau < T$, $\Omega \times (\tau, T) = Q_{\tau,T}$. The norm in $L^p(\Omega)$ is $\| \|_{p,\Omega}$ and the norm in $L^p(Q_{\tau,T})$ is $\| \|_{p,Q_{\tau,T}}$ for $1 \leq p \leq +\infty$.

Next we shall assume throughout this paper

(H0) Powers m_i verify $m_i > 1$, $i = 1..4$.

(H1) $\mu, \alpha, \nu, m, \lambda, \pi$ are nonnegative constants, $\lambda + \mu > 0$, $\alpha + m + \mu > 0$ and $0 \leq \pi \leq 1$.

(H2) $U_{i,0} \in C(\bar\Omega)$, $U_{i,0}(x) \geq 0$, $x \in \Omega$, $i = 1..4$.

(H3) $\gamma : I\!R_+^4 \longrightarrow I\!R_+$ is a locally lipschitz continuous function with polynomial growth and $\gamma(0, U_2, U_3, U_4) = 0$ on $I\!R_+^3$.

(H4) There exists nonnegative constants C_1, C_2 and $0 \leq r < 1$ such that

$$\gamma(U_1, U_2, U_3, U_4) \leq C_1 + C_2 U_1^r \text{ on } I\!R_+^4.$$

Remark 3.1 *In the limiting case* $\lambda+\mu=0$ *the equations for* U_3 *and* U_4 *do not depend on* U_2*, the equation for* U_3 *being a porous medium type equation as in* (3.1.4)*. This condition also implies* $\lambda=0$ *which is not relevant if one goes back to our motivating problem. In the limiting case* $\alpha+m+\mu=0$ *one could not get* $L^\infty(Q_{0,\infty})$ *bounds for* U_3*, but one still has global existence.*

The assumption $\gamma(0,U_2,U_3,U_4)=0$ is required to make sure that the nonnegative orthant $\mathbb{R}^4_+$ is forward invariant by (3.1.1) ; this is a natural assumption for our motivating problem : no new exposed individuals when there is no susceptible ones. $(H4)$ removes mass action incidence terms ; in that case one can also get global existence results, but no $L^\infty(Q_{0,\infty})$ bounds for U_2 and U_3.

3.2.2 Main results

System (3.1.1) is degenerate : when $U_i=0$ the equation for U_i degenerates into first order equation. Hence classical solutions cannot be expected for Problem (3.1.1) – (3.1.3). A suitable notion of generalized solutions is required : we adopt the notion of weak solution introduced in Oleinik *et al* [51].

Definition 3.1 *A quadruple* (U_1,U_2,U_3,U_4) *of nonnegative and continuous functions* U_i : $\Omega\times[0,+\infty)\to[0,+\infty)$, $i=1..4$*, is a weak solution of Problem* (3.1.1) − (3.1.3) *in* Q_T, $T>0$ *if for each* $i=1..4$ *and for each* $\varphi_i\in C^1(\bar{Q}_T)$*, such that* $\dfrac{\partial\varphi_i}{\partial\eta}=0$ *on* $\partial\Omega\times(0,T)$.

(i) ∇U^{m_i} *exists in the sense of distribution and* $\nabla U_i^{m_i}\in L^2(Q_T)$;

(ii) U_i *verifies the identity*

$$\begin{aligned}&\int_\Omega U_i(x,T)\varphi_i(x,T)dx+\int_{Q_T}\nabla U_i^{m_i}\nabla\varphi_i(x,t)dxdt\\&=\int_{Q_T}(\partial_t\varphi_iU_i-f_i\varphi_i)(x,t)dxdt+\int_\Omega U_{i,0}(x)\varphi_i(x,0)dx.\end{aligned}\tag{3.2.5}$$

We are now ready to state our first result.

Theorem 3.1 *For each quadruple of continuous nonnegative initial functions* $(U_{1,0},U_{2,0},U_{3,0},U_{4,0})$ *there exists a unique weak solution* (U_1,U_2,U_3,U_4) *of Problem* (3.1.1) − (3.1.3) *on* Q_∞ *; furthermore*

(i) For all $i=1..3$, $U_i\in L^1\cap L^\infty(Q_\infty)$ *and* $\nabla U_i^{m_i}$, $\partial_tU_i^{m_i}\in L^2(Q_{\tau,\infty})$, $\tau>0$ *;*

(ii) $U_4\in L^1\cap L^\infty(Q_T)$ *and* $\nabla U_4^{m_4}$, $\partial_tU_4^{m_4}\in L^2(Q_{\tau,T})$, $\tau>0$.

The proof is found in Section §3.4.

Now we look at the large time behavior of weak solutions.

Theorem 3.2 *There exist nonnegative constants* U_1^*, U_4^* *such that*

$$\begin{array}{rcl} U_2(.,t), U_3(.,t) & \longrightarrow & 0 \ \ , \ \ U_1(.,t) \longrightarrow U_1^* \ \ in \ \ C(\bar\Omega) \ \ as \ \ t \longrightarrow +\infty \\ and\ \overline{U_4}(t) & \longrightarrow & U_4^* \ \ in \ \ L^p(\Omega) \ forall\ p \geq 1 \ \ as \ \ t \longrightarrow +\infty; . \end{array}$$

moreover, if $\nu > 0$ *then* $U_1^* = 0$.

The proof is found in Section §3.5.

Remark 3.2 *In the non degenerate case* $m_4 = 1$ *one has that* $U_4(.,t) \longrightarrow U_4^*$ *in* $C(\bar\Omega)$. *More generally, this still holds provided* U_4 *lies in* $L^\infty(Q_\infty)$, *the proof being similar to the one for* U_1 *when* $\nu = 0$, *see subsection §§3.5.2.*

3.3 Auxiliary problem and a priori estimates

In this section we consider an auxiliary problem depending on a small parameter ε, with $0 < \varepsilon \leq 1$. Namely let us introduce in $\Omega \times (0, +\infty)$ the quasilinear nondegenerate initial and boundary value problem

$$(3.3.6) \qquad \left\{ \begin{array}{l} \partial_t U_1 - \Delta d_1(U_1) = -\gamma((U_1 - \varepsilon)^+, U_2, U_3, U_4) - \nu(U_1 - \varepsilon), \\ \partial_t U_2 - \Delta d_2(U_2) = \gamma((U_1 - \varepsilon)^+, U_2, U_3, U_4) - (\lambda + \mu)(U_2 - \varepsilon), \\ \partial_t U_3 - \Delta d_3(U_3) = \lambda\pi(U_2 - \varepsilon) - (\alpha + m + \mu)(U_3 - \varepsilon), \\ \partial_t U_4 - \Delta d_3(U_4) = (1 - \pi)\lambda(U_2 - \varepsilon) + \alpha(U_3 - \varepsilon) + \nu(U_1 - \varepsilon). \end{array} \right.$$

$$(3.3.7) \qquad \left\{ \begin{array}{l} U_{i,\varepsilon}(x, 0) = U_{i,0,\varepsilon}(x) \geq 0, \ \ x \in \Omega; \\ \dfrac{\partial d_i(U_{i,\varepsilon})}{\partial \eta}(x, t) = 0, \ \ x \in \partial\Omega, \ \ t > 0, \end{array} \right. \qquad i = 1..4.$$

Herein $(r)^+$ is the nonnegative part of the real number r; for each $i = 1..4$ $d_i : I\!R \longrightarrow (\frac{\varepsilon}{2}, +\infty)$ is a smooth and increasing functions with

$$(3.3.8) \qquad d_i(u) = u^{m_i}, \ \ \varepsilon \leq u;$$

$(U_{i,0,\varepsilon})_{i=1..4}$ is a quadruple of smooth functions over $\bar\Omega$ such that

$$(3.3.9) \qquad \left\{ \begin{array}{l} U_{i,0,\varepsilon}(x) \geq \varepsilon, \ \ x \in \Omega, \ \ 0 < \varepsilon \leq 1; \\ \displaystyle\int_\Omega (U_{i,0,\varepsilon}(x) - \varepsilon)dx = \int_\Omega U_{i,0}(x)dx \\ U_{i,0,\varepsilon} \longrightarrow U_{i,0} \ \text{ in } C(\bar\Omega), \ \text{ as } \ \varepsilon \longrightarrow 0; \end{array} \right. \qquad i = 1..4;$$

we refer to [3, 35] for a construction of such a set of initial data. From standard results, i.e. [42] or [56], local existence and uniqueness of a quadruple $(U_{1,\varepsilon}, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})$, a classical solution of (3.3.6) – (3.3.7) in some maximal interval $[0, T_{max,\varepsilon})$ is granted.

Looking at the equation for $U_{i,\varepsilon}$ it is checked that ε is a subsolution, thus $0<\varepsilon\le U_{i,\varepsilon}(x,t),\ x\in\Omega,\ 0<t<T_{max,\varepsilon}$. Next, from the maximum principle and the nonnegativity of γ,ν and $U_{1,\varepsilon}-\varepsilon$, it follows $U_{1,\varepsilon}(x,t)\le\|U_{1,\varepsilon,0}\|_{\infty,\Omega},\ \ x\in\Omega,\ \ 0<t<T_{max,\varepsilon}$. As a consequence one has

$$\left\{\begin{array}{l} 0<\varepsilon\le U_{1,\varepsilon}(x,t)\le\|U_{1,\varepsilon,0}\|_{\infty,\Omega},\ \ x\in\Omega,\ \ t<T_{max,\varepsilon} \\ 0<\varepsilon\le U_{i,\varepsilon}(x,t),\ \ x\in\Omega,\ \ t<T_{max,\varepsilon},\ \ i=2..4 \end{array}\right. \tag{3.3.10}$$

Then one can apply results in [25] to show global existence, i.e. $T_{max,\varepsilon}=+\infty$, of a classical solution for (3.3.6) − (3.3.7). Using (3.3.8) and (3.3.10) this yields global existence for the initial and boundary value problem

$$\left\{\begin{array}{l} \partial_t U_1-\Delta U_1^{m_1}=-\gamma(U_1-\varepsilon,U_2,U_3,U_4)-\nu(U_1-\varepsilon), \\ \partial_t U_2-\Delta U_2^{m_2}=\gamma(U_1-\varepsilon,U_2,U_3,U_4)-(\lambda+\mu)(U_2-\varepsilon), \\ \partial_t U_3-\Delta U_3^{m_3}=\lambda\pi(U_2-\varepsilon)-(\alpha+m+\mu)(U_3-\varepsilon), \\ \partial_t U_4-\Delta U_4^{m_4}=(1-\pi)\lambda(U_2-\varepsilon)+\alpha(U_3-\varepsilon)+\nu(U_1-\varepsilon). \end{array}\right. \tag{3.3.11}$$

in $\Omega\times(0,+\infty)$, together with (3.3.7).

We derive a priori estimates. First, using the L^1 property of $U_{1,0,\varepsilon}$ in (3.3.9) and the nonnegativity of $U_{1,\varepsilon}-\varepsilon$, a straightforward integration of the equation for $U_{1,\varepsilon}$ over $\Omega\times(0,+\infty)$ gives :

$$\int_{Q_T}(\gamma(U_{1,\varepsilon}-\varepsilon,U_{2,\varepsilon},U_{3,\varepsilon},U_{4,\varepsilon})+\nu(U_{1,\varepsilon}-\varepsilon))(x,t)dxdt\le\int_\Omega U_{1,0}(x)dx. \tag{3.3.12}$$

In what follows T is a positive number, $M_1,..,M_n$ are positive constants independent of T and $\varepsilon, 0<\varepsilon\le 1$, and $F_1,..,F_n$ are non decreasing functions of T independent of $\varepsilon, 0<\varepsilon\le 1$.

Lemma 3.1 *There exists a constant M_1 and nondecreasing function F_1, independent of $\varepsilon, 0<\varepsilon\le 1$ such that*

$$0<\varepsilon\le U_{i,\varepsilon}(x,t)\le M_1,\ x\in\Omega, t>0, i=1..3; \tag{3.3.13}$$

$$\varepsilon\le U_{4,\varepsilon}(x,t)\le F_1(T),\quad x\in\Omega,\quad 0<t<T. \tag{3.3.14}$$

Proof. The estimate for $U_{1,\varepsilon}$ follows from (3.3.10) and the choice of $(U_{1,0,\varepsilon})_{\varepsilon>0}$. Multiplying the equation for $U_{2,\varepsilon}$ by $p\,(U_{2,\varepsilon}-\varepsilon)^{p-1},\ p\ge 1$, and integrating over Ω one has

$$\begin{array}{l} \dfrac{d}{dt}\|U_{2,\varepsilon}(.,t)-\varepsilon\|^p_{p,\Omega}+p(\lambda+\mu)\,\|U_{2,\varepsilon}(.,t)-\varepsilon\|^p_{p,\Omega} \\ \qquad\le p\displaystyle\int_\Omega\gamma(U_{1,\varepsilon}-\varepsilon,U_{2,\varepsilon},U_{3,\varepsilon},U_{4,\varepsilon})(U_{2,\varepsilon}-\varepsilon)^{p-1}(x,t)dx; \end{array}$$

keeping in mind $\lambda+\mu>0$ from $(H1)$, one gets from Young's inequality

$$\frac{d}{dt}\|U_{2,\varepsilon}(.,t)-\varepsilon\|^p_{p,\Omega}\le(\frac{1}{\lambda+\mu})^{p-1}\int_\Omega[\gamma(U_{1,\varepsilon}-\varepsilon,U_{2,\varepsilon},U_{3,\varepsilon},U_{4,\varepsilon})]^p(x,t)dx. \tag{3.3.15}$$

A further integration over $(0,T)$ leads to

$$\|U_{2,\varepsilon}(.,T)-\varepsilon\|_{p,\Omega}^p \leq \|U_{2,0,\varepsilon}-\varepsilon\|_{p,\Omega}^p + (\frac{1}{\lambda+\mu})^{p-1}\int_{Q_T}[\gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})]^p(x,t)dxdt.$$

Using the already known L^∞ estimate for $U_{1,\varepsilon}$, assumption $(H4)$ and inequality (3.3.12) one arrives at : for each $T>0$

$$\|U_{2,\varepsilon}(.,T)-\varepsilon\|_{p,\Omega}^p \leq \|U_{2,0,\varepsilon}-\varepsilon\|_{p,\Omega}^p + (\frac{1}{\lambda+\mu})^{p-1}(C_1+C_2M_1^r)^{p-1}\|U_{1,0}\|_{1,\Omega}. \tag{3.3.16}$$

To conclude, one observes that $U_{2,\varepsilon}-\varepsilon$ being continuous on $\bar{\Omega}\times[0,+\infty)$ it follows

$$\lim_{p\to+\infty}\|U_{2,\varepsilon}(.,t)-\varepsilon\|_{p,\Omega} = \|U_{2,\varepsilon}(.,t)-\varepsilon\|_{\infty,\Omega}.$$

Hence for some constant M_2 independent of $\varepsilon, 0<\varepsilon\leq 1$, one gets

$$0<\varepsilon\leq U_{2,\varepsilon}(x,t)\leq M_2,\quad x\in\Omega,\quad t>0. \tag{3.3.17}$$

Now, integrating the equation for $U_{2,\varepsilon}$ over $\Omega\times[0,T)$, using the L^1 property of $U_{2,0,\varepsilon}$ in (3.3.9), the nonnegativity of $U_{2,\varepsilon}-\varepsilon$ and (3.3.12) one has for $0<\varepsilon\leq 1$

$$(\lambda+\mu)\int_{Q_T}(U_{2,\varepsilon}-\varepsilon)(x,t)dxdt \leq \int_\Omega[U_{1,0,\varepsilon}(x)+U_{2,0,\varepsilon}(x)]dx. \tag{3.3.18}$$

The estimate for $U_{3,\varepsilon}$ follows from computations similar to the ones for $U_{2,\varepsilon}$ above, carried over the equation for $U_{3,\varepsilon}$ and getting help from (3.3.18) and from the positivity of $\alpha+m+\mu$.

Along the same lines, from the equation for $U_{3,\varepsilon}$ one gets for $0<\varepsilon\leq 1$

$$(\alpha+m+\mu)\int_{Q_T}(U_{3,\varepsilon}-\varepsilon)dxdt \leq \int_\Omega[U_{1,0,\varepsilon}(x)+U_{2,0,\varepsilon}(x)+U_{3,0,\varepsilon}(x)]dx. \tag{3.3.19}$$

Hence, going back to the equation for $U_{4,\varepsilon}$ one can derive the a priori estimate upon multiplying it by $p\,(U_{2,\varepsilon}-\varepsilon)^{p-1},\ p\geq 1$ and using (3.3.18) – (3.3.19). □

Lemma 3.2 *There exists constants $M_{i,3},\ i=1..3$ and a nondecreasing function F_2, independent of $\varepsilon, 0<\varepsilon\leq 1$ such that*

$$\int_{Q_T}\|\nabla U_{i,\varepsilon}^{m_i}\|^2(x,t)dxdt \leq M_{i,3},\quad T>0,\quad i=1..3; \tag{3.3.20}$$

$$\int_{Q_T}\|\nabla U_{4,\varepsilon}^{m_i}\|^2(x,t)dxdt \leq F_2(T),\quad T>0. \tag{3.3.21}$$

Proof. The estimate for $\nabla U_{1,\varepsilon}^{m_1}$ is obtained upon multiplying the equation for $U_{1,\varepsilon}$ by $U_{1,\varepsilon}^{m_1}$, integrating over $\Omega \times (0,T)$ and using the nonnegativity of γ and $U_{1,\varepsilon} - \varepsilon$. One finds

$$M_{1,3} = \frac{1}{m_1+1}\int_\Omega U_{1,\varepsilon}^{m_1+1}(x,0)dx$$

Proceedings along the same lines for $U_{2,\varepsilon}$ one gets

$$\frac{1}{m_2+1}\int_\Omega U_{2,\varepsilon}^{m_2+1}(x,T)dx + \int_{Q_T} \|\nabla U_{2,\varepsilon}^{m_2}(x,t)\|^2 dxdt \leq$$
$$\frac{1}{m_2+1}\int_\Omega U_{2,\varepsilon}^{m_2+1}(x,0)dx + \int_{Q_T} \gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})[U_{2,\varepsilon}]^{m_2}(x,t)dxdt.$$

Using the properties of $U_{2,0,\varepsilon}$, the uniform estimate for $U_{2,\varepsilon}$ in Lemma 3.1 and the L^1 estimate for γ in (3.3.12) we obtain

$$\int_{Q_T} \|\nabla U_{2,\varepsilon}^{m_2}(x,t)\|^2 dxdt \leq M_{2,3}, \quad T>0.$$

A similar computation supplies the estimate for $U_{3,\varepsilon}$. The estimate for $U_{4,\varepsilon}$ then follows.
□

Lemma 3.3 *For all $t > 0$*
(3.3.22)

$$\|\nabla U_{1,\varepsilon}^{m_1}(.,t)\|_{2,\Omega}^2 \leq \frac{2}{t(m_1+1)}\int_\Omega U_{1,0,\varepsilon}^{m_1+1}(x)dx$$
$$+m_1^2\nu^2\|U_{1,0}\|_{\infty,\Omega}^{m_1-1}\int_{Q_{\frac{t}{2},t}} (U_{1,\varepsilon}-\varepsilon)^2(x,s)dxds$$
$$+m_1^2\|U_{1,0}\|_{\infty,\Omega}^{m-1}(C_1 + C_2\|U_{1,0}\|_{\infty,\Omega}^r)\int_{Q_{\frac{t}{2},t}} \gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})(x,s)dxds$$

Proof. Let us multiply by $\partial_t U_{1,\varepsilon}^{m_1}$ the equation for $U_{1,\varepsilon}$ and integrate over $\Omega\times(\tau,t)$, $\frac{t}{2} \leq \tau \leq t$; then one finds
(3.3.23)

$$(\frac{2}{m_1+1})^2\int_{Q_{\tau,t}} (\partial_t U_{1,\varepsilon}^{\frac{m_1+1}{2}})^2(x,s)dxds + \|\nabla U_{1,\varepsilon}^{m_1}(.,t)\|_{2,\Omega}^2$$
$$\leq \int_{Q_{\tau,t}} (-\gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon}) - \nu(U_{1,\varepsilon}-\varepsilon))\partial_t U_{1,\varepsilon}^{m_1}(x,s)dxds + \|\nabla U_{1,\varepsilon}^{m_1}(.,\tau)\|_{2,\Omega}^2.$$

Next, for any suitably smooth and nonnegative function U and any $m > 1$ one gets

$\partial_t U^m = \dfrac{2m}{m+1} U^{\frac{m-1}{2}} \partial_t U^{\frac{m+1}{2}}$ so that

$$
\begin{aligned}
&\int_{Q_{\tau,t}} (-\gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon}) - \nu(U_{1,\varepsilon}-\varepsilon))\partial_t U_{1,\varepsilon}^{m_1}(x,s)dxds \\
&\quad \le \frac{2}{(m_1+1)^2}\int_{Q_{\frac{t}{2},t}} (\partial_t U_{1,\varepsilon}^{(\frac{m_1+1}{2})})^2(x,s)dxds \\
&\quad +m_1^2\nu^2 \int_{Q_{\frac{t}{2},t}} [(U_{1,\varepsilon}-\varepsilon)U_{1,\varepsilon}^{\frac{m-1}{2}}]^2(x,s)dxds \\
&\quad +m_1^2 \int_{Q_{\frac{t}{2},t}} [\gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})U_{1,\varepsilon}^{\frac{m-1}{2}}]^2(x,s)dxds.
\end{aligned} \tag{3.3.24}
$$

The last term on the right handside of this inequality is bounded from above by

$$m_1^2\|U_{1,\varepsilon}\|_{\infty,Q_\infty}^{m-1}\|\gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})\|_{\infty,Q_\infty}\int_{Q_{\frac{t}{2},t}} \gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})(x,s)dxds.$$

putting this estimate in (3.3.23) one obtains
(3.3.25)

$$
\begin{aligned}
&\|\nabla U_{1,\varepsilon}^{m_1}(.,t)\|_{2,\Omega}^2 \le \|\nabla U_{1,\varepsilon}^{m_1}(.,\tau)\|_{2,\Omega}^2 + m_1^2\nu^2\|U_{1,0}\|_{\infty,\Omega}^{m_1-1}\int_{Q_{\frac{t}{2},t}} (U_{1,\varepsilon}-\varepsilon)^2(x,s)dxds \\
&+m_1^2\|U_{1,0}\|_{\infty,\Omega}^{m_1-1}(C_1 + C_2\|U_{1,0}\|_{\infty,\Omega}^r)\int_{Q_{\frac{t}{2},t}} \gamma(U_{1,\varepsilon}-\varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})(x,t)dxdt.
\end{aligned}
$$

Integrating this inequality in τ over $(\frac{t}{2}, t)$ and using the explicit value for $M_{1,3}$ found in the proof of Lemma 3.2 we deduce the desired result. □

Lemma 3.4 *There exists a constant M_1 and non decreasing function F_1, independent of $\varepsilon, 0 < \varepsilon \le 1$ such that*

$$\int_{Q_T} |(U_{i,\varepsilon}^{m_i})_t|^2(x,t)dxdt \le M_2, \quad T > 0, \quad i = 1..3; \tag{3.3.26}$$

$$\int_{Q_T} |(U_{4,\varepsilon}^{m_i})_t|^2(x,t)dxdt \le F_2(T), \quad T > 0. \tag{3.3.27}$$

Proof. The estimate for $U_{1,\varepsilon}$ is immediatly deduced from (3.3.23) keeping in mind that

$$|(U_{1,\varepsilon}^{m_1})_t|^2(x,t) \le \frac{{m_1}^2}{2}\|U_{1,\varepsilon}\|_{\infty,\Omega}^{m_1-1}(U_{1,\varepsilon}^{\frac{m_1+1}{2}})_t^2(x,t).$$

And one can establish such estimates for $U_{2,\varepsilon}, U_{3,\varepsilon}$ and $U_{4,\varepsilon}$ in the same way. □

3.4 Existence and uniqueness : proofs

In this section we supply a quick proof of Theorem 3.1.

3.4.1 Existence

Let us fix $T > 0$. From the estimates established in the previous section one has : for each $i = 1..4$ $(U_{i,\varepsilon} - \varepsilon)_{0<\varepsilon\leq 1}$ and $(\nabla U_{i,\varepsilon}^{m_i})_{0<\varepsilon\leq 1}$ are respectively bounded in $L^2(Q_T)$ and $(L^2(Q_T))^N$. Then there exists two sequences which one still denotes $(U_{i,\varepsilon} - \varepsilon)_{0<\varepsilon\leq 1}$ and $(\nabla U_{i,\varepsilon}^{m_i})_{0<\varepsilon\leq 1}$ such that for $i = 1..4$ as $\varepsilon \to 0$: $(U_{i,\varepsilon} - \varepsilon)_{0<\varepsilon\leq 1}$ is weakly convergent to some U_i in $L^2(Q_T)$ and $(\nabla U_{i,\varepsilon}^{m_i})_{0<\varepsilon\leq 1}$ is weakly convergent to some V_i in $(L^2(Q_T))^N$. On the other hand $(U_{i,\varepsilon})_{0<\varepsilon\leq 1}$ is bounded in $L^\infty(Q_T)$; using a weak formulation of the equation for $U_{i,\varepsilon}$ one can invoke the results in Di Benedetto [21] to get : $(U_{i,\varepsilon})_{0<\varepsilon\leq 1}$ is a relatively compact subset of $C(\overline{\Omega} \times [0,T])$. It follows that actually $(U_{i,\varepsilon} - \varepsilon)_{0<\varepsilon\leq 1}$ is convergent to U_i in $C(\overline{\Omega} \times [0,T])$ and $(U_{i,\varepsilon}^{m_i})_{0<\varepsilon\leq 1}$ is convergent to $U_i^{m_i}$ in $C(\overline{\Omega} \times [0,T])$. As a first consequence one has : $V_i = \nabla U_i^{m_i}$; next one also has :

$$\gamma(U_{1,\varepsilon} - \varepsilon, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon}) \;\to\; \gamma(U_1, U_2, U_3, U_4) \;\text{ in }\; C(\overline{\Omega} \times [0,T]) \;\text{ as }\; \varepsilon \to 0.$$

From standard arguments one may conclude that the quadruple (U_1, U_2, U_3, U_4) is a desirable weak solution.

The regularity results for $\nabla U_i^{m_i}$ and $\partial_t U_i^{m_i}$ follow from the a priori estimates in Lemma 3.2 and Lemma 3.4.

3.4.2 Uniqueness

Assume there exists two quadruples $(U_{j,1}, U_{j,2}, U_{j,3}, U_{j,4})_{j=1,2}$, both weak solutions of Problem (3.1.1) – (3.1.3). They verify the integral identity, for $i = 1..4$

(3.4.28)
$$\int_\Omega (U_{1,i} - U_{2,i})(x,T)\varphi_i(x,T)dx + \int_{Q_T} \nabla(U_{1,i}^{m_i} - U_{2,i}^{m_i})\nabla\varphi_i(x,t)dxdt$$
$$= \int_{Q_T} [\partial_t\varphi_i(U_{1,i} - U_{2,i}) - (f_i(U_{1,1}, U_{1,2}, U_{1,3}, U_{1,4}) - f_i(U_{2,1}, U_{2,2}, U_{2,3}, U_{2,4}))\varphi_i](x,t)dxdt$$

for every $\varphi_i \in C^1(\bar{Q}_T)$, such that $\dfrac{\partial\varphi_i}{\partial\eta} = 0$ on $\partial\Omega \times (0,T)$ and $\varphi_i > 0$.

We follow an idea of [44] and introduce a function ψ_i as follows

(3.4.29)
$$\psi_i(x,t) = \begin{cases} \dfrac{U_{1,i}^{m_i} - U_{2,i}^{m_i}}{U_{1,i} - U_{2,i}} & \text{if } U_{1,i} \neq U_{2,i}, \\ 0 & \text{otherwise.} \end{cases} \quad i = 1..4.$$

Let us consider a sequence of smooth functions $(\psi_{i,\varepsilon})_{\varepsilon\geq 0}$ such that $\psi_{i,\varepsilon} \geq \varepsilon$, $\psi_{i,\varepsilon}$ is uniformly bounded in $L^\infty(Q_T)$ and

$$\lim_{\varepsilon\to 0} \|(\psi_{i,\varepsilon} - \psi_i)/\sqrt{\psi_{i,\varepsilon}}\|_{L^2(Q_T)} = 0.$$

For any $0 < \varepsilon \leq 1, \sigma > 0$ let us introduce the adjoint nondegenerate boundary value problem

(3.4.30)
$$\begin{cases} \partial_t\varphi_i + \psi_{i,\varepsilon}\Delta\varphi_i = 0 & \text{in } \Omega \times (0,T) \\ \dfrac{\partial\varphi_i}{\partial\eta}(x,t) = 0 & \text{in } \partial\Omega \times (0,T) \quad i = 1..4. \\ \varphi_i(x,T) = \chi_i & \text{in } \Omega \end{cases}$$

For any smooth χ_i with $0 \le \chi_i(x,t) \le 1,\ i = 1..4$, any $0 < \varepsilon \le 1$ and any $\sigma > 0$ this problem has unique classical solution $\varphi_{i,\varepsilon}$ such that see [44]

$$0 \le \varphi_{i,\varepsilon}(x,t) \le 1 \tag{3.4.31}$$

$$\int_{Q_T} \psi_{i,\varepsilon}(\Delta\varphi_{i,\varepsilon})^2 dxdt \le K_1, \tag{3.4.32}$$

If in (3.4.28) we replace φ_i by $\varphi_{i,\varepsilon}$, which is the solution of problem (3.4.30) with $\chi_i = sign((U_i - V_i)^+)$ we obtain.

$$\begin{aligned} &\int_\Omega (U_{1,i} - U_{2,i})^+(x,T)\varphi_{i,\varepsilon}(x,T)dx + \int_{Q_T} (\psi_i - \psi_{i,\varepsilon})(U_{1,i} - U_{2,i})\Delta\varphi_{i,\varepsilon} dxdt \\ &= \int_{Q_T} (f_i(U_{1,1}, U_{1,2}, U_{1,3}, U_{1,4}) - f_i(U_{2,1}, U_{2,2}, U_{2,3}, U_{2,4}))\varphi_{i,\varepsilon} dxdt \end{aligned} \tag{3.4.33}$$

Using the local lipschitz continuity of f_i and the properties of $\psi_{i,\varepsilon}$ and $\varphi_{i,\varepsilon}$ we deduce by letting $\varepsilon \to 0$

$$\int_\Omega (U_{1,i} - U_{2,i})^+(x,T)dx \le K \int_{Q_T} \sum_{i=1}^4 |U_{1,i} - U_{2,i}|(x,t)dxdt \tag{3.4.34}$$

In a similar fashion we establish an analogous inequality for $(U_i - V_i)^-$ and deduce

$$\int_\Omega \sum_{i=1}^4 |U_{1,i} - U_{2,i}|(x,T)dx \le K \int_{Q_T} \sum_{i=1}^4 |U_{1,i} - U_{2,i}|(x,t)dxdt \tag{3.4.35}$$

Uniqueness follows from Gronwall's Lemma.

3.5 Large time behavior : proofs

The semi-orbit $\{(U_1(.,t), U_2(.,t), U_3(.,t)), t \ge 0\}$ is relatively compact in $(C(\bar{\Omega}))^3$: it is actually bounded in $(L^\infty(Q_\infty))^3$ by (3.3.20) and then one may use a result of [21].

3.5.1 Case $\nu > 0$

A convergence and continuity argument allows to deduce from (3.3.12)

$$\int_{Q_T} \gamma(U_1, U_2, U_3, U_4)(x,t)dxdt + \nu \int_{Q_T} U_1\ (x,t)dxdt\ \le \|U_{1,0}\|_{1,\Omega},\ \ T > 0. \tag{3.5.36}$$

Hence $U_1 \in L^1(\Omega \times (0,+\infty))$ and there is a sequence $(\tau_j)_{j\ge 0}$ such that $\tau_j \longrightarrow +\infty$ as $j \longrightarrow +\infty$ and $\int_\Omega U_1(x,\tau_j)dx \longrightarrow 0$ as $j \longrightarrow +\infty$. Next, given any $t > \tau_j$, one has

$$0 \le \int_\Omega U_1(x,t)dx \le \int_\Omega U_1(x,\tau_j)dx; \tag{3.5.37}$$

actually such an identity holds for $U_{1,\varepsilon}$ from a straightforward integration over $\Omega \times (\tau_j, t)$ and is preserved upon letting $\varepsilon \longrightarrow 0$ because $U_{1,\varepsilon} \rightarrow U_1$ in $C^0(\overline{\Omega} \times (0, +\infty)$ as $\varepsilon \rightarrow 0$. This shows that $U_1(.,t) \longrightarrow 0$ in $L^1(\Omega)$ as $t \longrightarrow +\infty$ and also in $C(\bar{\Omega})$.

Then, along the same lines, from (3.3.18) and (3.3.19) one has for $T > 0$

$$\begin{aligned}(3.5.38) \qquad &(\lambda + \mu) \int_{Q_T} U_2\ (x,t)dxdt + (\alpha + m + \mu) \int_{Q_T} U_3\ (x,t)dxdt \\ &\leq 2\|U_{1,0}\|_{1,\Omega} + 2\|U_{2,0}\|_{1,\Omega} + \|U_{3,0}\|_{1,\Omega}.\end{aligned}$$

Again, for some sequence $(\tau_j)_{j\geq 0}$ such that $\tau_j \longrightarrow +\infty$ one has $\int_\Omega U_2(x,\tau_j)dx \longrightarrow 0$ as $j \longrightarrow +\infty$. Integrating over $\Omega \times (\tau_j, t)$ the equation in (3.3.11) for $U_{2,\varepsilon}$ and letting $\varepsilon \longrightarrow 0$ one finds

$$(3.5.39) \qquad 0 \leq \int_\Omega U_2(x,t)dx \ \leq \int_{\tau_j}^t \int_\Omega \gamma(U_1, U_2, U_3, U_4)(x,\tau)dxd\tau + \int_\Omega U_2(x,\tau_j)dx;$$

thus again $U_2(.,t) \longrightarrow 0$ in $L^1(\Omega)$ and in $C(\bar{\Omega})$ because γ lies in $L^1(\Omega \times (0, +\infty)$.

The conclusion for $U_3(.,t)$ is derived in the same fashion, using the third equation in (3.3.11).

Now we will steady the long time behavior of U_4, to do this let us consider for any $\tau > 0$ the following problem

$$(3.5.40) \qquad \begin{cases} \partial_t V - \Delta V^{m_4} = 0, & (x,t) \in \Omega \times (0, +\infty) \\ V(x,0) = U_4(x,\tau), & x \in \Omega; \\ \dfrac{\partial V^{m_4}}{\partial \eta}(x,t) = 0. & x \in \partial\Omega, \quad t > 0. \end{cases}$$

It is well known see [2] that $\lim\limits_{t \to +\infty} V(.,t) = \overline{V}(0) = \overline{U_4}(\tau)$ in $L^p(\Omega)$, for all $p \geq 1$, and in another hand from [11] we have for all $p \geq 1$

$$(3.5.41) \qquad \|U_4(x,\tau + h) - V(x,h)\|_{p,\Omega} \leq \int_h^{\tau+h} \|f_4(x,s)\|_{p,\Omega}\, ds,$$

with $f_4(x,t) = (1-\pi)\lambda U_2(x,t) + \alpha U_3(x,t) + \nu U_1(x,t)$. Set $\tau = h = \frac{t}{2}$, we can write

$$\begin{aligned}\|U_4(x,t) - \overline{U_4}(\frac{t}{2})\|_{p,\Omega} &\leq \|U_4(x,t) - V(x,\frac{t}{2})\|_{p,\Omega} + \|V(x,t) - \overline{U_4}(\frac{t}{2})\|_{p,\Omega}, \\ &\leq \int_{\frac{t}{2}}^t \|f_4(x,s)\|_{p,\Omega} ds + \|V(x,t) - \overline{U_4}(\frac{t}{2})\|_{p,\Omega};\ p \geq 1,\end{aligned}$$

since $f_4 \in L^1(Q_\infty) \cap L^\infty(Q_\infty)$ we deduce that $\lim_{t \to +\infty} \|U_4(x,t) - \overline{U_4}(\frac{t}{2})\|_{p,\Omega} = 0$, furthermore $f_4 \geq 0$ allow to show that $t \longrightarrow \overline{U_4}(t)$ is bounded and nondecreasing and then converges to some nonnegative constant U_4^* and this yields $\lim\limits_{t \to +\infty} U_4(.,t) = \lim\limits_{t \to +\infty} \overline{U_4}(t) = U_4^*$ in $L^p(\Omega)$ for all $p \geq 1$.

3.5.2 Case $\nu = 0$

The analysis of the behavior of $\{U_1(.,t), t > 0\}$ requires modifications because it is not known, and actually it is not true, that $U_1 \in L^1(\Omega \times (0, +\infty))$. Set

$$\bar{\phi}(t) = \frac{1}{mes(\Omega)} \int_\Omega \phi(x,t)dx;$$

then multiplying the equation for $U_{1,\varepsilon}$ in (3.3.12) by $\frac{1}{m_1} U_{1,\varepsilon}^{m_1 - 1}$ and integrating over $\Omega \times (\tau, \tau + t)$ yields

(3.5.42) $$\overline{U_{1,\varepsilon}^{m_1}}(\tau) \geq \overline{U_{1,\varepsilon}^{m_1}}(\tau + t) \geq 0, \tau > 0, t > 0;$$

so that upon letting $\varepsilon \longrightarrow 0$, the average $\bar{U}_1^{m_1}$ is a nonincreasing function of time. From the inequality of Poincaré-Wirtinger one can conclude to the existence of a constant $K(\Omega)$ such that for $t > 0$

(3.5.43) $$\|U_1^{m_1}(.,t) - \overline{U_1^{m_1}}(t)\|_{2,\Omega} \leq K(\Omega) \|\nabla U_1^{m_1}(.,t)\|_{2,\Omega}.$$

Now, one gets from Lemma 3.3 with $\nu = 0$ that

(3.5.44) $$\begin{aligned} &\|\nabla U_1^{m_1}(.,t)\|_{2,\Omega}^2 \leq \frac{2}{t(m_1+1)} \int_\Omega U_{1,0}^{m_1+1}(x)dx \\ &+ m_1^2 \|U_{1,0}\|_{\infty,\Omega}^{m-1} (C_1 + C_2 \|U_{1,0}\|_{\infty,\Omega}^r) \int_{Q_{\frac{t}{2},t}} \gamma(U_1, U_2, U_3, U_4)(x,s)dxds \end{aligned}$$

It follows that $\|\nabla U_1^{m_1}(.,t)\|_{2,\Omega} \longrightarrow 0$ as $t \longrightarrow +\infty$, so that $\|U_1^{m_1}(.,t) - \bar{U}_1^{m_1}(t)\|_{2,\Omega} \longrightarrow 0$ by (3.5.43). The monotonicity of $t \longrightarrow \overline{U_1^{m_1}}(t)$ yields $\lim_{t \to +\infty} U_1^{m_1}(.,t) = \lim_{t \to +\infty} \bar{U}_1^{m_1}(t) = U_1^*$ in $L^2(\Omega)$ and also in $C(\bar{\Omega})$.

3.5.3 An elementary spatially homogeneous system

Let us consider the system of ordinary differential equation

(3.5.45) $$\begin{cases} U_1' &= -\gamma(U_1, U_2, U_3, U_4) \\ U_2' &= \gamma(U_1, U_2, U_3, U_4) - \lambda U_2 - \mu U_2, \\ U_3' &= \lambda \pi U_2 - \alpha U_3 - \mu U_3 - m U_3, \\ U_4' &= (1-\pi)\lambda U_2 + \alpha U_3. \end{cases}$$

With $U_i(0) \geq 0, \ \ i = 1..4, \ \ U_1(0) > 0, \ \ U_3(0) \geq 0$ and,

$$\begin{cases} \lambda > 0, \ \ \alpha > 0, \ \ m \geq 0, \ \ \mu \geq 0, \\ \gamma(U_1, U_2, U_3, U_4) = \sigma(U_2, U_3, U_4)\, U_1, \end{cases}$$

γ having either a masse action or a proportionate mixing form : see the introduction.

Then $U_1(t) = U_1(0) exp\left(- \int_0^t \sigma(U_2, U_3, U_4)(\tau) d\tau \right)$ so that $U_1(t) \searrow U_1^* \geq 0$ as $t \longrightarrow +\infty$ and $U_1^* = 0$ if and only if $\int_0^{+\infty} \sigma(U_2, U_3, U_4)(\tau) d\tau = +\infty$.

Next $U_1 + U_2 = -(\lambda+\mu)U_2(t)$ and upon integrating over $(0,+\infty)$ one gets U_2 lies in $L^1(0,+\infty)$ so that $U_2(t) \longrightarrow 0$ as $t \longrightarrow +\infty$ because U_2' is bounded.

A similar argument yields U_3 lies in $L^1(0,+\infty)$ and $U_3(t) \longrightarrow 0$ as $t \longrightarrow +\infty$. Then one has $U_4(t) = U_4(0) + (1-\pi)\lambda \int_0^t U_2(\tau)d\tau + \alpha \int_0^t U_3(\tau)d\tau$. Here $U_4(t) \nearrow U_4^* > 0$ as $t \longrightarrow +\infty$.

