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THESE

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Sujet

PROBLEMES AUX LIMITES POUR CETAINES EDP NON LINEAIRES

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Notations principales

Notation	**Description**
$\cdot$	produit scalaire
$\lvert . \rvert$	norme euclidienne
$\times$	produit vectoriel
$\otimes$	produit tensoriel, $\quad (A \otimes B)_{ij} = A_i B_j$
∇	le gradient
Δ	le laplacien
$\nabla \cdot F = div\ F$	la divergence de F
$\nabla \times F = rot\ F$	le rotationnel de F
∇F	$(\nabla F)_{ij} = \frac{\partial F_j}{\partial x_i}$
ΔF	$\Delta F = (\Delta F_1, \Delta F_2, \Delta F_3)$
$\star$	produit de convolution
S^2	sphère unité de $\mathbb{R}^3$
$\Omega \setminus \omega$	complémentaire de ω dans Ω
$\overline{\omega}$	fermeture de ω dans $\mathbb{R}^d$
$\omega \subset\subset \Omega$	$\overline{\omega} \subset \Omega$
$\partial\Omega$	le bord de Ω
n	la normale au bord unitaire et sortante
$\dfrac{\partial}{\partial n}$	dérivée normale
$\lvert\Omega\rvert$	la mesure (de Lebesgue) de Ω
$\chi(\Omega)$	la fonction caractéristique de Ω
$supp\ u$	support de la fonction u
$\lvert . \rvert_X$	la norme dans l'espace X
$\langle ., . \rangle_{X',X}$	dualité entre X et X'

$C^k(X)$, $C^\infty(X)$	espace des fonctions $f : X \to \mathbb{R}$, k fois continûment différentiables ou indéfiniment différentiables
$C^k(X,Y)$, $C^\infty(X,Y)$	espace des fonctions $\Phi : X \to Y$, k fois continûment différentiables ou indéfiniment différentiables
$\mathcal{D}(\Omega)$	espace des fonctions indéfiniment différentiables à support dans Ω
$\mathcal{D}'(\Omega)$	dual de $\mathcal{D}(\Omega)$
a.e. ou p.p.	presque partout
$L^p(\Omega)$	espace de Lebesgue
$L^p_{loc}(\Omega)$	espace de Lebesgue des fonctions localement intégrables
$\lvert.\rvert_p$	norme dans $L^p(\Omega)$
$L^p(X,Y)$	espace des fonctions $f : X \to Y$ mesurables telles que $\int_X \|f(t)\|^p_Y dt < \infty$
$W^{m,p}(\Omega), H^m(\Omega)$	espaces de Sobolev
$W^{m,p}(X,Y)$	espace des fonctions f telles que $\dfrac{d^k f}{dt^k} \in L^p(X;Y), 0 \leq k \leq m$

Noter que

$$\begin{cases} (A \times B) \cdot C = A \cdot (B \times C) = -B \cdot (A \times C) \\ A \times (B \times C) = (A \cdot C)B - (A \cdot B)C \\ \nabla \times (\nabla u) = 0,\ \nabla \cdot (\nabla \times G) = 0,\ \nabla \cdot (\nabla u) = \Delta u \\ \nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - (\nabla \cdot \nabla)F = \nabla(\nabla \cdot F) - \Delta F. \end{cases}$$

Quelques rappels [1]

Espaces abstraits contenant le temps

• Soit B un espace de Banach réel de norme $\|.\|$.
Soit $v \in L^1(0,T;B)$, l'application F_v définie pour tout $\varphi \in \mathcal{D}(]0,T[)$ par

$$F_v(\varphi) = -\int_0^T v(t)\frac{d\varphi}{dt}dt$$

est linéaire continue de $\mathcal{D}(]0,T[)$ dans B, autrement dit $F_v \in \mathcal{D}'(]0,T[;B)$

Definition 0.0.1 *On dit que F_v est la dérivée de v (au sens faible) et on note* $F_v = v' = \dfrac{dv}{dt}$

En particulier si $v' \in L^1(0,T;B)$, on peut écrire pour tout $\varphi \in \mathcal{D}(]0,T[)$

$$\int_0^T \varphi(t)v'(t)dt = -\int_0^T \varphi'(t)v(t)dt.$$

Proposition 0.0.2 *Soit $W^{1,p}(0,T;B) = \{v \in L^p(0,T;B); v' \in L^p(0,T;B)\}$. On a $W^{1,p}(0,T;B) \subset C([0,T];B)$ et*

$$v(t) = v(s) + \int_s^t v'(\tau)d\tau, \quad \forall v \in W^{1,p}(0,T;B).$$

• Soit H un espace de Hilbert, V un espace de Hilbert dense dans H et V' son dual. En identifiant H à son dual, on a

$$V \subset H \subset V'.$$

[1]Voir dans *Partial differential equations* de L. C. Evans par exemple.

