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Sujet

Asymptotic modeling of elastostatic and elastodynamic Signorini problem with friction for thin plates

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Dedication

I dedicate my dissertation work to my family and many friends.

A special feeling of gratitude to

my loving parents, Aicha and Belkhier.

My sisters Fatma, Mebrouka, Oum-elkheir, Hadda, Saliha and Nacera,

My brothers Aissa, Ali, Salah, Mohamed and Abdelkader

My wife Kenza

My daughters Ritaj, Dharifa, Tasnim

My son Ahmed yassine

I also dedicate this dissertation to my many friends and family who have supported me throughout the process.

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General introduction

The contact problem is an important problem in computational mechanics. The modeling of unilateral contact problems of elastic bodies with friction is quite challenging and encounters some difficulties. Frictional contact problems are of crucial importance in many engineering branches, they involve contact and friction interactions between two or more deformable and/or rigid bodies in the Coulomb friction law. Despite their simple nature, contact problems with Coulomb friction are rather difficult to analyze. This is mainly due to the non-monotone, non-compact and even non-smooth character of the friction term in the weak formulation of the contact problem.

The problem of a unilateral contact with Coulomb friction attracted attention of many research workers both in engineering and mathematics. It is characterized by unilateral inequalities, describing the physical impossibility of tensile contact tractions (except under special circumstances) and of material inter-penetration. Additional inequalities and / or non-linearities are introduced when friction laws are taken into account. These complex boundary conditions can lead to problems with existence and uniqueness of quasi-static solution and to lack of convergence of numerical algorithms. In frictional problems, there can also be lack of stability, leading to stick-slip motion and frictional vibrations.

The first formulation of the problem (without friction) has been established by Signorini in 1933. He stated the prescribed boundary conditions and contact conditions for an elastic body against a rigid foundation. Its mathematical analysis is due to Fichera [44] using an equivalent minimizing problem. Some existence results for a class of problems are established by Duvaut and Lions [39] where they have pointed out an open problem of existence and uniqueness in the case of Coulomb friction law (local). In 1980, Néčas, Jarušek and Haslinger [81] have established only the existence of a solution under the condition that the friction coefficient is small enough. After that, more general results have been established by Jarušek [54], Kato [57], Eck and Jarušek [42]. R. Hassani, P.Hild and I.Ionescu [50] have found a sufficient condition for non uniqueness result. Recently, for dynamic case, M. T. Cao and P. Quintela [10] have established an existence theorem for a linear elastic Signorini problem without friction.

Thin structures are elastic bodies for which one dimension is small compared with other ones, standard exemples are plates, shells and rods. Under realistic mechanical hypotheses, some models are proposed by Kirchhoff, Love, Mindlin, Reissner, Koiter and Naghdi. A major advantage in the modeling of thin structure in linear elasticity is the possibility of the justification of the convergence of the 3D model towards the 2D model, which is not the case in general within the framework of nonlinear elasticity.

In 1979, Ciarlet and Destyunder [24] performed a mathematical justification of some of these models.

The unilateral contact problem of thin plate with Coulomb friction was treated by Dhia [36] using a penalty method.

In 2002-2003, J. C. Paumier [85, 83, 84], has studied the linearized elastostatic Signorini problem with Coulomb friction of a plate where he has proved that the three dimensional problem converges strongly to the solution of the two-dimensional Signorini problem without friction.

During its research task he raised some open questions:

1. how is it possible to get a lower-dimensional model including friction?
2. is the study of the quasi-static case possible?
3. is it possible to replace a part of the clamped condition by an unilateral one?
4. is this approach valid for shells and rods?
5. what happens in the non-linear case (von Kármán equations)?
6. what happens for other constitutive laws?

It is announced that A. Léger and B. Miara [70] generalized the work of J. C. Paumier to the case of linearized shallow shell but without friction, which gives a partial answer to the fourth open question. The study carried out by J.C. Paumier is the modeling of a Coulomb frictional unilateral contact problem between an elastic thin plate and rigid foundation, within the framework of linear elasticity, by a two-dimensional Signorini model without friction by using the method of convergence. One can find the same results by using the method of the asymptotic expansions, one obtains that with the first significant order the same model obtained by the method of convergence.

The objective of this thesis is to realize some extensions of the study of Paumier [85, 83, 84] in point of view material (nonlinear), boundary conditions (von Karman types), geometry (shallow shells) and to dynamic state. The method used is "convergence method" in case of linearized elastic material and the method of "formal asymptotic expansions" in case of non linear elastic material. This thesis is divided into four chapters. The first chapter is devoted to the recall of some results on asymptotic modeling of plates with some boundary conditions as in the literature of Ciarlet [20, 28, 25], L. Gratie [48] and Chacha, Ghezal and Bensayah [14, 13]. The second chapter presents the description of Signorini problem static and dynamic one, without/with friction in case of linear three dimensional elasticity. The third chapter deals with Signorini problem in case of von Karman conditions with Coulomb friction for plates which is the objective of the paper Chacha and Bensayah[12] after that we pass to the problem with generalized Marguerre-von Karman conditions for shallow shells with Coulomb friction which is the objective of the paper Bensayah, Chacha and Ghezal [5]. In the fourth chapter, we pass to the dynamic case but in linearized elasticity and without friction which is the object of the paper Bensayah, Chacha and Nicase [6]. Finally we end this thesis with a conclusion which contains general results, some perspectives and some open problems.

Chapter 1

Description of some boundary value problems for elastic plates

1.1 Introduction

Throughout this dissertation, we make the following conventions and notations: Greek indexes (except ε) belong to the set $\{1, 2\}$, Latin indexes belong to the set $\{1, 2, 3\}$, the symbols of differentiation $\partial_j^\varepsilon = \partial/\partial x_j^\varepsilon$, $\partial_j = \partial/\partial x_j$, δ_{ij} the Kroneker symbols, and the summation convention with respect to the repeated indexes is systematically used.

Also we make the following geometrical assumptions and mechanical hypothesis. Let $\Omega^\varepsilon = \omega \times]-\varepsilon, +\varepsilon[$, where ε is a small parameter, be an open bounded set from $\mathbb{R}^3$, such that ω is an open subset from $\mathbb{R}^2$ with Lipschitz boundary γ. We denote the lateral boundary of Ω^ε by $\Gamma_0^\varepsilon = \gamma \times [-\varepsilon, \varepsilon]$, the upper and the lower faces are denoted, respectively, by Γ_+^ε and Γ_-^ε. We suppose that Ω^ε is occupied by an elastic, homogeneous, isotropic body. In its natural configuration: a plate of thickness 2ε whose Lamè's constants are denoted $\lambda > 0, \mu > 0$ and assumed to be independent of ε. The plate is supposed to be subjected

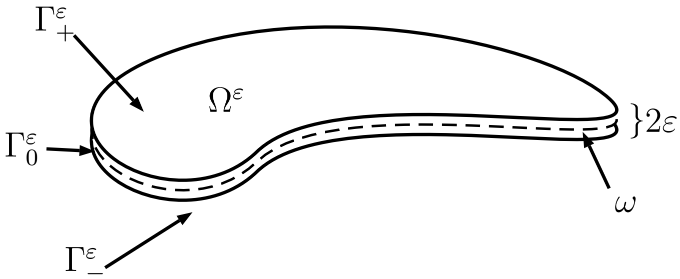

Figure 1.1: Geometric illustration of a thin plate.

to a body force of density f^ε, its lower and upper faces are subjected to a surface force of density g^ε. Note that we keep the same notation of the function to denote its trace.

1.2 Dirichlet-Neumann conditions for linearly elastic thin plates

In this section, we suppose that the material occupying the domain Ω^ε is linearized, the plate is totally clamped by its lateral boundary Γ_0^ε and the system is in static case. This situation is modeled by the following three-dimensional boundary value problem in terms of displacement $u^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^3$ such that $u^\varepsilon(x^\varepsilon) = (u_1^\varepsilon(x^\varepsilon), u_2^\varepsilon(x^\varepsilon), u_3^\varepsilon(x^\varepsilon))$ and

$$(CP^\varepsilon)\begin{cases}\text{Find } u^\varepsilon \text{ such that} & \\ -\partial_j^\varepsilon \sigma_{ij}^\varepsilon = f_i^\varepsilon \text{ in } \Omega^\varepsilon & (1.1)\\ \sigma_{ij}^\varepsilon n_j^\varepsilon = g_i^\varepsilon \text{ on } \Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon & (1.2)\\ u^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon & (1.3)\end{cases}$$

where

$$\sigma_{ij}^\varepsilon(u^\varepsilon) = a_{ijkl} e_{kl}^\varepsilon(u^\varepsilon) \tag{1.4}$$

are the components of the stress tensor, and also represent the constitutive equation of the elastic material,

$$e_{ij}^\varepsilon(u^\varepsilon) = \frac{1}{2}\left(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon\right) \tag{1.5}$$

being the components of the linearized deformation tensor. For homogeneous, isotropic material a_{ijkl} are constants independent of x^ε and verify:

$$a_{ijkl} = \lambda\delta_{ij}\delta_{kh} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk})(\text{ Hook's law }).$$

Hence

$$\sigma_{ij}^\varepsilon(u^\varepsilon) = \lambda e_{pp}^\varepsilon(u^\varepsilon)\delta_{ij} + 2\mu e_{ij}^\varepsilon(u^\varepsilon) \tag{1.6}$$

First, we rewrite the above boundary value problem $(C.\hat{P}^\varepsilon)$ in the weak form, by using Green's formula, we show that any smooth solution of the boundary value problem also satisfies the following variational problem

Proposition 1 *If u^ε is a solution of $(C.P^\varepsilon)$ then u^ε verifies the problem :*

$$(V.P^\varepsilon)\begin{cases} \text{Find } \ u^\varepsilon \in \vec{V}\left(\Omega^\varepsilon\right) \ \text{ such that} \\ a^\varepsilon(u^\varepsilon, v^\varepsilon) = L^\varepsilon(v^\varepsilon) \ \forall v^\varepsilon \in \vec{V}(\Omega^\varepsilon) \end{cases}$$

where

$$\begin{aligned} a^\varepsilon\left(u^\varepsilon, v^\varepsilon\right) &= \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon\left(u^\varepsilon\right) \partial_j^\varepsilon v_i^\varepsilon dx^\varepsilon \\ &= \int_{\Omega^\varepsilon} \left[\lambda e_{ii}^\varepsilon(u^\varepsilon) e_{jj}^\varepsilon\left(v^\varepsilon\right) + 2\mu \ e_{ij}^\varepsilon(u^\varepsilon) e_{ij}^\varepsilon\left(v^\varepsilon\right)\right] dx^\varepsilon \qquad (1.7) \\ L^\varepsilon\left(v^\varepsilon\right) &= \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \int_{\Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon} g_i^\varepsilon v_i^\varepsilon d\Gamma, \qquad (1.8) \end{aligned}$$

and

$$V(\Omega^\varepsilon) = \left\{ v^\varepsilon \in H^1\left(\Omega^\varepsilon\right) / v^\varepsilon = 0 \ on \ \Gamma_0^\varepsilon \ \right\},$$
$$\vec{V}(\Omega^\varepsilon) = V(\Omega^\varepsilon) \times V(\Omega^\varepsilon) \times V(\Omega^\varepsilon),$$

Note that the trace of the function v of $V(\Omega^\varepsilon)$ belongs to the space $H_{00}^{1/2}(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon)$, its dual space is denoted by $H^{-1/2}(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon)$. For more details see [64].

Remark 2 *For u^ε smooth enough, the problems $(C.P^\varepsilon)$ and $(V.P^\varepsilon)$ are equivalent.*

Remark 3 *Under the assumption that $f_i^\varepsilon \in L^2(\Omega^\varepsilon)$, $g_i^\varepsilon \in H^{-1/2}(\Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon)$ the problem $(V.P^\varepsilon)$ admits a unique solution due to Lax-Milgram lemma. This solution realize the minimum of the function*

$$F\left(v^\varepsilon\right) = \frac{1}{2} a^\varepsilon\left(v^\varepsilon, v^\varepsilon\right) - L^\varepsilon\left(v^\varepsilon\right), \forall v^\varepsilon \in \vec{V}\left(\Omega^\varepsilon\right).$$

1.2.1 Asymptotic study

Scalings on data

We follow here the same method as in [19, 20, 21, 28]. Let $\Omega = \omega \times]-1, +1[, \Gamma_\pm = \omega \times \{\pm 1\}, \Gamma_0 = \gamma \times [-1, +1]$. Let $x = (x_i) \in \bar{\Omega}$ denote a generic point in the set $\bar{\Omega}$.

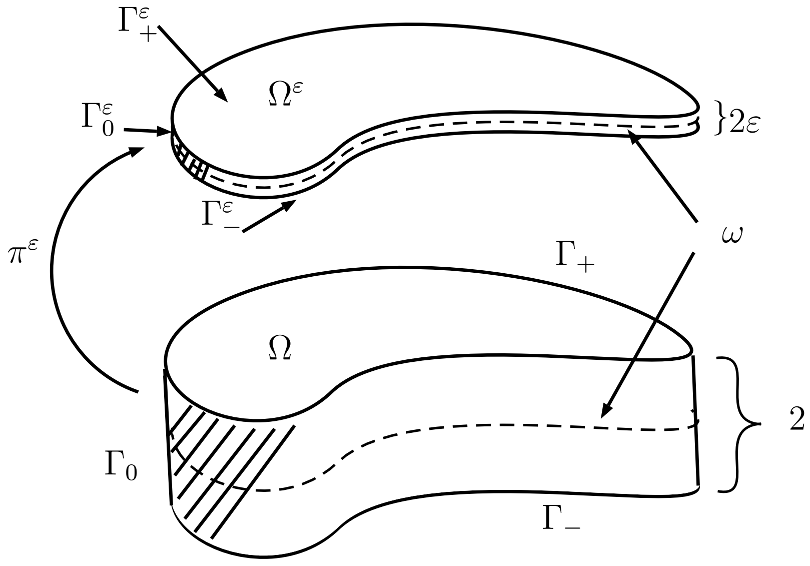

Figure 1.2: Transformation of the domain Ω^ε into a domain independent of ε.

We now transform the domain $\bar{\Omega}^\varepsilon$ having the thickness 2ε into a fixed domain $\bar{\Omega}$ independent of ε via the simple mapping: $\pi^\varepsilon : \bar{\Omega} \to \bar{\Omega}^\varepsilon$ where $x_\alpha^\varepsilon = x_\alpha, x_3^\varepsilon = \varepsilon x_3$ hence

$$\pi^\varepsilon(\Omega) = \Omega^\varepsilon,\ \pi^\varepsilon(\Gamma_\pm) = \Gamma_\pm^\varepsilon,\ \pi^\varepsilon(\Gamma_0) = \Gamma_0^\varepsilon,\ \partial_\alpha^\varepsilon = \partial_\alpha,\ \partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3.$$

See Figure1.2.

We introduce the scaled displacement $u(\varepsilon)$, the scaled test function $v(\varepsilon)$, the scaled stress tensor $\sigma(\varepsilon)$ such that for all $x^\varepsilon = \pi^\varepsilon(x)$

$$\left\{\begin{array}{c} u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x),\ u_3^\varepsilon(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x), \\ v_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 v_\alpha(x),\ v_3^\varepsilon(x^\varepsilon) = \varepsilon v_3(x), \\ \sigma_{\alpha\beta}^\varepsilon(x^\varepsilon) = \varepsilon^2\sigma_{\alpha\beta}(\varepsilon)(x),\ \sigma_{\alpha 3}^\varepsilon(x^\varepsilon) = \varepsilon^3\sigma_{\alpha 3}(\varepsilon)(x),\ \sigma_{33}^\varepsilon(x^\varepsilon) = \varepsilon^4\sigma_{33}(\varepsilon)(x). \end{array}\right.$$

We also introduce the scaling of the forces: $f_\alpha^\varepsilon = \varepsilon^2 f_\alpha,\ g_\alpha^\varepsilon = \varepsilon^3 g_\alpha,\ f_3^\varepsilon = \varepsilon^3 f_3,\ g_3^\varepsilon = \varepsilon^4 g_3$ where f_i and g_i are supposed independent of ε. Therefore we denote

$$V(\Omega) = \{v \in H^1(\Omega), v = 0 \ \ on \ \ \Gamma_0\}, \tag{1.9}$$

$$\vec{V}(\Omega) = V(\Omega) \times V(\Omega) \times V(\Omega) \tag{1.10}$$

The scaled variational problem

Using the upper assumptions and notations lead to the following result

Proposition 4 *If u^ε is solution of the problem $(V.P^\varepsilon)$ then $u(\varepsilon)$ solves the problem*

$$(SVP(\varepsilon)) \left\{\begin{array}{l} Find \ \ u(\varepsilon) \in \vec{V}(\Omega) \ such \ that \\ a^\varepsilon\left(u(\varepsilon), v\right) = \ L\left(v\right), \forall v \in \vec{V}(\Omega), \end{array}\right.$$

where

$$a^{\varepsilon}\left(u(\varepsilon), v\right) = \int_{\Omega} \sigma_{ij}(\varepsilon)\partial_j v_i dx, \tag{1.11}$$

$$L(v) = \int_{\Omega} f_i v_i dx + \int_{\Gamma_- \cup \Gamma_+} g_i v_i d\Gamma, \tag{1.12}$$

such that

$$\begin{cases} \sigma_{\alpha\beta}(\varepsilon) = \lambda e_{\gamma\gamma}(u\left(\varepsilon\right))\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(u\left(\varepsilon\right)) + \varepsilon^{-2}\lambda e_{33}(u\left(\varepsilon\right))\delta_{\alpha\beta} \\ \sigma_{\alpha 3}(\varepsilon) = \varepsilon^{-2} 2\mu e_{\alpha 3}(u\left(\varepsilon\right)) \\ \sigma_{33}(\varepsilon) = \varepsilon^{-4}\ \left(\lambda + 2\mu\right) e_{33}(u\left(\varepsilon\right)) + \varepsilon^{-2}\lambda e_{\gamma\gamma}(u\left(\varepsilon\right)) \end{cases} \tag{1.13}$$

This problem has a unique solution $u(\varepsilon)$ under the assumption that $f_i \in L^2(\Omega)$, $g_i \in H^{-1/2}(\Gamma_- \cup \Gamma_+)$, this solution realizes the minimum of the function

$$F(v) = \frac{1}{2} a^{\varepsilon}(v, v) - L(v), \forall v \in \vec{V}(\Omega).$$

Convergence theorem

First, we introduce the space of Kirchhoff-Love displacements defined by V_{KL} such that

$$V_{KL}\left(\Omega\right) = \left\{ v = (v_i) \in \vec{V}\left(\Omega\right)\ /\ \partial_i v_3 + \partial_3 v_i = 0 \right\} \tag{1.14}$$

Also it can be defined by

$$V_{KL}\left(\Omega\right) = \left\{ \begin{array}{c} v = (v_i) \in \left(\mathrm{H}^1\left(\Omega\right)\right)^3\ /\ v_\alpha = \eta_\alpha - x_3 \partial_\alpha \eta_3, v_3 = \eta_3 \text{ such that} \\ \eta_\alpha \in H_0^1\left(\omega\right), \eta_3 \in H_0^2\left(\omega\right) \end{array} \right\}$$

In addition this space is isomorph to the space

$$\vec{V}\left(\omega\right) = H_0^1\left(\omega\right) \times H_0^1\left(\omega\right) \times H_0^2\left(\omega\right).$$

Theorem 5 *Let $u(\varepsilon)$ be a solution of the problem $(SVP(\varepsilon))$, then*

$$u(\varepsilon) \to u(0), \quad in\ V_{KL}\left(\Omega\right)$$

where $u(0)$ satisfies the following problem:

$$(VP_{KL}(0)) \begin{cases} Find\ u(0) \in V_{KL}\left(\Omega\right),\ such\ that, \\ \displaystyle\int_{\Omega} \sigma_{\alpha\beta}(0)\partial_\beta v_\alpha dx = L\left(v\right), \forall v \in V_{KL}\left(\Omega\right), \end{cases}$$

where

$$\sigma_{\alpha\beta}(0) = \lambda^{\star} e_{\sigma\sigma}(u(0))\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(u(0)), \lambda^{\star} = \frac{2\lambda\mu}{\lambda + 2\mu}. \tag{1.15}$$

Proof. See [20, Theorem 1.4-1, p.34]. ■

Two dimensional reduced model

Remark 6 *From the definition of the space $V_{KL}(\Omega)$, look for $u(0)$ in $V_{KL}(\Omega)$ is similar to look for (ξ_1, ξ_2, ξ_3) in $\vec{V}(\omega)$ therefore we can reduce our three-dimensional problem to a two-dimensional problem.*

Theorem 7 *Let $u(0)$ be such that $u_\alpha(0) = \xi_\alpha - x_3\partial_\alpha\xi_3, u_3(0) = \xi_3$, where ξ_α, ξ_3 are smooth enough. If $u(0)$ is a solution of the problem $(VP_{KL}(0))$ then ξ_α, ξ_3 verify the following bi-dimensional problem $(BP(0))$*

$$\begin{cases} \text{Find } \xi_\alpha \in H_0^1(\omega), \xi_3 \in H_0^2(\omega) \text{ such that} \\ k\Delta^2\xi_3 = h_3^0 + h_1^1 + h_2^1 \\ -\partial_\beta n_{\alpha\beta} = h_\alpha^0 \end{cases}$$

where

$$k = \frac{8}{3}\mu\frac{\lambda+\mu}{\lambda+2\mu},\ h_i^0 = \int_{-1}^{+1} f_i dx_3 + g_i^- + g_i^+,\ h_i^1 = \int_{-1}^{+1} x_3\partial_i f_i dx_3 - \partial_i g_i^- - \partial_i g_i^+,$$
$$g_i^\pm = g_i(x_1, x_2, \pm 1), n_{\alpha\beta} = \frac{4\lambda\mu}{\lambda+2\mu} e_{\gamma\gamma}(\xi)\delta_{\alpha\beta} + 4\mu e_{\alpha\beta}(\xi)$$

Proof. Special case of [20, Theorem 1.7-2, p.66]. ■

1.3 Time dependent problem

1.3.1 Strong and weak Formulation of the problem

Using the assumptions in the Section 1.1 one can state the following classical elastodynamic problem

$$\begin{cases} \text{Find } u^\varepsilon := u^\varepsilon(x^\varepsilon, t) \text{ such that for all } t \geq 0 : \\ \rho^\varepsilon \dfrac{\partial^2 u_i^\varepsilon}{\partial t^2} - \partial_j^\varepsilon \sigma_{ij}^\varepsilon = f_i^\varepsilon \ in \ \Omega^\varepsilon \times]0, +\infty[& (1.16) \\ \sigma_{ij}^\varepsilon n_j^\varepsilon = g_i^\varepsilon \ \ on \, \Gamma_\pm^\varepsilon \times]0, +\infty[& (1.17) \\ u^\varepsilon = 0 \ \ on \ \Gamma_0^\varepsilon \times]0, +\infty[\,, & (1.18) \\ u^\varepsilon(.,0) = p^\varepsilon, \dot{u}^\varepsilon(.,0) = q^\varepsilon, & (1.19) \end{cases}$$

where $\sigma_{ij}^\varepsilon (u^\varepsilon)$ are defined by (1.4).

We rewrite the above boundary value problem in the following weak form, by using Green's formula, we show that any smooth solution of the boundary value problem also satisfies the following variational problem :

$$(VP^\varepsilon) \begin{cases} \text{Find } \ u^\varepsilon(t) \in \vec{V}(\Omega^\varepsilon), \ t \geq 0 \text{ such that} \\ \dfrac{\partial^2}{\partial t^2} \rho^\varepsilon \displaystyle\int_{\Omega^\varepsilon} u_i^\varepsilon v_i^\varepsilon dx^\varepsilon + a^\varepsilon(u^\varepsilon, v^\varepsilon) = L^\varepsilon(v^\varepsilon), \ \forall v^\varepsilon \in \vec{V}\left(\Omega^\varepsilon\right), \ t > 0 & (1.20) \\ u^\varepsilon(.,0) = p^\varepsilon, \dot{u}^\varepsilon(.,0) = q^\varepsilon, & (1.21) \end{cases}$$

where $a^\varepsilon \left(u^\varepsilon, v^\varepsilon\right)$, $L^\varepsilon \left(v^\varepsilon\right)$ and $\vec{V}(\Omega^\varepsilon)$ are defined respectively by (1.7), (1.8) and (1.10). Note that, under restrictive conditions on applied forces, body forces and initial data, this problem has a unique solution, see [55]. We can also consult [20, Section 1.14, p.113] and references therein.

1.3.2 Asymptotic study

In this subsection, we keep the same transformation and scalings on data and on unknowns as in the paragraph 1.2.1 and we add the assumption that there exist p and q independent of ε such that:

$$p_\alpha^\varepsilon = \varepsilon^2 p_\alpha, p_3^\varepsilon = \varepsilon p_3, q_\alpha^\varepsilon = \varepsilon^2 q_\alpha, q_3^\varepsilon = \varepsilon q_3$$

Inserting the upper scalings in the variational problem leads to the following proposition

Proposition 8 *The variational dynamic problem* (VP^ε) *is equivalent to the following scaled variational dynamic problem* $(SVP\,(\varepsilon))$*: Find* $u(\varepsilon)(t) \in \vec{V}\,(\Omega)\,, t \geq 0$ *such that*

$$\frac{\partial^2}{\partial t^2}\rho \int_\Omega u_3\,(\varepsilon)\,v_3 dx + \varepsilon^2 \frac{\partial^2}{\partial t^2}\rho \int_\Omega u_\alpha\,(\varepsilon)\,v_\alpha dx + \int_\Omega \sigma_{ij}\,(\varepsilon)\,\partial_j v_i dx = L\,(v)\,, \forall v \in \vec{V}\,(\Omega)\,, t > 0$$

$$u(\varepsilon)(.,0) = p, \dot{u}(\varepsilon)(.,0) = q$$

where

$$e_{ij}(u(\varepsilon)) = \frac{1}{2}\left(\partial_i u_j(\varepsilon) + \partial_j u_i(\varepsilon)\right)$$
$$L\left(v\right) = \int_{\Omega} f_i v_i dx + \int_{\Gamma_- \cup \Gamma_+} g_i v_i d\Gamma$$

1.3.3 Two-dimensional problem

Theorem 9 *Let $u(\varepsilon)$ be a solution of the dynamic problem $(SVP(\varepsilon))$, then*

$$u(\varepsilon) \to u(0) \text{ in } L^2(0,T,\, V_{KL}\left(\Omega\right)) \text{ as } \varepsilon \to 0,$$

where $u(0)$ satisfies the following dynamic problem:

$$(VP_{KL}(0))\begin{cases} Find\ u(0) \in V_{KL}\left(\Omega\right),\ such\ that, \\ \dfrac{\partial^2}{\partial t^2}\rho \displaystyle\int_{\Omega} u_3(0) v_3 dx + \int_{\Omega} \sigma_{\alpha\beta}(0)\partial_\beta v_\alpha dx = L\left(v\right), \forall v \in V_{KL}\left(\Omega\right), \\ u(0)(.,0) = p, \dot{u}(0)(.,0) = q \end{cases}$$

where $\sigma_{\alpha\beta}(0)$, $L\left(v\right)$ and $V_{KL}\left(\Omega\right)$ are defined respectively by (1.15), (1.12) and (1.14).

Proof. See [20, Thm. 1.14-2, p.115]. ■

In the next proposition, we re-write the problem $(VP_{KL}(0))$ in terms of ξ_α and ξ_3. Hence we get a two dimensional problem $(P^b\left(0\right))$ whose solutions are ξ_α and ξ_3. The vector field (ξ_i) represents the (scaled) displacement of the middle surface ω of the plate.

Proposition 10 *If $u(0)$ is a solution of $(VP_{KL}(0))$ such that $u_\alpha(0) = \xi_\alpha - x_3\partial_\alpha\xi_3$ and $u_3(0) = \xi_3$, with ξ_α, ξ_3 sufficiently smooth. Then ξ_α, ξ_3 verify with σ_{33}^0, at least formally, the two-dimensional boundary value problem :*

$$(P^b(0))\begin{cases} Find\ \ \xi_\alpha \in H^1\left(\omega\right), \xi_3 \in H_0^2\left(\omega\right),\ for\ \ a.e\ t \geq 0\ \ such\ that \\ 2\dfrac{\partial^2}{\partial t^2}\rho\xi_3 + \ k\Delta^2\xi_3 = h_1^1 + h_2^1 + h_3^0\ on\ \omega \times]0, +\infty[& (1.22) \\ -\partial_\beta n_{\alpha\beta} = h_\alpha^0 \quad on\ \ \omega \times]0, +\infty[& (1.23) \\ \xi_i(.,0) = \varphi_i, \dfrac{\partial \xi_i}{\partial t}(.,0) = \psi_i \end{cases}$$

where k, h_i^0, h_i^1, $g_i^\pm$ and $n_{\alpha\beta}$ are defined in the Theorem 7.

Proof. See [20, Thm. 1.14-3, p.117]. ■

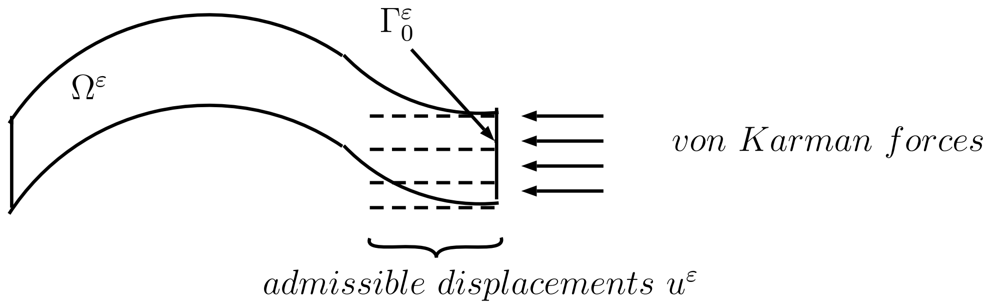

Figure 1.3: Description of von Karman forces and admissible displacements of the points of the boundary Γ_0^ε. See [32] or [19].

1.4 The von Karman equations for elastic plates

In this section, we replace the linear elastic body occupying the domain by a nonlinear one. And we apply on the lateral face forces with horizontal direction as described in the following paragraph.

1.4.1 Setting of the problem

Let Ω^ε defined as in the Section 1.1. We suppose that Ω^ε is occupied by a nonlinear, elastic, homogeneous, isotropic body. In its natural configuration: a plate of thickness 2ε whose Lamè's constants are denoted $\lambda > 0, \mu > 0$ and assumed to be independent of ε. The plate is supposed to be subjected to a body force of density $f^\varepsilon \in (L^2(\Omega^\varepsilon))^3$, its lower face subjected to a surface force of density $g^\varepsilon \in \left(L^2(\Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon)\right)^3$ and subjected, on Γ_0^ε to applied surface forces of "von Kármán's type" which are horizontal, and only their resultant $\left(\widetilde{F}_1^\varepsilon, \widetilde{F}_2^\varepsilon\right) \in (L^2(\gamma))^2$ after integration across the thickness is given along the boundary γ. Therefore, the displacements u^ε derived from this situation verify u_α^ε independent of x_3^ε and $u_3^\varepsilon = 0$ on Γ_0^ε which mean that the only horizontal displacements of equal direction and magnitude are allowed along each vertical segment of the lateral face Γ_0^ε. See Figure1.3. For more details on the von Kármán equations we return to [32] and [19].

Our aim is to find the asymptotic behavior of the equilibrium state of the plate Ω^ε

which is characterized by a displacement vector u^ε solution of the classical problem:

$$(CP^\varepsilon)\begin{cases} -\partial_j^\varepsilon \hat{\sigma}_{ij}^\varepsilon = f_i^\varepsilon \text{ in } \Omega^\varepsilon \\ u_\alpha^\varepsilon \text{ independent of } x_3^\varepsilon \text{ and } u_3^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon \\ \frac{1}{2\varepsilon}\int\limits_{-\varepsilon}^{+\varepsilon} \hat{\sigma}_{\alpha\beta}^\varepsilon.\nu_\beta^\varepsilon dx_3^\varepsilon = \widetilde{F}_\alpha^\varepsilon \text{ on } \gamma \\ \hat{\sigma}_{ij}^\varepsilon n_j^\varepsilon = g_i^\varepsilon \text{ on } \Gamma_-^\varepsilon \end{cases}$$

where: $\hat{\sigma}_{ij}^\varepsilon = \sigma_{ij}^\varepsilon + \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon$, $\sigma_{ij}^\varepsilon = \lambda E_{pp}^\varepsilon(u^\varepsilon)\delta_{ij} + 2\mu E_{ij}^\varepsilon(u^\varepsilon)$ the components of the stress tensor, $E_{ij}^\varepsilon(u^\varepsilon) = \frac{1}{2}(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon + \partial_i^\varepsilon u_k^\varepsilon \partial_j^\varepsilon u_k^\varepsilon)$ the components of the nonlinear strain tensor, $n^\varepsilon = (n_i^\varepsilon)$ is the unit outer normal vector along the boundary of the plate Ω^ε, $\nu^\varepsilon = (\nu_\alpha^\varepsilon)$ is the unit outer normal vector along the boundary of the set ω. To give a weak formulation of our problem we introduce some notations. Let

$$V(\Omega^\varepsilon) = \left\{v \in W^{1,4}(\Omega^\varepsilon)/\ v \text{ independent of } x_3^\varepsilon \text{ on } \Gamma_0^\varepsilon\right\},$$
$$V_0(\Omega^\varepsilon) = \left\{v \in W^{1,4}(\Omega^\varepsilon)/\ v = 0 \text{ on } \Gamma_0^\varepsilon\right\}$$
$$\vec{V}(\Omega^\varepsilon) = V(\Omega^\varepsilon) \times V(\Omega^\varepsilon) \times V_0(\Omega^\varepsilon)$$

Multiplying the system of equilibrium equations in (CP^ε) by functions v_i^ε and integrating over the set Ω^ε, using the Green formulas and the boundary conditions we obtain:

The variational formulation of the classical problem (CP^ε) is :

$$(VP^\varepsilon)\begin{cases} \text{Find } u^\varepsilon \in \vec{V}(\Omega^\varepsilon) \quad \text{such that:} \\ \displaystyle\int_{\Omega^\varepsilon} \hat{\sigma}_{ij}^\varepsilon \partial_j^\varepsilon v_i^\varepsilon dx^\varepsilon = L^\varepsilon(v^\varepsilon) + 2\varepsilon \int_\gamma \tilde{F}_\alpha^\varepsilon \tilde{v}_\alpha^\varepsilon d\gamma,\ \forall\ v^\varepsilon \in \vec{V}(\Omega^\varepsilon) \end{cases},$$

where: $L^\varepsilon(v^\varepsilon) = \displaystyle\int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \int_{\Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon} g_i^\varepsilon v_i^\varepsilon d\Gamma^\varepsilon$.

1.4.2 Asymptotic study

The scaled problem

Using the same transformation and scalings on data and on unknowns as in the paragraph 1.2.1 and we add the assumption on the von Kàrmàn forces. Then the scaled displacement $u(\varepsilon)$, the scaled test function $v(\varepsilon)$, the scaled stress tensor $\sigma(\varepsilon)$ satisfy:

$$\begin{cases} u_\alpha^\varepsilon \circ \pi^\varepsilon = \varepsilon^2 u_\alpha(\varepsilon),\quad u_3^\varepsilon \circ \pi^\varepsilon = \varepsilon u_3(\varepsilon), v_\alpha^\varepsilon \circ \pi^\varepsilon = \varepsilon^2 v_\alpha(\varepsilon),\ v_3^\varepsilon \circ \pi^\varepsilon = \varepsilon v_3(\varepsilon) \\ \sigma_{\alpha\beta}^\varepsilon \circ \pi^\varepsilon = \varepsilon^2 \sigma_{\alpha\beta}(\varepsilon),\ \sigma_{\alpha3}^\varepsilon \circ \pi^\varepsilon = \varepsilon^3 \sigma_{\alpha3}(\varepsilon),\ \sigma_{33}^\varepsilon \circ \pi^\varepsilon = \varepsilon^4 \sigma_{33}(\varepsilon) \end{cases} \tag{1.24}$$

We also introduce the scaling of the forces:

$$f_\alpha^\varepsilon \circ \pi^\varepsilon = \varepsilon^2 f_\alpha,\ f_3^\varepsilon \circ \pi^\varepsilon = \varepsilon^3 f_3, g_\alpha^\varepsilon \circ \pi^\varepsilon = \varepsilon^3 g_\alpha, g_3^\varepsilon \circ \pi^\varepsilon = \varepsilon^4 g_3, \tilde{F}_\alpha^\varepsilon = \varepsilon^2 \tilde{F}_\alpha(\varepsilon) \tag{1.25}$$

Then we obtain: $L^{\varepsilon}\left(v^{\varepsilon}\right)=\varepsilon^{5}L\left(v\right)$ with $L\left(v\right)=\displaystyle\int_{\Omega}f_iv_idx+\int_{\Gamma_-\cup\Gamma_+}g_iv_id\Gamma$ (1.26)

Therefore we denote by:

$$V\left(\Omega\right)=\left\{v\in W^{1,4}(\Omega),\ v \text{ independent of } x_3 \text{ on } \Gamma_0\right\}, \tag{1.27}$$

$$V_0(\Omega)=\left\{v\in W^{1,4}(\Omega),\ v=0 \text{ on } \Gamma_0\right\}, \tag{1.28}$$

$$\vec{V}(\Omega)=V(\Omega)\times V(\Omega)\times V_0(\Omega), \tag{1.29}$$

Using the upper assumptions and notations leads to the following proposition:

Proposition 11 *The variational problem* (VP^{ε}) *is equivalent to the following scaled variational problem* $(SVP\left(\varepsilon\right))$*:*

$$\begin{cases} \textit{Find } u\left(\varepsilon\right)\in\ \vec{V}\left(\Omega\right) \textit{ such that:} \\ \displaystyle\int_{\Omega}\sigma_{ij}\left(\varepsilon\right)\partial_jv_idx+\int_{\Omega}\sigma_{ij}\left(\varepsilon\right)\partial_iu_3\left(\varepsilon\right)\partial_jv_3dx+\varepsilon^2\int_{\Omega}\sigma_{ij}\left(\varepsilon\right)\partial_iu_\alpha\left(\varepsilon\right)\partial_jv_\alpha dx \\ \displaystyle=L\left(v\right)+\int_{\gamma}\tilde{F}_\alpha(\int_{-1}^{+1}v_\alpha dx_3)d\gamma,\forall v\in\vec{V}\left(\Omega\right) \end{cases}$$

1.4.3 The two-dimensional problem

We assume that the scaled displacement-stress $\left(u\left(\varepsilon\right),\sigma\left(\varepsilon\right)\right)$ admit a formal asymptotic expansion of the form:

$$\left(u\left(\varepsilon\right),\sigma\left(\varepsilon\right)\right)=\ (u^0,\sigma^0)+\ \varepsilon(u^1,\sigma^1)+\ \varepsilon^2(u^2,\sigma^2)+\ldots \tag{1.30}$$

We introduce the Kirchhoff-Love space of admissible displacements

$$V_{KL}\left(\Omega\right)=\left\{\begin{array}{c} v=(v_i)\,,\ v_\alpha=\eta_\alpha-x_3\partial_\alpha\eta_3,\ v_3=\eta_3 \text{ such that:} \\ \eta_\alpha\in H^1\left(\omega\right),\ \eta_3\in H_0^2\left(\omega\right)\end{array}\right\}$$

and the space

$$\mathbb{L}_s^2(\Omega)=\left\{\tau=(\tau_{ij})\in L^2(\Omega);\ \tau_{ij}=\tau_{ji}\right\}.$$

Substituting expansion (1.30) into the scaled variational problem $(SVP(\varepsilon))$, we obtain :

Proposition 12 *Assume that* $\partial_3u_3^0\in C^0\left(\bar{\Omega}\right)$ *then the leading term* (u^0,σ^0) *of the expansion (1.30) is solution of the problem* $(SVP(0))$*:*

$$\begin{cases} \textit{Find } (u^0,\sigma^0)\in\ V_{KL}\left(\Omega\right)\times\mathbb{L}_s^2(\Omega) \textit{ such that} \\ \displaystyle\int_{\Omega}\sigma_{\alpha\beta}^0\partial_\beta v_\alpha dx+\int_{\Omega}\sigma_{\alpha\beta}^0\partial_\alpha u_3^0\partial_\beta v_3dx=L\left(v\right)+\int_{\gamma}\tilde{F}_\alpha(\int_{-1}^{+1}v_\alpha dx_3)d\gamma,\forall v\in V_{KL}\left(\Omega\right), \end{cases}$$

where $\sigma_{\alpha\beta}^0=\frac{2\lambda\mu}{\lambda+2\mu}E_{\gamma\gamma}^0(u^0)\delta_{\alpha\beta}+2\mu E_{\alpha\beta}^0(u^0)$ *and* $E_{\alpha\beta}^0(u^0)=\frac{1}{2}(\partial_iu_j^0+\partial_ju_i^0+\partial_iu_3^0\partial_ju_3^0)$.

Next, we project the previous problem on the space $V(\omega)$.

Proposition 13 *Let $u^0 \in V_{KL}(\Omega)$ be such that $u^0_\alpha = \xi_\alpha - x_3\partial_\alpha\xi_3$ and $u^0_3 = \xi_3$, where ξ_α, ξ_3 sufficiently regulars. Then the problem $(SVP(0))$ can be formulated in the classical form as two-dimensional problem*:

$$(P^b(0)) \begin{cases} \text{Find } \xi \in (H^1_0(\omega))^2 \times H^2_0(\omega) \text{ such that} \\ k\Delta^2\xi_3 - \partial_\beta(n_{\alpha\beta}\partial_\alpha\xi_3) = h^1_1 + h^1_2 + h^0_3 \text{ on } \omega \\ -\partial_\beta n_{\alpha\beta} = h^0_\alpha \quad \text{on } \omega \\ n_{\alpha\beta}\nu_\beta = 2\tilde{F}_\alpha \quad \text{on } \gamma \end{cases}$$

where

$$n_{\alpha\beta} = \frac{4\lambda\mu}{\lambda + 2\mu}E^0_{\gamma\gamma}(\xi)\delta_{\alpha\beta} + 4\mu E^0_{\alpha\beta}(\xi), k = \frac{8}{3}\mu\frac{\lambda+\mu}{\lambda+2\mu},$$

$$h^0_i = \int_{-1}^{1} f_i dx_3 + g^-_i + g^+_i, h^1_i = \int_{-1}^{1} x_3\partial_i f_i dx_3 - \partial_i g^-_i - \partial_i g^+_i, g^\pm_i = g_i(x_1, x_2, \pm 1).$$

Proof. See [20, Thm. 5.4-2, p.384]. ■

We deduce that the displacement u^0 is characterized by a two dimensional problem. Then, our three-dimensional offers toward a two-dimensional problem.

1.5 The Generalized Marguerre-von Karman problem for shallow shells

The aim of this section is to extend the study of the von Karman problem for plates treated in the previous section to shallow shells subjected to von Karman forces type applied on only a part of the lateral boundary, this condition is called "Generalized Marguerre-von Karman condition".

Using the formal asymptotic expansion method, Ciarlet and Paumier [28] justified the Marguerre-von Kármán equations for shallow shells. Until 2001, Ciarlet and Gratie [25] generalized these equations for plates, after that Ciarlet, Gratie and Sabu [26] established an existence theorem for them. Next, in 2002 Gratie [48] formally extended in the same time the works [28] and [25] to generalized Marguerre-von Kármán equations for shallow shells, after that, Ciarlet and Gratie [31] gave the existence of solutions to this problem. For more details, one can consult [30].

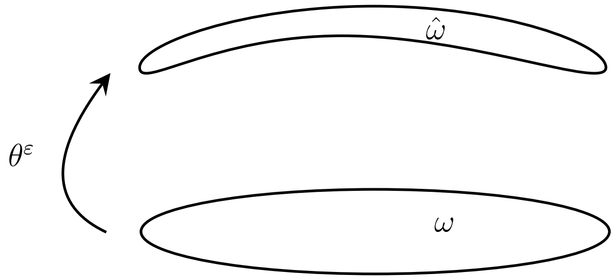

Figure 1.4: Illustration of the mapping θ^ε.

Setting of the problem

Let Ω^ε defined as in the Section 1.1. Let ω be a connected bounded open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, ω being locally on a single side of γ, we assume $0 \in \gamma$ and we denote by $\gamma(y)$ the arc joining 0 to the point $y \in \gamma$. Let γ_1 be a relatively open subset of γ such that $length\gamma_1 > 0$ and $length\gamma_2 > 0$, where $\gamma_2 = \gamma\backslash\gamma_1$. The unit outer normal vector (ν_α) and the unit tangent vector (τ_α) along the boundary γ are related by $\tau_1 = -\nu_2$ and $\tau_2 = \nu_1$. The outer normal and tangential derivative operators $\nu_\alpha\partial_\alpha$ and $\tau_\alpha\partial_\alpha$ along γ are denoted respectively by ∂_ν and ∂_τ.

For any $\varepsilon > 0$, let $\Omega^\varepsilon = \omega\times] - \varepsilon, \varepsilon[$, $\Gamma^\varepsilon_\pm = \omega \times \{\pm\varepsilon\}$ and $\theta^\varepsilon : \bar{\omega} \to \mathbb{R}$ is a function of class C^3 that satisfies $\theta^\varepsilon = \partial_\nu\theta^\varepsilon = 0$ on γ_1. See Figure1.4.

We define the mapping

$$\mathbf{\Theta}^\varepsilon : \bar{\Omega}^\varepsilon \to \mathbb{R}^3 : \mathbf{\Theta}^\varepsilon(x^\varepsilon) = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) + x_3^\varepsilon \mathbf{a}_3^\varepsilon(x_1, x_2),$$

for all $x^\varepsilon = (x_1, x_2, x_3^\varepsilon) \in \bar{\Omega}^\varepsilon$, where $\mathbf{a}_3^\varepsilon$ is a continuously varying unit vector normal to the middle surface $\mathbf{\Theta}^\varepsilon(\bar{\omega})$. For small enough ε, the mapping $\mathbf{\Theta}^\varepsilon : \bar{\Omega}^\varepsilon \to \mathbf{\Theta}^\varepsilon(\bar{\Omega}^\varepsilon)$ is a C^1- diffeomorphism (see [28]), and we suppose also that Θ^ε is orientation preserving i.e $\det\nabla^\varepsilon\Theta^\varepsilon(x^\varepsilon) > 0, \ \forall x^\varepsilon \in \bar{\Omega}^\varepsilon$. Let $\hat{\Omega}^\varepsilon = \mathbf{\Theta}^\varepsilon(\Omega^\varepsilon)$, $\hat{\gamma}_1^\varepsilon = \mathbf{\Theta}^\varepsilon(\gamma_1)$, $\hat{\Gamma}^\varepsilon_\pm = \mathbf{\Theta}^\varepsilon(\Gamma^\varepsilon_\pm)$. See Figure1.6. We denote by $\hat{x}^\varepsilon = \mathbf{\Theta}^\varepsilon(x^\varepsilon)$ a generic point in $\bar{\hat{\Omega}}^\varepsilon$, $(\hat{n}_i^\varepsilon)$ is the unit outer normal vector along the boundary of the set $\hat{\Omega}^\varepsilon$.