To conclude that $U_1^* > 0$ note that

– When $\gamma(U_1,U_2,U_3,U_4) = \gamma U_1 U_3$ then $\sigma(U_2,U_3,U_4) = \gamma U_3$ lies in $L^1(0,+\infty)$.

– When $\gamma(U_1,U_2,U_3,U_4) = \gamma \dfrac{U_1U_3}{U_1+U_2+U_3+U_4}$ then $\sigma(U_2,U_3,U_4) = \gamma \dfrac{U_3}{U_1+U_2+U_3+U_4}$. Now $(U_1+U_2+U_3+U_4)(t) \longrightarrow U_1^* + U_4^*$ as $t \longrightarrow +\infty$ and $U_1^* + U_4^* > 0$, because $U_4^* > 0$ and $U_1^* \geq 0$; hence for $t \geq t_0$ one has

$$\frac{1}{2}(U_1^* + U_4^*) \leq (U_1+U_2+U_3+U_4)(t) \leq (U_3+U_4)(0)$$

which implies

$$\frac{U_3(t)}{(U_3+U_4)(0)} \quad \leq \quad \sigma(U_2,U_3,U_4)(t) \quad \leq \quad 2\frac{U_3(t)}{U_1^*+U_4^*}\,, \quad t \geq t_0.$$

As a conclusion $\sigma(U_2,U_3,U_4)$ lies in $L^1(0,+\infty)$ and $U_1^* > 0$.

Last when $m = \mu = 0$, $\;U_1^* + U_4^* = (U_1+U_2+U_3+U_4)(0)$.

Chapitre 4

Degenerate diffusive SEIR model with logistic population control

4.1 Introduction

In this chapter we shall be concerned with a degenerate parabolic system of the form

(4.1.1)

$$\begin{cases} \partial_t U_1 - \Delta U_1^{m_1} = & -\gamma(U_1,U_2,U_3,U_4) + \sum_{i=1}^4 b_{1i}U_i + \delta U_4 - \nu U_1 - (k_1 P + m_1)U_1 \\ & +F_1(x,t) = \ f_1(x,t,U_1,U_2,U_3,U_4), \\ \partial_t U_2 - \Delta U_2^{m_2} = & \gamma(U_1,U_2,U_3,U_4) + b_{22}U_2 - (k_2 P + m_2 + \lambda + \mu)U_2 \\ & +F_2(x,t) = \ f_2(x,t,U_1,U_2,U_3,U_4), \\ \partial_t U_3 - \Delta U_3^{m_3} = & b_{33}U_3 + \lambda\pi U_2 - (k_3 P + \alpha + m_3 + m + \mu)U_3 \\ & +F_3(x,t) = \ f_3(x,t,U_1,U_2,U_3,U_4), \\ \partial_t U_4 - \Delta U_4^{m_4} = & b_{44}U_4 + (1-\pi)\lambda U_2 + \alpha U_3 + \nu U_1 - \delta U_4 - (k_4 P + m_4)U_4 \\ & +F_4(x,t) = \ f_4(x,t,U_1,U_2,U_3,U_4). \end{cases}$$

in $\Omega \times (0,+\infty)$, subject to the initial conditions

$$U_i(x,0) = U_{i,0}(x) \geq 0, \quad x \in \Omega; \qquad i = 1..4. \tag{4.1.2}$$

and to the Neumann boundary conditions

$$\frac{\partial U_i^{m_i}}{\partial \eta}(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \qquad i = 1..4. \tag{4.1.3}$$

Herein, Ω is an open, bounded and connected domain in $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$; Δ is the Laplace operator in $\mathbb{R}^N$. Powers m_i verify $m_i > 1$, $i = 1..4$. Finally P is the total mass of the population $P = \sum_{i=1}^{4} U_i$, and F_i, $i = 1,..,4$ are nonnegative and continuous function on $\Omega \times (0,+\infty)$.

In the spatially homogeneous case this problem can be reduced to one of the models of propagation of an epidemic disease devised in Kermack and McKendricks [40], namely

$$\begin{cases} S' &= -\gamma SI, \\ I' &= +\gamma SI - \lambda I, \\ R' &= +\lambda I. \end{cases}$$

This basic model served as a starting point for many further developments, both from epidemiological or mathematical point of view see Busenberg and Cooke [16] or Capasso [18] and their references. Thus, system (4.1.1) leads to so-called $(S - E - I - R)$ models : $U_1 = S$ is the distribution of susceptible individuals in a given population, $\gamma(S, E, I, R)$ is the incidence term or number of susceptible individuals infected by contact with an infective individual $U_3 = I$ per time unit and becoming exposed $U_2 = E$, while $U_4 = R$ is the density of removed or resistant (immune) individuals. Then $b_{i,j}$ (resp. m_i) is the natural birth-rate (resp. death-rate), λ (resp. α) is the inverse of the duration of the exposed stage (resp. infective stage) or rate at which exposed individuals enter the infective class (resp. infective individuals who do not die from the disease recover), m is the additional mortality due to infection in the infective class, immunity is lost at rate δ, F_i represents an eventually source term and the quadratic term accounts for the damping of growth due to resource limitation of the habitat or environment. The last two parameters are control parameters : first ν is a vaccination rate ; next, for a population of animals, as it is considered here as in Anderson *et al* [10], Fromont *et al* [28], Courchamp *et al* [20] or Langlais and Suppo [43], μ is an elimination rate of exposed and infective individuals. Lastly, as it is suggested by the FeLV, a retrovirus of domestic cats (*Felis catus*) see [28], one also introduces a parameter π measuring the proportion of exposed individuals which actually develop the disease after the exposed stage, the remaining proportion $1 - \pi$ becoming resistant.

The nonlinear incidence term γ takes various forms as it can be found from the literature ; at least two of them are widely used in applications

$$\gamma(S, E, I, R) = \begin{cases} \gamma SI, & [10,\ 18,\ 40], \quad \text{mass action in [16, 18], or pseudo-mass action in [39, 22].} \\ \gamma \dfrac{SI}{S + E + I + R}, & [20,\ 28,\ 43], \quad \text{proportionate mixing in [16] or true mass action in [39, 22].} \end{cases}$$

We refer to De Jong *et al*, [39] and Diekmann *et al* [22] for a discussion supporting the second one in populations of varying size and Fromont *et al* [29] for a specific discussion in the case of a cat population. See Capasso and Serio [17] and Capasso [18] for more general incidence terms.

System (4.1.1)-(4.1.3) is uniformly parabolic in the region $D = \bigcap_{i=1}^{4} [U_i \neq 0]$ and degenerate into first order equations on $Q_T \setminus D$. Note that degenerate diffusion is a good approach in modeling slow diffusion of individuals in the spatial spread of an epidemic disease, see Okubo [50].

A mathematical analysis of the model of Kermack and McKendricks for spatially structured populations with linear diffusion, i.e. $m_i = 1,\ i = 1..4$, is performed in Webb [58]. Nonlinear but nondegenerate diffusion terms are introduced in Fitzgibbon *et al* [25]. Global existence and large time behavior results are derived therein. Homogeneous Neumann boundary conditions correspond to isolated populations.

A comprehensive analysis of generic $(S - E - I - R)$ models with linear diffusion is initiated in Fitzgibbon and Langlais [26] and Fitzgibbon *et al* [27]. These models include a logistic effect on the demography, yielding $L^1(\Omega)$ a priori estimates on solutions independent of the initial data for large time ; this allows to use a bootstrapping argument to show global existence and exhibit a global attractor in $(C(\overline{\Omega}))^4$.

For degenerate reaction-diffusion equations, the case of mass action incidence was studied by Aliziane and Moulay [7] and they established the long time behavior of the solution of the SIS model, Aliziane and Langlais [6] studied the SEIR model without logistic effect on the demography and they established global existence result of the solution and the long time behavior of the solution. Finally Hadjadj *et al* [35] studied the case where the source term depends on gradient of solution, they resolved the problem of existence of globally bounded weak solutions or blow-up, depending on the relations between the parameters that appear in the problem.

This paper is organized as follows : in Section §4.2 notion of a weak solution is introduced and we state our mean results, in Section §4.3 we will construct our solution as a limit of solutions of quasilinear and nondegenerate problems depending on a parameter ε, derive uniform a priori estimates on these solutions, and prove existence, uniqueness and regularity results in Section §4.4. In Section §**??** we prove the existence of periodic solution of (4.1.1) − (4.1.3) under periodic assumption on F. Finally in the last section we obtain the existence of a global attractor.

4.2 Main results

4.2.1 Basic assumptions and notations

Herein, Ω is an open, bounded and connected domain of the N-dimensional Euclidian space $\mathbb{R}^N,\ N \geq 1$, with a smooth boundary $\partial\Omega$, a $(N-1)$-dimensional manifold so that locally Ω lies on one side of $\partial\Omega$; $x = (x_1, ..., x_N)$ is the generic element of $\mathbb{R}^N$. Next we shall denote the gradient with respect to x by ∇ and the Laplace operator in $\mathbb{R}^N$ by Δ, $sign_\varepsilon$ is a smooth approximation of the function signum ($sign$), finally if r is a real number then we set $r^+ = sup(r, 0),\ \ r^- = sup(-r, 0)$.

Then we set $\Omega \times (0, T) = Q_T$ and for $0 \leq \tau < T$, $\Omega \times (\tau, T) = Q_{\tau,T}$. The norm in $L^p(\Omega)$ is $\|\ \|_{p,\Omega}$ and the norm in $L^p(Q_{\tau,T})$ is $\|\ \|_{p,Q_{\tau,T}}$ for $1 \leq p \leq +\infty$.

Next we shall assume throughout this paper
(H0) $U_{i,0} \in C(\bar{\Omega}),\quad U_{i,0}(x) \geq 0,\ x \in \Omega,\qquad i = 1..4.$
(H1) Powers m_i verify $m_i > 1,\ i = 1..4.$

(H2) $\mu, \alpha, \nu, m, \lambda, \pi, b_{ii}, b_{1i}, k_i$, $i = 1, .., 4$ are nonnegative constants, $k_i > 0$, $i = 1, .., 4$ and $0 \leq \pi \leq 1$.
(H3) $\gamma : \mathbb{R}_+^4 \longrightarrow \mathbb{R}_+$ is a locally lipschitz continuous function with polynomial growth and $\gamma(0, U_2, U_3, U_4) = 0$ on $\mathbb{R}_+^3$.
(H4) There exists nonnegative constants C_1, C_2 and $0 \leq r \leq 1$ such that

$$\gamma(U_1, U_2, U_3, U_4) \leq (C_1 + C_2 \sum_{i=1}^{4} U_i^r) \text{ on } \mathbb{R}_+^4.$$

(H5) F_i, $i = 1, .., 4$ are nonnegative continuous and bounded function on $\Omega \times (0, +\infty)$.

Remark 4.1 *The assumption $\gamma(0, U_2, U_3, U_4) = 0$ is a natural assumption for our motivating problem : no new exposed individuals when there is no susceptible ones.* $(H4)$ *removes mass action incidence terms.*

4.2.2 Main results

System (4.1.1) is degenerate : when $U_i = 0$ the equation for U_i degenerates into first order equation. Hence classical solutions cannot be expected for Problem $(4.1.1) - (4.1.3)$. A suitable notion of generalized solutions is required. We adopt the notion of weak solution introduced in Oleinik *et al* [51].

Definition 4.1 *A quadruple (U_1, U_2, U_3, U_4) of nonnegative and continuous functions U_i : $\Omega \times [0, +\infty) \rightarrow [0, +\infty)$, $i = 1..4$, is a weak solution of Problem* (4.1.1) - (4.1.3) *in Q_T, $T > 0$ if for each $i = 1..4$ and for each $\varphi_i \in C^1(\bar{Q}_T)$, such that $\dfrac{\partial \varphi_i}{\partial \eta} = 0$ on $\partial\Omega \times (0, T)$.*
(i) ∇U^{m_i} exists in the sense of distribution and $\nabla U_i^{m_i} \in L^2(Q_T)$,
(ii) U_i verifies the identity

$$\begin{aligned} &\int_\Omega U_i(x,T)\varphi_i(x,T)dx + \int_{Q_T} \nabla U_i^{m_i} \nabla \varphi_i(x,t)dxdt \\ &= \int_{Q_T} (\partial_t \varphi_i U_i - f_i \varphi_i)(x,t)dxdt + \int_\Omega U_{i,0}(x)\varphi_i(x,0)dx, \end{aligned} \tag{4.2.4}$$

We are now ready to state our first result.

Theorem 4.1 *For each quadruple of continuous nonnegative initial functions $(U_{1,0}, U_{2,0}, U_{3,0}, U_{4,0})$ there exists a unique weak solution (U_1, U_2, U_3, U_4) of Problem $(4.1.1) - (4.1.3)$ on Q_∞*
i) $U_{i,0} \in C((0, +\infty); \bar{\Omega}) \cap L^\infty(Q_\infty)$, and $U_i^{m_i} \in H^1(Q_{\tau,T})$ for all , $0 < \tau < T$, $i = 1..4$.
ii) There exists a nonnegative constant K such that
(4.2.5)

$$\int_\Omega \sum_{i=1}^{4} |U_{1,i} - U_{2,i}|(x,t)dx \leq (1 + Kte^{Kt}) \int_\Omega |U_{1,i,0} - U_{2,i,0}|(x)dx, \text{ for all } t > 0.$$

where $U_{j,i}$ is solution of $(4.1.1) - (4.1.3)$ *with initial data $U_{j,i,0}$.*

The proof is found in Section §4.4.

Now we look at the existence of periodic nonnegative solution of (4.1.1).

Theorem 4.2 *Assume*

(*HP*) *There exists a positive constant* T^* *so that* $F_i(x,t+T^*) = F_i(x,t)$.

Then there exists a solution (U_1, U_2, U_3, U_4) *to* (4.1.1), (4.1.3) *so that for* $t \geq 0$, $x \in \Omega$, *we have*

$$U_i(x,t+T^*) = U_i(x,t), \ i = 1,..,4.$$

The proof is found in Section §4.5.

4.3 Auxiliary problem and a priori estimates

In this section we consider an auxiliary problem depending on a small parameter ε, with $0 < \varepsilon \leq 1$. Namely let us introduce in $\Omega \times (0,+\infty)$ the quasilinear nondegenerate initial and boundary value problem

$$(4.3.6)\quad \begin{cases} \partial_t U_1 - \Delta d_1(U_1) = & -\gamma((U_1-\varepsilon)^+, U_2, U_3, U_4)) + \sum_{i=1}^{4} b_{1i}(U_i-\varepsilon) + \delta(U_4-\varepsilon) \\ & -\nu(U_1-\varepsilon) - (k_1(P-4\varepsilon)+m_1)(U_1-\varepsilon) + F_1(x,t), \\ \partial_t U_2 - \Delta d_2(U_2) = & \gamma((U_1-\varepsilon)^+, U_2, U_3, U_4) + b_{21}(U_2-\varepsilon) \\ & -(k_2(P-4\varepsilon)+m_2+\lambda+\mu)(U_2-\varepsilon) + F_2(x,t), \\ \partial_t U_3 - \Delta d_3(U_3) = & b_{31}(U_3-\varepsilon) + \lambda\pi(U_2-\varepsilon) \\ & -(k_3(P-4\varepsilon)+\alpha+m_3+\mu)(U_3-\varepsilon) + F_3(x,t), \\ \partial_t U_4 - \Delta d_3(U_4) = & b_{41}(U_4-\varepsilon) + (1-\pi)\lambda(U_2-\varepsilon) + \alpha(U_3-\varepsilon) + \nu(U_1-\varepsilon) \\ & -\delta(U_4-\varepsilon) - (k_4(P-4\varepsilon)+m_4)(U_4-\varepsilon) + F_4(x,t). \end{cases}$$

$$(4.3.7)\quad \begin{cases} U_{i,\epsilon}(x,0) = U_{i,0,\epsilon}(x) \geq 0, \quad x \in \Omega; \\ \dfrac{\partial d_i(U_{i,\varepsilon})}{\partial\eta}(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \end{cases} \qquad i = 1..4.$$

Herein $(r)^+$ is the nonnegative part of the real number r; for each $i = 1..4$
$d_i : \mathbb{R} \longrightarrow (\frac{\varepsilon}{2}, +\infty)$ is a smooth and increasing functions with

$$(4.3.8)\quad d_i(u) = u^{m_i}, \ \ \varepsilon \leq u;$$

$(U_{i,0,\varepsilon})_{i=1..4}$ is a quadruple of smooth functions over $\bar{\Omega}$ such that

$$(4.3.9)\quad \begin{cases} U_{i,0,\varepsilon}(x) \geq \varepsilon, \quad x \in \Omega, \quad 0 < \varepsilon \leq 1; \\ \int_\Omega (U_{i,0,\varepsilon}(x)-\varepsilon)dx = \int_\Omega U_{i,0}(x)dx \qquad i = 1..4; \\ U_{i,0,\varepsilon} \longrightarrow U_{i,0} \ \text{ in } C(\bar{\Omega}), \ \text{ as } \ \varepsilon \longrightarrow 0; \end{cases}$$

we refer to [3] for a construction of such a set of initial data. From standard results, i.e. [42] or [56], local existence and uniqueness of a quadruple $(U_{1,\varepsilon}, U_{2,\varepsilon}, U_{3,\varepsilon}, U_{4,\varepsilon})$, a classical solution of (4.3.6) - (4.3.7) in some maximal interval $[0, T_{max,\varepsilon})$ is granted.

Looking at the equation for $U_{i,\varepsilon}$ it is checked that $([\varepsilon,\ +\infty[)^4$ is an invariant region (see [56]), thus $0 < \epsilon \leq U_{i,\varepsilon}(x,t),\ x \in \Omega,\ 0 < t < T_{max,\varepsilon}$. As a consequence $U_{i,\varepsilon}$ is the solution of the initial and boundary value problem

$$(4.3.10)\quad \begin{cases} \partial_t U_1 - \Delta U_1^{m_1} = & -\gamma((U_1-\varepsilon), U_2, U_3, U_4)) + \sum_{i=1}^{4} b_{1i} U_i + \delta(U_4 - \varepsilon) \\ & -\nu(U_1-\varepsilon) - (k_1 P + m_1)(U_1 - \varepsilon) + F_1(x,t), \\ \partial_t U_2 - \Delta U_2^{m_2} = & \gamma((U_1-\varepsilon), U_2, U_3, U_4) + b_{21}(U_2-\varepsilon) \\ & -(k_2 P + m_2 + \lambda + \mu)(U_2 - \varepsilon) + F_2(x,t), \\ \partial_t U_3 - \Delta U_3^{m_3} = & b_{31}(U_3-\varepsilon) + \lambda\pi(U_2-\varepsilon) - (k_3 P + \alpha + m_3 + \mu)(U_3-\varepsilon) \\ & +F_3(x,t), \\ \partial_t U_4 - \Delta U_4^{m_4} = & b_{41}(U_4-\varepsilon) + (1-\pi)\lambda(U_2-\varepsilon) + \alpha(U_3-\varepsilon) + \nu(U_1-\varepsilon) \\ & -\delta(U_4-\varepsilon) - (k_4 P + m_4)(U_4-\varepsilon) + F_4(x,t). \end{cases}$$

in $\Omega \times (0, +\infty)$, together with (4.3.7) .

Lemma 4.1 *The solution of* (4.3.10) *subject to* (4.3.7) *is global (that is $T_{\max,\varepsilon} = \infty$) and there exist a constant C independent of ε, $0 < \varepsilon < 1$, such that*

$$(4.3.11)\qquad \|u_{i\varepsilon}(t,.)\|_{L^\infty} \leq C\left(\|u_0\|_{L^\infty}\right), \quad \textit{for all}\ \ t > 0,\ \ i = 1,..,4.$$

Moreover, there exists a positive function F not depending on ε and on u_0 such that

$$(4.3.12)\qquad \|u_{i\varepsilon}(t,.)\|_{L^\infty} \leq F(\xi) \qquad \textit{for all}\ t \geq \xi > 0,$$

Proof.

Let us multiply each equation in $U_{i,\varepsilon}$ by $U_{i,\varepsilon}^{p-1}$,integrate over Ω and use $(H4)$ we get

$$(4.3.13)\quad \begin{cases} \frac{1}{p}\frac{d}{dt}\int_\Omega U_{1,\varepsilon}^p \leq & \int_\Omega \sum_{i=1}^{4} b_{1i} U_{i,\varepsilon} U_{1,\varepsilon}^{p-1} + \delta U_{4,\varepsilon} U_{1,\varepsilon}^{p-1} - \nu U_{1,\varepsilon}^p - (k_1 P_\varepsilon + m_1) U_{1,\varepsilon}^p + F_1(x,t) U_{1,\varepsilon}^{p-1}, \\ \frac{1}{p}\frac{d}{dt}\int_\Omega U_{2,\varepsilon}^p \leq & \int_\Omega (C_1 + C_2 \sum_{i=1}^{4} U_{i,\varepsilon}^r) U_{2,\varepsilon}^{p-1} + b_{21} U_{2,\varepsilon}^p - (k_2 P_\varepsilon + m_2 + \lambda + \mu) U_{2,\varepsilon}^p + F_2(x,t) U_{2,\varepsilon}^{p-1}, \\ \frac{1}{p}\frac{d}{dt}\int_\Omega U_{3,\varepsilon}^p \leq & \int_\Omega b_{31} U_{3,\varepsilon}^p + \lambda\pi U_{2,\varepsilon} U_{3,\varepsilon}^{p-1} - (k_3 P_\varepsilon + \alpha + m_3 + \mu) U_{3,\varepsilon}^p + F_3(x,t) U_{3,\varepsilon}^{p-1}, \\ \frac{1}{p}\frac{d}{dt}\int_\Omega U_{4,\varepsilon}^p \leq & \int_\Omega b_{41} U_{4,\varepsilon}^p + (1-\pi)\lambda U_{2,\varepsilon} U_{4,\varepsilon}^{p-1} + \alpha U_{3,\varepsilon} U_{4,\varepsilon}^{p-1} + \nu U_{1,\varepsilon} U_{4,\varepsilon}^{p-1} - \delta U_{4,\varepsilon} \\ & -(k_4 P_\varepsilon + m_4) U_{4,\varepsilon}^p + F_4(x,t) U_{4,\varepsilon}^{p-1}. \end{cases}$$

Now by Hölder and Jensen and Young inequalities one can deduce

$$\int_\Omega U_{i,\varepsilon}^r U_{2,\varepsilon}^{p-1}(x,t)dx \le \left\{(1-r)|\Omega|^{\frac{1}{p}} + r\|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega}\right\} \|U_{2,\varepsilon}(\cdot,t)\|_{p,\Omega}^{p-1}.$$

$$\int_\Omega U_{i,\varepsilon}^{p+1} \ge \left(\frac{1}{|\Omega|}\right)^{\frac{1}{p}} \left(\int_\Omega U_{i,\varepsilon}^p\right)^{\frac{p+1}{p}}$$

and
(4.3.14)

$$\begin{cases} \frac{1}{p}\frac{d}{dt}\int_\Omega U_{1,\varepsilon}^p \le & \left\{\sum_{i=1}^4 b_{1i}\|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega} - (\nu+m_1)\|U_{1,\varepsilon}(\cdot,t)\|_{p,\Omega} + \delta\|U_{4,\varepsilon}(\cdot,t)\|_{p,\Omega}\right. \\ & \left. +\|F_1(\cdot,t)\|_{p,\Omega} - k_1|\Omega|^{\frac{-1}{p}}\|U_{1,\varepsilon}(\cdot,t)\|_{p,\Omega}^2\right\} \|U_{1,\varepsilon}(\cdot,t)\|_{p,\Omega}^{p-1} \\ \frac{1}{p}\frac{d}{dt}\int_\Omega U_{2,\varepsilon}^p \le & \left\{(C_1+(1-r)C_2)|\Omega|^{\frac{1}{p}} + (b_{22}-m_2-\lambda-\mu)\|U_{2,\varepsilon}(\cdot,t)\|_{p,\Omega}\right. \\ & \left. +rC_2\sum_{i=1}^4 \|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega} + \|F_2(\cdot,t)\|_{p,\Omega} - k_1|\Omega|^{\frac{-1}{p}}\|U_{2,\varepsilon}(\cdot,t)\|_{p,\Omega}^2\right\} \|U_{2,\varepsilon}(\cdot,t)\|_{p,\Omega}^{p-1} \\ \frac{1}{p}\frac{d}{dt}\int_\Omega U_{3,\varepsilon}^p \le & \{(b_{33}-\alpha-m_3)\|U_{3,\varepsilon}(\cdot,t)\|_{p,\Omega} + \lambda\pi\|U_{2,\varepsilon}(\cdot,t)\|_{p,\Omega} + \|F_3(\cdot,t)\|_{p,\Omega} \\ & \left. -k_3|\Omega|^{\frac{-1}{p}}\|U_{3,\varepsilon}(\cdot,t)\|_{p,\Omega}^2\right\} \|U_{3,\varepsilon}(\cdot,t)\|_{p,\Omega}^{p-1} \\ \frac{1}{p}\frac{d}{dt}\int_\Omega U_{4,\varepsilon}^p \le & \{(b_{44}-\delta-m_4)\|U_{4,\varepsilon}(\cdot,t)\|_{p,\Omega} + (1-\pi)\lambda\|U_{2,\varepsilon}(\cdot,t)\|_{p,\Omega} + \alpha\|U_{3,\varepsilon}(\cdot,t)\|_{p,\Omega} \\ & \left. +\nu\|U_{1,\varepsilon}(\cdot,t)\|_{p,\Omega} + \|F_4(\cdot,t)\|_{p,\Omega} - k_4|\Omega|^{\frac{-1}{p}}\|U_{4,\varepsilon}(\cdot,t)\|_{p,\Omega}^2\right\} \|U_{4,\varepsilon}(\cdot,t)\|_{p,\Omega}^{p-1}. \end{cases}$$

Adding these inequalities and use jensen's and young inequalities another time to get

$$\frac{d}{dt}\sum_{i=1}^4 \|U_{4,\varepsilon}(\cdot,t)\|_{p,\Omega} \le B_{0,p} - B_{1,p}\left(\sum_{i=1}^4 \|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega}\right)^2, \tag{4.3.15}$$

with

$$B_{0,p} = (C_1+(1-r)C_2)|\Omega|^{\frac{1}{p}} + \sum_{i=1}^4 \sup_t \|F_4(\cdot,t)\|_{p,\Omega} + 2\frac{\left(\sum_{i=1}^4(b_{1,i}+b_{ii}-m_i)+rC_2\right)^2}{\min_i(k_i)}|\Omega|^{\frac{1}{p}},$$

$$B_{1,p} = \frac{\min_i(k_i)|\Omega|^{\frac{-1}{p}}}{8}.$$

Finally let $y(t) = \sum_{i=1}^4 \|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega}$, and $B_0 = \lim_{p\to+\infty} B_{0,p}$ and $B_1 = \lim_{p\to+\infty} B_{1,p}$ then $y(t)$ then $y(t)$ verifies

$$y'(t) \le \quad B_{0,p} - B_{1,p}y^2,$$

and by standard argument see [24, Lemma 1] we get

$$y(t) \le \left(\frac{B_{0,p}}{B_{1,p}}\right)^{\frac{1}{2}} + \frac{B_{1,p}}{t}. \tag{4.3.16}$$

and

$$y(t) \leq \max\left(y(0), \left(\frac{B_{0,p}}{B_{1,p}}\right)^{\frac{1}{2}}\right).$$

Going back to the definition of $y(t)$ one can find

$$\sum_{i=1}^{4} \|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega} \leq \max\left(\sum_{i=1}^{4} \|U_{i,0,\varepsilon}\|_{p,\Omega}, \left(\frac{B_{0,p}}{B_{1,p}}\right)^{\frac{1}{2}}\right).$$

To conclude, one observes that $U_{i,\varepsilon}$ being continuous on $\bar{\Omega} \times [0, T_{max,\varepsilon})$ it follows

$$\lim_{p\to+\infty} \|U_{i,\varepsilon}(\cdot,t)\|_{p,\Omega} = \|U_{i,\varepsilon}(\cdot,t)\|_{\infty,\Omega}.$$

Hence

$$\sum_{i=1}^{4} \|U_{i,\varepsilon}(\cdot,t)\|_{\infty,\Omega} \leq \max\left(\sum_{i=1}^{4} \|U_{i,0,\varepsilon}\|_{\infty,\Omega}, \left(\frac{B_0}{B_1}\right)^{\frac{1}{2}}\right), \tag{4.3.17}$$

and $T_{max,\varepsilon} = +\infty$. □

Remark 4.2 *Estimation* (4.3.16) *implies that for each* $\eta > 0$ *there exists a constant* $C(\eta)$ *independent on initial data such that*

$$\sum_{i=1}^{4} \|U_{i,\varepsilon}(\cdot,t)\|_{\infty,\Omega} \leq C(\eta), \text{ for all } t \geq \eta > 0 \tag{4.3.18}$$

Lemma 4.2 *For all* $T > 0$ *there exists a nondecreasing function* C_1 *independent of* ε, $0 < \varepsilon < 1$ *such that*

$$\int_{Q_T} U_{i,\varepsilon}^2(x,T)dx + \int_{Q_T} |\nabla U_{i,\varepsilon}^{m_i}|^2(x,t)dxdt \leq C_1(T), \quad T > 0, \quad i = 1..4; \tag{4.3.19}$$

Proof. The first term is bounded as an immediate consequence of Lemma 4.1 because $U_{i,\varepsilon}$ is uniformly bounded from below independently of ε. The boundeness of the second term is obtained by multiplying the equation for $U_{i,\varepsilon}$ by $U_{i,\varepsilon}^{m_i}$ and integrating over $\Omega \times (0,T)$ and use the same artifices as in the proof of Lemma 4.1. □

Lemma 4.3 *For all* $T > 0$ *there exists a nondecreasing function* C_1 *independent of* ε, $0 < \varepsilon < 1$ *such that*

$$\int_{Q_T} (\partial_t U_{i,\varepsilon}^{\frac{m_i+1}{2}})^2(x,t)dxdt + \int_{\Omega} |\nabla U_{i,\varepsilon}^{m_i}|^2(x,T)dx \leq C_1(T), \quad T > 0, \quad i = 1..4; \tag{4.3.20}$$

Proof. Let us multiply by $\partial_t U_{i,\varepsilon}^{m_i}$ the equation for $U_{i,\varepsilon}$ and integrate over $\Omega\times(\tau,T)$, $0<\tau<T$; then one finds

$$(\frac{2}{m_i+1})^2\int_{Q_{\tau,T}}(\partial_t U_{i,\varepsilon}^{\frac{m_i+1}{2}})^2(x,s)dxds+\frac{1}{2}\|\nabla U_{i,\varepsilon}^{m_i}(.,T)\|_{2,\Omega}^2$$
$$\leq\int_{Q_{\tau,T}}f_i(U_{1,\varepsilon},U_{2,\varepsilon},U_{3,\varepsilon},U_{4,\varepsilon})\partial_t U_{i,\varepsilon}^{m_i}(x,s)dxds+\frac{1}{2}\|\nabla U_{1,\varepsilon}^{m_i}(.,\tau)\|_{2,\Omega}^2.$$

By Lemma 4.1 f_i is style bounded and we can use Young's inequality to get

$$\int_{Q_{\tau,T}}f_i(U_{1,\varepsilon},U_{2,\varepsilon},U_{3,\varepsilon},U_{4,\varepsilon})\partial_t U_{i,\varepsilon}^{m_i}(x,s)dxds+\frac{1}{2}\|\nabla U_{i,\varepsilon}^{m_i}(.,T)\|_{2,\Omega}^2$$
$$\leq\frac{2}{(m_i+1)^2}\int_{Q_{\tau,T}}(\partial_t U_{i,\varepsilon}^{\frac{m_i+1}{2}})^2(x,s)dxds+\frac{Tm_i^2}{2}|\Omega|\|f_i\|_{\infty,\Omega\times(0,\infty)}\|U_{i,\varepsilon}^{m_i}\|_{\infty,\Omega\times(0,T)}^{m_i-1}.$$

Reporting this inequality into the previous and integrating in τ over $(0,T)$, and Lemma 4.3 follows by Lemma 4.2. □

4.4 Existence and continuous dependence on data

In this section we supply a quick proof of Theorem 4.1.

4.4.1 Existence

Let us fix $T>0$. From the estimates established in the previous section one has : for each $i=1..4$ $(U_{i,\varepsilon})_{0<\varepsilon\leq 1}$ and $(\nabla U_{i,\varepsilon}^{m_i})_{0<\varepsilon\leq 1}$ are respectively bounded in $L^2(Q_T)$ and $(L^2(Q_T))^N$. Then there exists two sequences which one still denotes $(U_{i,\varepsilon})_{0<\varepsilon\leq 1}$ and $(\nabla U_{i,\varepsilon}^{m_i})_{0<\varepsilon\leq 1}$ such that for $i=1,..,4$ as $\varepsilon\to 0$: $(U_{i,\varepsilon})_{0<\varepsilon\leq 1}$ is weakly convergent to some U_i in $L^2(Q_T)$ and $(\nabla U_{i,\varepsilon}^{m_i})_{0<\varepsilon\leq 1}$ is weakly convergent to some V_i in $(L^2(Q_T))^N$.
On the other hand $(U_{i,\epsilon})_{0<\epsilon\leq 1}$ is bounded in $L^\infty(Q_T)$; using a weak formulation of the equation for $U_{i,\epsilon}$ one can invoke the results in Di Benedetto [21] to get : $(U_{i,\varepsilon})_{0<\varepsilon\leq 1}$ is a relatively compact subset of $C(\overline{\Omega}\times(0,T])$. It follows that actually $(U_{i,\varepsilon})_{0<\epsilon\leq 1}$ is convergent to U_i in $C(\overline{\Omega}\times(0,T])$ and $(U_{i,\varepsilon}^{m_i})_{0<\varepsilon\leq 1}$ is convergent to $U_i^{m_i}$ in $C(\overline{\Omega}\times(0,T])$.
As a first consequence one has : $V_i=\nabla U_i^{m_i}$; next one also has :

$$\gamma(U_{1,\varepsilon}-\varepsilon,U_{2,\varepsilon},U_{3,\varepsilon},U_{4,\varepsilon})\ \to\ \gamma(U_1,U_2,U_3,U_4)\ \text{ in }\ C(\overline{\Omega}\times(0,T])\ \text{ as }\ \epsilon\ \to 0.$$

From standard arguments one may conclude that the quadruple (U_1,U_2,U_3,U_4) is a desirable weak solution. Note that all estimates in Lemmas 4.1–4.3 still valid for (U_1,U_2,U_3,U_4) by passing to limit as ε goes to zero.

The regularity results for $\nabla U_i^{m_i}$ and $\partial_t U_i^{m_i}$ follow from the a priori estimates in Lemma 4.2 and Lemma 4.3.

4.4.2 Uniqueness and continuous dependence on data

Assume there exists two quadruples $(U_{j,1}, U_{j,2}, U_{j,3}, U_{j,4})_{j=1,2}$, both weak solutions of Problem (4.1.1) – (4.1.3) with initial data $(U_{j,1,0}, U_{j,2,0}, U_{j,3,0}, U_{j,4,0})_{j=1,2}$. They verify the integral identity, for $i = 1..4$

(4.4.21)

$$\int_\Omega (U_{1,i} - U_{2,i})(x,T)\varphi_i(x,T)dx + \int_{Q_T} \nabla(U_{1,i}^{m_i} - U_{2,i}^{m_i})\nabla\varphi_i(x,t)dxdt$$
$$= \int_{Q_T} [\partial_t\varphi_i(U_{1,i} - U_{2,i}) - (f_i(U_{1,1}, U_{1,2}, U_{1,3}, U_{1,4}) - f_i(U_{2,1}, U_{2,2}, U_{2,3}, U_{2,4}))\varphi_i]dxdt$$
$$+ \int_\Omega (U_{1,i,0} - U_{2,i,0})(x)\varphi_i(x,0)dx$$

for every $\varphi_i \in C^1(\bar{Q}_T)$, such that $\dfrac{\partial\varphi_i}{\partial\eta} = 0$ on $\partial\Omega \times (0,T)$ and $\varphi_i > 0$.

We follow an idea of [44] and introduce a function ψ_i as follows

$$\psi_i(x,t) = \begin{cases} \dfrac{U_{1,i}^{m_i} - U_{2,i}^{m_i}}{U_{1,i} - U_{2,i}} & \text{if } U_{1,i} \neq U_{2,i}, \\ 0 & \text{otherwise.} \end{cases} \quad i = 1..4.$$

Let us consider a sequence of smooth functions $(\psi_{i,\varepsilon})_{\varepsilon\geq 0}$ such that $\psi_{i,\varepsilon} \geq \varepsilon$, $\psi_{i,\varepsilon}$ is uniformly bounded in $L^\infty(Q_T)$ and

$$\lim_{\varepsilon\to 0} \|(\psi_{i,\varepsilon} - \psi_i)/\sqrt{\psi_{i,\varepsilon}}\|_{L^2(Q_T)} = 0.$$

For any $0 < \varepsilon \leq 1$ let us introduce the adjoint nondegenerate boundary value problem

$$\text{(4.4.22)} \qquad \begin{cases} \partial_t\varphi_i + \psi_{i,\varepsilon}\Delta\varphi_i = 0 & \text{in } \Omega \times (0,T) \\ \dfrac{\partial\varphi_i}{\partial\eta}(x,t) = 0 & \text{in } \partial\Omega \times (0,T) \quad i = 1..4. \\ \varphi_i(x,T) = \chi_i & \text{in } \Omega \end{cases}$$

For any smooth χ_i with $0 \leq \chi_i(x,t) \leq 1$, $i = 1..4$, any $0 < \varepsilon \leq 1$ this problem has unique classical solution $\varphi_{i,\varepsilon}$ such that see [44]

$$0 \leq \varphi_{i,\varepsilon}(x,t) \leq 1$$

$$\int_{Q_T} \psi_{i,\varepsilon}(\Delta\varphi_{i,\varepsilon})^2 dxdt \leq K_1,$$

If in (4.4.21) we replace φ_i by $\varphi_{i,\varepsilon}$, which is the solution of problem (4.4.22) with $\chi_i = sign((U_i - V_i)^+)$ we obtain.$\chi_1(x) = \chi_{1,\varepsilon}(x) = sign_\varepsilon^+(S_1 - S_2)(x,T)$

$$\int_\Omega (U_{1,i} - U_{2,i})^+(x,T)\varphi_{i,\varepsilon}(x,T)dx + \int_{Q_T} (\psi_i - \psi_{i,\varepsilon})(U_{1,i} - U_{2,i})\Delta\varphi_{i,\varepsilon}dxdt$$
$$= \int_{Q_T} (f_i(U_{1,1}, U_{1,2}, U_{1,3}, U_{1,4}) - f_i(U_{2,1}, U_{2,2}, U_{2,3}, U_{2,4}))\varphi_{i,\epsilon}dxdt$$
$$+ \int_\Omega (U_{1,i,0} - U_{2,i,0})(x)\varphi_{i,\varepsilon}(x,0)dx$$

Using the local lipschitz continuity of f_i and the properties of $\psi_{i,\epsilon}$ and $\varphi_{i,\varepsilon}$ we deduce by letting $\epsilon \to 0$

$$\int_\Omega (U_{1,i} - U_{2,i})^+(x,T)dx \leq K \int_{Q_T} \sum_{i=1}^4 |U_{1,i} - U_{2,i}| + \int_\Omega |U_{1,i,0} - U_{2,i,0}|(x)dx$$

In a similar fashion we establish an analogous inequality for $(U_i - V_i)^-$ and deduce by Gronwall's Lemma.

$$\int_\Omega \sum_{i=1}^4 |U_{1,i} - U_{2,i}|(x,T)dx \leq (1 + KTe^{KT}) \int_\Omega |U_{1,i,0} - U_{2,i,0}|(x)dx. \quad (4.4.23)$$

Uniqueness is immediately deduced. □

4.5 Existence of Periodic solution

We need the following periodicity assumption upon our model :

(HP) There exists a T^* so that $F_i(x, t + T^*) = F_i(x, t)$.

By periodic solution with period T^*, we mean a weak solution of (4.1.1) satisfying (4.1.3) so that for all $t \geq 0,\ x \in \Omega,\ U_i(x, t + T^*) = U_i(x, t),\ i = 1, .., 4.$

In order to proof theorem 4.2 we need the following variant of the Schauder's Fixed Point Theorem which is given in [32].

Theorem 4.3 *(Schauder's Fixed Point). Let X be a Banach space, $K \subset X$ be a convex set in X and $J : K \longrightarrow K$ be a continuous mapping such that the image $J(K)$ is precompact. Then J has a fixed point in K.*

In the present context, let $X = (L^2(\Omega))^4$ and

$$K = \left\{ U = (U_1, U_2, U_3, U_4); U_i \in L^2(\Omega),\ U_i \geq 0 \text{ such that } \sum_{i=1}^4 U_i(x) \leq B \text{ a. e. } x \in \Omega. \right\}$$

with $B = \left(\dfrac{B_0}{B_1}\right)^{\frac{1}{2}}$, found in the proof of Lemma 4.1, K is a convex set in X.