On note $\|.\|$, $|.|$ et $\|.\|_*$ les normes respectives dans V, H et V', $(.,.)$ le produit scalaire dans H et $\langle .,. \rangle$ la dualité entre V et V'.

Proposition 0.0.3 *Soit* $W = \{v \in L^2(0,T;V); v' = \frac{dv}{dt} \in L^2(0,T;V')\}$ *alors*
1/ W *est un espace de Hilbert pour la norme*

$$\|v\|_W^2 = \int_0^T (\|v(t)\|^2 + \|v'(t)\|_*^2) dt$$

2/ $\mathcal{D}(]0,T[;V)$ *est dense dans* W
3/ W *s'injecte continûment dans* $C([0,T];H)$.
4/ *l'application* $t \mapsto |v(t)|^2$ *est absolument continue et*

$$\frac{d}{dt}|v(t)|^2 = 2\langle v'(t), v(t) \rangle$$

5/ *Pour* $u, v \in W$, *on a*

$$\int_0^T (\langle u', v \rangle + \langle v', u \rangle) dt = (u(T), v(T)) - (u(0), v(0))$$

• Nous allons maintenant rappeler un résultat de compacité. On se donne B_0, B_1 deux espaces de Banach réflexifs, B un espace de Banach tels que $B_0 \subset B \subset B_1$ et l'injection de B_0 dans B est compacte. On note

$$W = \{v \in L^{p_0}(0,T;B_0); v' \in L^{p_1}(0,T;B_1)\}$$

avec $0 < p_0, p_1 < \infty$, alors muni de la norme

$$\|v\|_W = \|v\|_{L^{p_0}(0,T;B_0)} + \|v'\|_{L^{p_1}(0,T;B_1)},$$

W est un espace de Banach et on a

Proposition 0.0.4 *L'injection de* W *dans* $L^{p_0}(0,T;B)$ *est compacte.*

L'inégalité de Gronwall

• 1 $^{\text{ère}}$ forme : Soit η une fonction positive, absolument continue sur $[0,T]$ telle que

$$\eta'(t) \leq \alpha(t)\eta(t) + \beta(t) \quad \text{p.p. } t \in [0,T]$$

où α et β sont des fonctions positives intégrables sur $[0,T]$. Alors

$$\eta(t) \leq \left(\eta(0) + \int_0^t \beta(s)ds\right) e^{\int_0^t \alpha(s)ds}.$$

• 2 $^{\text{ème}}$ forme : Soit η une fonction positive, intégrable sur $[0,T]$ telle que

$$\eta(t) \leq a + b\int_0^t \eta(s)ds \quad \text{p.p. } t \in [0,T]$$

avec $a, b \geq 0$. Alors

$$\eta(t) \leq a(1 + bte^{bt}).$$

Présentation générale

Les travaux rassemblés dans cette thèse sont relatifs à des modèles posés en ferromagnétisme, piézoélectricité ainsi qu'en physique des plasmas. Ces domaines ont suscité un intérêt croissant chez les chercheurs en raison des multiples applications dans l'industrie et notamment en microtechnologie. Une partie de la thèse sera consacrée à chacun de ces domaines. Afin d'en faciliter la consultation au lecteur, nous avons préféré faire suivre chaque partie des références qui y sont indiquées. Il nous a paru également utile de compléter ces références par une bibliographie plus générale, elle contient essentiellement des analyses mathématiques et numériques récentes sur le ferromagnétisme et la piézoélectricité.

Partie I : Couplage inter-couches de matériaux ferromagnétiques

Notre étude porte sur des structures F/N/F formées de deux couches cylindriques ferromagnétiques (F) séparées par une couche formée d'un métal non magnétique (N). La dynamique de l'aimantation dans le matériau ferromagnétique obéit à l'équation de Landau-Lifshitz-Gilbert couplée aux équations de Maxwell pour décrire le champ électromagnétique.