Following the definition proposed by Ciarlet and Paumier [28], we say that a shell is shallow if there exists a function $\theta \in C^3(\bar{\omega})$ independent of ε such that $\theta^\varepsilon(x_1, x_2) = \varepsilon\theta(x_1, x_2)$, for all $(x_1, x_2) \in \bar{\omega}$. See Figure1.5.

Consider a nonlinearly elastic shallow shell occupying in its reference configuration the set $\bar{\hat{\Omega}}^\varepsilon$, with thickness 2ε, its constituting material is a St Venant-Kirchhoff material with Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$.

The shell is subjected to vertical body forces of density $\hat{f}^\varepsilon = (0, 0, \hat{f}_3^\varepsilon)$ in its interior $\hat{\Omega}^\varepsilon$ and to vertical surface forces of density $\hat{g}^\varepsilon = (0, 0, \hat{g}_3^\varepsilon)$ on its upper and lower faces $\hat{\Gamma}^\varepsilon_+$ and $\hat{\Gamma}^\varepsilon_-$. On the portion $\mathbf{\Theta}^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ of its lateral face, the shell is subjected to horizontal forces of von Kàrmàn type $(\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0)$, of the form introduced by Ciarlet [29],

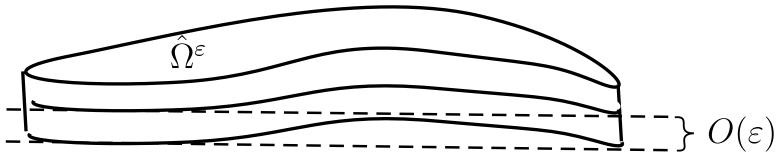

Figure 1.5: Graphic illustration of a shallow shell

the remaining portion $\mathbf{\Theta}^\varepsilon(\gamma_2 \times [-\varepsilon, \varepsilon])$ being free.

The problem consists of finding the displacement $\hat{u}^\varepsilon$ which satisfies the problem:

$$(C\hat{P}^\varepsilon) \begin{cases} -\hat{\partial}^\varepsilon_j(\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k\hat{u}^\varepsilon_i) = \hat{f}^\varepsilon_i \text{ in } \hat{\Omega}^\varepsilon \\ (\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k\hat{u}^\varepsilon_i)\hat{n}^\varepsilon_j \circ \Theta^\varepsilon = 0 \text{ on } \gamma_2 \times [-\varepsilon, \varepsilon] \\ (\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k\hat{u}^\varepsilon_i)\hat{n}^\varepsilon_j \circ \Theta^\varepsilon = \hat{g}^\varepsilon_i \circ \Theta^\varepsilon \text{ on } \Gamma^\varepsilon_- \\ \hat{u}^\varepsilon_\alpha \text{ independent of } \hat{x}^\varepsilon_3 \text{ and } \hat{u}^\varepsilon_3 = 0 \quad \text{on } \Theta^\varepsilon\left(\gamma_1 \times [-\varepsilon, \varepsilon]\right) \quad , \\ \dfrac{1}{2\varepsilon}\displaystyle\int_{-\varepsilon}^{\varepsilon} \left\{ \left(\hat{\sigma}^\varepsilon_{\alpha\beta} + \hat{\sigma}^\varepsilon_{k\beta}\hat{\partial}^\varepsilon_k\hat{u}^\varepsilon_\alpha \right) \circ \Theta^\varepsilon \right\} \nu_\beta dx^\varepsilon_3 = \hat{h}^\varepsilon_\alpha \circ \Theta^\varepsilon \text{ on } \gamma_1 \end{cases}$$

where
$\hat{\sigma}^\varepsilon_{ij} = \lambda \hat{E}^\varepsilon_{pp}\left(\hat{u}^\varepsilon\right)\delta_{ij} + 2\mu \hat{E}^\varepsilon_{ij}\left(\hat{u}^\varepsilon\right)$ (the components of stress tensor)
$\hat{E}^\varepsilon_{ij}\left(\hat{u}^\varepsilon\right) = \dfrac{1}{2}(\hat{\partial}^\varepsilon_i\hat{u}^\varepsilon_j + \hat{\partial}^\varepsilon_j\hat{u}^\varepsilon_i + \hat{\partial}^\varepsilon_i\hat{u}^\varepsilon_k\hat{\partial}^\varepsilon_j\hat{u}^\varepsilon_k)$ (the components of nonlinear strain tensor)
We consider the following functional spaces

$$\begin{aligned} V(\hat{\Omega}^\varepsilon) &= \left\{ \hat{v} \in W^{1,4}(\hat{\Omega}^\varepsilon)/\hat{v} \textit{ independent of } \hat{x}^\varepsilon_3 \textit{ on } \Theta^\varepsilon(\gamma_1 \times]-\varepsilon, +\varepsilon[) \right\} \\ V_0(\hat{\Omega}^\varepsilon) &= \left\{ \hat{v} \in W^{1,4}(\hat{\Omega}^\varepsilon)/\hat{v} = 0 \textit{ on } \Theta^\varepsilon(\gamma_1 \times]-\varepsilon, +\varepsilon[) \right\} \\ \vec{V}(\hat{\Omega}^\varepsilon) &= V(\hat{\Omega}^\varepsilon) \times V(\hat{\Omega}^\varepsilon) \times V_0(\hat{\Omega}^\varepsilon), \end{aligned}$$

Multiplying the system of equilibrium equations in $(C\hat{P}^\varepsilon)$ by functions $\hat{v}^\varepsilon_i$ and integrating over the set $\hat{\Omega}^\varepsilon$, after that using the Green formula and the boundary conditions we obtain the following variational formulation of the problem $(C\hat{P}^\varepsilon)$:

$$(V\hat{P}^\varepsilon) \begin{cases} \textit{Find } \hat{u}^\varepsilon \in \vec{V}(\hat{\Omega}^\varepsilon) \textit{ such that} \\ \hat{A}^\varepsilon(\hat{u}^\varepsilon, \hat{v}^\varepsilon) = \hat{L}^\varepsilon(\hat{v}^\varepsilon) + \displaystyle\int_{\hat{\gamma}_1} (\int_{-\varepsilon}^{\varepsilon} (\hat{v}^\varepsilon_\alpha \circ \Theta^\varepsilon) dx^\varepsilon_3) \hat{h}^\varepsilon_\alpha d\hat{\gamma} \forall\ \hat{v}^\varepsilon \in \vec{V}(\hat{\Omega}^\varepsilon) \end{cases}$$

where

$$\hat{A}^\varepsilon(\hat{u}^\varepsilon, \hat{v}^\varepsilon) = \int_{\hat{\Omega}^\varepsilon} (\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k\hat{u}^\varepsilon_i)\hat{\partial}^\varepsilon_j\hat{v}^\varepsilon_i d\hat{x}^\varepsilon,$$

$$\hat{L}^\varepsilon(\hat{v}^\varepsilon) = \int_{\hat{\Omega}^\varepsilon} \hat{f}^\varepsilon_i \hat{v}^\varepsilon_i d\hat{x}^\varepsilon + \int_{\hat{\Gamma}^\varepsilon_- \cup \hat{\Gamma}^\varepsilon_+} \hat{g}^\varepsilon_i \hat{v}^\varepsilon_i d\hat{\Gamma}^\varepsilon$$

In order to transform the problem $(V.\hat{P}^\varepsilon)$ into problem posed over the cylindrical domain Ω^ε, we use the one to one mapping $(\boldsymbol{\Theta}^\varepsilon)^{-1}$ and the following relations obtained from this transformation

$$\hat{\partial}_j^\varepsilon \hat{v}_i^\varepsilon = b_{kj}^\varepsilon(x^\varepsilon)\,\partial_k^\varepsilon v_i^\varepsilon(x^\varepsilon)\,, d\hat{x}^\varepsilon = |\det \nabla^\varepsilon \Theta^\varepsilon(x^\varepsilon)|\, dx^\varepsilon = \delta^\varepsilon dx^\varepsilon, d\hat{\Gamma}^\varepsilon = \delta^\varepsilon \beta^\varepsilon d\Gamma^\varepsilon,$$

where

$$\nabla^\varepsilon \Theta^\varepsilon(x^\varepsilon) = (\partial_j^\varepsilon \Theta_i^\varepsilon(x^\varepsilon)), \delta^\varepsilon(x^\varepsilon) = \det \nabla^\varepsilon \Theta^\varepsilon(x^\varepsilon),$$
$$b_{ij}^\varepsilon(x^\varepsilon) = (\{\nabla^\varepsilon \Theta^\varepsilon(x^\varepsilon)\}^{-1})_{ij}\ \forall x^\varepsilon \in \bar{\Omega}^\varepsilon,$$
$$\beta^\varepsilon(x^\varepsilon) = \{b_{3i}(x^\varepsilon) b_{3i}(x^\varepsilon)\}^{\frac{1}{2}}\ \forall x^\varepsilon \in (\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon).$$

We define the following functional spaces related to Ω^ε:

$$\begin{array}{rcl} V(\Omega^\varepsilon) & = & \left\{v^\varepsilon \in W^{1,4}(\Omega^\varepsilon)/v^\varepsilon \ independent\ of\ x_3^\varepsilon\ on\ \gamma_1 \times]-\varepsilon,+\varepsilon[\right\} \\ V_0(\Omega^\varepsilon) & = & \left\{v^\varepsilon \in W^{1,4}(\Omega^\varepsilon)/v^\varepsilon = 0\ on\ \gamma_1 \times]-\varepsilon,+\varepsilon[\right\}, \\ \vec{V}(\Omega^\varepsilon) & = & V(\Omega^\varepsilon) \times V(\Omega^\varepsilon) \times V_0(\Omega^\varepsilon) \end{array}$$

Then by a simple computation, we obtain

Proposition 14 *Suppose that ε is small enough. Then the variational problem $(V\hat{P}^\varepsilon)$ is equivalent to the following variational problem :*

$$(VP^\varepsilon)\left\{\begin{array}{l} Find\ u^\varepsilon \in \vec{V}(\Omega^\varepsilon),\ \ such\ that, \\ A^\varepsilon(u^\varepsilon, v^\varepsilon) = L^\varepsilon(v^\varepsilon) + \displaystyle\int_{\gamma_1} h_\alpha^\varepsilon \{\int_{-\varepsilon}^{\varepsilon} v_\alpha^\varepsilon dx_3^\varepsilon\} d\gamma, \forall v^\varepsilon \in \vec{V}(\Omega^\varepsilon), \end{array}\right.$$

where

$$A^\varepsilon(u^\varepsilon, v^\varepsilon) = \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon b_{kj}^\varepsilon \partial_k^\varepsilon v_i^\varepsilon \delta^\varepsilon dx^\varepsilon + \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon b_{ki}^\varepsilon \partial_k^\varepsilon u_l^\varepsilon b_{mj}^\varepsilon \partial_m^\varepsilon v_l^\varepsilon \delta^\varepsilon dx^\varepsilon$$
$$L^\varepsilon(v^\varepsilon) = \int_{\Omega^\varepsilon} f_3^\varepsilon v_3^\varepsilon \delta^\varepsilon dx^\varepsilon + \int_{\Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon} g_3^\varepsilon v_3^\varepsilon \delta^\varepsilon \beta^\varepsilon d\Gamma^\varepsilon,$$

$u_i^\varepsilon = \hat{u}_i^\varepsilon \circ \Theta^\varepsilon, \sigma_{ij}^\varepsilon = \hat{\sigma}_{ij}^\varepsilon \circ \Theta^\varepsilon, G_i^\varepsilon = \hat{G}_i^\varepsilon \circ \Theta^\varepsilon, n_i^\varepsilon = \hat{n}_i^\varepsilon \circ \Theta^\varepsilon, f_i^\varepsilon = \hat{f}_i^\varepsilon \circ \Theta^\varepsilon\ ,\ g_i^\varepsilon = \hat{g}_i^\varepsilon \circ \Theta^\varepsilon\ ,\ h_\alpha^\varepsilon = \hat{h}_\alpha^\varepsilon \circ \Theta^\varepsilon.$

1.5.1 Asymptotic study

The scaled problem

In this subsection we use the same transformation π^ε described in Section1.1 to define the domain Ω independent of ε. See Figure1.6. Next, we introduce the scaled displacement

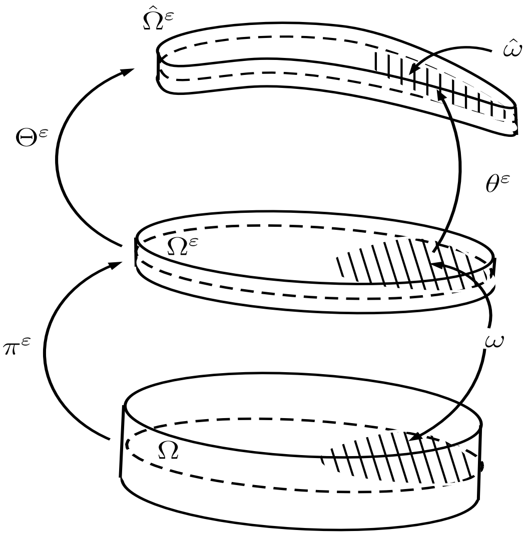

Figure 1.6: Transformation of the domain $\hat{\Omega}^\varepsilon$ into a cylindrical domain independent of ε.

$u(\varepsilon)$, test function $v(\varepsilon)$ and stress tensor $\sigma(\varepsilon)$ for all $x^\varepsilon = \pi^\varepsilon(x)$ as follows:

$$\begin{cases} u^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), u^\varepsilon_3(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x), v^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 v_\alpha(\varepsilon)(x), v^\varepsilon_3(x^\varepsilon) = \varepsilon v_3(\varepsilon)(x), \\ \sigma^\varepsilon_{\alpha\beta}(x^\varepsilon) = \varepsilon^2\sigma_{\alpha\beta}(\varepsilon)(x), \sigma^\varepsilon_{\alpha 3}(x^\varepsilon) = \varepsilon^3\sigma_{\alpha 3}(\varepsilon)(x), \sigma^\varepsilon_{33}(x^\varepsilon) = \varepsilon^4\sigma_{33}(\varepsilon)(x). \end{cases}$$

We also introduce the scalings: $f^\varepsilon_3 = \varepsilon^3 f_3, g^\varepsilon_3 = \varepsilon^4 g_3$ and $h^\varepsilon_\alpha = \varepsilon^2 h_\alpha$ where f_3, g_3 and h_α are supposed independent of ε. Therefore we denote:

$$V(\Omega) = \left\{v \in W^{1,4}(\Omega)/v \text{ independent of } x_3 \text{ on } \gamma_1 \times]-1,+1[\right\}$$
$$V_0(\Omega) = \left\{v \in W^{1,4}(\Omega)/v = 0 \text{ on } \gamma_1 \times]-1,+1[\right\}$$
$$\vec{V}(\Omega) = V(\Omega) \times V(\Omega) \times V_0(\Omega).$$

The use of the above assumptions and notations, we obtain the result:

Proposition 15 *For ε small enough the scaled solution of the problem (VP^ε) solves the problem ($SVP(\varepsilon)$):*

$$\begin{cases} \text{Find } u(\varepsilon) \in \vec{V}(\varepsilon)(\Omega) \text{ such that,} \\ A^\theta(u(\varepsilon), v) = L(v) + 2\int_{\gamma_1} h_\alpha v_\alpha dx_3 d\gamma + \varepsilon^2 r_1, \forall v \in \vec{V}(\Omega), \end{cases}$$

where

$$A^\theta(u(\varepsilon), v) = \int_\Omega \sigma_{ij}(\varepsilon)\gamma^\theta_{ij}(v)dx + \int_\Omega \sigma_{ij}(\varepsilon)\partial^\theta_i u_3(\varepsilon)\partial^\theta_j v_3 dx,$$

$$L(v) = \int_\Omega f_3 v_3 dx + \int_{\Gamma_-\cup\Gamma_+} g_3 v_3 d\Gamma,$$

$\partial^\theta_3 v = \partial_3 v, \gamma^\theta_{ij}(v) = \frac{1}{2}\left(\partial^\theta_i v_j + \partial^\theta_j v_i\right)$, *$r_1$ is a uniformly bounded function with respect to ε.*

1.5.2 The two-dimensional problem

We assume that the scaled displacement-stress $(u(\varepsilon), \sigma(\varepsilon))$ admit a formal asymptotic expansion of the form:

$$(u(\varepsilon), \sigma(\varepsilon)) = (u^0, \sigma^0) + \varepsilon(u^1, \sigma^1) + \varepsilon^2(u^2, \sigma^2) + ... \tag{1.31}$$

We introduce the space of Kirchhoff-Love admissible displacement

$$V_{KL}(\Omega) = \left\{ \begin{array}{l} v = (v_i)/v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3, v_3 = \eta_3 \textit{ such that} \\ \eta_\alpha \in H^1(\omega), \eta_3 \in H^2(\omega), \eta_3 = \partial_\nu\eta_3 = 0 \textit{ on } \gamma_1 \end{array} \right\} \tag{1.32}$$

Substituting expansion (1.31) into the scaled variational problem $(SVP(\varepsilon))$, we obtain:

Proposition 16 *Assume that* $\partial_3 u_3^0 \in C^0(\bar{\Omega})$ *then the leading term* (u^0, σ^0) *of the expansion* (1.31) *is a solution of the problem* $(SVP(0))$*:*

$$(SVP(0)) \left\{ \begin{array}{l} Find \ \ (u^0, \sigma^0) \in \ V_{KL}(\Omega) \times L_s^2(\Omega) \textit{ such that} : \\ \displaystyle\int_\Omega \sigma_{\alpha\beta}^0 \partial_\beta v_\alpha dx + \int_\Omega \sigma_{\alpha\beta}^0 \partial_\alpha (u_3^0 + \theta)\partial_\beta v_3 dx = L(v) + 2\int_{\gamma_1} h_\alpha v_\alpha d\gamma, \forall v \in V_{KL}(\Omega) \end{array} \right.$$

where

$$\sigma_{\alpha\beta}^0 = \frac{2\lambda\mu}{\lambda + 2\mu} E_{\sigma\sigma}^0(u^0)\delta_{\alpha\beta} + 2\mu E_{\alpha\beta}^0(u^0),$$

$$E_{\alpha\beta}^0(u^0) = \frac{1}{2}\left(\partial_\alpha u_\beta^0 + \partial_\beta u_\alpha^0 + \partial_\alpha u_3^0 \partial_\beta u_3^0 + \partial_\alpha\theta\partial_\beta u_3^0 + \partial_\beta\theta\partial_\alpha u_3^0\right).$$

We deduce from the following proposition that the leading term (u^0, σ^0) is characterized by a two dimensional problem.

Proposition 17 *If* u^0 *is a solution of the problem* $(SVP(0))$ *such that* $u_\alpha^0 = \xi_\alpha - x_3\partial_\alpha\xi_3$ *and* $u_3^0 = \xi_3$, ξ_α, ξ_3 *sufficiently regular. Then* ξ_α, ξ_3 *verify the two-dimensional problem* $(P^b(0))$*:*

$$(P^b(0)) \left\{ \begin{array}{l} Find \ \xi_\alpha \in H^1(\omega), \xi_3 \in H^2(\omega), \textit{ such that} \\ -\partial_{\alpha\beta} m_{\alpha\beta} - n_{\alpha\beta}\partial_{\alpha\beta}(\xi_3 + \theta)) = h_3^0 \textit{ in } \omega \\ \partial_\beta n_{\alpha\beta} = 0 \textit{ in } \omega, \\ \xi_3 = \partial_\nu \xi_3 = 0 \textit{ on } \gamma_1, \\ n_{\alpha\beta}\nu_\beta = 2h_\alpha \textit{ on } \gamma_1 \\ \partial_\alpha m_{\alpha\beta}\nu_\beta + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta) = 0 \textit{ on } \gamma_2, \\ m_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \textit{ on } \gamma_2, \\ n_{\alpha\beta}\nu_\beta = 0 \textit{ on } \gamma_2, \end{array} \right.$$

where

$$m_{\alpha\beta} = -\frac{1}{3}\{\frac{4\lambda\mu}{\lambda+2\mu}\Delta\xi_3\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\xi_3\},$$

$$n_{\alpha\beta} = 2\lambda^\star E^0_{\gamma\gamma}(\xi)\delta_{\alpha\beta} + 4\mu E^0_{\alpha\beta}(\xi), \lambda^\star = \frac{2\lambda\mu}{\lambda+2\mu}$$

$$E^0_{\alpha\beta}(\xi) = \frac{1}{2}\left(\partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha + \partial_\alpha\theta\partial_\beta\xi_3 + \partial_\beta\theta\partial_\alpha\xi_3 + \partial_\alpha\xi_3\partial_\beta\xi_3\right),$$

$$h^0_i = \int_{-1}^{1} f_i dx_3 + g_i^- + g_i^+; g_i^\pm = g_i(x_1, x_2, \pm 1).$$

Proof. See [48, Theorem 5]. ■

1.5.3 The associated generalized Marguerre-von Kármán equations

We can rewrite the two-dimensional boundary value problem $(P^b(0))$ as generalized Marguerre-von Kármán equations which depends on the Airy function Φ, the vertical component ξ_3 of the displacement field of the middle surface of the shallow shell as follows:

Proposition 18 *Assume that the set ω is simply-connected and that its boundary γ is smooth enough, and let $\xi = (\xi_i)$ be a solution $(P^b(0))$ with the regularity $\xi_\alpha \in H^3(\omega)$, $\xi_3 \in H^4(\omega)$. Then*

a) The functions $\tilde{h}_\alpha : \gamma \times [0,T] \to \mathbb{R}$ defined by :

$$\tilde{h}_\alpha = h_\alpha \text{ on } \gamma_1 \times [0,T] \text{ and } \tilde{h}_\alpha = 0 \text{ on } \gamma_2 \times [0,T],$$

are in the space $H^{\frac{3}{2}}(\gamma)$ and satisfy the compatibility conditions :

$$\int_\gamma \tilde{h}_1 d\gamma = \int_\gamma \tilde{h}_2 d\gamma = \int_\gamma (x_1\tilde{h}_2 - x_2\tilde{h}_1) d\gamma = 0.$$

b) Furthermore, there exists a function $\Phi \in H^4(\omega)$, uniquely defined by the relations $\Phi(0) = \partial_1\Phi(0) = \partial_2\Phi(0) = 0$, such that

$$n_{11} = 2\partial_{22}\Phi,\ n_{12} = n_{21} = -2\partial_{12}\Phi,\ n_{22} = 2\partial_{11}\Phi.$$

c) Finally, the pair $(\xi_3, \Phi) \in H^4(\omega) \times H^4(\omega)$*, satisfies the following problem*

$$\begin{cases} k\Delta^2\xi_3 = 2\left[\Phi, \xi_3 + \theta\right] + h_3^0 \;\; in\ \omega, \\ \Delta^2\Phi = -\frac{\mu(3\lambda+2\mu)}{2(\lambda+\mu)}\left[\xi_3, \xi_3 + 2\theta\right] \; in\ \omega, \\ \xi_3 = \partial_\nu \xi_3 = 0 \;\; on\ \gamma_1, \\ m_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \quad on\ \gamma_2, \\ \partial_\alpha m_{\alpha\beta}\nu_\beta + \partial_\tau\left(m_{\alpha\beta}\nu_\alpha\tau_\beta\right) = 0 \ on\ \gamma_2, \\ \Phi = \Phi_0 \ and\ \partial_\nu\Phi = \Phi_1 \ on\ \gamma, \end{cases}$$

where

$$k = \frac{8}{3}\mu\frac{\lambda+\mu}{\lambda+2\mu},$$

$$\Phi_0(y) = -y_1\int_{\gamma(y)}\tilde{h}_2 d\gamma + y_2\int_{\gamma(y)}\tilde{h}_1 d\gamma + \int_{\gamma(y)}(x_1\tilde{h}_2 - x_2\tilde{h}_1)d\gamma,$$

$$\Phi_1(y) = -\nu_1\int_{\gamma(y)}\tilde{h}_2 d\gamma + \nu_2\int_{\gamma(y)}\tilde{h}_1 d\gamma, y = (y_1, y_2) \in \gamma,$$

$$[\Phi, \xi] = \partial_{11}\Phi\partial_{22}\xi + \partial_{22}\Phi\partial_{11}\xi - 2\partial_{12}\Phi\partial_{12}\xi.$$

Proof. See [48, Theorem 6]. ■

1.6 Time dependent problem of generalized Marguerre-von Karman shallow shells

In a recent work in the static case, Gratie [48] has generalized the classical Marguerre-von Kàrmàn equations studied by Ciarlet and Paumier [28], where only a portion of the lateral face of the shallow shell is subjected to boundary conditions of von Kàrmàn type, while the remaining portion is subjected to boundary conditions of free edge. Then Ciarlet and Gratie [31] have established an existence theorem for these equations. In [14], we extended formally these studies to dynamic case. More precisely, we considered a three-dimensional dynamic model for a nonlinearly elastic shallow shells with a specific class of boundary conditions of generalized Marguerre-von Kàrmàn type. Using techniques from formal asymptotic analysis, we showed that the scaled three-dimensional solution still leads to two-dimensional dynamic boundary value problem called the dynamic equations of generalized Marguerre-von Kàrmàn shallow shells. In this section, we establish the existence of solutions to these equations using compactness method of Lions [76].

The first part of this section concerns the formal derivation of the two-dimensional dynamic model for thin elastic shallow shells of generalized Marguerre-von Kàrmàn type starting from the three-dimensional nonlinear elastodynamics problem. To this end we follow the same techniques of Gratie [48], this part is detailed in [14]. The second part concerns the study of existence of solutions to the problem obtained in the first part which generalizes the study carried out by Ciarlet and Gratie [31].

Setting of the problem

In this section, we keep the same situation as in the paragraph 1.5. We suppose that the system is dynamic. Next, we define the space

$$\mathbf{V}(\hat{\Omega}^\varepsilon)=\left\{\begin{array}{c}\hat{v}^\varepsilon=(\hat{v}_i^\varepsilon)\in W^{1,4}(\hat{\Omega}^\varepsilon;\mathbb{R}^3);\hat{v}_\alpha^\varepsilon \text{ independent of } \hat{x}_3^\varepsilon \text{ and } \hat{v}_3^\varepsilon=0\\ \text{on } \mathbf{\Theta}^\varepsilon(\gamma_1\times[-\varepsilon,\varepsilon])\end{array}\right\}.$$

The unknown in the three-dimensional formulation is the displacement field $\hat{u}^\varepsilon=(\hat{u}_i^\varepsilon)(\hat{x}^\varepsilon,t)$, where the functions $\hat{u}_i^\varepsilon$ are their Cartesian components. The unknown $\hat{u}^\varepsilon$ satisfies the following three-dimensional boundary value problem

$$(C.\hat{P}^\varepsilon)\left\{\begin{array}{l}\hat{\rho}^\varepsilon\frac{\partial^2\hat{u}_i^\varepsilon}{\partial t^2}-\hat{\partial}_j^\varepsilon\left(\hat{\sigma}_{ij}^\varepsilon+\hat{\sigma}_{kj}^\varepsilon\hat{\partial}_k^\varepsilon\hat{u}_i^\varepsilon\right)=\hat{f}_i^\varepsilon \text{ in } \hat{\Omega}^\varepsilon\times]0,+\infty[,\\ \left\{\begin{array}{l}\hat{u}_\alpha^\varepsilon \text{ independent of } \hat{x}_3^\varepsilon \text{ and } \hat{u}_3^\varepsilon=0 \text{ on } \mathbf{\Theta}^\varepsilon\left(\gamma_1\times[-\varepsilon,\varepsilon]\right)\times]0,+\infty[\\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{(\hat{\sigma}_{\alpha\beta}^\varepsilon+\hat{\sigma}_{k\beta}^\varepsilon\hat{\partial}_k^\varepsilon\hat{u}_\alpha^\varepsilon)\circ\mathbf{\Theta}^\varepsilon\}\nu_\beta dx_3^\varepsilon=\hat{h}_\alpha^\varepsilon\circ\mathbf{\Theta}^\varepsilon \text{ on } \gamma_1\times]0,+\infty[\end{array}\right.\\ (\hat{\sigma}_{ij}^\varepsilon+\hat{\sigma}_{kj}^\varepsilon\hat{\partial}_k^\varepsilon\hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon\circ\mathbf{\Theta}^\varepsilon=0 \text{ on } (\gamma_2\times[-\varepsilon,\varepsilon])\times]0,+\infty[,\\ (\hat{\sigma}_{ij}^\varepsilon+\hat{\sigma}_{kj}^\varepsilon\hat{\partial}_k^\varepsilon\hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon\circ\mathbf{\Theta}^\varepsilon=\hat{g}_i^\varepsilon\circ\mathbf{\Theta}^\varepsilon \text{ on } \left(\Gamma_+^\varepsilon\cup\Gamma_-^\varepsilon\right)\times]0,+\infty[,\\ \hat{u}^\varepsilon\left(\hat{x}^\varepsilon,0\right)=\hat{\mathbf{p}}^\varepsilon \text{ and } \frac{\partial\hat{u}^\varepsilon}{\partial t}\left(\hat{x}^\varepsilon,0\right)=\hat{\mathbf{q}}^\varepsilon \text{ in } \hat{\Omega}^\varepsilon,\end{array}\right.,$$

where

$$\left\{\begin{array}{rcl}\hat{\sigma}_{ij}^\varepsilon&=&\lambda^\varepsilon\hat{E}_{pp}^\varepsilon(\hat{u}^\varepsilon)\delta_{ij}+2\mu^\varepsilon\hat{E}_{ij}^\varepsilon(\hat{u}^\varepsilon),\\ \hat{E}_{ij}^\varepsilon(\hat{u}^\varepsilon)&=&\frac{1}{2}(\hat{\partial}_i^\varepsilon\hat{u}_j^\varepsilon+\hat{\partial}_j^\varepsilon\hat{u}_i^\varepsilon+\hat{\partial}_i^\varepsilon\hat{u}_m^\varepsilon\hat{\partial}_j^\varepsilon\hat{u}_m^\varepsilon),\\ \hat{\rho}^\varepsilon&:&\text{the mass density},\\ \hat{\mathbf{p}}^\varepsilon,\hat{\mathbf{q}}^\varepsilon&:&\text{the given initial data.}\end{array}\right. \tag{1.33}$$

First, we rewrite the above boundary value problem $(C.\hat{P}^\varepsilon)$ in the weak form, by using Green's formula, we show that any smooth solution of the boundary value problem also satisfies the following variational problem

$$(V.\hat{P}^\varepsilon)\left\{\begin{array}{l}\text{Find } \hat{u}^\varepsilon(\hat{x}^\varepsilon,t)\in\mathbf{V}(\hat{\Omega}^\varepsilon)\ \forall t\geq 0, \text{ such that,}\\ \frac{d^2}{dt^2}\left\{\hat{\rho}^\varepsilon\int_{\hat{\Omega}^\varepsilon}\hat{u}_i^\varepsilon\hat{v}_i^\varepsilon d\hat{x}^\varepsilon\right\}+\int_{\hat{\Omega}^\varepsilon}(\hat{\sigma}_{ij}^\varepsilon+\hat{\sigma}_{kj}^\varepsilon\hat{\partial}_k^\varepsilon\hat{u}_i^\varepsilon)\hat{\partial}_j^\varepsilon\hat{v}_i^\varepsilon d\hat{x}^\varepsilon=\int_{\hat{\Omega}^\varepsilon}\hat{f}_3^\varepsilon\hat{v}_3^\varepsilon d\hat{x}^\varepsilon\\ +\int_{\hat{\Gamma}_+^\varepsilon\cup\hat{\Gamma}_-^\varepsilon}\hat{g}_3^\varepsilon\hat{v}_3^\varepsilon d\hat{\Gamma}^\varepsilon+\int_{\hat{\gamma}_1^\varepsilon}\{\int_{-\varepsilon}^{\varepsilon}\left(\hat{v}_\alpha^\varepsilon\circ\mathbf{\Theta}^\varepsilon\right)dx_3^\varepsilon\}\hat{h}_\alpha^\varepsilon d\hat{\gamma}^\varepsilon,\\ \forall\hat{v}^\varepsilon\in\mathbf{V}(\hat{\Omega}^\varepsilon),\forall t>0,\\ \hat{u}^\varepsilon\left(\hat{x}^\varepsilon,0\right)=\hat{\mathbf{p}}^\varepsilon \text{ and } \frac{\partial\hat{u}^\varepsilon}{\partial t}\left(\hat{x}^\varepsilon,0\right)=\hat{\mathbf{q}}^\varepsilon \text{ in } \hat{\Omega}^\varepsilon.\end{array}\right.$$

In order to transform the problem $(V.\hat{P}^\varepsilon)$ into problem posed over the cylindrical domain Ω^ε, we use the one to one mapping $(\boldsymbol{\Theta}^\varepsilon)^{-1}$ and the following relations obtained from this transformation

$$\left\{\begin{array}{rcl} \hat{\partial}_j^\varepsilon \hat{v}_i^\varepsilon & = & b_{kj}^\varepsilon\left(x^\varepsilon\right)\partial_k^\varepsilon v_i^\varepsilon\left(x^\varepsilon\right), \\ d\hat{x}^\varepsilon & = & \delta^\varepsilon dx^\varepsilon, \\ d\hat{\Gamma}^\varepsilon & = & \delta^\varepsilon \beta^\varepsilon d\Gamma^\varepsilon, \end{array}\right. \tag{1.34}$$

where

$$\left\{\begin{array}{rcl} \nabla^\varepsilon \boldsymbol{\Theta}^\varepsilon(x^\varepsilon) & = & (\partial_j^\varepsilon \Theta_i^\varepsilon(x^\varepsilon))\ \forall x^\varepsilon \in \bar{\Omega}^\varepsilon, \\ \delta^\varepsilon(x^\varepsilon) & = & \det \nabla^\varepsilon \boldsymbol{\Theta}^\varepsilon(x^\varepsilon)\ \forall x^\varepsilon \in \bar{\Omega}^\varepsilon, \\ b_{ij}^\varepsilon(x^\varepsilon) & = & (\{\nabla^\varepsilon \boldsymbol{\Theta}^\varepsilon(x^\varepsilon)\}^{-1})_{ij}\ \forall x^\varepsilon \in \bar{\Omega}^\varepsilon, \\ \beta^\varepsilon(x^\varepsilon) & = & \{b_{3i}(x^\varepsilon)b_{3i}(x^\varepsilon)\}^{\frac{1}{2}}\ \forall x^\varepsilon \in (\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon). \end{array}\right.$$

Let there be a given C^1-diffeomorphism $\boldsymbol{\Theta}^\varepsilon$ that satisfies the orientation-preserving condition. Then the variational problem $(V.\hat{P}^\varepsilon)$ is equivalent to the following variational problem

$$(P^\varepsilon)\left\{\begin{array}{l} \text{Find } u^\varepsilon(x^\varepsilon,t) \in \mathbf{V}(\Omega^\varepsilon)\ \forall t \geq 0, \text{ such that,} \\ \frac{d^2}{dt^2}\left\{\rho^\varepsilon \int_{\Omega^\varepsilon} u_i^\varepsilon v_i^\varepsilon \delta^\varepsilon dx^\varepsilon\right\} + \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon b_{kj}^\varepsilon \partial_k^\varepsilon v_i^\varepsilon \delta^\varepsilon dx^\varepsilon \\ + \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon b_{ki}^\varepsilon \partial_k^\varepsilon u_l^\varepsilon b_{mj}^\varepsilon \partial_m^\varepsilon v_l^\varepsilon \delta^\varepsilon dx^\varepsilon = \int_{\Omega^\varepsilon} f_3^\varepsilon v_3^\varepsilon \delta^\varepsilon dx^\varepsilon + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_3^\varepsilon v_3^\varepsilon \delta^\varepsilon \beta^\varepsilon d\Gamma^\varepsilon \\ + \int_{\gamma_1} h_\alpha^\varepsilon \{\int_{-\varepsilon}^{\varepsilon} v_\alpha^\varepsilon dx_3^\varepsilon\} d\gamma, \forall v^\varepsilon \in \mathbf{V}(\Omega^\varepsilon), \forall t > 0, \\ u^\varepsilon(x^\varepsilon,0) = \mathbf{p}^\varepsilon \text{ and } \frac{\partial u^\varepsilon}{\partial t}(x^\varepsilon,0) = \mathbf{q}^\varepsilon \text{ in } \Omega^\varepsilon, \end{array}\right.$$

where

$$u_i^\varepsilon = \hat{u}_i^\varepsilon \circ \boldsymbol{\Theta}^\varepsilon,\ f_i^\varepsilon = \hat{f}_i^\varepsilon \circ \boldsymbol{\Theta}^\varepsilon,\ g_i^\varepsilon = \hat{g}_i^\varepsilon \circ \boldsymbol{\Theta}^\varepsilon,\ h_\alpha^\varepsilon = \hat{h}_\alpha^\varepsilon \circ \boldsymbol{\Theta}^\varepsilon,\ p_i^\varepsilon = \hat{p}_i^\varepsilon \circ \boldsymbol{\Theta}^\varepsilon,\ q_i^\varepsilon = \hat{q}_i^\varepsilon \circ \boldsymbol{\Theta}^\varepsilon.$$

1.6.1 Asymptotic analysis

The scaled problem

In the sequel we follow Ciarlet [19]. We first transform (P^ε) into a problem posed over an open set independent of ε. Accordingly to Section1.1, we recall $\Omega = \omega \times]-1,1[$, $\Gamma_\pm = \omega \times \{\pm 1\}$ and to any point $x \in \bar{\Omega}$, we associate the point $x^\varepsilon \in \bar{\Omega}^\varepsilon$ by the bijection $\pi^\varepsilon : x = (x_1,x_2,x_3) \in \bar{\Omega} \to x^\varepsilon = (x_1,x_2,\varepsilon x_3) \in \bar{\Omega}^\varepsilon$. See Figure1.6.
Next we define the functional spaces

$$\mathbf{V}(\Omega^\varepsilon) = \left\{\begin{array}{c} v^\varepsilon = (v_i^\varepsilon) \in W^{1,4}(\Omega^\varepsilon;\mathbb{R}^3); v_\alpha^\varepsilon \text{ independent of } x_3^\varepsilon \text{ and } v_3^\varepsilon = 0 \\ \text{on } \gamma_1 \times [-\varepsilon,\varepsilon] \end{array}\right\}, \tag{1.35}$$

$$\mathbf{L}_s^2(\Omega^\varepsilon) = \left\{\tau^\varepsilon = (\tau_{ij}^\varepsilon); \tau_{ij}^\varepsilon = \tau_{ji}^\varepsilon, \tau_{ij}^\varepsilon \in L^2(\Omega^\varepsilon)\right\} \tag{1.36}$$

To the functions u^ε, $v^\varepsilon \in \mathbf{V}(\Omega^\varepsilon)$ and $\sigma^\varepsilon \in \mathbf{L}_s^2(\Omega^\varepsilon)$, we associate the scaled functions $u(\varepsilon)$, v and $\sigma(\varepsilon)$ defined by

$$\begin{cases} u_\alpha^\varepsilon(x^\varepsilon,t) = \varepsilon^2 u_\alpha(\varepsilon)(x,t), u_3^\varepsilon(x^\varepsilon,t) = \varepsilon u_3(\varepsilon)(x,t), \\ v_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 v_\alpha(x), v_3^\varepsilon(x^\varepsilon) = \varepsilon v_3(x), \\ \sigma_{\alpha\beta}^\varepsilon(x^\varepsilon,t) = \varepsilon^2\sigma_{\alpha\beta}(\varepsilon)(x,t), \sigma_{\alpha 3}^\varepsilon(x^\varepsilon,t) = \varepsilon^3\sigma_{\alpha 3}(\varepsilon)(x,t), \\ \sigma_{33}^\varepsilon(x^\varepsilon,t) = \varepsilon^4\sigma_{33}(\varepsilon)(x,t), \end{cases} \tag{1.37}$$

for all $x^\varepsilon = \pi^\varepsilon x \in \bar{\Omega}^\varepsilon$. Then the appropriate functional spaces become

$$\mathbf{V}(\Omega) = \left\{ \begin{array}{c} v = (v_i) \in W^{1,4}(\Omega;\mathbb{R}^3); v_\alpha \text{ independent of } x_3 \text{ and } v_3 = 0 \\ \text{on } \gamma_1 \times [-1,1] \end{array} \right\}, \tag{1.38}$$

$$\mathbf{L}_s^2(\Omega) = \left\{\tau = (\tau_{ij}); \tau_{ij} = \tau_{ji}, \tau_{ij} \in L^2(\Omega)\right\} \tag{1.39}$$

After that, we make the following assumptions : there exists constants $\lambda > 0$, $\mu > 0$, $\rho > 0$ and for some $T > 0$, the functions $f_3 \in L^2(0,T;L^2(\Omega))$, $g_3 \in L^2(0,T;L^2(\Gamma_+ \cup \Gamma_-))$, $h_\alpha \in L^2(0,T;L^2(\gamma_1))$, $\theta \in C^3(\bar{\omega})$ independent of ε and $\mathbf{p}(\varepsilon) \in \mathbf{V}(\Omega)$, $\mathbf{q}(\varepsilon) \in L^2(\Omega;\mathbb{R}^3)$, such that

$$\begin{cases} \lambda^\varepsilon = \lambda, \mu^\varepsilon = \mu, \rho^\varepsilon = \varepsilon^2\rho, \\ f_3^\varepsilon(x^\varepsilon,t) = \varepsilon^3 f_3(x,t) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ g_3^\varepsilon(x^\varepsilon,t) = \varepsilon^4 g_3(x,t) \ \forall x^\varepsilon = \pi^\varepsilon x \in (\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon), \\ h_\alpha^\varepsilon(y_1,y_2,t) = \varepsilon^2 h_\alpha(y_1,y_2,t) \ \forall (y_1,y_2) \in \gamma_1, \\ \theta^\varepsilon(x_1,x_2) = \varepsilon\theta(x_1,x_2) \ \forall (x_1,x_2) \in \bar{\omega}, \\ p_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 p_\alpha(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ p_3^\varepsilon(x^\varepsilon) = \varepsilon p_3(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ q_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 q_\alpha(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ q_3^\varepsilon(x^\varepsilon) = \varepsilon q_3(\varepsilon)(x) \ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon. \end{cases} \tag{1.40}$$

Using the scalings (1.37) and the assumptions (1.40), we obtain

Proposition 19 *The scaled displacement field $u(\varepsilon) = (u_i(\varepsilon))$ satisfies the following variational problem*

$$(P(\varepsilon)) \begin{cases} \textit{Find } u(\varepsilon)(x,t) \in \mathbf{V}(\Omega) \ \forall t \in [0,T], \textit{ such that,} \\ A^t\left(u\left(\varepsilon\right),v\right) + B^\theta\left(\sigma\left(\varepsilon\right),v\right) + 2C^\theta\left(\sigma\left(\varepsilon\right),u\left(\varepsilon\right),v\right) = F\left(v\right) \\ +\varepsilon^2 R\left(\varepsilon;\sigma\left(\varepsilon\right),u\left(\varepsilon\right),v\right), \forall v \in \mathbf{V}(\Omega), \forall t \in \left]0,T\right[, \\ u\left(\varepsilon\right)(x,0) = \mathbf{p}\left(\varepsilon\right) \textit{ and } \frac{\partial u(\varepsilon)}{\partial t}(x,0) = \mathbf{q}\left(\varepsilon\right) \textit{ in } \Omega, \end{cases}$$

where

$A^t\left(u\left(\varepsilon\right),v\right) = -\frac{d^2}{dt^2}\left\{\rho\int_\Omega u_3\left(\varepsilon\right)v_3 dx\right\}$,

$B^\theta\left(\sigma\left(\varepsilon\right),v\right) = -\int_\Omega \sigma_{ij}\left(\varepsilon\right)\gamma_{ij}^\theta(v)dx$,

$C^{\theta}\left(\sigma\left(\varepsilon\right),u\left(\varepsilon\right),v\right)=-\frac{1}{2}\int_{\Omega}\sigma_{ij}\left(\varepsilon\right)\partial_i^{\theta}u_3\left(\varepsilon\right)\partial_j^{\theta}v_3dx,$
$F\left(v\right)=-\int_{\Omega}f_3v_3dx-\int_{\Gamma_+\cup\Gamma_-}g_3v_3d\Gamma-\int_{\gamma_1}h_{\alpha}\{\int_{-1}^{1}v_{\alpha}dx_3\}d\gamma,$
$\partial_{\alpha}^{\theta}v=\partial_{\alpha}v-\partial_{\alpha}\theta\partial_3v,\ \partial_3^{\theta}v=\partial_3v,\ \gamma_{ij}^{\theta}(v)=\frac{1}{2}\left(\partial_i^{\theta}v_j+\partial_j^{\theta}v_i\right).$

The limit three-dimensional problem

Assume that the scaled displacement-stress $(u(\varepsilon),\sigma(\varepsilon))$ admit a formal asymptotic expansion of the form:

$$(u(\varepsilon),\sigma(\varepsilon))=(u^0,\sigma^0)+\varepsilon(u^1,\sigma^1)+\varepsilon^2(u^2,\sigma^2)+\cdots, \tag{1.41}$$

with
$u^0=(u_i^0)\in\mathbf{V}\left(\Omega\right),\ \partial_3u_3^0\in C^0\left(\bar{\Omega}\right),\ u^p=(u_i^p)\in W^{1,4}(\Omega;\mathbb{R}^3)\ \forall p\geq 1,\ \sigma^0\in\mathbf{L}_s^2(\Omega).$
We also assume that when $\varepsilon\to 0$

$$\mathbf{p}\left(\varepsilon\right)\to\mathbf{p}^0\text{ in }\mathbf{V}(\Omega),\ \ \mathbf{q}\left(\varepsilon\right)\to\mathbf{q}^0\text{ in }L^2(\Omega;\mathbb{R}^3).$$

We substitute the formal asymptotic expansion (1.41) into the variational problem $(P(\varepsilon))$ and using techniques from asymptotic analysis, we prove that the leading term u^0 should satisfy the following scaled three-dimensional problem. First we define the following spaces

$$\mathbf{V}_{KL}(\Omega)=\left\{\begin{array}{c} v=(v_i)\in H^1(\Omega;\mathbb{R}^3);v_{\alpha}\text{ independent of }x_3\text{ and }v_3=0\\ \text{on }\gamma_1\times[-1,1],\partial_iv_3+\partial_3v_i=0\text{ in }\Omega\end{array}\right\}, \tag{1.42}$$

$$\mathbf{V}(\omega)=\left\{\eta=(\eta_i)\in H^1(\omega)\times H^1(\omega)\times H^2(\omega);\eta_3=\partial_{\nu}\eta_3=0\text{ on }\gamma_1\right\}. \tag{1.43}$$

Proposition 20 *The leading term $u^0\in\mathbf{V}_{KL}(\Omega)\ \forall t\in[0,T]$ (Kirchhoff-Love displacement field) is solution of the problem*

$$(P_{KL})\left\{\begin{array}{l} \textit{Find } u^0\in\mathbf{V}_{KL}(\Omega)\ \forall t\in[0,T],\ \textit{such that,}\\ \frac{d^2}{dt^2}\left\{\rho\int_{\Omega}u_3^0v_3dx\right\}+\int_{\Omega}\sigma_{\alpha\beta}^0\partial_{\beta}v_{\alpha}dx+\int_{\Omega}\sigma_{\alpha\beta}^0\partial_{\alpha}(u_3^0+\theta)\partial_{\beta}v_3dx=\\ \int_{\Omega}f_3v_3dx+\int_{\Gamma_+\cup\Gamma_-}g_3v_3d\Gamma+2\int_{\gamma_1}h_{\alpha}v_{\alpha}d\gamma,\forall v\in\mathbf{V}_{KL}(\Omega),\forall t\in\left]0,T\right[,\\ u^0\left(x,0\right)=\mathbf{p}^0\textit{ and }\frac{\partial u^0}{\partial t}\left(x,0\right)=\mathbf{q}^0\textit{ in }\Omega,\end{array}\right.$$

where

$$\left\{\begin{array}{l}\sigma_{\alpha\beta}^0=\frac{2\lambda\mu}{\lambda+2\mu}\bar{E}_{\sigma\sigma}^0(u^0)\delta_{\alpha\beta}+2\mu\bar{E}_{\alpha\beta}^0(u^0),\\ \bar{E}_{\alpha\beta}^0(u^0)=\frac{1}{2}(\partial_{\alpha}u_{\beta}^0+\partial_{\beta}u_{\alpha}^0+\partial_{\alpha}u_3^0\partial_{\beta}u_3^0+\partial_{\alpha}\theta\partial_{\beta}u_3^0+\partial_{\beta}\theta\partial_{\alpha}u_3^0).\end{array}\right. \tag{1.44}$$

The limit two-dimensional displacement problem

We use some techniques employed by Raoult [88], who assumed that the initial data p_3^0 and q_3^0 are independent of x_3 and smooth enough.