For $U_0 = (U_{1,0}, U_{2,0}, U_{3,0}, U_{4,0}) \in X$, let $J(U_0) = U(\cdot, T^*)$, with U solution of problem (4.1.1) – (4.1.3) . Then by lemma 4.2 and (4.3.17), we have $J(K) \subset K$ and by (4.4.23) there exists a constant C dependent only on B, k', T^* and , $|\Omega|$ with k' is the lipschitz constant of the vector field $(f_i)_i$ such that

$$\|J(U) - J(U')\|_X \leq C\|U - U'\|_X^{\frac{1}{4}} \text{ for all } U, U' \in K$$

and J is continuous from K into K.
Now Let $(U_n)_n$ be a bounded sequence in K, then by Lemma 4.2 and Lemma 4.3 for each $i = 1,..,4$, $J(U_n)_i^{m_i}$ is bounded in $H^1(\Omega)$, then there exists a sequence which still denoted U_n such $J(U_n)_i^{m_i}$ converges in $L^2(\Omega)$ and almost every where in Ω, finally thanks to Lebesgue dominate convergence theorem to deduce with (4.3.17) that $J(U_n)$ converges in X, and $J(K)$ is precompact. By schauder's fixed point theorem there exists $U^* \in K$ such that $J(U^*) = U^*$.

Now let $U(t, x)$ be the solution of of problem $(4.1.1) - (4.1.3)$ with $U_0 = U^*$ and set $V(t, x) = U(t+T^*, x)$ then U, V are solution of problem $(4.1.1)-(4.1.3)$ with same initial datas then by uniqueness $U(t, x) = U(t + T^*, x)$ and U is the desired periodic solution of (4.1.1) .

4.6 Global attractor

Let us consider the following problem

$$(4.6.24)\quad \begin{cases} \partial_t U_i - \Delta\left(|U_i|^{m_i} signU_i\right) = f_i(x, t, U_1, U_2, U_3, U_4), & (x, t) \in \Omega \times (0, +\infty) \\ \frac{\partial(|U_i|^{m_i} signU_i)}{\partial\eta}(x, t) = 0, & x \in \partial\Omega, t > 0, i = 1..4. \\ U_i(x, 0) = U_{i,0}(x), & x \in \Omega; i = 1..4. \end{cases}$$

Problem (4.6.24) admits a unique weak solution verifying (4.3.17), (4.3.18), (4.3.19), (4.3.20) and (4.4.23). The construction of the solution is obtained in the same manner as below with light modification see [35] for more details. This yields that the PDE system (4.6.24) defines a nonlinear semigroup $\{S(t)\}$ as follows $S(t)(U_{1,0}, U_{2,0}, U_{3,0}, U_{4,0}) = (U_1(t), U_2(t), U_3(t), U_4(t))$ and $S(0) = I$ the identity map. we have the a continuous dynamical system on the set of bounded vector valued function. we follow the general setting of [57, 1.1] to prove that there exist a global attractor $\mathcal{A}$ of the above dynamical system to which all the trajectories of this dynamical system will eventually converges, namely we have the following

Theorem 4.4 *Let $X = (L^\infty(\Omega))^4$ with the metric inherited from $L^2(\Omega)$ then the semi-group $\{S(t)\}_{t\geq 0}$ defined above posses a global attractor $\mathcal{A} \subset (H^1(\Omega) \cap L^\infty(\Omega))^4$.*

Proof. From (4.3.18) we can proof easily that $\|U(., t)\|_{L^2(\Omega)}$ and $\|\nabla U^{m_i}(., t)\|_{L^2(\Omega)}$ are bounded independently of the initial data for $t \geq \eta > 0$, and we see that $S(t)$ defined on $X = (L^\infty(\Omega))^4$ is a compact mapping on X with the L^2 norm and and admits an absorbing set in X which absorbs any bounded set B in X after some finite time. There fore, theorem [57, 1.1] can be applied to exhibit global attractor which is bounded in $(H^1(\Omega) \cap L^\infty(\Omega))^4$. □

Chapitre 5

Non linear reaction diffusion system of degenerate parabolic type

5.1 Introduction

The purpose of this chapter is to study a reaction-diffusion system of the type :

$$\partial_t(u_i) - \Delta(|u_i|^{\sigma_i} u_i) = f_i(u, \nabla u_i) \quad \text{in } (0,\infty)\times\Omega \qquad i = 1,..,d, \tag{5.1.1}$$

where u is the vector $u = (u_1, ..., u_d), d$ is an integer ≥ 1, $\sigma_i > 0$ and the reacting functions f_i have the following model form

$$f_i(u, \nabla u_i) = g_i(u) + \overrightarrow{b_i}\nabla(|u_i|^{m_i-1} u_i) \quad i = 1,..,d, \tag{5.1.2}$$

with $\overrightarrow{b_i} = \overrightarrow{b_i}(t,x) \in \mathbb{R}^N$, $m_i > 0$. We supplement this system with boundary conditions

$$u_i = 0 \;\text{ in } (0,\infty)\times\partial\Omega \quad i = 1,..,d, \tag{5.1.3}$$

and the initial data

$$u_i(0,.) = u_{i0} \qquad \text{in } \Omega \qquad i = 1,..,d, \tag{5.1.4}$$

Throughout this chapter we use the following notations

Let i and j be positive integers such that $1 \leq i,\ j \leq d$, T and τ be positive real numbers such that $T > \tau$, η is arbitrary positive real number, Ω is a bounded open set in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$, $\Delta := \sum_{k=1}^{N} \partial_k^2$ denotes the Laplace operator in Euclidean coordinates, ∇ is the gradient with respect to x and the outer normal on $\partial\Omega$ is denoted by $\nu = (\nu_1, \nu_2, ..., \nu_N)$, finally $Hess(u)$ is the hessian of u. In the following we will denote $(0,T)\times\Omega$ by Q_T, and $(\tau,T)\times\Omega$ by $Q_{\tau,T}$. The norm in

$L^p(\Omega)$, $p > 1$, will be written $\|.\|_p$ and we also make use of the sobolev spaces esspecially

$$W^{1,p}(\Omega) := \left\{u : \Omega \to \mathbb{R} \ / \ u \in L^p(\Omega) \text{ and } \nabla u \in (L^p(\Omega))^N \ \right\},$$
$$W_p^{1,2}(\Omega) := \left\{u : \Omega \to \mathbb{R} \ / \ u \in L^p(\Omega) \text{ and } \nabla u \in (L^2(\Omega))^N \ \right\},$$
$$W^{2,p}(\Omega) := \left\{u \in W^{1,p}(\Omega);\ Hess(u) \in (L^p(\Omega))^{N\times N}\right\},$$
$$W_p^{1,2}(Q_T) := \{u : Q_T \to \mathbb{R} \ / \ u \in L^p(]0,T[\,, W^{2,p}(\Omega)) \text{ and } u_t \in L^p(]0,T]\,, L^p(\Omega))\ \},$$
and
$$H(\Omega) := \left\{u = (u_1, u_2, ..., u_d) \ \ v_i : \Omega \to \mathbb{R} \ / \ \left\|\nabla\left(|u_i|_i^{\sigma_i} u\right)\right\|^2 \in L^2(Q_T) \qquad i = 1, d\right\}.$$

Once for all, we notice that the different constants (independent of ε) are denoted by the same latter C.

System (5.1.1) − (5.1.4), in the case $\overrightarrow{b_i} = \overrightarrow{0}$ has been studied extensively under various types of initial and boundary conditions by a large number of authors, see among others [3],[30],[31],[41],[44] and the literature therein.

This problem describes (in the case $\overrightarrow{b_i} = \overrightarrow{0}$) many phenomena, for example it describes non-stationary gas filtration in a porous medium (where u represents the density of the gas) or the diffusion in an biological population (u represents the density of the population) see [44]. Finally in [57] u can be treated as temperature vector of interacting components of a combustible mixture. In the case $\overrightarrow{b_i} \neq \overrightarrow{0}$ the system (5.1.1) − (5.1.4) arise in :

1) Population dynamics. In the following system

$$\begin{cases} S_t - \Delta S^m = -I(\gamma S - \delta) + \overrightarrow{b}\nabla S \\ I_t - \Delta I^n = I(\gamma S - \delta) + \alpha\overrightarrow{b}\nabla I \end{cases} \quad \text{in } (0,\infty)\times\Omega$$

S, I represent respectively (as cited in [3] in the case when $\overrightarrow{b} = \overrightarrow{0}$) the densities of susceptibles and infectives under the effect of certain natural mechanism represented by $\overrightarrow{b}$, γIS is the force of infection or incidence term, it represents the number of susceptible individuals S infected by contact with infective individuals I per time unit, finally δI is the number of infectives who become susceptibles after recovery.

2) Environmental purification

Suppose that a polluted river contains d suspensions with concentration u_i, $i = 1, 2, ..., d$. Then we obtain the following equations

$$\frac{\partial u_i}{\partial t} - \gamma_i \frac{\partial^2 u_i}{\partial z^2} = F_i(u) - \frac{\partial(wu_i)}{\partial z}$$

where z measures distance along the river, w is the velocity of the water.

The following results are well known. First, in the work of Galaktionov [30], it is proved that the global existence of nonnegative solutions of the boundary value problem (5.1.1) − (5.1.4) in the case when $d = 1$ and $f(u, \nabla u) = u^\beta$, depends on a relation between σ(the power in diffusion term), β, N and the data u_0,where $u_0 \geq 0$.

In [31] the authors considered the system (5.1.1) − (5.1.4) with : $d = 2$, $g_1(u_1, u_2) = (u_2)^p$; $g_2(u_1, u_2) = (u_1)^q$; $\overrightarrow{b_i} = \overrightarrow{0}$; They proved that the above system has a global nonnegative solution, for arbitrary nonnegative initial functions $u_{i0} \in L^{\sigma i+2}(\Omega)$, if $1 \leq p < \sigma_2+1$ and $1 \leq q < \sigma_1+1$. For the limit cases $p = \sigma_2+1$ or $q = \sigma_1+1$ they established that the global solvability of the system depends on the spatial structure of Ω.

In [44] Madallena generalized the preceding work by proving the existence of global nonnegative weak solutions for a reaction-diffusion system $(5.1.1)-(5.1.4)$, for arbitrary nonnegative initial functions $u_{i0} \in L^{\infty}(\Omega)$, such that the functions f_i satisfy in the domain $u_j \geq 0$ the following conditions

- $f_i(0) = 0$,
- $f_i(u) \geqslant 0$ for every $u = (u_1, u_2, ..., u_d)$ such that $u_i = 0$ that is f_i is quasi-positive,
- $f_i(u) \leq \sum\limits_{1\leq j\leq d} c_{ij} u_j^{\alpha_{ij}} + c_i$ where $c_{ij}, c_i > 0$ and $0 < \alpha_{ij} < \sigma_j + 1$.

Moreover, existence of nonnegative mild solution for nonnegative initial data in $L^{\sigma_i+2}(\Omega)$, when $f_i = \sum\limits_{1\leq j\leq d} c_{ij} u_j^{\alpha_{ij}}$ and $\alpha_{ij} < \sigma_j + 1$, is studied in [41], and it is proved also that if $\alpha_{ij} = \sigma_j + 1$ solutions may blow-up in finite time.

In this chapter we generalize the preceding works, by supposing dependence on the gradient in the reacting terms, that is namely the system $(5.1.1)-(5.1.4)$. This chapter is organized as follows. In the next section we introduce a weak solution concept and we state our main results on existence, uniqueness, asymptotic behavior and blow-up. In section 5.3, which is the core of the remainder, we prove that one can pass from L^{σ_i+1} bounds to an L^{∞} one, under various boundary conditions. To derive the L^{∞} bounds we use the Moser-type iteration technique of Alikakos (see [2]), for a single equation (in the case $\overrightarrow{b_i} = \overrightarrow{0}$) and developed by Dung (see [24]), in the case $0 < \sigma_i < 1$. It should be noted that this section has the advantage that, generally it is hard or almost impossible to establish L^{∞} bounds directly from the equation.

Moreover we prove that the solution is more regular than the initial data (to be more precise, we prove that if $u_{i0} \in L^{\sigma_i+2}(\Omega)$ and $\|u_i(t,.)\|_{L^{\sigma_i+1}(\Omega)} \leq C(\xi)$ for all $t \geq \xi > 0$ where C is an independent constant of the initial conditions, then $\|u_i(t,.)\|_{L^{\infty}(\Omega)} \leq C(\xi)$ for all $t \geq \xi > 0$). Thus we obtain uniform estimates with respect to the initial data u_0, which will in particular imply the existence of a bounded set that absorbs all solutions. Such a set is called the absorbing set, which plays an important role in proving the existence of the global attractor.

In section 5.4, it will be established that if the initial data belongs to $\prod\limits_{i=1}^{d} L^{\sigma_i+2}(\Omega)$ then under appropriate growth conditions on g_i, problem $(5.1.1)-(5.1.4)$ has a global weak solution $u(t) = (u_1(t), u_2(t), ..., u_d(t))$ $(u_i(t) = u_i(t,x))$, which belongs to $(L^{\infty}(\Omega))^d$ for each $t \geq \xi > 0$ and we prove that if the initial data is bounded, problem $(5.1.1)-(5.1.4)$ has a unique global weak solution, which is bounded for any $t \geq 0$. Therefore the system $(5.1.1)-(7.1.2)$ generates a nonlinear semigroup

$$S(t) : (L^{\infty}(\Omega))^d \longrightarrow (L^{\infty}(\Omega))^d \quad \forall t \geq 0,$$

where $S(t)(u_0) = u(t)$ is a solution of the problem $(5.1.1)-(5.1.4)$, so we shall study the global behavior of solutions mainly terms of this semi-group. In section 5.6, we concentrate on the determination of conditions ensuring the existence of global attractor, which cannot give a complete description of the asymptotic behavior near $t = \infty$, but at least the existence of global attractor gives some informations on the asymptotic behavior of solutions (all solutions remain in a compact set after some finite time). In the Final section, we prove that in the limit case $(f_i(u, \nabla u_i) = \sum_{j=1}^{d} c_{ij} u_j^{\alpha_{ij}} + \overrightarrow{b_i} \nabla (u_i^{m_i}))$, the global solvability depends on the spatial structure of Ω, more precisely, we prove that there exist thick domains Ω such that all (nontrivial) positive weak solutions of $(5.1.1) - (5.1.4)$ blow up in finite time, while they exist globally and decay uniformly to zero as $t \to \infty$ if Ω is small.

Remark 5.1 *In practice, it is most important to consider a positive initial data but we will assume that it is arbitrary for mathematical considerations. For simplicity when investigating the limit case we may assume without loss of generality that* $u_{i0} \geq 0$ *in* Ω

5.2 Statements of main results

The following assumptions will be made throughout this chapter, for all $i = 1, 2, ..., d$

(H_1) $1 < m_i < \sigma_i + 1$,

(H_2) $g_i(0) = 0$,

(H_3) g_i and $\overrightarrow{b_i}$ are locally lipschitz in there arguments,

(H_4) there exist positive constants L_i, α_{ij} with $\alpha_{ij} < \sigma_j + 1$ such that

$$\left\| \overrightarrow{b_i} \right\| \leq L_i, \ |g_i(u)| \leq L_i(\sum_{j=1}^{d} |u_j|^{\alpha_{ij}} + 1),$$

(H_5) $u_{i0} \in L^{\sigma_i+2}(\Omega)$.

Equation (5.1.1) is degenerate parabolic at the points where u_i vanishes. Therefore the problem $(5.1.1) - (5.1.4)$, in general, has no classical solutions. The weak solution is defined as follows

Definition 5.1 *A function* $(u_1, u_2, ..., u_d)$ *is said to be a weak solution of problem* (5.1.1)–(5.1.4) *on* Q_T *if for all* $i = 1, 2, ..., d$

1. $u_i \in C((0, T]; L^2(\Omega))$,
2. $\nabla(|u_i|^{\sigma_i} u_i)$ exists in the sense of distributions in Q_T and $\nabla(|u_i|^{\sigma_i} u_i) \in (L^2(Q_T))^N$,
3. u_i satisfies the identity

$$\int_{\Omega} u_i(x,T)\varphi_i(x,T)dx - \int_{Q_T} \varphi_{it} u_i dxdt + \int_{Q_T} \nabla(|u_i|^{\sigma_i} u_i)\nabla\varphi_i dxdt$$
$$= \int_{Q_T} (g_i(u_1, u_2)\varphi_i - \overrightarrow{b_i}\nabla\varphi_i |u_i|^{m_i-1} u_i - div\left(\overrightarrow{b_i}\right) \varphi_i |u_i|^{m_i-1} u_i)dxdt$$
$$+ \int_{\Omega} u_{i0}(x)\varphi_i(0,x)dx$$

for every $\varphi_i \in C^1(\overline{Q_T})$ *such that* $\varphi_i = 0$ *on* $(0, T) \times \partial\Omega$.

We shall say that u is a global weak solution of problem $(5.1.1) - (7.1.2)$ if u is a weak solution on Q_T for all $T > 0$. For blow-up of solutions we mean that the solution is defined in $(0, T)$, $0 < T < \infty$, and at that time T we have,

$$\lim_{t \nearrow T} \|u(t, .)\|_{L^\infty(\Omega)} = +\infty$$

With respect to global existence and uniqueness our main result is the following

Theorem 5.1 *Under the above assumptions, there exists a global weak solution* $u = (u_1, u_2, ..., u_d)$ *of the problem* $(5.1.1) - (5.1.4)$ *, which has the property*

$$\|u_i(., t)\|_{L^\infty(\Omega)} \leq F(\xi) \quad \text{for all } t \geq \xi > 0,$$

And if $u_{i0} \in L^\infty(\Omega)$ *then* u *is unique in the class of bounded solutions and has the property*

$$\|u_i(., t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \geq 0,$$

where $F(\xi)$ *is a positive function depending only on* ξ *and* C *is a positive constant depending only on* u_0.

Moreover, if the initial data is positive and the functions g_i *are quasi-positives then* $(u_1, u_2, ..., u_d)$ *is positive.*

The proof is found in sections 5.4 and 5.5.

Theorem 5.1 establishes the existence of nonlinear semi-group $S(t)$, which maps $(L^\infty(\Omega))^d$ into $(L^\infty(\Omega))^d$ such that $S(t)u_0 = u(t, .)$, and plays a basic role in the study of the asymptotic behavior of solutions.

Theorem 5.2 *The semi-group* $S(t)_{t\geq 0}$ *associated to the system* $(5.1.1) - (5.1.4)$ *possesses a global attractor* $\mathcal{A}$ *which is bounded in* $\mathbb{C}^\alpha$, *compact in* $(L^\infty(\Omega))^d$.

A proof of the above result is given in section 5.6.

Finally, in section 5.7, we present the global existence and blow-up results, depending on the range of the parameters in the limit case

Theorem 5.3 *Let* $f_i(u, \nabla u_i) = c_{i0} + \sum_{j=1}^{d} c_{ij} u_j^{\sigma_j + 1} + \overrightarrow{b_i} \nabla \left(u_i^{\sigma_i + 1} \right)$

1. *If* $2d \max_{i,j=1,d} c_{ij} + \max_{i=1,d} \left\| \overrightarrow{b_i} \right\| (\lambda + 1) < 2\lambda$ *(*λ *is the first eigenvalue of the Laplacian with zero Dirichlet data on* $\partial\Omega$*) then for every positive initial data in* $(L^\infty(\Omega))^d$ *there exists a global weak solution of* $(5.1.1) - (5.1.4)$ *-tending to zero if* $c_{i0} = 0$*- which is unique, positive and bounded.*
2. *If* $\overrightarrow{b_i}$ is independent of t, $\overrightarrow{b_i} \in C^\infty(\overline{\Omega})$ *and* if $c_{ii} > \lambda_i$, *(*λ_i *is the first eigenvalue of* $-\Delta\psi(x) + \overrightarrow{b_i} \nabla \psi(x)$ *with zero Dirichlet data on* $\partial\Omega$*) then any nonnegative (nontrivial) weak solution of* $(5.1.1) - (5.1.4)$ *blows up in finite time.*

5.3 L^{∞} regularity

In this section we give a basic result of L^{∞} regularity for weak solutions of (5.1.1) − (5.1.4). More precisely, we have the following theorem :

Theorem 5.4 *Let* $(u_1, u_{2,...}, u_d)$ *be a weak solution of the problem* (5.1.1)−(5.1.4).*Assume that there exists a positive continuous function* F_1 *not depending on* u_0 *such that :*

(5.3.5) $$\|u_i(t,.)\|_{L^{\sigma_i+1}(\Omega)} \leq F_1(\xi), \text{ for all } t \in [\xi, T_{\max}), \ i = 1,..,d$$

then there exists a positive continuous function F_∞ *not depending on* u_0 *such that :*

(5.3.6) $$\|u_i(t,.)\|_{L^{\infty}(\Omega} \leq F_\infty(\xi), \text{ for all } t \in [\xi, T_{\max}), \ i = 1,..,d.$$

Moreover, if there exists a positive number $C_1(u_0)$ *such that*

(5.3.7) $$\|u_i(t,.)\|_{L^{\sigma_i+1}(\Omega)} \leq C_1(u_0) \text{ for all } t \in [0, T_{\max}), \ i = 1,..,d$$

then there exists a positive number $C_\infty(u_0)$ *such that :*

(5.3.8) $$\|u_i(t,.)\|_{L^{\infty}(\Omega)} \leq C_\infty(u_0) \text{ for all } t \in [0, T_{\max}), \ i = 1,..,d.$$

The proof of the last theorem is obtained by obvious modification of the techniques of Dung [24], and the following two lemmas serve as the main ingredients.

Lemma 5.1 *Suppose that the nonnegative function* y *is absolutely continuous, and satisfies for almost all* t *the inequality*

(5.3.9) $$y' + \theta y^{\nu} \leq \delta \qquad \text{with} : \nu > 1, \theta > 0, \delta \geq 0$$

then for all $t > 0$ *we have*

(5.3.10) $$y(t) \leq \left(\frac{\delta}{\theta}\right)^{\frac{1}{\nu}} + \left(\theta\,(\nu - 1)\,t\right)^{\frac{-1}{\nu-1}} .$$

In particular, if $\lim_{t\to 0^+} y(t) = y(0)$ *is finite,* (5.3.10) *becomes*

(5.3.11) $$y(t) \leq \max\left\{ y(0), \left(\frac{\delta}{\theta}\right)^{\frac{1}{\nu}} \right\} \qquad \text{for all } t \geq 0.$$

The proof can be found in [57, page 167].

Lemma 5.2 *Let* $p \in [1, 2)$ *and* $r \in \left[p, 2\frac{N+1}{N}\right)$, *then for any given* $\eta > 0$, *there exist positive constants* $c(\eta), q$ *depending only on* p *and* r *such that*

$$\int_\Omega |u|^r \leq \eta \left(\int_\Omega \|\nabla u\|^2 \, dx + \|u\|^2_{L^p(\Omega)} \right) + c(\eta) \|u\|^q_{L^p(\Omega)} .$$

for any $u \in W_p^{1,2}(\Omega)$. *Where :*

$$q = \frac{2r(1-\tau)}{2 - r\tau}, \text{ with} : \ \tau = \frac{\frac{1}{p} - \frac{1}{r}}{\frac{1}{p} + \frac{1}{N} - \frac{1}{2}}.$$

In proving local existence for degenerate equations as (5.1.1) – (5.1.4) one standard approach is to approximate the problem with a sequence of nondegenerate problems which can be solved in a classical sense. To do that we consider

• an increasing sequence of positive numbers $(R_\varepsilon)_\varepsilon$ such that : $\lim\limits_{\varepsilon\to 0} R_\varepsilon = +\infty$.

•$\psi_\varepsilon \in C_c^\infty(\mathbb{R}^+)$ such that , $0 \leq \psi_\varepsilon \leq 1$, and :

$$\psi_\varepsilon(r) = \begin{cases} 1 & \text{if } |r| \leq R_\varepsilon \ . \\ 0 & \text{if } |r| \geq R_\varepsilon + 1 \ . \end{cases}$$

• smooth functions $g_{i\varepsilon}$ such that :

$g_{i\varepsilon}(r_1, r_2, ..., r_d) = g_i(r_1, r_2, ..., r_d)\psi_\varepsilon(|r_1| + |r_2| + ... + |r_d|)$.

•$\phi_\varepsilon(r) = (|r| + \varepsilon)$ for all $r \in \mathbb{R}$.

• a sequence $(u_{0\varepsilon}) = (u_{10\varepsilon}, u_{20\varepsilon}, ..., u_{d0\varepsilon}) \in (C_c^\infty(\Omega))^d$ (which is uniformly bounded in L^∞ if $u_{i0} \in L^\infty$) such that $(u_{i0\varepsilon})_\varepsilon$ tends to u_{i0} in $L^{\sigma_i+2}(\Omega)$.

Consider the following regularizing problems

$$\partial_t(u_{i\varepsilon}) - (\sigma_i + 1)div(\phi_\varepsilon^{\sigma_i}(u_{i\varepsilon})\nabla u_{i\varepsilon}) = g_{i\varepsilon}(u_\varepsilon) + \overrightarrow{b_i}\nabla(|u_{i\varepsilon}|^{m_i-1} u_{i\varepsilon}) \text{ in } Q_T, \tag{5.3.12}$$

subject to Dirichlet boundary conditions

$$u_{i\varepsilon} = 0 \text{ on } (0, T) \times \partial\Omega, \tag{5.3.13}$$

and initial conditions

$$u_{i\varepsilon}(0, .) = u_{i0\varepsilon} \text{ in } \Omega \tag{5.3.14}$$

By [42, Theorem 7.4], there is $T_{\max,\varepsilon} > 0$ such that the problem (5.3.12) – (5.3.14) has a unique maximal solution $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon}, ..., u_d) \in \left(W_q^{1,2}(Q_{T_{\max,\varepsilon}})\right)^d$ for all $1 \leq q < \infty$.

Moreover, with the additional conditions

(H_6) $u_{i0} \geq 0$ $i = 1, 2, ..., d$,

(H_7) g_i is quasi-positive that is $g_i(u) \geqslant 0$ for every $u = (u_1, u_2, ..., u_d)$ such that $u_i = 0$ and $u_j \geq 0$ for $i \neq j$.

we can prove that u_ε is classical and positive, see [35]. In order to prove Theorem 5.4 it suffices to prove

Proposition 2 *Suppose that there exists a positive continuous function F_1 not depending on ε and u_0 such that*

$$\|u_{i\varepsilon}(t, .)\|_{L^{\sigma_i+1}(\Omega)} \leq F_1(\xi), \text{ for all } t \in [\xi, T_{\max}) , \tag{5.3.15}$$

then there exists a positive continuous function F_∞ not depending on ε and u_0 such that

$$\|u_{i\varepsilon}(t, .)\|_{L^\infty(\Omega)} \leq F_\infty(\xi), \quad \text{for all } t \in [\xi, T_{\max}) . \tag{5.3.16}$$

Alternatively, if there exists a positive finite constant $C_1(u_0)$ not depending on ε such that

$$\|u_{i\varepsilon}(t, .)\|_{L^{\sigma_i+1}(\Omega)} \leq C_1(u_0) \text{ for all } t \in [0, T_{\max}) , \tag{5.3.17}$$

then there exists a finite positive constant $C_\infty(u_0)$ not depending on ε such that

$$\|u_{i\varepsilon}(t, .)\|_{L^\infty(\Omega)} \leq C_\infty(u_0) \text{ for all } t \in [0, T_{\max}) . \tag{5.3.18}$$

To prove this Proposition we prove at first the following lemmas

Lemma 5.3 *Assuming* (5.3.15)*, there exists a positive continuous function* F_2 *not depending on* ε *and* u_0 *such that*

$$\|u_{i\varepsilon}(t,.)\|_{L^{\sigma_i+2}(\Omega)} \leq F_2(\xi), \text{ for all } t \in [\xi, T_{\max}) . \tag{5.3.19}$$

If (5.3.17) *is satisfied then there exists a finite positive constant* $C_2(u_0)$ *not depending on* ε *such that*

$$\|u_{i\varepsilon}(t,.)\|_{L^{\sigma_i+2}(\Omega)} \leq C_2(u_0) \text{ for all } t \in [0, T_{\max}) . \tag{5.3.20}$$

Proof. For simplicity , we omit the index ε.

Multiplying (5.3.12) by $|u_i|^{\sigma_i} u_i$,and integrating over Ω, we obtain the following inequality with the help of Young's inequality

$$\begin{aligned}\frac{1}{\sigma_i+2}\frac{d}{dt}\int_\Omega |u_i|^{\sigma_i+2}\,dx + \int_\Omega \|\nabla\left(|u_i|^{\sigma_i} u_i\right)\|^2 dx \\ \leq C(\eta)\sum_{j=1}^{d}\int_\Omega |u_j|^{\sigma_j+1+\theta}\,dx + \eta\int_\Omega \|\nabla\left(|u_i|^{\sigma_j} u_i\right)\|^2 dx + C(\eta),\end{aligned}$$

where $\theta \leq \sigma_j + 1$

From Lemma 5.2, if we take into account assumptions on α_{ij} and m_i, we find

$$\begin{aligned}\frac{d}{dt}\int_\Omega |u_i|^{\sigma_i+2}\,dx + C\int_\Omega \|\nabla\left(|u_i|^{\sigma_i} u_i\right)\|^2 dx \\ \leq 2\eta\sum_{j=1}^{d}\int_\Omega \|\nabla\left(|u_j|^{\sigma_j} u_j\right)\|^2 dx + C(\eta)\sum_{j=1}^{d}\left(\int_\Omega |u_j|^{\sigma_j+1}\,dx\right)^q + C(\eta),\end{aligned}$$

Adding these inequalities, we obtain that for η sufficiently small

$$\frac{d}{dt}\sum_{i=1}^{d}\int_\Omega |u_i|^{\sigma_i+2}\,dx + C\sum_{i=1}^{d}\int_\Omega \|\nabla\left(|u_i|^{\sigma_i} u_i\right)\|^2 dx \leq C\sum_{i=1}^{d}\left(\int_\Omega |u_i|^{\sigma_i+1}\,dx\right)^q + C \tag{5.3.21}$$

Assuming (5.3.15), (5.3.21) can be written in the following form

$$\frac{d}{dt}\sum_{i=1}^{d}\int_\Omega |u_i|^{\sigma_i+2}\,dx + C\int_\Omega \|\nabla\left(|u_i|^{\sigma_i} u_i\right)\|^2 dx \leq C(\xi) \text{ for all } t \geq \xi > 0 \tag{5.3.22}$$

On the other hand, the Hölder and Young inequalities, imply

$$\int_\Omega |u_i|^{\sigma_i+2}\,dx \leq C\left(\int_\Omega |u_i|^{2(\sigma_i+1)}\,dx\right)^{\frac{\sigma_i+2}{2(\sigma_i+1)}} \leq C\left(\int_\Omega |u_i|^{2(\sigma_i+1)}\,dx + 1\right)^{\frac{\gamma+1}{2}}$$

where: $\gamma = \max\limits_{1\leq i\leq d}\frac{1}{(\sigma_i+1)}$.

then from Lemma 5.2 and Jensen's inequality, (5.3.22) becomes

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^d |u_i|^{\sigma_i+2}\,dx + C\left(\int_\Omega \sum_{i=1}^d |u_i|^{\sigma_i+2}\,dx\right)^{\frac{2}{\gamma+1}} \leq C(\xi) \text{ for all } t \geq \xi > 0. \tag{5.3.23}$$

alternatively if (5.3.17) is satisfied, we obtain

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^d |u_i|^{\sigma_i+2}\,dx + C_{13}\left(\int_\Omega \sum_{i=1}^d |u_i|^{\sigma_i+2}\right)^{\frac{2}{\gamma+1}} \leq C(u_0) \text{ for all } t \geq 0. \tag{5.3.24}$$

Finally, by putting $y(t) = \int\limits_\Omega \sum_{i=1}^d |u_i|^{\sigma_i+2}\,dx$ in (5.3.23) and (5.3.24), Lemma 5.1 implies the desired result. □

We now prove inductively that $u_{i\varepsilon}$ is bounded in L^p for each $p \geq \sigma_i + 1$.

Lemma 5.4 *Let $p \geq \sigma_i + 1$, assuming* (5.3.15)*, there exists a positive function F_p not depending on u_0 and ε such that*

$$\|u_{i\varepsilon}(t,.)\|_{L^p(\Omega)} \leq F_p(\xi) \text{ for all } t \in [\xi, T_{\max,\varepsilon}). \tag{5.3.25}$$

If (5.3.17) *is given, then there exists a positive constant $C_p(u_0)$ not depending on ε such that*

$$\|u_{i\varepsilon}(t,.)\|_{L^p(\Omega)} \leq C_p(u_0) \text{ for all } t \in [0, T_{\max,\varepsilon}). \tag{5.3.26}$$

Proof. Let $r_k \geq 1$

Multiplying (5.3.12) by $|u_i|^{r_k(\sigma_i+1)-1}u_i$, and integrating over Ω, we obtain the following with the help of Young's inequality

(5.3.27)

$$\begin{aligned}\frac{1}{r_k(\sigma_1+1)+1}\frac{d}{dt}\int_\Omega |u_j|^{r_k(\sigma_1+1)+1}\,dx + \frac{4r_k}{(1+r_k)^2}\int_\Omega \left\|\nabla\left(|u_j|^{\frac{(\sigma_1+1)(r_k+1)}{2}-1}u_j\right)\right\|^2 dx \\ \leq C(\eta)\sum_{j=1}^d \int_\Omega |u_j|^{r_k(\sigma_j+1)+\theta}\,dx + \eta\int_\Omega \left\|\nabla\left(|u_j|^{\frac{(\sigma_1+1)(r_k+1)}{2}-1}u_j\right)\right\|^2 dx + C(\eta)\end{aligned}$$

where $\theta \leq \sigma_j + 1$.

In order to estimate $\int_\Omega |u_j|^{r_k(\sigma_j+1)+\theta}\,dx$ we construct the following sequences :

$$r_k = \lambda^k, \quad p_{ik} = \frac{2(r_{k-1}(\sigma_i+1)+1)}{(\sigma_i+1)(1+r_k)} \text{ and } \nu_{ik} = \frac{((\sigma_i+1)1+r_k)}{1+r_k(\sigma_i+1)}.$$

where $1 < \lambda < 1 + \min\limits_{i=1,..,d}\dfrac{1}{\sigma_i+1}$

It is obvious that $1 < p_{ik} < 2$ for all $i = 1,..,d$.

setting $w_i = |u_i|^{(\sigma_i+1)\frac{r_k+1}{2}-1} u_i$, and applying Lemma 5.2, we can estimate $\int_\Omega |u_j|^{r_k(\sigma_j+1)+\theta}\,dx = \int_\Omega |u_j|^{\frac{2\left(r_k(\sigma_j+1)+\theta\right)}{(r_k+1)(\sigma_j+1)}}\,dx$ in term of $\|w_i\|_{L^{p_{ik}}}$ and $\|\nabla w_i\|_{L^2}$. Hence (5.3.27) becomes

$$\frac{d}{dt}\int_\Omega |w_i|^{\frac{2}{\nu_{ik}}}\,dx+(1-\eta)\int_\Omega \|\nabla w_i\|^2\,dx \le \eta\sum_{j=1}^{d}\int_\Omega \|\nabla w_j\|^2\,dx+C(\eta)\sum_{j=1}^{d}\|w_j\|^{q_k}_{L^{p_{ik}}(\Omega)}+C(\eta)$$

When we summing up these inequalities over i, we find

$$\begin{aligned}\frac{d}{dt}\int_\Omega \sum_{i=1}^{d}|w_i|^{\frac{2}{\nu_{ik}}}\,dx + (1-2d\eta)\int_\Omega \sum_{i=1}^{d}\|\nabla w_i\|^2\,dx \\ \le 2d\eta\int_\Omega\sum_{i=1}^{d}\|\nabla w_i\|^2\,dx + C(\eta)\sum_{i=1}^{d}\|w_i\|^{q_k}_{L^{p_{ik}}(\Omega)} + C(\eta)\end{aligned} \tag{5.3.28}$$

We will prove by induction on $k \geqslant 1$, that :

$$\|w_i\|_{L^{p_{ik}}(\Omega)} < F_p(\xi) \quad \text{for all } t \ge \xi > 0\ . \tag{5.3.29}$$

assuming (5.3.29) for some k, (5.3.28) becomes

$$\frac{d}{dt}\int_\Omega\sum_{i=1}^{d}|w_i|^{\frac{2}{\nu_{ik}}}\,dx + C\int_\Omega\sum_{i=1}^{d}\|\nabla w_i\|^2\,dx \le F_p(\xi) \tag{5.3.30}$$

Combining Hölder, Sobolev and Young's inequalities, we get :

$$\left(\int_\Omega |w_i|^{\frac{2}{\nu_{ik}}}\,dx\right)^{\nu_k} \le C\int_\Omega \|\nabla w_i\|^2\,dx + C \text{ ,where } \nu_k = \min_{i=1,2}(\nu_{ik}) \tag{5.3.31}$$

Letting $y_k(t) = \int_\Omega \sum\limits_{1\le i\le 2}|w_i|^{\frac{2}{\nu_{ik}}}\,dx = \|w_i\|_{L^{p_{ik}}}$, and inserting (5.3.31) into (5.3.30), we find

$$\frac{d}{dt}y_k(t) + Cy_k(t)^{\nu_k} \le C$$

Consequently, Lemma 5.1 implies that (5.3.29) will be satisfied for $k+1$. The lemma now follows by applying Lemma 5.3. □

Next, in order to show that the solution u_ε is uniformly bounded, we make use of the following lemma.

Lemma 5.5 *For any $\lambda \ge 1$,there exist positive constants : d_0, d_1, d_2, τ and τ' with τ and τ' not depending on λ such that if* (5.3.15) *is satisfied then for all $t \geqslant \xi > 0$ we have*

$$\frac{d}{dt}\int_\Omega\sum_{1\le i\le 2}|v_i|^{\gamma_i+\lambda}\,dx + d_0\int_\Omega\left\|\nabla\left(|v_i|^{\frac{\lambda-1}{2}}v_i\right)\right\|^2 dx \le d_1(\xi)\lambda^\tau\int_\Omega\sum_{1\le i\le 2}|v_i|^{\gamma_i+\lambda}\,dx + d_2\lambda^\tau,$$

and if (5.3.17) *is satisfied then for all* $t \geqslant 0$ *we have :*

$$\frac{d}{dt}\int_\Omega \sum_{1\le i\le 2} |v_i|^{\gamma_i+\lambda}\,dx + d_0\int_\Omega \left\|\nabla\left(|v_i|^{\frac{\lambda-1}{2}} v_i\right)\right\|^2 dx \le d_1(\|u_0\|)\lambda^\tau \int_\Omega \sum_{1\le i\le 2} |v_i|^{\gamma_i+\lambda}\,dx + d_2\lambda^\tau,$$

where $v_i = |u_i|^{\sigma_i} u_i$ *and* $\gamma_i = \frac{1}{\sigma_i+1}$.

Proof. Multiplying (5.3.12) by $|u_i|^{\lambda(\sigma_i+1)-1} u_i$, and integrating over Ω, we can proceed exactly as we did in the proof of the Lemma 5.4 to obtain :

$$(5.3.32)\quad \frac{d}{dt}\int_\Omega |v_i|^{\gamma_i+\lambda}\,dx + d_3\int_\Omega \left\|\nabla(|v_i|^{\frac{\lambda-1}{2}} v_i)\right\|^2 dx \le d_4\lambda\int_\Omega \sum_{1\le i\le 2} |v_i|^{\lambda+\alpha}\,dx + d_5\lambda^2 + C$$

where $\alpha \ge 1$. By using Hölder's inequality, and the fact that :

$$|v_i|^{\lambda+\alpha} = |v_i|^{\frac{h(1+\lambda)(\alpha-\gamma_i)}{h(\alpha-\gamma_i)+e}}\,|v_i|^{\frac{(\alpha-\gamma_i)(h(\alpha-1)+e)}{h(\alpha-\gamma_i)+e}}\,|v_i|^{\frac{e(\gamma_i+\lambda)}{h(\alpha-\gamma_i)+e}}$$

where e is positive number, and $h > 0$ to be chosen below, we get :

$$(5.3.33)\qquad \int_\Omega |v_i|^{\alpha+\lambda}\,dx \le \left(\int_\Omega |v_i|^{\frac{h(1+\lambda)}{h-2}}\,dx\right)^{P_i}\left(\int_\Omega |v_i|^{p^*}\,dx\right)^{R}\left(\int_\Omega |v_i|^{\lambda+\gamma_i}\,dx\right)^{Q_i}$$

with :

$$P_i = \frac{(h-2)(\alpha-\gamma_i)}{h(\alpha-\gamma_i)+e}, Q_i = \frac{e}{h(\alpha-\gamma_i)+e}, R_i = \frac{2(\alpha-\gamma_i)}{h(\alpha-\gamma_i)+e}, p^* = \frac{1}{2}(h(\alpha-1)+e).$$

Three cases are possible :

• First, consider the case $N > 2$ in this case it suffices to choose $h = N$, next we use the compactness of the imbedding $W^{1,2} \hookrightarrow L^{\frac{2h}{h-2}}$ to obtain :

$$(5.3.34)\qquad \int_\Omega |v_i|^{\frac{h(1+\lambda)}{h-2}}\,dx \le C(\Omega)\left[\int_\Omega \left\|\nabla(|v_i|^{\frac{(\lambda-1)}{2}} v_i)\right\|^2 dx + \int_\Omega |v_i|^{1+\lambda}\,dx\right]^{\frac{h}{h-2}}$$

• Next, we consider the case $N = 2$ one can obtain (5.3.34), by choosing $h > 2$, and using the compactness of the imbedding $W^{1,2} \hookrightarrow L^q$ for all $q \ge 2$.