Dans le chapitre 1, nous donnons une introduction dans laquelle nous décrivons les propriétés des matériaux ferromagnétiques ainsi que les équations qui régissent la magnétisation, les conditions aux limites et les différentes énergies associées à la magnétisation de ces matériaux.

En approximation magnétostatique, le champ magnétique induit H est donné par

$$H = \nabla\varphi, \quad \nabla \cdot (\nabla\varphi + \chi(\Omega)M) = 0 \text{ dans } \mathbb{R}^+ \times \mathbb{R}^3 \tag{1}$$

où $\chi(\Omega)$ est le fonction caractéristique de Ω alors que la magnétisation M nulle dans (N) satisfait dans le domaine ferromagnétique (F) à

$$\begin{cases} \partial_t M - \alpha M \times \partial_t M = -(1+\alpha^2) M \times \mathcal{H}(M) \\ M(0,x) = M_0(x),\ |M_0(x)|^2 = 1 \end{cases} \tag{2}$$

où le champ magnétique effectif $\mathcal{H}(M)$ est donné par

$$\mathcal{H}(M) = A\Delta M - \nabla_M \psi(M) + \nabla\varphi. \tag{3}$$

Le système d'équations (1)-(2) est complété par des conditions aux limites.

Dans le chapitre 2, nous considérons l'équation de Landau-Lifshitz-Gilbert complétée par des conditions aux limites sur les interfaces associées à l'énergie d'échange biquadratique

$$\mathcal{E}_{bq}(M) = \frac{K_{bq}}{2} \int (1 - (M^+ \cdot M^-)^2) dx$$

où $M^\pm$ représentent l'aimantation aux interfaces $\Gamma^\pm$ du matériau ferromagnétique. La condition de couplage s'exprime par une condition aux limites de type Neumann donnée sur $\Gamma^\pm$ par

$$M^\pm \times (A(\frac{\partial M}{\partial n})^\pm + K_{bq}(|M^\mp|^2 M^\pm - (M^+ \cdot M^-)M^\mp)) = 0 \tag{4}$$

et ailleurs, nous posons

$$M \times A\frac{\partial M}{\partial n} = 0 \tag{5}$$

La difficulté principale dans ce problème réside dans la donnée aux limites de couplage des deux interfaces qui est non linéaire et à croissance cubique. Pour résoudre ce problème, nous proposons une méthode basée sur plusieurs approximations. D'abord nous pénalisons la condition de saturation $|M| = 1$ satisfaite par la magnétisation dans le domaine ferromagnétique, puis on introduit une régularisation et une autre pénalisation de la condition aux limites pour avoir $|M^\pm| \leq 1$ sur les interfaces. Finalement, le problème aux limites obtenu est résolu en introduisant une régularisation elliptique en temps pour absorber le terme $M \times \partial_t M$. Toutes ces transformations sont faites par l'intermédiaire de petits paramètres positifs ν, η et ε. En utilisant une technique standard basée sur un argument de monotonicité, on montre l'existence de solutions $V^\varepsilon_{\eta,\nu}$ du problème approché et on conclut grâce aux estimations uniformes sur $V^{\varepsilon}_{\eta,\nu}$ déduites de l'inégalité d'énergie que la limite de $V^{\varepsilon}_{\eta,\nu}$, lorsqu'on fait tendre successivement les paramètres ε, η et ν vers 0, est une solution globale du modèle considéré. On notera que la pénalisation introduite pour obtenir l'aimantation $|M^{\pm}| \leq 1$ sur les interfaces est cruciale dans le passage à la limite lorsque $\nu \to 0$.

Dans le chapitre 3, on s'intéresse aux équations introduites par Zhang, Levy et Fert modélisant le renversement de la magnétisation dans un matériau en couches F/N/F par un courant polarisé en spins injecté perpendiculairement au domaine F/N/F. La magnétisation locale du matériau ferromagnétique provoque une accumulation de spins dans l'espaceur (N) et les interfaces F/N et N/F. L'équation de Landau-Lifshitz-Gilbert est alors couplée à l'équation non linéaire de la chaleur satisfaite par le champ d'accumulation de spins m, le couplage étant dû à l'énergie d'interaction de contact $-\int JM \cdot m$. Dans ce cas le champ magnétique effectif devient

$$\mathcal{H}(M) = A\Delta M - \nabla_M \psi(M) + \nabla\varphi + Jm. \tag{6}$$

L'accumulation de spins quant à elle, est régie par l'équation

$$\frac{1}{d^2}\partial_t m - \partial_x(\mathcal{A}(\langle M\rangle)(\partial_x m)) + \frac{1}{\lambda_J^2} m \times \langle M\rangle + \frac{1}{\lambda_{\mathrm{sf}}^2} m = -\frac{\beta}{d^2}\partial_x(j_e\langle M\rangle) \tag{7}$$