First, we show in the next proposition that (P_{KL}) is in sense of two-dimensional problem posed over the two-dimensional domain $\bar{\omega}$.

Proposition 21 *The components of the leading term $u^0 = (u_i^0)$ are of the form $u_\alpha^0 = \zeta_\alpha - x_3\partial_\alpha\zeta_3$ and $u_3^0 = \zeta_3$ with $\zeta = (\zeta_i) \in \mathbf{V}(\omega)$ $\forall t \in [0,T]$, where the field ζ satisfies the following limit scaled two-dimensional displacement problem*

$$(P(\omega))\begin{cases} 2\rho\int_\omega \frac{\partial^2\zeta_3}{\partial t^2}\eta_3 d\omega - \int_\omega m_{\alpha\beta}(\nabla^2\zeta_3)\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega \bar{N}_{\alpha\beta}\partial_\alpha(\zeta_3+\theta)\partial_\beta\eta_3 d\omega \\ +\int_\omega \bar{N}_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = \int_\omega \mathrm{p}_3\eta_3 d\omega + 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma, \forall\eta \in \mathbf{V}(\omega), \forall t \in]0,T[, \\ \zeta_3(.,0) = p_3^0 \text{ and } \frac{\partial\zeta_3}{\partial t}(.,0) = q_3^0 \text{ in } \omega, \end{cases}$$

where

$m_{\alpha\beta}(\nabla^2\zeta_3) = -\frac{1}{3}\left\{\frac{4\lambda\mu}{\lambda+2\mu}\Delta\zeta_3\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\zeta_3\right\}$,

$\bar{N}_{\alpha\beta} = \frac{4\lambda\mu}{\lambda+2\mu}\bar{E}_{\sigma\sigma}^0(\zeta)\,\delta_{\alpha\beta} + 4\mu\bar{E}_{\alpha\beta}^0(\zeta)$,

$\bar{E}_{\alpha\beta}^0(\zeta) = \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha + \partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3 + \partial_\alpha\zeta_3\partial_\beta\zeta_3)$,

$\mathrm{p}_3 = \int_{-1}^1 f_3 dx_3 + g_3(.,+1) + g_3(.,-1)$.

Next, we write the two-dimensional boundary value problem in an equivalent variational problem $(\bar{P}(\omega))$. We equate to zero all the factors of η_α, η_3, and $\partial_\nu\eta_3$ in their respective domains of integration, we obtain

Proposition 22 *Assume that the boundary γ is smooth enough. Then any smooth enough solution $\zeta = (\zeta_i)$ of the variational problem $(P(\omega))$ is also a solution of the following two-*

dimensional displacement problem

$$(\bar{P}(\omega))\begin{cases} \textit{Find } \zeta \in \mathbf{V}(\omega)\ \forall t \in [0,T], \textit{ such that,} \\ 2\rho\frac{\partial^2\zeta_3}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta}(\nabla^2\zeta_3) - \bar{N}_{\alpha\beta}\partial_{\alpha\beta}\left(\zeta_3+\theta\right) = \mathrm{p}_3 \textit{ in } \omega\times\,]0,T[\,, \\ \partial_\beta\bar{N}_{\alpha\beta} = 0 \textit{ in } \omega\times\,]0,T[\,, \\ \zeta_3 = \partial_\nu\zeta_3 = 0 \textit{ on } \gamma_1\times\,]0,T[\,, \\ \bar{N}_{\alpha\beta}\nu_\beta = 2h_\alpha \textit{ on } \gamma_1\times\,]0,T[\,, \\ m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\alpha\nu_\beta = 0 \textit{ on } \gamma_2\times\,]0,T[\,, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\beta + \partial_\tau\left(m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\alpha\tau_\beta\right) = 0 \textit{ on } \gamma_2\times\,]0,T[\,, \\ \bar{N}_{\alpha\beta}\nu_\beta = 0 \textit{ on } \gamma_2\times\,]0,T[\,, \\ \zeta_3\left(.,0\right) = p_3^0 \textit{ and } \frac{\partial\zeta_3}{\partial t}\left(.,0\right) = q_3^0 \textit{ in } \omega. \end{cases}$$

1.6.2 Dynamic equations of generalized Marguerre-von Kàrmàn shallow shells

We rewrite the two-dimensional boundary value problem $(\bar{P}(\omega))$ as dynamic equations of generalized Marguerre-von Kàrmàn shallow shells as follows:

Proposition 23 *Assume that the set ω is simply-connected and that its boundary γ is smooth enough. Let $\zeta = (\zeta_i)$ be a solution of $(\bar{P}(\omega))$ with the regularity*
$\zeta_\alpha \in H^3(\omega)$, $\zeta_3 \in H^4(\omega)$ $\forall t \in [0,T]$.
Then

a) *The functions $\tilde{h}_\alpha : \gamma\times[0,T] \to \mathbb{R}$ defined by :*

$$\tilde{h}_\alpha = h_\alpha \textit{ on } \gamma_1\times[0,T] \textit{ and } \tilde{h}_\alpha = 0 \textit{ on } \gamma_2\times[0,T],$$

are in the space $H^{\frac{3}{2}}(\gamma)$ and satisfy the compatibility conditions :

$$\int_\gamma \tilde{h}_1 d\gamma = \int_\gamma \tilde{h}_2 d\gamma = \int_\gamma (x_1\tilde{h}_2 - x_2\tilde{h}_1)d\gamma = 0.$$

b) *Furthermore, there exists a function $\Phi \in H^4(\omega)$, uniquely defined by the relations $\Phi(0) = \partial_1\Phi(0) = \partial_2\Phi(0) = 0$, such that*

$$\bar{N}_{11} = 2\partial_{22}\Phi,\ \bar{N}_{12} = \bar{N}_{21} = -2\partial_{12}\Phi,\ \bar{N}_{22} = 2\partial_{11}\Phi.$$

c) *Finally, the pair* $(\zeta_3, \Phi) \in H^4(\omega) \times H^4(\omega)$ $\forall t \in [0,T]$*, satisfies the following scaled dynamic equations of generalized Marguerre-von Kármán shallow shells*

$$(P)\begin{cases} 2\rho\frac{\partial^2\zeta_3}{\partial t^2} + \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3 = 2\,[\Phi, \zeta_3+\theta] + \mathrm{p}_3 \ \text{in } \omega \times]0,T[\,, \\ \Delta^2\Phi = -\frac{\mu(3\lambda+2\mu)}{2(\lambda+\mu)}\,[\zeta_3, \zeta_3+2\theta] \ \text{in } \omega \times]0,T[\,, \\ \zeta_3 = \partial_\nu\zeta_3 = 0 \ \text{on } \gamma_1\times]0,T[\,, \\ m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\alpha\nu_\beta = 0 \ \text{on } \gamma_2\times]0,T[\,, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\beta + \partial_\tau\left(m_{\alpha\beta}(\nabla^2\zeta_3)\nu_\alpha\tau_\beta\right) = 0 \ \text{on } \gamma_2 \times]0,T[\,, \\ \Phi = \Phi_0 \ \text{and } \partial_\nu\Phi = \Phi_1 \ \text{on } \gamma \times]0,T[\,, \\ \zeta_3\,(.,0) = p_3^0 \ \text{and } \frac{\partial\zeta_3}{\partial t}\,(.,0) = q_3^0 \ \text{in } \omega, \end{cases}$$

where

$\Phi_0(y) = -y_1\int_{\gamma(y)}\tilde{h}_2 d\gamma + y_2\int_{\gamma(y)}\tilde{h}_1 d\gamma + \int_{\gamma(y)}(x_1\tilde{h}_2 - x_2\tilde{h}_1)d\gamma,$

$\Phi_1(y) = -\nu_1\int_{\gamma(y)}\tilde{h}_2 d\gamma + \nu_2\int_{\gamma(y)}\tilde{h}_1 d\gamma,\ y = (y_1, y_2) \in \gamma,$

$[\Phi, \zeta] = \partial_{11}\Phi\partial_{22}\zeta + \partial_{22}\Phi\partial_{11}\zeta - 2\partial_{12}\Phi\partial_{12}\zeta.$

Proof. See [14]. ∎

1.6.3 Existence theory

The asymptotic analysis carried out in the first part is purely formal. In what follows, we establish the existence of solutions to the dynamic equations of generalized Marguerre-von Kàrmàn shallow shells. We first deduce that they are equivalent to another variational problem $(\mathcal{P})$, we then solve this problem, by adapting a compactness method.

We use the following Lemma

Lemma 1 *If* $(\xi, \eta, \chi) \in [H^2(\omega)]^3$ *such that*

$$\xi = \partial_\nu\xi = 0 \ \text{on } \gamma_1 \ \text{and } \chi = \partial_\nu\chi = 0 \ \text{on } \gamma_2,$$

then

$$\int_\omega [\xi, \eta]\chi d\omega = \int_\omega [\chi, \eta]\xi d\omega, \tag{1.45}$$

where

$$[\xi, \eta] \ = \ \partial_{11}(\partial_{22}\eta.\xi) + \partial_{22}(\partial_{11}\eta.\xi) - 2\partial_{12}(\partial_{12}\eta.\xi).$$

Proof.
Since $\overline{C^\infty(\bar{\omega})} = H^2(\omega)$, let the functions ξ, η, and χ in $C^\infty(\bar{\omega})$. Integrating by parts, we obtain

$$\int_\omega [\xi,\eta]\chi d\omega - \int_\omega [\chi,\eta]\xi d\omega = \int_\gamma \chi\{\partial_{22}\eta\partial_1\xi\nu_1 + \partial_{11}\eta\partial_2\xi\nu_2 - \partial_{12}\eta\partial_2\xi\nu_1 - \partial_{12}\eta\partial_1\xi\nu_2\}d\gamma$$
$$- \int_\gamma \xi\{\partial_{22}\eta\partial_1\chi\nu_1 + \partial_{11}\eta\partial_2\chi\nu_2 - \partial_{12}\eta\partial_2\chi\nu_1 - \partial_{12}\eta\partial_1\chi\nu_2\}d\gamma.$$

If $\xi = \partial_\nu\xi = 0$ on γ_1 and $\chi = \partial_\nu\chi = 0$ on γ_2, consequently

$$\int_\omega [\xi,\eta]\chi d\omega - \int_\omega [\chi,\eta]\xi d\omega = 0.$$

■

Theorem 24 *Assume that the set ω is simply-connected and that the functions $\tilde{h}_\alpha \in L^2(\gamma)$ $\forall t \in [0,T]$ satisfy the compatibility conditions. Let $\chi \in H^2(\omega)$ be the unique solution in the sense of distributions of*

$$\begin{cases} \Delta^2\chi = 0 \ \ in \ \omega, \\ \chi = \Phi_0 \ \ and \ \partial_\nu\chi = \Phi_1 \ \ on \ \gamma, \\ \Phi_0 \in H^{3/2}(\gamma), \Phi_1 \in H^{1/2}(\gamma) \end{cases} \tag{1.46}$$

and let

$$E = \frac{\mu\,(3\lambda+2\mu)}{\lambda+\mu},\ \xi = \sqrt{E}\zeta_3,\ \widetilde{\theta} = \sqrt{E}\theta,\ f = \sqrt{E}\mathrm{p}_3,\ \widetilde{\Phi} = \Phi - \chi, \tag{1.47}$$

$$V(\omega) = \left\{\eta \in H^2(\omega); \eta = \partial_\nu\eta = 0 \ \ on \ \gamma_1\right\}. \tag{1.48}$$

The pair $(\zeta_3, \Phi) \in H^4(\omega) \times H^4(\omega)$ $\forall t \in [0,T]$, satisfies the scaled dynamic equations of generalized Marguerre-von Kármán shallow shells in the sense of distributions, if and only if, the pair $(\xi, \widetilde{\Phi}) \in V(\omega) \times H_0^2(\omega)$ $\forall t \in [0,T]$, satisfies

$$(\mathcal{P})\begin{cases} 2\rho\frac{\partial^2\xi}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta}(\nabla^2\xi) = 2[\widetilde{\Phi} + \chi, \xi + \widetilde{\theta}] + f \;\; in \;\, \omega\times]0,T[, \\ \Delta^2\widetilde{\Phi} = -\frac{1}{2}[\xi, \xi + 2\widetilde{\theta}] \;\; in \;\, \omega\times]0,T[, \\ \xi = \partial_\nu\xi = 0 \;\; on \;\, \gamma_1\times]0,T[, \\ m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\nu_\beta = 0 \;\; on \;\, \gamma_2\times]0,T[, \\ \partial_\alpha m_{\alpha\beta}(\nabla^2\xi)\nu_\beta + \partial_\tau(m_{\alpha\beta}(\nabla^2\xi)\nu_\alpha\tau_\beta) = 0 \;\; on \;\, \gamma_2\times]0,T[, \\ \widetilde{\Phi} = \partial_\nu\widetilde{\Phi} = 0 \;\; on \;\, \gamma\times]0,T[, \\ \xi(.,0) = \xi_0(.) \;\; and \;\, \frac{\partial\xi}{\partial t}(.,0) = \xi_1(.) \;\; in \;\, \omega. \end{cases}$$

Proof. By the classical elliptic theory, there exists a unique function $\chi \in H^2(\omega)$ such that
$\Delta^2\chi = 0$ in ω, $\chi = \Phi_0$ and $\partial_\nu\chi = \Phi_1$ on γ (see [20, Theorem 5.6-1]),
and let $\widetilde{\Phi} = \Phi - \chi$. Obviously,

$$\begin{cases} \Delta^2\widetilde{\Phi} = \Delta^2\Phi \text{ in } \omega \times\,]0,T[\,, \\ \widetilde{\Phi} = \partial_\nu\widetilde{\Phi} = 0 \text{ on } \gamma\times\,]0,T[\,. \end{cases}$$

Using the functions ξ, $\widetilde{\theta}$, f and $\widetilde{\Phi}$ defined in (1.47), the scaled dynamic equations of generalized Marguerre-von Kármán shallow shells presented in proposition 23 is equivalent to the scaled problem $(\mathcal{P})$. ■

Theorem 25 *Assume $f \in L^2(0,T;L^2(\omega))$, $\xi_0 \in V(\omega)$ and $\xi_1 \in L^2(\omega)$, then there exists a solution $(\xi, \widetilde{\Phi})$ to the problem $(\mathcal{P})$, such that*

$$\begin{cases} \xi \in L^\infty(0,T;V(\omega)), \\ \frac{\partial\xi}{\partial t} \in L^\infty(0,T;L^2(\omega)), \\ \widetilde{\Phi} \in L^\infty(0,T;H_0^2(\omega)). \end{cases} \tag{1.49}$$

Proof. Denote by G_2 the inverse of Δ^2 with homogenous Dirichlet boundary condition in ω (the Green operator), we write

$$\widetilde{\Phi} = -\frac{1}{2}G_2\left[\xi, \xi + 2\widetilde{\theta}\right] \text{ in } \omega \times\,]0,T[\,,$$

then

$$2\rho\frac{\partial^2\xi}{\partial t^2} - \partial_{\alpha\beta}m_{\alpha\beta}\left(\nabla^2\xi\right) = 2\left[-\frac{1}{2}G_2\left[\xi, \xi + 2\widetilde{\theta}\right] + \chi, \xi + \widetilde{\theta}\right] + f \text{ in } \omega \times\,]0,T[\,.$$

From (1.49), we get

$$\left[\widetilde{\Phi} + \chi, \xi + \widetilde{\theta}\right] \in L^\infty(0, T; L^1(\omega)),$$

and for the first equation in $(\mathcal{P})$, we have

$$\frac{\partial^2 \xi}{\partial t^2} \in L^\infty(0, T; H^{-1}(\omega)),$$

so that the initial conditions make sense.
Step 1: (Faedo-Galerkin approximation)

Let w_i, $i \geq 1$ denote an orthonormal basis of the Hilbert space $V(\omega)$ and let V_m denote, for each integer $m \geq 1$, the subspace of $V(\omega)$ spanned by the functions w_i, $1 \leq i \leq m$.

We construct the Faedo-Galerkin approximation $\xi_m(t)$ of a solution in the form

$$\xi_m(t) = \sum_{i=1}^{m} \alpha_{im}(t) w_i.$$

So the function $\xi_m(t)$ is the solution of the following approximate problem

$$(\mathcal{P}_m) \begin{cases} 2\rho \int_\omega \frac{\partial^2 \xi_m(t)}{\partial t^2} w_j d\omega \ - \int_\omega \partial_{\alpha\beta} m_{\alpha\beta} \left(\nabla^2 \xi_m(t)\right) w_j d\omega = \\ 2 \int_\omega \left[-\frac{1}{2} G_2 \left[\xi_m(t), \xi_m(t) + 2\widetilde{\theta}\right] + \chi, \xi_m(t) + \widetilde{\theta}\right] w_j d\omega + \int_\omega f w_j d\omega, \\ 1 \leq j \leq m \text{ in } \omega \times]0, T[\,, \\ \xi_m(t) = \partial_\nu \xi_m(t) = 0 \text{ on } \gamma_1 \times]0, T[\,, \\ m_{\alpha\beta} \left(\nabla^2 \xi_m(t)\right) \nu_\alpha \nu_\beta = 0 \text{ on } \gamma_2 \times]0, T[\,, \\ \partial_\alpha m_{\alpha\beta} \left(\nabla^2 \xi_m(t)\right) \nu_\beta + \partial_\tau \left(m_{\alpha\beta} \left(\nabla^2 \xi_m(t)\right) \nu_\alpha \tau_\beta\right) = 0 \text{ on } \gamma_2 \times]0, T[\,, \\ \xi_m\left(.,0\right) = \xi_{0m}(.) \text{ and } \frac{\partial \xi_m}{\partial t}\left(.,0\right) = \xi_{1m}(.) \text{ in } \omega, \end{cases}$$

and we have

$$\xi_{0m} \in V_m \text{ and } \xi_{0m} \to \xi_0 \text{ in } V(\omega), \ \ \xi_{1m} \in V_m \text{ and } \xi_{1m} \to \xi_1 \text{ in } L^2(\omega).$$

We define

$$\widetilde{\Phi}_m(t) = -\frac{1}{2} G_2 \left[\xi_m(t), \xi_m(t) + 2\widetilde{\theta}\right] \text{ in } \omega \times]0, T[\,, \tag{1.50}$$

then, we obtain

$$\Delta^2 \widetilde{\Phi}_m(t) = -\frac{1}{2} \left[\xi_m(t), \xi_m(t) + 2\widetilde{\theta}\right] \text{ in } \omega \times]0, T[\,, \tag{1.51}$$

$$\widetilde{\Phi}_m(t) \in H_0^2(\omega), \tag{1.52}$$

and we rewrite the first equation of $(\mathcal{P}_m)$ as

$$2\rho \int_\omega \frac{\partial^2 \xi_m(t)}{\partial t^2} w_j d\omega + a(\xi_m(t), w_j) - 2 \int_\omega \left[\widetilde{\Phi}_m(t), \xi_m(t) + \widetilde{\theta}\right] w_j d\omega \ = $$

$$2 \int_\omega \left[\chi, \xi_m(t) + \widetilde{\theta}\right] w_j d\omega + \int_\omega f w_j d\omega \ , \ 1 \leq j \leq m \ \text{ in } \omega \times]0, T[\,, \tag{1.53}$$

where

$$a(\xi,\eta) = \frac{2E}{3(1-\sigma^2)}\int_\omega [\Delta\xi\Delta\eta - (1-\sigma)\{\partial_{11}\xi\partial_{22}\eta + \partial_{22}\xi\partial_{11}\eta - 2\partial_{12}\xi\partial_{12}\eta\}]d\omega.$$

The constants $E > 0$ and $\sigma \in]0, \frac{1}{2}[$ are respectively the Young's modulus and the Poisson's coefficient of the constitutive elastic material of the shallow shells.

In general $\widetilde{\Phi}_m(t)$ is not in V_m, one assures the existence of $\xi_m(t)$ and therefore of $\widetilde{\Phi}_m(t)$ in an interval $[0, t_m]$, $t_m > 0$ (see [76, Theorem 4.1]).

Step 2: (A priori estimates)

Multiplying $\frac{d\alpha_{jm}(t)}{dt}$ on both sides of (1.53) and summing up from j, we obtain that

$$\begin{aligned} 2\rho\int_\omega \frac{\partial^2\xi_m(t)}{\partial t^2}\frac{\partial\xi_m(t)}{\partial t}d\omega + a(\xi_m(t), \frac{\partial\xi_m(t)}{\partial t}) \\ -2\int_\omega \left[\widetilde{\Phi}_m(t), \xi_m(t) + \widetilde{\theta}\right]\frac{\partial\xi_m(t)}{\partial t}d\omega = 2\int_\omega \left[\chi, \xi_m(t) + \widetilde{\theta}\right]\frac{\partial\xi_m(t)}{\partial t}d\omega \\ + \int_\omega f\frac{\partial\xi_m(t)}{\partial t}d\omega \text{ in } \omega\times]0,T[. \end{aligned} \tag{1.54}$$

We have

$$2\rho\int_\omega \frac{\partial^2\xi_m(t)}{\partial t^2}\frac{\partial\xi_m(t)}{\partial t}d\omega = \rho\frac{d}{dt}\int_\omega |\frac{\partial\xi_m(t)}{\partial t}|^2 d\omega = \rho\frac{d}{dt}||\frac{\partial\xi_m(t)}{\partial t}||^2_{0,\omega},$$

and since a is elliptic, we conclude that there exists a constant $\alpha > 0$, such that

$$a(\xi_m(t), \xi_m(t)) \geq \alpha||\xi_m(t)||^2_{V(\omega)},$$

thus

$$a(\xi_m(t), \frac{\partial\xi_m(t)}{\partial t}) = \frac{d}{2dt}a(\xi_m(t), \xi_m(t)) \geq \frac{\alpha}{2}\frac{d}{dt}||\xi_m(t)||^2_{V(\omega)}.$$

Since $\widetilde{\Phi}_m(t) \in H^2_0(\omega)$ and by [20, Theorem 5.8-2], we infer

$$\int_\omega \left[\widetilde{\Phi}_m(t), \xi_m(t) + \widetilde{\theta}\right]\frac{\partial\xi_m(t)}{\partial t}d\omega = \int_\omega \left[\frac{\partial\xi_m(t)}{\partial t}, \xi_m(t) + \widetilde{\theta}\right]\widetilde{\Phi}_m(t)d\omega.$$

Using (1.51), we get

$$\begin{aligned} \frac{\partial}{\partial t}\Delta^2\widetilde{\Phi}_m(t) &= \Delta^2\frac{\partial\widetilde{\Phi}_m(t)}{\partial t} \\ &= -\frac{1}{2}\frac{\partial}{\partial t}\left[\xi_m(t), \xi_m(t) + 2\widetilde{\theta}\right] \\ &= -\frac{1}{2}\left[\frac{\partial\xi_m(t)}{\partial t}, \xi_m(t) + 2\widetilde{\theta}\right] - \frac{1}{2}\left[\xi_m(t), \frac{\partial\xi_m(t)}{\partial t}\right] \\ &= -\left[\frac{\partial\xi_m(t)}{\partial t}, \xi_m(t) + \widetilde{\theta}\right], \end{aligned}$$

thus

$$\begin{aligned} -2\int_{\omega}\left[\frac{\partial \xi_m(t)}{\partial t}, \xi_m(t)+\widetilde{\theta}\right]\widetilde{\Phi}_m(t)d\omega &= 2\int_{\omega}[\Delta^2\frac{\partial\widetilde{\Phi}_m(t)}{\partial t}]\widetilde{\Phi}_m(t)d\omega \\ &= 2\int_{\omega}[\Delta\frac{\partial\widetilde{\Phi}_m(t)}{\partial t}][\Delta\widetilde{\Phi}_m(t)]d\omega \\ &= \frac{d}{dt}\int_{\omega}|\Delta\widetilde{\Phi}_m(t)|^2 d\omega \\ &= \frac{d}{dt}||\Delta\widetilde{\Phi}_m(t)||^2_{0,\omega}. \end{aligned}$$

Since $\frac{\partial \xi_m(t)}{\partial t} \in V(\omega)$ i.e $\frac{\partial \xi_m(t)}{\partial t} = \partial_\nu[\frac{\partial \xi_m(t)}{\partial t}] = 0$ on γ_1 and $\chi = \partial_\nu \chi = 0$ on γ_2, then applying Lemma 1, gives

$$\begin{aligned} 2\int_{\omega}\left[\chi, \xi_m(t)+\widetilde{\theta}\right]\frac{\partial \xi_m(t)}{\partial t}d\omega &= 2\int_{\omega}\left[\frac{\partial \xi_m(t)}{\partial t}, \xi_m(t)+\widetilde{\theta}\right]\chi d\omega \\ &= -2\int_{\omega}\Delta^2\frac{\partial\widetilde{\Phi}_m(t)}{\partial t}\chi d\omega, \end{aligned}$$

and we have

$$\int_{\omega}\Delta^2\frac{\partial\widetilde{\Phi}_m(t)}{\partial t}.\chi d\omega = \frac{d}{dt}\int_{\omega}\Delta^2\widetilde{\Phi}_m(t).\chi d\omega - \int_{\omega}\Delta^2\widetilde{\Phi}_m(t).\frac{\partial \chi}{\partial t}d\omega.$$

From (1.46) and (1.52), it follows that

$$\frac{d}{dt}\int_{\omega}\Delta^2\widetilde{\Phi}_m(t).\chi d\omega = \frac{d}{dt}\int_{\omega}\widetilde{\Phi}_m(t).\Delta^2\chi d\omega = 0,$$

and since the function χ is independent of t, so that

$$\int_{\omega}\Delta^2\widetilde{\Phi}_m(t).\frac{\partial \chi}{\partial t}d\omega = 0,$$

thus

$$\int_{\omega}\Delta^2\frac{\partial\widetilde{\Phi}_m(t)}{\partial t}.\chi d\omega = 0.$$

Then (1.54) can be written now as

$$\frac{d}{dt}\{\rho||\frac{\partial \xi_m(t)}{\partial t}||^2_{0,\omega} + \frac{1}{2}a(\xi_m(t), \xi_m(t)) + ||\Delta\widetilde{\Phi}_m(t)||^2_{0,\omega}\} = \int_{\omega} f\frac{\partial \xi_m(t)}{\partial t}d\omega,$$

integrating from 0 to t, we obtain

$$\int_0^t \frac{d}{d\tau}\{\rho||\frac{\partial \xi_m(\tau)}{\partial \tau}||^2_{0,\omega} + \frac{1}{2}a(\xi_m(\tau), \xi_m(\tau)) + ||\Delta\widetilde{\Phi}_m(\tau)||^2_{0,\omega}\}d\tau = \int_0^t\{\int_{\omega} f\frac{\partial \xi_m(\tau)}{\partial \tau}d\omega\}d\tau,$$

hence, there exists constants $C_1 > 0$ and $C_2 > 0$ such that

$$\rho||\frac{\partial \xi_m(t)}{\partial t}||^2_{0,\omega} + \frac{\alpha}{2}||\xi_m(t)||^2_{V(\omega)} + ||\Delta\widetilde{\Phi}_m(t)||^2_{0,\omega} \leq C_1 \int_0^t ||f||^2_{0,\omega} d\tau$$
$$+C_2 \int_0^t ||\frac{\partial \xi_m(\tau)}{\partial \tau}||^2_{0,\omega} d\tau + \rho||\frac{\partial \xi_m(0)}{\partial t}||^2_{0,\omega} + \frac{\alpha}{2}||\xi_m(0)||^2_{V(\omega)} + ||\Delta\widetilde{\Phi}_m(0)||^2_{0,\omega}.$$

Since

$$\Delta^2\widetilde{\Phi}_m(0) = -\frac{1}{2}\left[\xi_m(0), \xi_m(0) + 2\widetilde{\theta}\right],$$

then, there exists a constant $C_3 > 0$ such that

$$||\Delta\widetilde{\Phi}_m(0)||_{0,\omega} \leq C_3.$$

Thus, there exists a constant $C_4 > 0$ such that

$$\rho||\frac{\partial \xi_m(t)}{\partial t}||^2_{0,\omega} + \frac{\alpha}{2}||\xi_m(t)||^2_{V(\omega)} + ||\Delta\widetilde{\Phi}_m(t)||^2_{0,\omega} \leq C_4 + C_2 \int_0^t ||\frac{\partial \xi_m(\tau)}{\partial \tau}||^2_{0,\omega} d\tau,$$

for all $t \in [0, T]$, these imply that $t_m = T$.

Then, via Gronwall's inequality, we conclude that

$$\xi_m(t) \in L^\infty(0, T; V(\omega)), \tag{1.55}$$

$$\frac{\partial \xi_m(t)}{\partial t} \in L^\infty(0, T; L^2(\omega)), \tag{1.56}$$

$$\widetilde{\Phi}_m(t) \in L^\infty(0, T; H^2_0(\omega)). \tag{1.57}$$

Step 3: (Passing to the limit)
From (1.55)-(1.57), we observe that there exists $\xi_n(t)$ and $\widetilde{\Phi}_n(t)$ such that (weak convergence is denoted $\rightharpoonup$)

$$\xi_n(t) \rightharpoonup \xi(t) \text{ in } L^\infty(0, T; V(\omega)) \text{ weak}*,$$

$$\frac{\partial \xi_n(t)}{\partial t} \rightharpoonup \frac{\partial \xi(t)}{\partial t} \text{ in } L^\infty(0, T; L^2(\omega)) \text{ weak}*,$$

$$\widetilde{\Phi}_n(t) \rightharpoonup \widetilde{\Phi}(t) \text{ in } L^\infty(0, T; H^2_0(\omega)) \text{ weak} * .$$

According to the Rellich-Kondrachoff theorem [77, Chap 1, Theorem 16.1], the compact imbedding of $H^2(\omega\times]0, T[)$ into $L^2(\omega\times]0, T[)$ implies that

$$\xi_n(t) \rightarrow \xi(t) \text{ in } L^2(\omega\times]0, T[). \tag{1.58}$$

Let ϕ_j, $1 \leq j \leq j_0$ are functions of $C^1([0,T])$ such that

$$\phi_j(T) = 0 \text{ and } \psi = \sum_{j=1}^{j_0} \phi_j \otimes w_j. \tag{1.59}$$

For $m = n > j_0$, we obtain

$$\begin{aligned} 2\rho \int_\omega \frac{\partial^2 \xi_n(t)}{\partial t^2} \psi(t) d\omega + a(\xi_n(t), \psi(t)) - 2 \int_\omega \left[\widetilde{\Phi}_n(t), \xi_n(t) + \widetilde{\theta} \right] \psi(t) d\omega &= \\ 2 \int_\omega \left[\chi, \xi_n(t) + \widetilde{\theta} \right] \psi(t) d\omega + \int_\omega f \psi(t) d\omega \text{ in } \omega \times]0, T[\,, \end{aligned}$$

thus

$$\begin{aligned} 2\rho \int_0^T \{ \int_\omega \frac{\partial^2 \xi_n(t)}{\partial t^2} \psi(t) d\omega \} dt + \int_0^T a(\xi_n(t), \psi(t)) dt \\ -2 \int_0^T \{ \int_\omega \left[\widetilde{\Phi}_n(t), \xi_n(t) + \widetilde{\theta} \right] \psi(t) d\omega \} dt = 2 \int_0^T \{ \int_\omega \left[\chi, \xi_n(t) + \widetilde{\theta} \right] \psi(t) d\omega \} dt \\ + \int_0^T \{ \int_\omega f \psi(t) d\omega \} dt \text{ in } \omega \times]0, T[\,, \end{aligned}$$

we have

$$\begin{aligned} \int_0^T \{ \int_\omega \frac{\partial^2 \xi_n(t)}{\partial t^2} \psi(t) d\omega \} dt &= - \int_0^T \{ \int_\omega \frac{\partial \xi_n(t)}{\partial t} \frac{\partial \psi(t)}{\partial t} d\omega \} dt \\ + \int_\omega \frac{\partial \xi_n(T)}{\partial t} \psi(T) d\omega - \int_\omega \frac{\partial \xi_n(0)}{\partial t} \psi(0) d\omega &= - \int_0^T \{ \int_\omega \frac{\partial \xi_n(t)}{\partial t} \frac{\partial \psi(t)}{\partial t} d\omega \} dt \\ &- \int_\omega \xi_{1n} \psi(0) d\omega. \end{aligned}$$

Since $\psi(T) = 0$, we also obtain

$$\begin{aligned} -2\rho \int_0^T \{ \int_\omega \frac{\partial \xi_n(t)}{\partial t} \frac{\partial \psi(t)}{\partial t} d\omega \} dt + \int_0^T a(\xi_n(t), \psi(t)) dt &- \\ 2 \int_0^T \{ \int_\omega \left[\widetilde{\Phi}_n(t), \xi_n(t) + \widetilde{\theta} \right] \psi(t) d\omega \} dt &= \\ 2 \int_0^T \{ \int_\omega \left[\chi, \xi_n(t) + \widetilde{\theta} \right] \psi(t) d\omega \} dt + \int_0^T \{ \int_\omega f \psi(t) d\omega \} dt &+ \\ 2\rho \int_\omega \xi_{1n} \psi(0) d\omega \text{ in } \omega \times]0, T[\,. \end{aligned} \tag{1.60}$$

From (1.52), we get

$$\int_0^T \{ \int_\omega \left[\widetilde{\Phi}_n(t), \xi_n(t) + \widetilde{\theta} \right] \psi(t) d\omega \} dt = \int_0^T \{ \int_\omega \left[\widetilde{\Phi}_n(t), \psi(t) \right] (\xi_n(t) + \widetilde{\theta}) d\omega \} dt,$$

we have

$$[\widetilde{\Phi}_n(t), \psi(t)] \rightharpoonup [\widetilde{\Phi}(t), \psi(t)] \text{ in } L^2(\omega \times]0,T[),$$

and since $\xi_n(t) \to \xi(t)$ in $L^2(\omega\times]0,T[)$, we obtain

$$\begin{aligned} \int_0^T \{\int_\omega \left[\widetilde{\Phi}_n(t), \xi_n(t) + \widetilde{\theta}\right] \psi(t) d\omega\} dt &\to \int_0^T \{\int_\omega \left[\widetilde{\Phi}(t), \psi(t)\right] (\xi(t) + \widetilde{\theta}) d\omega\} dt \\ &= \int_0^T \{\int_\omega \left[\widetilde{\Phi}(t), \xi(t) + \widetilde{\theta}\right] \psi(t) d\omega\} dt. \end{aligned}$$

We have

$$[\chi, \xi_n(t) + \widetilde{\theta}] \rightharpoonup [\chi, \xi(t) + \widetilde{\theta}] \text{ in } L^2(\omega\times]0,T[),$$

thus

$$\int_0^T \{\int_\omega \left[\chi, \xi_n(t) + \widetilde{\theta}\right] \psi(t) d\omega\} dt \to \int_0^T \{\int_\omega \left[\chi, \xi(t) + \widetilde{\theta}\right] \psi(t) d\omega\} dt.$$

Then passing to the limit in (1.60), we obtain

$$\begin{aligned} -2\rho \int_0^T \{\int_\omega \frac{\partial \xi(t)}{\partial t} \frac{\partial \psi(t)}{\partial t} d\omega\} dt + \int_0^T a(\xi(t), \psi(t)) dt &- \\ 2 \int_0^T \{\int_\omega \left[\widetilde{\Phi}(t), \xi(t) + \widetilde{\theta}\right] \psi(t) d\omega\} dt &= \\ 2 \int_0^T \{\int_\omega \left[\chi, \xi(t) + \widetilde{\theta}\right] \psi(t) d\omega\} dt + \int_0^T \{\int_\omega f \psi(t) d\omega\} dt &+ \\ 2\rho \int_\omega \xi_1 \psi(0) d\omega \text{ in } \omega \times]0,T[, & \end{aligned} \tag{1.61}$$

for all ψ of the form (1.59).

Passing to the limit, we deduce that (1.61) still true for all $\psi(t) \in L^2(0,T;V(\omega))$ such that $\frac{\partial \psi(t)}{\partial t} \in L^2(0,T;L^2(\omega))$ and $\psi(T) = 0$[1].

Then $(\xi, \widetilde{\Phi})$ satisfies

$$2\rho \frac{\partial^2 \xi}{\partial t^2} - \partial_{\alpha\beta} m_{\alpha\beta} \left(\nabla^2 \xi\right) = 2 \left[\widetilde{\Phi} + \chi, \xi + \widetilde{\theta}\right] + f \text{ in } \omega \times]0,T[,$$

and

$$\frac{\partial \xi}{\partial t}(0) = \xi_1.$$

[1]This comes from the density of functions of the form (1.59) in the space of functions $\psi(t) \in L^2(0,T;V(\omega))$ such that $\frac{\partial \psi(t)}{\partial t} \in L^2(0,T;L^2(\omega))$ with $\psi(T) = 0$ see [39, 77].

Taking into account (1.55) and (1.56), then applying [76, Lemma 1.2], we deduce that

$$\xi_n(0) \rightharpoonup \xi(0) \text{ in } L^2(\omega),$$

and we obtain

$$\xi_n(0) = \xi_{0n} \to \xi_0 \text{ in } V(\omega),$$

thus

$$\xi(0) = \xi_0.$$

It remains to be shown

$$\Delta^2 \widetilde{\Phi} = -\frac{1}{2}\left[\xi, \xi + 2\widetilde{\theta}\right] \text{ in } \omega \times]0, T[\,,$$

noting that

$$[\xi_n(t), \xi_n(t) + 2\widetilde{\theta}] \to [\xi(t), \xi(t) + 2\widetilde{\theta}] \text{ in } D'(\omega \times]0, T[),$$

if $\phi \in D(\omega \times]0, T[)$, we obtain

$$[\phi, \xi_n(t) + 2\widetilde{\theta}] \rightharpoonup [\phi, \xi(t) + 2\widetilde{\theta}] \text{ in } L^2(\omega \times]0, T[),$$

and from (1.58), we deduce that

$$\begin{aligned}
\int_0^T \{\int_\omega \left[\xi_n(t), \xi_n(t) + 2\widetilde{\theta}\right] \phi d\omega\} dt &= \int_0^T \{\int_\omega \left[\phi, \xi_n(t) + 2\widetilde{\theta}\right] \xi_n(t) d\omega\} dt \\
&\to \int_0^T \{\int_\omega \left[\phi, \xi(t) + 2\widetilde{\theta}\right] \xi(t) d\omega\} dt \\
&= \int_0^T \{\int_\omega \left[\xi(t), \xi(t) + 2\widetilde{\theta}\right] \phi d\omega\} dt,
\end{aligned}$$

and passing to the limit in (1.50) for $m = n$, we have

$$\widetilde{\Phi}(t) = -\frac{1}{2} G_2 \left[\xi(t), \xi(t) + 2\widetilde{\theta}\right] \text{ in } \omega \times]0, T[\,,$$

thus

$$\Delta^2 \widetilde{\Phi} = -\frac{1}{2}\left[\xi, \xi + 2\widetilde{\theta}\right] \text{ in } \omega \times]0, T[\,.$$

This completes the proof. ■

Conclusion

The application of the asymptotic expansions method to the three-dimensional nonlinear elastodynamic shallow shells, with a specific class of boundary conditions of generalized Marguerre-von Kàrmàn type, shows that the leading term of the asymptotic expansions is characterized by two-dimensional dynamic problem which depends on the Airy function Φ and the vertical component ζ_3 of the displacement field of the middle surface of the shallow shell. We then establish an existence theorem for these equations by means of a compactness method.

Note that, in the case $\gamma = \gamma_1$, the dynamic equations of generalized Marguerre-von Kàrmàn shallow shells reduce to the dynamic equations of classical Marguerre-von Kàrmàn shallow shells. If the function $\theta \equiv 0$ in $\bar{\omega}$, we recover the dynamic equations of generalized von Kàrmàn plate.

Chapter 2

Signorini problem in three dimensional elasticity

Introduction

Let Ω be an open bounded subset of $\mathbb{R}^3$ occupied by an elastic, isotropic homogeneous linear material. The boundary of Ω is divided into three parts $\Gamma_0, \Gamma_1, \Gamma_2$ ($\text{mes}(\Gamma_2) > 0$). Suppose that this material goes into contact with a rigid foundation on Γ_0 and subjected to a surface force g on Γ_1 and a volume force f in Ω. See Figure2.1 Suppose that the system is in static state and the contact on Γ_0 is with Signorini conditions. Our objective is to find the displacements of the points of $\bar{\Omega}$.

2.1 Frictionless contact

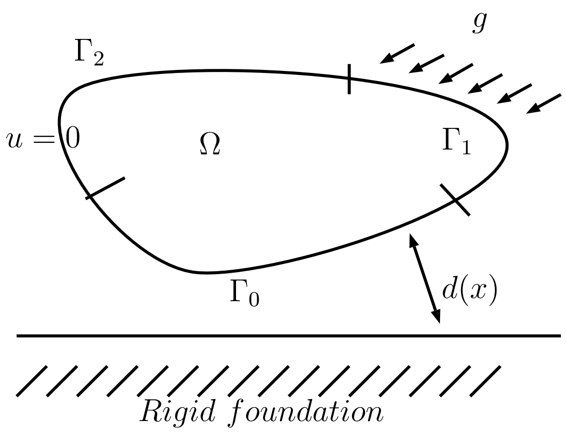

Figure 2.1: A body occupying a domain Ω goes in contact against a rigid foundation.

2.1.1 Classical problem C.P

The previous phenomenon is interpreted by the following problem: Find u such that

$$\begin{cases} -\mathrm{div}\sigma(u) = f \text{ in } \Omega & (2.1) \\ \sigma(u)n = g \text{ on } \Gamma_1 & (2.2) \\ u = 0 \text{ on } \Gamma_2 & (2.3) \\ u_N \le d \ , \sigma_N \le 0 \ , \sigma_N(u_N - d) = 0 \ , \sigma_T = 0 \text{ on } \Gamma_0 & (2.4) \end{cases}$$

Equation (2.1) is the equilibrium equation of the system where $\sigma(u) = (\sigma_{ij}(u))$ is the stress tensor and $\sigma_{ij}(u) = a_{ijkl}e_{kl}(u)$ where $e_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ and $e(u) = (e_{ij}(u))$ is the linearized strain tensor where $\partial_i u_j = \frac{\partial u_j}{\partial x_i}$. We note by $\sigma(u) = Ae(u)$, this is called the constitutive law of the elastic body.

Assume that the coefficients $a_{ijkl} \in L^\infty(\Omega)$ verify the property of symmetry and the ellipticity i.e.,

$$\begin{aligned} a_{ijkl} &= a_{jikl} = a_{klij} \\ \exists\, M > 0 \ , \ a_{ijkl}\, e_{ij}(u)\, e_{kl}(u) &\ge M e_{ij}(u)\, e_{ij}(u) \ , \forall e_{ij} = e_{ji}. \end{aligned}$$

Equations (2.2) and (2.3) are the conditions imposed on the edges respectively Γ_1 and Γ_2. The conditions (2.4) are called the Signorini conditions, $u_N = un$ denotes the normal component of displacement and n is the outward normal.

$u_N \le d$ (contact condition).

$\sigma_N = (\sigma(u)n)$ is the normal component of the force applied to a section with normal n.

$\sigma_N(u_N - d) = 0$ means detachment or contact.

$\sigma_T = 0$ means no friction, no shear.

d is the gap function defined on Γ_0 measured in the normal direction and assumed in $H^{\frac{1}{2}}(\Gamma_0)$.

2.1.2 Variational problem (V.P)

We denote

$$\mathbf{H}^1(\Omega) = \left(H^1(\Omega)\right)^3,$$
$$\mathbf{L}^2(\Omega) = \left(L^2(\Omega)\right)^3,$$

KV

$$V = \left\{v \in \mathbf{H}^1(\Omega) \ / \ v = 0 \text{ on } \ \Gamma_2\right\},$$

and

$$K = \{v \in V \ / \ v_n \leq 0 \text{ on } \ \Gamma_0\}$$

is a convex closed subset in V.

Theorem 26 *If u is a solution of $(C.P)$ then u verifies the problem $(V.P)$:*

$$\begin{cases} \textit{Find } u \in K \textit{ such that}: \\ a(u,v) = l(v) + \langle \sigma_N, v_N \rangle, \ \forall v \in V & (2.5) \\ \langle \sigma_N, v_N - u_N \rangle \geq 0, \ \forall v \in K & (2.6) \end{cases}$$

where

$$a(u,v) = \int_\Omega \sigma(u) : e(v)\, dx,$$
$$l(v) = \int_\Omega f v dx + \int_{\Gamma_1} \mathbf{g} v d\Gamma,$$
$$\langle \sigma_N, v_N \rangle = \int_{\Gamma_0} \sigma_N v_N d\Gamma,$$

Proof. Weak formulation of (2.1):
Let $v \in K$, the equation (2.1) gives

$$\int_\Omega -\text{div}\sigma(u)\, v dx = \int_\Omega f v dx \tag{2.7}$$

We have

$$\int_\Omega \text{div}\sigma(u)\, v dx = \int_{\partial\Omega} \sigma(u)\, n v \ d\Gamma - \int_\Omega \sigma(u) : e(v)\, dx \tag{2.8}$$

with $\sigma(u) : e(v) = \sigma_{ij}(u) \ e_{ij}(v)$.

Combining (2.8) with (2.7) and using (2.2) and (2.3) we find (2.5).
Weak formulation of the unilateral contact condition or complementarity condition: we have

$$\begin{aligned} \langle \sigma_N, v_N - u_N \rangle &= \langle \sigma_N, v_N - d + d - u_N \rangle \\ &= \langle \sigma_N, v_N - d \rangle + \langle \sigma_N, d - u_N \rangle \\ &= \langle \sigma_N, v_N - d \rangle \ \geq 0 \end{aligned}$$

hence (2.6). ■

Theorem 27 *Assume that the solution u is regular enough, then u is a solution of* $(C.P)$ *if and only if u is solution of* $(V.P)$.