• Lastly, we consider the case $N = 1$, we choose $h > 2$ and we use the compactness of the imbedding $W^{1,2} \hookrightarrow L^\infty$, to obtain (5.3.34).

Since p^* is independent of λ, by reporting (5.3.34) in (5.3.33), and by using the fact that $\frac{h}{h-2}P_i + Q_i = 1$, Young's inequality gives us :

(5.3.35)

$$d_4\lambda\int_\Omega |v_i|^{\alpha+\lambda}\,dx \le \eta\left[\int_\Omega \left\|\nabla(|v_i|^{\frac{(\lambda-1)}{2}} v_i)\right\|^2 dx + \int_\Omega |v_i|^{1+\lambda}\,dx\right] + C(\eta)\lambda^\tau\int_\Omega v_i^{\gamma_i+\lambda}dx$$

where $\tau = \max_{i=1,d}\left\{\frac{1}{Q_i}\right\}$for all $\alpha \ge 1$. In particular, for $\alpha = 1$,we have :

$$(5.3.36)\qquad \int_\Omega |v_i|^{1+\lambda}\,dx \le \eta\int_\Omega \left\|\nabla(|v_i|^{\frac{(\lambda-1)}{2}} v_i)\right\|^2 dx + C(\eta)\lambda^\tau\int_\Omega v_i^{\gamma_i+\lambda}dx$$

Assuming (5.3.15), and inserting (5.3.35) and (5.3.36) in (5.3.32), we obtain that for η sufficiently small

$$\frac{d}{dt}\int_\Omega |v_i|^{\gamma_i+\lambda}\,dx + d_0\int_\Omega \left\|\nabla(|v_i|^{\frac{\lambda-1}{2}}v_i)\right\|^2 dx \leq d_1(\xi)\lambda^\tau\int_\Omega\sum_{1\leq i\leq 2}|v_i|^{\gamma_i+\lambda}\,dx + d_2(\xi)\lambda^2.$$

for all $t \geq \xi > 0$. Exactly in the same way we can prove that if (8.1.3) is given we have (5.3.37)

$$\frac{d}{dt}\int_\Omega |v_i|^{\gamma_i+\lambda}\,dx + d_0\int_\Omega \left\|\nabla(|v_i|^{\frac{\lambda-1}{2}}v_i)\right\|^2 dx \leq d_1(u_0)\lambda^\tau\int_\Omega\sum_{1\leq i\leq 2}|v_i|^{\gamma_i+\lambda}\,dx + d_2(u_0)\lambda^2.$$

for all $t \geq 0$. This completes the proof of the lemma. □

As a final preparation we state the

Lemma 5.6 *Let $\lambda_k = 2^k$, $k \in \mathbb{N}$, t and μ be positive constants such that : $t - \frac{\mu}{\lambda_k} > 0$. then there exist positive constants $\varkappa$ and $C_0(\mu)$ such that*

(5.3.38)
$$y_k(t) \leq U_k(t,\mu),$$

where

$$y_k(t) = \int_\Omega \sum_{1\leq i\leq k}|u_i|^{(\sigma_i+1)(\lambda_k+\gamma_i)}\,dx \qquad ,k\geq 1.$$

$$U_k(t,\mu) = C_0(\mu)\lambda_k^{\varkappa}(\sup_{s\geq t} y_{k-1}(s)+1)^{s_k}$$

with

$$s_k = \frac{\delta+\lambda_{k+1}}{\delta+\lambda_k}$$

where $\delta = \min_{1\leq i\leq 2}\{h-\gamma_i(h-2)\} > 0$, and $h = N$ if $N > 2$ $h = 2$ if $N < 2$.

Proof. let us construct the following sequences

$$Q_{ik} = \frac{h(\lambda_k+1)-(h-2)(\gamma_i+1)}{h(\lambda_k+1)-(h-2)(\gamma_i+1)}\ ,\ P_{ik} = 1-Q_{ik}\ ,\ \overline{P_{ik}} = \frac{h}{h-2}P_{ik}$$

$$s_{ik} = \frac{Q_{ik}}{1-\overline{P_{ik}}} = \frac{h-(h-2)\gamma_i+\lambda_{k+1}}{h-(h-2)\gamma_i+\lambda_k} > 1$$

Hence Hölder's inequality implies that

$$\int_\Omega |v_i|^{\lambda_k+\gamma_i}\,dx \leq \left(\int_\Omega |v_i|^{\frac{h(1+\lambda_k)}{h-2}}\,dx\right)^{P_{ik}}\left(\int_\Omega |v_i|^{\lambda_{k-1}+\gamma_i}\,dx\right)^{Q_{ik}}$$

From which, with the help of Sobolev's inequality, we obtain

$$\int_\Omega |v_i|^{\lambda_k+\gamma_i}\,dx \leq C\left(\int_\Omega\left\|\nabla(|v_i|^{\frac{\lambda_k-1}{2}}v_i\right\|^2 dx + \int_\Omega |v_i|^{\lambda_k+\gamma_i}\,dx\right)^{\overline{P_{ik}}}\left(\int_\Omega |v_i|^{\lambda_{k-1}+\gamma_i}\,dx\right)^{Q_{ik}}$$

hence the Young's inequality guarantees that

$$c\lambda^{\tau_2} \int_\Omega |v_i|^{\lambda_k+\gamma_i} dx \leq \frac{d_0}{2} \int_\Omega \left\| \nabla(|v_i|^{\frac{\lambda_k-1}{2}} v_i \right\|^2 dx + c^{'} \lambda^{\tau_3} \left(\int_\Omega |v_i|^{\lambda_k+\gamma_i} dx \right)^{s_{ik}} \tag{5.3.39}$$

The remaining part of the proof follows from the proof of [24, lemma 4]. □

Proof of Proposition 2

1 . Suppose that (5.3.15) *is given :*

Let ξ and $\xi^{'}$ be tow positive reals such that : $\xi^{'} > \xi > 0$. We put $\mu = \frac{\xi^{'}-\xi}{2}$; $t_0 = \frac{\xi^{'}+\xi}{2} > \xi$; $t_k = t_{k-1} - \frac{\mu}{\lambda_k}$. From (5.3.38) we have :

$$1 + \sup_{t \geq t_{k-1}} y_k(t) \leq C_0 \lambda_k^\sigma \left(1 + \sup_{t \geq t_k} y_{k-1}(t) \right)^{s_k}$$

Letting $K_\xi = \max_{i=1,2} \sup_{t \geq \xi} \left(\int_\Omega |v_i|^{\gamma_i+1} dx + 1 \right)$, we deduce that

$$\sup_{t \geq t_{k-1}} y_k(t) \leq C_0^{A_k} 2^{\sigma B_k} K_\xi^{C_k}$$

where

$A_k = 1 + s_k + s_k s_{k-1} + ... + s_k s_{k-1} ... s_1.$
$B_k = k + (k-1)s_k + (k-2) s_k s_{k-1} + .. + s_k s_{k-1} ... s_1.$
$C_k = s_k s_{k-1} ...... s_1 = \frac{\delta+\lambda_{k+1}}{\delta+\lambda_1}$

For completing the proof it suffices to see that A_k and B_k are of order 2^k as k tends to $+\infty$. Hence we find :

$$\sup_{t \geq t_0} y_k(t) \leq \sup_{t \geq t_{k-1}} y_k(t) \leq C 2^k 2^{c 2^k} K_\xi^{\frac{\delta+\lambda_{k+1}}{\delta+\lambda 0}} \tag{5.3.40}$$

By taking the $\frac{1}{\gamma_i+2^k}$ power of both sides of (5.3.40) and passing to the limit as k tends to $+\infty$, we obtain :

$$\sup_{t \geq \xi^{'}} \|v_i(t,.)\|_{L^\infty(\Omega)} \leq \lim_{k \to \infty} \sup_{t \geq t_0} (y_k(t))^{\frac{1}{\gamma_i+\lambda_k}} \leq C C_0^{2^k} 2^{\varkappa 2^k} K_\xi^{\frac{\delta+\lambda_{k1}}{\delta+\lambda_1}}.$$

2 . Suppose that (5.3.17) *is given :* We need the following Lemma due to Alikakos [2].

Lemma 5.7 *Let* ω *a nonnegative function defined on* $(0,\infty) \times \Omega$, *satisfying the differential inequality :*

$$\frac{\partial}{\partial t} \int_\Omega |\omega|^{\lambda_k+\gamma} \leq -\varepsilon_k \int_\Omega |\omega|^{\lambda_k+\gamma} + (a_k+\varepsilon_k) c_k \left[\sup_{t \geq 0} \int_\Omega |\omega|^{\lambda_{k-1}+\gamma} \right]^{p_k} \quad k = 1, 2, ...$$

where : a_k, ε_k, c_k *are respectively of order* $\dfrac{1}{2^k}, 2^{\alpha k}, 2^k$ *as* k *tends to infinity,* α *is a positive constant , and* $(\lambda_{k-1}+1)p_k \leq \lambda_k + 1$*, then there exists a positive constant* a *such that :*

$$\sup_{t\geq 0} \|\omega(t,.)\|_{L^\infty} \leq a 2^{2(\alpha+2)} K$$

where : $K \geq \max\left\{1, \sup_{t\geq 0} \|\omega(t,.)\|_{L^{\sigma+1}}, \|\omega(0,.)\|_{L^\infty}\right\}$

Now combining (5.3.37) and (5.3.39) we obtain :

$$\frac{d}{dt}\int_\Omega |v_i|^{\lambda_k+\gamma} \leq (-2c + d_1(u_0))\,\lambda_k^{\tau_2}\int_\Omega |v_i|^{\lambda_k+\gamma} + C\lambda_k^{\tau_3}\left[\sup_{t\geq 0}\int_\Omega |v_i|^{\lambda_{k-1}+\gamma}\right]^{s_{ik}}$$

$k = 1, 2, ..., \infty$. which completes the proof with the aid of Lemma 5.7.

Remark 5.2 *The results of this section can be extended to the following cases :*

Case 1 .

$$\begin{cases} \partial_t u_i - \Delta(|u_i|^{\sigma_i} u_i) = f_i(t,x,u,\nabla u_i) & in\ ]0,\infty[\times\Omega \\ \dfrac{\partial}{\partial\nu}(|u_i|^{\sigma_i} u_i)\, u_i \leq 0 & on]0,\infty[\times\partial\Omega \\ u(0,.) = u_{i0}\ , u_{i0} \in L^\infty(\Omega) & in\quad \Omega \end{cases}$$

with

- $\sigma_i > 0$
- $|f_i(t,x,u,\xi)| \leq k_1 \sum_{1\leq j\leq d} |u_j|^{\alpha_j} + k_2 \|\xi\|^{\delta_i} + k_3$ *, where*
- $k_l \geq 0$ *;* $l = 1, 3$ *;* $\alpha_i \in \left[0, \sigma_i + 1 + \frac{\sigma_i+2}{N}\right[$ *;* $\delta_i \in \left[0, \frac{\sigma_i+1}{\sigma_i}\right[$.

Case 2 .

$$\begin{cases} \partial_t(u_i) - \Delta(|u_i|^{\sigma_i} u_i) = g_i(t,x,u) + \overrightarrow{b_i}\nabla(|u_i|^{m_i-1} u_i) & in\ ]0,\infty[\times\Omega \\ \left.\begin{array}{l} \dfrac{\partial}{\partial\nu}(u_i\,|u|^{\sigma_i})\, u_i \leq 0 \\ or \\ \sum_{j=1}^{N} u_i\left[\frac{\partial}{\partial x_j}(|u_i|^{\sigma_i} u_i) + b_{ij}\,|u_i|^{m_i-1} u_i\nu_j\right] \leq 0 \end{array}\right\} & on\ ]0,\infty[\times\partial\Omega \\ u_i(0,.) = u_{i0} & in\ \Omega \end{cases}$$

with

- $\sigma_i > 0$
- *There exist* $\alpha_j \in \left[0, \sigma_j + 1 + \dfrac{\sigma_j+2}{N}\right[$ *such that for* $(t,x) \in \mathbb{R}^+\times\Omega$*, and* $u = (u_1, u_2, ..., u_d)$ *we have*

$$|g_i(t,x,u)| \leq k_1 \sum_{1\leq j\leq d} u_j^{\alpha_j} + k_2$$

for some positive constants k_1, k_2.

- $m_i \in \left[0, (\sigma_i+1)\,\frac{N+1}{N}\right[$.

5.4 Global existence

In order to prove the global existence we prove at first the following energy estimates

Lemma 5.8 *Suppose that the assumptions* $(H_1) - (H_5)$ *are satisfied. Then the solution* u_ε *of* (5.3.12) − (5.3.14) *is global (that is* $T_{\max,\varepsilon} = \infty$*) and there exists a positive function* F *not depending on* ε *and on* u_0 *such that*

(5.4.41) $$\|u_{i\varepsilon}(t,.)\|_{L^\infty} \le F(\xi) \qquad \textit{for all } t \ge \xi > 0,$$

Moreover, if $u_0 \in (L^\infty(\Omega))^d$ *then there exists a positive constant* C *not depending on* ε *such that*

(5.4.42) $$\|u_{i\varepsilon}(t,.)\|_{L^\infty} \le C(\|u_0\|_{L^\infty}) \qquad \textit{for all} \quad t \ge 0,$$

Proof. Since Proposition 2 is proved, it is enough to show that there is a positive function F_0 such that :

$$\|u_{i\varepsilon}(t,.)\|_{L^{\sigma_i+2}(\Omega)} \le F_0(\xi) \text{ for all } t \ge \xi > 0,$$

and if $u_0 \in (L^\infty(\Omega))^d$ then there is a positive constant C_0 such that

$$\|u_{i\varepsilon}(t,.)\|_{L^{\sigma_i+2}(\Omega)} \le C_0(u_0) \text{ for all } t \ge 0.$$

Multiplying (5.3.12) by $|u_{i\varepsilon}|^{\sigma_i} u_{i\varepsilon}$, integrating over Ω and taking into account that $\alpha_{ij} < \sigma_j + 1$ and $m_i < \sigma_i + 1$, we obtain the following, with the help of Young and Poincaré inequalities, for all $\eta > 0$

$$\frac{1}{\sigma_i+2}\int_\Omega \partial_t |u_{i\varepsilon}|^{\sigma_i+2} dx + \int_\Omega \|\nabla(|u_{i\varepsilon}|^{\sigma_i} u_{i\varepsilon})\|^2 dx \le \eta \sum_{j=1}^{d} \int_\Omega \|\nabla(|u_j|^{\sigma_j} u_j)\|^2 dx + C(\eta)$$

Adding these inequalities, we find

(5.4.43) $$\int_\Omega \sum_{i=1}^{d} \partial_t |u_{i\varepsilon}|^{\sigma_i+2} dx + C(1-d\eta)\int_\Omega \sum_{i=1}^{d} \left\|\nabla(|u_{i\varepsilon}|^{\sigma i} u_{i\varepsilon})\right\|^2 dx \le C(\eta)$$

By choosing η small enough in the last inequality, and using the Poincaré inequality we have

(5.4.44) $$\frac{d}{dt}\int_\Omega \sum_{i=1}^{d} |u_{i\varepsilon}|^{\sigma_i+2} dx + C\int_\Omega \sum_{i=1}^{d} |u_{i\varepsilon}|^{\,2(\sigma_i+1)} dx \le C$$

Using Hölder's inequality for the second term in the left side, we find :
(5.4.45)
$$\sum_{i=1}^{d}\int_\Omega |u_{i\varepsilon}|^{2(\sigma_i+1)} dx \ge C\sum_{i=1}^{d}\left(\int_\Omega |u_{i\varepsilon}|^{\sigma_i+2} dx\right)^{2\frac{(\sigma_i+1)}{\sigma_i+2}} \ge C\sum_{i=1}^{d}\left(\int_\Omega |u_{i\varepsilon}|^{\sigma_i+2} dx\right)^{\nu}$$

where $\nu > 1$ depending on σ_i.

Inserting (5.4.45) into (5.4.44) and writing $y = \sum_{i=1}^{d} \int_\Omega |u_{i\varepsilon}|^{\sigma_i+2} dx$, we obtain the following, after the use of Jensen's inequality

$$\frac{d}{dt}y(t) + Cy(t)^\nu \le C$$

which completes the proof with the aid of Lemma 5.1. □

Integrating the differential inequality (5.4.43) over $[0,T]$, and choosing η sufficiently small, we obtain

$$\int_\Omega |u_{i\varepsilon}|^{\sigma_i+2}(T,x)dx + \int_{Q_T} \left\|\nabla(|u_{i\varepsilon}|^{\sigma i} u_{i\varepsilon})\right\|^2 dxdt \le C(T) \quad i = 1, d$$

Using the uniform estimate (5.4.42), multiplying (5.3.12) by $\phi(u_{i\varepsilon})^{\sigma_i} u_{i\varepsilon}$ and integrating over Q_T, we get

$$\int_0^T \int_\Omega \|\phi(u_{i\varepsilon})^{\sigma_i}\nabla u_{i\varepsilon}\|^2 dxdt \le C(T)\ i = 1, d.$$

By compactness arguments, it follows that there exists a function u_i and a subsequence of $u_{i\varepsilon}$, which we still denoted by $u_{i\varepsilon}$ such that

$$\begin{aligned}
&(|u_{i\varepsilon}| + \varepsilon)^{\sigma_i} \nabla u_{i\varepsilon} \longrightarrow |u_i|^{\sigma_i} \nabla u_i \quad \text{weakly in } L^2(Q_T),\\
&|u_{i\varepsilon}|^{\sigma_i} u_{i\varepsilon} \longrightarrow |u_i|^{\sigma_i} u_i \text{ in the strong topology of } L^2(Q_T),\\
&u_{i\varepsilon}(t,.) \longrightarrow u_i(t,.) \text{ almost every where in } \Omega,\\
&|u_{i\varepsilon}|^{m_i-1} u_{i\varepsilon} \longrightarrow |u_i|^{m_i-1} u_i \text{ in the strong topology of } L^2(Q_T),\\
&g_{i\varepsilon}(u_\varepsilon) \longrightarrow g_i(u) \text{ almost every where in } Q_T.
\end{aligned}$$

Hence, the dominated convergence theorem guarantees that
$g_{i\varepsilon}(u_\varepsilon) \longrightarrow g_i(u)$ in the strong topology of $L^2(Q_T)$,
Since u_ε is a smooth solution of (5.3.12) − (5.3.14), it satisfies clearly

$$\begin{aligned}
&\int_\Omega u_{i\varepsilon}(x,T)\varphi_i(x,T)dx - \int_{Q_T} \varphi_{it} u_{i\varepsilon} dxdt + \int_{Q_T} \nabla(|u_{i\varepsilon}|^{\sigma_i} u_{i\varepsilon})\nabla\varphi_i dxdt\\
&\quad = \int_{Q_T} (g_{i\varepsilon}(u_{i\varepsilon}, u_{2\varepsilon})\varphi_i + \overrightarrow{b_i}\nabla\varphi_i |u_{i\varepsilon}|^{m_i-1} u_{i\varepsilon})dxdt + \int_\Omega u_{i0\varepsilon}(x)\varphi_i(0,x)dx
\end{aligned}$$

for any test function φ_i. From here, passing to the limit when ε tends to zero we obtain that $u = (u_1, u_2, ..., u_d)$ is indeed a weak solution in the sense of our definition.

Finally from the fact that, for all $t \ge \xi > 0$, $\|u_{i\varepsilon}(t,.)\|_{L^\infty(\Omega)}$ is uniformly bounded, then we can extract a subsequence still denoted $(u_{i\varepsilon}(t,.))_{0<\varepsilon<1}$ such that as ε tends to 0 $(u_{i\varepsilon}(t,.))_{0<\varepsilon<1}$ is weakly convergent to $u_i(t,.)$ in $L^p(\Omega)$ for every finite $p \ge 1$. Hence due to [23], one can extract a subsequence $(\omega_{i\varepsilon}(t,.))_{0<\varepsilon<1}$ of convex combination of element of $u_{i\varepsilon}(t,.)$ such that $\omega_{i\varepsilon}(t,.) \longrightarrow u_i(t,.)$ weakly in $L^p(Q_T)$, and almost every where in Ω. From the fact just proved it follows that

$$u_i \in L^\infty_{loc}(\xi,\infty; L^\infty(\Omega)) \qquad i = 1, 2, ..., d.$$

Moreover, if $u_0 \in (L^\infty(\Omega))^d$ one finds that

$$u_i \in L^\infty_{loc}(0,\infty; L^\infty(\Omega)) \qquad i = 1,2,...,d.$$

5.5 Uniqueness

In this section we consider the question of uniqueness of bounded solution. We will always assume that
(H_8) $\quad u_{i0} \in L^\infty(\Omega) \qquad i = 1,2,...,d.$

Theorem 5.5 *If, in addition to* $(H_1)-(H_4)$*,* $u_{i0} \in L^\infty(\Omega)$ *, then* u *is unique in the class of bounded functions.*

Proof. The proof is a straight forward extension of that given in [9] in special situation.

Suppose on the contrary that there exist tow weak solutions $u = (u_1, u_2, ..., u_d)$ and $\hat{u} = (\hat{u}_1, \hat{u}_2, ..., \hat{u}_d)$ of problem (5.1.1) – (5.1.4) such that : $u, \hat{u} \in (L^\infty(Q_T))^d$ that is there exist a positive constant $M(T)$ and a set $J \subset \{1,2,..,d\}$ such that :

$$\left(\int_{Q_T} |u_i - \hat{u}_i|^2 \, dxdt\right)^{\frac{1}{2}} > M(T) \text{ if } i \in J, \text{ and } u_i = \hat{u}_i \text{ if } i \notin J, \tag{5.5.46}$$

we argue from this to a contradiction by constructing suitable test functions,to do this let us introduce a function $\Psi_i \in L^\infty(Q_T)$ such that

$$\Psi_i = \begin{cases} \dfrac{|u_i|^{\sigma_i} u_i - |\hat{u}_i|^{\sigma_i} \hat{u}_i}{u_i - \hat{u}_i} & \text{if } u_i \neq \hat{u}_i \\ 0 & \text{otherwise} \end{cases}$$

We consider a sequence of functions $\{\Psi_{i\varepsilon}\}$, such that

$$\begin{aligned} &i)\ \Psi_{i\varepsilon} \in L^\infty(Q_T), \\ &ii)\ \varepsilon \le \Psi_{i\varepsilon} \le \|\Psi_i\|_{L^\infty(Q_T)} + \varepsilon, \\ &iii)\ \frac{\Psi_{i\varepsilon} - \Psi_i}{\sqrt{\Psi_{i\varepsilon}}} \longrightarrow 0 \text{ in } L^\infty(Q_T). \end{aligned}$$

We consider also the adjoint non-degenerate boundary value problem

$$\begin{cases} \partial_t \varphi_{i\varepsilon} + \Psi_{i\varepsilon} \Delta \varphi_{i\varepsilon} = 0 & \text{in } Q_T \\ \varphi_{i\varepsilon} = 0 & \text{on } (0,T) \times \partial\Omega \\ \varphi_{i\varepsilon} = \varkappa_i & \text{in } \Omega \times (t = T) \end{cases} \tag{5.5.47}$$

For any smooth function $\varkappa_i$, with $0 \le \varkappa_i \le 1$, the problem (5.5.47) has a unique solution $\varphi_{i\varepsilon} \in C^\infty(Q_T)$ satisfying

$$\begin{aligned} &i)\ 0 \le \varphi_{i\varepsilon} \le 1, \\ &ii)\ \int_{Q_T} \Psi_{i\varepsilon} \left(\Delta \varphi_{i\varepsilon}\right)^2 \le C, \\ &iii)\ \sup_{0 \le t \le T} \int_\Omega \|\nabla \varphi_{i\varepsilon}\|^2 \le C. \end{aligned}$$

where the constant C depends only on $\varkappa_i$.

It is obvious that the difference $u_i - \hat{u}_i$ satisfies the following equality

$$\begin{aligned}\int_\Omega (u_i-\hat{u}_i)\varphi_i(x,T)dx + (\sigma_i+1)\int_{Q_T} \nabla\left[u_i^{\sigma_i}u_i - |\hat{u}_i|^{\sigma_i}\hat{u}_i\right]\nabla\varphi_i dxdt \\ = \int_{Q_T}(u_i-\hat{u}_i)\varphi_{it}(x,t)dxdt + \int_{Q_T}(g_i(u)-g_i(\hat{u}))\varphi_i(x,t)dxdt \\ + \int_{Q_T}\overrightarrow{b_i}\nabla\varphi_i\left[|u_i|^{m_i-1}u_i - |\hat{u}_i|^{m_i-1}\hat{u}_i\right]\end{aligned} \tag{5.5.48}$$

for every $\varphi_i \in C^1(\overline{Q_T}), \varphi_i = 0$ on $(0,T)\times\partial\Omega$.

Setting $\varphi_i = \varphi_{i\varepsilon}$ and $\varkappa_i = sign_\varepsilon(u_i-\hat{u}_i)^+$ in (5.5.48) when $sign_\varepsilon$ is a regular approximation of the function $sign$, we obtain :

$$\begin{aligned}\int_\Omega (u_i-\hat{u}_i)^+(x,T)dx + \int_{Q_T}\Delta\varphi_{i\varepsilon}\left(\Psi_{i\varepsilon}-\Psi_i\right)(u_i-\hat{u}_i)dxdt \\ = \int_{Q_T}(g_i(u)-g_i(\hat{u}))\varphi_{i\varepsilon}(x,t) + \int_{Q_T}\overrightarrow{b_i}\nabla\varphi_i\left[|u_i|^{m_i-1}u_i - |\hat{u}_i|^{m_i-1}\hat{u}_i\right]\end{aligned}$$

Using the local Lipschitz continuity of the functions g_i and $|z|^{m_i}z$, and the fact that u_ε is uniformly bounded, and letting $\varepsilon \longrightarrow 0$, we obtain the following inequality after the use of Hölder's inequality

$$\int_\Omega (u_i-\hat{u}_i)^+(x,T)dx \le C\int_{Q_T}\sum_{j=1}^{d}|u_j-\hat{u}_j|\,dxdt + C(T)\left(\int_{Q_T}|u_i-\hat{u}_i|^2dxdt\right)^{\frac{1}{2}} \tag{5.5.49}$$

but if $i \in J$ we have :

$$\left(\int_{Q_T}|u_i-\hat{u}_i|^2dxdt\right)^{\frac{1}{2}} \le \frac{\int_{Q_T}|u_i-\hat{u}_i|^2dxdt}{M(T)} \le C(T)\int_{Q_T}|u_i-\hat{u}_i|\,dxdt \tag{5.5.50}$$

Combining (5.5.49), (5.5.50) and assumption (5.5.46) we find :

$$\begin{aligned}\int_\Omega (u_i-\hat{u}_i)^+(x,T)dx &\le (C+C(T))\int_{Q_T}\sum_{j=1}^{d}|u_j-\hat{u}_j|\,dxdt \\ &\le (C+C(T))\int_{Q_T}\sum_{j\in J}|u_j-\hat{u}_j|\,dxdt\end{aligned}$$

By putting the sum over $j \in J$,, we have :

$$\int_\Omega\sum_{j\in J}(u_j-\hat{u}_j)^+(x,T)dx \le d(C+C(T))\int_{Q_T}\sum_{j\in J}|u_j-\hat{u}_j|\,dxdt \tag{5.5.51}$$

In a similar way we establish that if we put $\varkappa_i = sgn_\varepsilon(u_i-\hat{u}_i)^-$, we find :

$$\int_\Omega\sum_{j\in J}(u_j-\hat{u}_j)^-(x,T)dx \le d(C+C(T))\int_{Q_T}\sum_{j=1}^{d}|u_j-\hat{u}_j|\,dxdt \tag{5.5.52}$$

By summing up (5.5.51) and (5.5.52), we obtain :

$$\int_\Omega \sum_{j\in J} |u_j - \hat{u}_j| (x,T)dx \leq 2d(C + C(T)) \int_{Q_T} \sum_{j\in J} |u_j - \hat{u}_j| \, dxdt$$

We may apply Gronwall's lemma to conclude. □

5.6 Global attractor

We are now in a position to give the proof of Theorem 5.2. We start the definition

Definition 5.2 *Let $S(t)$ be a semigroup of operators acting in a real Banach space E. A set $K \subset E$ is said to be an attracting set for $S(t)$, if for all $B \subset E$*

$$\lim_{t\longrightarrow\infty} d(S(t)(B), K) = 0$$

A set $B_0 \subset E$ is said to be an absorbing set for $S(t)$, if for all $B \subset E$, there exists a real $T = T(B_0) > 0$, such that for all $t \geq T$ $S(t)B \subset B_0$.

we also need the following indispensable tool see [57]

Theorem 5.6 *Let $S(t)$ be a semigroup of operators acting in a real Banach space E and suppose for each $t \geq 0$ the operator $S(t)$ is continuous. If $(S(t))_{t\geq 0}$ has a compact attracting set $\mathcal{K}$, then this semigroup has a global attractor $\mathcal{A}$ where $\mathcal{A} \subseteq \mathcal{K}$.*

Remark 5.3 *Since an absorbing set is also an attracting set, a semigroup $S(t)$ of continuous operators having a compact absorbing set $\mathcal{B}_0$ has by Theorem 5.6 a global attractor $\mathcal{A}$ which is the $\omega-$limit set for $\mathcal{B}_0$.*

The theorem 5.1 allows us to define the operators $S(t) : u_0 \rightarrow u(t) \quad t \geq 0$. These operators enjoy the semigroup property and the continuity property as it will be proved in the following. First we establish, as in [24] with some modification, some properties of the semigroup $S(t)$.

Theorem 5.7 *Given $u_0, \hat{u}_0 \in (L^1(\Omega))^d$, let u and $\hat{u}$ be limit of the smooth solutions of the approximated problem $(5.3.12) - (5.3.14)$ with $\varepsilon = \dfrac{1}{n}$ and initial data $u_{0n}, \hat{u}_{0n}$ respectively (that is u and $\hat{u}$ are the weak solutions), where $u_{0n} \rightarrow u_0$, $\hat{u}_{0n} \rightarrow \hat{u}_0$ in $L^1(\Omega)$. Then there exists a positive function C such that :*

$$\sum_{i=1}^{d} \|u_i(t) - \hat{u}_i(t)\|_{L^1(\Omega)} \leq C(t) \sum_{i=1}^{d} \|u_{i0} - \hat{u}_{i0}\|_{L^1(\Omega)}$$

that is the operator $S(t)$ is continuous from $(L^1(\Omega), \|.\|_1)^d$ to itself.

Proof. The difference $u_{in} - \hat{u}_{in}$ satisfies the following equality

$$\begin{aligned}(u_{in} - \hat{u}_{in})_t - (\sigma_i + 1)\left[div\left(|u_{in}| + \tfrac{1}{n}\right)^{\sigma_i} \nabla u_{in} - \left(|\hat{u}_{in}| + \tfrac{1}{n}\right)^{\sigma_i} \nabla \hat{u}_{in}\right] \\ = g_{in}(u_n) - g_{in}(\hat{u}_n) + \overrightarrow{b_i} \nabla \left[|u_{in}|^{m_i-1} u_{in} - |\hat{u}_{in}|^{m_i-1} \hat{u}_{in}\right]\end{aligned} \tag{5.6.53}$$

Multiplying (5.6.53) by $sng(u_{in} - \hat{u}_{in})$, and integrating over Q_t, we get :

(5.6.54)

$$\begin{aligned}&\int_{Q_t} |u_{in} - \hat{u}_{in}|_t - \int_{Q_t} sng(u_{in} - \hat{u}_{in}) \overrightarrow{b_i} \nabla \left[|u_{in}|^{m_i-1} u_{in} - |\hat{u}_{in}|^{m_i-1} \hat{u}_{in}\right] dxds \\ &-(\sigma_i + 1) \int_0^t \int_\Omega sng(u_{in} - \hat{u}_{in}) \left[div\left(|u_{in}| + \frac{1}{n}\right)^{\sigma_i} \nabla u_{in} - \left(|\hat{u}_{in}| + \frac{1}{n}\right)^{\sigma_i} \nabla \hat{u}_{in}\right] \\ &\leq \int_0^t \|g_{in}(u_n(s)) - g_{in}(\hat{u}_n(s))\|_{L^1(\Omega)} ds\end{aligned}$$

Concerning the second integral in the left side of above inequality, we use the fact that $|z|^{m_i-1} z$ is an increasing function, to obtain

$$\begin{aligned}&\int_{Q_t} sign(u_{in} - \hat{u}_{in}) \overrightarrow{b_i} \nabla \left(|u_{in}|^{m_i-1} u_{in} - |\hat{u}_{in}|^{m_i-1} \hat{u}_{in}\right) dxds \\ &= \int_{Q_t} \overrightarrow{b_i} \nabla \left||u_{in}|^{m_i-1} u_{in} - |\hat{u}_{in}|^{m_i-1} \hat{u}_{in}\right| dxds \\ &= \int_{Q_t} div(\overrightarrow{b_i}) \left||u_{in}|^{m_i-1} u_{in} - |\hat{u}_{in}|^{m_i-1} \hat{u}_{in}\right| dxds\end{aligned}$$

Since the function $|z|^{m_i-1} z$ is locally Lipschitz the uniform estimates (5.4.42) gives

$$\int_{Q_t} sign(u_{in} - \hat{u}_{in}) \overrightarrow{b_i} \nabla \left(|u_{in}|^{m_i-1} u_{in} - |\hat{u}_{in}|^{m_i-1} \hat{u}_{in}\right) dxds \leq C \int_{Q_t} |u_{in} - \hat{u}_{in}| \tag{5.6.55}$$

We use semi-groups theory to prove that the third integral in the left side of (5.6.54) is positive. Moreover, since g_i is locally Lipschitz the uniform estimate (5.4.42), combined with (5.6.54) and (5.6.55), gives

$$\|u_{in}(t) - \hat{u}_{in}(t)\|_{L^1(\Omega)} \leq \|u_{i0n} - \hat{u}_{i0n}\|_{L^1(\Omega)} + R \int_0^t \sum_{k=1}^d \|u_{kn}(s) - \hat{u}_{kn}(s)\|_{L^1(\Omega)} ds$$

Finally, summing up these inequalities over i, and applying Gronwall's lemma, we get

$$\sum_{i=1}^d \|u_{in}(t) - \hat{u}_{in}(t)\|_{L^1(\Omega)} \leq C(t) \sum_{i=1}^d \|u_{i0n} - \hat{u}_{i0n}\|_{L^1(\Omega)} .$$

After this we pass to limit as $n \to \infty$, we obtain the desired result. □

By using the dominated theorem, we can extend the preceding lemma as follows

Lemma 5.9 *For all $t > 0$, The operator $S(t)$ is continuous from $\left(L^p(\Omega), \|.\|_p\right)^d$ to itself for all $p \geq 1$.*

Now, in order to obtain the existence of an absorbing set in $\mathbb{C}^{\alpha}(\Omega)$, we use the estimate (5.4.42) which, by virtue of the results of [54], implies that there is a positive constant C depends only on ξ and $0 < \alpha < 1$ such that

$$\|u_i(.,t)\|_{L^{\infty}(\Omega)} \leq C(\xi) \quad \textit{for all } t \geq \xi > 0,$$

That is the semigroup possesses an absorbing set $\mathcal{B}_0$ in $\mathbb{C}^{\alpha}(\Omega)$.

Proof of theorem 5.2

Finally, all the assumptions of theorem 5.6 are satisfied and we deduce from this theorem the existence of a global attractor $\mathcal{A}$ for the system (5.1.1) – (5.1.4). Hence theorem 5.2 is proved.

5.7 The Limit cases

We are going to see now that in the limit case ($f_i(u, \nabla u_i) = \sum_{j=1}^{d} c_{ij} u_j^{\sigma_j+1} + \overrightarrow{b_i} \nabla \left(u_i^{\sigma_i+1}\right)$) depending on the relation between parameters $c_{ij}, \lambda_i, \lambda$, we get globally bounded weak solution or blowing up solutions. More precisely we prove that

1. If Ω is small, in an appropriate sense, all positive weak solutions of (5.1.1) – (5.1.4) are global.
2. If Ω is sufficiently large, all positive weak solutions of (5.1.1) – (5.1.4) blow-up (i.e. become unbounded) in finite time.

Hence we deduce that large domains ($\lambda < 1$ which is equivalent to $\lambda_i < 0$) are more unstable than small domains ($\lambda \geq 1$).

Throughout this section we suppose that (H_2), (H_3), (H_6), (H_7) and (H_8) are satisfied.

5.7.1 Global Existence

Let us consider the problem :

$$\text{(5.7.56)} \qquad \begin{cases} \partial_t(u_i) - \Delta(|u_i|^{\sigma_i} u_i) = g_i(u) + \overrightarrow{b_i} \nabla(|u_i|^{m_i-1} u_i) & \text{in }]0,\infty[\times \Omega \\ u_i = 0 & \text{on }]0,\infty[\times \partial\Omega \\ u_i(0,.) = u_{i0} & \text{in } \Omega \end{cases}$$

We suppose that

(H_9) there exist positive constants $c_{ij}, \alpha_{ij}, L_i \geq 0$ such that for all $u_1, u_2 \geq 0$ we have

$$|g_i(u)| = c_{i0} + \sum_{j=1}^{d} c_{ij} u_j^{\alpha_{ij}}$$
$$\left\|\overrightarrow{b_i}\right\| \leq L_i$$

Finally, we suppose that

$$(H_{10})\begin{cases} 1 \,.\ \alpha_{ij} < \sigma_j + 1,\ m_i = \sigma_i + 1 \text{ and } \left\|\overrightarrow{b_i}\right\| < 2\dfrac{\lambda}{\lambda+1} \forall i,j = 1,...;d \\ \text{or} \\ 2 \,.\ \exists\ j_0 \in \{1,...,d\} \,/\, \alpha_{ij_0} = \sigma_{j_0} + 1,\ m_i < \sigma_i + 1 \text{ and } c_{ij_0} < \lambda\ \forall i = 1,...;d \\ \text{or} \\ 3 \,.\alpha_{ij} = \sigma_j + 1\ ,\ m_i < \sigma_i + 1; \text{and } d \max\limits_{i,j=1,d} c_{ij} < \lambda\ \forall i,j = 1,...;d \\ \text{or} \\ 4 \,.\ \alpha_{ij} = \sigma_j + 1,\ m_i = \sigma_i + 1; \text{and } \ 2d \max\limits_{i,j=1,d} c_{ij} + \max\limits_{i=1,d} \left\|\overrightarrow{b_i}\right\| (\lambda+1) < 2\lambda \forall i,j = 1,...;d \end{cases}$$

Theorem 5.8 *Let all assumptions of this section be fulfilled. then the problem* (5.7.56) *has a unique global positive weak solution* $(u_1, u_2, ..., u_d)$ *such that*

$$\|u_i(.,t)\|_{L^\infty(\Omega)} \leq F(\xi) \quad \text{for all } t \geq \xi > 0 \qquad i = 1,2,...,d,$$

and

$$\|u_i(.,t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \geq 0, \qquad i = 1,2,...,d,$$

where $F(\xi)$ *is a positive function not depending on* u_0 *and* C *is a positive constant depending only on* u_0. *Moreover the semigroup* $S(t)$ *corresponding to the system* (5.7.56) *possesses a global attractor. Finally, in the fourth case in* (H_{10}), *if we assume that* $c_{i0} = 0$ $\forall i = 1,...;d$, *then the solution* u *tends to zero as* t *tends to infinity.*

In proving the existence of global weak solution, we find apriori estimates for smooth solutions of problem (5.3.12) − (5.3.14) and proceed as in section 4. Without loss of generality, we give the details only in the fourth case of (H_{10}).

Lemma 5.10 *For all* $T > 0$, *there exist a positive function* F *not depending on* ε *such that*

(5.7.57) $$\|u_{i\varepsilon}(T)\|_{L^\infty(\Omega)}\,,\ \left\|\nabla(u_{i\varepsilon}^{\sigma_i+1})\right\|^2_{L^2(Q_T)} \leq F(T).$$

Moreover, in the fourth case of (H_{10}), *if we assume that* $c_{i0} = 0$, *then*

(5.7.58) $$\left\|\nabla(u_{i\varepsilon}^{\sigma_i+1})\right\|^2_{L^2(Q_T)} \leq C$$

with C *is a positive constant independent of* T.