$x \in [-L, l]$ où l'axe Ox représente la direction de circulation du courant injecté j_e, $\langle M\rangle$ est la moyenne de M sur la base du cylindre formant la structure et la matrice $\mathcal{A}(\langle M\rangle)$ est définie par

$$\mathcal{A}(\langle M\rangle)(\xi) = \xi - \beta\beta'(\langle M\rangle \cdot \xi)\langle M\rangle, \ \xi \in \mathbb{R}^3. \tag{8}$$

Bien sûr le problème est complété par la donnée de conditions initiale et aux limites satisfaites par m et M.

Pour montrer l'existence de solutions de ce problème, nous suivons la méthode introduite dans le chapitre 2. En particulier une régularisation est nécessaire dans le terme de diffusion de l'équation (7) ainsi que la pénalisation de la condition de saturation $|M| = 1$ satisfaite par la magnétisation. On notera qu'en raison de la condition aux limites linéaire, une régularisation hyperbolique de (2) est plus adaptée au cas présent. Le problème approché est alors résolu en utilisant un argument de point fixe et les inégalités d'énergie permettent de conclure à l'existence de solutions globales. Cette méthode permet aussi de résoudre d'autres modèles de retournement de l'aimantation par l'injection d'un courant, nous en donnons quelques uns pour conclure.

Partie II. Retournement de domaines dans une tige piézoélectrique mince

Dans cette partie, on s'intéresse à un modèle 1-D de matériaux piézoélectriques proposé par F. Davì. On commence par une introduction générale au chapitre 1 puis dans le chapitre 2, on établit un résultat d'existence de solutions de ce modèle mathématique. Le système est composé d'une équation de la chaleur décrivant la dynamique du nombre f de dipôles électriques et d'une équation des ondes modélisant la dynamique du champ des déplacements u

$$\partial_t f - \partial_x^2 f = -\partial_x u \text{ dans } (0,T)\times\mathbb{R} \tag{9}$$

$$\partial_t^2 u - \partial_x[(1+f)\partial_x u] = 0 \text{ dans } (0,T)\times\mathbb{R} \tag{10}$$

le déplacement électrique étant donné dans ce cas par $\partial_x u$.

L'énergie associée à l'équation des ondes est donnée par $E(t) = (|\partial_t u|_2^2 + |(1+f)^{1/2}\partial_x u|_2^2)^{1/2}$ et elle vérifie

$$\frac{dE^2(t)}{dt} = \int_{\mathbb{R}} \partial_t f |\partial_x u|^2 \, dx$$

de sorte qu'il n'y a pas de conservation de l'énergie. Néanmoins l'énergie est décroissante si $\partial_t f(t,x) \leq 0$ p.p. de même que la positivité de l'énergie est assurée si $f(t,x) \geq 0$ p.p. Comme le signe de $\partial_x u$ n'est pas fixé alors la solution f de l'équation de la chaleur est telle que $f(t,x) \geq 0$ (resp $\partial_t f(t,x) \leq 0$) pour un temps fini dépendant de $E^2(0)$ lorsqu'on suppose la condition initiale $f(0,x) \geq F_0 > 0$ (resp $\partial_t f(0,x) \leq -F_0$) p.p. C'est la raison pour laquelle, on arrive à montrer un résultat d'existence locale et non globale en temps. Notons que pour résoudre l'équation des ondes (10), il est commode d'y introduire le terme $-\varepsilon\partial_x^2[\partial_t u]$ qui représente une viscosité artificielle. En utilisant la méthode variationnelle, on montre l'existence et l'unicité d'une solution $(f^\varepsilon, u^\varepsilon)$ pour tout $\varepsilon > 0$ fixé, ces solutions étant toutes définies sur un intervalle fini $[0,T]$ indépendant de ε. Pour conclure, on passe à la limite lorsque $\varepsilon \to 0$ et on obtient une solution locale du problème considéré.