Proof. Taking in (2.5) $v = \varphi$ for all $\varphi \in (D(\Omega))^3$ (since v remains in V). It is found that

$$a(u, \varphi) = l(\varphi), \forall \varphi \in (D(\Omega))^3 \tag{2.9}$$

where, using the Green formula in (2.9), we find

$$\int_\Omega (-\text{div}\sigma(u) - f)\, \varphi dx = 0$$

therefore

$$-\text{div}\sigma(u) - f = 0 \quad \text{a.e in } \Omega \tag{2.10}$$

then we have (2.1). For (2.2), we take $v = \varphi$ for all $\varphi \in (D(\Omega \cup \Gamma_2))^3$ in (2.5), and taking into account (2.10), we find $\int_{\Gamma_2} (\sigma(u) n - g)\, \varphi\, d\Gamma = 0$ whence (2.2). We take $v = \varphi$ in (2.5) for all $\varphi \in (D(\Omega \cup \Gamma_0))^3$ with (2.9) we find $\langle \sigma_T, \varphi_T \rangle = 0$ whence $\sigma_T = 0$ on Γ_0.

We take $v = u + \varphi$ in (2.6) with $\varphi \in (D(\Omega \cup \Gamma_0))^3$ and $\varphi_N \leq 0$ on Γ_0, we find that $\langle \sigma_N, \varphi_N \rangle \geq 0$ giving $\sigma_N \leq 0$ on Γ_0. After that we take $v_N = d$ then $v_N = 2v_N - d$ in (2.6), we obtain $\langle \sigma_N, u_N - d \rangle = 0$. Since $\sigma_N (u_N - d) \geq 0$ then $\sigma_N (u_N - d) = 0$ on Γ_0 therefore (2.4) is proved. ■

2.1.3 Existence and uniqueness

Theorem 28 *If* $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_1)$ *then the problem* $(V.P)$ *admits a unique solution in* **V**.

Proof. see [10] and [21] ■

2.2 Frictional contact with Tresca law

We assume that exerts a force on an elastic body that comes into contact with a rigid foundation. Note that if the tangential force exceeds a certain threshold, the body loses its resistance and goes on slip. This phenomenon is interpreted by the law of Tresca.

So this law imposes on the contact area the following conditions :

$$\begin{aligned}
&|\sigma_T| \leq s, s \text{ is the threshold of friction}\\
&|\sigma_T| < s \longrightarrow u_T = 0\\
&|\sigma_T| = s \longrightarrow \exists \delta > 0, u_T = -\delta \sigma_T
\end{aligned}$$

2.2.1 Classical problem (V.P)

The strong formulation of the problem is : find u such that

$$\begin{cases}
-\text{div}\sigma(u) = f \quad \text{in} \quad \Omega & (2.11)\\
\sigma(u)\, n = g \text{ on } \Gamma_1 & (2.12)\\
u = 0 \text{ on } \Gamma_2 & (2.13)\\
u_N \leq d \ , \sigma_N \leq 0 \ , \sigma_N (u_N - d) = 0 \text{ on } \Gamma_0 & (2.14)\\
|\sigma_T| < s \longrightarrow u_T = 0 \text{ on } \Gamma_0 & (2.15)\\
|\sigma_T| = s \longrightarrow \exists \delta > 0, u_T = -\delta \sigma_T \text{ on } \Gamma_0 & (2.16)
\end{cases}$$

2.2.2 Variational problem (V.P)

Theorem 29 *If u is a solution of $(C.P)$ then u verifies the problem $(V.P)$*

$$\begin{cases}
\text{Find } u \in K \text{ such that} & \\
a(u,v) = l(v) + \langle \sigma_N, v_N \rangle + \langle \sigma_T, v_T \rangle \ , \ \forall v \in V & (2.17)\\
\langle \sigma_N, v_N - u_N \rangle \geq 0, \ \forall v \in K & (2.18)\\
\langle \sigma_T, v_T - u_T \rangle + \langle s, |v_T| - |u_T| \rangle \geq 0 \ , \forall v_T \in V_T & (2.19)
\end{cases}$$

where $V_T = \left\{ v \in \left(H^1(\Omega)\right)^3 \ / \ v = 0 \ on \ \Gamma_2 \right\}$

Proof. From (2.11), we find

$$\int_\Omega -\text{div}\sigma(u)\ v dx = \int_\Omega f \ v dx, \forall v \in V \quad (2.20)$$

On the other hand, using the Green formula, we have

$$\begin{aligned}
\int_\Omega -\text{div}\sigma(u)\, v dx = \int_\Omega \sigma(u) : e(v)\, dx - \int_{\Gamma_0} \sigma(u)\, n v d\Gamma \\
- \int_{\Gamma_2} \sigma(u)\, n v d\Gamma - \int_{\Gamma_1} \sigma(u)\, n v d\Gamma \quad (2.21)
\end{aligned}$$

Using (2.12)-(2.13) in (2.21), and taking in account (2.20), we find (2.17). To prove (2.18), we have

$$\begin{aligned}\langle \sigma_N, v_N - u_N \rangle &= \langle \sigma_N, v_N - d + d - u_N \rangle \\ &= \langle \sigma_N, v_N - d \rangle + \langle \sigma_N, d - u_N \rangle \\ &= \langle \sigma_N, v_N - d \rangle \ \geq 0\end{aligned}$$

which is the weak formulation of Signorini condition. To prove the weak formulation of the friction conditions, we have

$$-\int_{\Gamma_0} \sigma_T v_T d\Gamma \leq \int_{\Gamma_c} |\sigma_T| \, |v_T| \, d\Gamma \leq \int_{\Gamma_0} s \, |v_T| \, d\Gamma \tag{2.22}$$

For $u_T = 0$ or $u_T = -\delta\sigma_T$ we have

$$\int_{\Gamma_0} \sigma_T u_T d\Gamma = \int_{\Gamma_0} -\delta \, |\sigma_T|^2 \, d\Gamma = -\int_{\Gamma_c} |\sigma_T| \, |u_T| \, d\Gamma = \int_{\Gamma_0} s \, |u_T| \, d\Gamma \tag{2.23}$$

At the end, from (2.22) and (2.23), we deduce (2.19). ■

Theorem 30 *Assume that u is regular enough. Then u is solution of $(C.P)$ if and only if u is solution of $(V.P)$.*

Proof. See Duvaut-Lions [39] ■

2.2.3 Existence and uniqueness

We need some tools to establish the existence and uniqueness of the solution of the variational problem. Let

$$j(v) = \int_{\Gamma_0} s \, |v_T| \, d\Gamma, \ v \in K$$

and

$$I(v) = \frac{1}{2} a(u, v) - l(v), v \in K.$$

We have

- j is convex in K, non linear and non differentiable.
- I is strictly convex and Gateaux differentiable: $I'(u)\, u = a(u, u) - l(u)$, $u \neq 0$.

Corollary 31 *Let $u \in K$. u is a solution of the variational problem $(V.P)$ if and only if u is a solution of the variational inequality:*

$$a(u, v-u) - l(v-u) + j(v) - j(u) \geq 0, \forall v \in K \tag{2.24}$$

Proof. Suppose that u, the solution of the variational problem, is regular enough. Equation (2.17) gives

$$a(u, u) = l(u) + \langle \sigma_N, v_N - u_N \rangle + \langle \sigma_T, v_T - u_T \rangle. \tag{2.25}$$

To obtain (2.24), it suffices to insert (2.18) and (2.19) in (2.17). Conversely, suppose that u is sufficiently regular and satisfies (2.24). Introducing the test function $v = u \pm \varphi$, with $\varphi \in (D(\Omega))^3$, in the inequality (2.24) then we obtain (2.17). After inserting (2.11) and (2.12) in (2.24), we find

$$\langle \sigma_N, v_N - u_N \rangle + \langle \sigma_T, v_T - u_T \rangle + \langle s, |v_T| - |u_T| \rangle \geq 0, \forall v_T \in V \tag{2.26}$$

we take $v_T = u_T$ then $v_N = u_N$ in (2.25), we obtain respectively (2.18) and (2.19). ■

Corollary 32 *The following two problems are equivalent:*

$$(i) \begin{cases} \text{Find } u \text{ minimizing} \\ F(v) = I(v) + j(v), \ \forall v \in K. \end{cases} \tag{2.27}$$

$$(ii) \quad \begin{cases} \text{Find } u \text{ in } K \text{ such as} \\ a(u, v-u) - l(v-u) + j(v) - j(u) \geq 0, \forall v \in K. \end{cases} \tag{2.28}$$

Proof. We first show that (i) implies (ii). Let u satisfying (i), we have: $\forall v \in K$, $u + t(v-u) \in K$,$\forall t \in]0,1[$. So we have

$$\begin{aligned} F(u) &\leq F(u + t(v-u)) \\ &\leq I(u + t(v-u)) + j(u + t(v-u)) \end{aligned}$$

where:

$$I(u) + j(u) \leq I(u + t(v-u)) + (1-t) j(u) + t\, j(v)$$

therefore

$$\frac{I(u + t(v-u)) - I(u)}{t} + j(v) - j(u) \geq 0$$

Letting t goes to zero, we find (ii).

Now we show the converse. Suppose I is convex and G-differentiable, then $I'(u)w = a(u, w) - l(w)$, $w \neq 0$, so $I(v) \geq I(u) + I'(u)(v-u)$ for all $v \in V$ hence u satisfies $ii)$.

When combined with (ii), we find:

$$I(v) \geq I(u) + j(u) - j(v)$$

whence

$$I(u) + j(u) \leq I(v) + j(v), \forall v \in K$$

as $u \in K$, it follows that:

$$F(u) \leq F(v), \forall v \in K$$

hence (ii). ■

Remark 33 *This corollary allows us to pass from the variational inequality problem to a minimization problem.*

Corollary 34 *Suppose that $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_1)$, $s \in L^2(\Gamma_0)$. So the minimization problem (2.27) admits at least one solution in K.*

Proof. To establish the existence of u, it suffices to show that F is weakly $l.s.c$ and coercive on K. I is a convex function and G-differentiable on K, it is weakly $l.s.c$ on K. So it suffices to show that j is $l.s.c.$ For this it suffices to show that its epigraph is closed. Which is easy to prove and this is due to the continuity of j.

Since the sum of two $l.s.c$ functions is $l.s.c$ function, then F is $l.s.c.$

It remains to establish the coercivity. According to the coercivity of $a(.,.)$, the linearity of $l(.)$ and the continuity of the trace mapping from $\mathbf{H}^1(\Omega)$ into $L^2(\Gamma)$, we have

$$|I(v)| \geq c\,\|v\|^2_{\mathbf{H}^1(\Omega)} - c'\,\|v\|^2_{\mathbf{H}^1(\Omega)}$$

and

$$j(v) = \int_{\Gamma_0} s\,|v_T|\,d\Gamma \leq \int_{\Gamma_0} s\,|v|\,d\Gamma \leq c''\,\|v\|_{\mathbf{H}^1(\Omega)} \text{ for } s \in L^2(\Gamma_0)$$

where $\lim F(v) = +\infty$ when $\|v\|_{\mathbf{H}^1(\Omega)} \longrightarrow \infty$ therefore F is coercive. Moreover, if F is strictly convex, this solution is unique. ■

Theorem 35 *For $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_1)$,and $s \in L^2(\Gamma_0)$ .The problem $(V.P)$ admits at least one solution.*

Proof. Just use the corollaries 31, 32 and 34. ■

2.3 Frictional contact with Coulomb's law

Besides the Tresca law, the experience shows that the threshold s is proportional to the normal component of the force exerted on the contact area, *i.e*, there exist a constant $\kappa > 0$ such that $s = \kappa |\sigma_N|$, κ is called the coefficient of friction which depends on the elastic and the rigid foundation. Hence the *Coulomb law* which is interpreted by the conditions

$$\begin{aligned}
&|\sigma_T| \leq \kappa |\sigma_N| \\
&|\sigma_T| < \kappa |\sigma_N| \longrightarrow u_T = 0 \\
&|\sigma_T| = \kappa |\sigma_N| \longrightarrow \exists \delta > 0, u_T = -\delta \sigma_T.
\end{aligned}$$

2.3.1 Classical problem (C.P)

Find u such that

$$\begin{cases}
-\text{div}\sigma(u) = f \ \text{ in } \ \Omega & (2.29)\\
\sigma(u)\, n = g \text{ on } \ \Gamma_1 & (2.30)\\
u = 0 \text{ on } \ \Gamma_2 & (2.31)\\
u_N \leq d \ \ , \sigma_N \leq 0 \ , \sigma_N (u_N - d) = 0 \text{ on } \Gamma_0 & (2.32)\\
|\sigma_T| < \kappa |\sigma_N| \longrightarrow u_T = 0 \text{ on } \Gamma_c & (2.33)\\
|\sigma_T| = \kappa |\sigma_N| \longrightarrow \exists \delta > 0, u_T = -\delta \sigma_T \text{ on } \Gamma_0 & (2.34)
\end{cases}$$

2.3.2 Variational problem (V.P)

Theorem 36 *If u is a solution of $(C.P)$ then u verifies the problem $(V.P)$*

$$\begin{cases}
\textit{Find } u \in K \textit{ such that}: \\
a(u, v) = l(v) + \langle \sigma_N, v_N \rangle + \langle \sigma_T, v_T \rangle \ , \forall v \in V & (2.35)\\
\langle \sigma_N, v_N - u_N \rangle \geq 0, \ \forall v \in K & (2.36)\\
\langle \sigma_T, v_T - u_T \rangle + \langle \kappa |\sigma_N| \, , |v_T| - |u_T| \rangle \geq 0 \ , \forall v_T \in V_T & (2.37)
\end{cases}$$

where $V_T = \left\{ v \in \left(H^1(\Omega)\right)^2 \ / \ v = 0 \ \textit{on} \ \ \Gamma_2 \right\}$

Proof. From (2.29), we find

$$\int_\Omega -\text{div}\sigma(u) \ v dx = \int_\Omega f \ v dx, \forall v \in V \tag{2.38}$$

On the other hand, using the Green formula, we have

$$\int_{\Omega} -\text{div}\sigma\,(u)\,vdx = \int_{\Omega} \sigma\,(u) : e\,(v)\,dx - \int_{\Gamma_0} \sigma\,(u)\,nvd\Gamma$$
$$- \int_{\Gamma_2} \sigma\,(u)\,nvd\Gamma - \int_{\Gamma_1} \sigma\,(u)\,nvd\Gamma. \tag{2.39}$$

Using (2.29)-(2.32) in (2.39), and introducing the result in (2.38), we find (2.35). We have

$$\begin{aligned} \langle \sigma_N, v_N - u_N \rangle &= \langle \sigma_N, v_N - d + d - u_N \rangle \\ &= \langle \sigma_N, v_N - d \rangle + \langle \sigma_N, d - u_N \rangle \\ &= \langle \sigma_N, v_N - d \rangle \ \geq 0 \end{aligned}$$

hence (2.36).
Weak formulation of the friction conditions: We have

$$-\int_{\Gamma_0} \sigma_T v_T d\Gamma \leq \int_{\Gamma_c} |\sigma_T|\,|v_T|\,d\Gamma \leq \int_{\Gamma_0} \kappa\,|\sigma_T|\,|v_T|\,d\Gamma \tag{2.40}$$

and $u_T = 0$ or $u_T = -\delta\sigma_T$ we have:

$$\int_{\Gamma_0} \sigma_T u_T d\Gamma = \int_{\Gamma_0} -\delta\,|\sigma_T|^2\,d\Gamma = -\int_{\Gamma_c} |\sigma_T|\,|u_T|\,d\Gamma = \int_{\Gamma_0} \kappa\,|\sigma_N|\,|u_T|\,d\Gamma \tag{2.41}$$

From (2.40) and (2.41), we deduce (2.37). ■

Theorem 37 *Assume that u is regular enough. Then u is a solution of $(C.P)$ if and only if u is a solution of $(V.P)$.*

Proof. see [39] ■

2.3.3 Existence theorem

Corollary 38 *The problem $(V.P)$ is equivalent to the following inequality :*

$$\textit{Find} \ \ u \ \textit{in} \ K \ \ \textit{such} \ \ \textit{as}:$$
$$a\,(u, v-u) - l\,(v-u) + j\,(v) - j\,(u) \geq \ 0 \quad \forall v \in K. \tag{2.42}$$

where

$$j\,(v) = \int_{\Gamma_0} \kappa\,|\sigma_N|\,|v_T|\,d\Gamma.$$

Proof. At first, we have from (2.35) that

$$a\,(u, v-u) = l\,(v-u) + \langle \sigma_N, v_N - \overline{u}_N \rangle + \langle \sigma_T, v_T - u_T \rangle \tag{2.43}$$

After inserting (2.29) and (2.30) in (2.43), we find

$$\langle \sigma_N, v_N - u_N \rangle + \langle \sigma_T, v_T - u_T \rangle + \langle \kappa\,|\sigma_N|\,, |v_T| - |u_N| \rangle \geq 0, \forall v_T \in V_T, \tag{2.44}$$

we take $v_T = u_T$ then $v_N = u_N$ in (2.44), we obtain respectively (2.36) and (2.37). For the converse, it suffices to insert (2.36) and (2.37) in (2.43). ■

Theorem 39 *Suppose that* $f_i \in L^{\,2}(\Omega)$, $g_i \in L^{\,2}(\Gamma_1)$, *and* $|\kappa|_{L^{\,\infty}(\Gamma_0)} < \kappa_0$. *For* κ_0 *small enough, the problem (V.P) admits at least one solution in* V.

We can see Nečas, Jarusek and Haslinger [81] for an existence result to a two-dimensional problem where they assume that the coefficient of friction is small enough. Recently, Eck and Jarusek [41] gave a demonstration using the penalty method.

2.4 Time dependent Signorini problem

In this section we present the same mechanical situation as in previous chapter but in the dynamic state. We consider the following problem

2.4.1 Classic problem C.P

$$\begin{cases} \text{Find } u \text{ such that} \\ \rho\ddot{u} - div\sigma(u) = f \quad \text{in } \Omega & (2.45) \\ \sigma(u)n = g \;\; \text{on } \Gamma_1 & (2.46) \\ u = 0 \;\; \text{on } \Gamma_2 & (2.47) \\ u_N \leq d\;, \sigma_N \leq 0, \sigma_N(u_N - d) = 0, \sigma_T = 0 \;\; \text{on } \Gamma_0 & (2.48) \\ u(x,0) = u_0,\; \dot{u}(x,0) = u_1 \;\; \text{in } \Omega. & (2.49) \end{cases}$$

2.4.2 Variational problem (V.P)

Let $H^1(\Omega) = (H^1(\Omega))^3, L^2(\Omega) = (L^2(\Omega))^3$

Theorem 40 *If u is a solution of the problem (C.P), then u verifies the problem (V.P):*

$$\begin{cases} \text{Trouver } u \in K, t \geq 0 \quad \text{such that} \\ \rho\langle \ddot{u}, v\rangle + a(u,v) = L(v) + < \sigma_N, v_N >, \forall v \in V & (2.50) \\ < \sigma_N, v_N - u_N > \geq 0, \quad \forall v \in K & (2.51) \\ u(x,0) = u_0; \dot{u}(x,0) = u_1 \text{ in } \Omega & (2.52) \end{cases}$$

where

$$a(u,v) = \int_\Omega \sigma(u) : e(u)dx, L(v) = \int_\Omega fvdx + \int_{\Gamma_1} gvd\Gamma,$$

$$< \sigma_N, v_N >= \int_{\Gamma_0} \sigma_N v_N d\Gamma \quad (\text{duality sens}).$$

Proof. *Weak formulation of (2.45):*
Let $v \in K, (2.45)$ then

$$\int_\Omega (\rho\ddot{u} - div\sigma(u))vdx = \int_\Omega fvdx \tag{2.53}$$

We have

$$\int_\Omega div\sigma(u)vdx = \int_{\partial\Omega} \sigma(u)nvd\Gamma - \int_\Omega \sigma(u) : e(v)dx, \tag{2.54}$$

with $\sigma(u) : e(v) = \sigma_{ij}(u) : e_{ij}(v)$. We introduce (2.54) in (2.53) and using (2.46) and (2.47) we find (2.49).
Weak formulation of the Signorini problem:
We have $\forall v \in K$,

$$\begin{aligned} \langle \sigma_N, v_N - u_N\rangle &= \langle \sigma_N, v_N - d + d - u_N\rangle \\ &= \langle \sigma_N, v_N - d\rangle + \langle \sigma_N, d - u_N\rangle \\ &= \langle \sigma_N, v_N - d\rangle \geq 0. \end{aligned}$$

hence (2.51). ■

Theorem 41 *We suppose that the solution u is regular enough, then u is a solution of (C.P) if and only if u is a solution of (V.P).*

Proof. Inserting (2.50) $v = \varphi$ for all $\varphi \in (D(\Omega))^3$ (because v stills in V), we have

$$\rho\langle \ddot{u}, \varphi\rangle + a(u,\varphi) = L(v), \ \forall \varphi \in (D(\Omega))^3 \tag{2.55}$$

Using the generalized Green formula in (2.55), we obtain:

$$\int (\rho\ddot{u} - div\sigma(u) - f)\varphi dx = 0$$

then

$$\rho\ddot{u} - div\sigma(u) - f = 0 \quad a.e \ \text{ on } \Omega \tag{2.56}$$

hence (2.45). To obtain (2.46), we take $v = \varphi$ for all $\varphi \in (D(\Omega \cup \Gamma_2))^3$ in (2.50) and taking in account (2.56), we find: $\int_{\Gamma_2} (\sigma(u)n - g)\varphi d\Gamma = 0$ hence (2.46). We take $v = \varphi$ in (2.50) with $\varphi \in (D(\Omega \cup \Gamma_0))^3$, taking in account (2.55) leads to $\langle \sigma_T, \varphi_T \rangle = 0 \ \forall \sigma_T$ then $\sigma_T = 0$ on Γ_0 *a.e.* After that, we choose $v = u + \varphi$ in (2.51) with $\varphi \in (D(\Omega \cup \Gamma_0))^3$ and $\varphi_N \leq 0$ on Γ_0 we find: $\langle \sigma_N, \varphi_N \rangle \geq 0$ which shows that $\sigma_N \leq 0$ on Γ_0. Taking $v_N = 0$ then $v_N = 2u_N - d$ in (2.51), we obtain $\langle \sigma_N, u_N - d \rangle = 0$ and since $\sigma_N(u_N - d) \geq 0$ then $\sigma_N(u_N - d) = 0$ on Γ_0. Then finally we have (2.48). ■

The associated variational inequality to the problem (V.P)

The variational problem (V.P) is equivalent to the following variational inequality:

$$\begin{cases} \text{Find } u \text{ such that} \\ \rho \int_\Omega \ddot{u}(v-u)\,dx + \int_\Omega \Lambda^{-1} e\,(u) : e\,(v-u)\,dx \geq L\,(v-u)\,\forall v \in K \\ u(x,0) = u_0; \dot{u}(x,0) = u_1 \text{ in } \Omega \end{cases} \tag{2.57}$$

For more details see [11]. To simplify the computations, we assume that $d = 0$.

2.4.3 Existence of solutions

Theorem 42 *Under the following conditions:*

$$f \in W^{2,\infty}\left(0, T; L^2(\Omega)\right),$$
$$g \in W^{2,\infty}\left(0, T; L^2(\Gamma)\right),$$
$$\textit{and } u_0, u_1 \in H^1(\Omega) \ \textit{ with } div\sigma(u_0) \in L^2(\Omega).$$

Then the problem $(C.P)$ *has at least one solution* $(u, \sigma(u))$ *verifying*

- $u \in L^\infty(0, T; K)$, $\dot{u} \in L^\infty\left(0, T; \left[L^2(\Omega)\right]^3\right)$, *and* $\ddot{u} \in D'\left(0, T; \left[L^2(\Omega)\right]^3\right)$.
- $\sigma(u) \in D'(0, T; E_{ad}(g)) \cap L^\infty\left(0, T; \left[L^2(\Omega)\right]^{3\times 3}\right)$.

The proof of the theorem is divided into five steps and is integrally taken from [9].

Step 1. The time discretization of the problem $(V.P)$

First, suppose that the interval time $[0, T]$ is divided into I equal sub-intervals. We define the following problem in the time $t = t_i$.

Problem $(P1^i)$

$$\text{Find } u^i \in V,\ \dot{u}^i \in \left[H^1(\Omega)\right]^3 \text{ and } \ddot{u}^i \in \left[L^2(\Omega)\right]^3 \text{ such that}$$

$$\int_\Omega \rho\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right)\left(v - u^i\right)dx + \int_\Omega \Lambda^{-1} e\left(\frac{u^i + u^{i-1}}{2}\right) : e\left(v - u^i\right)dx$$
$$\geq L^i\left(v - u^i\right), \forall v \in K \tag{2.58}$$

The following relations between sub-intervals of displacement, velocity and acceleration is given by the Newmark method (see Huges [52]) with parameters $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{2}$:

$$u^i = u^{i-1} + \Delta t \dot{u}^{i-1} + \frac{\Delta t^2}{2}\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}, \tag{2.59}$$

$$\dot{u}^i = \dot{u}^{i-1} + \Delta t \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2} \tag{2.60}$$

From (2.59), we have the acceleration in time t_i :

$$\ddot{u}^i = \frac{4\left[u^i - u^{i-1} - \Delta t \dot{u}^{i-1} - \frac{\Delta t^2}{4}\ddot{u}^{i-1}\right]}{\Delta t^2} \tag{2.61}$$

which allows us to write the variational inequality (2.58) only in terms of displacement. Thus, we consider the following algorithm:

- At initial instant $u^0 = u(0) = u_0$ and $\dot{u}^0 = \dot{u}(0) = u_1$, with $div\Lambda^{-1}e(u^0) \in L^2(\Omega)$, then the acceleration $\ddot{u}^0$ is calculated from the equilibrium equation:

$$\ddot{u}^0 = \frac{1}{\rho}\left(f^0 + div\Lambda^{-1}e\left(u^0\right)\right) \tag{2.62}$$

With $f^0 = f(0)$. Noting that $\ddot{u}^0 \in \left(L^2(\Omega)\right)^3$.

For each time step t_i, for given $u^{i-1}, \dot{u}^{i-1}$ and $\ddot{u}^{i-1}$, the displacement u^i is obtained as the solution of the following variational inequality

Problem $(P2^{\mathbf{i}})$ **:**

$$\text{Find } u^i \in K \text{ such that}$$

$$\int_\Omega \rho u^i.\left(v - u^i\right)dx + \frac{\Delta t^2}{4}\int_\Omega \Lambda^{-1}e\left(u^i\right) : e\left(v - u^i\right)dx \geq \int_\Omega \rho\left[u^{i-1} + \Delta t \dot{u}^{i-1}\right].\left(v - u^i\right)dx$$
$$- \frac{\Delta t^2}{4}\int_\Omega \Lambda^{-1}e\left(u^{i-1}\right) : e\left(v - u^i\right)dx + \frac{\Delta t^2}{2}L^i\left(v - u^i\right) \qquad \forall v \in K. \tag{2.63}$$

Next, $\ddot{u}^i$ and $\dot{u}^i$ are calculated by (2.61) and (2.60). Once we know u^i, $\dot{u}^i$ and $\ddot{u}^i$ in t_i we repeat the process for the next time step.

Theorem 43 *Let $u^{i-1} \in K$, $\dot{u}^{i-1} \in H^1(\Omega)$ and $\ddot{u}^{i-1} \in L^2(\Omega)$. Then there exists a unique solution u^i of the problem (**P2**^{i}). In addition, $\dot{u}^i$ and $\ddot{u}^i$ constructed from relations (2.60) and (2.61) verify:*

$$\dot{u}^i \in H^1(\Omega), \ddot{u}^i \in L^2(\Omega), \text{ and } \ddot{u}^i + \ddot{u}^{i-1} \in H^1(\Omega), \quad 1 \leq i \leq I \tag{2.64}$$

Proof. Assuming the following bilinear form b, continuous and coercive on $V \times V$:

$$b(u,v) = \int_\Omega \rho u.v dx + \frac{\Delta t^2}{4} \int_\Omega \Lambda^{-1} e(u) : e(v) \, dx \tag{2.65}$$

and the linear mapping defined on V by:

$$\begin{aligned} L(v) = \frac{\Delta t^2}{2} L^i(v) + \int_\Omega \rho \left[u^{i-1} + \Delta t \dot{u}^{i-1} \right] v dx \\ - \frac{\Delta t^2}{4} \int_\Omega \Lambda^{-1} \varepsilon \left(u^{i-1} \right) : e(v) \, dx. \end{aligned}$$

The existence and uniqueness result is obtained by applying Stampacchia theorem. Now, as $u^i, u^{i-1}, \dot{u}^{i-1}$ belong to $H^1(\Omega)$ from (2.59) we get that $\ddot{u}^i + \ddot{u}^{i-1} \in H^1(\Omega)$. Thus, since $\ddot{u}^{i-1} \in L^2(\Omega)$, and that $\ddot{u}^i \in L^2(\Omega)$, we obtain from (2.60) that $\dot{u}^i \in H^1(\Omega)$. ■

Remark 44 *Assumptions on the initial conditions ensure that $u_0 \in H^1(\Omega)$, $\dot{u}^0 \in H^1(\Omega)$ et $\ddot{u}^0 \in L^2(\Omega)$. Therefore, Theorem 43 guarantees that the displacement, velocity and acceleration fields are calculated with the previous algorithm for all $0 \leq i \leq I$.*

Corollary 45 *Suppose that $u^{i-1} \in H^1(\Omega)$, $\dot{u}^{i-1} \in H^1(\Omega)$ and $\ddot{u}^{i-1} \in L^2(\Omega)$ are known. If $(u^i, \dot{u}^i, \ddot{u}^i)$ is solution of the problem $(P1^i)$. Then, u^i is a solution of the problem $(P2^i)$, $1 \leq i \leq I$.*
Conversely, if u^i is the solution of problem $(P2^i)$ and $\dot{u}^i$ and $\ddot{u}^i$ are defined by (2.60) and (2.61) then $(u^i, \dot{u}^i, \ddot{u}^i)$ is solution of the problem $(P1^i)$.

The properties of u^i

First, to simplify the notation, we denote:

$$h^i = \frac{u^i + u^{i-1}}{2}, 1 \leq i \leq I \tag{2.66}$$

Corollary 46 *If u^i is a solution of $(P2^i)$ and $\sigma(h^i) = \Lambda^{-1}e(h^i)$. Then*

$$\rho\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2} - div\sigma\left(h^i\right) = f^i \quad on \quad \Omega \tag{2.67}$$

where $\ddot{u}^i$ is obtained by (2.59) and (2.60), $f^i = f(t_i)$ and h^i is given by (2.66). Also $\sigma(h^i) \in E_{ad}(g^i)$.

Proof. The result is a direct consequence of the relation (2.2.12) and the proposition 2.2.5 in ([9]). ■

Corollary 47 *If $(u^i, \dot{u}^i, \ddot{u}^i)$ is a solution of the problem $(P1^i)$ and $\sigma(h^i) = \Lambda^{-1}e(h^i), 1 \leq i \leq I$. Then (2.67) holds, $\sigma(h^i) \in E_{ad}(g^i)$ and*

$$\langle \pi_n\left(\sigma\left(h^i\right)\right)|_{\Gamma_1}, u_n^i|_{\Gamma_1}\rangle_{\Gamma_1} = 0. \tag{2.68}$$

Proof. The proof is trivial, using Proposition (2.2.7) in [9]. ■

Step 2. Approximations of the solution of the problem (P)

First approximation

In this paragraph, we construct by the mean of $\{u^i, \dot{u}^i, \ddot{u}^i\}$ $i = 0, 1, ...I$, a function $h^I(t)$ defined on $[0, T]$ such that h^I is of the class $C^1([0, T])$ and of class C^2 on (t_{i-1}, t_i) $i = 1, 2, ...I$. To this end, we define the function h^I as follows:

$$h^I = u^{i-1} + \dot{u}^{i-1}(t - t_{i-1}) + \frac{\ddot{u}^{i-1} + \ddot{u}^i}{4}(t - t_{i-1})^2, \forall t \in [t_{i-1}, t_i) \tag{2.69}$$

and at the instant $t = T$, $h^I(T) = u^I, \dot{h}^I(T) = \dot{u}^I$. This choice guarantees the compatibility of $h^I(t)$ with Newmark scheme. The function $h^I(t)$ verifies the following properties:

- $h^I(t) \in H^1(\Omega), \forall t \in [0, T]$.
- $$\dot{h}^I(t) = \dot{u}^{i-1} + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}(t - t_{i-1}),\ t \in [t_{i-1}, t_i) \tag{2.70}$$
- $\dot{h}^I(t_i) = \dot{u}^i$, for all $i = 0, ..I$,
- $\ddot{h}^I \in L^\infty(0, T; H^1(\Omega))$, such that:

$$\ddot{h}^I = \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}, t \in [t_{i-1}, t_i). \tag{2.71}$$

- At time t_i by (2.59) and (2.60), we obtain for $i = 1, ..I$

$$\lim_{t \to t_i} h^I(t) = u^{i-1} + \dot{u}^i \Delta t + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{4} \Delta t^2 = u^i = h^I(t_i), \qquad (2.72)$$

$$\lim_{t \to t_i} \dot{h}^I(t) = \dot{u}^{i-1} + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2} \Delta t = \dot{u}^i = \dot{h}^I(t_i), \qquad (2.73)$$

then $h^I(t) \in C(0, T; H^1(\Omega))$.

Remark 48 *Note that $h^I(t_k) \in K$, but there is no guarantee that $h^I(t) \in K$ for all $t \in [0, T]$.*

Another approximations

Now we define below other approximations of a solution of the problem (P) which are also convergent when $I \longrightarrow \infty$:

$$\begin{cases} l^I(t) = u^{i-1} + \dfrac{u^i - u^{i-1}}{\Delta t}(t - t_{i-1}), \quad \forall t \in [t_{i-1}, t_i) & (2.74) \\ h^I_*(t) = h^i = \dfrac{u^i + u^{i-1}}{2}, \quad \forall t \in [t_{i-1}, t_i) & (2.75) \\ h^I_\#(t) = \dot{u}^i, \quad \forall t \in [t_{i-1}, t_i) & (2.76) \\ u^I_*(t) = u^i, \quad \forall t \in [t_{i-1}, t_i) & (2.77) \end{cases}$$

Remark 49 *Note that in this case $l^I(t), h^I_*(t), u^I_*(t)$ belong to K for all $t \in [0, T]$.*

Step 3. A prior estimations

Lemma 2 *If $(u^i, \dot{u}^i, \ddot{u}^i)$ is the solution of problem $(P1^i)$, $1 \le i \le I$ and h^I defined by (2.69). Then, we have:*

$$\rho \int_{t_{i-1}}^{t_i} \frac{1}{2} \frac{d}{dt} \| \dot{h}^I(t) \|^2_{L^2(\Omega)} dt + \int_{t_{i-1}}^{t_i} \frac{1}{2} \frac{d}{dt} a\left(h^I(t), h^I(t)\right) dt \le L^i\left(u^i - u^{i-1}\right) \qquad (2.78)$$

where

$$a(u, v) = \int_\Omega \Lambda^{-1} e(u) : e(v)\, dx.$$

Proof. Taking $v = u^{i-1}$ in (2.58) gives:

$$\int_\Omega \rho \left(\ddot{u}^i + \ddot{u}^{i-1}\right) . \frac{u^i - u^{i-1}}{2} dx + \int_\Omega \Lambda^{-1} e\left(u^i + u^{i-1}\right) : e\left(\frac{u^i - u^{i-1}}{2}\right) dx$$
$$\leq L^i \left(u^i - u^{i-1}\right), \tag{2.79}$$

First, we rewrite the first member of equation (2.79) in terms of $h^I(t)$. Using (2.59), we can show that displacement verifies

$$\frac{u^i - u^{i-1}}{2} = \frac{\Delta t}{2}\dot{u}^{i-1} + \frac{\Delta t^2}{4}\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}, \tag{2.80}$$

$$u^i + u^{i-1} = 2u^{i-1} + \Delta t \dot{u}^{i-1} + \frac{\Delta t^2}{2}\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2} \tag{2.81}$$

Substituting these expressions in the first member of equation (2.79) we obtain

$$\rho\frac{\Delta t}{2}\int_\Omega \left(\ddot{u}^i + \ddot{u}^{i-1}\right).\dot{u}^{i-1} dx + \rho\frac{\Delta t^2}{4}\int_\Omega \left(\ddot{u}^i + \ddot{u}^{i-1}\right).\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) dx$$
$$+\Delta t \int_\Omega \Lambda^{-1} e\left(u^{i-1}\right) : e\left(\dot{u}^{i-1}\right) dx + \frac{\Delta t^2}{2}\int_\Omega \Lambda^{-1} e\left(u^{i-1}\right) : e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) dx$$
$$+\frac{\Delta t^2}{2}\int_\Omega \Lambda^{-1} e\left(\dot{u}^{i-1}\right) : e\left(\dot{u}^{i-1}\right) dx + \frac{\Delta t^3}{4}\int_\Omega \Lambda^{-1} e\left(\dot{u}^{i-1}\right) : e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) dx$$
$$+\frac{\Delta t^4}{4}\int_\Omega \Lambda^{-1} e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}\right) : e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) dx$$
$$+\frac{\Delta t^3}{4}\int_\Omega \Lambda^{-1} e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) : e\left(\dot{u}^{i-1}\right) dx \leq L^i\left(u^i - u^{i-1}\right) \tag{2.82}$$

Now, the first member of this expression is

$$\rho\int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt} \parallel \dot{h}^I(t) \parallel^2_{L^2(\Omega)} dt + \int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt} a\left(h^I(t), h^I(t)\right) dt.$$

Indeed, the definition $h^I(t)$, $t \in [t_{i-1}, t_i)$ shows that:

$$\parallel \dot{h}^I(t) \parallel^2_{L^2(\Omega)} = \int_\Omega \left(\dot{u}^{i-1} + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}(t - t_{i-1})\right).\left(\dot{u}^{i-1} + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}(t - t_{i-1})\right) dx,$$

and

$$\frac{1}{2}\frac{d}{dt} \parallel \dot{h}^I(t) \parallel^2_{L^2(\Omega)} = \int_\Omega \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}.\left(\dot{u}^{i-1} + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}(t - t_{i-1})\right) dx.$$

Thus,

$$\int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt} \parallel \dot{h}^I(t) \parallel_{L^2(\Omega)}^2 dt = \Delta t \int_\Omega \dot{u}^{i-1}.\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2} dx + \frac{\Delta t^2}{2}\int_\Omega \left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right)^2 dx \tag{2.83}$$

also, on $[t_{i-1}, t_i)$:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} a\left(h^I(t), h^I(t)\right) &= \int_\Omega \Lambda^{-1} e\left(\dot{h}^I(t)\right) : e\left(h^I(t)\right) dx \\
&= \int_\Omega \Lambda^{-1} e\left(\dot{u}^{i-1} + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}(t - t_{i-1})\right) : \\
&\quad e\left(u^{i-1} + \dot{u}^{i-1}(t - t_{i-1}) + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}(t - t_{i-1})^2\right) dx \\
&= \int_\Omega \Lambda^{-1} e\left(\dot{u}^{i-1}\right) : e\left(u^{i-1}\right) dx + \int_\Omega \Lambda^{-1} e\left(\dot{u}^{i-1}\right) : e\left(\dot{u}^{i-1}\right)(t - t_{i-1}) dx \\
&\quad + \int_\Omega \Lambda^{-1} e\left(\dot{u}^{i-1}\right) : e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}\right)(t - t_{i-1})^2 dx \\
&\quad + \int_\Omega \Lambda^{-1} e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) : e\left(u^{i-1}\right)(t - t_{i-1}) dx \\
&\quad + \int_\Omega \Lambda^{-1} e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) : e\left(\dot{u}^{i-1}\right)(t - t_{i-1})^2 dx \\
&\quad + \int_\Omega \Lambda^{-1} e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) : e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}\right)(t - t_{i-1})^3 dx.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt} a\left(h^I(t), h^I(t)\right) dt &= \Delta t \int_\Omega \Lambda^{-1} e\left(\dot{u}^{i-1}\right) : e\left(u^{i-1}\right) dx \\
+ \frac{\Delta t^2}{2}\int_\Omega \Lambda^{-1} e\left(\dot{u}^{i-1}\right) : e\left(\dot{u}^{i-1}\right) dx &+ \frac{\Delta t^3}{3}\int_\Omega \Lambda^{-1} e\left(\dot{u}^{i-1}\right) : e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}\right) dx \\
&+ \frac{\Delta t^2}{2}\int_\Omega \Lambda^{-1} e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) : e\left(u^{i-1}\right) dx \\
&+ \frac{\Delta t^3}{3}\int_\Omega \Lambda^{-1} e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}\right) : e\left(\dot{u}^{i-1}\right) dx \\
&+ \frac{\Delta t^4}{4}\int_\Omega \Lambda^{-1} e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) : e\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}\right).
\end{aligned} \tag{2.84}$$

And from (2.82) - (2.84), we conclude the result. ■

Proposition 50 *Let $h^I(t)$ defined by (2.59). Then :*

- $\|h(t_k)\|_{H^1(\Omega)}$ *is bounded by a constant independent of I and $k, 1 \leq k \leq I$.*
- h^I *and* $\dot{h}^I$ *are bounded in $C(0,T;L^2(\Omega))$ by a constant independent of I.*

Proof. For all k such that $1 \leq k \leq I$ we have the following expression:

$$\sum_{i=1}^{k} \rho \int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt} \parallel \dot{h}^I(t) \parallel^2_{L^2(\Omega)} dt = \frac{\rho}{2}\left(\parallel \dot{h}^I(t_k) \parallel^2_{L^2(\Omega)} - \parallel \dot{h}^I(0) \parallel^2_{L^2(\Omega)} \right)$$

$$\sum_{i=1}^{k} \rho \int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt} a\left(h^I(t), h^I(t)\right) dt = \frac{1}{2}\left(a\left(h^I(t_k), h^I(t_k)\right) - a\left(h^I(0), h^I(0)\right)\right)$$

Then, by Lemma 2 we deduce that for $1 \leq k \leq I$

$$\frac{1}{2}\left(\rho \parallel \dot{h}^I(t_k) \parallel^2_{L^2(\Omega)} + a\left(h^I(t_k), h^I(t_k)\right)\right) \leq \frac{1}{2}\left(\parallel \dot{h}^I(0) \parallel^2_{L^2(\Omega)} + a\left(h^I(0), h^I(0)\right)\right) + \sum_{i=1}^{k} L^i\left(u^i - u^{i-1}\right). \tag{2.85}$$

Now, we obtain an upper bound for the second term on the right side of (2.85). As

$$u^i - u^{i-1} = h^I(t_i) - h^I(t_{i-1}) \tag{2.86}$$

we deduce that:

$$\sum_{i=1}^{k} L^i\left(u^i - u^{i-1}\right) = \sum_{i=1}^{k} L^i(h^I(t_i) - h^I(t_{i-1}))$$
$$= \left(\sum_{i=1}^{k-1} \left(L^i - L^{i+1}\right)\left(h^I(t_i)\right) + L^k(h^I(t_k)) - L^1(h^I(0)) \right). \tag{2.87}$$

We have also that:

$$\begin{aligned} \left|L^i(h(t_k))\right| &\leq \left| \int_\Omega f^i h(t_k) \right| + \left| \int_{\Gamma_1} g^i h(t_k) \right| \\ &\leq \left\|f^i\right\|_2 \|h(t_k)\|_2 + \left\|g^i\right\|_{2,\Gamma_1} \|h(t_k)\|_{H^{1/2}} \\ &\leq (\|f\|_{L^\infty(0,T;L^2(\Omega))} + c\,\|g\|_{L^\infty(0,T;L^2(\Gamma_1))}\, \|h(t_k)\|_{H^1(\Omega)} \\ &\leq C_1 \|h(t_k)\|_{H^1(\Omega)} \end{aligned} \tag{2.88}$$

and

$$\begin{aligned}\left|L^i\left(h\left(t_k\right)\right)-L^{i+1}\left(h\left(t_k\right)\right)\right| &\leq \left\|f^i-f^{i+1}\right\|_2\left\|h\left(t_k\right)\right\|_2+\left\|g^i-g^{i+1}\right\|_2\left\|h\left(t_k\right)\right\|_{H^{1/2}} \\ &\leq \left(\left\|\dot{f}\right\|_{L^\infty(0,T;L^2(\Omega))}+c\left\|\dot{g}\right\|_{L^\infty(0,T;L^2(\Omega))}\right)\left\|h\left(t_k\right)\right\|_{H^1} \\ &\leq C_2\left\|h\left(t_k\right)\right\|_{H^1} \qquad (2.89)\end{aligned}$$

Then, by taking the absolute value of (2.87), using (2.88) and (2.89) and applying Hlder's inequality we get:

$$\left|\sum_{i=1}^{k} L^i\left(u^i-u^{i-1}\right)\right| \leq C_1\Delta t\sum_{i=i}^{k-1}\| h^I\left(t_i\right)\|_{H^1}+C_2\left(\| h^I\left(t_k\right)\|_{H^1}+\| h^I\left(0\right)\|_{H^1}\right),$$

where C_1, C_2 are positive constants . Thus, from (2.85), it follows that:

$$\begin{aligned}\left(\| \dot{h}^I\left(t_k\right)\|^2_{L^2(\Omega)}+\frac{1}{\rho}a\left(h^I\left(t_k\right),h^I\left(t_k\right)\right)\right) &\leq \left(\| \dot{h}^I\left(0\right)\|^2_{H^1}+\frac{1}{\rho}a\left(h^I\left(0\right),h^I\left(0\right)\right)\right) \\ &+\frac{2}{\rho}\left(C_2\Delta t\sum_{i=1}^{k-1}\| h^I\left(t_i\right)\|_{H^1}+C_1\left(\| h^I\left(t_k\right)\|_{H^1}+\| h^I\left(0\right)\|_{H^1}\right)\right) \qquad (2.90)\end{aligned}$$

Now, since the bilinear form $a(..)$ Is coercive, the above equation allows us to conclude that there are positive constants $C_1, C_2, 3$, which do not depend on I or k such that:

$$\| h^I\left(t_k\right)\|^2_{H^1(\Omega)}\leq C_1+C_2\| h^I\left(t_k\right)\|_{H^1(\Omega)}+C_3\Delta t\sum_{i=1}^{k-1}\| h^I\left(t_i\right)\|_{H^1(\Omega)}, \qquad (2.91)$$

for $1\leq k\leq I$. Without loss of generality, we assume that $C_1>1$, the previous inequality implies that:

$$\| h^I\left(t_k\right)\|_{H^1(\Omega)}\leq C_1+C_2+C_3\Delta t\sum_{i=1}^{k-1}\| h^I\left(t_i\right)\|_{H^1(\Omega)} \qquad (2.92)$$

Applying discrete Gronwall Lemma (see Lions [75]), we get that :

$$\| h^I\left(t_k\right)\|_{H^1(\Omega)}\leq Ce^T, C\in\mathbf{R}^+ \qquad (2.93)$$

Then $h^I\left(t_k\right)$ is bounded in $H^1\left(\Omega\right)$ by a constant independent of I and k.