Proof. Multiplying (5.3.12) by $u_{i\varepsilon}^{\sigma_i+1}$, adding them together, and integrating over Q_T, we obtain the following with the help of Cauchy-Schwartz inequality

$$\sum_{i=1}^{d} \frac{1}{\sigma_i+2} \int_\Omega u_{i\varepsilon}^{\sigma_i+2}(T)dx + \sum_{i=1}^{d} \int_{Q_T} \left\|\nabla(u_{i\varepsilon}^{\sigma_i+1})\right\|^2 dxdt$$
$$\leq \sum_{i,j=1}^{d} c_{ij} \int_{Q_T} u_{j\varepsilon}^{2(\sigma_j+1)} dxdt + \sum_{i=1}^{d} \int_{Q_T} \frac{\left\|\overrightarrow{b_i}\right\|}{2} \left(u_{i\varepsilon}^{2(\sigma_i+1)} + \left\|\nabla u_{i\varepsilon}^{\sigma_i+1}\right\|^2\right) dxdt$$
$$+ \sum_{i=1}^{d} \eta \int_{Q_T} u_{i\varepsilon}^{2(\sigma_i+1)} dxdt + \sum_{i=1}^{d} \frac{1}{\sigma_i+2} \int_\Omega u_{i0\varepsilon}^{\sigma_i+2} dx + C(\eta,T) \sum_{i=1}^{d} c_{i0}^2$$

Letting $M = \max\limits_{i,j=1,d} c_{ij}$ and $b = \max\limits_{i=1,d} \left\| \overrightarrow{b_i} \right\|$ and applying Poincaré's inequality, we get

$$\sum_{i=1}^{d} \frac{1}{\sigma_i + 2} \int_\Omega u_{i\varepsilon}^{\sigma_i+2}(T)dx + \left(\frac{2\lambda - b\lambda - 2dM - b}{2\lambda} - \eta \right) \sum_{i=1}^{d} \int_{Q_T} \left\| \nabla (u_{i\varepsilon}^{\sigma_i+1}) \right\|^2 dxdt \leq C(T)$$

and

$$\sum_{i=1}^{d} \frac{1}{\sigma_i + 2} \int_\Omega u_{i\varepsilon}^{\sigma_i+2}(T)dx + \left(\frac{2\lambda - b\lambda - 2dM - b}{2\lambda} - \eta \right) \sum_{i=1}^{d} \int_{Q_T} \left\| \nabla (u_{i\varepsilon}^{\sigma_i+1}) \right\|^2 dxdt \leq C$$

where C is independent of T if $c_{i0} = 0$ for all $i = 1, ...; d.$

For η small enough we obtain

(5.7.59) $$\left\| u_{i\varepsilon}(T) \right\|_{L^{\sigma_i+2}(\Omega)}, \left\| \nabla (u_{i\varepsilon}^{\sigma_i+1}) \right\|_{L^2(Q_T)}^2 \leq C(T),$$

and

(5.7.60) $$\left\| u_{i\varepsilon}(T) \right\|_{L^{\sigma_i+2}(\Omega)}, \left\| \nabla (u_{i\varepsilon}^{\sigma_i+1}) \right\|_{L^2(Q_T)}^2 \leq C,$$

if $c_{i0} = 0 \quad i = 1, ..., d.$

and hence Theorem 5.4, proves the claim. □

Remark 5.4 *As a conclusion of* (5.7.60) *and Poincaré's inequality, we emphasize that if* $c_{i0} = 0$ *then* $\left\| u_{i\varepsilon}^{\sigma_i+1} \right\|_{L^2(Q_T)}$ *is uniformly bounded with respect to* T*, that is* $\left\| u_{i\varepsilon}^{\sigma_i+1} \right\|_{L^2(Q_\infty)}$ *and then* $\left\| f_{i\varepsilon}(u_\varepsilon, \nabla u_{i\varepsilon}) \right\|_{L^2(Q_\infty)}$ *are bounded. Thus* $\left\| f_{i\varepsilon}(u_\varepsilon, \nabla u_{i\varepsilon}) \right\|_{L^2(Q_{\frac{t}{2},t})}$ *tends to zero as* $t \to \infty$.

Lemma 5.11 *There is a positive constant* C *such that for all* $t > 0$ *we have the following inequality*

(5.7.61) $$\left\| \nabla u_{i\varepsilon}^{\sigma_i+1}(t) \right\|_{2,\Omega} \leq \frac{2}{t} C + \int_{Q_{\frac{t}{2},t}} f_{i\varepsilon}^2(u_\varepsilon, \nabla u_{i\varepsilon}) ds \quad \forall\ i = 1, ..., d.$$

which implies that the solution tends to zero as t *tends to* ∞ *if* $c_{i0} = 0$

Proof. Let $\tau \in \left[\frac{t}{2}, t \right]$, where $t > 0$. Multiplying (5.3.12) by $\left(u_{i\varepsilon}^{\sigma_i+1} \right)_t$, and by integrating the obtained result over $\Omega \times [\tau, t]$, we obtain

(5.7.62) $$\begin{aligned} I &= \left(\frac{2}{\sigma_i + 2} \right)^2 \int_{Q_{\frac{t}{2},t}} (\partial_t (u_{i\varepsilon}^{\frac{\sigma_i+1}{2}}))^2 dsdx + \left\| \nabla u_{i\varepsilon}^{\sigma_i+1}(., t) \right\|_{2,\Omega}^2 \\ &\leq \left\| \nabla u_{i\varepsilon}^{\sigma_i+1}(., \tau) \right\|_{2,\Omega}^2 + \int_{Q_{\frac{t}{2},t}} \partial_t (u_{i\varepsilon}^{\sigma_i+1}) f_{i\varepsilon}(u_{1\varepsilon}, u_{2\varepsilon}, \nabla u_{i\varepsilon}) dsdx \end{aligned}$$

The Cauchy-Schwartz inequality, gives us

$$(5.7.63)\qquad I \le \left(\frac{2}{\sigma_i+2}\right)^2 \int_{Q_{\frac{t}{2},t}} (\partial_t(u_{i\varepsilon}^{\frac{\sigma_i+1}{2}}))^2 dsdx + \left\|\nabla u_{i\varepsilon}^{\sigma_i+1}(.,\tau)\right\|_{2,\Omega}^2 + C_1 \int_{Q_{\frac{t}{2},t}} u_{i\varepsilon}^{\sigma_i} f_{i\varepsilon}^2(u_{1\varepsilon}, u_{2\varepsilon}, \nabla u_{i\varepsilon}) dxds$$

By combining estimates (5.7.62) and (5.7.63) , it becomes

$$(5.7.64)\qquad \left\|\nabla u_{i\varepsilon}^{\sigma_i+1}(.,t)\right\|_{2,\Omega}^2 \le \left\|\nabla u_{i\varepsilon}^{\sigma_i+1}(.,\tau)\right\|_{2,\Omega}^2 + C_2 \int_{Q_{\frac{t}{2},t}} f_{i\varepsilon}^2(u_{1\varepsilon}, u_{2\varepsilon}, \nabla u_{i\varepsilon}) dxds$$

Integrating the estimate (5.7.64) on τ over $\left[\frac{t}{2}, t\right]$, we have

$$\frac{t}{2}\left\|\nabla u_{i\varepsilon}^{\sigma_i+1}(.,t)\right\|_{2,\Omega}^2 \le \int_{Q_{\frac{t}{2},t}} \left\|\nabla u_{i\varepsilon}^{\sigma_i+1}(.,\tau)\right\|_{2,\Omega}^2 + C_2\frac{t}{2}\int_{Q_{\frac{t}{2},t}} f_{i\varepsilon}^2(u_{1\varepsilon}, u_{2\varepsilon}, \nabla u_{i\varepsilon}) dsdx$$

this completes the proof. □

5.7.2 Blow-up Results

In the following we assume that

$$\overrightarrow{b_i} \text{ is independent of } t, \quad \overrightarrow{b_i} \in \left(C^\infty(\overline{\Omega})\right)^N$$

and

$$f_i(u, \nabla u_i) = c_{i0} + \sum_{j=1}^{d} c_{ij} u_j^{\alpha_{ij}} + \overrightarrow{b_i} \nabla\left(u_i^{m_i}\right).$$

In this subsection we prove the finite time blow-up results in theorem **??**.
A crucial role is played here by the first eigenvalue of the Dirichlet problem

$$\begin{cases} -\Delta\psi_i(x) + \overrightarrow{b_i}(x)\nabla\psi_i(x) = \lambda_i\psi_i(x) & \text{in } \Omega \\ \psi_i(x) = 0 & \text{on } \partial\Omega \end{cases}$$

Denote by λ_i the first eigenvalue and by $\psi_i(x)$ the corresponding eigenfunction with the normalization $\psi_i(x) > 0$ in Ω and $\|\psi_i\|_{L^1} = 1$ (see [8]). It is well known that λ_i increases as the size of the domain Ω decreases (see [20]).

Theorem 5.9 *Suppose* $c_{ii} > \lambda_i$, *then any positive (nontrivial) weak solution of* (5.1.1) – (5.1.4) *blows up in finite time.*

Proof. We multiply the equation defining u_i by ψ_i, add them together and integrate over $(0,t)\times\Omega$ to obtain

$$\begin{aligned}(5.7.65)\qquad &\sum_{i=1}^{d}\int_\Omega u_i(t)\psi_i dx+\lambda_i\sum_{i=1}^{d}\int_{Q_t} u_i^{\sigma_i+1}(s)\psi_i dxdt\\ &=\sum_{i,j=1}^{d} c_{ij}\int_{Q_t} u_j^{\sigma_j+1}(s)\psi_i dxdt+\sum_{i=1}^{d}\left(\int_\Omega u_{i0}\psi_i dx+C(t)c_{i0}\right)\end{aligned}$$

But

$$\sum_{i,j=1}^{d} c_{ij}u_j^{\sigma_j+1}(t)\psi_i\geq Mu_i^{\sigma_i+1}(t)\psi_i$$

where $M=\max\limits_{i=1,d} c_{ii}$

On the other hand, Hölder's inequality gives

$$\int_\Omega u_i^{\sigma_i+1}(t)\psi_i dx\geq\left(\int_\Omega u_i(s)\psi_i dx\right)^{\sigma_i+1}$$

Inserting this into (8.12), and writing $g(s)=\sum\limits_{i=1}^{d}\left(\int_\Omega u_i(s)\psi_i dx\right)$ and $\sigma=\min\limits_{i=1,d}\sigma_i$, we obtain

$$g(t)\geq(M-\lambda_i)\int_0^t (g(s))^{\sigma+1}\,ds+C$$

which shows that there exists a finite time T^* such that

$$\lim_{t\nearrow T^*} g(t)=+\infty$$

hence u blows-up in finite time. □

Conclusion et problèmes ouverts

Dans ce travail, on a étudié des systèmes de réaction-diffusion paraboliques non-linéaires dégénérés. En utilisant la technique de compacité, on a établi l'existence globale en temps de solutions faibles dans un sens bien précis, où les deux propriétés suivantes sont satisfaites :

1. La positivité des solutions est préservée au cours du temps.
2. La masse totale est contrôlée au cours du temps.

On a établi aussi quelques résultats de régularité et d'unicité de solutions faibles. Enfin dans la dernière partie de ce travail on s'est intéressé à l'étude de la possibilité d'existence d'un attracteur global et au comportement asymptotique.

Il reste plusieurs problèmes ouverts, et on peut les résumer dans les cas suivants :

- L'étude des systèmes lorsque, la dépendence en le gradient est non affine.
- les conditions au bord ne sont pas homogènes.
- les conditions au bord ne sont pas linéaires.
- Les systèmes couplés suivants :

1. Les seconds membres sont de la forme :

$$f_i(u_1, u_2, \nabla u_1, \nabla u_2)$$

2. Les termes de diffusion sont plus généraux

$$\begin{cases} \partial_t u_1 - \Delta(\varphi(u_1)) = f_1(u_1, u_2, \nabla u_1, \nabla u_2) \\ \partial_t u_1 - \Delta(\phi(u_2)) = f_2(u_1, u_2, \nabla u_1, \nabla u_2) \end{cases}$$

3.

$$\begin{cases} \partial_t u_1 - div(a(u_1)\nabla u_2) = f_1(u_1, u_2, \nabla u_1, \nabla u_2) \\ \partial_t u_1 - div(a(u_2)\nabla u_1) = f_2(u_1, u_2, \nabla u_1, \nabla u_2) \end{cases}$$

4.

$$\begin{cases} \partial_t u_1 - \Delta(\varphi(u_1)) = f_1(u_1, u_2, \nabla u_1, \nabla u_2) \\ \partial_t u_1 - \Delta(\phi(u_2)) - \Delta(\psi(u_1)) = f_2(u_1, u_2, \nabla u_1, \nabla u_2) \end{cases}$$

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Deuxième partie

Implémentation numérique d'un problème de contrôlabilité exacte et approchée en dynamique des populations

Chapitre 6

Introduction

In this part, we are interested by population dynamics control problem, it acts to determine the process allowing to direct the evolution of a population towards a state specified in advance. The application of this kind of control can be made to improve the output of the basins of fish breeding or the control of the propagation of an epidemic. Interior control, in a theoretical population, can be made by vaccination, sterilization, migration or well elimination.

We consider a linear model describing the dynamics of a single species population with age dependence and spatial structure.

Let $u = u(x,t,a)$ be the distribution of individuals having age $a \geq 0$ at time $t \geq 0$ and location x in Ω, where Ω is a bounded open set in $I\!R^d$, $d \in \{1,2,3\}$, having a suitably smooth boundary Γ. Let $\beta(a) \geq 0$ be the natural fertility-rate and $\mu(a) \geq 0$ be the natural death-rate of individuals having age a, and let $A_\dagger$ be the maximum life expectancy of an individual. We assume that the flux of population takes the form $k\nabla u(x,t,a)$ with $k > 0$ and ∇ is the gradient vector with respect to the spatial variable. Under these conditions the evolution of the distribution u is governed by the partial differential equation

$$\partial_t u + \partial_a u - k\Delta_x u + \mu(a)u = 0,\ (x,t,a) \in \Omega \times (0,T) \times (0,A_\dagger)\,; \tag{6.0.1}$$

see [9], [7]. The birth-process is given by the renewal equation

$$u(x,t,0) = \int_0^{A_\dagger} \beta(a)u(x,t,a)da,\ x \in \Omega,\ t > 0. \tag{6.0.2}$$

We assume closed boundary conditions

$$u\,(x,t,a) = 0,\ x \in \Gamma,\ t > 0,\ 0 < a < A_\dagger. \tag{6.0.3}$$

The exact controllability question is : Can the age and space specific density of population reach a given distribution $u_d(x,a)$ at time T upon choosing a suitable migration function ? More precisely, let ω be a non empty open subdomain of Ω and let χ_ω be the characteristic function of ω. Let $u_d(x,a)$ be a given age and space dependent distribution of individuals. Given a positive and finite time T, can the solution to

$$(6.0.4)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} - K\Delta_x u + \mu(a)u = v(t,a,x)\chi_\omega, & \text{in } (0,T)\times(0,A)\times\Omega, \\ u(0,a,x) = u_0(a,x), & \text{on } (0,A)\times\Omega, \\ u(t,0,x) = \displaystyle\int_0^A \beta(a)u(t,a,x)da, & \text{on } (0,T)\times\Omega, \\ \dfrac{\partial u}{\partial \eta} = 0, & \text{on } (0,T(\times(0,A)\times\partial\Omega. \end{cases}$$

verify

$$(6.0.5)\qquad u(x,T,a) = u_d(x,a), x \in \Omega, 0 < a < A_\dagger,$$

upon selecting a suitable control v corresponding to an external supply or to a removal of individuals on the subdomain ω?

In practical applications one would like to approximate numerically a suitable control $v \in L^2(\omega \times (0,T) \times (0,A_\dagger))$, given u_0 and u_d. There are two difficulties : (a) Exact controllability do not always hold and (b) when it holds the control is not unique. Is proved see [2] that when the birth function has a support in $(0,A_\dagger)$, all the stationary states are exactly controllable. We assume here that for the pair $\{u_0, u_d\}$ exact controllability holds and we look to the optimal control v having minimum norm in $L^2(\omega \times (0,T) \times (0,A_\dagger))$.

Assumptions and definitions

Let $A_\dagger$ be a finite positive number ; one assumes the fertility rate β and the mortality rate μ verify

H_1 $\quad \beta : [0,A_\dagger] \to [0,+\infty)$ continuous,

$\mathbf{H}_2$ $\quad \mu : [0,A_\dagger) \to [0,+\infty)$ continuous; $\int_0^{A_\dagger} \mu(a)da = +\infty$.

The second condition in $(\mathbf{H}_2)$ means that each individual in the population dies before age $A_\dagger$.

Let Ω be a bounded open set in $I\!R^d$ with a smooth boundary Γ so that locally Ω lies on one side of Γ. Next, ω is an open subset of Ω having a positive d-dimensional Lebesgue measure, such that $\omega \subset \overline{\omega} \subset \Omega$; χ_ω is the characteristic function of ω. The initial data u_0 verifies $u_0 \in L^2(\Omega\times(0,A_\dagger))$. Last, an admissible control v on $\omega\times(0,T)\times(0,A_\dagger)$ is a control $v \in L^2(\omega \times (0,T) \times (0,A_\dagger))$. we will use the following notations $Q = (0,T) \times (0,A) \times \Omega$, $Q_\omega = (0,T) \times (0,A) \times \omega$, $Q_a = (0,A) \times \Omega$, $Q_t = (0,T) \times \Omega$ and finally $\Gamma = (0,A) \times \partial\Omega$.

For (6.0.4) and the associated backward problem we use the notion of solutions considered in [6], built on the first order Sobolev space $H^1(\Omega)$ and its topological dual space $[H^1(\Omega)]'$. These are measurable functions $u : \Omega \times (0,T) \times (0,A_\dagger) \to I\!R$ such that $u \in L^2((0,T) \times (0,A_\dagger) ; H^1(\Omega))$ and $\partial_t u + \partial_a u \in L^2((0,T) \times (0,A_\dagger) ; [H^1(\Omega)]')$, weak solutions of $(6.0.4)_1$ and $(6.0.4)_4$ in the usual integral form. Then, the initial conditions in $(6.0.4)_2$ and $(6.0.4)_3$ make sense in $L^2(\Omega \times (0,A_\dagger))$ and $L^2(\Omega \times (0,T))$.

As soon as u_0 is smooth enough, e.g. $u_0 = 0$, solutions are smoother, i.e. $\partial_t u + \partial_a u$ and Δu are members of $L^2(\Omega \times (0,T) \times (0,A_\dagger))$. A smoother u_0 would make routine calculations much easier, but backward problems have non smooth conditions at $t = T$. In any case most computations can be first carried out with sufficiently smooth solutions and completed via a density argument.

Chapitre 7

First formulation of the controllability problem

7.1 Continuous formulation

Let $\varepsilon > 0$ be a fixed number, and $\mathcal{K} = L^2(Q_\omega)$for v in K and u the solution of (6.0.4) we introduce the functional

$$J_1(v) = \int_{Q_a} (u(T,a,x) - u_d(a,x))^2 dadx + \frac{\varepsilon}{2} \int_{Q_\omega} v(t,a,x)^2 dtdadx. \tag{7.1.1}$$

We are concerned by the following minimization problem

$$\min_{v \in \mathcal{K}} J_1(v) \tag{7.1.2}$$

under the set of constraints

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} - K\Delta_x u + \mu u = v\chi_\omega, & in\ (0,T) \times (0,A) \times \Omega, \\ u(0,a,x) = u_0(a,x), 0 \ < \ a \ , \ x \in \Omega, & (x,a) \in]0,A[, \times \Omega \\ u(t,0,x) = \displaystyle\int_0^A \beta(a) \times u(t,a,x)\, da, & (x,t) \in]0,T[\times \Omega, \\ \dfrac{\partial u}{\partial \eta} = 0 & on \quad \Gamma =]0,T[\times]0,A[\times \partial\Omega, \end{cases} \tag{7.1.3}$$

it is easy to proof the following

Theorem 7.1 *Since J_1 is strictly convex and $J_1 \longrightarrow +\infty$ as $||v|| \longrightarrow +\infty$, the minimization problem* (7.1.2) - (7.1.3) *has a unique solution.*

7.1.1 Calculation of $J_1'(v)$

In order to compute $J_1'(v)$, we shall use (Formal) perturbation analysis, which can be made rigorous by using for example the method described in [8]. Let us consider $v \in \mathcal{U}$

and a small perturbation δv of v. We have then with the Obvious notation,
(7.1.4)

$$\begin{aligned}\delta J(v) &= \int_{Q_\omega} J_1'(v)\delta v(t,a,x)dxdtda \\ &= \varepsilon\int_{Q_\omega} v(t,a,x)\delta v(t,a,x)dtdadx + 2\int_{Q_a}(u(T,a,x)-u_d(a,x))\delta u(T,a,x)dadx.\end{aligned}$$

where δu is the solution of (6.0.4) with $v=\delta v$ and $\delta u_0 = 0$. Now let us introduce a reasonable smooth function $P\in L^2((0,T)\times(0,A)\times\Omega)$ such that $P(T,a,x) = -2(u(T,a,x)-u_d(a,x))$, multiplying both side the equation of δu by P and integrating over $Q = (0,T)\times(0,A)\times\Omega$ one get

$$\begin{aligned}-\int_{Q_t} P(T,a,x)\delta u(T,a,x)dadx &= \int_{Q_a} P(t,A,x)\delta u(t,A,x)dtdx \\ &+\int_Q\left[-\frac{\partial P}{\partial t}-\frac{\partial P}{\partial a}-K\Delta_x P+\mu(a)P-P(t,0,x)\beta(a)\right]\delta u(t,a,x)dtdadx \\ &-\int_{Q_\omega} P(t,a,x)\delta v(t,a,x)dtdadx + K\int_\Gamma \frac{\partial P}{\partial\eta}\delta udtdad\Gamma\end{aligned}$$

then (7.1.4) became

$$\begin{aligned}\int_{Q_\omega} J_1'(v)\delta v(t,a,x)dxdtda &= \int_{Q_\omega}\left[\varepsilon v(t,a,x)-P(t,a,x)\right]\delta v(t,a,x)dtdadx \\ &+\int_Q\left[-\frac{\partial P}{\partial t}-\frac{\partial P}{\partial a}-K\Delta_x P+\mu(a)P-P(t,0,x)\beta(a)\right]\delta u(t,a,x)dtdadx \\ &+\int_{Q_a} P(t,A,x)\delta u(t,A,x)dtdx + K\int_\Gamma \frac{\partial P}{\partial\eta}\delta udtdad\Gamma\end{aligned}$$

then we deduce firstly P is the solution of the backward problem
(7.1.5)

$$\begin{cases} -\dfrac{\partial P}{\partial t}-\dfrac{\partial P}{\partial a}-K\Delta_x P+\mu(a)P(t,a,x)=\beta(a)p(t,0,x) & (t,a,x)\in Q \\ p(T,a,x) = -2(u(T,a,x)-u_d(a,x)), & (x,a)\in\Omega\times]0,A[\\ p(t,A,x)=0 & (x,t)\in\Omega\times]0,T[\\ \dfrac{\partial p}{\partial\eta}=0 & on \quad \Gamma=\partial\Omega\times]0,T[\times]0,A[\end{cases}$$

and secondly

$$J_1'(v) = \varepsilon v(t,a,x) - P(t,a,x)$$

7.1.2 Optimality conditions

Let v be a solution of problem (7.1.2), and let us denote by u (resp. P) the corresponding solution of the state system (6.0.4) (resp. the backward system (7.1.5)). It follows from previous subsection that $J_1'(v)=0$ is equivalent to the following optimality system :

$$\varepsilon v(t,a,x) = P(t,a,x)$$

the backward equation
(7.1.6)

$$\begin{cases} -\dfrac{\partial P}{\partial t} - \dfrac{\partial P}{\partial a} - K\Delta_x P + \mu(a)P(t,a,x) = \beta(a)p(t,0,x) & (t,a,x) \in Q \\ p(T,a,x) = -2(u(T,a,x) - u_d(a,x)), & (x,a) \in \Omega\times]0,A[\\ p(t,A,x) = 0 & (x,t) \in \Omega\times]0,T[\\ \dfrac{\partial p}{\partial \eta} = 0 & on \quad \Gamma = \partial\Omega\times]0,T[\times]0,A[\end{cases}$$

the state equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} - K\Delta_x u + \mu(a)u = v(t,a,x)\chi_\omega, & \text{in } (0,T)\times(0,A)\times\Omega, \\ u(0,a,x) = u_0(a,x), & \text{on } (0,A)\times\Omega, \\ u(t,0,x) = \displaystyle\int_0^A \beta(a)u(t,a,x)da, & \text{on } (0,T)\times\Omega, \\ \dfrac{\partial u}{\partial \eta} = 0, & \text{on } (0,T(\times(0,A)\times\partial\Omega. \end{cases}$$

Conversely, it can be shown (see [8]) that these equations characterize v as the solution of control problem (7.1.2) The above optimality conditions will play a crucial role concerning the iterative solution of the control problem (7.1.2)

7.1.3 Existence and uniqueness result

7.1.3.1 The state equation

In order to show existence of solution of problem (6.0.4), we introduce the following problem :
(7.1.7)

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} - k\Delta_x u + (\lambda + \overline{\mu})u = (\overline{\mu} - \mu)w + v\ \chi_\omega\ e^{(-\lambda t)}, & \text{in } (0,T)\times(0,A)\times\Omega, \\ u(0,a,x) = u_0(a,x), & \text{on } (0,A)\times\Omega, \\ u(t,0,x) = \displaystyle\int_0^A \beta(a)u(t,a,x)da, & \text{on } (0,T)\times\Omega, \\ \dfrac{\partial u}{\partial \eta} = 0, & \text{on } (0,T(\times(0,A)\times\partial\Omega. \end{cases}$$

where λ is a real positive number, $w \in H = L^2(]0,T[\times]0,A[\times\Omega)$.

Theorem 7.2 *Let T the application defined from H to H such that for all $w \in H$, $u = T(w)$ the solution of problem* (7.1.7)*, then if* $\lambda > \frac{1}{2}\int_0^A \beta(a)^2\, da$*, T is contracting and* (6.0.4) *has a unique solution.*

Proof.

Let $w_1,\ w_2,\ u_1,\ u_2$ in H such that $T(w_1) = u_1$ and $T(w_2) = u_2$ and let $u = u_1 - u_2$ and $w = w_1 - w_2$ then w is the solution of the following problem

$$\begin{cases} u_t + u_a - K\Delta_x u + (\lambda + \overline{\mu})u = (\overline{\mu} - \mu)w & x \in \Omega, 0 < t\ ,\ 0 < a < A \\ u(0,a,x) = 0 & x \in \Omega, 0 < a < A \\ u(t,0,x) = \int_0^A \beta(a) w(t,a,x)\, da. & x \in \Omega,\ 0 < t \\ \dfrac{\partial u}{\partial \eta} = 0 \qquad sur & \partial\Omega \times]0,T[\times]0,A[\end{cases}$$

multiplying the first equation by u and integrating over $Q =]0,T[\times]0,A[\times\Omega$ one gets

$$(\lambda + \overline{\mu}) \int_Q u^2(t,a,x) dt da dx \ \leq\ \int_Q (\overline{\mu} - \mu).w.u\ \ dt da dx + \frac{1}{2}\int_0^T \int_\Omega [\int_0^A \beta(a).w\ da]^2\ \ dt dx$$

Using Cauchy-Schwartz inequality to get

$$(\lambda + \frac{\overline{\mu}}{2})\ ||u||^2_{L^2(Q)}\ \leq\ (\frac{\overline{\mu}}{2} + \frac{1}{2}\int_0^A \beta^2(a) da)\ ||w||^2_{L^2(Q)}.$$

and

$$||T(w_1) - T(w_2)||^2_{L^2(Q)}\ \leq\ (\frac{\overline{\mu} + \int_0^A \beta^2(a) da}{2\lambda + \overline{\mu}})\ ||w_1 - w_2||^2_{L^2(Q)}$$

it follows that if $\lambda\ >\ \frac{1}{2}\int_0^A \beta^2(a) da$ T is a contracting mapping then there exists a unique function $u \in H$ such that $u = Tu$ and u solution of the problem :

$$\begin{cases} u_t + u_a - k\Delta_x u + \lambda u + \mu u = v\chi_\omega e^{-\lambda t} & x \in \Omega\ ,\ 0\ <\ t\ ,\ 0\ <\ a\ <\ A, \\ u(0,a,x) = u_0(a,x) & x \in \Omega\ ,\ 0\ <\ a\ <\ A \\ u(t,0,x) = \int_0^A \beta(a) u(t,a,x)\, da. & x \in \Omega\ ,\ 0 < t \\ \dfrac{\partial u}{\partial \eta} = 0 \qquad sur & \partial\Omega \times]0,T[\times]0,A[\end{cases}$$

Now let us consider $\tilde{u}(t,a,x) = e^{\lambda t} u(t,a,x)$ then u is the desired solution of problem (6.0.4).

□

7.1.3.2 The backward equation

Theorem 7.3 *Let β and μ two continuous and nonnegative functions on $]0, A[$ and $f(a,x) = 2(u(T,a,x) - ud(a,x))$ a function in $L^2(]0, A[\times\Omega)$ then problem (7.1.6) has a unique solution in $L^2(Q)$.*

Proof.

Let $\tilde{P}(t,a,x) = e^{\lambda t} \times \tilde{P}(t,a,x)$, where $\lambda > 0$ to be chosen then $\tilde{P}$ is solution of

$$\begin{cases} -\dfrac{\partial \tilde{P}}{\partial t} - \dfrac{\partial \tilde{P}}{\partial a} - K\Delta_x \tilde{P} + \mu(a)\tilde{P}(t,a,x) = \beta(a)\tilde{P}(t,0,x) & \text{in } (0,T)\times(0,A)\times\Omega, \\ \tilde{P}(T,a,x) = e^{\lambda t} f(a,x)\ , & \text{on } (0,A)\times\Omega, \\ \tilde{P}(t,A,x) = 0 & \text{on } (0,T)\times\Omega, \\ \dfrac{\partial \tilde{P}}{\partial \eta} = 0 & on \quad \Gamma = \partial\Omega\times]0,T[\times]0,A[\end{cases}$$

define the map F by

$$\begin{array}{llll} F: & L^2(\Omega\times]0,T[) & \longrightarrow & L^2(\Omega\times]0,T[) \\ & \phi(x,t) & \longrightarrow & \tilde{P}(x,t,0) \end{array}$$

Consider the problem

$$\begin{cases} -\dfrac{\partial \tilde{P}}{\partial t} - \dfrac{\partial \tilde{P}}{\partial a} - K\Delta_x \tilde{P} + \mu(a)\tilde{P}(t,a,x) = \phi(x,t)\beta(a) & \text{in } (0,T)\times(0,A)\times\Omega, \\ \tilde{P}(T,a,x) = e^{\lambda t} f(a,x)\ , & \text{on } (0,A)\times\Omega, \\ \tilde{P}(t,A,x) = 0 & \text{on } (0,T)\times\Omega, \\ \dfrac{\partial \tilde{P}}{\partial \eta} = 0 & \text{on} \quad \Gamma = \partial\Omega\times]0,T[\times]0,A[\end{cases}$$

Let ϕ_1, ϕ_2, $\tilde{P}_1$ and $\tilde{P}_2$ in $L^2(\Omega\times]0,T[)$. Set $\phi(x,t) = \phi_1(x,t) - \phi_2(x,t)$ and $\tilde{P}(t,0,x) = \tilde{P}_1(t,0,x) - \tilde{P}_2(t,0,x)$, then $(\phi, \tilde{P})$ is solution of :
(7.1.8)

$$\begin{cases} -\dfrac{\partial \tilde{P}}{\partial t} - \dfrac{\partial \tilde{P}}{\partial a} - K\Delta_x \tilde{P} + \mu(a)\tilde{P}(t,a,x) = \phi(x,t)\beta(a), & \text{in } (0,T)\times(0,A)\times\Omega, \\ \tilde{P}(T,a,x) = 0, & \text{on } (0,A)\times\Omega, \\ \tilde{P}(t,A,x) = 0, & \text{on } (0,T)\times\Omega, \\ \dfrac{\partial \tilde{P}}{\partial \eta} = 0 & \text{on} \quad \Gamma = \partial\Omega\times]0,T[\times]0,A[\end{cases}$$

multiply (7.1.2) by $\tilde{P}$ and integrating over Q we get :

$$\int_Q \tilde{P}_t\tilde{P}dtdadx - \int_Q \tilde{P}_a\tilde{P}\ dtdadx - K\int_Q \Delta\tilde{P}\tilde{P}\ dtdadx + (\mu+\lambda)\int_Q \tilde{P}^2\ dtdadx$$
$$= \int_Q \phi(x,t)\beta(a)\tilde{P}(t,a,x)\ dtdadx.$$

the integration by parts gives

$$\tfrac{1}{2}\int_0^A\int_\Omega \tilde{P}^2(0,a,x)\ dadx + \tfrac{1}{2}\int_0^T\int_\Omega \tilde{P}^2(t,0,x)\ dtdx + K\int_Q |\nabla\tilde{P}|^2 + \mu\int_Q \tilde{P}^2(t,a,x)\ dtdadx$$
$$+\lambda\int_Q \tilde{P}^2(t,a,x)\ dtdadx = \int_Q \phi(x,t)\beta(a)\tilde{P}(t,a,x)\ dtdadx.$$

using Young inequality we get

$$\begin{aligned}\frac{1}{2}\int_0^T\int_\Omega \tilde{P}^2(t,0,x)(t,0,x)\ dtdx &\leq \frac{1}{\lambda}\int_\phi^2 \phi(t,x)^2\beta^2(a)\ dtdadx\\ &\leq \frac{\int_0^A \beta^2(a)\ da}{\lambda}\int_0^T\int_\Omega \phi^2(x,t)\ dtdx\end{aligned}$$

and then

$$\|F(\phi_1)-F(\phi_2)\|^2_{L^2(\Omega\times]0,T[)} \leq \frac{2\int_0^A \beta^2(a)\ da}{\lambda}\|\phi_{1-},\phi_2\|^2_{L^2(\Omega\times]0,T[)}$$

it is enough to take $\lambda \geq 2\int_0^A \beta^2(a)\ da$, to show that F is contracting and then admits a unique fixed point in $L^2(\Omega\times]0,T[)$. □

7.2 Discrete formulation

7.2.1 Discretization of the state equation

In this section we will take $\Omega = (0,d)$, $A < A_\dagger$ and assume that T is finite, we introduce a time discretization step Δt, defined by $\Delta t = \frac{T}{N}$, a age discretization step Δa, defined by $\Delta a = \frac{A}{M}$ and a space discretization step Δx, defined by $\Delta x = \frac{d}{R}$ where N, M and R are positive integer and as

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a}(t,a,x) = \lim_{h\longrightarrow 0}\frac{u(t+h,a+h,x)-u(t,a,x)}{h}$$

we will impose the following condition $h = \Delta a\left(=\frac{A}{M}\right) = \Delta t\left(=\frac{T}{N}\right)$. Now let with $\beta^j \simeq \beta(jh)$, $\mu^j \simeq \mu(a_j)$, $u_0^{j,k} \simeq u(0,j,k)$ and suppose (by the homogeneous Neumann condition) that $u(t,a,0) = u(t,a,\Delta x)$ (i.e. $u^{i,j,0} = u^{i,j,1}$) and $u(t,a,R\Delta x) = u(t,a,(R+1)\Delta x)$ (i.e. $u^{i,j,R} = u^{i,j,R+1}$) finally we use trapeze method to approximate integrals then problem (6.0.4) can be discretized by the following problem :

$$\begin{cases} \dfrac{u^{i+1,j+1,k}-u^{i,j,k}}{h} - K\dfrac{u^{i+1,j+1,k+1}-2u^{i+1,j+1,k}+u^{i+1,j+1,k-1}}{(\Delta x)^2} + \mu^{j+1}u^{i+1,j+1,k} = v^{i+1,j+1,k}\chi_\omega,\\ u^{0,j,k} = u_0^{j,k},\\ u^{i,0,k} = h\displaystyle\sum_{j=1}^{M}\beta^j\ u^{i,j,k}.\end{cases}$$

For $i = 1, ..., N$, $j = 1, ..., M$, we define the vector $U^{i,j} = (u^{i,j,k})_{k=1}^{R-1}$ then the above problem correspond to solve the following linear system :

$$A^j \, U^{i+1,j+1} = F^{i+1,j+1} + U^{i,j}$$

where $F_k^{i+1,j+1} = hv^{i+1,j+1,k}\gamma_k$, $\gamma_k = \chi_\omega(k\Delta x)$ and

$$A^{j+1} = I + \frac{Kh}{(\Delta x)^2} D + h\mu^{j+1} I,$$

I is the unit matrix and D is the tridiagonal symmetric and positive definite matrix given by :

$$D = \begin{pmatrix} 1 & -1 & 0 & \cdots & & & 0 \\ -1 & 2 & -1 & 0 & \cdots & & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & & 0 & -1 & 2 & -1 \\ 0 & \cdots & & & 0 & -1 & 1 \end{pmatrix}$$

then A^j is tridiagonal symmetric and positive definite matrix and the discrete problem has a unique solution.

7.2.2 Formulation of problem

Let $\Delta = h + \Delta x$, we approximate then problem (7.1.2) by the following finite-dimensional minimization problem :

(7.2.9) $$\min_{v \in \mathcal{U}^\Delta} J_1^\Delta(V),$$

with

$$\mathcal{U}^\Delta = R^{M \times N \times R}$$

$$V = v^{i,j,k}, i = 1, .., N, j = 1, .., M, k = 1, .., R$$

$$J_1^\Delta(V) = \sum_{j=1}^{N} \sum_{k=1}^{R} (u^{N,j,k} - u_d^{j,k})^2 + \frac{\varepsilon}{2} \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k=1}^{R} \left(v^{i,j,k}\right)^2 \chi_\omega(k\Delta x)$$

$$u_d^{j,k} = u_d(jh, k\Delta x), j = 1, .., M, k = 1, .., R$$

subject to the following constraint (discretization of problem (6.0.4))

$$
(7.2.10)\qquad
\begin{aligned}
&A^j\, U^{i+1,j+1} = F^{i+1,j+1} + U^{i,j}, i = 0,..,N-1, j = 0,..,M-1,\\
&U^{i,0} = h\, \sum_{j=1}^{M} \beta^j\, U^{i,j}, i = 1,..,N,\\
&U^{0,j} = U_0^j, j = 0,..,M.
\end{aligned}
$$

with $U_0^{j,k} = u_0^{j,k}$

7.2.3 Optimality conditions

In order to calculate the gradient of J_1^{Δ} we define the discrete Lagrangian by :

$$
\begin{aligned}
\mathcal{L}(u,v,P) \;&= J_1^{\Delta}(v) + \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} < A^{j+1}U^{i+1,j+1} - F^{i+1,j+1} - U^{i,j}, P^{i,j} >\\
&+ \sum_{j=0}^{M} < U^{0,j} - U_0^j, P^{0,j} > + \sum_{i=1}^{N} < U^{i,0} - h\, \sum_{j=1}^{M} \beta^j\, U^{i,j}, P^{i,0} >
\end{aligned}
$$

it can be written as follows

$$
\begin{aligned}
\mathcal{L}(U,V,P) \;&= \sum_{i=1}^{N}\sum_{j=1}^{M} < A^j P^{i-1,j-1} - h\beta^j P^{i,0} - P^{i,j}, U^{i,j} > - \sum_{j=0}^{M} < U_0^j, P^{0,j} >\\
&+ \sum_{j=1}^{M} \left\| U^{N,j} - U_d^j \right\|^2 + \sum_{i=1}^{N} < P^{i,M}, U^{i,M} > + \sum_{j=1}^{M} < P^{N,j}, U^{N,j} >\\
&\frac{\varepsilon}{2}\sum_{i=1}^{M}\sum_{j=1}^{N} \left\| V^{i,j} \right\|^2 - \sum_{i=1}^{N}\sum_{j=1}^{M} < V^{i,j}, P^{i-1,j-1} >
\end{aligned}
$$

then if (U,V,P) is the optimum of $\mathcal{L}$ then :

$$
\frac{\partial \mathcal{L}}{\partial U} = \frac{\partial \mathcal{L}}{\partial V} = 0.
$$

at this point then .

$$
\begin{aligned}
< \frac{\partial \mathcal{L}}{\partial U}(U,V,P), H > \;&= \sum_{i=1}^{N}\sum_{j=1}^{M} < A^j P^{i-1,j-1} - h\beta^j P^{i,0} - P^{i,j}, H^{i,j} > + \sum_{i=1}^{N} < P^{i,M}, H^{i,M} >\\
&+ \sum_{j=1}^{M} < P^{N,j} + 2(U^{N,j} - U_d^j), H^{N,j} >= 0
\end{aligned}
$$

which implies that P is solution of the following linear system

$$
(7.2.11)\qquad
\left\{
\begin{array}{ll}
A^j P^{i-1,j-1} = h\beta^j P^{i,0} + P^{i,j}, & i = 1,..,N, j = 1,..,M,\\
P^{i,M} = 0, & i = 1,..,N,\\
P^{N,j} = -2(U^{N,j} - U_d^j), & j = 0,..,M.
\end{array}
\right.
$$

and this is exactly a discretization of the backward problem (7.1.5).

The calculus of the gradient gives

$$< \frac{\partial \mathcal{L}}{\partial V}(U,V,P), H > \ = \sum_{i=1}^{M}\sum_{j=1}^{N} < \varepsilon V^{i,j} - P^{i-1,j-1}, H^{i,j} >,$$

which implies

$$(\nabla J_1^{\Delta})^{i,j,k} \ = \ \left(\varepsilon v^{i,j,k} - p^{i-1,j-1,k}\right)\chi_\omega(k\Delta x),$$

and $(\nabla J_1^{\Delta})^{i,j,k} \ = 0$ implies

$$\varepsilon v^{i,j,k}\chi_\omega(k\Delta x) = p^{i-1,j-1,k}\chi_\omega(k\Delta x).$$

7.2.4 Algorithm

Control minimizing the discrete function cost (7.2.9), is that which enables us to arrive at the desired density u_d with the smallest norm will be given by the following Conjugate Gradient algorithm :

Let $V_0^{i,j,k} \in L^2(]0,T[\times]0,A[\times\Omega)$, arbitrarily given

$g_0^{i,j,k} = \nabla J(V_0^{i,j,k})$,

$d_0^{i,j,k} = -g_0^{i,j,k}$.