Partie III. Le problème aux limites de Vlasov-Poisson-Fokker-Planck

Cette dernière partie est consacrée à l'étude du système de Vlasov-Poisson-Fokker-Planck qui décrit la dynamique de la fonction de distribution f des particules dans un plasma, en tenant compte des collisions entre les particules. Là aussi nous commençons par rappeler au chapitre 1 les équations cinétiques des plasmas puis dans le chapitre suivant, un résultat d'existence de solutions est établi pour le problème aux limites de Vlasov-Poisson-Fokker-Planck. L'équation de Vlasov-Fokker-Planck s'écrit

$$\partial_t f + v \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_v f - \sigma \Delta_v f = 0 \ \text{ dans } \mathbb{R}^+ \times \omega \times \mathbb{R}^d \tag{11}$$

où le potentiel électrique Φ vérifie l'équation de Poisson

$$\Delta_x \Phi(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, dv \ \text{ dans } \mathbb{R}^+ \times \omega \tag{12}$$

ω étant un domaine borné de $\mathbb{R}^d$. Pour compléter le système d'équations (11)-(12), on suppose que $\Phi = 0$ sur ∂w, $f(0,x,v) = f_0(x,v)$ dans $\omega \times \mathbb{R}^d$ et

$$f(t,x,v) = g(t,x,v) \ \text{ dans } \mathbb{R}^+ \times \Gamma_- \tag{13}$$

avec $\Gamma_- = \left\{(x,v) \in \partial\omega \times \mathbb{R}^d, v \cdot \nu(x) < 0\right\}$. Cette condition aux limites est de type absorbant et correspond à l'injection de particules au bord de $\partial\omega$ suivant la fonction de distribution g.

On démontre un résultat d'existence globale d'une solution faible du problème en dimension 1 et 2. La méthode repose sur une analyse détaillée de l'équation linéaire de Vlasov-Fokker-Planck (à Φ fixé) ainsi que sur les approximations successives du point fixe. En particulier le taux de décroissance des solutions par rapport à la vitesse joue un rôle important. En effet, si on pose $F = (1 + |v|^2)^{\gamma/2} f$ et $G(t) = Max(1, |F(t)|_{L^\infty(\omega \times \mathbb{R}^d)})$, on montre que

$$G(t) \leq A_1 + A_2 \int_0^t G(t) dt + A_3 \int_0^t (G(s))^{\frac{d-1}{\gamma}} (G(s))^{1-\frac{1}{\gamma}} ds \tag{14}$$

où $A_i, i = 1,2,3$ sont des constantes positives. Or l'équation différentielle

$$\dot{\alpha}(t) = (A_2 + A_3)\alpha(t)^{\frac{d-2}{\gamma}+1}, \ \alpha(0) = A_1 \tag{15}$$

n'a de solution globale que lorsque $d \leq 2$, c'est la raison pour laquelle notre méthode aboutit à un résultat d'existence globale de solutions en dimensions 1 et 2 et seulement locale en dimension 3. Le point intéressant est que nous obtenons pour les solutions des estimations uniformes par rapport au coefficient de diffusion σ. Par conséquent, en faisant tendre σ vers 0, on obtient des solutions du problème aux limites de Vlasov-Poisson qui décrit la dynamique de la fonction de distribution en absence de collisions entre les particules.

Première partie

Couplage inter-couches de matériaux ferromagnétiques

Chapitre 1

Introduction au ferromagnétisme

Le magnétisme est connu depuis l'antiquité mais les premières études quantitatives dans ce domaine n'ont été effectuées que vers la fin du 18 ème siècle. Ce fut C.A. de Coulomb qui mesura pour la première fois les forces s'exerçant entre deux charges magnétiques. En 1820, H. Orstead observa l'existence d'un champ magnétique autour d'un conducteur traversé par un courant électrique. Cette découverte fut à la base du fondement de la théorie de l'électromagnétisme qui fut élaborée par la suite par A.M. Ampère, M. Faraday et J. Maxwell.

1.1 Les matériaux magnétiques

Dans la matière, les atomes sont regroupés en "domaines magnétiques" qui sont des régions dans lesquelles les spins des atomes sont tous alignés. L'orientation de chaque domaine est en général aléatoire et les champs magnétiques créés dans les domaines se compensent mutuellement de sorte que le champ magnétique résultant est nul.

Un matériau est dit magnétique lorsqu'il réagit à la présence d'un aimant. Sous l'effet d'un champ magnétique extérieur, la matière se réorganise à l'intérieur du matériau magnétique, les atomes tendant à aligner leurs spins.