It is easy to obtain from (2.90) that $\dot{h}^I\left(t_k\right)$ is bounded in $L^2\left(\Omega\right)$ by a constant independent of I and k. Therefore, since $\dot{h}^I$ is piecewise linear

$$\| \dot{h}^I\left(t\right)\|^2_{L^2(\Omega)}\leq\max\left\{\| \dot{h}^I\left(t_{i-1}\right)\|^2_{L^2(\Omega)},\| \dot{h}^I\left(t_i\right)\|^2_{L^2(\Omega)}\right\}\quad\forall t\in\left[t_{i-1},t_i\right].$$

Then $\dot{h}^{I}(t)$ is bounded in $L^2(\Omega)$, and as it is continuous, then it is bounded in $C(0,T;L^2(\Omega))$. Then

$$h^I(t) = h^I(0) + \int_0^t \dot{h}^I(s)\,ds$$

hence $h^I(t)$ is bounded in $C(0,T;L^2(\Omega))$ by a constant independent of I for all $t \in (0,T)$ ■

Corollary 51 *Let h^I defined by (2.59). Then, there exist subsequences indexed by I, such that, when $I \to +\infty$ we obtain the following convergences:*

$$h^I \rightharpoonup h \, weak^* \text{ in } L^\infty\left(0,T;L^2(\Omega)\right), \tag{2.94}$$

$$\dot{h}^I \rightharpoonup \dot{h} \;\; weak^* \text{ in } L^\infty\left(0,T;L^2(\Omega)\right) \tag{2.95}$$

for some h in $L^\infty(0,T;L^2(\Omega))$.

Another a prior estimations

Proposition 52 *Let $l^I(t), h_*^I(t), h_\#^I(t)$ et $u_*^I(t)$ defined by (2.74)-(2.77) respectively, then :*

i) $\| h_*^I(t) \|_{\mathbf{H}^1(\Omega)}$ and $\| u_*^I(t) \|_{\mathbf{H}^1(\Omega)}$ are bounded by a constant independent of I for all $t \in [0,T]$.

ii) $\| l^I(t) \|_{\mathbf{H}^1(\Omega)}, \| \dot{l}^I(t) \|_{\mathbf{L}^2(\Omega)}$ and $\| h_\#^I(t) \|_{\mathbf{L}^2(\Omega)}$ are bounded by a constant independent of I for all $t \in [0,T]$.

iii) $\ddot{h}^I$ is bounded in $L^\infty(0,T;H^{-1}(\Omega))$.

Proof. From the definition of h^I we have $u^i = h^I(t_i)$ for all $0 \le i \le I$, then h_*^I can be expressed by :

$$h_*^I(t) = \frac{h^I(t_i) + h^I(t_{i-1})}{2}, \quad \forall t \in [t_{i-1}, t_i) \tag{2.96}$$

and

$$u_*^I(t) = h^I(t_i), \quad \forall t \in [t_{i-1}, t_i) \tag{2.97}$$

also $h_*^I \in L^\infty(0,T;H^1(\Omega))$ sice $u^i \in H^1(\Omega)$ for all i, and

$$\| h_*^I(t) \|_{H^1(\Omega)} = \| \frac{h^I(t_i)+h^I(t_{i-1})}{2} \|_{H^1(\Omega)}$$
$$\leq \frac{1}{2}\left(\| h^I(t_i) \|_{H^1(\Omega)} + \| h^I(t_{i-1}) \|_{H^1(\Omega)}\right), \quad \forall t \in [t_{i-1},t_i) \tag{2.98}$$

Therefore, by the Proposition 50 $h_*^I(t)$ is bounded in $L^\infty(0,T;H^1(\Omega))$ by a constant independent of I. In the same way, $u_*^I(t)$ is bounded in $L^\infty(0,T;H^1(\Omega))$ by a constant independent of I. Now, since l^I is piecewise linear,

$$\| l^I(t) \|_{H^1(\Omega)} \leq \left\{ \| l^I(t_{i-1}) \|_{H^1(\Omega)}, \| l^I(t_i) \|_{H^1(\Omega)} \right\} \quad \forall t \in [t_{i-1},t_i],$$

and since $l^I(t_k) = h^I(t_k)$ and $\| h^I(t_k) \|_{H^1(\Omega)}$ is bounded by a constant independent of I and k, $\|l^I(t)\|_{H^1(\Omega)}$ is bounded by a constant independent of I. To show that $\dot{l}^I$ is bounded, we express in terms of $\dot{h}^I$:

$$\dot{l}^I(t) = \frac{u^i + u^{i-1}}{\Delta t},$$

which can be written, using (2.80) and (2.70) as :

$$\dot{l}^I(t) = \dot{u}^{i-1} + \frac{\Delta t}{2}\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2} = \frac{\dot{h}^I(t_{i-1}) + \dot{h}^I(t_i)}{2}, \forall t \in [t_{i-1},t_i).$$

Then

$$\|\dot{l}^I(t)\|_{L^2(\Omega)} = \left\| \frac{\dot{h}^I(t_{i-1}) + \dot{h}^I(t_i)}{2} \right\|_{L^2(\Omega)} \tag{2.99}$$

Thus, again by Proposition 50, $\| \dot{l}^I(t) \|_{\mathbf{L}^2(\Omega)}$ is bounded by a constant independent of I for all $t \in (0,T)$. In the same way :

$$\| h_\#^I(t) \|_{L^2(\Omega)} = \| \dot{h}^I(t_i) \|_{L^2(\Omega)}, \forall t \in [t_{i-1},t_i).$$

Finally, the boundedness of $\ddot{h}^I$ in $L^\infty(0,T;H^{-1}(\Omega))$ is obtained as a direct consequence of the previous boundedness. Equation (2.67) can be written in terms of $\ddot{h}^I$ and h_*^I as :

$$\rho \ddot{h}^I - div\sigma\left(h_*^I\right) = f_0^I \text{ in } \Omega \tag{2.100}$$

such that : $f_0^I(t) = f_0^i$ for all $t \in [t_{i-1},t_i)$. ■

Step 4. Weak convergence of approximate solutions to the same limit

Corollary 53 *Let $l^I(t), h_*^I(t), h_\#^I(t)$ and $u_*^I(t)$ defined by (2.74)–(2.77) respectively. Then, there exist extracted subsequences (still indexed by I) such that :*

$$l^I \rightharpoonup l weak^* \text{ in } L^\infty\left(0,T;H^1(\Omega)\right), \tag{2.101}$$

$$\dot{l}^I \rightharpoonup \dot{l} \text{ weak}^* \text{ in } L^\infty\left(0,T;L^2(\Omega)\right), \tag{2.102}$$

$$h_*^I \rightharpoonup h_* weak^* \text{ in } L^\infty\left(0,T;H^1(\Omega)\right), \tag{2.103}$$

$$u_*^I \rightharpoonup u_* weak^* \text{ in } L^\infty\left(0,T;H^1(\Omega)\right), \tag{2.104}$$

$$h_\#^I \rightharpoonup h_\# \text{ weak}^* \text{ in } L^\infty\left(0,T;L^2(\Omega)\right), \tag{2.105}$$

$$\ddot{h}^I \rightharpoonup \ddot{h} \text{ weak}^* \text{ in } L^\infty\left(0,T;H^1(\Omega)\right). \tag{2.106}$$

when $I \to \infty$ and have the same limit.

Proof. The proof is similar to the one of Corollary 51. ■

Corollary 54 *Let $\dot{h}^I$ and l^I defined by (2.69) and (2.74) respectively. Then, there exist extracted sub sequences (still indexed by I) verify, when $I \to +\infty$:*

$$l^I \to l \text{ in } C\left([0,T];\mathbf{H}^\beta(\Omega)\right) \cap C_s\left([0,T];\mathbf{H}^1(\Omega)\right), \ 0 \le \beta < 1, \tag{2.107}$$

$$\dot{h}^I \to \dot{h} \text{in } C\left([0,T];\mathbf{H}^\alpha(\Omega)\right) \cap C_s\left([0,T];\mathbf{L}^2(\Omega)\right), \ -1 \le \alpha < 0, \tag{2.108}$$

eventually after a modification on a set of zero measure.

Proof. The proof of this result is a consequence of the boundedness of $\dot{h}^I, \ddot{h}^I$, l and $\dot{l}$ and Lemma 8.1, page 297 of [77], Corollary 4, p. 85 of [93]. ■

Uniqueness of the limit

In this section, we will show that all the limits in (2.94), (2.103), (2.106) and (2.107) are equal : $h = l = h_* = u_*$. First, from (2.75) and using Barrow formula for C^1

functions on $[t_{i-1}, t_i]$, we have:

$$\left\|h^I(t) - h^I_*(t)\right\|_{L^2(\Omega)} = \left\|\frac{h^I(t)}{2} + \frac{h^I(t)}{2} - \frac{h^I(t_{i-1})}{2} - \frac{h^I(t_i)}{2}\right\|_{L^2(\Omega)}$$
$$= \left\|\frac{1}{2}\int_{t_{i-1}}^{t} \dot{h}^I(s)\,ds - \frac{1}{2}\int_{t}^{t_i} \dot{h}^I(s)\,ds\right\|_{L^2(\Omega)} \leq \left\|\frac{1}{2}\int_{t_{i-1}}^{t} \dot{h}^I(s)\,ds\right\|_{L^2(\Omega)}$$
$$+ \left\|\frac{1}{2}\int_{t}^{t_i} \dot{h}^I(s)\,ds\right\| \leq \Delta t \left\|\dot{h}^I(s)\right\|_{L^\infty(0,T;L^2(\Omega))} \to 0, \tag{2.109}$$

as $I \to +\infty$. Then h and h_* are equal in $L^\infty(0,T;L^2(\Omega))$, and since $h_* \in L^\infty(0,T;H^1(\Omega))$ also $h \in L^\infty(0,T;H^1(\Omega))$. Similarly, we prove that h and u_* are equal in $H^1(\Omega)$. In addition, as $\dot{l}$ is bounded in $L^\infty(0,T;L^2(\Omega))$,

$$\left\|l^I(t) - u^{I\prime}_*(t)\right\|_{L^2(\Omega)} = \left\|l^I(t) - l^I(t_i)\right\|_{L^2(\Omega)}$$
$$= \left\|\int_{t}^{t_i} \dot{l}^I(s)\,ds\right\|_{L^2(\Omega)} \leq \Delta t \left\|\dot{l}^I\right\|_{L^\infty(0,T;L^2(\Omega))} \to 0, \tag{2.110}$$

when $I \to +\infty$. Similarly, we show that $\dot{h}$ and $h_\#$ coincide in $L^2(\Omega)$. Then, $l = u_* = h_* = h$. Henceforth, we denote this limit by u. In summary, we have shown the following convergences:

Theorem 55 *Let h^I, l^I, h^I_*, $h^I_\#$ and u^I_* given by (2.69),(2.74),(2.75),(2.76) and (2.77) respectively, then there exists u such that:*

$$\begin{aligned}
h^I &\rightharpoonup u \text{ weak}^* \text{ in } L^\infty\left(0,T;L^2(\Omega)\right),\\
\dot{h}^I &\rightharpoonup \dot{u} \text{ weak}^* \text{ in } L^\infty\left(0,T;L^2(\Omega)\right)\\
\dot{l}^I &\rightharpoonup u \text{ weak}^* \text{ in } L^\infty\left(0,T;H^1(\Omega)\right)\\
l^I &\rightharpoonup \dot{u} \text{ weak}^* \text{ in } L^\infty\left(0,T;L^2(\Omega)\right)\\
h^I_* &\rightharpoonup u \text{ weak}^* \text{ in } L^\infty\left(0,T;H^1(\Omega)\right)\\
u^I_* &\rightharpoonup u \text{ weak}^* \text{ in } L^\infty\left(0,T;H^1(\Omega)\right)\\
h^I_\# &\rightharpoonup \dot{u} \text{ weak}^* \text{ in } L^\infty\left(0,T;L^2(\Omega)\right)\\
\sigma\left(h^I_*\right) &\rightharpoonup \sigma(u) \text{ weak}^* \text{ in } L^\infty\left(0,T;L^2(\Omega)\right).
\end{aligned}$$

Furthermore, we have the following limits :

$$\lim_{I\to+\infty}\left(h^I - h_*^I\right) = 0, \tag{2.111}$$

$$\lim_{I\to+\infty}\left(h^I - u_*^I\right) = 0, \tag{2.112}$$

$$\lim_{I\to+\infty}\left(l^I - u_*^I\right) = 0, \tag{2.113}$$

strongly in $L^\infty\left(0,T;L^2\left(\Omega\right)\right)$.

Corollary 56 *Let h^I and l^I are given by (2.69) and (2.74) respectively. Then there exists u such that*

$$\dot{h}^I \to \dot{u} \text{ in } C\left(0,T;H^\alpha\left(\Omega\right)\right)\cap C_s\left(\left[0,T\right];L^2\left(\Omega\right)\right), \quad -1\leq\alpha\leq 0, \tag{2.114}$$

$$l^I \to \ u \text{ in } C\left(\left[0,T\right];H^\beta\left(\Omega\right)\right)\cap C_s\left(\left[0,T\right];H^1\left(\Omega\right)\right),\ 0\leq\beta\leq 1. \tag{2.115}$$

Proof. From Corollary 54 and uniqueness of the weak limit. ■

Theorem 57 *Let h_*^I and u_*^I are given by* (2.75) *and* (2.77) *respectively. Then*

$$h_*^I \ - u_*^I \quad\to\quad 0 \quad in \quad D'\left(0,T;H^1\left(\Omega\right)\right), \tag{2.116}$$

$$l^I \ - u_*^I \quad\to\quad 0 \quad in \quad L^\infty\left(0,T;H^r\left(\Omega\right)\right), \quad 0<r<1, \tag{2.117}$$

when $I\to+\infty$.

Proof. Let $\varphi\in D(0,T)$. Let $I\geq I_0$, where I_0 is such that the support of φ included in $[\delta_0, T-\delta_0]$ with $\delta_0 = T/2^{I_0}$, so that $supp(\varphi)\subset[\delta, T-\delta]$, $\delta = T/I$. We have

$$\begin{aligned}
\int_0^T\left(h_*^I \ - u_*^I\right)\varphi dt &= \sum_{i=0}^{I-1}\int_{t_i}^{t_{i+1}}\left(h_*^I\ \left(t\right) - u_*^I\ \left(t\right)\right)\varphi\left(t\right)dt\\
&= \sum_{i=0}^{I-1}\int_{t_i}^{t_{i+1}}\left(u^{i-1}-u^i\right)\theta_I\left(t\right)\varphi\left(t\right)dt\\
&= \sum_{i=1}^{I-1}\int_{t_i}^{t_{i+1}}u^i\left(\varphi_\delta-\varphi\right)\left(t\right)dt,
\end{aligned}$$

with $\varphi_\delta = \varphi(t+\delta)$ and $|\varphi - \varphi_\delta| \le c\delta$ with $c = \max\left|\frac{d\varphi}{dt}\right|$. Therefore,

$$\begin{aligned}
\left\|\int_0^T \left(h_*^I - u_*^I\right)\varphi dt\right\|_{H^1(\Omega)} &\le c\delta^2 \sum_{i=1}^{I-1} \left\|u^I\right\|_{H^1(\Omega)} \\
&\le c\delta^2 \left((I-1)\sum_{i=1}^{I-1} \left\|u^I\right\|_{H^1(\Omega)}^2\right)^{1/2} \\
&\le c\delta^2 \left((I-1)^2 C\right)^{1/2} \\
&\le \hat{c}\delta^2 I \\
&= \hat{c}\delta^2 T/\delta = \hat{c}\delta T \to 0 \text{ si } \delta \to 0,
\end{aligned}$$

where C and $\hat{c}$ are positive constants. To show (2.117) we use the following convergences :

$$\begin{aligned}
l^I &\rightharpoonup u \quad \text{weak}^* \text{ in } L^\infty\left(0,T;\mathbf{H}^1(\Omega)\right) \\
\dot{l}^I &\rightharpoonup \dot{u} \quad \text{weak}^* \text{ in } L^\infty\left(0,T;\mathbf{L}^2(\Omega)\right)
\end{aligned}$$

Let $0 < r = \theta\alpha + (1-\theta)\beta < 1$, $r \neq 1/2$. There exists a set $A \subset [0,T]$ such that $mes(A) = 0$ and for any $t_1, t_2 \in [0,T]\setminus A$, with $t_1 \le t_2$

$$\begin{aligned}
\left\|l^I(t_2) - l^I(t_1)\right\|_{\mathbf{H}^r(\Omega)} &\le M_\theta \left\|l^I(t_2) - l^I(t_1)\right\|_{L^2(\Omega)}^\theta \left\|l^I(t_2) - l^I(t_1)\right\|_{H^1(\Omega)}^{1-\theta} \\
&\le M_\theta \left(\int_{t_1}^{t_2} \left\|\dot{l}^I(t)\right\|_{L^2(\Omega)} dt\right)^\theta \\
&\le M_\theta (t_2 - t_1)^{\theta/2}. \qquad (2.118)
\end{aligned}$$

In particular, for any $t \in [t_{i-1}, t_i)\setminus A$, we have :

$$\left\|u_*^I(t) - l^I\ (t)\right\|_{H^r(\Omega)} = \left\|l^I\ (t_i) - l^I\ (t)\right\|_{H^r(\Omega)} \le M_\theta (t_i - t)^{\theta/2} \to 0,$$

when $I \to +\infty$, ■

Theorem 58 *Let u the limit defined in the Theorem 55, then:*

$$u \in L^\infty(0,T;K) \qquad (2.119)$$

$$\dot{u} \in L^\infty\left(0,T;L^2(\Omega)\right) \qquad (2.120)$$

$$\ddot{u} \in D'\left(0,T;L^2(\Omega)\right) \qquad (2.121)$$

$$\sigma(u) \in D'(0,T;H(div)) \cap L^\infty\left(0,T;L^2(\Omega)\right) \qquad (2.122)$$

Proof. According to the relation 2.58 we have

$$\rho \int_\Omega \ddot{h}^I v dx + a\left(h_*^I, v\right) \geq L^i(v) \quad \forall v \in K$$

this inequality remains true for $v(t) \in L^2(0,T;V)\,; v(t) \in K$ *p.p.* i.e.,;

$$\rho \int_0^T \int_\Omega \ddot{h}^I v(t)\, dxdt + \int_0^T a\left(h_*^I, v(t)\right) dt \geq \int_0^T L_*^I(v(t))\, dt, \quad \forall v \in L^2(0,T;V)$$

hence, we choose $v(t) = \varphi(t)\,\omega(t)$ such that $\varphi(t) \in D(0,T)\,;\ \omega(t) \in D(\Omega)$. Integrating by parts, we find

$$-\rho \int_0^T \int_\Omega \dot{h}^I \dot{v}(t)\, dxdt + \int_0^T a\left(h_*^I, v(t)\right) dt \geq \int_0^T \int_\Omega f_*^I v(t) dxdt,$$

Next, taking $v(t) = -\varphi(t)\,\omega(t)$, we find

$$\rho \int_0^T \int_\Omega \ddot{h}^I v(t)\, dxdt + \int_0^T a\left(h_*^I, v(t)\right) dt = \int_0^T \int_\Omega f_*^I v(t)\, dt, \tag{2.123}$$

We can show also that, when $I \to \infty$

$$\int_0^T a\left(h_*^I, v(t)\right) dt \to \int_0^T a(u, v(t))\, dt \tag{2.124}$$

and that

$$\int_0^T \int_\Omega f_*^I v(t)\, dxdt \to \int_0^T \int_\Omega f v(t)\, dxdt. \tag{2.125}$$

As a consequence

$$-\rho \int_0^T \int_\Omega \dot{u}\dot{v}(t)\, dxdt + \int_0^T a(u, v(t))\, dt = \int_0^T \int_\Omega f v(t)\, dxdt \quad \text{for } v \in D((0,T)\times\Omega)$$

To prove (2.119), we have that $u_*^I = u \in K$ and since $u_*^I \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;H^1(\Omega))$ then $u_*^I \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;K)$ hence (2.119). We have (2.120) directly from $\dot{h}^I \overset{*}{\rightharpoonup} \dot{u}$ in $L^\infty(0,T;L^2(\Omega))$ using the test function $v(t) \in D(0,T;K)$ we have

$$\rho \int_0^T \int_\Omega \ddot{u} v(t)\, dxdt - \int_0^T div\sigma(u)\, v(t)\, dt = \int_0^T \int_\Omega f v(t)\, dxdt.$$

$$\rho\ddot{u} - div\sigma(u) - f = 0 \ a.e \ sur \ Q = (0,T)\times\Omega. \tag{2.126}$$

As $\dot{u} \in L^\infty(0,T;L^2(\Omega))$, then $\ddot{u} \in D'(0,T;L^2(\Omega))$

$$\ddot{u} = \frac{1}{\rho}(div\sigma(u) + f) \quad \in L^\infty\left(0,T;H^{-1}(\Omega)\right)$$

$$\ddot{u} \in D'\left(0,T;L^2(\Omega)\right) \cap L^\infty\left(0,T;H^{-1}(\Omega)\right) \quad \text{hence (2.121)}$$

Since $\sigma\left(h_*^I\right) \to \sigma(u)$ in $L^\infty(0,T;L^2(\Omega))$ then $\sigma(u) \in L^\infty(0,T;L^2(\Omega))$ and $\sigma(u) \in H(div)$ implies that $\sigma(u) \in D'(0,T;H(div))$ hence $\sigma(u) \in L^\infty(0,T;L^2(\Omega)) \cap D'(0,T;H(div))$ therefore (2.122). ■

Step 5. Characterization of u by the problem $(V.P)$

In this step, we will show that the weak limit u is a weak solution of the problem (P). For this, it suffices to show that u is a solution of the problem:

$$(V.P)\begin{cases}\rho\langle\ddot{u},v\rangle+a(u,v)=L(v)+\langle\sigma_n(u),v_n\rangle \quad \forall v\in V\\ \langle\sigma_n(u),v_n-u_n\rangle\geq 0,\ \forall v\in K.\\ u(x,0)=u_0,\ \dot{u}(x,0)=u_1.\end{cases}$$

We consider the following problem:

$$\begin{cases}\rho\left\langle\ddot{h}^I,v\right\rangle+a\left(h^i,v\right)=L^i(v)+\left\langle\sigma_n\left(h^i\right),v_n\right\rangle \quad \forall v\in V\\ \left\langle\sigma_n\left(h^i\right),v_n-u_n^i\right\rangle\geq 0,\ \forall v\in K\end{cases}$$

which comes from the properties of solutions of the problem $(P1^i)$. Using functions h_*^I and u_*^I, we define the above problem is the gap $[0,T]$ as a result [1]

$$\begin{cases}\rho\displaystyle\int_0^T\left\langle\ddot{h}^I,v(t)\right\rangle dt+\int_0^T a\left(h_*^I,v(t)\right)dt=\int_0^T L_*^I(v(t))\,dt+\int_0^T\left\langle\sigma_n\left(h_*^I\right),v_n(t)\right\rangle dt\\ \qquad\forall v\in L^1(0,T;V)\\ \displaystyle\int_0^T\left\langle\sigma_n\left(h_*^I\right),v_n(t)-u_{*n}^I\right\rangle dt\geq 0,\ \forall v\in L^1(0,T;K)\end{cases}$$

such that

$$L_*^I(v)=\int_\Omega f_*^I v\,dx+\int_{\Gamma_1}g_*^I v\ d\Gamma \ \text{ where } \ f_*^I(t)=f^i,\ g_*^I(t)=g^i \ \text{ on } \left]t_{i-1},t_i\right].$$

For $v(t)\in D(0,T;V)$ we have

$$\int_0^T\left\langle\ddot{h}^I,v(t)\right\rangle dt=-\int_0^T\left\langle\dot{h}^I,\dot{v}(t)\right\rangle dt \ \text{ au sens de } D'(0,T).$$

And since

$$\dot{h}^I\overset{*}{\rightharpoonup}\dot{u} \text{ in } L^\infty\left(0,T;L^2(\Omega)\right)$$
$$\text{i.e }\int_0^T\left\langle\dot{h}^I,g(t)\right\rangle dt\to\int_0^T\langle\dot{u},g(t)\rangle\,dt \quad \forall g\in L^1\left(0,T,L^2(\Omega)\right)$$

and the fact that $v(t)\in L^1(0,T,L^2(\Omega))$, then we can pass to the limit on I, we obtain:

$$\int_0^T\int_\Omega\dot{h}^I\dot{v}(t)\,dxdt\to\int_0^T\int_\Omega\dot{u}\dot{v}(t)\,dxdt,$$
$$\int_0^T a\left(h_*^I,v(t)\right)dt\to\int_0^T a(u,v(t))\,dt,$$

[1] We define this problem in order to pass to the limit weak star u.

$$\int_0^T \int_\Omega L_*^I (v(t))\, dxdt \to \int_0^T \int_\Omega L(v(t))\, dxdt.$$

Therefore we have

$$\rho \int_0^T \langle \ddot{u}, v(t) \rangle \, dt + \int_0^T a(u, v(t))\, dt = \int_0^T L(v(t))\, dt + \lim_{I\to\infty} \int_0^T \left\langle \sigma_n \left(h_*^I\right), v_n(t) \right\rangle dt \tag{2.127}$$

the form :

$$\lambda_n : v_n(t) \to \rho \int_0^T \langle \ddot{u}, v(t) \rangle \, dt + \int_0^T a(u, v(t))\, dt \ - \int_0^T L(v(t))\, dt$$

defines a continuous linear form on $D\left(0, T; H^{1/2}(\Gamma_0)\right)$. Then

$$\rho \int_0^T \langle \ddot{u}, v(t) \rangle \, dt + \int_0^T a(u, v(t))\, dt \ = \int_0^T L(v(t))\, dt + \int_0^T \langle \lambda_n, v_n(t) \rangle \, dt \quad \forall v \in D(0, T; V)$$

Furthermore

$$\int_0^T \left\langle \sigma_n \left(h_*^I\right), v_n(t) - u_{*n}^I \right\rangle dt \geq 0, \quad \forall v \in L^1(0, T; K)$$

then

$$\int_0^T \left\langle \sigma_n \left(h_*^I\right), v_n(t) \right\rangle dt \geq \int_0^T \left\langle \sigma_n \left(h_*^I\right), u_{*n}^I \right\rangle dt \quad \forall v \in L^1(0, T; K).$$

Hence, we have from (2.127) for $v = u_*^I$ in $D'\left(0, T; H^{-1/2}(\Gamma_0)\right)$, that

$$\begin{aligned}
\int_0^T \langle \lambda_n, v_n(t) \rangle \, dt \quad &\geq \lim_{I\to\infty} \int_0^T \left\langle \sigma_n \left(h_*^I\right), u_{*n}^I(t) \right\rangle dt \\
&\geq \lim_{I\to\infty} \left(\int_0^T \left\langle \ddot{h}^I, u_*^I \right\rangle dt + \int_0^T a\left(h_*^I, u_*^I\right) dt \ - \int_0^T L_*^I\left(u_*^I\right) dt \right)
\end{aligned}$$

On the other hand, we have

$$\int_0^T \left\langle \ddot{h}^I, u_*^I \right\rangle dt \to \int_0^T \langle \ddot{u}, u \rangle \, dt \quad \text{au sens } D'(0, T)$$

and

$$\int_0^T a\left(h_*^I, u_*^I\right) dt \ = \int_0^T a\left(h_*^I - u_*^I, u_*^I\right) dt + \int_0^T a\left(u_*^I, u_*^I\right) dt$$

hence from (2.116)

$$\int_0^T a\left(h_*^I - u_*^I, u_*^I\right) dt \to 0 \text{ au sens } D'(0, T)$$

and from the lower semi continuity of the norm defined by $a(.,.)$, we have

$$\lim_{I\to\infty}\int_0^T a\left(u_*^I,u_*^I\right)dt \ \geq \int_0^T a\left(u,u\right)dt$$

and

$$\int_0^T L_*^I\left(u_*^I\right)dt \to \int_0^T L\left(u\right)dt$$

Therefore

$$\int_0^T \left\langle \lambda_n, v_n\left(t\right)\right\rangle dt \geq \rho\int_0^T \left\langle \ddot{u},u\right\rangle dt + \int_0^T a\left(u,u\right)dt \ - \int_0^T L\left(u\right)dt = \int_0^T \left\langle \lambda_n, u_n\right\rangle dt$$

hence

$$\int_0^T \left\langle \lambda_n, v_n\left(t\right) - u_n\right\rangle dt \geq 0, \quad \text{in } D'\left(0,T;K\right)$$

Finally, we obtain

$$\begin{cases} \rho\int_0^T \left\langle \ddot{u},v\left(t\right)\right\rangle dt + \int_0^T a\left(u,v\left(t\right)\right)dt \ = \int_0^T L\left(v\left(t\right)\right)dt + \int_0^T \left\langle \lambda_n, v_n\right\rangle dt \quad \forall v\in D\left(0,T;V\right) \\ \int_0^T \left\langle \lambda_n, v_n\left(t\right)-u_n\right\rangle dt \geq 0, \quad \forall v\in D\left(0,T;K\right) \end{cases}$$

Remark 59 *Under regularity conditions, we can show that* $\lambda_n = \sigma_n\left(u\right)$.

Theorem 60 *Let* u *the limit presented in the Theorem 55. Then* u *verifies the initial condition 2.49.*

Proof. According to the Corollary 56

$$l^I \to u \quad \text{ en } \ C\left(\left[0,T\right];L^2\left(\Omega\right)\right).$$

Then, as $l^I\left(0\right) = u_0$ for all I, we can pass to the limit and obtain that : $u\left(0\right) = u_0$.
Furthermore, since:

$$\dot{h}^I \to \dot{u} \qquad \text{in } \ C_s\left(\left[0,T\right];L^2\left(\Omega\right)\right).$$

Then

$$\int_\Omega \dot{h}^I\left(0\right)vdx \rightharpoonup \int_\Omega \dot{u}\left(0\right).vdx, \quad \forall v \in L^2\left(\Omega\right).$$

As $\dot{h}^I\left(0\right) = u_1$ for all I, we conclude that:

$$\int_\Omega \dot{u}\left(0\right)vdx = \int_\Omega u_1 vdx, \quad \forall v \in L^2\left(\Omega\right)$$

and the initial conditions are fulfilled. ■

Conclusion

In this section, we presented an existence result of a weak solution of dynamic Signorini problem without friction in case of domain occupied by homogeneous and isotropic body. Among the issues encountered are those related to the regularity of the solution and the uniqueness of the solution. There was another issue related to extensions of the problem and the study of the same problem but with friction. And other issues related to the material:

- elastic non-linear.
- anisotropic.
- non homogeneous.

Chapter 3

Asymptotic modeling of a Coulomb frictional Signorini problem for the von Kármán plates

3.1 Introduction

We study in this chapter the asymptotic modeling of Coulomb frictional unilateral contact problem between an elastic nonlinear von Kármán plate and a rigid obstacle. To this end we use a formal asymptotic expansions method in terms of the half-thickness of the plate as the parameter. The leading term of the asymptotic expansion is characterized by two-dimensional von Kármán plate problem with Signorini conditions but without friction.

Our objective in this chapter is to answer the fifth open question by a formal asymptotic analysis. So we generalized the study of J.C. Paumier to the non linear plate of von

Kármán but by using a formal asymptotic expansions method.

3.2 Setting of the problem

We recall the following mechanical and geometrical assumptions introduced in the Section1.1. Let $\Omega^\varepsilon = \omega \times]-\varepsilon, +\varepsilon[$, where ε is a small parameter, be an open bounded set from $\mathbb{R}^3$, such that ω is an open subset from $\mathbb{R}^2$ with Lipschitz boundary γ. We denote the lateral boundary of Ω^ε by $\Gamma_0^\varepsilon = \gamma \times [-\varepsilon, \varepsilon]$, the upper and the lower faces are denoted, respectively, by Γ_+^ε and Γ_-^ε. We suppose that Ω^ε is occupied by a nonlinear, elastic, homogeneous, isotropic body. In its natural configuration: a plate of thickness 2ε whose Lamès constants are denoted $\lambda > 0, \mu > 0$ and assumed to be independent of ε. The plate is supposed to be subjected to a body force of density $f^\varepsilon \in (L^2(\Omega^\varepsilon))^3$, its lower face subjected to a surface force of density $g^\varepsilon \in \left(L^2(\Gamma_-^\varepsilon)\right)^3$ and submitted, on Γ_0^ε to applied surface forces of "von Kármán's type" which are horizontal, and only their resultant $\left(\widetilde{F}_1^\varepsilon, \widetilde{F}_2^\varepsilon\right) \in (L^2(\gamma))^2$ after integration across the thickness is given along the boundary γ. Therefore, the displacements u^ε derived from this situation verify u_α^ε independent of x_3^ε and $u_3^\varepsilon = 0$ on Γ_0^ε which mean that the only horizontal displacements of equal direction and magnitude are allowed along each vertical segment of the lateral face Γ_0^ε. For more details on the von Kármán equations we return to [32] and [19]. We suppose that the upper face Γ_+^ε of the plate is in unilateral Coulomb frictional contact with a rigid foundation. Let Λ denote the frictional coefficient, $\Theta^\varepsilon = \{x^\varepsilon \in \mathbb{R}^3 \ / \ (x_1^\varepsilon, x_2^\varepsilon) \in \omega \ , \ x_3^\varepsilon \geq \varepsilon d\}$ the foundation domain, where d (≥ 0) is the gap function defined on Γ_+^ε which describes the distance between the upper face and the rigid foundation measured in the normal direction. The function $\overline{v}$ denotes the trace of v on Γ_+^ε and $\underline{v}$ denotes the one of v on Γ_-^ε or simply by v if there is no confusion.

Our aim is to find the asymptotic behavior of the equilibrium state of the plate Ω^ε which is characterized by a displacement vector u^ε solution of the classical problem:

$$(CP^\varepsilon) \begin{cases} -\partial_j^\varepsilon \hat{\sigma}_{ij}^\varepsilon = f_i^\varepsilon \ \text{in}\, \Omega^\varepsilon \\ u_\alpha^\varepsilon \text{ independent of } x_3^\varepsilon \text{ and } u_3^\varepsilon \ = 0 \text{ on } \Gamma_0^\varepsilon \\ \dfrac{1}{2\varepsilon} \int_{-\varepsilon}^{+\varepsilon} \hat{\sigma}_{\alpha\beta}^\varepsilon . \nu_\beta^\varepsilon dx_3^\varepsilon = \widetilde{F}_\alpha^\varepsilon \text{ on } \gamma \\ \hat{\sigma}_{ij}^\varepsilon n_j^\varepsilon = g_i^\varepsilon \text{ on } \Gamma_-^\varepsilon \\ \bar{u}_3^\varepsilon \leq \varepsilon d, \ \hat{\sigma}_{33}^\varepsilon \leq 0, \ \hat{\sigma}_{33}^\varepsilon (\bar{u}_3^\varepsilon - \varepsilon d) = 0 \ \text{ on } \Gamma_+^\varepsilon \ \text{ (the Signorini conditions)} \\ \ |\hat{\sigma}_T^\varepsilon| < \Lambda |\hat{\sigma}_{33}^\varepsilon| \Rightarrow u_T^\varepsilon = 0 \quad \text{on } \Gamma_+^\varepsilon \\ \ |\hat{\sigma}_T^\varepsilon| = \Lambda |\hat{\sigma}_{33}^\varepsilon| \Rightarrow \exists \delta > 0, \, u_T^\varepsilon = -\delta \hat{\sigma}_T^\varepsilon, \ \hat{\sigma}_T^\varepsilon = (\hat{\sigma}_{\alpha 3}^\varepsilon) \quad \text{on } \Gamma_+^\varepsilon, \end{cases}$$

where: $\hat{\sigma}_{ij}^\varepsilon = \sigma_{ij}^\varepsilon + \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon$, $\sigma_{ij}^\varepsilon = \lambda E_{pp}^\varepsilon(u^\varepsilon)\, \delta_{ij} + 2\mu E_{ij}^\varepsilon(u^\varepsilon)$ the components of stress tensor, $E_{ij}^\varepsilon(u^\varepsilon) = \dfrac{1}{2}(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon + \partial_i^\varepsilon u_k^\varepsilon \partial_j^\varepsilon u_k^\varepsilon)$ the components of the nonlinear train tensor,

$n^\varepsilon = (n_i^\varepsilon)$ is the unit outer normal vector along the boundary of the plate Ω^ε, $\nu^\varepsilon = (\nu_\alpha^\varepsilon)$ is the unit outer normal vector along the boundary of the set ω and the subscripts T to the tangential components. To give a weak formulation of our problem we introduce some notations. Let

$$V(\Omega^\varepsilon) = \left\{v \in W^{1,4}(\Omega^\varepsilon)/v \text{ independent of } x_3^\varepsilon \text{ on } \Gamma_0^\varepsilon\right\},$$
$$V_0(\Omega^\varepsilon) = \left\{v \in W^{1,4}(\Omega^\varepsilon)/v = 0 \text{ on } \Gamma_0^\varepsilon\right\}$$
$$\vec{V}(\Omega^\varepsilon) = V(\Omega^\varepsilon) \times V(\Omega^\varepsilon) \times V_0(\Omega^\varepsilon),$$
$$K(\Omega^\varepsilon) = \left\{v \in V_0(\Omega^\varepsilon)/\bar{v} \le \varepsilon d \text{ on } \Gamma_+^\varepsilon\right\},$$
$$\vec{K}(\Omega^\varepsilon) = V(\Omega^\varepsilon) \times V(\Omega^\varepsilon) \times K(\Omega^\varepsilon).$$

Multiplying the system of equilibrium equations in (CP^ε) by functions v_i^ε and integrating over the set Ω^ε, using the Green formulas and the boundary conditions we obtain:
The variational formulation of the classical problem (CP^ε) is :

$$(VP^\varepsilon)\begin{cases} \text{Find } u^\varepsilon \in \vec{K}(\Omega^\varepsilon) \quad \text{such that:} \\ \int_{\Omega^\varepsilon} \hat{\sigma}_{ij}^\varepsilon \partial_j^\varepsilon v_i^\varepsilon dx^\varepsilon = L^\varepsilon(v^\varepsilon) + 2\varepsilon \int_\gamma \tilde{F}_\alpha^\varepsilon \tilde{v}_\alpha^\varepsilon d\gamma + \langle \hat{\sigma}_{i3}^\varepsilon, \bar{v}_i^\varepsilon\rangle,\ \forall\ v^\varepsilon \in \vec{V}(\Omega^\varepsilon) \\ \langle \hat{\sigma}_{33}^\varepsilon, \bar{v}_3^\varepsilon - \bar{u}_3^\varepsilon\rangle \ge 0,\ \forall\ \ v_3^\varepsilon \in K(\Omega^\varepsilon) \\ \ \langle \hat{\sigma}_{\alpha 3}^\varepsilon, \bar{v}_\alpha^\varepsilon - \bar{u}_\alpha^\varepsilon\rangle + \langle \Lambda\left|\hat{\sigma}_{33}^\varepsilon\right|, \left|\bar{v}_T^\varepsilon\right| - \left|\bar{u}_T^\varepsilon\right|\rangle \ge 0,\ \forall\ v_\alpha^\varepsilon \in V(\Omega^\varepsilon) \end{cases},$$

where: $L^\varepsilon(v^\varepsilon) = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \int_{\Gamma_-^\varepsilon} g_i^\varepsilon \underline{v}_i^\varepsilon d\Gamma^\varepsilon$ and $\left\langle \hat{\sigma}_{i3}^\varepsilon, \overline{\phi}_i^\varepsilon \right\rangle = \int\limits_{\Gamma_+^\varepsilon} \hat{\sigma}_{i3}^\varepsilon \cdot \overline{\phi}_i^\varepsilon d\Gamma^\varepsilon$

3.3 Asymptotic study

3.3.1 The scaled problem

We make the same transformation and scalings as in paragraph 1.2.1. Let let $\Omega = \omega \times]-1,+1[$, $\Gamma_- = \omega \times \{-1\}$, $\Gamma_+ = \omega \times \{+1\}$, $\Gamma_0 = \partial\omega \times [-1,+1]$. Let $x = (x_i) \in \overline{\Omega}$ denote a generic point in the set $\overline{\Omega}$. We now transform the domain Ω^ε having the thickness 2ε into fixed domain Ω independent of ε via the simple mapping:

$$\pi^\varepsilon : x^\varepsilon \in \Omega^\varepsilon \to x \in \Omega, \text{ where } x_\alpha = x_\alpha^\varepsilon,\ x_3 = x_3^\varepsilon/\varepsilon. \tag{3.1}$$

Hence

$$\pi^\varepsilon(\Omega^\varepsilon) = \Omega,\ \pi^\varepsilon(\Gamma_\pm^\varepsilon) = \Gamma_-,\ \pi^\varepsilon(\Gamma_+^\varepsilon) = \Gamma_+,\ \pi^\varepsilon(\Gamma_0^\varepsilon) = \Gamma_0,\ \partial_\alpha^\varepsilon = \partial_\alpha \text{ and } \partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3. \tag{3.2}$$

We introduce the scaled displacement $u(\varepsilon)$, the scaled test function $v(\varepsilon)$, the scaled stress tensor $\sigma(\varepsilon)$ and the scaled contact condition as follow:

$$\begin{cases} u_\alpha^\varepsilon \circ \pi^\varepsilon = \varepsilon^2 u_\alpha(\varepsilon), \quad u_3^\varepsilon \circ \pi^\varepsilon = \varepsilon u_3(\varepsilon), v_\alpha^\varepsilon \circ \pi^\varepsilon = \varepsilon^2 v_\alpha(\varepsilon), \; v_3^\varepsilon \circ \pi^\varepsilon = \varepsilon v_3(\varepsilon) \\ \sigma_{\alpha\beta}^\varepsilon \circ \pi^\varepsilon = \varepsilon^2 \sigma_{\alpha\beta}(\varepsilon), \; \sigma_{\alpha 3}^\varepsilon \circ \pi^\varepsilon = \varepsilon^3 \sigma_{\alpha 3}(\varepsilon), \; \sigma_{33}^\varepsilon \circ \pi^\varepsilon = \varepsilon^4 \sigma_{33}(\varepsilon), u_3(\varepsilon) \le d \end{cases} \tag{3.3}$$

We also introduce the scaling of the forces:

$$f_\alpha^\varepsilon \circ \pi^\varepsilon = \varepsilon^2 f_\alpha, f_3^\varepsilon \circ \pi^\varepsilon = \varepsilon^3 f_3, \; g_\alpha^\varepsilon \circ \pi^\varepsilon = \varepsilon^3 g_\alpha, g_3^\varepsilon \circ \pi^\varepsilon = \varepsilon^4 g_3, \tilde{F}_\alpha^\varepsilon = \varepsilon^2 \tilde{F}_\alpha \tag{3.4}$$

Then we obtain

$$L^\varepsilon(v^\varepsilon) = \varepsilon^5 L(v) \;\text{ with }\; L(v) = \int_\Omega f_i v_i dx + \int_{\Gamma_-} g_i \underline{v}_i d\Gamma \tag{3.5}$$

Therefore we denote by:

$$V(\Omega) = \left\{v \in W^{1,4}(\Omega), \; v \text{ independent of } x_3 \text{ on } \Gamma_0\right\}, \tag{3.6}$$
$$V_0(\Omega) = \left\{v \in W^{1,4}(\Omega), \; v = 0 \text{ on } \Gamma_0\right\}, \tag{3.7}$$
$$\vec{V}(\Omega) = V(\Omega) \times V(\Omega) \times V_0(\Omega), \tag{3.8}$$
$$K(\Omega) = \{v \in V_0(\Omega), \; \bar{v} \le d \;\text{ on }\; \Gamma_+\}, \tag{3.9}$$
$$\vec{K}(\Omega) = V(\Omega) \times V(\Omega) \times K(\Omega) \tag{3.10}$$

Using the upper assumptions and notations (3.1)-(3.10) lead to the following:

Proposition 61 *The variational problem* (VP^ε) *is equivalent to the following scaled variational problem* $(SVP(\varepsilon))$*:*

$$\begin{cases} \textit{Find } \; u(\varepsilon) \in \vec{K}(\Omega) \quad \textit{such that:} \\ \int_\Omega \sigma_{ij}(\varepsilon)\partial_j v_i dx + \int_\Omega \sigma_{ij}(\varepsilon)\partial_i u_3(\varepsilon)\partial_j v_3 dx + \varepsilon^2 \int_\Omega \sigma_{ij}(\varepsilon)\partial_i u_\alpha(\varepsilon)\partial_j v_\alpha dx = L(v) + \\ + \langle \hat{\sigma}_{33}(\varepsilon), \bar{v}_3 \rangle \; + \int_\gamma \tilde{F}_\alpha (\int_{-1}^{+1} v_\alpha dx_3) d\gamma \; + \langle \sigma_{\alpha 3}(\varepsilon), \bar{v}_\alpha \rangle + \varepsilon^2 \langle \sigma_{k3}(\varepsilon)\partial_k u_\alpha, \bar{v}_\alpha \rangle, \forall v \in \vec{V}(\Omega) \\ \langle \hat{\sigma}_{33}(\varepsilon), \bar{v}_3 - \bar{u}_3(\varepsilon) \rangle \;\; \ge 0, \; \forall v_3 \in K(\Omega) \\ \langle \sigma_{\alpha 3}(\varepsilon), \bar{v}_\alpha - \bar{u}_\alpha(\varepsilon) \rangle + \varepsilon \langle \Lambda |\hat{\sigma}_{33}(\varepsilon)|, |\bar{v}_T| - |\bar{u}_T(\varepsilon)| \rangle \\ \qquad\qquad + \varepsilon^2 \langle \sigma_{k3}(\varepsilon)\partial_k u_\alpha(\varepsilon), \bar{v}_\alpha - \bar{u}_\alpha(\varepsilon) \rangle \ge 0, \; \forall \; v_\alpha \in V(\Omega) \end{cases}$$

3.3.2 The two-dimensional problem

We assume that the scaled displacement-stress $(u(\varepsilon), \sigma(\varepsilon))$ admit a formal asymptotic expansion of the form:

$$(u(\varepsilon), \sigma(\varepsilon)) = (u^0, \sigma^0) + \varepsilon(u^1, \sigma^1) + \varepsilon^2(u^2, \sigma^2) + \dots \tag{3.11}$$

We introduce the Kirchhoff-Love space of admissible displacements

$$V_{KL}(\Omega) = \left\{ v = (v_i),\ v_\alpha = \eta_\alpha - x_3 \partial_\alpha \eta_3,\ v_3 = \eta_3 \text{ such that: } \eta_\alpha \in H^1(\omega),\ \eta_3 \in H_0^2(\omega) \right\}$$

and the space

$$\mathbb{L}_s^2(\Omega) = \left\{ \tau = (\tau_{ij}) \in L^2(\Omega);\ \tau_{ij} = \tau_{ji} \right\}.$$

Substituting expansion (3.11) into the scaled variational problem $(SVP(\varepsilon))$, we obtain :

Proposition 62 *Assume that $\partial_3 u_3^0 \in C^0(\bar{\Omega})$ then the leading term (u^0, σ^0) of the expansion (3.11) is solution of the problem $(SVP(0))$:*

$$\begin{cases} \textit{Find } (u^0, \sigma^0) \in V_{KL}(\Omega) \cap \vec{K}(\Omega) \times \mathbb{L}_s^2(\Omega) \textit{ such that:} \\ \int_\Omega \sigma_{\alpha\beta}^0 \partial_\beta v_\alpha dx + \int_\Omega \sigma_{\alpha\beta}^0 \partial_\alpha u_3^0 \partial_\beta v_3 dx = L(v) + \langle \sigma_{33}^0, \bar{v}_3 \rangle + \int_\gamma \tilde{F}_\alpha (\int_{-1}^{+1} v_\alpha dx_3) d\gamma, \forall v \in V_{KL}(\Omega) \\ \langle \sigma_{33}^0, \bar{v}_3 - \bar{u}_3^0 \rangle \geq 0, \forall v_3 \in K(\Omega) \end{cases},$$

where $\sigma_{\alpha\beta}^0 = \frac{2\lambda\mu}{\lambda + 2\mu} E_{\gamma\gamma}^0(u^0) \delta_{\alpha\beta} + 2\mu E_{\alpha\beta}^0(u^0)$ *and* $E_{\alpha\beta}^0(u^0) = \frac{1}{2}(\partial_i u_j^0 + \partial_j u_i^0 + \partial_i u_3^0 \partial_j u_3^0)$.