$V_n^{i,j,k}$ being known, then

$g_n^{i,j,k} = \nabla J(V_n^{i,j,k})$.

$$\delta_n^{i,j,k} = \frac{||g_n^{i,j,k}||^2}{(d_n^{i,j,k})^t A d_n^{i,j,k}}.$$

$$V_{n+1}^{i,j,k} = V_n^{i,j,k} - \delta_n^{i,j,k} d_n^{i,j,k}$$

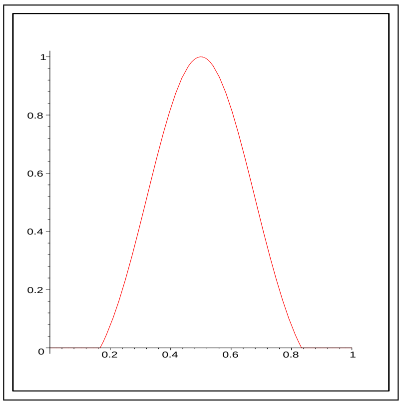
$$\mu_n^{i,j,k} = \frac{||g_{n+1}^{i,j,k}||^2}{||g_n^{i,j,k}||^2}.$$

$$d_{n+1}^{i,j,k} = -g_{n+1}^{i,j,k} + \mu_n^{i,j,k} d_n^{i,j,k}.$$

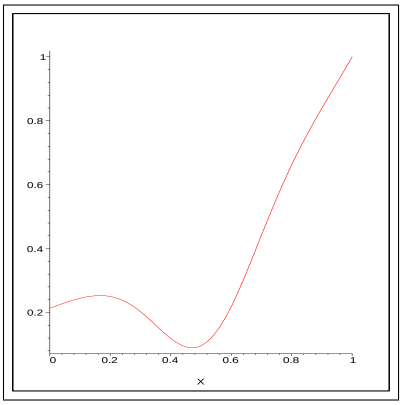
7.3 Numerical results

In this section we will give some numerical results let us consider $\omega = (\frac{3}{8}, \frac{5}{8})$, with initial data $U_0(x,a) = 10^4\, x^2\,(1.-x)^2\,(y+0.01)^2\,(y-0.8)^2$ and a control $v(x,a) = (4\,a\,(1.-$

$a))(4\,x\,(1.-x))$, also we use the following birth and death rates



Fertilityβ

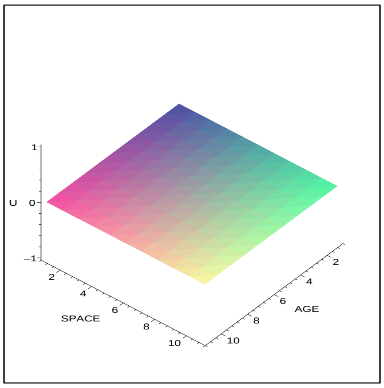


Mortality μ

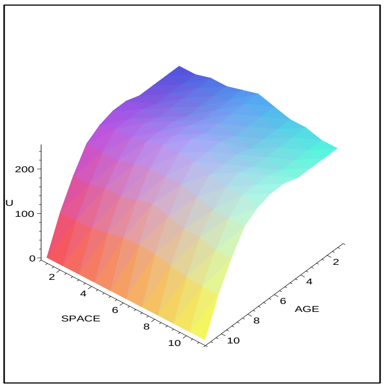
For these datas we try to find the optimal control by setting $U_d = U(T,a,x)$. Tests are used by varying proportionally U_d $(k\,U_d\,)$ and U_0 $(k\,U_0)$.

7.3.1 Exact controllability

We take $N = M = R = 10$ and



U_0



U_D

7.3.1.1 Dependence on the parameter ε

ε	$\|U-UD\|$	$\|V\|$	$\|\nabla J\|$	$J(V)$
0	2.10^{-5}	16861.79	1.5610^{-4}	4.0210^{-8}
10-^{9}	2.10^{-5}	16861.79	1.5610^{-4}	4.0210^{-8}
10^{-8}	2.10^{-5}	16861.79	1.5610^{-4}	4.0210^{-8}
10^{-7}	2.10^{-5}	16861.79	1.5610^{-4}	4.0210^{-8}
10^{-6}	2.10^{-5}	16861.79	1.5610^{-4}	4.0210^{-8}
10^{-5}	2.10^{-5}	16861.79	1.5610^{-4}	4.0210^{-8}
10^{-4}	2.10^{-5}	16861.79	1.5610^{-4}	4.0210^{-8}
10^{-3}	2.10^{-5}	16861.79	1.5610^{-4}	4.0210^{-8}
10^{-2}	2.10^{-5}	16861.79	1.5610^{-4}	4.0210^{-8}
10^{-1}	2.10^{-5}	16861.79	1.5610^{-4}	4.0210^{-8}

7.3.1.2 Dependence on the control domain

We use a 10,10,10 discretization in space, time and age

ω	$\{x_0, ..., x_{10}\}$	$\{x_1, ..., x_9\}$	$\{x_2, ..., x_8\}$	$\{x_3, ..., x_7\}$	$\{x_4, ..., x_6\}$
$\|e\| = \|U - UD\|$	$2.41.10^{-5}$	$2.42.10^{-5}$	$2.02.10^{-5}$	$1.04.10^{-5}$	2.10^{-5}
$\|V\|$	12092.86	12092.86	13524.65	13626.20	16861.79
$\|\nabla J\|$	$4.03\ 10^{-4}$	$3.95\ 10^{-4}$	$4.41\ 10^{-4}$	$2.09\ 10^{-4}$	$1.56\ 10^{-4}$
$J(V)$	5.8410^{-8}	5.8710^{-8}	4.1010^{-8}	1.0810^{-8}	4.0210^{-8}

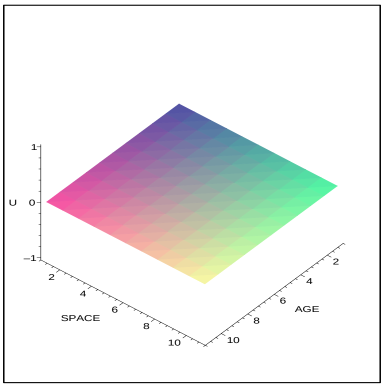
7.3.1.3 Dependence on the discretization

Age discretization	10	20	30	40	50
Time discretization	10	20	30	40	50
Space discretization	10	20	30	40	50
$\|e\| = \|U - UD\|$	$1.01.10^{-5}$	$4.67.10^{-6}$	$2.69.10^{-6}$	$1.48.10^{-6}$	$1.12.10^{-6}$
$\|V\|$	16861.79	25905.18	28960.48	30492.13	31412.13
$\|\nabla J\|$	$1.41.10^{-4}$	$1.68.10^{-4}$	$1.51.10^{-4}$	$9.33.10^{-5}$	$8.56.10^{-5}$
$J(V)$	$1.03.10^{-8}$	$8.72.10^{-9}$	$6.56.10^{-9}$	$3.52.10^{-9}$	$3.15.10^{-9}$

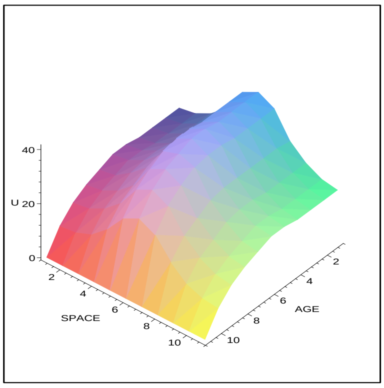
7.3.1.4 Dependence on the time control

	$\|U - UD\|$	$\|V\|$	$\|\nabla J\|$	$J(V)$
$T = A_\dagger$	$4.67.10^{-6}$	25905.18	$1.68.10^{-4}$	$8.72.10^{-9}$
$T = \frac{3A_\dagger}{2}$	$4.67.10^{-6}$	31727.24	$1.84.10^{-4}$	$8.72.10^{-9}$
$T = \frac{2A_\dagger}{3}$	$4.49.10^{-6}$	31727.24	$2.39.10^{-4}$	$1.21.10^{-8}$

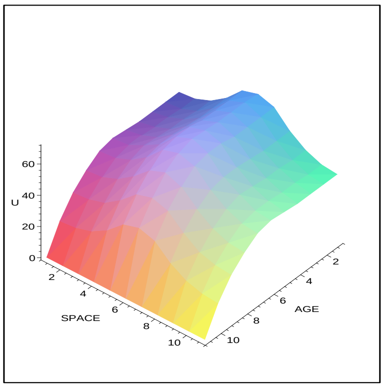
Remark 7.1 *In the case* $T \simeq \frac{A_\dagger}{2}$ *the algorithm don't work*



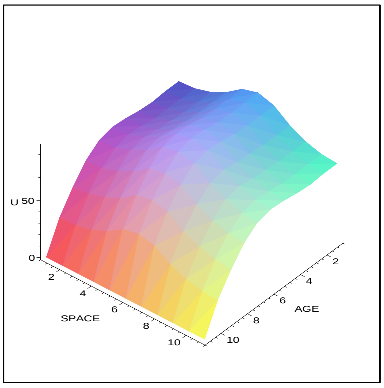
$t = 0$



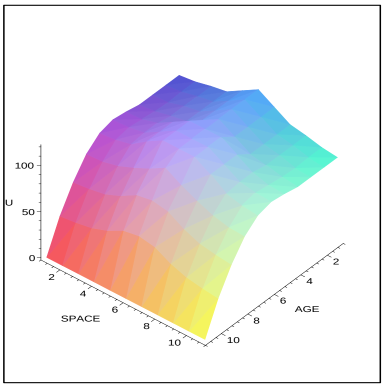
$t = 1$



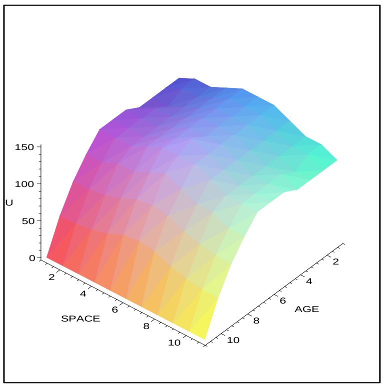
$t = 2$



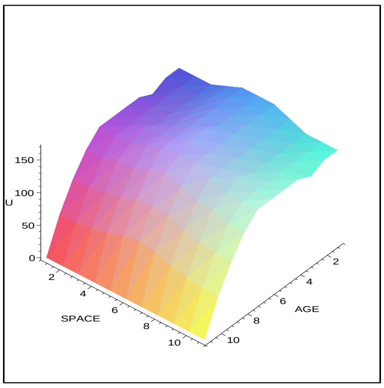
$t = 3$



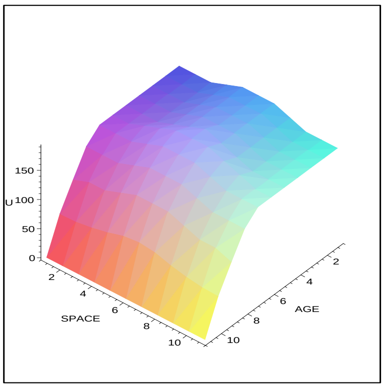
$t = 4$



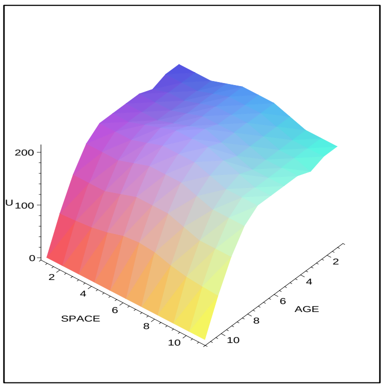
$t = 5$



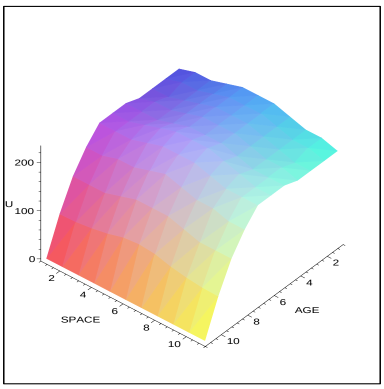
$t = 6$



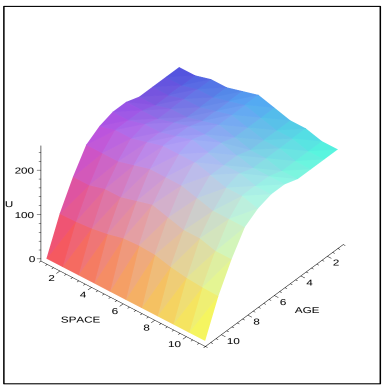
$t = 7$



$t = 8$

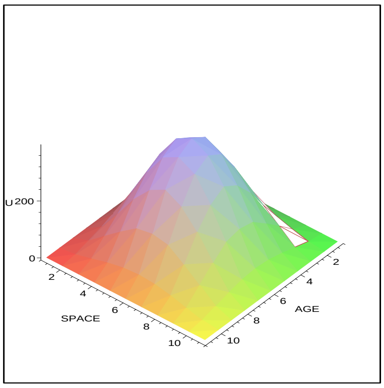


$t = 9$



$t = T$

7.3.2 Approximate controllability



U_0 U_D

7.3.2.1 Dependence on the parameter ε

ε	$\|U-UD\|$	$\|V\|$	$\|\nabla J\|$	$J(V)$
10-^{20}	$5.301\ 10^{-3}$	8980.011	$6.243\ 10^{-2}$	$2.810\ 10^{-3}$
10-^{11}	$5.301\ 10^{-3}$	8980.011	$6.2439\ 10^{-2}$	$2.810\ 10^{-3}$
10-^{10}	$5.301\ 10^{-3}$	8980.010	$6.242\ 10^{-2}$	$2.810\ 10^{-3}$
10^{-9}	$5.302\ 10^{-3}$	8980.010	$6.244\ 10^{-2}$	$2.811\ 10^{-3}$
10^{-8}	$5.299\ 10^{-3}$	8980.011	$6.242\ 10^{-2}$	$2.807\ 10^{-3}$
10^{-7}	$5.279\ 10^{-3}$	8980.005	$6.236\ 10^{-2}$	$2.787\ 10^{-3}$
10^{-6}	$5.119\ 10^{-3}$	8979.966	$6.239\ 10^{-2}$	$2.621\ 10^{-3}$
10^{-5}	$4.395\ 10^{-3}$	8979.575	$6.235\ 10^{-2}$	$1.932\ 10^{-3}$

7.3.2.2 Dependence on the control domain

We use a 10,10,10 discretization in space, time and age

ω	$\{x_0, ..., x_{10}\}$	$\{x_1, ..., x_9\}$	$\{x_2, ..., x_8\}$	$\{x_3, ..., x_7\}$	$\{x_4, ..., x_6\}$
$\|e\| = \|U-UD\|$	$7.19.10^{-3}$	$7.19.10^{-3}$	$6.39.10^{-3}$	$7.12.10^{-3}$	$1.07.10^{-2}$
$\|V\|$	2308.145	2308.212	2251.115	2061.943	1684.492
$\|\nabla J\|$	0.2397860	0.2337012	0.2368054	0.2059831	0.2886783
$J(V)$	$5.16.10^{-3}$	$5.17.10^{-3}$	$4.08.10^{-3}$	$5.07.10^{-3}$	$1.14.10^{-3}$

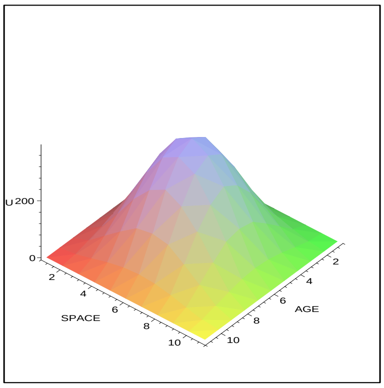
7.3.2.3 Dependence on the discretization

Age discretization	10	20	30	40	50
Time discretization	10	20	30	40	50
Space discretization	10	20	30	40	50
$\|e\| = \|U-UD\|$	$1.77.10^{-2}$	$1.12.10^{-2}$	$1.07.10^{-2}$	$4.97.10^{-3}$	$8.48.10^{-4}$
$\|V\|$	1755.05	2697.60	3020.56	3179.68	3138.09
$\|\nabla J\|$	0.124	0.246	0.291	0.174	$9.97.10^{-2}$
$J(V)$	$3.14.10^{-2}$	$5.09.10^{-2}$	0.103	$3.95.10^{-2}$	$1.80.10^{-3}$

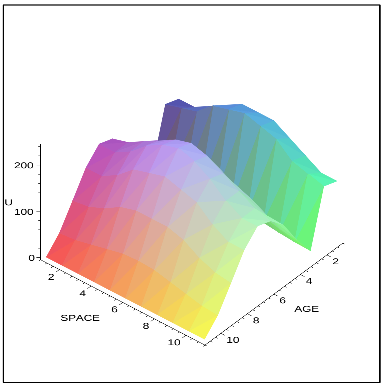
7.3.2.4 Dependence on the time control

	$\|U-UD\|$	$\|V\|$	$\|\nabla J\|$	$J(V)$
$T=A_\dagger$	$4.25\ 10^{-3}$	$2.58\ 10^{+3}$	$2.22\ 10^{-1}$	$7.24\ 10^{-3}$
$T=\frac{3A_\dagger}{2}$	$4.25\ 10^{-3}$	$3.16\ 10^{+3}$	$2.45\ 10^{-1}$	$7.24\ 10^{-3}$
$T=\frac{A_\dagger}{2}$	$3.62\ 10^{-3}$	$1.82\ 10^{+3}$	$1.54\ 10^{-1}$	$5.25\ 10^{-3}$

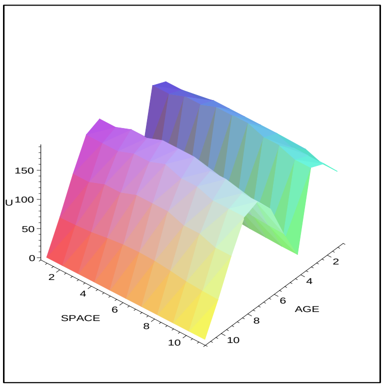
Remark 7.2 *When* $T \simeq \frac{A_\dagger}{2}$ *the algorithm don't work*



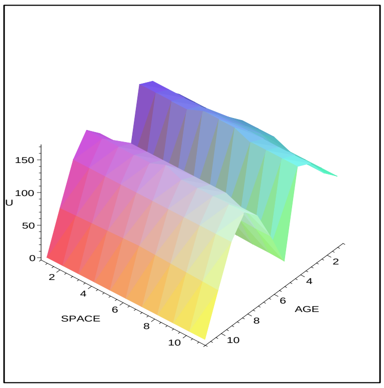
$t = 0$



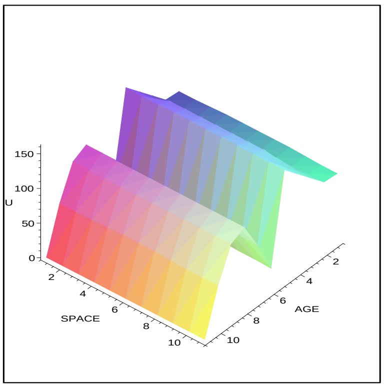
$t = 1$



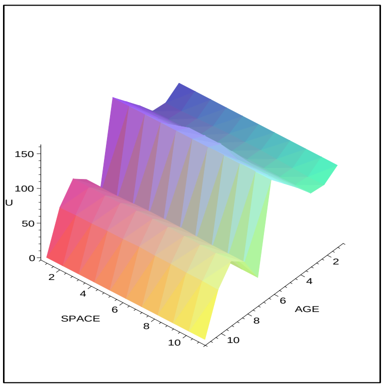
$t = 2$



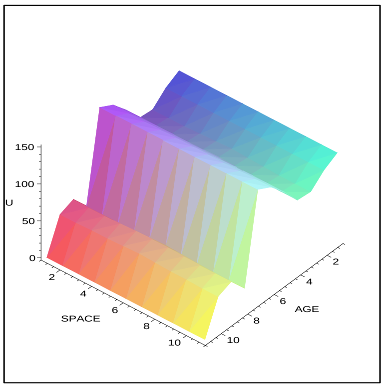
$t = 3$



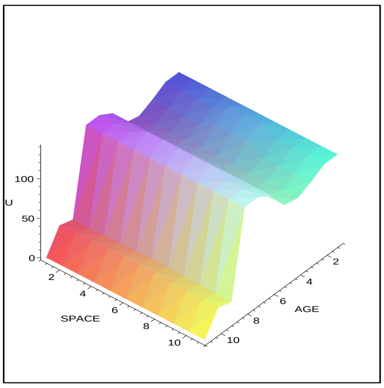
$t = 4$



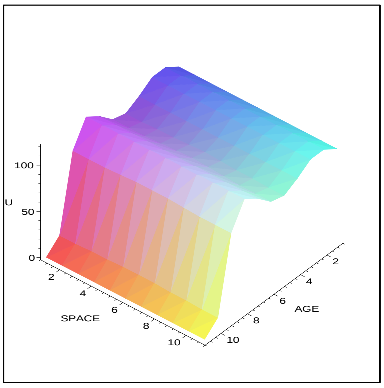
$t = 5$



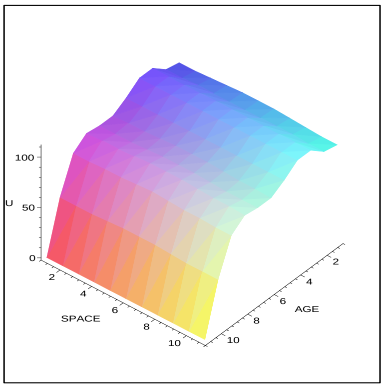
$t = 6$



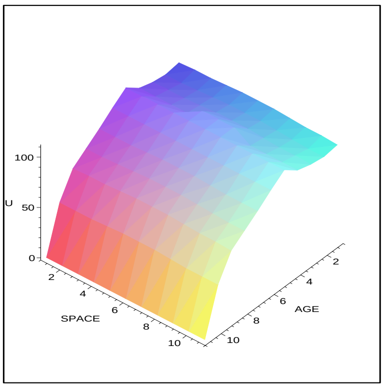
$t = 7$



$t = 8$

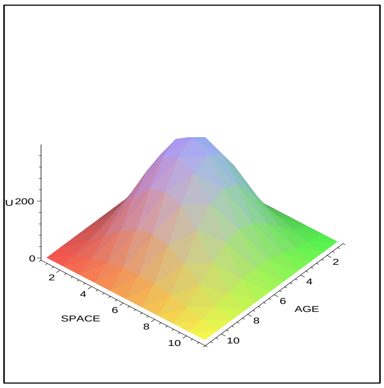


$t = 9$

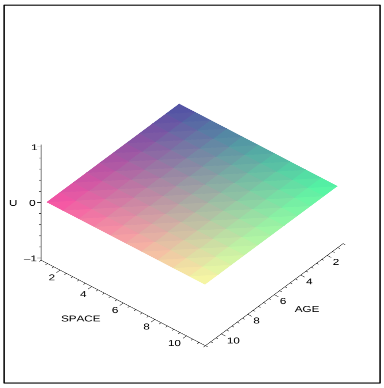


$t = T$

7.3.3 Null controllability



U_0



U_D

The algorithm work only For large time control T . For a 10,10 discretization in space and age the objective function is obtained for 300 time steps, and for any values of $\varepsilon \leq 10^{-1}$ we obtain the following results

7.3.3.1 Dependence on the parameter ε

ε	$\|U - UD\|$	$\|V\|$	$\|\nabla J\|$	$J(V)$
0	1.07 10^{-3}	0.1686	3.0142371 10^{-2}	1.1410^{-4}
10-9	1.07 10^{-3}	0.1686	3.0142371 10^{-2}	1.1410^{-4}
10^{-8}	1.07 10^{-3}	0.1686	3.0142371 10^{-2}	1.1410^{-4}
10^{-7}	1.07 10^{-3}	0.1686	3.0142380 10^{-2}	1.1410^{-4}
10^{-6}	1.07 10^{-3}	0.1686	3.0142497 10^{-2}	1.1410^{-4}
10^{-5}	1.07 10^{-3}	0.1686	3.0143654 10^{-2}	1.1410^{-4}
10^{-4}	1.07 10^{-3}	0.1686	3.0155335 10^{-2}	1.1410^{-4}
10^{-3}	1.07 10^{-3}	0.1686	3.0272234 10^{-2}	1.1410^{-4}
10^{-2}	1.07 10^{-3}	0.1686	3.1457566 10^{-2}	1.1410^{-4}
10^{-1}	1.07 10^{-3}	0.1686	4.4436604 10^{-2}	1.1410^{-4}

7.3.3.2 Dependence on the control domain

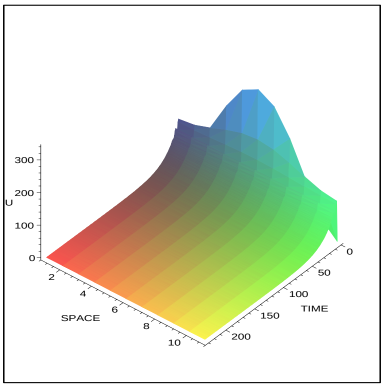
We use a 10,10 discretization in space and age and $\varepsilon = 0$, the following table is obtained after 230 time step

ω	$\{x_0, ..., x_{10}\}$	$\{x_1, ..., x_9\}$	$\{x_2, ..., x_8\}$	$\{x_3, ..., x_7\}$	$\{x_4, ..., x_6\}$
$\|e\| = \|U - UD\|$	$6.88.10^{-4}$	$2.157.10^{-3}$	$1.423.10^{-3}$	$1.686.10^{-3}$	$1.070.10^{-3}$
$\|V\|$	0.162	0.225	$2.309.10^{-2}$	0.206	0.168
$\|\nabla J\|$	$8.344.10^{-2}$	$8.118.10^{-2}$	$7.153.10^{-2}$	$6.041.10^{-2}$	$3.014.10^{-2}$
$J(V)$	$1.246.10^{-4}$	$1.246.10^{-4}$	$1.246.10^{-4}$	$1.246.10^{-4}$	$1.146.10^{-4}$

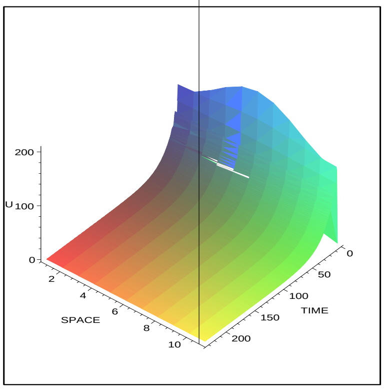
7.3.3.3 Dependence on the discretization

Age discretization	10	20	30	40
Space discretization	230	500	800	1000
Time step convergence	10	20	30	40
$\|e\| = \|U - UD\|$	$1.07.10^{-4}$	$4.381.10^{-4}$	$1.973.10^{-4}$	$4.477.10^{-4}$
$\|V\|$	$1.68.10^{-2}$	$1.29.10^{-3}$	$9.65.10^{-3}$	$7.62.10^{-3}$
$\|\nabla J\|$	$3.01.10^{-3}$	$3.621.10^{-2}$	$2.436.10^{-2}$	$7.346.10^{-2}$
$J(V)$	$1.14.10^{-6}$	$7.677.10^{-5}$	$3.505.10^{-5}$	$3.207.10^{-4}$

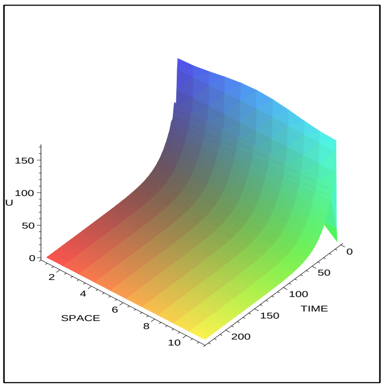
The following graphs describe evolution of $u(t, x, a)$ for $a = 0$ to $10(A_\dagger)$



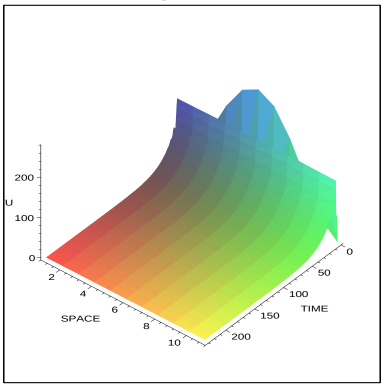
$Age = 0$



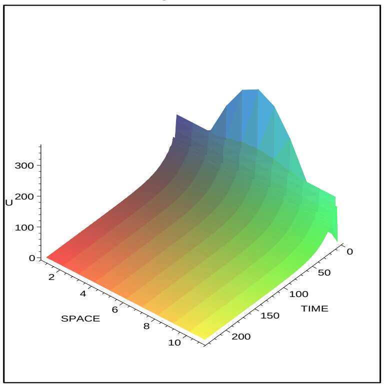
$Age = 1$



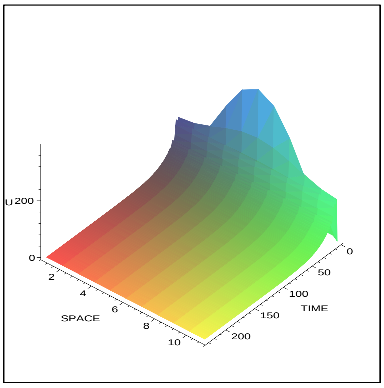
$Age = 2$



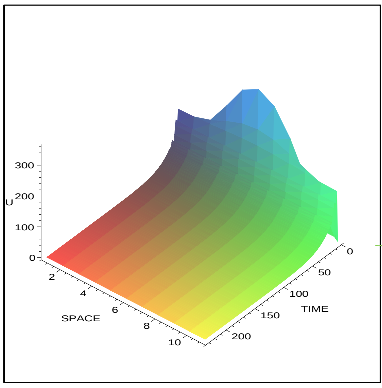
$Age = 3$



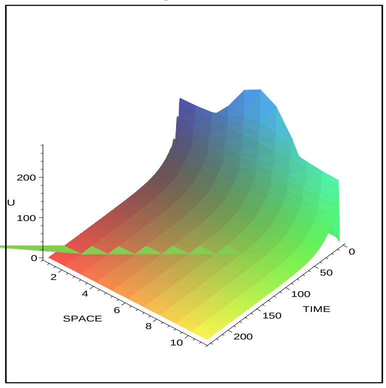
$Age = 4$



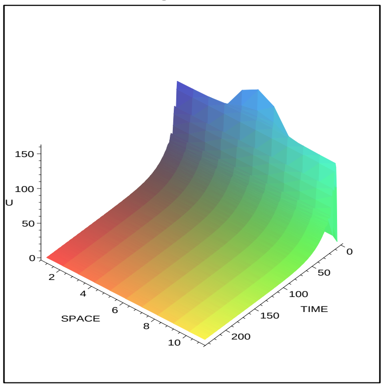
$Age = 5$



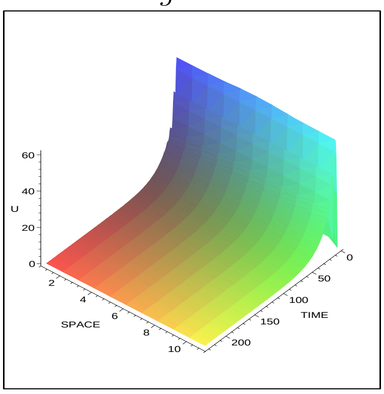
$Age = 6$



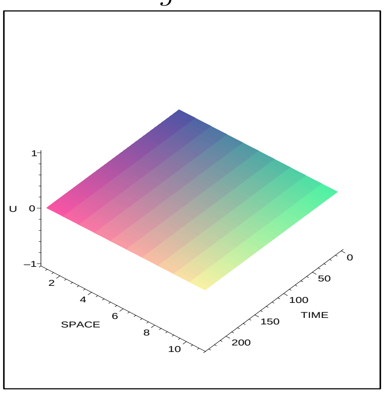
$Age = 7$



$Age = 8$



$Age = 9$



$Age = A_\dagger$

Chapitre 8

Second formulation of the controllability problem

8.1 Continuous formulation

Let $\varepsilon > 0$ be a fixed number, for g and u_0 in $\mathcal{U} = L^2(Q_a)$ and ρ the solution of

$$(8.1.1)\qquad \begin{cases} -\dfrac{\partial \rho}{\partial t} - \dfrac{\partial \rho}{\partial a} - K\Delta_x \rho + \mu(a)\rho = \rho(t,0,x)\beta(a) & \text{(x,t,a)} \in \Omega\times]0,T[\times]0,A[\\ \rho(T,a,x) = g(a,x) & \text{a} > 0\ ,\ x \in \Omega \\ \rho(t,A,x) = 0 & \text{t} > 0\ ,\ x \in \Omega \\ \dfrac{\partial \rho}{\partial \eta} = 0 & \text{sur} \quad \partial\Omega\times]0,T[\times]0,A[\end{cases}$$

we introduce the functional

$$(8.1.2)\qquad \begin{aligned} J_2(g) &= \frac{1}{2}\int_Q \rho^2(t,a,x)\chi_\omega dxdadt + \int_{Q_a} u_0(a,x)\rho(0,a,x)dxda \\ &+\varepsilon\left(\int_{Q_a} g^2(x,a)dxda\right)^{\frac{1}{2}} - \int_{Q_a} u_d(x,a)g(x,a)dxda. \end{aligned}$$

We are concerned by the following minimization Problem

$$(8.1.3)\qquad \min_{g\in\mathcal{U}} J_2(g).$$

The existence of the solution of (8.1.3) is established in [2, 3]

8.1.1 Calculation of $J_2'(g)$

As in section 7.1.1, let us consider g,(resp. u_0) $\in \mathcal{U}$ and a small perturbation δg (resp. δu_0) of g,(resp. u_0) then
(8.1.4)

$$\begin{aligned}\delta J_2(g) &= \int_{Q_a} J_2'(g)\delta g(a,x)dxda \\ &= \int_Q \rho(t,a,x)\chi_\omega \delta\rho dxdadt + \frac{\varepsilon}{\left(\int_{Q_a} g^2(x,a)dxda\right)^{\frac{1}{2}}}\int_{Q_\omega} g(t,a,x)\delta g(t,a,x)dtdadx \\ &+ \int_{Q_a} u_0(a,x)\delta\rho(0,a,x)dxda - \int_{Q_a} u_d(x,a)\delta g(x,a)dxda.\end{aligned}$$

where $\delta\rho$ is the solution of (8.1.1) with $g = \delta g$. Now let us introduce a reasonable smooth function $u \in L^2((0,T)\times(0,A)\times\Omega)$ such that $u(0,a,x) = u_0(a,x)$, multiplying both side the equation of $\delta\rho$ by u and integrating over $Q = (0,T)\times(0,A)\times\Omega$ one get

$$\begin{aligned}\int_{Q_a} u_0(a,x)\delta\rho(0,a,x)dadx &= \int_{Q_a} u(T,a,x)\delta g(a,x)dadx - K\int_\Gamma \frac{\partial u}{\partial\eta}\delta\rho dtdad\Gamma \\ &+ \int_{Q_t} \left[u(t,a,x)\beta(a) - u(t,0,x)\right]\delta\rho(t,0,x)dxdt \\ &- \int_Q \left[\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} - K\Delta_x u + \mu(a)u\right]\delta\rho(t,a,x)\end{aligned}$$

then (8.1.4) became

$$\begin{aligned}\int_{Q_\omega} J_2'(g)\delta g(t,a,x)dxdtda &= \int_Q \left(\rho\chi_\omega - \left[\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} - K\Delta_x u + \mu(a)u\right]\right)\delta\rho(t,a,x)dtdadx \\ &-K\int_\Gamma \frac{\partial u}{\partial\eta}\delta\rho dtdad\Gamma + \int_{Q_t}\left[u(t,a,x)\beta(a) - u(t,0,x)\right]\delta\rho(t,0,x)dxdt \\ &+ \int_{Q_a}\left[u(T,a,x) + \frac{\varepsilon g(t,a,x)}{\left(\int_{Q_a} g^2(x,a)dxda\right)^{\frac{1}{2}}} - u_d(x,a)\right]\delta g(a,x)dadx\end{aligned}$$

then $\delta J_2(g) = 0$ implies :

firstly u is the solution of the state problem

$$(8.1.5)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} - K\Delta_x u + \mu(a)u = \rho(t,a,x)\chi_\omega, & \text{in } (0,T)\times(0,A)\times\Omega, \\ u(0,a,x) = u_0(a,x), & \text{on } (0,A)\times\Omega, \\ u(t,0,x) = \displaystyle\int_0^A \beta(a)u(t,a,x)da, & \text{on } (0,T)\times\Omega, \\ \dfrac{\partial u}{\partial\eta} = 0, & \text{on } (0,T(\times(0,A)\times\partial\Omega. \end{cases}$$

and secondly

$$J_2'(g) = u(T,a,x) + \frac{\varepsilon g(t,a,x)}{\left(\int_{Q_a} g^2(x,a)dxda\right)^{\frac{1}{2}}} - u_d(x,a)$$

8.1.2 Optimality conditions

Let g be a solution of problem (8.1.3), and let us denote by u (resp. ρ) the corresponding solution of the state system (8.1.5) (resp. the backward system (8.1.1)). It follows from previous subsection that $J_2'(g) = 0$ is equivalent to the following optimality system :

$$\frac{\varepsilon g(t,a,x)}{\left(\int_{Q_a} g^2(x,a)dxda\right)^{\frac{1}{2}}} = u_d(x,a) - u(T,a,x)$$

the state equation

$$(8.1.6)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial a} - K\Delta_x u + \mu(a)u = \rho(t,a,x)\chi_\omega, & \text{in } (0,T)\times(0,A)\times\Omega, \\ u(0,a,x) = u_0(a,x), & \text{on } (0,A)\times\Omega, \\ u(t,0,x) = \displaystyle\int_0^A \beta(a)u(t,a,x)da, & \text{on } (0,T)\times\Omega, \\ \dfrac{\partial u}{\partial \eta} = 0, & \text{on } (0,T(\times(0,A)\times\partial\Omega. \end{cases}$$

the backward equation

$$(8.1.7)\qquad \begin{cases} -\dfrac{\partial \rho}{\partial t} - \dfrac{\partial \rho}{\partial a} - K\Delta_x\rho + \mu(a)\rho = \rho(t,0,x)\beta(a) & \text{(x,t,a)} \in \Omega\times]0,T[\times]0,A[\\ \rho(T,a,x) = g(a,x) & \text{a} > 0,\ x\in\Omega \\ \rho(t,A,x) = 0 & \text{t} > 0,\ x\in\Omega \\ \dfrac{\partial \rho}{\partial \eta} = 0 & \text{sur} \quad \partial\Omega\times]0,T[\times]0,A[\end{cases}$$

and one can deduce with identification with (6.0.4) that

$$v(t,a,x) = \rho(t,a,x)\chi_\omega$$

8.2 Discrete formulation

Let us use the following discretization of the problem (8.1.1)

$$(8.2.8)\qquad \begin{cases} A^{j-1}\rho^{i-1,j-1} = h\beta^j\rho^{i,0} + \rho^{i,j}, & i = 1,..,N, j = 1,..,M, \\ \rho^{i,M} = 0, & i = 0,..,N, \\ \rho^{N,j} = g^j & j = 0,..,M-1. \end{cases}$$

such that $g^i = (g^{i,k})_k$ and $\rho^{i,j} = (\rho^{i,j,k})_k$ and define the following function

$$\begin{aligned} J_2^{\Delta}(g) &= \frac{1}{2}h^2\Delta x \sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\sum_{k=0}^{R}(\rho^{i,j,k})^2\gamma_k + h\Delta x \sum_{k=0}^{R}\sum_{i=0}^{M}\rho^{j,0,k}u_0^{j,k} \\ &+\varepsilon\sqrt{(h\Delta x)}(\sum_{k=0}^{R}\sum_{j=0}^{M-1}(g^{j,k})^2)^{\frac{1}{2}} - h\Delta x\sum_{k=0}^{R}\sum_{j=0}^{M-1}u_d^{j,k}g^{j,k}. \end{aligned}$$

and we are concerned by the following minimization problem

$$\min_{v\in\mathcal{U}^{\Delta}} J_1^{\Delta}(V), \tag{8.2.9}$$

with

$$\mathcal{U}^{\Delta} = \mathbb{R}^{M\times R}$$

we proof the following

Theorem 8.1 *The minimization problem* (8.2.9) *has a unique solution* $\{g^{i,k}\}_{i,k}$.