On distingue cinq types de magnétisme : le diamagnétisme, le paramagnétisme, le ferromagnétisme, le ferrimagnétisme et l'antiferromagnétisme. Les matériaux ferromagnétiques sont caractérisés par une aimantation spontanée due au fait que les domaines ont plus ou moins la même orientation. Le moment magnétique résultant est assez élevé, raison pour laquelle il est employé en industrie pour réaliser des aimants permanents. Exemples : le cobalt, le nickel, le fer et ses alliages.

Enfin indiquons que les matériaux ferromagnétiques sont sujets à l'hystérésis : lorsque le champ magnétique extérieur est supprimé, l'aimantation du matériau ne revient pas à son état initial. Par ailleurs au-delà d'une certaine température appelée température de Curie, un matériau ferromagnétique devient paramagnétique.

Dans la suite, nous nous intéresserons uniquement aux matériaux ferromagnétiques.

1.2 Les matériaux ferromagnétiques

Un marériau ferromagnétique maintenu à une température en-dessous de celle de Curie possède une magnétisation (ou aimantation) spontanée et il est composé de domaines où la magnétisation est uniforme. Les limites entre ces domaines sont appelées des parois de Bloch. Ainsi en plaçant un matériau ferromagnétique dans un champ magnétique extérieur H, les parois vont se déplacer car les moments des spins des atomes tendent à s'aligner dans la même direction et le même sens que le champ extérieur H de manière à renforcer celui-ci. Il en résulte un champ magnétique B qui peut arriver à saturation lorsqu'on augmente H. Lorsqu'on supprime le champ extérieur, le matériau conserve partiellement sa structure ordonnée et donc une partie de son aimantation, en raison d'interactions entre les moments magnétiques des atomes (résultant d'un processus quantique) et du phénomène d'hystérésis. L'aimantation qui reste lorsqu'on ramène H à 0 est dite rémanente. L'excitation magnétique nécessaire pour ramener le champ B à zéro est appelée champ coercitif. On distingue deux types de matériaux ferromagnétiques :

les matériaux doux ou mous qui présentent un cycle d'hystérésis étroit, un champ rémanent B_r et un champ coercitif H_c faibles et peu de pertes par hystérésis. De plus ils s'aimantent et se désaimantent facilement,

les matériaux durs qui présentent un cycle d'hystérésis large, un champ rémanent B_r et un champ coercitif H_c importants ainsi que des pertes par hystérésis importantes. De plus ils s'aimantent et se désaimantent très difficilement.

Les matériaux doux sont employés pour réaliser des circuits magnétiques dans les moteurs, les générateurs et les transformateurs par exemple, alors que les matériaux ferromagnétiques durs sont utilisés pour la réalisation d'aimants permanents ou le stockage d'informations sous forme magnétique (bandes magnétiques, disques durs et disquettes informatiques).

1.3 Les équations de Maxwell

L'électromagnétisme est fondé sur les équations de Maxwell. Ces 4 équations établies par le physicien J. Maxwell en 1873 décrivent les caractéristiques du champ électrique et du champ magnétique. Elles permettent d'expliquer l'ensemble des lois de l'élecromagnétisme, de la physique atomique ou de l'optique.
Soient E et B les champs électrique et magnétique respectivement. Leurs inductions correspondantes D et H vérifient dans un milieu continu les relations suivantes ([25, 7])

$$\begin{cases} \partial_t D - \nabla \times H = -J \\ \partial_t B + \nabla \times E = 0 \\ \nabla \cdot B = 0, \ \nabla \cdot D = \rho \end{cases} \tag{1.1}$$

où ρ est la densité de charge. Les champs électrique et magnétique ainsi que leurs inductions sont liés par des lois de comportement

$$B = \mu_0(H + P_m) \tag{1.2}$$

$$D = \varepsilon_0 E + P_e \tag{1.3}$$

où P_e et P_m désignent respectivement la polarisation électrique et la polarisation magnétique, $\mu_0 = 4\pi 10^{-7}$ étant la constante de perméabilité magnétique dans le vide et $\varepsilon_0 \simeq \frac{1}{36\pi}10^{-9}$ la permittivité électrique. Dans le vide, on a $P_e = P_m = 0$. Dans un milieu linéaire, la polarisation électrique s'exprime par $P_e = \varepsilon_0 \chi_e E$ où χ_e est la matrice des susceptibilités électriques ; celle-ci est constante lorsque le milieu est homogène et vaut $\chi_{e,0} I$ dans le cas d'un milieu isotrope avec $\chi_{e,0} > 0$ (I étant la matrice identité). Dans un milieu linéaire homogène et isotrope, on écrit