We deduce that the leading term (u^0, σ^0) is characterized by an unilateral contact problem without friction.

Proposition 63 *Let $u^0 \in V_{KL}(\Omega) \cap \vec{K}(\Omega)$ be such that $u_\alpha^0 = \xi_\alpha - x_3 \partial_\alpha \xi_3$ and $u_3^0 = \xi_3$, where ξ_α, ξ_3 sufficiently regulars. Then the problem $(SVP(0))$ can be formulated in the classical form as two-dimensional problem*:

$$(P^b(0)) \begin{cases} \textit{Find } (\xi, \sigma_{33}^0) \in (H_0^1(\omega))^2 \times K_0(\omega) \times H^{-2}(\omega),, \textit{ such that} \\ k\Delta^2 \xi_3 - \partial_\beta(n_{\alpha\beta} \partial_\alpha \xi_3) = h_1^1 + h_2^1 + h_3^0 + \sigma_{33}^0 \textit{ on } \omega \\ -\partial_\beta n_{\alpha\beta} = h_\alpha^0 \textit{ on } \omega \\ n_{\alpha\beta} \nu_\beta = 2\tilde{F}_\alpha \textit{ on } \gamma \\ \sigma_{33}^0 (d - \xi_3) = 0 \textit{ and } \sigma_{33}^0 \leq 0 \textit{ in } \omega \end{cases}$$

where

$$K_0(\omega) = \left\{ v \in H_0^2(\omega), v \leq d \right\},$$

$$n_{\alpha\beta} = \frac{4\lambda\mu}{\lambda + 2\mu} E^0_{\gamma\gamma}(\xi)\delta_{\alpha\beta} + 4\mu E^0_{\alpha\beta}(\xi), k = \frac{8}{3}\mu\frac{\lambda+\mu}{\lambda+2\mu},$$

$$h_i^0 = \int_{-1}^{1} f_i dx_3 + g_i^-, h_i^1 = \int_{-1}^{1} x_3 \partial_i f_i dx_3 - \partial_i g_i^-, g_i^- = g_i(x_1, x_2, -1).$$

We deduce that the displacement u^0 is characterized by a two dimensional problem without friction. Then, our three-dimensional Signorini problem with Coulomb friction offers toward a two-dimensional problem without friction.

Remark 64 *The loss of the friction term in $(SVP(0))$ and $(P^b(0))$ results owing to the fact that the friction force behaves like $O(\varepsilon^3)$ whereas the contact pressure force scales as $O(\varepsilon^4)$. Since the two measures are connected by Coulomb law via $|\hat{\sigma}_{\alpha 3}(\varepsilon)| \leq \varepsilon\Lambda |\hat{\sigma}_{33}(\varepsilon)|$. Therefore, at least formally when ε tends towards zero, the friction force must be canceled. In the absence of unilateral contact the problem $(P^b(0))$ is reduced to nonlinear von Kármán plate model.*

3.4 Extended study for generalized Marguerre-von Kármán shallow shells

In the previous part of this chapter, we have studied the asymptotic modeling of Coulomb frictional unilateral contact problem between an elastic nonlinear von Kármán plate and a rigid obstacle. The main result obtained is that the leading term of the asymptotic expansion is characterized by a two-dimensional Signorini problem but without friction. In this section, we extend this study to the case of a shallow shell under generalized Marguerre-von Kármán conditions.

In the case of linearly thin elastic structures, Paumier [83] studied the asymptotic modeling of Signorini problem with Coulomb friction in the Kirchhoff-Love theory of plates by using a convergence method. In the same way but for the frictionless case, Léger and Miara [70, 71] extended the study to the elastic shallow shell. More recently, Ben Belgacem and all. [4] modeled the obstacle problem without friction for Naghdi shell. In the nonlinear case, Chacha and Bensayah [12] studied the asymptotic modeling of a Coulomb frictional Signorini problem for the von Kármán plates using the formal asymptotic expansion method. Remind that by the same mean, Ciarlet and Paumier [28] justified the Marguerre-von Kármán equations for shallow shells. Untill 2001, Ciarlet and Gratie [25] generalized these equations for plates, after that Ciarlet, Gratie and Sabu [26] established an existence theorem for them. Next, in 2002 Gratie [48] formally extended in the same time the works [28] and [25] to generalized Marguerre-von Kármán equations for shallow shells, after that, Ciarlet and Gratie [31] gave the existence of solutions to this problem. For more details, one can consult [30]. The aim of this section is to extend the study carried out in [12] to the nonlinear generalized Marguerre-von Kármán shallow shell derived and analyzed by Gratie [48].

3.4.1 Setting of the problem

Let $\Omega^\varepsilon = \omega \times]-\varepsilon, +\varepsilon[, \varepsilon > 0$, be an open bounded domain from $\mathbb{R}^3$, such that ω is an open subset from $\mathbb{R}^2$, with smooth boundary γ. We denote the lateral boundary of Ω^ε by $\Gamma_0^\varepsilon = \gamma \times [-\varepsilon, \varepsilon]$, the upper and the lower faces are denoted, respectively, by Γ_+^ε and Γ_-^ε. Let $\theta^\varepsilon : \bar{\omega} \to \mathbb{R}$ be a function of the class $\mathcal{C}^3$. The reference configuration of the shell is $\{\hat{\Omega}^\varepsilon\}^-$, where $\hat{\Omega}^\varepsilon = \Theta^\varepsilon(\Omega^\varepsilon)$, $\hat{x}^\varepsilon = \Theta^\varepsilon(x^\varepsilon)$, $\Theta^\varepsilon(x^\varepsilon) = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) + x_3^\varepsilon a_3^\varepsilon(x_1, x_2)$ for all $x^\varepsilon = (x_1, x_2, x_3^\varepsilon) \in \bar{\Omega}^\varepsilon$ and a_3^ε is a unit vector normal to the middle surface $\Theta^\varepsilon(\bar{\omega})$ of the shell. Following the definition proposed by Ciarlet and Paumier [28], we say that a shell is shallow if there exists a function $\theta \in \mathcal{C}^3(\bar{\omega})$ independent of ε such that $\theta^\varepsilon(x_1, x_2) = \varepsilon\theta(x_1, x_2), \forall (x_1, x_2) \in \bar{\omega}$. For ε small enough, the mapping $\Theta^\varepsilon : \bar{\Omega}^\varepsilon \to \Theta^\varepsilon(\bar{\Omega}^\varepsilon)$ is a $\mathcal{C}^1$-diffeomorphism see [28] and we suppose also that Θ^ε is orientation preserving i.e $\det \nabla^\varepsilon \Theta^\varepsilon(x^\varepsilon) > 0,\ \forall x^\varepsilon \in \bar{\Omega}^\varepsilon$. We suppose that $\hat{\Omega}^\varepsilon$ is occupied by a nonlinear, elastic, homogeneous, isotropic body. In its natural configuration: a shallow shell of thickness 2ε whose Lamès constants are denoted $\lambda > 0, \mu > 0$ and assumed to be independent of ε. The shallow shell is supposed to be subjected to applied body forces of density $\hat{f}^\varepsilon \in (L^2(\hat{\Omega}^\varepsilon))^3$, its lower face $\hat{\Gamma}_-^\varepsilon = \Theta^\varepsilon(\Gamma_-^\varepsilon)$ subjected to a surface forces of density $\hat{g}^\varepsilon \in (L^2(\hat{\Gamma}_-^\varepsilon))^3$ such that $\hat{f}_\alpha^\varepsilon = \hat{g}_\alpha^\varepsilon = 0$ and to applied surface forces of *von Kármán's* type $\hat{h}_\alpha^\varepsilon \in L^2(\hat{\gamma}_1^\varepsilon)$ only on a portion $\Theta^\varepsilon(\gamma_1 \times [-\varepsilon, \varepsilon])$ of its lateral face $\hat{\Gamma}_0^\varepsilon$, where $\hat{\gamma}_1^\varepsilon = \Theta^\varepsilon(\gamma_1)$, $\gamma = \gamma_1 \cup \gamma_2$, $length(\gamma_1) > 0$, $length(\gamma_2) > 0$. We suppose also that this shell is in unilateral contact

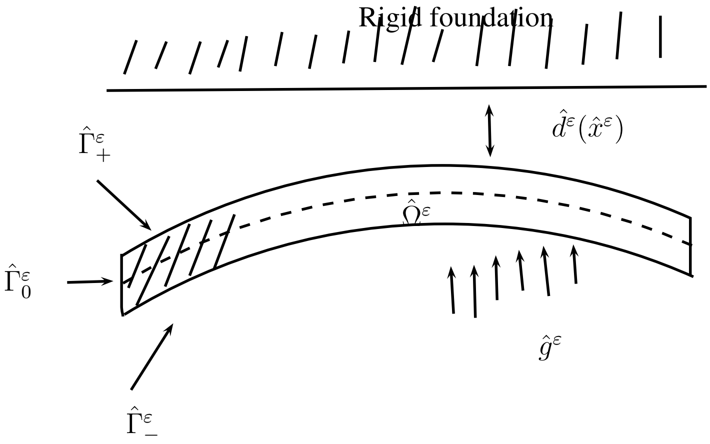

Figure 3.1: A shallow shell goes in contact against a rigid foundation.

with Coulomb friction at the upper face $\hat{\Gamma}^\varepsilon_+ = \Theta^\varepsilon(\Gamma^\varepsilon_+)$ against a rigid foundation. See Figure3.1. The contact condition is expressed by $\hat{u}^\varepsilon_N \leq \hat{d}^\varepsilon$, $\hat{u}^\varepsilon_N = \hat{u}^\varepsilon_i \hat{n}^\varepsilon_i$ where $\hat{d}^\varepsilon(\geq 0)$ is the gap function defined on $\hat{\Gamma}^\varepsilon_+$ which describes the distance between the upper face and the foundation measured in the normal direction and $\hat{n}^\varepsilon = (\hat{n}^\varepsilon_i)$ is the unit outer normal vector along the boundary of the shell $\hat{\Omega}^\varepsilon$, Λ its frictional coefficient. We suppose, also that the system is in static case. $\nu = (\nu_\alpha)$ and $\tau = (\tau_\alpha)$ are respectively the unit outer normal and the unit tangential vector such that $\tau_1 = -\nu_2$ and $\tau_2 = \nu_1$ along the boundary of the set ω. The outer normal and tangential derivative operators $\nu_\alpha \partial_\alpha$ and $\tau_\alpha \partial_\alpha$ along γ are denoted ∂_ν and ∂_τ.

The problem consists of finding the displacement $\hat{u}^\varepsilon$ and the force $\hat{G}^\varepsilon$ which satisfy the problem:

$$(C\hat{P}^\varepsilon)\begin{cases} -\hat{\partial}^\varepsilon_j(\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i) = \hat{f}^\varepsilon_i \text{ in } \hat{\Omega}^\varepsilon \\ (\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i)\hat{n}^\varepsilon_j \circ \Theta^\varepsilon = 0 \text{ on } \gamma_2 \times [-\varepsilon, \varepsilon] \\ (\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i)\hat{n}^\varepsilon_j \circ \Theta^\varepsilon = \hat{g}^\varepsilon_i \circ \Theta^\varepsilon \text{ on } \Gamma^\varepsilon_- \\ \hat{u}^\varepsilon_\alpha \text{ independent of } \hat{x}^\varepsilon_3 \text{ and } \hat{u}^\varepsilon_3 = 0 \quad \text{on } \Theta^\varepsilon\left(\gamma_1 \times [-\varepsilon, \varepsilon]\right) \\ \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \left\{ \left(\hat{\sigma}^\varepsilon_{\alpha\beta} + \hat{\sigma}^\varepsilon_{k\beta}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_\alpha\right) \circ \Theta^\varepsilon \right\} \nu_\beta dx^\varepsilon_3 = \hat{h}^\varepsilon_\alpha \circ \Theta^\varepsilon \text{ on } \gamma_1 \\ \hat{u}^\varepsilon_N \leq \hat{d}^\varepsilon, \hat{G}^\varepsilon_N \leq 0, \hat{G}^\varepsilon_N \left(\hat{u}^\varepsilon_N - \hat{d}^\varepsilon\right) = 0 \; on \; \hat{\Gamma}^\varepsilon_+ \; (Signorini\, conditions) \\ \left|\hat{G}^\varepsilon_T\right| < \Lambda \left|\hat{G}^\varepsilon_N\right| \Rightarrow \hat{u}^\varepsilon_T = 0 \; on \; \hat{\Gamma}^\varepsilon_+ \\ \left|\hat{G}^\varepsilon_T\right| = \Lambda \left|\hat{G}^\varepsilon_N\right| \Rightarrow \exists \delta > 0, \, \hat{u}^\varepsilon_T = -\delta \hat{G}^\varepsilon_T \; on \; \hat{\Gamma}^\varepsilon_+ \end{cases},$$

where
$\hat{G}^\varepsilon_i = (\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i)\hat{n}^\varepsilon_j, \hat{G}^\varepsilon_N = \hat{G}^\varepsilon_i \hat{n}^\varepsilon_i, \hat{G}^\varepsilon_T = \hat{G}^\varepsilon - \hat{G}^\varepsilon_N \hat{n}^\varepsilon$

$\hat{\sigma}_{ij}^{\varepsilon} = \lambda \hat{E}_{pp}^{\varepsilon}(\hat{u}^{\varepsilon})\,\delta_{ij} + 2\mu \hat{E}_{ij}^{\varepsilon}(\hat{u}^{\varepsilon})$ (the components of stress tensor)
$\hat{E}_{ij}^{\varepsilon}(\hat{u}^{\varepsilon}) = \frac{1}{2}(\hat{\partial}_i^{\varepsilon}\hat{u}_j^{\varepsilon} + \hat{\partial}_j^{\varepsilon}\hat{u}_i^{\varepsilon} + \hat{\partial}_i^{\varepsilon}\hat{u}_k^{\varepsilon}\hat{\partial}_j^{\varepsilon}\hat{u}_k^{\varepsilon})$ (the components of nonlinear strain tensor)
We consider the following functional spaces

$$\begin{array}{rcl} V(\hat{\Omega}^{\varepsilon}) & = & \left\{\hat{v} \in W^{1,4}(\hat{\Omega}^{\varepsilon})/\hat{v} \text{ } independent\ of\ \hat{x}_3^{\varepsilon}\ on\ \Theta^{\varepsilon}(\gamma_1 \times]-\varepsilon, +\varepsilon[)\right\} \\ V_0(\hat{\Omega}^{\varepsilon}) & = & \left\{\hat{v} \in W^{1,4}(\hat{\Omega}^{\varepsilon})/\hat{v} = 0\ on\ \Theta^{\varepsilon}(\gamma_1 \times]-\varepsilon, +\varepsilon[)\right\} \\ \vec{V}(\hat{\Omega}^{\varepsilon}) & = & V(\hat{\Omega}^{\varepsilon}) \times V(\hat{\Omega}^{\varepsilon}) \times V_0(\hat{\Omega}^{\varepsilon}), \end{array}$$

and the convex closed set

$$\vec{K}(\hat{\Omega}^{\varepsilon}) = \left\{\hat{v} \in \vec{V}(\hat{\Omega}^{\varepsilon})/\hat{v}_N \leq \hat{d}^{\varepsilon}\ on\ \hat{\Gamma}_+^{\varepsilon}\right\}$$

Multiplying the system of equilibrium equations in $(C\hat{P}^{\varepsilon})$ by functions $\hat{v}_i^{\varepsilon}$ and integrating over the set $\hat{\Omega}^{\varepsilon}$, after that using the Green formula and the boundary conditions we obtain the following variational formulation of the problem $(C\hat{P}^{\varepsilon})$:

$$(V\hat{P}^{\varepsilon}) \begin{cases} Find\ (\hat{u}^{\varepsilon}, \hat{G}^{\varepsilon}) \in \vec{K}(\hat{\Omega}^{\varepsilon}) \times (L^2(\hat{\Gamma}_+^{\varepsilon}))^3\ such\ that \\ \hat{A}^{\varepsilon}(\hat{u}^{\varepsilon}, \hat{v}^{\varepsilon}) = \hat{L}^{\varepsilon}(\hat{v}^{\varepsilon}) + \int_{\hat{\gamma}_1}(\int_{-\varepsilon}^{\varepsilon}(\hat{v}_{\alpha}^{\varepsilon} \circ \Theta^{\varepsilon})dx_3^{\varepsilon})\hat{h}_{\alpha}^{\varepsilon}d\hat{\gamma} + \left\langle \hat{G}_i^{\varepsilon}, \hat{v}_i^{\varepsilon}\right\rangle\ \forall\ \hat{v}^{\varepsilon} \in \vec{V}(\hat{\Omega}^{\varepsilon}) \\ \left\langle \hat{G}_N^{\varepsilon}, \hat{v}_N^{\varepsilon} - \hat{u}_N^{\varepsilon}\right\rangle \geq 0\,,\ \forall \hat{v}^{\varepsilon} \in \vec{K}(\hat{\Omega}^{\varepsilon}) \\ \left\langle \hat{G}_T^{\varepsilon}, \hat{v}_T^{\varepsilon} - \hat{u}_T^{\varepsilon}\right\rangle + \left\langle \Lambda\left|\hat{G}_N^{\varepsilon}\right|, |\hat{v}_T^{\varepsilon}| - |\hat{u}_T^{\varepsilon}|\right\rangle \geq 0,\ \forall\ \hat{v}^{\varepsilon} \in \vec{V}(\hat{\Omega}^{\varepsilon}) \end{cases}$$

where

$$\hat{A}^{\varepsilon}(\hat{u}^{\varepsilon}, \hat{v}^{\varepsilon}) = \int_{\hat{\Omega}^{\varepsilon}}(\hat{\sigma}_{ij}^{\varepsilon} + \hat{\sigma}_{kj}^{\varepsilon}\hat{\partial}_k^{\varepsilon}\hat{u}_i^{\varepsilon})\hat{\partial}_j^{\varepsilon}\hat{v}_i^{\varepsilon}d\hat{x}^{\varepsilon},$$

$$\hat{L}^{\varepsilon}(\hat{v}^{\varepsilon}) = \int_{\hat{\Omega}^{\varepsilon}}\hat{f}_i^{\varepsilon}\hat{v}_i^{\varepsilon}d\hat{x}^{\varepsilon} + \int_{\hat{\Gamma}_-^{\varepsilon}}\hat{g}_i^{\varepsilon}\hat{v}_i^{\varepsilon}d\hat{\Gamma}^{\varepsilon}$$

and

$$\left\langle \hat{G}_i^{\varepsilon}, \hat{\phi}_i^{\varepsilon}\right\rangle = \int_{\hat{\Gamma}_+^{\varepsilon}}\hat{G}_i^{\varepsilon}\hat{\phi}_i^{\varepsilon}d\hat{\Gamma}^{\varepsilon}$$

In order to transform the problem $(V.\hat{P}^{\varepsilon})$ into problem posed over the cylindrical domain Ω^{ε}, we use the one to one mapping $(\mathbf{\Theta}^{\varepsilon})^{-1}$ and the following relations obtained from this transformation

$$\hat{\partial}_j^{\varepsilon}\hat{v}_i^{\varepsilon} = b_{kj}^{\varepsilon}(x^{\varepsilon})\,\partial_k^{\varepsilon}v_i^{\varepsilon}(x^{\varepsilon})\,, d\hat{x}^{\varepsilon} = |\det \nabla^{\varepsilon}\Theta^{\varepsilon}(x^{\varepsilon})|\,dx^{\varepsilon} = \delta^{\varepsilon}dx^{\varepsilon}, d\hat{\Gamma}^{\varepsilon} = \delta^{\varepsilon}\beta^{\varepsilon}d\Gamma^{\varepsilon},$$

where

$$\nabla^{\varepsilon}\Theta^{\varepsilon}(x^{\varepsilon}) = (\partial_j^{\varepsilon}\Theta_i^{\varepsilon}(x^{\varepsilon})), \delta^{\varepsilon}(x^{\varepsilon}) = \det \nabla^{\varepsilon}\Theta^{\varepsilon}(x^{\varepsilon}),$$

$$b_{ij}^\varepsilon(x^\varepsilon) = (\{\nabla^\varepsilon \Theta^\varepsilon(x^\varepsilon)\}^{-1})_{ij} \ \forall x^\varepsilon \in \bar{\Omega}^\varepsilon,$$

$$\beta^\varepsilon(x^\varepsilon) = \{b_{3i}(x^\varepsilon) b_{3i}(x^\varepsilon)\}^{\frac{1}{2}} \ \forall x^\varepsilon \in (\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon).$$

We define the following functional spaces related to Ω^ε:

$$\begin{array}{rcl} V(\Omega^\varepsilon) & = & \left\{ v^\varepsilon \in W^{1,4}(\Omega^\varepsilon) / v^\varepsilon \ \textit{independent of} \ x_3^\varepsilon \ \textit{on} \ \gamma_1 \times \left]-\varepsilon, +\varepsilon\right[\right\} \\ V_0(\Omega^\varepsilon) & = & \left\{ v^\varepsilon \in W^{1,4}(\Omega^\varepsilon) / v^\varepsilon = 0 \ \textit{on} \ \gamma_1 \times \left]-\varepsilon, +\varepsilon\right[\right\}, \\ \vec{V}(\Omega^\varepsilon) & = & V(\Omega^\varepsilon) \times V(\Omega^\varepsilon) \times V_0(\Omega^\varepsilon) \end{array}$$

and the convex closed set

$$\vec{K}(\Omega^\varepsilon) = \left\{ v^\varepsilon \in \vec{V}(\Omega^\varepsilon) / v_N^\varepsilon \le d^\varepsilon \ \textit{on} \ \Gamma_+^\varepsilon \right\}, d^\varepsilon = \hat{d}^\varepsilon \circ \Theta^\varepsilon$$

Then by a simple computation, we obtain

Proposition 65 *Suppose that ε is small enough. Then the variational problem $(V\hat{P}^\varepsilon)$ is equivalent to the following variational problem :*

$$(VP^\varepsilon) \left\{ \begin{array}{l} Find \ (u^\varepsilon, G^\varepsilon) \in \vec{K}(\Omega^\varepsilon) \times (L^2(\Gamma_+^\varepsilon))^3, \ such \ that, \\ A^\varepsilon(u^\varepsilon, v^\varepsilon) = L^\varepsilon(v^\varepsilon) + \int_{\gamma_1} h_\alpha^\varepsilon \{\int_{-\varepsilon}^{\varepsilon} v_\alpha^\varepsilon dx_3^\varepsilon\} d\gamma + \langle G_i^\varepsilon, v_i^\varepsilon \rangle, \forall v^\varepsilon \in \vec{V}(\Omega^\varepsilon), \\ \langle G_N^\varepsilon, v_N^\varepsilon - u_N^\varepsilon \rangle \ge 0, \ \forall v^\varepsilon \in \vec{K}(\Omega^\varepsilon) \\ \langle G_T^\varepsilon, v_T^\varepsilon - u_T^\varepsilon \rangle + \langle \Lambda \left| G_N^\varepsilon \right|, \left| v_T^\varepsilon \right| - \left| u_T^\varepsilon \right| \rangle \ge 0, \ \forall \, v^\varepsilon \in \vec{V}(\Omega^\varepsilon) \end{array} \right.$$

where :

$$A^\varepsilon(u^\varepsilon, v^\varepsilon) = \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon b_{kj}^\varepsilon \partial_k^\varepsilon v_i^\varepsilon \delta^\varepsilon dx^\varepsilon + \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon b_{ki}^\varepsilon \partial_k^\varepsilon u_l^\varepsilon b_{mj}^\varepsilon \partial_m^\varepsilon v_l^\varepsilon \delta^\varepsilon dx^\varepsilon$$

$$L^\varepsilon(v^\varepsilon) = \int_{\Omega^\varepsilon} f_3^\varepsilon v_3^\varepsilon \delta^\varepsilon dx^\varepsilon + \int_{\Gamma_-^\varepsilon} g_3^\varepsilon v_3^\varepsilon \delta^\varepsilon \beta^\varepsilon d\Gamma^\varepsilon, \langle G_i^\varepsilon, v_i^\varepsilon \rangle = \int_{\Gamma_+^\varepsilon} G_i^\varepsilon v_i^\varepsilon \delta^\varepsilon \beta^\varepsilon d\Gamma^\varepsilon$$

$u_i^\varepsilon = \hat{u}_i^\varepsilon \circ \Theta^\varepsilon, \sigma_{ij}^\varepsilon = \hat{\sigma}_{ij}^\varepsilon \circ \Theta^\varepsilon, G_i^\varepsilon = \hat{G}_i^\varepsilon \circ \Theta^\varepsilon, n_i^\varepsilon = \hat{n}_i^\varepsilon \circ \Theta^\varepsilon, f_i^\varepsilon = \hat{f}_i^\varepsilon \circ \Theta^\varepsilon$, $g_i^\varepsilon = \hat{g}_i^\varepsilon \circ \Theta^\varepsilon$, $h_\alpha^\varepsilon = \hat{h}_\alpha^\varepsilon \circ \Theta^\varepsilon$.

3.4.2 Asymptotic study

The scaled problem

We keep the same process of transformation and scalings as in the paragraph 1.2.1. We introduce the scaled displacement $u(\varepsilon)$, test function $v(\varepsilon)$ and stress tensor $\sigma(\varepsilon)$ for all

$x^\varepsilon = \pi^\varepsilon(x)$ as follows:

$$\begin{cases} u^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), u^\varepsilon_3(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x), v^\varepsilon_\alpha(x^\varepsilon) = \varepsilon^2 v_\alpha(\varepsilon)(x), v^\varepsilon_3(x^\varepsilon) = \varepsilon v_3(\varepsilon)(x) \\ \sigma^\varepsilon_{\alpha\beta}(x^\varepsilon) = \varepsilon^2\sigma_{\alpha\beta}(\varepsilon)(x), \sigma^\varepsilon_{\alpha 3}(x^\varepsilon) = \varepsilon^3\sigma_{\alpha 3}(\varepsilon)(x), \sigma^\varepsilon_{33}(x^\varepsilon) = \varepsilon^4\sigma_{33}(\varepsilon)(x) \end{cases}$$

Noting that the unit normal $\hat{n}^\varepsilon$ on $\hat{\Gamma}^\varepsilon_+$ reads $\hat{n}^\varepsilon = (-\partial^\varepsilon_1\theta^\varepsilon + O(\varepsilon^3), -\partial^\varepsilon_2\theta^\varepsilon + O(\varepsilon^3), 1 + O(\varepsilon^2))$. If we pose $G_i = \sigma_{ij}n^\theta_j$ such that $n^\theta = (-\partial_1\theta, -\partial_2\theta, 1)$ then a simple computation gives:

$$\begin{aligned} G^\varepsilon_\alpha &= \varepsilon^3 G_\alpha + O(\varepsilon^5), \\ G^\varepsilon_3 &= \varepsilon^4 G_3 + \varepsilon^4\sigma_{ij}n^\theta_j\partial^\theta_i u_3 + O(\varepsilon^6) \end{aligned}$$

then

$$\begin{aligned} v^\varepsilon_T &= (\varepsilon^2(v_1 - v_3 n^\theta_1) + O(\varepsilon^4), \varepsilon^2(v_2 - v_3 n^\theta_2) + O(\varepsilon^4), O(\varepsilon^3)), \\ v^\varepsilon_N &= \varepsilon v_N(\varepsilon), v_N(\varepsilon) = v_3 n^\theta_3 + O(\varepsilon^2) \end{aligned}$$

and

$$G^\varepsilon_T = (\varepsilon^3 G_1 + O(\varepsilon^5), \varepsilon^3 G_2 + O(\varepsilon^5), \varepsilon^4(G_3 - G_i n^\theta_i) + O(\varepsilon^6))$$

We also introduce the scalings: $f^\varepsilon_3 = \varepsilon^3 f_3, g^\varepsilon_3 = \varepsilon^4 g_3, h^\varepsilon_\alpha = \varepsilon^2 h_\alpha$ and $d^\varepsilon = \varepsilon d(\varepsilon)$ where f_3, g_3 and h_α supposed independent of ε.

Therefore we denote:

$$\begin{aligned} V(\Omega) &= \left\{v \in W^{1,4}(\Omega)/v \textit{ independent of } x_3 \textit{ on } \gamma_1 \times]-1,+1[\right\} \\ V_0(\Omega) &= \left\{v \in W^{1,4}(\Omega)/v = 0 \textit{ on } \gamma_1 \times]-1,+1[\right\} \\ \vec{V}(\Omega) &= V(\Omega) \times V(\Omega) \times V_0(\Omega), \\ \vec{K}(\varepsilon)(\Omega) &= \left\{v \in \vec{V}(\Omega)/v_N(\varepsilon) \le d(\varepsilon) \textit{ on } \Gamma_+\right\} \\ L^2_s(\Omega) &= \{\tau = (\tau_{ij}) \in L^2(\Omega);\ \tau_{ij} = \tau_{ji}\}. \end{aligned}$$

Using the assumptions and notations above we obtain the result:

Proposition 66 *For ε small enough the scaled solution of the problem (VP^ε) solves the problem $(SVP(\varepsilon))$:*

$$\begin{cases} \textit{Find } (u(\varepsilon), G(\varepsilon)) \in \vec{K}(\varepsilon)(\Omega) \times (L^2(\Gamma_+))^3 \textit{ such that}, \\ A^\theta(u(\varepsilon), v) = L(v) + 2\int_{\gamma_1} h_\alpha v_\alpha dx_3 d\gamma + \langle G_i(\varepsilon), v_i\rangle + \int_{\Gamma_+}\sigma_{ij}(\varepsilon)n^\theta_j\partial^\theta_i u_3(\varepsilon)v_3 d\Gamma + \varepsilon^2 r_1, \forall v \in \vec{V}(\Omega), \\ \left\langle G_i(\varepsilon)n^\theta_i + \sigma_{ij}(\varepsilon)n^\theta_j\partial^\theta_i u_3(\varepsilon), v_3 - u_3(\varepsilon)\right\rangle + \varepsilon^2 r_2 \ge 0\,, \forall v \in \vec{K}(\varepsilon)(\Omega) \\ \left\langle G_\alpha(\varepsilon), (v_\alpha - u_\alpha(\varepsilon)) - (v_3 - u_3(\varepsilon))n^\theta_\alpha\right\rangle + \varepsilon r_3 \ge 0, \forall\, v \in \vec{V}(\Omega) \end{cases}$$

where

$$A^{\theta}(u(\varepsilon),v)=\int_{\Omega}\sigma_{ij}\left(\varepsilon\right)\gamma_{ij}^{\theta}(v)dx+\int_{\Omega}\sigma_{ij}\left(\varepsilon\right)\partial_i^{\theta}u_3\left(\varepsilon\right)\partial_j^{\theta}v_3dx,$$

$$L(v)=\int_{\Omega}f_3v_3dx+\int_{\Gamma_-}g_3v_3d\Gamma,$$

$$\langle G_i(\varepsilon),v_i\rangle=\int_{\Gamma_+}G_i(\varepsilon)v_id\Gamma,\partial_{\alpha}^{\theta}v=\partial_{\alpha}v-\partial_{\alpha}\theta\partial_3v$$

$\partial_3^{\theta}v=\partial_3v,\gamma_{ij}^{\theta}(v)=\frac{1}{2}\left(\partial_i^{\theta}v_j+\partial_j^{\theta}v_i\right)$, r_i *are uniformly bounded functions in with respect to* ε.

Proof. First, we infer from assumption $\theta^{\varepsilon}(x_1,x_2)=\varepsilon\theta(x_1,x_2)$ for all $(x_1,x_2)\in\bar{\omega}$ with $\theta\in\mathcal{C}^3(\bar{\omega})$ that, for $\varepsilon_0>0$ small enough,
$b_{\alpha\beta}^{\varepsilon}(x^{\varepsilon})=\delta_{\alpha\beta}+\varepsilon^2r_{\alpha\beta}(\varepsilon;x_1,x_2)$, $b_{\alpha3}^{\varepsilon}(x^{\varepsilon})=\varepsilon(\partial_{\alpha}\theta+\varepsilon^2r_{\alpha3}(\varepsilon;x_1,x_2))$, $b_{3\beta}^{\varepsilon}(x^{\varepsilon})=-\varepsilon(\partial_{\beta}\theta+\varepsilon^2r_{3\beta}(\varepsilon;x_1,x_2))$, $b_{33}^{\varepsilon}(x^{\varepsilon})=1+\varepsilon^2r_{33}(\varepsilon;x_1,x_2)$, $\delta^{\varepsilon}(x^{\varepsilon})=1+\varepsilon^2r_{\delta}(\varepsilon;x_1,x_2)$, for all $x^{\varepsilon}\in\bar{\Omega}^{\varepsilon}$, and $\beta^{\varepsilon}(x^{\varepsilon})=1+\varepsilon^2r_{\beta}(\varepsilon;x_1,x_2)$, for all $x^{\varepsilon}\in\Gamma_+^{\varepsilon}\cup\Gamma_+^{\varepsilon}$, where the real-valued functions r_{ij}, r_{δ}, r_{β} are bounded. (For details see [28, Theorem 3.1]).
Next, we insert the above equalities with the change of variables, we obtain,

$$\int_{\Omega^{\varepsilon}}\sigma_{ij}^{\varepsilon}b_{kj}^{\varepsilon}\partial_k^{\varepsilon}v_i^{\varepsilon}\delta^{\varepsilon}dx^{\varepsilon}=\varepsilon^5\int_{\Omega}\sigma_{ij}(\varepsilon)\gamma_{ij}^{\theta}(v)dx+\varepsilon^7\rho_1(\varepsilon;\sigma(\varepsilon),v)$$

$$\int_{\Omega^{\varepsilon}}\sigma_{ij}^{\varepsilon}b_{ki}^{\varepsilon}\partial_k^{\varepsilon}u_l^{\varepsilon}b_{mj}^{\varepsilon}\partial_m^{\varepsilon}v_l^{\varepsilon}\delta^{\varepsilon}dx^{\varepsilon}=\varepsilon^5\int_{\Omega}\sigma_{ij}(\varepsilon)\partial_i^{\theta}u_3(\varepsilon)\partial_j^{\theta}v_3dx+\varepsilon^7\rho_2(\varepsilon;\sigma(\varepsilon),u(\varepsilon),v)$$

$$\int_{\Omega^{\varepsilon}}f_i^{\varepsilon}v_i^{\varepsilon}\delta^{\varepsilon}dx^{\varepsilon}+\int_{\Gamma_-^{\varepsilon}}g_i^{\varepsilon}v_i^{\varepsilon}\delta^{\varepsilon}\beta^{\varepsilon}d\Gamma^{\varepsilon}=\varepsilon^5(\int_{\Omega}f_iv_idx+\int_{\Gamma_-}g_iv_id\Gamma)+\varepsilon^7\rho_3(\varepsilon;v)$$

$$\langle G_i^{\varepsilon},v_i^{\varepsilon}\rangle=\varepsilon^5\langle G_i(\varepsilon),v_i\rangle+\varepsilon^7\rho_4(\varepsilon;G(\varepsilon),u(\varepsilon),v)$$

where there exists a constant c_1 such that, for all $u(\varepsilon)\in\vec{K}(\varepsilon)(\Omega)$, $v\in\vec{V}(\Omega)$, $\sigma(\varepsilon)\in L_s^2(\Omega)$ and $G(\varepsilon)\in(L^2(\Gamma_+))^3$,

$$\sup_{0\le\varepsilon\le\varepsilon_0}\left|\rho_1\left(\varepsilon;\sigma(\varepsilon),v\right)\right|\le c_1\left|\sigma(\varepsilon)\right|_{0,\Omega}\left|v\right|_{1,\Omega},$$

$$\sup_{0\le\varepsilon\le\varepsilon_0}\left|\rho_2\left(\varepsilon;\sigma(\varepsilon),u(\varepsilon),v\right)\right|\le c_1\left|\sigma(\varepsilon)\right|_{0,\Omega}\left|u(\varepsilon)\right|_{1,4,\Omega}\left|v\right|_{1,4,\Omega},$$

$$\sup_{0\le\varepsilon\le\varepsilon_0}\left|\rho_3\left(\varepsilon;v\right)\right|\le c_1\left\|v\right\|_{1,\Omega},$$

$$\sup_{0\le\varepsilon\le\varepsilon_0}\left|\rho_4(\varepsilon;G(\varepsilon),u(\varepsilon),v)\right|\le c_1(\left\|G(\varepsilon)\right\|_{-\frac{1}{2},\Gamma_+}+\left\|G(\varepsilon)\right\|_{-\frac{1}{2},\Gamma_+}\left\|u(\varepsilon)\right\|_{\frac{1}{2},\Gamma_+})\left\|v\right\|_{\frac{1}{2},\Gamma_+}.$$

Diving by ε^5 and combining the above estimates, we get

$$\begin{aligned}\sup_{0\le\varepsilon\le\varepsilon_0}|r_1| &\le c_1\Big((1+|\sigma(\varepsilon)|_{0,\Omega}+|\sigma(\varepsilon)|_{0,\Omega}\,|u(\varepsilon)|_{1,4,\Omega})\,\|v\|_{1,4,\Omega}\\ &+ (\|G(\varepsilon)\|_{-\frac{1}{2},\Gamma_+}+\|G(\varepsilon)\|_{-\frac{1}{2},\Gamma_+}\,\|u(\varepsilon)\|_{\frac{1}{2},\Gamma_+})\,\|v\|_{\frac{1}{2},\Gamma_+}\Big).\end{aligned}$$

For finding what the unilateral contact conditions become, we use the relations

$$\langle G_N^\varepsilon, v_N^\varepsilon-u_N^\varepsilon\rangle=\varepsilon^5\left\langle G_\alpha(\varepsilon)n_\alpha^\theta+G_3(\varepsilon),v_3-u_3(\varepsilon)\right\rangle+\varepsilon^7 r_2,$$

$$\langle G_T^\varepsilon, v_T^\varepsilon-u_T^\varepsilon\rangle+\langle\Lambda\,|G_N^\varepsilon|\,,|v_T^\varepsilon|-|u_T^\varepsilon|\rangle=\varepsilon^5\left\langle G_\alpha(\varepsilon),(v_\alpha-u_\alpha(\varepsilon))-(v_3-u_3(\varepsilon))n_\alpha^\theta\right\rangle+\varepsilon^6 r_3,$$

where there exists two constants c_2 and c_3 such that for all $u(\varepsilon)\in\vec{K}(\varepsilon)(\Omega)$, $v\in\vec{K}(\varepsilon)(\Omega)$ and $G(\varepsilon)\in(L^2(\Gamma_+))^3$,

$$\sup_{0\le\varepsilon\le\varepsilon_0}|r_2|\le c_2(\|G(\varepsilon)\|_{-\frac{1}{2},\Gamma_+}+\|G(\varepsilon)\|_{-\frac{1}{2},\Gamma_+}\,\|u(\varepsilon)\|_{\frac{1}{2},\Gamma_+})(\|u(\varepsilon)\|_{\frac{1}{2},\Gamma_+}+\|v\|_{\frac{1}{2},\Gamma_+}),$$

and for all $u(\varepsilon)\in\vec{K}(\varepsilon)(\Omega)$, $v\in\vec{V}(\Omega)$ and $G(\varepsilon)\in(L^2(\Gamma_+))^3$,

$$\sup_{0\le\varepsilon\le\varepsilon_0}|r_3|\le c_3(\|G(\varepsilon)\|_{-\frac{1}{2},\Gamma_+}+\|G(\varepsilon)\|_{-\frac{1}{2},\Gamma_+}\,\|u(\varepsilon)\|_{\frac{1}{2},\Gamma_+})(\|u(\varepsilon)\|_{\frac{1}{2},\Gamma_+}+\|v\|_{\frac{1}{2},\Gamma_+}).$$

■

The two-dimensional problem

We assume that the scaled displacement-stress $(u(\varepsilon),\sigma(\varepsilon))$ admit a formal asymptotic expansion of the form:

$$(u(\varepsilon),\sigma(\varepsilon))=(u^0,\sigma^0)+\varepsilon(u^1,\sigma^1)+\varepsilon^2(u^2,\sigma^2)+... \tag{3.12}$$

then

$$G_i(\varepsilon)=G_i^0+\varepsilon G_i^1+\varepsilon^2 G_i^2+...,\ with\ G_i^k=\sigma_{ij}^k n_j^\theta.$$

We introduce the space of Kirchhoff-Love admissible displacement

$$V_{KL}(\Omega)=\left\{\begin{array}{c} v=(v_i)\,/v_\alpha=\eta_\alpha-x_3\partial_\alpha\eta_3, v_3=\eta_3\ such\ that\\ \eta_\alpha\in H^1(\omega)\,,\eta_3\in H^2(\omega)\,,\eta_3=\partial_\nu\eta_3=0\ on\ \gamma_1\end{array}\right\} \tag{3.13}$$

and the convex closed set

$$\vec{K}(\Omega)=\left\{v\in\vec{V}(\Omega)/v_3\le d\ on\ \Gamma_+\right\}\text{ with } d(\varepsilon)=d+O(\varepsilon).$$

Substituting expansion (4.12) into the scaled variational problem $(SVP(\varepsilon))$, we obtain:

Proposition 67 *Assume that* $\partial_3 u_3^0 \in C^0\left(\bar{\Omega}\right)$ *then the leading term* (u^0, σ^0) *of the expansion* (4.12) *is a solution of the problem* $(SVP(0))$*:*

$$\begin{cases} Find\ \ (u^0, \sigma^0, G_3^0) \in\ \ (V_{KL}(\Omega) \cap \vec{K}(\Omega)) \times L_s^2(\Omega) \times L^2(\Gamma_+)\ such\ that: \\ \int_\Omega \sigma_{\alpha\beta}^0 \partial_\beta v_\alpha dx + \int_\Omega \sigma_{\alpha\beta}^0 \partial_\alpha (u_3^0 + \theta) \partial_\beta v_3 dx = L(v) + \langle G_3^0, v_3 \rangle + 2\int_{\gamma_1} h_\alpha v_\alpha d\gamma, \forall v \in V_{KL}(\Omega) \\ \langle G_3^0, v_3 - u_3^0 \rangle \geq 0, \forall v \in \vec{K}(\Omega) \end{cases}$$

where

$$\sigma_{\alpha\beta}^0 = \frac{2\lambda\mu}{\lambda + 2\mu} E_{\sigma\sigma}^0(u^0)\delta_{\alpha\beta} + 2\mu E_{\alpha\beta}^0(u^0),$$

$$E_{\alpha\beta}^0(u^0) = \frac{1}{2}\left(\partial_\alpha u_\beta^0 + \partial_\beta u_\alpha^0 + \partial_\alpha u_3^0 \partial_\beta u_3^0 + \partial_\alpha \theta \partial_\beta u_3^0 + \partial_\beta \theta \partial_\alpha u_3^0\right),$$

$$G_3^0 = -\sigma_{31}^0 \partial_1 \theta - \sigma_{32}^0 \partial_2 \theta + \sigma_{33}^0.$$

Proof. We introduce the formal series expansions of the scaled displacement and the scaled stresses into the variational problem $(SVP(\varepsilon))$ and cancel the successive powers of ε, until we can fully identify the leading term. ■

We deduce from the following proposition that the leading term (u^0, σ^0) is characterized by an unilateral contact problem without friction.

Proposition 68 *If* u^0 *is a solution of the problem* $(SVP(0))$ *such that* $u_\alpha^0 = \xi_\alpha - x_3 \partial_\alpha \xi_3$ *and* $u_3^0 = \xi_3$*,* ξ_α, ξ_3 *sufficiently regular. Then* ξ_α, ξ_3 *verify the two-dimensional problem* $(P^b(0))$*:*

$$(P^b(0)) \begin{cases} Find\ \xi_\alpha \in H^1(\omega), \xi_3 \in H^2(\omega), \xi_3 \leq d, G_3^0 \in L^2(\omega)\ such\ that \\ -\partial_{\alpha\beta} m_{\alpha\beta} - n_{\alpha\beta} \partial_{\alpha\beta}(\xi_3 + \theta)) = h_3^0 + G_3^0\ in\ \omega \\ \partial_\beta n_{\alpha\beta} = 0\ in\ \omega, \\ \xi_3 = \partial_\nu \xi_3 = 0\ on\ \gamma_1, \\ n_{\alpha\beta}\nu_\beta = 2h_\alpha\ on\ \gamma_1 \\ \partial_\alpha m_{\alpha\beta}\nu_\beta + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta) = 0\ on\ \gamma_2, \\ m_{\alpha\beta}\nu_\alpha\nu_\beta = 0\ on\ \gamma_2, \\ n_{\alpha\beta}\nu_\beta = 0\ on\ \gamma_2 \\ G_3^0(d - \xi_3) = 0\ in\ \omega, G_3^0 \leq 0\ in\ \omega \end{cases}$$

where

$$m_{\alpha\beta} = -\frac{1}{3}\{\frac{4\lambda\mu}{\lambda+2\mu}\Delta\xi_3\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\xi_3\},$$

$$n_{\alpha\beta} = 2\lambda^\star E^0_{\gamma\gamma}(\xi)\delta_{\alpha\beta} + 4\mu E^0_{\alpha\beta}(\xi), \lambda^\star = \frac{2\lambda\mu}{\lambda+2\mu}$$

$$E^0_{\alpha\beta}(\xi) = \frac{1}{2}\left(\partial_\alpha\xi_\beta + \partial_\beta\xi_\alpha + \partial_\alpha\theta\partial_\beta\xi_3 + \partial_\beta\theta\partial_\alpha\xi_3 + \partial_\alpha\xi_3\partial_\beta\xi_3\right),$$

$$h^0_i = \int_{-1}^{1} f_i dx_3 + g_i^-; g_i^- = g_i(x_1, x_2, -1).$$

Proof. The proof will be divided into 3 steps.