Proof. J_2^{Δ} verifies

$$\liminf_{\|g^{i,k}\|_2\longrightarrow+\infty}\frac{J^h(g^{i,k})}{\|g^{i,k}\|_2} \geq \varepsilon\sqrt{(h\Delta x)},$$

otherwise there is a sequence $\{g_n^{i,k}\}_n$ such that $\|g_n^{i,k}\|_2 \longrightarrow +\infty$ as $n \longrightarrow +\infty$ and

$$\liminf_{n\longrightarrow+\infty}\frac{J^h(g_n^{i,k})}{\|g_n^{i,k}\|_2} < \varepsilon\sqrt{(h\Delta x)}. \tag{8.2.10}$$

Set $\widetilde{g}_n^{i,k} = \frac{g_n^{i,k}}{\|g_n^{i,k}\|_2}$, and let $\widetilde{\rho}_n^{i,j,k}$ the solution of (8.2.8) with $g^{i,k} = \widetilde{g}_n^{i,k}$, but the sequence $\{\widetilde{g}_n^{i,k}\}_n$ is bounded in $\mathbb{R}^{M\times(R+1)}$ we can extract a subsequence which style noted $\widetilde{g}_n^{i,k}$ such that $\widetilde{g}_n^{i,k}$ converges to $\widetilde{g}^{i,k}$ and then $\widetilde{\rho}_n^{i,j,k}$ converges to $\widetilde{\rho}^{i,j,k}$ solution of (8.2.8) with $g^{i,k} = \widetilde{g}^{i,k}$, furthermore if $\rho_n^{i,j,k}$ is the solution of (8.2.8) with $g^{i,k} = g_n^{i,k}$ we have $\widetilde{\rho}_n^{i,j,k} = \frac{\rho_n^{i,k}}{\|g_n^{i,k}\|_2}$ and

$$\begin{aligned} \frac{J_2^{\Delta}(g_n^{i,k})}{\|g_n^{i,k}\|_2} = J_2^{\Delta}(g) &= \frac{1}{2}h^2\Delta x\|g_n^{i,k}\|_2\sum_{j=0}^{N-1}\sum_{i=0}^{M-1}\sum_{k=0}^{R}(\widetilde{\rho}_n^{i,j,k})^2\gamma_k + h\Delta x\sum_{k=0}^{R}\sum_{i=1}^{M}\widetilde{\rho}_n^{j,0,k}u_0^{j,k} \\ &+\varepsilon\sqrt{(h\Delta x)} - h\Delta x\sum_{k=1}^{R}\sum_{i=1}^{M}u_d^{i,k}\widetilde{g}_n^{i,k}. \end{aligned}$$

therefore so that (8.2.10) is true it is necessary that $\widetilde{\rho}_n^{i,j,k}\gamma_k = 0,\ k = 0, ..., R$ which means $\widetilde{\rho}^{i,j,k}\gamma_k = 0,\ k = 0, ..., R$, and by induction we deduce that : $\widetilde{\rho}^{i,j,k} = 0,\ k = 0, ..., R$ and hence $\widetilde{g}^{i,k} = 0,\ k = i, ..., R$ which is in contradiction with the construction of $\widetilde{g}^{i,k}$.

Finally J^h is continuous coercive and strictly convex and then have a unique minimum $\widetilde{g}^{i,k}$. □

8.2.1 Optimality conditions

In order to calculate the gradient of J_2^{Δ} we define the discrete lagrangian by :

$$\begin{aligned}\mathcal{L}(\rho, g, P) &= J_2^{\Delta}(g) + h\Delta x \sum_{i=1}^{N}\sum_{j=1}^{M} < h\beta^j \rho^{i,0} + \rho^{i,j} - A^{j-1}\rho^{j-1,i-1,j-1}, P^{i,j} > \\ &+ h\Delta x \sum_{j=0}^{M-1} < g^j - \rho^{N,j}, P^{N,j} > - h\Delta x \sum_{i=0}^{N} < \rho^{i,M}, P^{i,M} >\end{aligned}$$

it can be written as follows

$$\begin{aligned}\mathcal{L}(\rho, g, P) &= h\Delta x \left\{ \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} < -A^j P^{j+1,i+1,j+1} + P^{i,j}, \rho^{i,j} > + \sum_{j=0}^{M} < u_0^{j,k} - P^{0,j}, \rho^{0,j} > \right\} \\ &+ h\Delta x \left\{ \sum_{i=1}^{N} < h \sum_{j=1}^{M} \beta^j P^{i,j} - P^{i,0}, \rho^{i,0} > + \sum_{j=0}^{M} < P^{N,j} - u_d^j, g^j > \right\} \\ &+ \varepsilon\sqrt{(h\Delta x)}(\sum_{k=1}^{R}\sum_{j=0}^{M}(g^{ji,k})^2)^{\frac{1}{2}} + \frac{1}{2}h^2\Delta x \sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\sum_{k=0}^{R}(\rho^{i,j,k})^2\gamma_k.\end{aligned}$$

then if (ρ, g, P) is the optimum of $\mathcal{L}$ then :

(8.2.11)
$$\frac{\partial \mathcal{L}}{\partial \rho} = \frac{\partial \mathcal{L}}{\partial g} = 0.$$

at this point

Then for all vector $H \in \left(\mathbb{R}^{(R+1)}\right)^{(N+1)\times(M+1)}$.

$$\begin{aligned}< \frac{\partial \mathcal{L}}{\partial \rho}(\rho, g, P), H > &= h\Delta x \sum_{i=0}^{N-1}\sum_{j=0}^{M-1} < -A^j P^{j+1,i+1,j+1} + P^{i,j} + h\rho^{i,j,k}\gamma_k., H^{i,j} > \\ &+ h\Delta x \sum_{j=0}^{M} < U_0^i - P^{0,j}, H^{0,j} > \\ &+ h\Delta x \sum_{i=1}^{N} < h \sum_{j=1}^{M} \beta^j P^{i,j} - P^{i,0}, H^{i,0} >= 0\end{aligned}$$

then (8.2.11) implies the following linear system

(8.2.12)
$$\begin{cases} A^j P^{j+1,i+1,j+1} = P^{i,j} + h\rho^{i,j,k}\gamma_k, & i = 0,..,N-1, j = 0,..,M-1, \\ P^{i,0} = h \sum_{j=1}^{M} \beta^j P^{i,j}, & i = 1,..,N, \\ P^{0,j} = U_0^i & j = 0,..,M. \end{cases}$$

and this is exactly a discretization of the state problem (6.0.4).

The calculus of the gradient gives

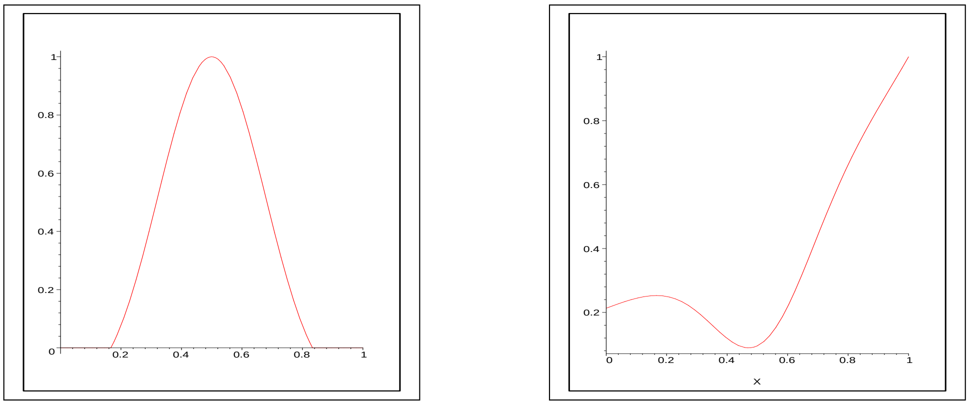
$$< \frac{\partial \mathcal{L}}{\partial V}(\rho, g, P), H > \;\; = h\Delta x \sum_{j=1}^{M} < \frac{\varepsilon g^j}{\sqrt{(h\Delta x)}(\sum_{k=1}^{R}\sum_{j=1}^{M}(g^{j,k})^2)^{\frac{1}{2}}} + P^{N,j} - u_d^j, H^j >,$$

which implies

$$(\nabla J_2^{\Delta})^{j,k}(g) = \frac{\varepsilon\sqrt{(h\Delta x)}g^{j,k}}{(\sum_{k=1}^{R}\sum_{i=1}^{M}(g^{j,k})^2)^{\frac{1}{2}}} + h\Delta x(P^{N,j,k} - u_d^{j,k}),$$

8.3 Numerical results

In this section we will give some numerical results let us consider $\omega = (\frac{3}{8}, \frac{5}{8})$, with initial data $U_0(x,a) = 10^4\, x^2\,(1.-x)^2\,(y+0.01)^2\,(y-0.8)^2$ and a control $v(x,a) = (4\,a\,(1.-a))(4\,x\,(1.-x))$, also we use the following birth and death rates

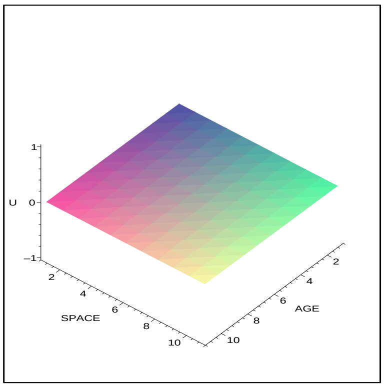


Fertilityβ Mortality μ

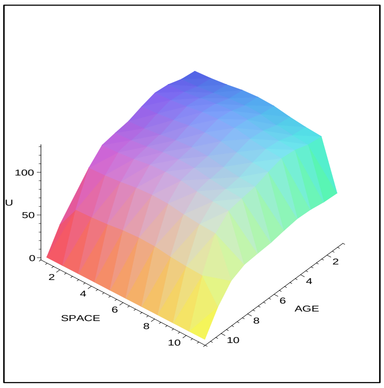
For these datas we try to find the optimal control by setting $U_d = U(T,a,x)$. Tests are used by varying proportionally U_d $(k\,U_d\,)$ and U_0 $(k\,U_0)$.

8.3.1 Exact controllability

We take $N = M = R = 10$ and



U_0



U_D

8.3.1.1 Dependence on the parameter ε

ε	$\|U-UD\|$	$\|g\|$	$\|\nabla J\|$	$J(g)$
0	$1.011\ 10^{-2}$	$1.864\ 10^{-2}$	$6.217\ 10^{-3}$	$5.100\ 10^{-2}$
10^{-20}	$1.011\ 10^{-2}$	$1.864\ 10^{-2}$	$6.217\ 10^{-3}$	$5.100\ 10^{-2}$
10^{-10}	$1.011\ 10^{-2}$	$1.864\ 10^{-2}$	$6.217\ 10^{-3}$	$5.100\ 10^{-2}$
10^{-4}	$1.011\ 10^{-2}$	$1.864\ 10^{-2}$	$6.222\ 10^{-3}$	$5.100\ 10^{-2}$
10^{-3}	$1.011\ 10^{-2}$	$1.864\ 10^{-2}$	$6.235\ 10^{-3}$	$5.100\ 10^{-2}$
10^{-2}	$1.011\ 10^{-2}$	$1.864\ 10^{-2}$	$8.331\ 10^{-3}$	$5.100\ 10^{-2}$
10^{-1}	$1.011\ 10^{-2}$	$1.864\ 10^{-2}$	$9.631\ 10^{-3}$	$5.102\ 10^{-2}$

8.3.1.2 Dependence on the control domain

We use a 10,10,10 discretization in space, time and age

ω	$\{x_0,...,x_{10}\}$	$\{x_1,...,x_9\}$	$\{x_2,...,x_8\}$	$\{x_3,...,x_7\}$	$\{x_4,...,x_6\}$
$\|e\|=\|U-UD\|$	$1.209\ 10^{-4}$	$1.214\ 10^{-4}$	$1.223\ 10^{-4}$	$1.217\ 10^{-4}$	$1.011\ 10^{-2}$
$\|g\|$	$1.864\ 10^{-2}$	$1.864\ 10^{-2}$	$1.864\ 10^{-2}$	$1.864\ 10^{-2}$	$1.864\ 10^{-2}$
$\|\nabla J\|$	$6.217\ 10^{-3}$	$6.217\ 10^{-3}$	$6.217\ 10^{-3}$	$6.217\ 10^{-3}$	$6.217\ 10^{-3}$
$J(V)$	$5.100\ 10^{-2}$	$5.100\ 10^{-2}$	$5.100\ 10^{-2}$	$5.100\ 10^{-2}$	$5.100\ 10^{-2}$

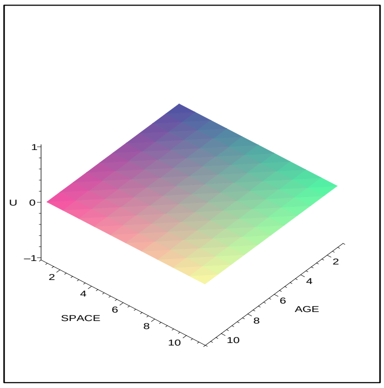
8.3.1.3 Dependence on the discretization

Age discretization	10	20	30	40	50
Time discretization	10	20	30	40	50
Space discretization	10	20	30	40	50
$\|e\|=\|U-UD\|$	$1.011\ 10^{-2}$	$1.038\ 10^{-2}$	$1.012\ 10^{-2}$	$1.104\ 10^{-2}$	$1.159\ 10^{-2}$
$\|g\|$	$1.864\ 10^{-2}$	$1.531\ 10^{-2}$	$1.373\ 10^{-2}$	$1.272\ 10^{-2}$	$1.200\ 10^{-2}$
$\|\nabla J\|$	$6.217\ 10^{-3}$	$4.910\ 10^{-3}$	$4.326\ 10^{-3}$	$4.390\ 10^{-3}$	$4.048\ 10^{-3}$
$J(g)$	$5.100\ 10^{-2}$	$5.555\ 10^{-2}$	$5.703\ 10^{-2}$	$5.772\ 10^{-2}$	$5.814\ 10^{-2}$

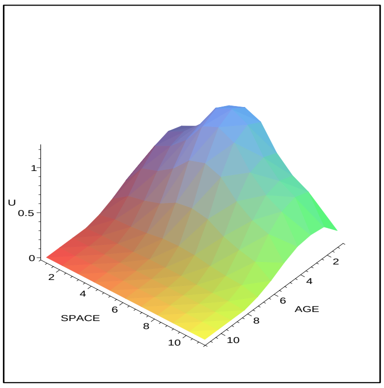
8.3.1.4 Dependence on the time control

	$\|U - UD\|$	$\|g\|$	$\|\nabla J\|$	$J(g)$
$T = A_\dagger$	$1.038\ 10^{-2}$	$1.531\ 10^{-2}$	$1.373\ 10^{-2}$	$5.555\ 10^{-2}$
$T = \dfrac{3A_\dagger}{2}$	$1.038\ 10^{-2}$	$1.531\ 10^{-2}$	$1.373\ 10^{-2}$	$4.272\ 10^{-2}$

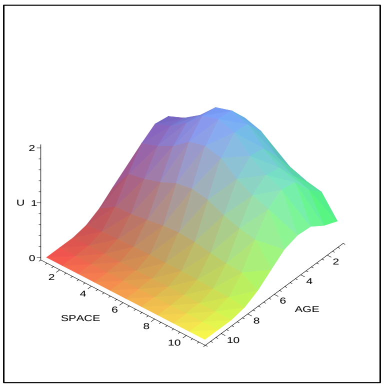
Remark 8.1 *In the case $T < A_\dagger$ the algorithm don't work*



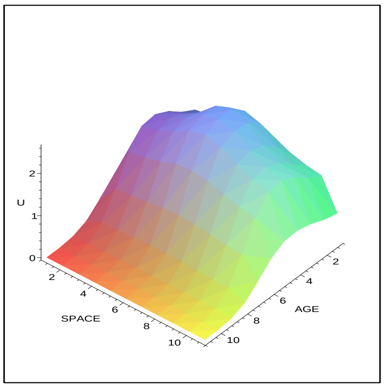
$t = 0$



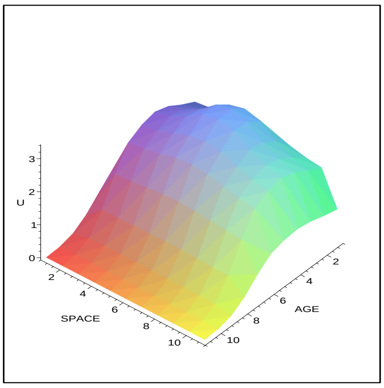
$t = 1$



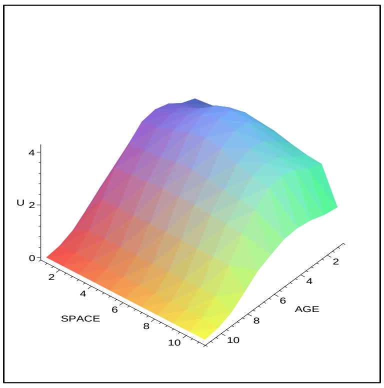
$t = 2$



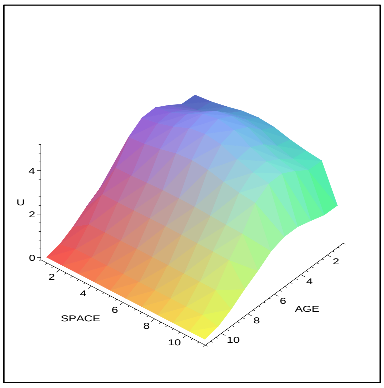
$t = 3$



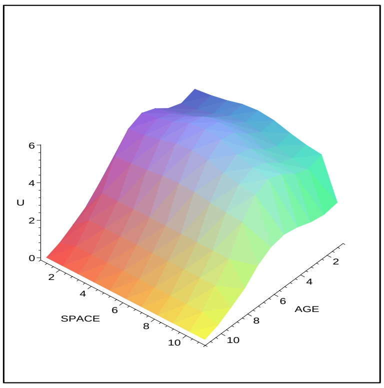
$t = 4$



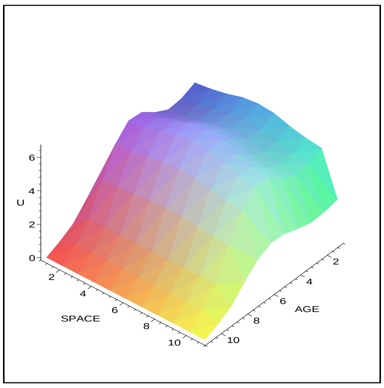
$t = 5$



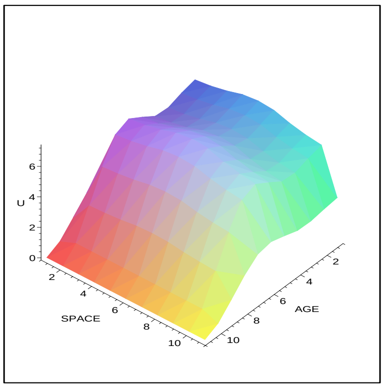
$t = 6$



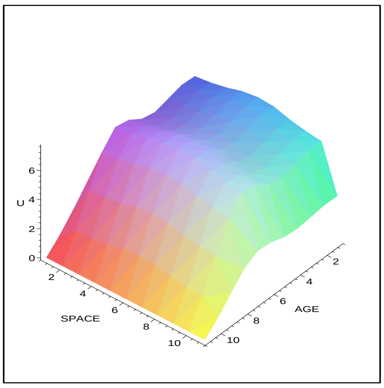
$t = 7$



$t = 8$

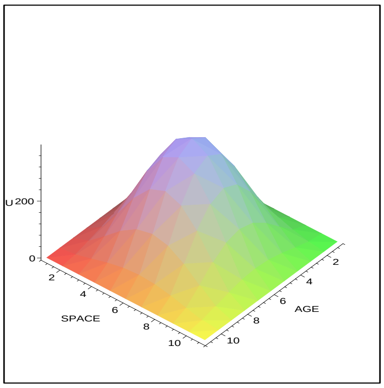


$t = 9$

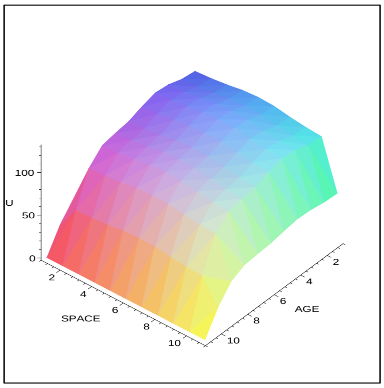


$t = T$

8.3.2 Approximate controllability



U_0



U_D

8.3.2.1 Dependence on the parameter ε

ε	$\|U - UD\|$	$\|V\|$	$\|\nabla J\|$	$J(V)$
10^{-10}	$3.344\ 10^{-2}$	320.73	$3.343\ 10^{-4}$	14041.73
10^{-4}	$3.344\ 10^{-2}$	320.73	$3.343\ 10^{-4}$	14041.75
10^{-3}	$3.344\ 10^{-2}$	320.73	$3.351\ 10^{-4}$	14041.78
10^{-2}	$3.344\ 10^{-2}$	320.73	$3.417\ 10^{-4}$	14041.79
10^{-1}	$3.344\ 10^{-2}$	320.73	$4.130\ 10^{-4}$	14041.93

8.3.2.2 Dependence on the control domain

We use a 10,10,10 discretization in space, time and age

ω	$\{x_0, ..., x_{10}\}$	$\{x_1, ..., x_9\}$	$\{x_2, ..., x_8\}$	$\{x_3, ..., x_7\}$	$\{x_4, ..., x_6\}$
$\|e\| = \|U - UD\|$	$3.329\ 10^{-2}$	$3.336\ 10^{-2}$	$3.338\ 10^{-2}$	$3.341\ 10^{-2}$	$3.344\ 10^{-2}$
$\|g\|$	320.735	320.735	320.735	320.735	320.735
$\|\nabla J\|$	$4.130\ 10^{-4}$	$4.130\ 10^{-4}$	$4.130\ 10^{-4}$	$4.130\ 10^{-4}$	$4.130\ 10^{-4}$
$J(V)$	14041.93	14041.93	14041.93	14041.93	14041.93

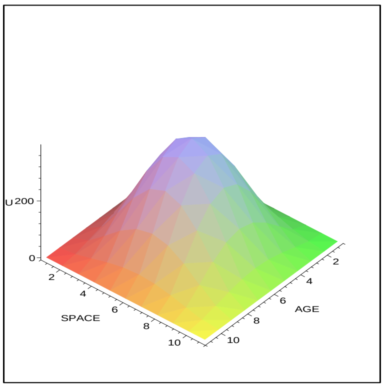
8.3.2.3 Dependence on the discretization

Age discretization	10	20	30	40	50
Time discretization	10	20	30	40	50
Space discretization	10	20	30	40	50
$\|e\| = \|U - UD\|$	$3.344\ 10^{-2}$	$2.923\ 10^{-2}$	$2.771\ 10^{-2}$	$3.288\ 10^{-2}$	$3.124\ 10^{-2}$
$\|g\|$	320.735	306.574	301.852	299.492	298.076
$\|\nabla J\|$	$4.130\ 10^{-4}$	$1.156\ 10^{-4}$	$6.026\ 10^{-5}$	$4.288\ 10^{-5}$	$3.070\ 10^{-5}$
$J(g)$	14041.93	14903.83	15215.03	15350.72	15441.94

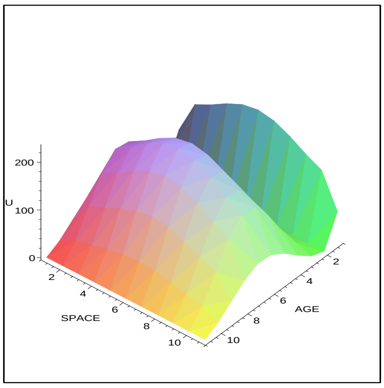
8.3.2.4 Dependence on the time control

	$\|U-UD\|$	$\|g\|$	$\|\nabla J\|$	$J(g)$
$T=A_\dagger$	$3.344\ 10^{-2}$	306.574	$1.156\ 10^{-4}$	14903.83
$T=\frac{3A_\dagger}{2}$	$3.344\ 10^{-2}$	306.574	$1.156\ 10^{-4}$	13183.42
$T=2A_\dagger$	$2.923\ 10^{-2}$	306.574	$1.156\ 10^{-4}$	11656.33

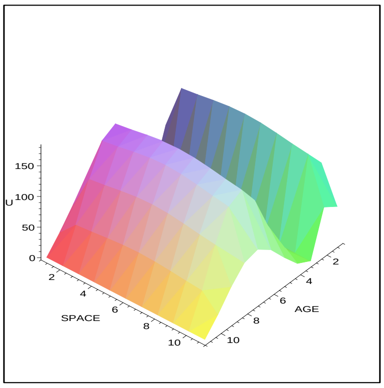
Remark 8.2 *In the case $T<A_\dagger$ the algorithm don't work*



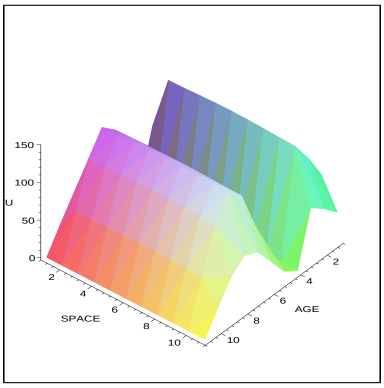
$t = 0$



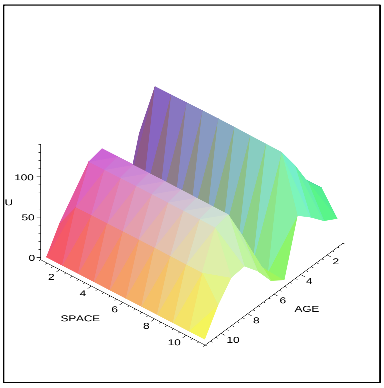
$t = 1$



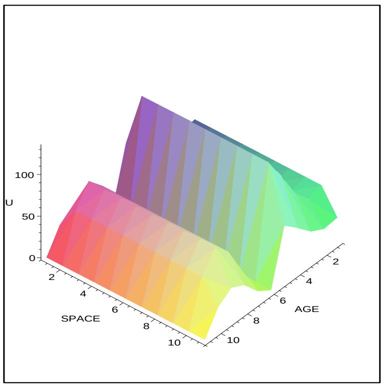
$t = 2$



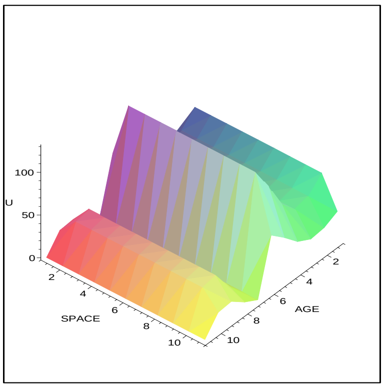
$t = 3$



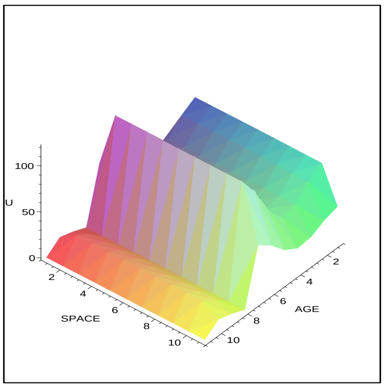
$t = 4$



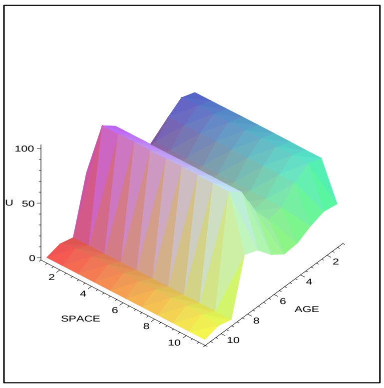
$t = 5$



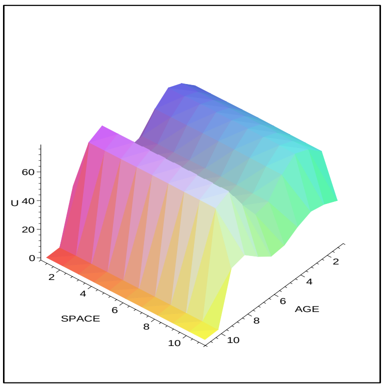
$t = 6$



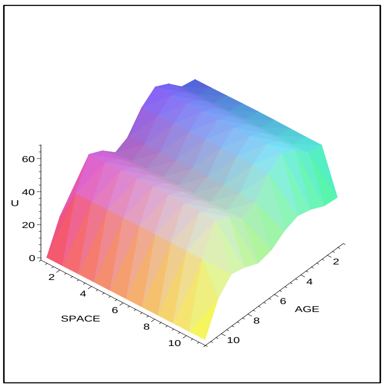
$t = 7$



$t = 8$

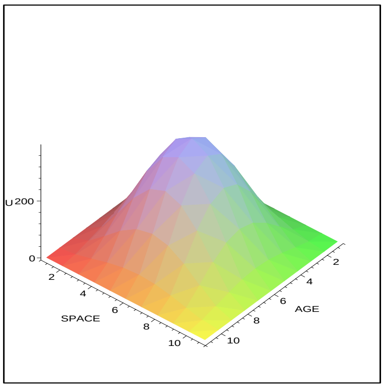


$t = 9$

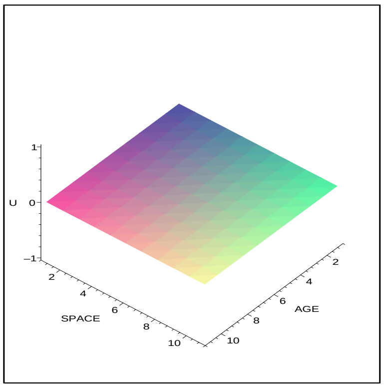


$t = T$

8.3.3 Null controllability



U_0

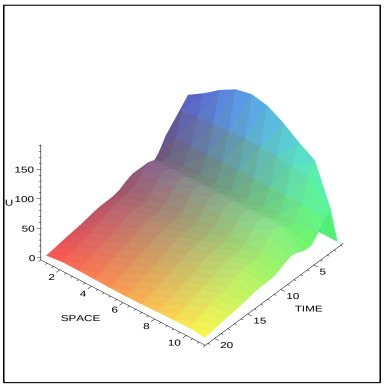


U_D

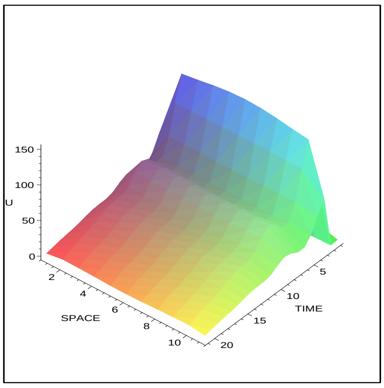
The algorithm work only For large time control T . For a 10,10 discretization in space and age the objective function is obtained for 230 time steps, and for any values of $\varepsilon \leq 10^{-1}$ we obtain the following results

$\|U - UD\|$ =4.013 10^{-2}	$\|g\|$ =3.478 10^{-8}	$\|\nabla J\|$ =1.001 10^{-7}	$J(g)$ =2.102 10^{-8}

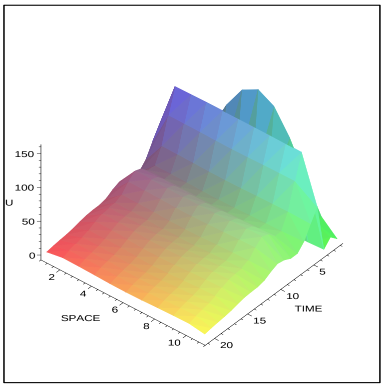
These results remain the same ones for various choice of the control domain ω. The following graphs describe evolution of $u(t, x, a)$ for $a = 0$ to $10(A_\dagger)$



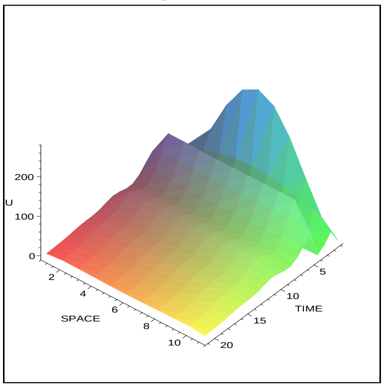
$Age = 0$



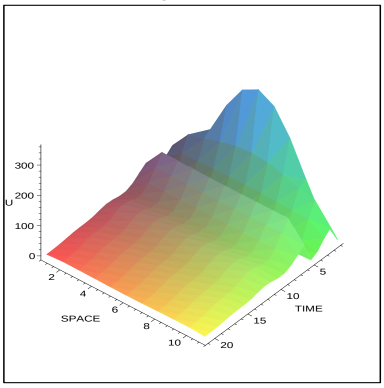
$Age = 1$



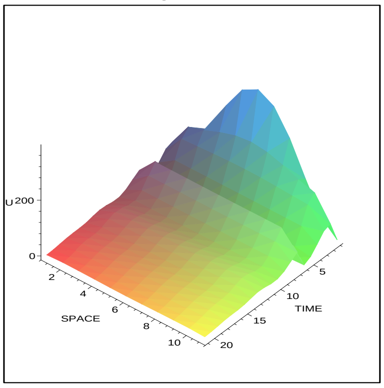
$Age = 2$



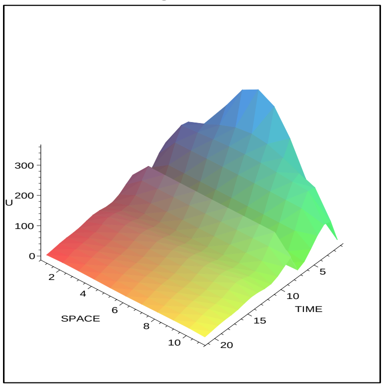
$Age = 3$



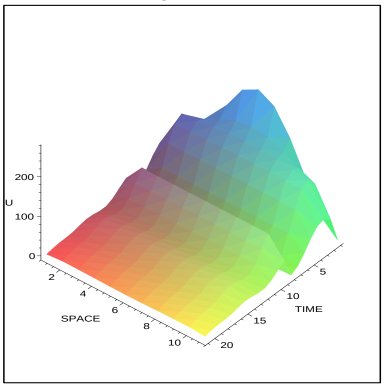
$Age = 4$



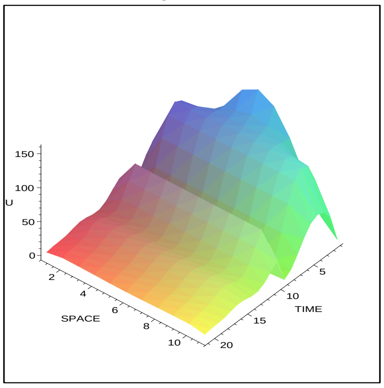
$Age = 5$



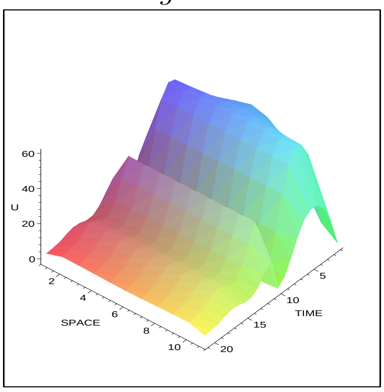
$Age = 6$



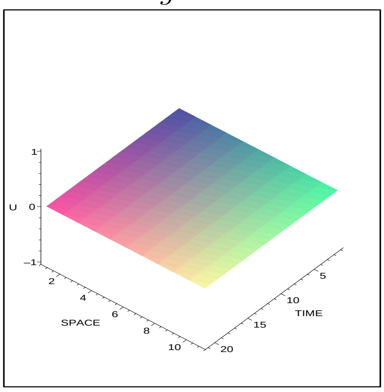
$Age = 7$



$Age = 8$



$Age = 9$



$Age = 10$

Bibliographie

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Troisième partie

Un Algorithme Multi-Niveau d'Identification d'un Canal en Communication Numérique

Chapitre 9

Un Algorithme Multi-Niveau d'Identification d'un Canal en Communication Numérique

Presented here is a multilevel algorithm for parameters identification of a channel with phase jitter and additive white noise. Two levels of the algorithm are considered. The first one, is an off-line algorithm based on Forward Backward strategy introduced by Baum and Welsh and extended by Liporace to multivariate case. It estimates iteratively the parameters using reestimation transformation by taking a bloc of data. The second level is a recursive algorithm. It consists of the evaluation of the time average value of the parameter conditionally on the observation. This is possible by replacing in the first algorithm smoothing by filtering and which makes recursive the reestimation transformation in the Forward-Backward algorithm. This new recursive algorithm is a non-linear truncation of the Birkoff Ergodic theorem. Convergence analysis and demonstrations results are given. The results are obtained by defining a new space, new model and new probability law and by using martingales and contiguity properties. Simulation results showing convergence and stability of our multilevel algorithm are also given.

9.1 Introduction

Dans ce chapitre nous développons deux méthodes d'estimation des paramètres d'un système non linéaire non contrôlé, excité par des entrées non directement observables. La première est une méthode itérative d'estimation hors ligne des paramètres. La deuxième est un algorithme adaptatif d'éstimation en-ligne des paramètres. Le système est modélisé par une chaîne de Markov cachée. Nous considérons dans ce chapitre une application de ces deux algorithmes au problème de la modélisation et d'identification d'une chaîne de transmission numérique comportant un canal à phase aléatoire et à bruit additif blanc gaussien.

Dans les systèmes de transmission numérique de données, simultanément à la détection des symboles d'information transmise, par un algorithme de lissage ou de filtrage non linéaire, il est nécessaire d'estimer les paramètres du modèle considéré. Cette identification du modèle peut s'effectuer de façon itérative dans une phase d'apprentissage ou récursivement au cours de la transmission pour suivre les variations lentes des paramètres du modèle.

Un système de transmission sur fréquence porteuse est représenté par son équivalent en bande de base. Lorsque la valeur de la fréquence est grande par rapport à la largeur de la bande du signal modulé, la décomposition est unique. Mais l'observation est une fonction non linéaire des variations de la phase apportée par le canal de transmission. Pour une modulation numérique à sauts de phase, une maximisation de la fonction de vraisemblance conjointe de la phase et des données a été proposé par O. Macchi [11] et qui a résolu le problème par une approche de programmation dynamique. L'estimation des paramètres du modèle peut être effectuée par un algorithme de Kalman étendu si on linéarise l'équation d'observation autour de la valeur de la phase estimée à l'instant précédent. La phase de l'observation est alors un paramètre du modèle. Cette approche n'est possible que pour de bons rapports signaux à bruits ($> 15\ dB$) et des variations lentes de la phase.

Dans ce chapitre, nous présentons deux algorithmes d'identification des paramètres d'une chaîne de Markov cachée. En première partie, nous présentons l'algorithme progressif-rétrograde proposé par Baum et Welche et par Liporace [9] au cas multivariables. On effectue une estimation de paramètres du modèle au sens du maximum d'une fonction de réestimation basée sur la vraisemblance conjointe de la phase et des données. En deuxième partie, nous présentons un algorithme adaptatif de type gradient stochastique basée sur la forme filtre de l'estimateur de l'état du modèle.

Les modèles de Markov cachés constituent maintenant une méthode bien développée sur le plan de la théorie et des applications et une littérature abondante et croissante est publiée à ce sujet : [3, 1, 9, 2, 8, 15, 14, 4, 5, 16, 13]. Introduits pour la première fois par Baum et Welche comme outil de modélisation et d'estimation des processus doublement stochastiques à deux composantes : une cachée, c'est-à-dire non accessible, qui représente l'état et qui est la réalisation d'une suite de variables aléatoires, l'autre observable, qui est la réalisation d'un processus aléatoire constituant la suite des observations. Pour une étude détaillée des modèles de Markov cachés, le lecteur est invité à consulter [14]. Koukhi a développé l'application des algorithmes d'identification des modèles de Markov cachés au contrôle stochastique et adaptatifs des robots mobiles et manipulateurs [7, 6]. Nous allons utiliser des modèles de Markov cachés discrets, c'est-à-dire que le nombre d'états cachés, ainsi que l'ensemble des symboles « observés » sont finis et dénombrables.

Dans la deuxième section nous présenterons la modélisation de la modulation et la définition de la chaîne de Markov cachée associée. La troisième section étudie l'algorithme de lissage non linéaire et d'estimation hors-ligne des paramètres. La section quatre décrit la procédure d'estimation des paramètres ainsi que l'analyse de la convergence en suivant l'approche donnée dans [9]. La section cinq considère les relations de filtrage progressive et rétrograde, qui constituent un puissant outil d'identification par les modèles de Markov cachés. Dans la section six nous introduisons l'algorithme adaptatif. Cet algorithme est de la classe des algorithmes du gradient stochastiques. Dans la septième section nous développons l'analyse de la convergence de l'algorithme adaptatif. Dans la section huit, nous donnerons des simulations préliminaires de ces algorithmes pour le problème d'identification des paramètres du canal de transmission considéré. Nous terminons dans la section neuf par des conclusions et des perspectives à court terme de développement de ces algorithmes. Enfin des simulations démontrent l'efficacité de nos algorithmes d'identification des paramètres.

9.2 Modélisation de la modulation et définition de la chaîne de Markov cachée

Le modèle de la modulation est une chaîne de Markov cachée. Cette chaîne est une représentation d'état à temps discret fini et à valeurs discrètes et finies du canal de transmission. Dans ce modèle les états sont représentes par le vecteur $X_n = (a_n, \theta_n), n \in (1, N)$ où

$$a_n = 2u_n - 1 \quad \text{et} \quad \theta_n = \theta_{n-1} + d\theta_n + \Delta\theta \tag{9.2.1}$$

L'équation d'observation est :

$$Y_n = a_n A e^{j\theta_n} + b_n \quad \text{pour} \quad n = 1, .., N \tag{9.2.2}$$

u_n est un symbole produit par la source, a_n est un symbole utilisé par l'émetteur (ex $u_n = 0$ ou 1 et $a_n = -1$ ou $+1$).