Step 1. First, we show that $(SVP(0))$ is in a sense a two-dimensional problem, posed over the middle surface $\bar{\omega}$ of the shell.

$$-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega n_{\alpha\beta}\partial_\alpha(\xi_3+\theta)\partial_\beta\eta_3 d\omega + \int_\omega n_{\alpha\beta}\partial_\beta\eta_\alpha d\omega = \int_\omega (h^0_3 + G^0_3)\eta_3 d\omega + 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma, \forall\eta\in V(\omega),$$

where

$$V(\omega) = \{\eta = (\eta_i) \in H^1(\omega)\times H^1(\omega)\times H^2(\omega); \eta_3 = \partial_\nu\eta_3 = 0 \; sur \; \gamma_1\}.$$

It is known that $v = (v_i) \in V_{KL}(\Omega)$ if and only if there exists $\eta = (\eta_i) \in V(\omega)$ such that $v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3$ and $v_3 = \eta_3$ (see [20, Théorème 1.4-4]). The same proof works for Gratie [48, Theorem 3]. In $(SVP(0))$, we take test-functions $v = (-x_3\partial_1\eta_3, -x_3\partial_2\eta_3, \eta_3)$, with $\eta_3 \in H^2(\omega)$ and $\eta_3 = \partial_\nu\eta_3 = 0$ on γ_1. Next we take $v = (\eta_1, \eta_2, 0)$, with $\eta_\alpha \in H^1(\omega)$. The first choice yields

$$\int_\Omega -x_3\sigma^0_{\alpha\beta}\partial_{\alpha\beta}\eta_3 dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(\xi_3+\theta)\partial_\beta\eta_3 dx = \int_\Omega f_3\eta_3 dx + \int_{\Gamma_-} g_3\eta_3 d\Gamma + \left\langle G^0_3, \eta_3\right\rangle$$

The second choice yields

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\eta_\alpha dx = 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma$$

Using Fubini's formula to the above integrals, we get

$$\int_\Omega -x_3\sigma^0_{\alpha\beta}\partial_{\alpha\beta}\eta_3 dx = -\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 d\omega, \tag{3.14}$$

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha(\xi_3+\theta)\partial_\beta\eta_3 dx = \int_\omega n_{\alpha\beta}\partial_\alpha(\xi_3+\theta)\partial_\beta\eta_3 d\omega, \tag{3.15}$$

$$\int_\Omega f_3\eta_3 dx + \int_{\Gamma_-} g_3\eta_3 d\Gamma + \left\langle G^0_3, \eta_3\right\rangle =$$

$$\int_\omega \left(\int_{-1}^1 f_3 dx_3 + g_3^- + G^0_3(x_1,x_2,+1)\right)\eta_3 d\omega, \tag{3.16}$$

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\eta_\alpha dx = 2\int_{\gamma_1} h_\alpha\eta_\alpha d\gamma, \tag{3.17}$$

where

$$G^0_3(.,+1) = -\sigma^0_{31}(.,+1)\partial_1\theta - \sigma^0_{32}(.,+1)\partial_2\theta + \sigma^0_{33}(.,+1)$$

Step 2. Applying Green formulas, we obtain

$$\int_\omega \left[-\partial_{\alpha\beta}m_{\alpha\beta} - \partial_\beta\left(n_{\alpha\beta}\partial_\alpha\left(\xi_3+\theta\right)\right) - \left(h^0_3 + G^0_3\right)\right]\eta_3 d\omega -$$

$$\int_\omega \left(\partial_\beta n_{\alpha\beta}\right)\eta_\alpha d\omega + \int_\gamma \left(n_{\alpha\beta}\nu_\beta - 2\tilde{h}_\alpha\right)\eta_\alpha d\gamma - \int_{\gamma_2} m_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3 d\gamma +$$

$$\int_{\gamma_2} \left\{\left[\partial_\alpha m_{\alpha\beta} + n_{\alpha\beta}\partial_\alpha\left(\xi_3+\theta\right)\right]\nu_\beta + \partial_\tau\left(m_{\alpha\beta}\nu_\alpha\tau_\beta\right)\right\}\eta_3 d\gamma = 0$$

for all $\eta = (\eta_\alpha, \eta_3) \in V(\omega)$, and the functions $\tilde{h}_\alpha : \gamma \to \mathbb{R}$ defined by :

$$\tilde{h}_\alpha = h_\alpha \;\; on \;\; \gamma_1 \; and \; \tilde{h}_\alpha = 0 \;\; on \;\; \gamma_2$$

So that, all the factors of η_α, η_3, and $\partial_\nu\eta_3$ in the above integrals vanish in their respective domains of integration. (For more details we refer the reader to [48, Theorem 5])

Step 3. It remains to prove the unilateral contact conditions. To this end, substitute the test function $v = d$ after that $v = 2\xi_3 - d$, with $\xi_3 \in H^2(\omega)$ into the inequality in the problem $(SVP(0))$, then we obtain

$$G^0_3(d - \xi_3) = 0 \; in \; \omega$$

Taking into account

$$\left\langle G^0_3, \eta_3 - d\right\rangle \geq 0, \text{ for all } \eta \in \vec{K}\left(\Omega\right),$$

we obtain

$$G_3^0 \leq 0 \text{ in } \omega.$$

■

Computation of σ_{i3}^0 in case $\gamma_1 = \gamma$

In the squel, we compute the components σ_{i3}^0. In order to realize this, we suppose that $\gamma_1 = \gamma$ which the case of Marguerre-von Kàrmàn conditions.

Computation of $\sigma_{\alpha 3}^0$

In the identification process of factors of powers $\varepsilon^k, k = 0, 1, 2, 3, ...$ we obtain at the order ε^0 the equation

$$\begin{aligned} \int_\Omega \sigma_{ij}^0 \gamma_{ij}^\theta (v)\, dx + \int_\Omega \sigma_{ij}^0 \partial_i^\theta u_3^0 \partial_j^\theta v_3 dx &= \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma \\ &+ \int_{\Gamma_+} G_3^0 v_3 d\Gamma + 2 \int_\gamma h_\alpha \left\{ \int_{-1}^1 v_\alpha dx_3 \right\} d\gamma \quad (3.18) \end{aligned}$$

The term on the left hand side of the equation verifies

$$\int_\Omega \sigma_{\alpha\beta}^0 \gamma_{\alpha\beta}^\theta (v)\, dx = \frac{1}{2} \int_\Omega \sigma_{\alpha\beta}^0 (\partial_\alpha v_\beta + \partial_\beta v_\alpha - \partial_\alpha \theta \partial_3 v_\beta - \partial_\beta \theta \partial_3 v_\alpha) dx$$

$$\int_\Omega \sigma_{\alpha 3}^0 \gamma_{\alpha 3}^\theta (v)\, dx = \frac{1}{2} \int_\Omega \sigma_{\alpha 3}^0 (\partial_\alpha v_3 + \partial_3 v_\alpha - \partial_\alpha \theta \partial_3 v_3) dx$$

$$\int_\Omega \sigma_{33}^0 \gamma_{33}^\theta (v)\, dx = \int_\Omega \sigma_{33}^0 \partial_3 v_3 dx$$

$$\int_\Omega \sigma_{\alpha\beta}^0 \partial_\alpha^\theta u_3^0 \partial_\beta^\theta v_3 dx = \int_\Omega \sigma_{\alpha\beta}^0 (\partial_\alpha u_3^0 - \partial_\alpha \theta \partial_3 u_3^0)(\partial_\beta v_3 - \partial_\beta \theta \partial_3 v_3) dx$$

$$\int_\Omega \sigma_{\alpha 3}^0 \partial_\alpha^\theta u_3^0 \partial_3^\theta v_3 dx = \int_\Omega \sigma_{\alpha 3}^0 (\partial_\alpha u_3^0 - \partial_\alpha \theta \partial_3 u_3^0) \partial_3 v_3 dx$$

$$\int_\Omega \sigma_{3\alpha}^0 \partial_3^\theta u_3^0 \partial_\alpha^\theta v_3 dx = \int_\Omega \sigma_{3\alpha}^0 \partial_3 u_3^0 (\partial_\alpha v_3 - \partial_\alpha \theta \partial_3 v_3) dx$$

$$\int_\Omega \sigma_{33}^0 \partial_3^\theta u_3^0 \partial_3^\theta v_3 dx = \int_\Omega \sigma_{3\alpha}^0 \partial_3 u_3^0 \partial_3 v_3 dx$$

The equation (3.18) with $v_3 = 0$ yields

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha v_\beta + \int_\Omega \sigma^0_{\alpha 3}\partial_3 v_\alpha - \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\theta\partial_3 v_\alpha = 2\int_\gamma h_\alpha\left\{\int_{-1}^{1} v_\alpha dx_3\right\}d\gamma \tag{3.19}$$

for all $v_\alpha \in H^1(\Omega)$ independent of x_3 on Γ_0.
On other hand

$$\int_\gamma h_\alpha\left\{\int_{-1}^{1} v_\alpha dx_3\right\}d\gamma = \frac{1}{2}\int_\Omega n_{\alpha\beta}\partial_\beta v_\alpha dx, \text{ for all } v_\alpha \text{ independent of } x_3 \text{ on } \Gamma_0$$

then (3.19) is *formally* equivalent to the following boundary value problem

$$\left\{\begin{array}{rcl} \partial_3\sigma^0_{\alpha 3} &=& \partial_3\sigma^0_{\alpha\beta}\partial_\beta\theta - \partial_\beta\sigma^0_{\alpha\beta} \text{ in } \Omega \\ \sigma^0_{\alpha 3} &=& \sigma^0_{\alpha\beta}(.,+1)\partial_\beta\theta \text{ on } \Gamma_+ \\ \sigma^0_{\alpha 3} &=& \sigma^0_{\alpha\beta}(.,-1)\partial_\beta\theta \text{ on } \Gamma_- \end{array}\right. \tag{3.20}$$

Noting that $\sigma^0_{\alpha\beta} = \frac{1}{2}n_{\alpha\beta} + \frac{3}{2}x_3 m_{\alpha\beta}$ and from the Proposition 3.3 that $\partial_\beta n_{\alpha\beta} = 0$ which makes the compatibility condition $\int_{-1}^{+1}\partial_\beta\sigma^0_{\alpha\beta}dx_3 = 0$ satisfied. Then, the explicit expressions of $\sigma^0_{\alpha 3}$ are given by

$$\sigma^0_{\alpha 3} = \frac{3}{4}(1 - x_3^2)\partial_\beta m_{\alpha\beta} + \sigma^0_{\alpha\beta}\partial_\beta\theta$$

Computation of σ^0_{33}

We take $v_\alpha = 0$ in the equation (3.18). As $\partial_3 u^0_3 = 0$, we get

$$\begin{aligned}\int_\Omega \sigma^0_{\alpha 3}(\partial_\alpha v_3 - \partial_\alpha\theta\partial_3 v_3)dx + \int_\Omega \sigma^0_{33}\partial_3 v_3 dx + \int_\Omega \sigma^0_{\alpha\beta}\partial_\alpha u^0_3(\partial_\beta v_3 - \partial_\beta\theta\partial_3 v_3)dx + \\ + \int_\Omega \sigma^0_{\alpha 3}\partial_\alpha u^0_3\partial_3 v_3 dx = \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g_3 v_3 d\Gamma + \int_{\Gamma_+} G^0_3 v_3 d\Gamma\end{aligned} \tag{3.21}$$

thus we see that it is *formally* equivalent to the following boundary condition problem

$$\left\{\begin{array}{rcl} -\partial_3\sigma^0_{33} &=& -\partial_3\sigma^0_{\alpha 3}\partial_\alpha\theta + \partial_\alpha\sigma^0_{\alpha 3} + \partial_\beta(\sigma^0_{\alpha\beta}\partial_\alpha\xi_3) \\ & - & \partial_3\sigma^0_{\alpha\beta}\partial_\alpha\xi_3\partial_\beta\theta + \partial_3\sigma^0_{\alpha 3}\partial_\alpha\xi_3 + f_3 \text{ in } \Omega \\ \sigma^0_{33} &=& G^0_3 + \sigma^0_{\alpha\beta}(.,+1)\partial_\alpha\theta\partial_\beta\theta \text{ on } \Gamma_+ \\ \sigma^0_{33} &=& -g^-_3 + \sigma^0_{\alpha\beta}(.,-1)\partial_\alpha\theta\partial_\beta\theta \text{ on } \Gamma_- \end{array}\right. \tag{3.22}$$

such that G^0_3 verifies with ξ_3 on Γ_+ the condition

$$G^0_3(d - \xi_3) = 0, G^0_3 \leq 0$$

Taking in account that

$$\int_{-1}^{x_3} \partial_3 \sigma^0_{\alpha 3} \partial_\alpha \xi_3 dx_3 = \frac{3}{4}(1-x_3^2)\partial_\beta m_{\alpha\beta}\partial_\alpha \xi_3 + \frac{3}{2} x_3 m_{\alpha\beta}\partial_\beta \theta \partial_\alpha \xi_3 + \frac{3}{2} m_{\alpha\beta}\partial_\beta \theta \partial_\alpha \xi_3$$

$$\int_{-1}^{x_3} \partial_3 \sigma^0_{\alpha 3} \partial_\alpha \theta dx_3 = \frac{3}{4}(1-x_3^2)\partial_\beta m_{\alpha\beta}\partial_\alpha \theta + \frac{3}{2} x_3 m_{\alpha\beta}\partial_\beta \theta \partial_\alpha \theta + \frac{3}{2} m_{\alpha\beta}\partial_\beta \theta \partial_\alpha \theta$$

$$\int_{-1}^{x_3} \partial_3 \sigma^0_{\alpha\beta} \partial_\alpha \xi_3 \partial_\beta \theta dx_3 = \frac{3}{2} x_3 m_{\alpha\beta}\partial_\alpha \xi_3 \partial_\beta \theta + \frac{3}{2} m_{\alpha\beta}\partial_\alpha \xi_3 \partial_\beta \theta$$

$$\begin{aligned}\int_{-1}^{x_3} \partial_\alpha \sigma^0_{\alpha 3} dx_3 = {} & \frac{1}{4}(3x_3 - x_3^3)\partial_{\alpha\beta} m_{\alpha\beta} + \frac{1}{2} x_3 n_{\alpha\beta}\partial_{\alpha\beta}\theta + \frac{3}{4} x_3^2 \partial_\alpha m_{\alpha\beta}\partial_\beta \theta \\ & + \frac{3}{4} x_3^2 m_{\alpha\beta}\partial_{\alpha\beta}\theta + \frac{1}{2}\partial_{\alpha\beta} m_{\alpha\beta} - \frac{3}{4}\partial_\alpha m_{\alpha\beta}\partial_\beta \theta - \frac{3}{4} m_{\alpha\beta}\partial_{\alpha\beta}\theta \\ & + \frac{1}{2} n_{\alpha\beta}\partial_{\alpha\beta}\theta\end{aligned}$$

$$\begin{aligned}\int_{-1}^{x_3} \partial_\beta (\sigma^0_{\alpha\beta}\partial_\alpha \xi_3) dx_3 = {} & \frac{1}{2} x_3 n_{\alpha\beta}\partial_{\alpha\beta}\xi_3 + \frac{3}{4} x_3^2 \partial_\beta m_{\alpha\beta}\partial_\alpha \xi_3 + \frac{3}{4} x_3^2 m_{\alpha\beta}\partial_{\alpha\beta}\xi_3 \\ & + \frac{1}{2} n_{\alpha\beta}\partial_{\alpha\beta}\xi_3 - \frac{3}{4}\partial_\beta m_{\alpha\beta}\partial_\alpha \xi_3 - \frac{3}{4} m_{\alpha\beta}\partial_{\alpha\beta}\xi_3\end{aligned}$$

Then

$$\begin{aligned}\sigma^0_{33} = {} & -\frac{1}{4} x_3 (1-x_3^2)\partial_{\alpha\beta} m_{\alpha\beta} + \frac{3}{4}(1-x_3^2) m_{\alpha\beta}\partial_{\alpha\beta}(\xi_3 + \theta) + \frac{3}{4}(1-x_3^2)\partial_\beta m_{\alpha\beta}\partial_\alpha \theta \\ & + \frac{1}{2}(1+x_3)\int_{-1}^{1} f_3 dy_3 - \int_{-1}^{x_3} f_3 dy_3 + \frac{1}{2}(1+x_3) G_3^0 - \frac{1}{2}(1-x_3) g_3^- + \sigma^0_{\alpha 3}\partial_\alpha \theta\end{aligned}$$

3.4.3 Generalized Marguerre-von Kármán equations with Signorini conditions

We can rewrite the two-dimensional boundary value problem $(P^b(0))$ as generalized Marguerre-von Kármán equations with Signorini conditions which depends on the Airy function Φ, the vertical component ξ_3 of the displacement field of the middle surface of the shallow shell and G_3^0 as follows:

Proposition 69 *Assume that the set ω is simply-connected and that its boundary γ is smooth enough, and let $\xi = (\xi_i)$ be a solution $(P^b(0))$ with the regularity $\xi_\alpha \in H^3(\omega)$, $\xi_3 \in H^4(\omega)$. Then* *a) The functions $\tilde{h}_\alpha$ are in the space $H^{\frac{3}{2}}(\gamma)$ and satisfy the compatibility conditions :*

$$\int_\gamma \tilde{h}_1 d\gamma = \int_\gamma \tilde{h}_2 d\gamma = \int_\gamma (x_1\tilde{h}_2 - x_2\tilde{h}_1) d\gamma = 0.$$

b) Furthermore, there exists a function $\Phi \in H^4(\omega)$, uniquely defined by the relations $\Phi(0) = \partial_1\Phi(0) = \partial_2\Phi(0) = 0$, such that

$$n_{11} = 2\partial_{22}\Phi,\ n_{12} = n_{21} = -2\partial_{12}\Phi,\ n_{22} = 2\partial_{11}\Phi.$$

c) Finally, the pair $(\xi_3, \Phi, G_3^0) \in H^4(\omega) \times H^4(\omega) \times L^2(\omega)$, satisfies the following problem

$$\begin{cases} k\Delta^2\xi_3 = 2\left[\Phi, \xi_3 + \theta\right] + h_3^0 + G_3^0 \ \ in\ \omega, \\ \Delta^2\Phi = -\frac{\mu(3\lambda+2\mu)}{2(\lambda+\mu)}\left[\xi_3, \xi_3 + 2\theta\right] \ in\ \omega, \\ \xi_3 = \partial_\nu\xi_3 = 0 \ \ on\ \gamma_1, \\ m_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \ \ \ on\ \gamma_2, \\ \partial_\alpha m_{\alpha\beta}\nu_\beta + \partial_\tau\left(m_{\alpha\beta}\nu_\alpha\tau_\beta\right) = 0\ on\ \gamma_2, \\ \Phi = \Phi_0\ and\ \partial_\nu\Phi = \Phi_1\ on\ \gamma, \\ G_3^0(d - \xi_3) = 0\ in\ \omega, G_3^0 \leq 0\ in\ \omega, \end{cases}$$

where

$$k = \frac{8}{3}\mu\frac{\lambda+\mu}{\lambda+2\mu}, G_3^0 = -\sigma_{31}^0\partial_1\theta - \sigma_{32}^0\partial_2\theta + \sigma_{33}^0,$$

$$\Phi_0(y) = -y_1\int_{\gamma(y)}\tilde{h}_2 d\gamma + y_2\int_{\gamma(y)}\tilde{h}_1 d\gamma + \int_{\gamma(y)}(x_1\tilde{h}_2 - x_2\tilde{h}_1)d\gamma,$$

$$\Phi_1(y) = -\nu_1\int_{\gamma(y)}\tilde{h}_2 d\gamma + \nu_2\int_{\gamma(y)}\tilde{h}_1 d\gamma, y = (y_1, y_2) \in \gamma,$$

$$[\Phi, \xi] = \partial_{11}\Phi\partial_{22}\xi + \partial_{22}\Phi\partial_{11}\xi - 2\partial_{12}\Phi\partial_{12}\xi.$$

Proof. The proof is divided into three steps.

Step 1. The regularity of the function ξ_i implies that $n_{\alpha\beta} \in H^2(\omega)$ and $n_{\alpha\beta}\nu_\beta = 2\tilde{h}_\alpha$ *on* γ. Hence the functions $\tilde{h}_\alpha$ belong to the space $\in H^{\frac{3}{2}}(\gamma)$ and satisfy the compatibility conditions (see [25, Theorem 4]).

Step 2. Since the set ω is simply-connected and by using the generalized Poincaré theorem (see [89, Theorem VI, p.59],[25, Theorem 7]), the equation $\partial_\beta n_{\alpha\beta} = 0$ in ω imply that there exist distributions $\psi_\alpha \in D'(\omega)$, unique up to the addition of constants, such that $n_{1\alpha} = 2\partial_2\psi_\alpha,\ n_{2\alpha} = -2\partial_1\psi_\alpha$.

The equation $n_{12} = n_{21}$ implies that $\partial_\alpha\psi_\alpha = 0$. Another application of the same result shows that there exist a distribution $\Phi \in D'(\omega)$, unique up to the addition of polynomials of degree ≤ 1, such that $\psi_1 = \partial_2\Phi,\ \psi_2 = -\partial_1\Phi$, so that $n_{11} = 2\partial_{22}\Phi,\ n_{12} = n_{21} = -2\partial_{12}\Phi,\ n_{22} = 2\partial_{11}\Phi\ in\ \omega$.

Step 3. Since $n_{\alpha\beta}\partial_{\alpha\beta}(\xi_3 + \theta) = 2[\Phi, \xi_3 + \theta]$, we have

$$-\partial_{\alpha\beta}m_{\alpha\beta} = k\Delta^2\xi_3 = 2\left[\Phi, \xi_3 + \theta\right] + h_3^0 + G_3^0\ \ in\ \omega.$$

Since $\Delta^2\Phi = \frac{1}{2}\Delta n_{\alpha\alpha}$ and $\partial_{\alpha\beta}n_{\alpha\beta} = 0$, so that

$$\Delta^2\Phi = -\frac{\mu\left(3\lambda + 2\mu\right)}{2\left(\lambda + \mu\right)}\left[\xi_3, \xi_3 + 2\theta\right]\ \ in\ \omega.$$

■

Conclusion

The result obtained in this chapter is similar to that of [84] and [12] that the leading term u^0 of the asymptotic expansion of displacements field is characterized by two dimensional problem without friction. Thus if we consider the work of Léger and Miara [70] but with Coulomb friction, we affirm that we obtain the same result formally.

At the end, we deduce that the displacement u^0 is characterized by a two dimensional problem without friction. Then, our three-dimensional Signorini problem with Coulomb friction offers toward a two-dimensional problem without friction. The loss of frictional densities in $SVP(0)$ and $P^b(0)$ is due to the fact that the friction force behaves as $O(\varepsilon^3)$ whereas the pressure force behaves as $O(\varepsilon^4)$ therefore, at least formally, via the Coulomb law $|\tilde{G}_T^\varepsilon| \leq \Lambda|\tilde{G}_N^\varepsilon|$, when ε tends towards 0 the friction force must be canceled. The question which stands here is how to involve the friction force in the lower dimensional problem and, in the absence of convergence and the existence of asymptotic expansion, is it possible to obtain an algorithm which allows us to compute the higher terms in the asymptotic expansion?

Chapter 4

Asymptotic modeling of Signorini problem for thin elastic plates. Dynamic case.

Introduction

The aim of this chapter is to extend the work of Paumier [84] to a dynamic state problem but without friction by using at first the formal asymptotic expansion method and then the convergence method. First, we give the strong formulation of the three-dimensional contact problem. Next, we rewrite the problem in a weak form. Using a convenient scaling of the unknowns and data, we get the scaled variational problem. For the first method, called the *displacement-stress approach*, we insert the formal asymptotic expansion of the unknowns in the scaled variational problem. In that way we characterize the problem solved by the leading term (u^0, σ^0) of the expansion. For the second method, called the *convergence method*, we prove that a subsequence of the sequence $(u(\varepsilon), \sigma(\varepsilon))$ has a weak star limit denoted $(u(0), \sigma(0))$ which solves the same two dimensional problem than (u^0, σ^0).

4.1 Setting of the problem

In this work, we use the following conventions and notations; Greek indexes belong to the set $\{1, 2\}$, Latin indexes belong to the set $\{1, 2, 3\}$, the symbols of differentiation $\partial_i = \partial/\partial x_i$, $\partial_i^\varepsilon = \partial/\partial x_i^\varepsilon$, $\frac{\partial v}{\partial t} = \dot{v}$, $\frac{\partial^2 v}{\partial t^2} = \ddot{v}$, δ_{ij} the Kronecker symbols, and the summation convention with respect to the repeated indexes is used. Let $\omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz boundary γ. We consider a plate as a three-dimensional body, occupying the volume $\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$, where ε is small parameter $(0 < \varepsilon \leq 1)$. We denote the lateral boundary by $\Gamma_0^\varepsilon = \gamma \times [-\varepsilon, \varepsilon]$, the upper and the lower faces respectively by $\Gamma_+^\varepsilon = \omega \times \{+\varepsilon\}$ and $\Gamma_-^\varepsilon = \omega \times \{-\varepsilon\}$. We denote by $\bar{v}$ the trace of v on Γ_+^ε and by $\underline{v}$ the trace of v on Γ_-^ε, or simply by v if there is no confusion. We restrict ourselves to the case of an isotropic and homogeneous elastic body with Lamé constants $\lambda > 0$, $\mu > 0$ in its natural configuration and having ρ^ε as a volume density. This plate is subjected to body force f^ε on $\Omega^\varepsilon \times]0, +\infty[$ and to surface force g^ε on $\Gamma_-^\varepsilon \times]0, +\infty[$ and is in unilateral contact on Γ_+^ε with a rigid obstacle which occupies the domain $O^\varepsilon = \{x^\varepsilon \in \mathbb{R}^3 / \, (x_1^\varepsilon, x_2^\varepsilon) \in \omega, x_3^\varepsilon > \varepsilon\}$. The contact condition is defined by the inequality $\bar{v}_3 \leq 0$. We assume that this system is in dynamic state and the contact is without friction.

4.2 Strong and weak Formulation of the problem

Using the above assumptions one can state the following classical elastodynamic Signorini problem without friction:

$$\begin{cases} \text{Find } \ (u^\varepsilon, \sigma^\varepsilon) := (u^\varepsilon(x^\varepsilon, t), \sigma^\varepsilon(u^\varepsilon)), t \geq 0 \text{ such that} \\ \rho^\varepsilon \dfrac{\partial^2 u_i^\varepsilon}{\partial t^2} - \partial_j^\varepsilon \sigma_{ij}^\varepsilon = f_i^\varepsilon \ in \ \Omega^\varepsilon \times]0, +\infty[\\ \sigma_{ij}^\varepsilon n_j^\varepsilon = g_i^\varepsilon \ \ on \, \Gamma_-^\varepsilon \times]0, +\infty[\\ u^\varepsilon = 0 \ \ on \ \Gamma_0^\varepsilon \times]0, +\infty[\, , \end{cases}$$

with Signorini boundary conditions

$$\bar{u}_3^\varepsilon \leq 0, \sigma_{33}^\varepsilon \leq 0, \sigma_{33}^\varepsilon \bar{u}_3^\varepsilon = 0 \, on \, \Gamma_+^\varepsilon \times]0, +\infty[\, .$$

The contact is without friction that is interpreted by $\sigma_{\alpha 3}^\varepsilon = 0$ on $\Gamma_+^\varepsilon \times]0, +\infty[$, and finally the initial conditions are

$$u^\varepsilon(.,0) = p^\varepsilon, \dot{u}^\varepsilon(.,0) = q^\varepsilon,$$

where $\sigma_{ij}^\varepsilon \left(u^\varepsilon\right) = \lambda e_{pp}^\varepsilon(u^\varepsilon)\delta_{ij} + 2\mu e_{ij}^\varepsilon(u^\varepsilon)$ are the components of the stress tensor, and also represent the constitutive equation of the elastic material, $e_{ij}^\varepsilon(u^\varepsilon) = \frac{1}{2}\left(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon\right)$ being the components of the linearized deformation tensor.

The vector field $\sigma_{ij}^\varepsilon n_j^\varepsilon$ represents the force acting on the surface section ds whose the unit outward normal vector n^ε. The quantity $\sigma_N^\varepsilon = \sigma_{ij}^\varepsilon n_j^\varepsilon n_i^\varepsilon$ is the component of the pression force and $\sigma_T^\varepsilon = \sigma^\varepsilon n^\varepsilon - \sigma_N^\varepsilon n^\varepsilon$ is the friction force. In our case, $\sigma_N^\varepsilon = \sigma_{33}^\varepsilon$ and $\sigma_T^\varepsilon = (\sigma_{13}^\varepsilon, \sigma_{23}^\varepsilon, 0)$ on Γ_+^ε. Note that we keep the same notation of the function to denote its trace.

We rewrite the above boundary value problem in the following weak form, by using Green's formula, we show that any smooth solution of the boundary value problem also satisfies the following variational problem :

$$(VP^\varepsilon) \begin{cases} \text{Find } \ (u^\varepsilon(t), \sigma^\varepsilon(t)) \in \vec{K}(\Omega^\varepsilon) \times \mathbb{L}_s^2(\Omega^\varepsilon), \ t \geq 0 \text{ such that} \\ \dfrac{\partial^2}{\partial t^2} \rho^\varepsilon \displaystyle\int_{\Omega^\varepsilon} u_i^\varepsilon v_i^\varepsilon dx^\varepsilon + a^\varepsilon(u^\varepsilon, v^\varepsilon) = L^\varepsilon(v^\varepsilon) + \langle \sigma_{33}^\varepsilon, \bar{v}_3^\varepsilon \rangle \, , \ \forall v^\varepsilon \in \vec{V}\left(\Omega^\varepsilon\right), \ t > 0 \\ \langle \sigma_{33}^\varepsilon, v_3^\varepsilon - u_3^\varepsilon \rangle \geq 0, \forall v_3^\varepsilon \in K\left(\Omega^\varepsilon\right), t > 0 \\ \sigma_{ij}^\varepsilon \left(u^\varepsilon\right) = \lambda e_{pp}^\varepsilon(u^\varepsilon)\delta_{ij} + 2\mu e_{ij}^\varepsilon(u^\varepsilon) \\ u^\varepsilon(.,0) = p^\varepsilon, \dot{u}^\varepsilon(.,0) = q^\varepsilon, \end{cases}$$

where

$$
\begin{aligned}
a^{\varepsilon}\left(u^{\varepsilon}, v^{\varepsilon}\right) &= \int_{\Omega^{\varepsilon}}\left[\lambda e_{ii}^{\varepsilon}(u^{\varepsilon}) e_{jj}^{\varepsilon}\left(v^{\varepsilon}\right)+2 \mu\, e_{ij}^{\varepsilon}(u^{\varepsilon}) e_{ij}^{\varepsilon}\left(v^{\varepsilon}\right)\right] d x^{\varepsilon} \\
e_{ij}^{\varepsilon}(u^{\varepsilon}) &= \frac{1}{2}\left(\partial_i^{\varepsilon} u_j^{\varepsilon}+\partial_j^{\varepsilon} u_i^{\varepsilon}\right) \\
L^{\varepsilon}\left(v^{\varepsilon}\right) &= \int_{\Omega^{\varepsilon}} f_i^{\varepsilon} v_i^{\varepsilon} d x^{\varepsilon}+\int_{\Gamma_{-}^{\varepsilon}} g_i^{\varepsilon} \underline{v}_i^{\varepsilon} d \Gamma, \\
&\langle\sigma_{33}^{\varepsilon}, \phi_3^{\varepsilon}\rangle \text{ means the duality pairing on } \Gamma_{+}^{\varepsilon}.
\end{aligned}
$$

and

$$
\begin{aligned}
V(\Omega^{\varepsilon}) &= \left\{v^{\varepsilon} \in H^1\left(\Omega^{\varepsilon}\right) / v^{\varepsilon}=0 \text{ on } \Gamma_0^{\varepsilon}\right\}, K(\Omega^{\varepsilon})=\left\{v^{\varepsilon} \in V(\Omega^{\varepsilon}) / \bar{v}_3^{\varepsilon} \leq 0\right\}, \\
\vec{V}(\Omega^{\varepsilon}) &= V(\Omega^{\varepsilon}) \times V(\Omega^{\varepsilon}) \times V(\Omega^{\varepsilon}), \vec{K}\left(\Omega^{\varepsilon}\right)=V(\Omega^{\varepsilon}) \times V(\Omega^{\varepsilon}) \times K(\Omega^{\varepsilon}),
\end{aligned}
$$

$$
\mathbb{L}_s^2(\Omega^{\varepsilon})=\{\tau^{\varepsilon}=(\tau_{ij}^{\varepsilon}) \in (L^2(\Omega^{\varepsilon}))^9 ;\ \tau_{ij}^{\varepsilon}=\tau_{ji}^{\varepsilon}\}.
$$

Theorem 70 *As described in [10], under the assumptions*

$$
f_i^{\varepsilon} \in W^{2, \infty}(0, T, L^2(\Omega^{\varepsilon})), g_i^{\varepsilon} \in W^{2, \infty}(0, T, L^2(\Gamma_{-}^{\varepsilon}) \cap H^{-1/2}(\Gamma_{-}^{\varepsilon}))
$$

and the initial conditions p_i^{ε}, q_i^{ε} are in $H^1(\Omega^{\varepsilon})$ with $\operatorname{div}\sigma^{\varepsilon}(p^{\varepsilon}) \in (L^2(\Omega^{\varepsilon}))^3$, the problem (VP^{ε}) admits a solution u^{ε} verifying

$$
u^{\varepsilon} \in L^{\infty}(0, T, \vec{K}\left(\Omega^{\varepsilon}\right)), \dot{u}^{\varepsilon} \in L^{\infty}(0, T,(L^2(\Omega^{\varepsilon}))^3) \text{ and } \ddot{u}^{\varepsilon} \in \mathcal{D}^{\prime}(0, T,(L^2(\Omega^{\varepsilon}))^3).
$$

The stress tensor $\sigma^{\varepsilon}(u^{\varepsilon})$ belongs to $\mathcal{D}'(0, T, E_{ad}(g^{\varepsilon})) \cap L^{\infty}(0, T,(L^2(\Omega^{\varepsilon}))^9)$ with

$$
E_{ad}(g^{\varepsilon})=\left\{\begin{array}{c}
\tau^{\varepsilon} \in \mathbb{L}_s^2(\Omega^{\varepsilon});\ \operatorname{div} \tau^{\varepsilon} \in(L^2(\Omega^{\varepsilon}))^3, \tau_{\alpha 3}^{\varepsilon}=0 \text{ and } \\
\tau_{33}^{\varepsilon} \leq 0 \text{ on } \Gamma_{+}^{\varepsilon} ; \tau^{\varepsilon} n^{\varepsilon}=g^{\varepsilon} \text{ on } \Gamma_{-}^{\varepsilon}
\end{array}\right\}
$$

The duality pairing $\langle\sigma_{33}^{\varepsilon}, \phi_3^{\varepsilon}\rangle$ on Γ_{+}^{ε} can be expressed as an integral on Γ_{+}^{ε}. Indeed, we have $\sigma_{33}^{\varepsilon} \in L^2(\Omega^{\varepsilon})$ and $\partial_3^{\varepsilon} \sigma_{33}^{\varepsilon} \in L^2(\Omega^{\varepsilon})$ then $\sigma_{33}^{\varepsilon} \in L^2(\Gamma_{+}^{\varepsilon})$, in sense of trace, with $||\sigma_{33}^{\varepsilon}||_{L^2(\Gamma_{+}^{\varepsilon})} \leq C(||\sigma_{33}^{\varepsilon}||_{L^2(\Omega^{\varepsilon})}^2+||\partial_3^{\varepsilon} \sigma_{33}^{\varepsilon}||_{L^2(\Omega^{\varepsilon})}^2)^{1/2}$ (see [21] page 219). More general, elements of $H(\operatorname{div}, \Omega^{\varepsilon})=\{\tau^{\varepsilon}=(\tau_{ij}^{\varepsilon}) \in L^2(\Omega^{\varepsilon}) ; \operatorname{div} \tau^{\varepsilon} \in(L^2(\Omega^{\varepsilon}))^3\}$ have normal traces $\gamma_N(\sigma^{\varepsilon})$ on Γ_{+}^{ε}.

Proof. We give only the sketch of the proof, see [10]. The proof is divided into five steps.

Step 1 First, a regular partition of the interval $[0, T]$ is considered. we introduce the approximate problem (DVP^ε) at time $t = t_i, 1 \leq i \leq 2^I$:

$$\begin{cases} \text{find } u^{\varepsilon,i} \in \vec{K}(\Omega^\varepsilon), \dot{u}^{\varepsilon,i} \in (H^1(\Omega^\varepsilon))^3 \text{ and } \ddot{u}^{\varepsilon,i} \in (L^2(\Omega^\varepsilon))^3, \\ \rho^\varepsilon \int_{\Omega^\varepsilon} \left(\frac{\ddot{u}^{\varepsilon,i} + \ddot{u}^{\varepsilon,i-1}}{2}\right)\left(v^\varepsilon - u^{\varepsilon,i}\right) dx^\varepsilon + a^\varepsilon \left(\frac{u^{\varepsilon,i} + u^{\varepsilon,i-1}}{2}, v^\varepsilon - u^{\varepsilon,i}\right) \geq L^{\varepsilon,i}(v^\varepsilon - u^{\varepsilon,i}) \\ \forall v^\varepsilon \in \vec{K}(\Omega^\varepsilon) \end{cases}$$

where

$$L^{\varepsilon,i}(v) = \int_{\Omega^\varepsilon} f^{\varepsilon,i} v dx^\varepsilon + \int_{\Gamma^\varepsilon_-} g^{\varepsilon,i} v d\Gamma^\varepsilon$$

and the notation $\phi^{\varepsilon,i}$ stands for $\phi^{\varepsilon,i} := \phi^\varepsilon(t_i)$, $\dot{\phi}^\varepsilon := \frac{d}{dt}\phi^\varepsilon$ and $\ddot{\phi}^\varepsilon := \frac{d^2}{dt^2}\phi^\varepsilon$. The discrete displacement, velocity and acceleration are constructed via Newmark's method as follows:

$$u^{\varepsilon,i} = u^{\varepsilon,i-1} + \Delta t \dot{u}^{\varepsilon,i-1} + \frac{\Delta t^2}{2}\frac{\ddot{u}^{\varepsilon,i} + \ddot{u}^{\varepsilon,i-1}}{2}, \tag{4.1}$$

$$\dot{u}^{\varepsilon,i-1} = \dot{u}^{\varepsilon,i-1} + \Delta t \frac{\ddot{u}^{\varepsilon,i} + \ddot{u}^{\varepsilon,i-1}}{2}. \tag{4.2}$$

Lemma 3 *If at each time t_i, $u^{\varepsilon,i-1} \in \vec{K}(\Omega^\varepsilon)$, $\dot{u}^{\varepsilon,i-1} \in (H^1(\Omega^\varepsilon))^3$ and $\ddot{u}^{\varepsilon,i-1} \in (L^2(\Omega^\varepsilon))^3$ then the problem (DVP^ε) admits a unique solution.*

Proof. see [10]. ■

Step 2 Next, we construct the approximate functions:

$$h^{\varepsilon,I}(t) = u^{\varepsilon,i-1} + \dot{u}^{\varepsilon,i-1}(t - t_{i-1}) + \frac{\ddot{u}^{\varepsilon,i} + \ddot{u}^{\varepsilon,i-1}}{4}(t - t_{i-1})^2, \tag{4.3}$$

$$u^{\varepsilon,I}_\star(t) = u^{\varepsilon,i}, \forall t \in [t_{i-1}, t_i[, \tag{4.4}$$

$$h^{\varepsilon,I}_\star = \frac{u^{\varepsilon,i} + u^{\varepsilon,i-1}}{2}, h^{\varepsilon,2^I}(T) = u^{\varepsilon,2^I}, h^{\varepsilon,2^I}_t(T) = u^{\varepsilon,2^I}_t. \tag{4.5}$$

Step 3 After that, we treat the contact condition with a Lagrange multiplier whose orthogonality properties allow us to obtain the following a priori estimate:

$$\int_{t_{i-1}}^{t_i} \frac{d}{2dt}\left[\rho^\varepsilon \int_\Omega |\dot{h}^{\varepsilon,I}(t)|^2 dx^\varepsilon + a^\varepsilon(h^{\varepsilon,I}(t), h^{\varepsilon,I}(t))\right] dt \tag{4.6}$$

$$\leq L^{\varepsilon,i}(u^{\varepsilon,i} - u^{\varepsilon,i-1}), \forall t \in]t_{i-1}, t_i[. \tag{4.7}$$

Step 4 This estimate allows us to show the weak convergence of approximate solutions and that these limits are equal.

$$h^{\varepsilon,I}(t) \rightharpoonup u^{\varepsilon}(t), \dot{h}^{\varepsilon,I}(t) \rightharpoonup \dot{u}^{\varepsilon}(t) \text{ weak } * \text{ in } L^{\infty}(0,T,(L^2(\Omega^{\varepsilon}))^3) \tag{4.8}$$

$$h_{\star}^{\varepsilon,I}(t) \rightharpoonup u^{\varepsilon}(t), u_{\star}^{\varepsilon,I}(t) \rightharpoonup u^{\varepsilon}(t) \text{ weak } * \text{ in } L^{\infty}(0,T,(H^1(\Omega^{\varepsilon}))^3) \tag{4.9}$$

Step 5 Finally, we show that this limit is a solution of problem (VP^{ε}).

■

4.3 Asymptotic study

In this section, we follow [19] as in the paragraph1.2.1. First we transform the shell problem into a scaled problem posed over a set Ω independent of ε. After that, according to the basic Ansatz of the method of formal asymptotic expansions, we inject the formal expansion of the unknowns in the scaled variational problem. Finally, we identify the leading term of the formal expansion of the scaled displacement and the scaled stress tensor.

4.3.1 Assumptions on data

Let the mapping:

$$\begin{aligned} \pi^{\varepsilon} \quad &: \quad \Omega \to \Omega^{\varepsilon} \\ (x_1,x_2,x_3) \quad &\to \quad (x_1^{\varepsilon},x_2^{\varepsilon},x_3^{\varepsilon})\,/x_1^{\varepsilon}=x_1, x_2^{\varepsilon}=x_2, x_3^{\varepsilon}=\varepsilon x_3 \end{aligned}$$

hence

$$\begin{aligned} \pi^{\varepsilon}(\Omega) &= \Omega^{\varepsilon}; \Omega \ = \omega \times]-1,+1[\,; \pi^{\varepsilon}(\Gamma_{-}^{\varepsilon}) = \Gamma_{-} = \omega \times \{-1\} \\ \pi^{\varepsilon}(\Gamma_{-}^{\varepsilon}) &= \Gamma_{-} = \omega \times \{+1\}\,;\ \pi^{\varepsilon}(\Gamma_0^{\varepsilon}) = \Gamma_0 = \partial\omega \times [-1,+1] \end{aligned}$$

We assume that $\rho^{\varepsilon} = \varepsilon^2 \rho$ and make the change (scaling) of unknowns:

$$\begin{cases} u_\alpha^\varepsilon \circ \pi^\varepsilon = \varepsilon^2 u_\alpha(\varepsilon), \quad u_3^\varepsilon \circ \pi^\varepsilon = \varepsilon u_3(\varepsilon) \\ v_\alpha^\varepsilon \circ \pi^\varepsilon = \varepsilon^2 v_\alpha(\varepsilon), \quad v_3^\varepsilon \circ \pi^\varepsilon = \varepsilon v_3(\varepsilon) \\ \sigma_{\alpha\beta}(\varepsilon) = \varepsilon^{-2}\ \sigma_{\alpha\beta}^\varepsilon(u^\varepsilon), \ \sigma_{\alpha 3}(\varepsilon) = \varepsilon^{-3}\sigma_{\alpha 3}^\varepsilon(u^\varepsilon), \ \sigma_{33}(\varepsilon) = \varepsilon^{-4}\ \sigma_{33}^\varepsilon(u^\varepsilon) \end{cases}$$

The scaling of the contact condition is defined by $\bar{v}_3 \leq 0$. Then we denote

$$\begin{aligned} V(\Omega) &= \left\{v \in H^1(\Omega)/v = 0 \quad \text{on} \quad \Gamma_0\right\}, \vec{V}(\Omega) = V(\Omega) \times V(\Omega) \times V(\Omega) \\ K(\Omega) &= \{v \in V(\Omega)/\ \bar{v}_3 \leq 0 \quad \text{on} \quad \Gamma_+\}, \ \vec{K}(\Omega) = V(\Omega) \times V(\Omega) \times K(\Omega) \end{aligned}$$

$$\mathbb{L}_s^2(\Omega) = \{\tau = (\tau_{ij}) \in (L^2(\Omega))^{3\times 3};\ \tau_{ij} = \tau_{ji}\}.$$

For the forces we suppose that the following scaling holds: there exists f_i, g_i independent of ε such that

$$\begin{cases} f_\alpha^\varepsilon \ \circ \pi^\varepsilon = \varepsilon^2 f_\alpha, \ f_3^\varepsilon \ \circ \pi^\varepsilon = \varepsilon^3 f_3 \\ g_\alpha^\varepsilon \ \circ \pi^\varepsilon = \varepsilon^3 g_\alpha, \ g_3^\varepsilon \ \circ \pi^\varepsilon = \varepsilon^4 g_3 \end{cases}$$

We finally suppose that there exist p and q independent of ε such that:

$$p_\alpha^\varepsilon = \varepsilon^2 p_\alpha^0, p_3^\varepsilon = \varepsilon p_3^0, q_\alpha^\varepsilon = \varepsilon^2 q_\alpha^0, q_3^\varepsilon = \varepsilon q_3^0$$

The scaling of the differential operators is clearly governed by

$$\partial_\alpha^\varepsilon = \partial_\alpha, \partial_3^\varepsilon = \varepsilon^{-1}\partial_3$$

4.3.2 The scaled variational problem

Inserting the upper scalings in the variational problem lead to the following:

Proposition 71 *The variational problem (VP^ε) is equivalent to the following scaled variational problem :*

$$(SVP(\varepsilon)) \begin{cases} Find \ \ (u(\varepsilon)(t), \sigma(\varepsilon)) \in \vec{K}(\Omega) \times \mathbb{L}_s^2(\Omega), t \geq 0 \ such \ that \\ \dfrac{\partial^2}{\partial t^2}\rho \displaystyle\int_\Omega u_3(\varepsilon)\, v_3 dx + \varepsilon^2 \dfrac{\partial^2}{\partial t^2}\rho \int_\Omega u_\alpha(\varepsilon)\, v_\alpha dx + \int_\Omega \sigma_{ij}(\varepsilon)\, \partial_j v_i dx \\ = L(v) + \langle \sigma_{33}(\varepsilon), \bar{v}_3\rangle \ \forall v \in \vec{V}(\Omega), t > 0 & (4.10) \\ \langle \sigma_{33}(\varepsilon), \bar{v}_3 - \bar{u}_3(\varepsilon)\rangle \geq 0, \ \forall v_3 \in K(\Omega), t > 0 & (4.11) \\ u(\varepsilon)(.,0) = p^0, \dot{u}(\varepsilon)(.,0) = q^0 \end{cases}$$

with

$$\begin{cases} \sigma_{\alpha\beta}(\varepsilon) = \lambda e_{\gamma\gamma}(u\,(\varepsilon))\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(u\,(\varepsilon)) + \varepsilon^{-2}\lambda e_{33}(u\,(\varepsilon)) \\ \sigma_{\alpha 3}(\varepsilon) = \varepsilon^{-2} 2\mu e_{\alpha 3}(u\,(\varepsilon)) \\ \sigma_{33}(\varepsilon) = \varepsilon^{-4}\ \ (\lambda + 2\mu)\, e_{33}(u\,(\varepsilon)) + \varepsilon^{-2}\lambda e_{\gamma\gamma}(u\,(\varepsilon)) \end{cases}$$

where

$$e_{ij}(u(\varepsilon)) = \frac{1}{2}\left(\partial_i u_j(\varepsilon) + \partial_j u_i(\varepsilon)\right)$$
$$L\,(v) = \int_\Omega f_i v_i dx + \int_{\Gamma_-} g_i \underline{v}_i d\Gamma$$

Since the problem (VP^ε) has at least a solution u^ε then the problem $(SVP(\varepsilon))$ has at least a solution $u(\varepsilon)$ with the regularity

$$u(\varepsilon) \in L^\infty(0, T, \vec{K}\,(\Omega)), \dot{u}(\varepsilon) \in L^\infty(0, T, (L^2(\Omega))^3) \text{ and } \ddot{u}(\varepsilon) \in \mathcal{D}'(0, T, (L^2(\Omega))^3)$$

The tensor $\sigma(\varepsilon)$ belongs to $\mathcal{D}'(0, T, E_{ad}(g)) \cap L^\infty(0, T, (L^2(\Omega))^9)$ with

$$E_{ad}(g) = \{\tau \in \mathbb{L}^2_s(\Omega);\ \text{div}\tau \in (L^2(\Omega))^3, \tau_{\alpha 3} = 0 \text{ and } \tau_{33} \leq 0 \text{ on } \Gamma_+; \tau n = g \text{ on } \Gamma_-\}.$$

4.3.3 Two-dimensional problem

We assume that the scaled displacement-stress $(u(\varepsilon), \sigma(\varepsilon))$ admits a formal asymptotic expansion of the form:

$$(u(\varepsilon), \sigma(\varepsilon)) = (u^0, \sigma^0) + \varepsilon(u^1, \sigma^1) + \varepsilon^2(u^2, \sigma^2) + ..., \tag{4.12}$$
$$u^0 \in \vec{V}(\Omega), u^q \in (H^1(\Omega))^3, \sigma^0, \sigma^q \in \mathbb{L}^2_s(\Omega), q \in \{1, 2, \ldots\}, t > 0$$

We introduce the space of Kirchhoff-Love admissible displacement

$$\begin{aligned} V_{KL}\,(\Omega) &= \{v = (v_i) \in \left(H^1\,(\Omega)\right)^3, e_{i3}(v) = 0\} \\ &= \left\{ \begin{array}{c} v = (v_i) \in (H^1\,(\Omega))^3\ /\ v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3, v_3 = \eta_3 \text{ such that} \\ \eta_\alpha \in H^1_0\,(\omega)\,, \eta_3 \in H^2_0\,(\omega) \end{array} \right\} \end{aligned}$$

Suppose that there exist φ_i, ψ_i independent of x_3 such that $p^0_\alpha = \varphi_\alpha - x_3\partial_\alpha\varphi_3, p^0_3 = \varphi_3$, and $q^0_\alpha = \psi_\alpha - x_3\partial_\alpha\psi_3, q^0_3 = \psi_3$.

In the next proposition, we give the problem that characterizes (u^0, σ^0) the leading term in the expansion of the scaled displacement-stress $(u(\varepsilon), \sigma(\varepsilon))$.

Proposition 72 *The leading term* (u^0, σ^0) *of the expansion* (4.12) *is a solution of the problem :*

$$(VP^0_{KL})\begin{cases} \text{Find} \quad (u^0(t),\sigma^0(t)) \in \ (V_{KL}(\Omega)\cap \vec{K}(\Omega)) \times \mathbb{L}^2_s(\Omega),\ t\geq 0 \quad \text{such that} \\ \dfrac{\partial^2}{\partial t^2}\rho\displaystyle\int_\Omega u_3^0 v_3 dx + \int_\Omega \sigma^0_{ij}\partial_j v_i dx = L(v) + \left\langle \sigma^0_{33}, \bar{v}_3\right\rangle, \forall v \in \vec{V}(\Omega),\ t>0 & (4.13) \\ \left\langle \sigma^0_{33}, \bar{v}_3 - \bar{u}^0_3\right\rangle \geq 0, \forall v_3 \in K(\Omega),\ t>0 & (4.14) \\ \sigma^0_{\alpha\beta} = \lambda^* e_{\gamma\gamma}(u^0)\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(u^0) \ \text{with}\ \lambda^* = \dfrac{2\lambda\mu}{\lambda+2\mu} & (4.15) \\ u^0(.,0) = p^0, \dot{u}^0(.,0) = q^0 \end{cases}$$

where

$$e_{\alpha\beta}(u^0) = \frac{1}{2}(\partial_\alpha u^0_\beta + \partial_\beta u^0_\alpha).$$

Proof. The inverted constitutive equation reads $e^\varepsilon_{ij}(u^\varepsilon) = \lambda_1 \sigma^\varepsilon_{pp}\delta_{ij} + \dfrac{1}{2\mu}\sigma^\varepsilon_{ij}$ with $\lambda_1 = -\dfrac{\lambda}{2\mu(3\lambda+2\mu)}$. Then the scaled deformation tensor reads

$$\begin{cases} e_{\alpha\beta}(u(\varepsilon)) = \lambda_1\sigma_{\gamma\gamma}(\varepsilon)\delta_{\alpha\beta} + \dfrac{1}{2\mu}\sigma_{\alpha\beta}(\varepsilon) + \varepsilon^2\lambda_1\sigma_{33}(\varepsilon)\delta_{\alpha\beta}, \\ e_{\alpha 3}(u(\varepsilon)) = \varepsilon^2 \dfrac{1}{2\mu}\sigma_{\alpha 3}(\varepsilon), \\ e_{33}(u(\varepsilon)) = \varepsilon^2\lambda_1\sigma_{\gamma\gamma}(\varepsilon) + \varepsilon^4\dfrac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma_{33}(\varepsilon). \end{cases} \tag{4.16}$$

Inserting (4.12) in the system (4.16), we obtain

$$e_{\alpha\beta}(u^0) = -\lambda_1\sigma^0_{\gamma\gamma}\delta_{\alpha\beta} + \frac{1}{2\mu}\sigma^0_{\alpha\beta}, \tag{4.17}$$

$$e_{\alpha 3}(u^0) = 0, \tag{4.18}$$

$$e_{33}(u^0) = 0. \tag{4.19}$$

From (4.18) − (4.19) and the contact condition, we deduce that $u^0 \in V_{KL}(\Omega)\cap\vec{K}(\Omega))$. The equation (4.13) is obtained by inserting (4.12) in (4.10). By the same mean, we obtain (4.11). ■

In the next proposition, we re-write the problem (VP^0_{KL}) in terms of ξ_α and ξ_3. Hence we get a two dimensional problem $(P^b(0))$ whose solutions are ξ_α and ξ_3. The vector field (ξ_i) represents the (scaled) displacement of the middle surface ω of the plate.

Proposition 73 *If* (u^0, σ^0) *is a solution of* (VP^0_{KL}) *such that* $u^0_\alpha = \xi_\alpha - x_3\partial_\alpha\xi_3$ *and* $u^0_3 = \xi_3$*, with* ξ_α, ξ_3 *sufficiently smooth. Then* ξ_α, ξ_3 *verify with* σ^0_{33}*, at least formally, the two-dimensional boundary value problem :*

$$(P^b(0))\begin{cases} \text{Find } \xi_\alpha \in H^1(\omega), \xi_3 \in H^2_0(\omega), \xi_3 \leq 0, \text{ for a.e } t \geq 0 \text{ such that} \\ 2\dfrac{\partial^2}{\partial t^2}\rho\xi_3 + k\Delta^2\xi_3 = h^1_1 + h^1_2 + h^0_3 + \sigma^0_{33} \text{ on } \omega\times]0,+\infty[& (4.20) \\ -\partial_\beta n_{\alpha\beta} = h^0_\alpha \quad \text{on } \omega\times]0,+\infty[& (4.21) \\ \sigma^0_{33}\xi_3 = 0 \text{ in } \omega\times]0,+\infty[, \sigma^0_{33} \leq 0 \text{ in } H^{-2}(\omega) & (4.22) \\ \xi_i(.,0) = \varphi_i, \dfrac{\partial\xi_i}{\partial t}(.,0) = \psi_i \end{cases}$$

where

$$k = \frac{8}{3}\mu\frac{\lambda+\mu}{\lambda+2\mu},\ h^0_i = \int_{-1}^{+1} f_i dx_3 + g^-_i,\ h^1_i = \int_{-1}^{+1} x_3\partial_i f_i dx_3 - \partial_i g^-_i,$$

$$g^-_i = g_i(x_1,x_2,-1)\, n_{\alpha\beta} = \frac{4\lambda\mu}{\lambda+2\mu}e_{\gamma\gamma}(\xi)\delta_{\alpha\beta} + 4\mu e_{\alpha\beta}(\xi)$$

Proof. Let u^0 be a solution of the problem (VP^0_{KL}) then ξ_α, ξ_3 verify $\xi_\alpha \in H^1_0(\omega), \xi_3 \in H^2_0(\omega)$, $\xi_3 \leq 0$. Substituting

$$e_{\alpha\beta}(u^0) = e_{\alpha\beta}(\xi) - x_3\partial_{\alpha\beta}\xi_3$$

in (4.15), we obtain

$$\sigma^0_{\alpha\beta} = \frac{1}{2}n_{\alpha\beta} + \frac{3}{2}x_3 m_{\alpha\beta}$$

with

$$\begin{aligned} n_{\alpha\beta} &= \frac{4\lambda\mu}{\lambda+2\mu}e_{\gamma\gamma}(\xi)\delta_{\alpha\beta} + 4\mu e_{\alpha\beta}(\xi) \\ m_{\alpha\beta} &= -\frac{4}{3}\left(\frac{\lambda\mu}{\lambda+2\mu}\Delta\xi_3\delta_{\alpha\beta} + \mu\partial_{\alpha\beta}\xi_3\right) \end{aligned}$$

We take $v = (-x_3\partial_1\eta_3, -x_3\partial_2\eta_3, \eta_3)$ then the second term in the left hand side of the equilibrium equation in the problem (VP^0_{KL}) becomes

$$\begin{aligned} \int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx &= \int_\Omega -\frac{1}{2}\, n_{\alpha\beta}x_3\partial_{\alpha\beta}\eta_3 dx + \int_\Omega -\frac{3}{2}x_3^2 m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 dx \\ &= \int_\Omega x_3^2\left(\frac{2\lambda\mu}{\lambda+2\mu}\Delta\xi_3\Delta\eta_3 + 2\mu\partial_{\alpha\beta}\xi_3\partial_{\alpha\beta}\eta_3\right)dx \\ &= \frac{4}{3}\int_\omega\left(\frac{2\lambda\mu}{\lambda+2\mu}\Delta\xi_3\Delta\eta_3 + 2\mu\partial_{\alpha\beta}\xi_3\partial_{\alpha\beta}\eta_3\right)dx'. \end{aligned}$$

For $\eta_3 \in \mathcal{D}(\Omega)$ we have,

$$\begin{aligned}\int_\omega \Delta\xi_3 \Delta\eta_3 dx' &= \int_\omega \Delta^2\xi_3\eta_3 dx' \\ \int_\omega \partial_{\alpha\beta}\xi_3\partial_{\alpha\beta}\eta_3 dx' &= \int_\omega \Delta^2\xi_3\eta_3 dx'\end{aligned}$$

which, by density, still holds for elements of $H_0^2(\omega)$. Hence

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta v_\alpha dx = \frac{8}{3}\mu\frac{\lambda+\mu}{\lambda+2\mu}\int_\omega \Delta^2\xi_3\eta_3 dx'. \tag{4.23}$$

On the other hand, the right hand side of the equation becomes

$$\begin{aligned} L(v) + \langle \sigma^0_{33}, \bar{v}_3\rangle &= \int_\Omega f_\alpha v_\alpha dx + \int_\Omega f_3 v_3 dx + \int_{\Gamma_-} g^-_\alpha \underline{v}_\alpha d\Gamma + \int_{\Gamma_-} g^-_3 \underline{v}_3 d\Gamma \\ &+ \int_{\Gamma_+} \sigma^0_{33}\bar{v}_3 d\Gamma \\ &= \int_\omega \left\{ \int_{-1}^{+1} x_3\partial_\alpha f_\alpha dx_3 - \partial_\alpha g^-_\alpha \right\} \eta_3 dx' \\ &+ \int_\omega \left\{ \int_{-1}^{+1} f_3 dx_3 + g^-_3 \right\} \eta_3 dx' + \int_\omega \sigma^0_{33}\eta_3 dx' \\ &= \int_\omega \left(h^0_3 + h^1_\alpha + \sigma^0_{33}\right)\eta_3 dx' \end{aligned} \tag{4.24}$$

with $h^0_i = \int_{-1}^{+1} f_i dx_3 + g^-_i$, $h^1_i = \int_{-1}^{+1} x_3\partial_i f_i dx_3 - \partial_i g^-_i$, g^-_i trace of g_i on Γ_-. Then, from (4.23) and (4.24) we obtain (4.20), in sense of distributions. By taking $\bar{v}_3 = 0$ (resp. $\bar{v}_3 = 2u^0_3$) in the inequality in the problem (VP^0_{KL}) we find $\langle\sigma^0_{33}, \xi_3\rangle = 0$ (resp. $\langle\sigma^0_{33}, \eta_3\rangle \geq 0$, for all $\eta_3 \in H^2_0(\omega)$ with $\eta_3 \leq 0$, which leads to $\sigma^0_{33} \leq 0$ in $H^{-2}(\omega)$). This proves (4.22). Now, we take $v = (\eta_1, \eta_2, 0)$ in the left hand side of the equilibrium equation in the problem (VP^0_{KL}), we obtain

$$\int_\Omega \sigma^0_{\alpha\beta}\partial_\beta\eta_\alpha dx = \int_\Omega f_\alpha\eta_\alpha dx + \int_{\Gamma_-} g^-_\alpha\eta_\alpha d\Gamma, \forall \eta_1, \eta_2 \in H^1_0(\omega).$$

Therefore

$$\int_\omega n_{\alpha\beta}\partial_\beta\eta_\alpha dx' = \int_\omega h^0_\alpha\eta_\alpha dx', \forall \eta_1, \eta_2 \in H^1_0(\omega),$$

hence we find (4.21) in sense of distributions. ■

The next proposition is devoted to the characterization of σ^0_{i3}.

Proposition 74 *If* (u^0, σ^0) *is a solution of the problem* (VP^0_{KL}) *and* ξ_α, ξ_3 *verify the problem* $(P^b(0))$*, then* u^0 *and* σ^0 *are given by:*

$$\begin{cases} u^0_\alpha = \ \xi_\alpha - x_3\partial_\alpha\xi_3, \ u^0_3 = \xi_3, & (4.25) \\ \sigma^0_{\alpha\beta} = \dfrac{1}{2}\ n_{\alpha\beta} + \dfrac{3}{2}x_3 m_{\alpha\beta}, & (4.26) \\ \sigma^0_{\alpha 3} = \dfrac{3}{4}(1-x_3^2)\partial_\beta m_{\alpha\beta} + \dfrac{1}{2}(1+x_3)\displaystyle\int_{-1}^{+1} f_\alpha dy_3 - \int_{-1}^{x_3} f_\alpha dy_3 & \\ \qquad +\dfrac{1}{2}(x_3-1)g^-_\alpha. & (4.27) \end{cases}$$

Finally σ^0_{33} *satisfies in sense of distributions:*

$$\begin{cases} \partial_3\sigma^0_{33} = \rho\ddot{\xi}_3 - \partial_\alpha\sigma^0_{\alpha 3} - f_3 \text{ in } \Omega\times\left]0,+\infty\right[, & (4.28) \\ \sigma^0_{33} = -g^-_3 \ \text{ on } \Gamma_-\times\left]0,+\infty\right[, & (4.29) \\ \sigma^0_{33}\xi_3 = 0,\ \sigma^0_{33}\le 0 \quad \text{on } \Gamma_+\times\left]0,+\infty\right[. & (4.30) \end{cases}$$

Proof. The proof of this proposition follows essentially the same pattern as in [20, Thm 4.8-1, p.301] or in [19, Thm 1.7-1, p.38]. The equations $(4.25)-(4.26)$ are already obtained in the Proposition 73.

Using test function $v = (v_1, v_2, 0)$ in the equation (4.13), we obtain, *formally* that $\sigma^0_{\alpha 3}$ verifies in sense of distributions the quasi-static boundary value problem

$$\begin{cases} \partial_3\sigma^0_{\alpha 3} = -\partial_\beta\sigma^0_{\alpha\beta} - f_\alpha \text{ in } \ \Omega\times\left]0,+\infty\right[\\ \sigma^0_{\alpha 3} = -g^-_\alpha \ \text{ on } \ \Gamma_-\times\left]0,+\infty\right[\\ \sigma^0_{\alpha 3} = 0 \ \text{ on } \ \Gamma_+\times\left]0,+\infty\right[\end{cases}$$

Integrating the first equation over $[-1, x_3]$ and taking in account the boundary conditions, we obtain (4.27).

For the computation of σ^0_{33}, we take as test functions $v = (0,0,v_3)$ in the equation (4.13) then we obtain

$$\frac{\partial^2}{\partial t^2}\rho\int_\Omega u^0_3 v_3 dx + \int_\Omega \sigma^0_{33}\partial_3 v_3 dx = -\int_\Omega \sigma^0_{\alpha 3}\partial_\alpha v_3 dx + \int_\Omega f_3 v_3 dx$$
$$+\int_{\Gamma_-} g_3\underline{v}_3 d\Gamma + \int_{\Gamma_+}\sigma^0_{33}\bar{v}_3 d\Gamma, \forall v_3\in V(\Omega), t>0. \qquad (4.31)$$

On the other hand, by Green's formula and taking into account the cancellation on the boundary Γ_0, we have *formally* that

$$\int_\Omega \sigma^0_{33}\partial_3 v_3 dx = -\int_\Omega \partial_3\sigma^0_{33} v_3 dx + \int_{\Gamma_+}\sigma^0_{33}\bar{v}_3 d\Gamma$$
$$-\int_{\Gamma_-}\sigma^0_{33}\underline{v}_3 d\Gamma, \forall v_3\in V(\Omega), t>0 \qquad (4.32)$$

and

$$\int_{\Omega} \sigma^0_{\alpha 3} \partial_\alpha v_3 dx = -\int_{\Omega} \partial_\alpha \sigma^0_{\alpha 3} v_3 dx, \forall v_3 \in V(\Omega), t > 0. \tag{4.33}$$

Inserting (4.32) and (4.33) in (4.31) we find that

$$\frac{\partial^2}{\partial t^2}\rho \int_{\Omega} u^0_3 v_3 dx + \int_{\Omega} \partial_3 \sigma^0_{33} v_3 dx = -\int_{\Omega} \partial_\alpha \sigma^0_{\alpha 3} v_3 dx - \int_{\Omega} f_3 v_3 dx - \int_{\Gamma_-} g_3 \underline{v}_3 d\Gamma$$
$$- \int_{\Gamma_-} \sigma^0_{33} \underline{v}_3 d\Gamma, \forall v_3 \in V(\Omega)$$

We derive from the inequality (4.14) that σ^0_{33} must *formally* verifies (4.30). In summary the boundary value problem (4.28)-(4.30) is *formally* satisfied. ■

4.4 Convergence study

In this section, we suppose that $u(\varepsilon)$ is a solution of $(P(\varepsilon).V)$, the forces verify $f, \dot{f} \in L^\infty(0, T, L^2(\Omega))$ and $g, \dot{g} \in L^\infty(0, T, L^2(\Gamma_-))$ and we study the limit of the sequence $(u(\varepsilon))$, after that, we compare the results. To this end, we introduce the tensor $\kappa(\varepsilon, v)$ defined as following

$$\kappa_{\alpha\beta}(\varepsilon, v) = e_{\alpha\beta}(v), \kappa_{\alpha 3}(\varepsilon, v) = \varepsilon^{-1} e_{\alpha 3}(v), \kappa_{33}(\varepsilon, v) = \varepsilon^{-2} e_{33}(v) \tag{4.34}$$

Then, the bilinear form $a^\varepsilon(u(\varepsilon), v)$ can be rewritten as following

$$a^\varepsilon(u(\varepsilon), v) = \int_{\Omega} [\lambda \kappa_{ii}(\varepsilon) \kappa_{jj}(\varepsilon, v) + 2\mu \kappa_{ij}(\varepsilon) \kappa_{ij}(\varepsilon, v)] dx$$

where $\kappa(\varepsilon)$ denotes $\kappa(\varepsilon) := \kappa(\varepsilon, u(\varepsilon))$. The space $\mathbb{L}^2_s(\Omega)$ of symmetric, square integrable tensors equipped with the scalar product $\langle \sigma, \tau \rangle = \int_{\Omega} \sigma_{ij} \tau_{ij} dx$ is a real Hilbert space. As it is known, the norms

$$\|\sigma\|_A = \langle A\sigma, \sigma \rangle^{1/2}, \|\sigma\|_{0,\Omega} \tag{4.35}$$

are equivalent where $\langle A\sigma, \tau \rangle = \int_{\Omega} (\lambda \sigma_{ii} \tau_{jj} + 2\mu \sigma_{ij} \tau_{ij} dx)$.

Proposition 75 *If $u(\varepsilon)$ is a solution of the problem $(P(\varepsilon).V)$ then, for ε small enough, the sequences $(u(\varepsilon)), (\kappa_{ij}(\varepsilon))$ are bounded, respectively, in $L^\infty(0, T, \vec{V}(\Omega))$ and $L^\infty(0, T, \mathbb{L}^2_s(\Omega))$*

hence, there exists a subsequences also denoted by $(u(\varepsilon))$ *and* $(\kappa_{ij}(\varepsilon))$ *which admit a weak star limits respectively in* $L^\infty(0,T,\vec{V}(\Omega))$ *and* $L^\infty(0,T,\ \mathbb{L}^2_s(\Omega))$ *denoted respectively by* $(u(0))$ *and* $(\kappa_{ij}(0))$.

Proof. We suppose that the approximate solutions $h^{\varepsilon,I}$ and $h^{\varepsilon,I}_\star$ have the same scaling as the displacement u^ε, i.e., $h^{\varepsilon,I}_\alpha = \varepsilon^2 h^I_\alpha(\varepsilon)$, $h^{\varepsilon,I}_3 = \varepsilon h^I_3(\varepsilon)$, $h^{\varepsilon,I}_{\star\alpha} = \varepsilon^2 h^I_{\star\alpha}(\varepsilon)$ and $h^{\varepsilon,I}_{\star 3} = \varepsilon h^I_{\star 3}(\varepsilon)$. Since

$$h^{\varepsilon,I} \rightharpoonup u^\varepsilon \text{ weak } * \text{ in } L^\infty(0,T,(L^2(\Omega^\varepsilon))^3), \tag{4.36}$$

$$h^{\varepsilon,I}_\star \rightharpoonup u^\varepsilon \text{ weak } * \text{ in } L^\infty(0,T,(H^1(\Omega^\varepsilon))^3), \tag{4.37}$$

then

$$h^I(\varepsilon) \rightharpoonup u(\varepsilon) \text{ weak } * \text{ in } L^\infty(0,T,(L^2(\Omega))^3), \tag{4.38}$$

$$h^I_\star(\varepsilon) \rightharpoonup u(\varepsilon) \text{ weak } * \text{ in } L^\infty(0,T,(H^1(\Omega))^3), \tag{4.39}$$

The above scalings and the scalings of f^ε and g^ε leads to

$$\rho^\varepsilon \|\dot{h}^{\varepsilon,I}(t)\|^2_{(L^2(\Omega^\varepsilon)^3} = \rho\varepsilon^7 \sum_{\alpha=1}^{2} \|\dot{h}^I_\alpha(\varepsilon)\|^2_{L^2(\Omega)} + \rho\varepsilon^5 \|\dot{h}^I_3(\varepsilon)\|^2_{L^2(\Omega)}, \tag{4.40}$$

$$a^\varepsilon(h^{\varepsilon,I}(t), h^{\varepsilon,I}(t)) = \varepsilon^5 a(h^I(\varepsilon), h^I(\varepsilon)), \tag{4.41}$$

$$L^{\varepsilon,i}(v^\varepsilon) = \varepsilon^5 L^i(v(\varepsilon)), \tag{4.42}$$

where the forms a and L^i are independent of ε. Then the inequality (4.7) becomes:

$$\begin{aligned}
&\rho\varepsilon^2 \sum_{\alpha=1}^{2} \|\dot{h}^I_\alpha(\varepsilon)(t_k)\|^2_{L^2(\Omega)} + \rho\|\dot{h}^I_3(\varepsilon)(t_k)\|^2_{L^2(\Omega)} + a(h^I(\varepsilon)(t_k), h^I(\varepsilon)(t_k)) \\
&\qquad \le \rho\varepsilon^2 \sum_{\alpha=1}^{2} \|q^0_\alpha\|^2_{L^2(\Omega)} + \rho\|q^0_3\|^2_{L^2(\Omega)} + a(p^0,p^0) \\
&\qquad\quad + \sum_{i=1}^{k} L^i(h^I(\varepsilon)(t_i) - h^I(\varepsilon)(t_{i-1})),
\end{aligned} \tag{4.43}$$

As a consequence we have the following prior estimate:

$$\begin{aligned}
&\rho\varepsilon^2 \sum_{\alpha=1}^{2} \|\dot{h}^I_\alpha(\varepsilon)(t_k)\|^2_{L^2(\Omega)} + \rho\|\dot{h}^I_3(\varepsilon)(t_k)\|^2_{L^2(\Omega)} + \mu_1\|h^I(\varepsilon)(t_k)\|^2_{(H^1(\Omega))^3} \\
&\quad \le \mu_2 + \mu_3\|h^I(\varepsilon)(t_k)\|_{(H^1(\Omega))^3} + \mu_4 \sum_{i=1}^{k-1} \Delta t \|h^I(\varepsilon)(t_i)\|_{(H^1(\Omega))^3},
\end{aligned} \tag{4.44}$$

where $\mu_k, k = 1, 2, 3, 4$ are independent of ε and verify:

$$\mu_1 : \text{ coercivity constant of the form } a, \tag{4.45}$$

$$\begin{aligned}\mu_2 = \rho \sum_{\alpha=1}^{2} \|q_\alpha^0\|^2_{L^2(\Omega)} + \rho \|q_3^0\|^2_{L^2(\Omega)} + a(p^0, p^0) \\ + \|f\|_{L^\infty(0,T,(L^2(\Omega)^3)}.\|p^0\|_{(L^2(\Omega))^3} \\ + \|g\|_{L^\infty(0,T,(L^2(\Gamma_-))^3)}.\|p^0\|_{(L^2(\Gamma_-))^3},\end{aligned} \tag{4.46}$$

$$\mu_3 = \|f\|_{L^\infty(0,T,(L^2(\Omega)^3)} + \lambda_1 \|g\|_{L^\infty(0,T,(L^2(\Gamma_-))^3)} \tag{4.47}$$

$$\mu_4 = \|\dot{f}\|_{L^\infty(0,T,(L^2(\Omega)^3)} + \lambda_2 \|\dot{g}\|_{L^\infty(0,T,(L^2(\Gamma_-))^3)} \tag{4.48}$$

Without loss of generality, we suppose that $\mu_2 \geq 1$. Then we obtain

$$\mu_1 \|h^I(\varepsilon)(t_k)\|_{(H^1(\Omega))^3} \leq \mu_2 + \mu_3 + \mu_4 \sum_{i=1}^{k-1} \Delta t \|h^I(\varepsilon)(t_i)\|_{(H^1(\Omega))^3}. \tag{4.49}$$

Using Discrete Gronwall lemma yields

$$\mu_1 \|h^I(\varepsilon)(t_k)\|_{(H^1(\Omega))^3} \leq (\mu_2 + \mu_3) e^{T\mu_4/\mu_1}. \tag{4.50}$$

then also the sequence $(h^I_\star(\varepsilon)(t_k))$ is bounded independently of k and ε in $L^\infty(0, T, (H^1(\Omega))^3)$.

Since the norm is weakly lower semi-continuous, one has

$$\|u(\varepsilon)\|_{L^\infty(0,T,(H^1(\Omega))^3)} \leq \liminf_{I>0} \|h^I_\star(\varepsilon)(t)\|_{L^\infty(0,T,(H^1(\Omega))^3)} < C. \tag{4.51}$$

such that C independent of ε. The we conclude:

$$u(\varepsilon) \in L^\infty(0, T, \vec{V}(\Omega)) \text{ and } \dot{u}_3(\varepsilon), \varepsilon\dot{u}_\alpha(\varepsilon) \in L^\infty(0, T, L^2(\Omega)) \tag{4.52}$$

Using the continuity of the form $a^\varepsilon(.,.)$, the assumptions (4.34) and the equivalence of the norms (4.35), we get

$$\|e_{\alpha\beta}(u(\varepsilon))\|_{0,\Omega} \leq c, \|e_{\alpha 3}(u(\varepsilon))\|_{0,\Omega} \leq \varepsilon c, \|e_{33}(u(\varepsilon))\|_{0,\Omega} \leq \varepsilon^2 c, \tag{4.53}$$

for a.e $t \in]0, T]$. ■

Proposition 76 *The weak star limits $u(0)$ and $\kappa(0)$ verify the following*

$$u(0) \in V_{KL}(\Omega) \cap \vec{K}(\Omega), \text{ for } a.e\ t \in]0, T],$$

$$\kappa_{\alpha\beta}(0) = e_{\alpha\beta}(u(0)), \kappa_{\alpha 3}(0) = 0, \kappa_{33}(0) = \frac{-\lambda}{\lambda + 2\mu} e_{\alpha\alpha}(u(0)), \tag{4.54}$$

for $a.e\ t \in]0, T]$.

Proof. From (4.53) and the lower semi-continuity of the norm

$$\|e_{ij}(u(0))\|_{0,\Omega} \leq \liminf_{\varepsilon>0} \|e_{ij}(u(\varepsilon))\|_{0,\Omega}, \text{for } a.e\ t \in]0,T] \tag{4.55}$$

we deduce that $e_{33}(0) = e_{\alpha 3}(0) = 0$, for $a.e\ t \in]0,T]$.
From the weak closeness of $\vec{K}(\Omega)$, we deduce that

$$u(0) \in \vec{K}(\Omega),\ \text{for } a.e\ t \in]0,T]$$

Then

$$u(0) \in L^{\infty}(0,T,V_{KL}(\Omega) \cap \vec{K}(\Omega))$$

The equality $\kappa_{\alpha\beta}(0) = e_{\alpha\beta}(u(0))$ derives from $\kappa_{\alpha\beta}(u(\varepsilon)) \rightharpoonup \kappa_{\alpha\beta}(u(0))$ in $L^{\infty}(0,T,\ L^2(\Omega))$ weak star and $e_{\alpha\beta}(u(\varepsilon)) \rightharpoonup e_{\alpha\beta}(u(0))$ in $L^{\infty}(0,T,L^2(\Omega))$ and the uniqueness of the weak limit.
To prove $\kappa_{\alpha 3}(0) = 0\ a.e\ t \in]0,T]$, we choose as test function $\phi(t)(\varepsilon v_1, \varepsilon v_2, 0)$, $v_\alpha \in V(\Omega)$ and $\phi(t) \in \mathcal{D}(0,T)$ in (4.10) after that we integrate from 0 to T we obtain

$$\rho \int_0^T \phi(t) \int_\Omega \varepsilon^2 \ddot{u}_\alpha \varepsilon v_\alpha dx dt + \int_0^T \phi(t) a^\varepsilon(u(\varepsilon), \varepsilon v) dt = \int_0^T \phi(t) L(\varepsilon v) dt. \tag{4.56}$$

The term $\rho \int_0^T \phi(t) \int_\Omega \varepsilon^2 \ddot{u}_\alpha \varepsilon v_\alpha dx dt$ goes to 0 when $\varepsilon \to 0$. Indeed, integrating by parts we obtain $\rho \int_0^T \dot{\phi}(t) \int_\Omega \varepsilon^2 \dot{u}_\alpha \varepsilon v_\alpha dx dt \to 0$ when $\varepsilon \to 0$, reminding that $(\dot{\phi}(t) v_\alpha \in L^1(0,T,L^2(\Omega)))$.
The term

$$\begin{aligned}
\int_0^T \phi(t) a^\varepsilon(u(\varepsilon), \varepsilon v) dt &= \varepsilon \int_0^T \phi(t) \int_\Omega \lambda \kappa_{ii}(\varepsilon,u)\kappa_{jj}(\varepsilon,v) + 2\mu \kappa_{ij}(\varepsilon,u)\kappa_{ij}(\varepsilon,v) dx dt \\
&= \varepsilon \int_0^T \phi(t) \int_\Omega \lambda \kappa_{ii}(\varepsilon,u) e_{\beta\beta}(v) + 2\mu \kappa_{\alpha\beta}(\varepsilon,u) e_{\alpha\beta}(v) dx dt \\
&+ \int_0^T \phi(t) \int_\Omega 2\mu \kappa_{\alpha 3}(\varepsilon,u)(\partial_\alpha v_3 + \partial_3 v_\alpha) dx dt \\
&+ \frac{1}{\varepsilon} \int_0^T \phi(t) \int_\Omega (\lambda \kappa_{ii}(\varepsilon,u) + 2\mu \kappa_{33}(\varepsilon,u)) e_{33}(v) dx dt \\
&= \varepsilon \int_0^T \phi(t) \int_\Omega \lambda \kappa_{ii}(\varepsilon,u) e_{\beta\beta}(v) + 2\mu \kappa_{\alpha\beta}(\varepsilon,u) e_{\alpha\beta}(v) dx dt \\
&+ \int_0^T \phi(t) \int_\Omega 2\mu \kappa_{\alpha 3}(\varepsilon,u) \partial_3 v_\alpha dx dt,
\end{aligned}$$

due to the choice of the test function. We now pass to the limit in (4.56) and get

$$\lim_{\varepsilon \to 0} \int_0^T \phi(t) \int_\Omega 2\mu \kappa_{\alpha 3}(\varepsilon,u) \partial_3 v_\alpha dx dt = 0, \forall \phi \in \mathcal{D}(0,T)$$

Then

$$\int_{\Omega} 2\mu\kappa_{\alpha 3}(u(0))\partial_3 v_\alpha dx = 0 \ a.e \ for \ t \in]0,T], \forall v_\alpha \in V(\Omega).$$

Recalling that of [21, Theorem 3.4-1] states that any $w \in L^2(\Omega)$ such that $\int_\Omega w\partial_3 \, vdx = 0$ for all $v \in \mathcal{C}^\infty(\bar{\Omega})$ that satisfy $v = 0$ on $\gamma\times[-1,1]$ then $w = 0$, we deduce that $\kappa_{\alpha 3}(u(0)) = 0$ in Ω in sense of distribution for a.e $t \in]0,T]$. To prove $\kappa_{33}(0) = \frac{-\lambda}{\lambda+2\mu} e_{\alpha\alpha}(u(0))$ we use as test function $\phi(t)(0,0,\varepsilon^2 v_3), \bar{v}_3 = 0$, $v_3 \in V(\Omega)$ and $\phi(t) \in \mathcal{D}(0,T)$ in (4.10) after that we integrate from 0 to T in (4.10) then, we pass to the limit and obtain

$$\int_{\Omega} (\lambda\kappa_{ii}(0) + 2\mu\kappa_{33}(0))\partial_3 v_3 dx = 0 \ a.e \ for \ t \in]0,T], \forall v_3 \in V(\Omega).$$

The result follows. ■

Proposition 77 *The sequence* $(\sigma_{33}(\varepsilon))$ *verifies*

$$\sigma_{33}(\varepsilon) \rightharpoonup \sigma_{33}(0) \text{ in } L^\infty(0,T,H^{-2}(\omega)) \text{ weak star}$$

and verifies with $u(0)$ *the inequality*

$$\langle\sigma_{33}(0), u_3(0)\rangle \leq \lim_{\varepsilon\to 0}\langle\sigma_{33}(\varepsilon), u_3(\varepsilon)\rangle \text{ in } L^\infty(0,T).$$

Proof. We recall that among unknowns of the problem is the contact stress $\sigma_{33}(\varepsilon)$ on Γ_+. Now, we prove that also this quantity has weak limit when ε tends to zero.
We prove now that

$$\int_\Omega \ddot{u}_3(\varepsilon)v_3 dx \ \rightarrow \ \int_\Omega \ddot{u}_3(0)v_3 dx \ \text{ in } \mathcal{D}'(0,T)$$

Let $\phi \in \mathcal{D}(0,T)$, we have

$$\int_0^T \phi(t)\int_\Omega \ddot{u}_3(\varepsilon)v_3 dx = -\int_0^T\int_\Omega \dot{u}_3(\varepsilon)\dot{\phi}(t)v_3 dx.$$

From (4.52) we deduce that there exists $\chi \in L^\infty(0,T,L^2(\Omega))$ such that

$$\dot{u}_3(\varepsilon) \rightharpoonup \chi \text{ in } L^\infty(0,T,L^2(\Omega)) \text{ weak star}$$

i.e

$$\int_0^T (\dot{u}_3(\varepsilon), g(t))dt \rightarrow \int_0^T (\chi, g(t))dt, \forall g \in L^1(0,T,L^2(\Omega)).$$

For $g(x,t)=\phi(t)w(x)$ such that $\phi(t)\in\mathcal{D}(0,T)$ and $w\in L^2(\Omega)$, we have

$$\int_0^T(\dot{u}_3(\varepsilon),\phi(t)w(\cdot))dt=-\int_0^T(u_3(\varepsilon),\dot{\phi}(t)w(\cdot))dt.$$

On the other hand $u(\varepsilon)$ converges strongly to $u(0)$ in the space $L^\infty(0,T,L^2(\Omega))$. Therefore

$$\int_0^T(\dot{u}_3(\varepsilon),\phi(t)w(x))dt\to-\int_0^T(u_3(0),\dot{\phi}(t)w(\cdot))dt=\int_0^T(\dot{u}_3(0),\phi(t)w(\cdot))dt.$$

From the density of functions $\phi(t)w(x)$ in $L^1(0,T,L^2(\Omega))$ we deduce that

$$\int_0^T(\dot{u}_3(\varepsilon),g(t))dt\to\int_0^T(\chi,g(t))dt,\forall g\in L^1(0,T,(L^2(\Omega)))$$

with $\chi=\dot{u}_3(0)$. That means

$$\dot{u}_3(\varepsilon)\rightharpoonup\dot{u}_3(0)\text{ in }L^\infty(0,T,L^2(\Omega))\text{ weak star}$$

Therefore $\chi=\dot{u}_3(0)$ by virtue of the uniqueness of weak limit. Since $\dot{u}_3(\varepsilon)\rightharpoonup\dot{u}_3(0)$ in $L^\infty(0,T,L^2(\Omega))$ weak star and $\dot{\phi}v_3\in L^1(0,T,L^2(\Omega))$ then

$$\int_0^T\int_\Omega\dot{u}_3(\varepsilon)\dot{\phi}(t)v_3dx\to\int_0^T\int_\Omega\dot{u}_3(0)\dot{\phi}(t)v_3dx=-\int_0^T\int_\Omega\ddot{u}_3(0)\phi(t)v_3dx$$

From (4.10), we get for all $v\in V_{KL}(\Omega)$ and $\phi\in\mathcal{D}(0,T)$

$$\begin{aligned}\int_0^T\phi(t)\left\langle\sigma_{33}(\varepsilon),v_3\right\rangle dt=&-\int_0^T\phi(t)L(v)dt+\rho\int_0^T\phi(t)\left\langle\ddot{u}_3(\varepsilon),v_3\right\rangle dt\\&+\varepsilon^2\rho\int_0^T\phi(t)\left\langle\ddot{u}_\alpha(\varepsilon),v_\alpha\right\rangle dt+\int_0^T\phi(t)a^\varepsilon(u(\varepsilon),v)dt\end{aligned}$$

when ε goes to 0, we obtain that for all $v\in V_{KL}(\Omega)$ and $\phi\in\mathcal{D}(0,T)$

$$\int_0^T\phi(t)\left\langle\sigma_{33}(\varepsilon),v_3\right\rangle dt\to\int_0^T\phi(t)(a_*^0(u(0),v)-L(v)+\rho\left\langle\ddot{u}_3(0),v_3\right\rangle)dt$$

Then

$$\left\langle\sigma_{33}(\varepsilon),v_3\right\rangle\to a_*^0(u(0),v)-L(v)+\rho\left\langle\ddot{u}_3(0),v_3\right\rangle\ \text{in}\,\mathcal{D}'(0,T)$$

such that

$$a_*^0(u(0),v)=\int_\Omega\sigma_{\alpha\beta}(0)\partial_\beta v_\alpha dx,\sigma_{\alpha\beta}(0)=\lambda^*e_{\gamma\gamma}(u(0))\delta_{\alpha\beta}+2\mu e_{\alpha\beta}(u(0)),\lambda^*=\frac{2\lambda\mu}{\lambda+2\mu}$$

The map $v_3 \longrightarrow a^0_*(u(0), v) - L(v) + \rho \langle \ddot{u}_3(0), v_3\rangle$ defines a linear form on $H^2_0(\omega)$. Then the sequence $\sigma_{33}(\varepsilon)$ admits a weak star limit in $L^\infty(0, T, H^{-2}(\omega))$ denoted by $\sigma_{33}(0)$ which verifies the equation

$$\rho \langle \ddot{u}_3(0), v_3\rangle + a^0_*(u(0), v) = L(v) + \langle \sigma_{33}(0), v_3\rangle\,, \forall v \in V_{KL}(\Omega),\ a.e\ t \in]0, T[\qquad (4.57)$$

From (4.57), we deduce $\sigma_{33}(0)$ verifies with $u(0)$ the equation

$$\rho \langle \ddot{u}_3(0), u(0)\rangle + a^0_*(u(0), u(0)) = L(u(0)) + \langle \sigma_{33}(0), u_3(0)\rangle\ \ a.e\ t \in]0, T[. \qquad (4.58)$$

We prove now that

$$\langle \sigma_{33}(0), v_3 - u_3(0)\rangle \geq 0,\ \forall v_3 \in K(\Omega),\ a.e\ t \in]0, T[. \qquad (4.59)$$

To do this, we must verify that

$$\langle \sigma_{33}(0), u_3(0)\rangle \leq \lim_{\varepsilon \longrightarrow 0} \langle \sigma_{33}(\varepsilon), u_3(\varepsilon)\rangle\,,\ \text{for}\ \ a.e\ t \in]0, T[. \qquad (4.60)$$

then letting ε goes to 0, we obtain (4.59).
We take in (4.11) first $v_3 = 0$ and secondly $v_3 = 2u_3$ and obtain that $\langle \sigma_{33}(\varepsilon), u_3(\varepsilon)\rangle = 0$ then

$$\langle \sigma_{33}(\varepsilon), u_3(\varepsilon)\rangle \rightharpoonup 0\ in\ L^\infty(0, T)\ \text{weak star}. \qquad (4.61)$$

We have that

$$\|\kappa(0)\|^2_A = a^0_*(u(0), u(0))$$

hence for a.e $t \in]0, T[$

$$\begin{aligned} \langle \sigma_{33}(0), u_3(0)\rangle &= \|\kappa(0)\|^2_A - L(0) + \rho \langle \ddot{u}_3(0), u_3(0)\rangle \\ &\leq \liminf_{\varepsilon > 0} \left(\|K(\varepsilon)\|^2_A - L(\varepsilon) + \rho \langle \ddot{u}_3(\varepsilon), u_3(\varepsilon)\rangle\right) \\ &\leq \liminf_{\varepsilon > 0} \langle \sigma_{33}(\varepsilon), u_3(\varepsilon)\rangle \\ &\leq \lim_{\varepsilon \longrightarrow 0} \langle \sigma_{33}(\varepsilon), u_3(\varepsilon)\rangle \end{aligned}$$

Indeed, since $u_3(\varepsilon)$ converges weak star to $u_3(0)$ in $H^2_0(\Omega)$ for a.e $t \in]0, T[$ then converges strongly in $H^1_0(\Omega)$ and $\ddot{u}_3(\varepsilon)$ converges weak star to $\ddot{u}_3(0)$ in $H^{-1}(\Omega)$ for a.e $t \in]0, T[$ then, we have proved (4.60) therefore (4.59). For the initial conditions, $u(0)$ verifies

$$u_{|t=0}(0) = p(0), \dot{u}_{|t=0}(0) = q(0)$$

■

In summary we have proved the following theorem:

Theorem 78 *If $u(\varepsilon)$ is a solution of the problem $(P(\varepsilon).V)$ then*

$$u(\varepsilon) \rightharpoonup u(0) \text{ in } L^\infty(0,T,\vec{V}(\Omega)) \ weak - *$$
$$\sigma_{33}(\varepsilon) \rightharpoonup \sigma_{33}(0) \text{ in } L^\infty(0,T,H^{-2}(\omega)) \ weak - *$$

where $u(0)$ is a solution of the problem :

$$(VP_{KL}(0)) \begin{cases} Find \ \ u(0) \in \ V_{KL}(\Omega) \cap \vec{K}(\Omega) \, for \ \ a.e \ t \in [0,T] \, such \ that \\ \rho \langle \ddot{u}_3(0), v_3 \rangle + a^0_*(u(0),v) = L(v) + \langle \sigma_{33}(0), v_3 \rangle \, , \, \forall v \in V_{KL}(\Omega) \\ \langle \sigma_{33}(0), v_3 - u_3(0) \rangle \geq 0, \ \forall v_3 \in K(\Omega) \\ u_{|t=0}(0) = p(0), \dot{u}_{|t=0}(0) = q(0) \end{cases}$$

with

$$a^0_*(u(0),v) = \int_\Omega \sigma_{\alpha\beta}(0)\partial_\beta v_\alpha dx, \sigma_{\alpha\beta}(0) = \lambda^* e_{\gamma\gamma}(u(0))\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(u(0)), \ \lambda^* = \frac{2\mu\lambda}{\lambda + 2\mu}.$$

Notice that the solution (u^0, σ^0) of problem (VP^0_{KL}) from Proposition 72 is also a solution of problem $(VP_{KL}(0))$, but since the uniqueness of the solution of this last problem is not guaranteed we cannot conclude that $(u^0, \sigma^0) = (u(0), \sigma(0))$.

Conclusion

We have shown that the weak limit of the subsequence $(u(\varepsilon), \sigma_{33}(\varepsilon))$ of the three-dimensional unilateral problem is solution to a two-dimensional unilateral problem. Furthermore the solution of the other two-dimensional problem which derives from the asymptotic method is also solution of the one deduced from the convergence method. As there is no uniqueness result for this problem, we cannot conclude that the solutions coincide. Nevertheless is it possible to prove the strong convergence of the whole sequence by using the variational inequality (2.57)? Another open problem is to prove similar results for the time-dependent Signorini problem with Coulomb friction.

General conclusion

As general conclusion to this thesis is that in the asymptotic modeling nonlinear Signorini problem for plates or for extended geometry shallow shells, we get the same result that the leading term in the formal asymptotic expansions of the solution is characterized by a two dimensional problem without friction.

For linear shallow shells, we obtain the same result using convergence method as in the work (in progress) on " Asymptotic modeling of linear Signorini problem for shallow shells with Coulomb friction ". This is a partial answer of questions of Paumier in the introduction. The loss of friction force in the two dimensional problem does not still true for shells as shown in the work (in progress) on " Asymptotic modeling of Signorini problem with Coulomb friction for linear membrane shells"

In the case of dynamic Signorini problem without friction for linear shallow shells, we justify the reduced problem using convergence method as in the work (in progress) " Asymptotic modeling of linear Signorini problem for shallow shells. The dynamic case "

Some open questions

- Using Gamma convergence to justify the two-dimensional models in the nonlinear case.
- Existence of solutions of the problem pointed out in Proposition 63.
- Existence of solutions of the problem pointed out in Proposition 69.
- Existence of solutions of the dynamic Signorini problem with given friction force (Tresca Law) (in progress).
- Existence of solutions of a hyperbolic problem with obstacle (in progress).

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