θ_n est une suite de variables aléatoires telle que la densité de probabilité de θ_0 soit : $P(\theta_0) = \frac{1}{2\pi}$, i.e. θ_0 est équirépartie sur $[0, 2\pi]$.

La densité conditionnelle $P(\theta_n/\theta_{n-1})$ correspond à $\mathcal{N}(\theta_n + \Delta\theta, \sigma^2)$ si on néglige la masse en dehors de $[-\pi, \pi]$.

θ_n pour $n = 1, .., N$ est une marche aléatoire sur le cercle unité. Les incréments $d\theta_n$ forment une suite de variables aléatoires indépendantes et identiquement distribuées de moyenne $\Delta\theta$.

A est une constante réelle positive telle que $A^2 = 2E_b$ où E_b est l'énergie par bit du signal transmis.

b_n est une suite de variables aléatoires à valeurs complexes de variance $2N_0$.

$$b_n = b_{rn} + jb_{sn} \quad , \quad E(b_{rn}^2) = E(b_{sn}^2) = N_0 \quad \text{et} \quad E(b_{rn}b_{sn}) = 0.$$

Le vecteur de paramètres à estimer est donc défini par :

$$\Lambda = (A, N_0, \sigma_\theta^2, \Delta\theta) \tag{9.2.3}$$

9.3 Procédure de l'algorithme de réestimation hors-ligne des paramètres

Dans ce qui suit nous supposons que θ_n est convenablement quantifiée pour les besoins de calcul. La densité de probabilité conditionnelle de $Y_1^N = (Y_1, .., Y_N)$ étant donnée

$\theta_1^N = (\theta_1, ..., \theta_N)$ s'écrit

$$L(Y_1^N|\theta_1^N, \Lambda) = \prod_{n=1}^{N} P(Y_n|\theta_n, \Lambda) \tag{9.3.4}$$

La probabilité d'une suite (θ_1^N), étant donné le vecteur de paramètres Λ, est définie par

$$P(\theta_1^N|\Lambda) = P(\theta_0) \prod_{n=1}^{N} P(\theta_n|\theta_{n-1}, \Lambda). \tag{9.3.5}$$

La densité de probabilité de l'observation (Y_1^N) s'exprime par :

$$L(Y_1^N|\Lambda) = \sum_{\theta_0^N} L(Y_1^N|\theta_1^N, \Lambda) P(\theta_1^N) \tag{9.3.6}$$

ou en reportant (9.3.4) et (9.3.5) dans (9.3.6) on obtient

$$L(Y_1^N|\Lambda) = \sum_{\theta_0^n} P(\theta_0) \prod_{n=1}^{N} P(\theta_n|\theta_{n-1}, \Lambda) P(Y_n|\theta_n, \Lambda). \tag{9.3.7}$$

où $\sum_{\theta_0^n}$ représente la sommation sur tous les chemins possibles (θ_1^N) soit $(N+1)^M$ chemins différents si la phase θ_i est quantifiée sur M valeurs.

La fonction de vraisemblance de la phase θ_n à un instant n, compte tenu de l'observation Y_1^n avec $1 \leq n \leq N$, i.e. la loi jointe de Y_1^N et de θ_i peut s'exprimer à partir de deux fonctions calculées de façon récursive, l'une $\alpha(\theta_n|\Lambda)$, dans le sens direct (progressif) et l'autre $\beta(\theta_n|\Lambda)$ dans le sens rétrograde, dont on verra plus loin le calcul de leur expressions respectives.

$$L(Y_1^N, \theta_n|\Lambda) = \alpha(\theta_n|\Lambda) \beta(\theta_n|\Lambda) \tag{9.3.8}$$

Cette fonction de vraisemblance peut ainsi être calculée pour une valeur initiale du vecteur Λ pour estimer la valeur du vecteur des paramètres Λ. Etant donnée cette valeur initiale, l'algorithme donné dans la section suivante, permet une nouvelle estimation de Λ nommée Λ' qui réalise après convergence du processus itératif le calcul du maximum de la fonction $Q(\Lambda, \Lambda')$ comme fonction de Λ'.

9.4 Algorithme de réestimation hors-ligne des paramètres

Nous considérons la méthode proposée par Liporace[9] qui est basée sur l'utilisation de la fonction d'information de Kullback Liebler, qui définit une "distance" entre deux vecteurs de paramètres Λ et Λ' .

Cette fonction d'entropie est définie par :

$$Q(\Lambda,\Lambda') = \sum_{\theta_1^N} \left(L(Y_1^N,\theta_1^N|\Lambda)\log(L(Y_1^N,\theta_1^N|\Lambda')) \right) \tag{9.4.9}$$

l'utilisation de cette fonction auxiliaire repose sur les propriétés suivantes :

1. Si $Q(\Lambda,\Lambda') > Q(\Lambda,\Lambda)$ alors $L(Y_1^N|\Lambda') > L(Y_1^N|\Lambda)$;
2. Si $P(Y_n|\theta_n,\Lambda\ )$ est symétrique elliptique, Y_n est une somme de variables aléatoires gaussiennes, $Q(\Lambda\ ,\Lambda')$ possède un maximum, qui est de plus un point critique.
3. La valeur Λ du paramètre est un point critique de la fonction de vraisemblance si et seulement s'il est un point fixe de la transformation de réestimation.

$$\nabla_\Lambda L(Y_1^N|\Lambda) = \nabla_{\Lambda'} Q(\Lambda,\Lambda')|_{\Lambda'=\Lambda} \tag{9.4.10}$$

Ces propriétés sont facilement démontrables voir [9]. Dans la suite on se propose de montrer le théorème suivant qui est une adaptation de la démonstration du théorème de [9] par rapport à notre cas garantissant l'unicité et la globalité de la fonction Q.

Théorème 1 *La fonction de réestimation $Q(\Lambda,\Lambda')$ admet un unique point critique Λ' qui est l'unique maximum global.*

Démonstration. L'algorithme de réestimation des paramètres utilise l'information mutuelle entre les deux modèles Λ et Λ'. On cherche Λ' tel que $Q(\Lambda,\Lambda')$ soit maximale ou ce qui est équivalent à ce que sa dérivée en Λ' soit nulle. En utilisant la définition du modèle donné par les relations plus haut, la fonction de vraisemblance s'écrit :
(9.4.11)

$$L(Y_1^N,\theta_1^N|\Lambda) = P(\theta_0)\prod_{n=1}^{N}\frac{1}{\sqrt{2\pi}\sigma_\theta}\exp\left(\frac{-(\theta_n-\theta_{n-1})^2}{2\sigma_\theta^2}\right).\frac{1}{2\pi N_0}\exp\left(\frac{-\left|Y_n-a_nAe^{j\theta_n}\right|^2}{2N_0}\right)$$

Rappelons que le vecteur de paramètres Λ est défini par

$$\Lambda = (A,N_0,\sigma_\theta^2,\Delta\theta) \in (0,+\infty)^4.$$

La suite des symboles $(a_1,...,a_n)$ est estimée lorsque l'algorithme considéré converge. Pour simplifier les expressions obtenues, on peut considérer que l'observateur connaît la suite des symboles $(a_1,...,a_n)$, pour simplifier les calculs on supposera que $a_n = 1$, $n = 1..N$, l'algorithme est alors utilisé dans une période d'apprentissage.

Le logarithme de l'expression (9.4.11) est facile à obtenir ; en dérivant l'expression ainsi obtenue par à rapport chacune des composantes du vecteur Λ' on obtient les relations suivantes :

$$\frac{\partial\log(L(Y_1^N,\theta_1^N|\Lambda'))}{\partial A'} = \sum_{n=1}^{N}\frac{Re\left(e^{-j\theta_n}(Y_n-A'e^{j\theta_n})\right)}{2N_0'} \tag{9.4.12}$$

(9.4.13)
$$\frac{\partial \log(L(Y_1^N, \theta_1^N|\Lambda'))}{\partial N_0'} = \sum_{n=1}^{N} \left(\frac{-1}{N_0'} + \frac{\left|Y_n - A'e^{j\theta_n}\right|^2}{2N_0'^2} \right)$$

(9.4.14)
$$\frac{\partial \log(L(Y_1^N, \theta_1^N|\Lambda'))}{\partial \sigma_\theta'^2} = \sum_{n=1}^{N} \left(\frac{-1}{2\sigma_\theta'^2} + \frac{|\theta_n - \theta_{n-1} - \Delta'\theta|^2}{2\sigma_\theta'^4} \right)$$

(9.4.15)
$$\frac{\partial \log(L(Y_1^N, \theta_1^N|\Lambda'))}{\partial \Delta'\theta} = \sum_{n=1}^{N} \frac{(\theta_n - \theta_{n-1} - \Delta'\theta)}{\sigma_\theta'^2}$$

En portant la relation (9.4.12) dans (9.4.9) on obtient :

$$\sum_{n=1}^{N} \sum_{\theta_n} L\left(Y_1^N, \theta_1^N|\Lambda\right) \frac{Re\left(e^{-j\theta_n}(Y_n - A'e^{j\theta_n})\right)}{2N_0'} = 0$$

et en développant l'expression précédente on aura

(9.4.16)
$$A' = \frac{1}{N} \frac{\sum_{n=1}^{N} \sum_{\theta_n} Re\left(Y_n e^{-j\theta_n}\right) L\left(Y_1^N, \theta_n|\Lambda\right)}{L\left(Y_1^N|\Lambda\right)}$$

en notant que la loi de θ_n conditionnellement à l'observation Y_1^N peut s'exprimer à partir de la fonction de vraisemblance par :

$$P(\theta_n, Y_1^N, \Lambda) = \frac{L\left(Y_1^N, \theta_n|\Lambda\right)}{\sum_{\theta_n} L\left(Y_1^N, \theta_n|\Lambda\right)}$$

A' correspond à une estimation empirique de la moyenne de l'observation Y_n qui peut s'écrire :

$$A' = \frac{1}{N} \sum_{n=1}^{N} \sum_{\theta_n} Re\left(Y_n e^{-j\theta_n}\right) P\left(\theta_n|Y_1^N, \Lambda\right)$$

Des calculs équivalents permettent d'exprimer les estimations des autres paramètres :

(9.4.17)
$$N_0' = \frac{1}{2N} \frac{\sum_{n=1}^{N} \sum_{\theta_n} \left|Y_n - Ae^{j\theta_n}\right|^2 L\left(Y_1^N, \theta_n|\Lambda\right)}{L\left(Y_1^N|\Lambda\right)}$$

L'estimation du paramètre caractérisant la variance des variations de la phase de la porteuse s'exprime par :

(9.4.18)
$$\sigma\prime_\theta^2 = \frac{1}{N} \sum_{n=1}^{N} \sum_{\theta_n} \sum_{\theta_{n-1}} \frac{(\theta_n - \theta_{n-1} - \Delta\theta)^2 L\left(Y_1^N, \theta_n, \theta_{n-1}|\Lambda\right)}{\sum_{\theta_n} \sum_{\theta_{n-1}} L\left(Y_1^N, \theta_n, \theta_{n-1}|\Lambda\right)}$$

et la dérivée de fréquence s'exprime par :

$$\Delta\theta' = \frac{1}{N}\sum_{n=1}^{N}\sum_{\theta_n}\sum_{\theta_{n-1}}\frac{(\theta_n-\theta_{n-1})L\left(Y_1^N,\theta_n,\theta_{n-1}|\Lambda\right)}{\sum_{\theta_n}\sum_{\theta_{n-1}}L\left(Y_1^N,\theta_n,\theta_{n-1}|\Lambda\right)} \tag{9.4.19}$$

Le dénominateur est commun à l'ensemble des quatre expressions précédentes. En suivant l'idée de Liporace, ces paramètres déterminent l'unique point critique Λ' de $Q(\Lambda,\Lambda')$. Ce point critique est en fait un maximum, en effet en utilisant les relations (9.4.12)− (9.4.15) on calcule la hessienne H de $Q(\Lambda,\Lambda')$ au point critique Λ' de $Q(\Lambda,\Lambda')$ donné par les relations (9.4.16)− (9.4.19) et on obtient

$$\begin{aligned}
H_{11} &= -\frac{1}{2N'}\sum_{n=1}^{N}L\left(Y_1^N|\,\Lambda\right) && <0,\\
H_{22} &= -\frac{1}{2N'^3}\sum_{n=1}^{N}\sum_{\theta_n}L\left(Y_1^N,\theta_n|\Lambda\right)\left|Y_n-Ae^{j\theta_n}\right|^2 && <0,\\
H_{33} &= -\frac{1}{2\sigma_\theta^{6\prime}}\sum_{n=1}^{N}\sum_{\theta_n}\sum_{\theta_{n-1}}L\left(Y_1^N,\theta_n,\theta_{n-1}|\Lambda\right)(\theta_n-\theta_{n-1}-\Delta\theta)^2 && <0,\\
H_{44} &= -\frac{1}{2\sigma_\theta^{2\prime}}\sum_{n=1}^{N}\sum_{\theta_n}L\left(Y_1^N,\theta_n|\Lambda\right) && <0,\\
H_{i,j} &= 0, && \text{pour } i\neq j.
\end{aligned}$$

Par ces relation H est diagonale définie négative ce qui implique que le point critique est un maximum local.

Maintenant en calculant la limite de $Q(\Lambda,\Lambda')$ lorsque Λ' s'approche de la frontière de son domaine d'existence $(0,+\infty)^4$ on obtient $-\infty$. Ce qui montre en utilisant la continuité de $Q(\Lambda,\Lambda')$ par rapport à Λ' et l'unicité du point critique, que le point critique donné par les relations (9.4.16)− (9.4.19) est l'unique maximum de la fonction de réestimation $Q(\Lambda,\Lambda')$ [9].

9.5 Calcul récursif des vraisemblances progressive et rétrograde

La puissance des modèls de Markov cachés comme technique d'estimation et d'identification réside dans la stratégie du calcul récursif de la vraisemblance en l'exprimant comme produit d'une vraisemblance progressive et d'une vraisemblance rétrograde (notée Forward-Backward)[14]. Ce qui permet un gain considérable en temps de calcul.

Pour un instant n quelconque, on a les relations suivantes :

$$L(Y_1^N|\Lambda)=\sum_{\theta_0^n}P(Y_n,\theta_n|\Lambda),\ \forall n\in(1,...,N)$$

et

$$P(Y_1^n, Y_{n+1}^N, \theta_n|\Lambda) = P(Y_1^n, \theta_n|\Lambda)P(Y_{n+1}^N|Y_1^n, \theta_n, \Lambda) \tag{9.5.20}$$

Par suite des propriétés Markoviennes de la suite θ_n, la loi de Y_{n+1}^N sachant θ_n est indépendante de la fonction de vraisemblance conjointe de (Y_1^N, θ_n) :

$$L(Y_1^N, \theta_n|\Lambda) = \alpha(\theta_n|\Lambda)\beta(\theta_n|\Lambda) \tag{9.5.21}$$

Les deux fonctions de vraisemblances directe et rétrograde s'expriment par :

$$\alpha\left(\theta_n|\Lambda\right) = P(Y_1^n, \theta_n|\Lambda) \tag{9.5.22}$$

$$\beta\left(\theta_n|\Lambda\right) = P\left(Y_{n+1}^N|\theta_n, \Lambda\right) \tag{9.5.23}$$

Pour calculer la vraisemblance de toutes les variables pour $n = 1, \ldots, N$, il faut calculer les deux fonctions directe et rétrograde de façon récursive en utilisant les propriétés suivantes :

9.5.1 Vraisemblance progressive

Les vraisemblances progressives s'expriment par :

$$\begin{aligned}\alpha\left(\theta_n|\Lambda\right) &= \sum_{\theta_{n-1}} P\left(Y_1^n, \theta_n, \theta_{n-1}|\Lambda\right) \\ &= \sum_{\theta_{n-1}} P\left(Y_1^{n-1}, \theta_{n-1}|\Lambda\right) P\left(Y_n, \theta_n|Y_1^{n-1}, \theta_{n-1}, \Lambda\right) \\ &= \sum_{\theta_{n-1}} P\left(Y_1^{n-1}, \theta_{n-1}|\Lambda\right) P\left(Y_n, \theta_n|\theta_{n-1}, \Lambda\right)\end{aligned} \tag{9.5.24}$$

et si b_n et θ_n sont indépendantes on obtient :

$$\alpha\left(\theta_n|\Lambda\right) = \sum_{\theta_{n-1}} P\left(Y_1^{n-1}, \theta_{n-1}|\Lambda\right) P\left(Y_n|\theta_n, \Lambda\right) P\left(\theta_n|\theta_{n-1}, \Lambda\right). \tag{9.5.25}$$

Soit la relation récursive sur le filtre direct :

$$\alpha\left(\theta_n|\Lambda\right) = \sum_{\theta_{n-1}} \alpha\left(\theta_{n-1}|\Lambda\right) P\left(Y_n|\theta_n, \Lambda\right) P\left(\theta_n|\theta_{n-1}, \Lambda\right) \tag{9.5.26}$$

9.5.2 Vraisemblance rétrograde

La relation (9.5.26) est utilisée dans des algorithmes de filtrage non-linéaires. La relation de récurrence rétrograde s'établit pour $n = 1, ..., N$ de la même façon en notant que :

$$\begin{aligned}
\beta\left(\theta_n|\Lambda\right) &= P\left(Y_{n+1}^N|\theta_n, \Lambda\right) \\
&= \sum_{\theta_{n+1}} P\left(Y_{n+1}, Y_{n+2}^N, \theta_{n+1}|\theta_n, \Lambda\right) \\
&= \sum_{\theta_{n-1}} P\left(Y_{n+2}^N|\theta_n, \theta_{n+1}, \Lambda\right) P\left(Y_{n+1}, \theta_{n+1}|\theta_n, \Lambda\right)
\end{aligned} \tag{9.5.27}$$

et si b_n et θ_n sont indépendantes on obtient :

$$\beta\left(\theta_n|\Lambda\right) = \sum_{\theta_{n+1}} P\left(Y_{n+2}^N|\theta_{n+1}, \Lambda\right) P\left(Y_{n+1}|\theta_{n+1}, \Lambda\right) P\left(\theta_{n+1}|\theta_n, \Lambda\right)$$

Soit la relation récursive sur le filtre rétrograde :

$$\beta\left(\theta_n|\Lambda\right) = \sum_{\theta_{n+1}} \beta\left(\theta_{n+1}|\Lambda\right) P\left(Y_{n+1}|\theta_{n+1}, \Lambda\right) P\left(\theta_{n+1}|\theta_n, \Lambda\right) \tag{9.5.28}$$

La valeur initiale à l'instant N peut être choisie de façon arbitraire, par exemple $\beta\left(\theta_N|\Lambda\right) = 1$. On peut, après normalisation, exprimer la probabilité à posteriori de la variable θ_n par :

$$P\left(\theta_n|Y_1^N, \Lambda\right) = \frac{\alpha\left(\theta_n|\Lambda\right)\beta\left(\theta_n|\Lambda\right)}{L\left(\theta_n|Y_1^N, \Lambda\right)}$$

Les relations (9.5.26) et (9.5.28) forment les relations de base, de notre algorithme appliquant les modèles de Markov cachés.

9.5.3 Instabilités numériques et normalisation

D'après les formules définissant les vraisemblances progressive et rétrograde, le calcul de ces dernières montre qu'elles tendent rapidement vers zéro lorsque N croit. Ces probabilités deviennent alors trop petites pour être représentées en machines (problème connu communément sous le nom de dépassement de capacité, underflow). Pour remédier à ce problème, une normalisation adéquate est nécessaire. Pour ce faire, on multiplie les probabilités progressive et rétrograde par un coefficient c_n de normalisation redéfinissant ces probabilités comme suit :

$$\hat{\alpha}\left(\theta_n|\Lambda\right) = c_{p_n}\alpha\left(\theta_n|\Lambda\right),\ n = 1, ..., N$$

$$\hat{\beta}\left(\theta_n|\Lambda\right) = c_{r_n}\beta\left(\theta_n|\Lambda\right),\ n = 1, ..., N$$

ce qui nous donne :

$$c_{p_n} = \left(\sum_{\theta_n} \alpha\left(\theta_n|\Lambda\right) \right)^{-1}, \ n = 1, ..., N$$

$$c_{r_n} = \left(\sum_{\theta_n} \beta\left(\theta_n|\Lambda\right) \right)^{-1}, \ n = 1, ..., N$$

Lors de la simulation, nous utiliserons les vraisemblances progressive et retrograde normalisées afin d'éviter les instabilités numériques [14].

9.6 Estimation adaptative

L'algorithme d'identification adaptative que nous présentons est nouveau, et il est de la forme des algorithmes de gradient stochastique. L'allure générale de l'algorithme pour le problème considéré est la suivante :

$$A_n = A_{n-1} - \frac{1}{n} \left(A_{n-1} - \sum_{\theta_n} Re\left(Y_n e^{-j\theta_n}\right) \alpha\left(\theta_n|\Lambda_{n-1}\right) \right) \tag{9.6.29}$$

où $\alpha\left(\theta_n|\Lambda_{n-1}\right)$ est la probabilité conditionnelle de θ_n étant donnée la suite d'observations Y_1^n qui s'exprime récursivement par la relation (9.5.26) de vraisemblance progressive jusqu'a n. La relation récursive d'estimation de la variance du bruit s'exprime par :

$$N_n = N_{n-1} - \frac{1}{2n} \left(N_{n-1} - \sum_{\theta_n} \left|Y_n - A_{n-1} e^{-j\theta_n}\right|^2 \alpha\left(\theta_n|\Lambda_{n-1}\right) \right). \tag{9.6.30}$$

L'estimation de la variance de la gigue de la phase est définie par :

$$\sigma^2_{\theta,n} = \sigma^2_{\theta,n-1} - \frac{1}{n} \left(\sigma^2_{\theta,n-1} - \sum_{\theta_n} \sum_{\theta_{n-1}} (\theta_n - \theta_{n-1} - \Delta\theta)^2 \alpha\left(\theta_n, \theta_{n-1}|\Lambda_{n-1}\right) \right). \tag{9.6.31}$$

La dérivée de fréquence est donnée par :

$$\Delta\theta_n = \Delta\theta_{n-1} - \frac{1}{n} \left(\Delta\theta_{n-1} - \sum_{\theta_n} \sum_{\theta_{n-1}} (\theta_n - \theta_{n-1}) \alpha\left(\theta_n, \theta_{n-1}|\Lambda_{n-1}\right) \right) \tag{9.6.32}$$

La connaissance de la probabilité conditionnelle $\alpha\left(\theta_n, \theta_{n-1}|\Lambda_{n-1}\right)$ permet à chaque instant n d'estimer la phase θ_n qui maximise cette probabilité et de détecter la suite des symboles transmis.

Pour un nombre voisin de 60 échantillons, nous avons pu remarquer que l'on n'observait pas de phénomène de convergence (cours terme) pour une valeur initiale très petite de la variance de gigue de phase de 0.01. Ce problème nous semble lié à la présence de sauts de cycles.

9.7 Analyse de la convergence (Théorèmes)

Pour étudier la convergence de l'algorithme récursif on va définir un nouvel espace, un nouveau modèle et une nouvelle loi de probabilité $\widehat{P}$ tels que les paramètres $\widehat{A}_n$, $\widehat{N}_n$, $\widehat{\sigma}_{\theta_n}$ et $\Delta\widehat{\theta}_n$ convergent vers les mêmes vraies valeurs que sous P.

La construction de ce nouveau modèle n'est qu'un un artifice mathématique permettant de démontrer la convergence de l'algorithme d'estimation adaptative des paramètres sous ce modèle, et l'absolue contiguïté entre ce nouveau modèle et le modèle initial qui garantit la convergence des paramètres sous le modèle initial.

9.7.1 Le nouveau modèle

En se référant à l'équation définissant le modèle de la modulation (section 9.2), l'équation d'observation peut s'écrire :

(9.7.33) $$Y_n = a_n A e^{j\theta_n} + \sqrt{4\pi N_0} W_n$$

$\widehat{P}$ est telle que :

(9.7.34) $$\widehat{E}(\widehat{P}(W_n \mid \theta_n)) = 1$$

9.7.2 Théorèmes

Théorème 2 *$\widehat{A}_n$, $\widehat{N}_n$, $\widehat{\sigma}_{\theta_n}$ et $\Delta\widehat{\theta}_n$ sont des martingales sous $\widehat{P}$, i.e.*

$$\begin{aligned}
\widehat{E}(A_n \mid \widehat{\Lambda}_{n-1}) &= A_{n-1}\\
\widehat{E}(N_n \mid \widehat{\Lambda}_{n-1}) &= N_{n-1}\\
\widehat{E}(\sigma_n^2 \mid \widehat{\Lambda}_{n-1}) &= \sigma_{n-1}^2\\
\widehat{E}(\Delta\theta_n \mid \widehat{\Lambda}_{n-1}) &= \Delta\theta_{n-1}
\end{aligned}$$

Démonstration. De simples calculs conduisent aux résultats suivants :
pour θ_n et θ_{n-1} fixé

(9.7.35) $$\widehat{E}(\widehat{\alpha}(\theta_n, \theta_{n-1}|\Lambda_{n-1})|\Lambda_{n-1}) = \widehat{P}(\theta_n, \theta_{n-1}|\Lambda_{n-1})$$

pour θ_n fixé

(9.7.36) $$\widehat{E}(b_n\widehat{\alpha}(\theta_n|\Lambda_{n-1})|\Lambda_{n-1}) = 0$$

et

(9.7.37) $$\widehat{E}(|b_n|^2\,\widehat{\alpha}(\theta_n|\Lambda_{n-1})|\Lambda_{n-1}) = N_{n-1}\widehat{P}(\theta_n|\Lambda_{n-1}).$$

Montrons maintenant que $\widehat{A}_n$ est une martingale sous $\widehat{P}$

$$\begin{aligned}\widehat{E}(A_n|\Lambda_{n-1}) &= A_{n-1} - \frac{1}{n}\left(A_{n-1} - \sum_{\theta_n} Re\widehat{E}((Y_n e^{-j\theta_n})\widehat{\alpha}(\theta_n|\Lambda_{n-1})|\Lambda_{n-1})\right)\\ &= A_{n-1} - \frac{1}{n}\left(A_{n-1} - \sum_{\theta_n} Re\left[A_{n-1}\widehat{E}(\widehat{\alpha}(\theta_n|\Lambda_{n-1})|\Lambda_{n-1})\right.\right.\\ &+ \left.\left. e^{-j\theta_n}\widehat{E}(b_n\widehat{\alpha}(\theta_n|\Lambda_{n-1})|\Lambda_{n-1})\right]\right)\end{aligned}$$

en utilisant les relations (9.7.35) – (9.7.37) on aura

$$\widehat{E}(A_n|\Lambda_{n-1}) = A_{n-1} - \frac{1}{n}\left(A_{n-1} - A_{n-1}\sum_{\theta_n}\widehat{P}(\theta_n|\Lambda_{n-1})\right) = A_{n-1}$$

de la même manière on montre que $\widehat{N}_n$, $\widehat{\sigma}_{\theta_n}$ et $\Delta\widehat{\theta}_n$ sont des martingales sous $\widehat{P}$.

Théorème 3 *$\widehat{A}_n$, $\widehat{N}_n$, $\widehat{\sigma}_{\theta_n}$ et $\Delta\widehat{\theta}_n$ sont de carrées intégrables sous $\widehat{P}$*

Démonstration :
On a

$$\widehat{E}(A_n^2|\Lambda_{n-1}) = \left(1-\frac{1}{n}\right)^2 A_{n-1}^2 - \frac{2}{n}\left(1-\frac{1}{n}\right)A_{n-1}\widehat{E}(K_n|\Lambda_{n-1}) + \frac{1}{n}\widehat{E}(K_n^2|\Lambda_{n-1})$$

avec

$$K_n = \sum_{\theta_n} Re(Y_n e^{-j\theta_n})\widehat{\alpha}(\theta_n|\Lambda_{n-1})$$

Par des calculs pénibles on montre que :

$$\widehat{E}(K_n|\Lambda_{n-1}) = A_{n-1} \tag{9.7.38}$$

et

$$\widehat{E}(K_n^2|\Lambda_{n-1}) = A_{n-1}^2 \tag{9.7.39}$$

par suite

$$\widehat{E}(A_n^2|\Lambda_{n-1}) = A_{n-1}^2 \tag{9.7.40}$$

donc A_n^2 est de carrée intégrable. On montre de même pour $\widehat{N}_n$, $\widehat{\sigma}_{\theta_n}$ et $\Delta\widehat{\theta}_n$.

Théorème 4 *$\widehat{A}_n$, $\widehat{N}_n$, $\widehat{\sigma}_{\theta_n}$ et $\Delta\widehat{\theta}_n$ convergent sous $\widehat{P}$.*

Démonstration. A_n, $\widehat{N}_n$, $\widehat{\sigma}_{\theta_n}$ et $\Delta\widehat{\theta}_n$ sont des martingales de carrée intégrable sous $\widehat{P}$ alors d'après un résultat de Neveu [12] elles sont donc convergentes sous $\widehat{P}$.

Théorème 5 *A_n, N_n, σ_{θ_n} et $\Delta\theta_n$ convergent sous P.*

Démonstration. En posant $L_n = \dfrac{dP_n}{d\widehat{P}_n}$ avec $P_n = P|_{\Lambda_n}$et $\widehat{P}_n = \widehat{P}|_{\Lambda_n}$et en calculant $\alpha = \dfrac{L_n}{L_{n-1}} = 1$, on en déduit que la série $\sum_{n\geq 1} E((1-\sqrt{\alpha_n})^2|\Lambda_{n-1}) < +\infty$ et d'après le théorème dû à Kabanov, Liptser et Shirayev ci-dessous P et $\widehat{P}$ sont absolument contiguës et par suite A_n, N_n, σ_{θ_n} et $\Delta\theta_n$ convergent sous P **[10]**.

Théorème 6 ***(Kabanov, Liptser, Shiryayev [10])*** *Etant donnée (Ω, F, P) un espace mesurable avec une σ-algèbre F_n, $n \geq 0$, P_n et $\widehat{P}_n$ des restrictions à F_n de P et $\widehat{P}$ i.e. $P_n = P|_{F_n}$ et $\widehat{P}_n = \widehat{P}|_{F_n}$. Supposons que $P_n << \widehat{P}_n$ (absolument contiguës) et posons $\alpha_n = \dfrac{L_n}{L_{n-1}}$ avec $L_n = \dfrac{dP_n}{d\widehat{P}_n}$, alors $P << \widehat{P}$ (absolument contiguës) ce qui est équivalent à*

$$\sum_{n\geq 1} E((1-\sqrt{\alpha_n})^2|\Lambda_{n-1}) < +\infty.$$

9.8 Etude de la complexité

La complexité des algorithmes proposés s'exprime en nombre d'opérations pour L échantillons observés. L'algorithme itératif nécessite $2M^2L$ opérations et $2ML$ mots mémoires par itération. L'algorithme récursif nécessite M^2L opérations entre multiplications et additions et M mots mémoires [14].

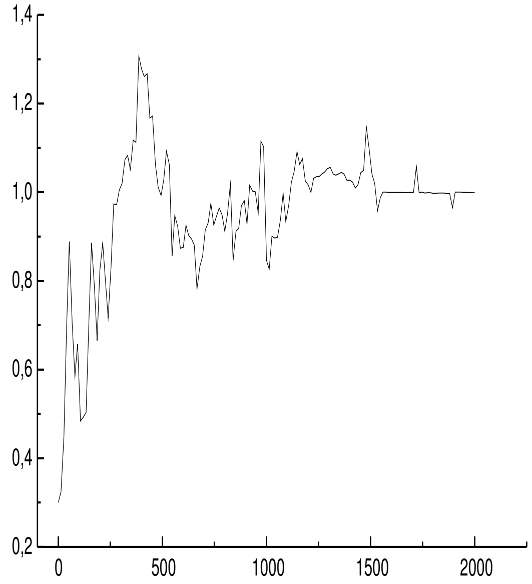
9.9 Résultats de simulation

Les simulations effectuées pour vérifier la validité de l'algorithme proposé, ont montré la robustesse de la méthode. Pour un vecteur initial $\Lambda_0 = (A_0 = 0.5,\ N_{00} = 0.1,\ \sigma_{\theta_0} = 2.$ et $\Delta\theta = 0.001)$, et un vecteur vraie valeur $\Lambda = (A = 1.414,\ N_0 = 10.,\ \sigma_\theta = 1.$ et $\Delta\theta = 0.003)$.

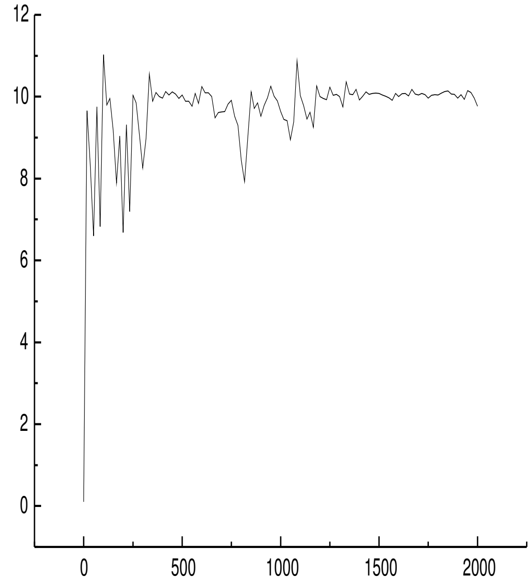
On remarque d'après les résultats obtenus que l'algorithme récursif est moins coûteux que l'algorithme itératif global. Notons que les difficultés rencontrés lors des simulations sont essentiellement dues aux problèmes de normalisation. Alors que l'algorithme récursif demande seulement la normalisation de $P(\theta_n|\theta_{n-1}, \Lambda_n)$ et de $\alpha(\theta_n)$, l'algorithme itératif global demande, en plus de ces normalisations la normalisation de $P(Y_n|\theta_n, \Lambda_n)$ et de $L(Y_1^N)$ ce qui augmente considérablement le temps de calcul qui sera d'autant plus grand que N est plus grand.

Les résultats fournis sur les courbes ci-dessous ont été effectuées sous Borland Builder C++ 5.0. L'algorithme récursif a convergé pour un bloc d'observations de 300 échantillons pour un même jeu de paramètres vrais valeurs. L'algorithme itératif global a convergé pour 48 échantillons au bout de 1000 itérations.

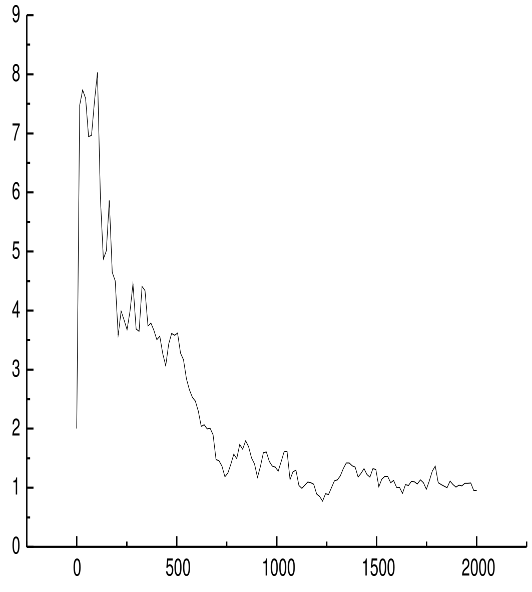
Simulation de l'algorithme itératif hors-ligne



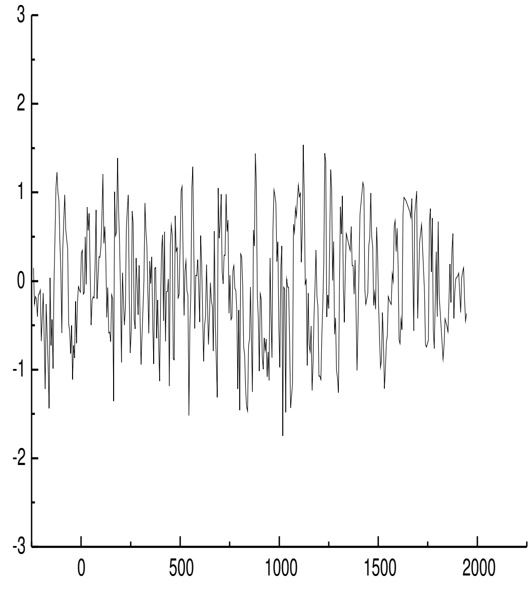
Amplitude A_n



Variance N_{0n}

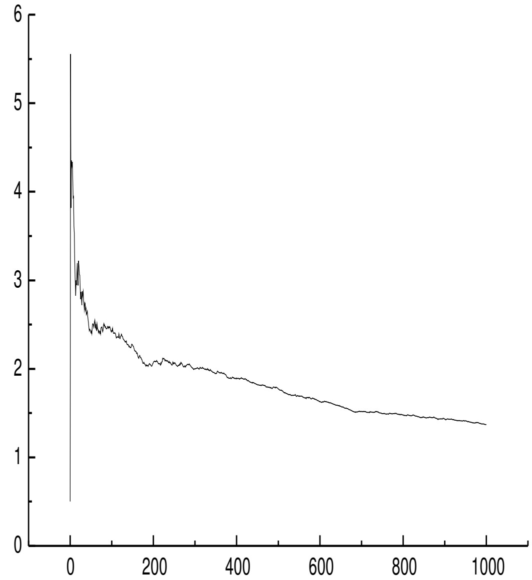


Variance σ_{0n}^2

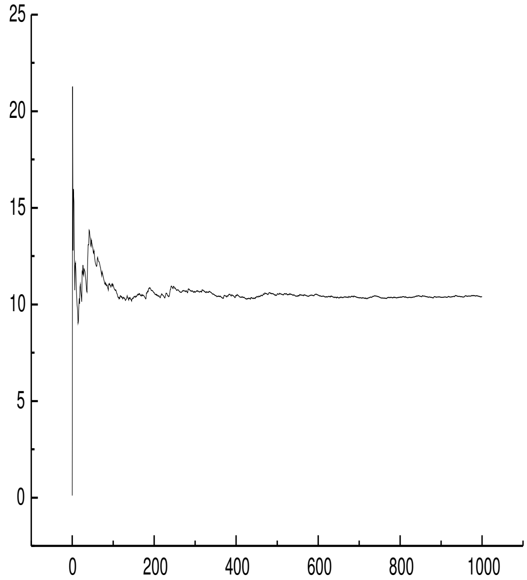


Dérivée de fréquence $\Delta\theta_n$

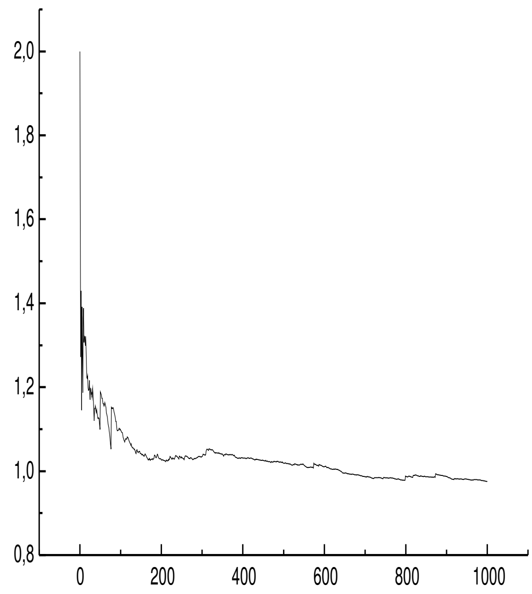
Simulation de l'algorithme adaptatif



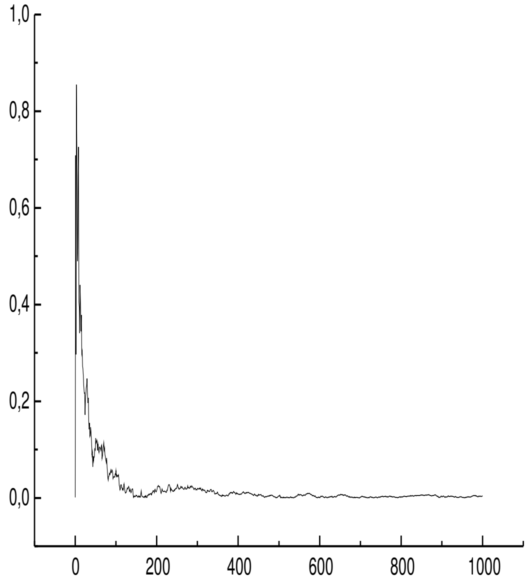
Amplitude A_n



Variance N_{0n}



Variance σ_{0n}^2



Dérivée de fréquence $\Delta\theta_n$

Conclusion et travaux futurs

L'estimation des paramètres de transmission d'une chaîne de transmission est effectuée par des algorithmes d'identification de chaînes de Markov cachées itératifs et récursifs. Ces algorithmes permettent, simultanément, d'identifier les paramètres du canal de transmission et de détecter les symboles d'information transmis. La complexité des algorithmes proposés est compensée par leur efficacité en ce sens qu'ils permettent une identification dans un environnement très bruité ($0dB$). Les propriétés de la gigue de phase sont estimées et utilisées pour la détection des données. L'algorithme récursif présenté fait mieux que le filtre de Kalman étendu qui est une linéarisation donc une approximation, notre algorithme traite les non-linéarités des systèmes dynamiques dans leur globalité. La convergence de ces algorithmes est établie et les études de simulation montrent aussi la convergence rapide de ces algorithmes.

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