$$D = \varepsilon_0(1 + \chi_{e,0})E = \varepsilon_0 \varepsilon_r E = \varepsilon E \tag{1.4}$$

où ε est la permittivité du milieu et ε_r est la permittivité relative. De même on a $P_m = \chi_m H$ où χ_m est la matrice des susceptibilités magnétiques avec χ_m constante dans un milieu homogène et $\chi_m = \chi_{m,0} I$ dans un milieu isotrope de sorte qu'en milieu linéaire homogène et isotrope, on ait

$$B = \mu_0(1 + \chi_{m,0})H = \mu_0 \mu_r H = \mu H \tag{1.5}$$

où μ est la perméabilité du milieu et μ_r est la perméabilité relative. Noter que contrairement à la susceptibilité électrique, la susceptibilité magnétique d'un milieu peut être négative, c'est le cas d'un milieu diamagnétique (cuivre, argent, or) alors que la susceptibilité est positive dans les milieux paramagnétiques, ferrimagnétiques ou ferromagnétiques. Lorsque $|\chi_{m,0}| << 1$ (milieux paramagnétiques ou diamagnétiques), on prend $\mu = \mu_0$ mais ce n'est pas le cas dans les milieux ferromagnétiques, la susceptibilité y est forte et non constante.

Un milieu réel est caractérisé par des propriétés diélectriques (liées à la susceptibilité électrique χ_e), des propriétés magnétiques (liées à la susceptibilité magnétique χ_m) et des propriétés conductrices. Pour ces dernières, on utilise la loi d'Ohm qui décrit le courant dû au déplacement de charges libres d'un métal sous l'action d'un champ électrique

$$J = \sigma E. \tag{1.6}$$

Ici J représente la densité de courant électrique et σ est la conductivité électrique. Dans la suite, nous considérerons uniquement le cas où $\sigma = 0$ (ce qui est pratiquement le cas pour les ferrites) et $\rho = 0$. Pour simplifier, nous travaillerons dans le cadre de l'approximation magnétostatique, c'est à dire que l'on négligera le champ électrique et la dépendance par rapport au temps de sorte que les équations de Maxwell (1.1) se réduisent à

$$\nabla \times H = 0, \quad \nabla \cdot B = 0 \tag{1.7}$$

En outre, nous désignerons par M la polarisation magnétique et on écrit la loi de comportement (1.2) sous la forme

$$B = \mu_0(H + M). \tag{1.8}$$

Dans un matériau ferromagnétique, la description de la magnétisation M fait appel à la théorie du micromagnétisme introduit par Landau et Lifshitz dans leurs travaux [20], W.F. Brown dans [5] puis développée par Aharoni [1] et Labrune et Miltrat [19]. Il ressort de cette théorie que l'aimantation M est liée au champ incident par une équation non linéaire appelée équation de Landau-Lifshitz (LL) ou de manière équivalente par l'équation de Landau-Lifshitz-Gilbert (LLG) que nous allons préciser dans le prochain paragraphe.

1.4 Les équations LL et LLG

En négligeant les effets mécaniques dus à la magnétisation et à température constante plus petite que la température de Curie, la polarisation magnétique M satisfait à l'équation de Landau-Lifshitz

$$\partial_t M = -\gamma M \times H_T - \alpha \frac{M}{|M|} \times (M \times H_T) \tag{1.9}$$

où H_T est le champ magnétique total ou effectif, $\gamma > 0$ est la constante gyromagnétique et α le facteur d'amortissement. Le premier terme du second membre dans (1.9) décrit la précession de M autour de H_T (ie un mouvement conique autour de H_T) avec une vitesse angulaire de $\gamma |H_T|$; quant au second, il tend à aligner M sur H_T. Il en découle que

$$\frac{1}{2} \frac{\partial |M|^2}{\partial t} = M \cdot \frac{\partial M}{\partial t} = 0$$

ce qui traduit la conservation de la norme de la magnétisation durant le processus. Cette propriété permet de réécrire l'équation de la magnétisation sous la forme suivante

$$\partial_t M = -\frac{\alpha^2 + \gamma^2}{\gamma} M \times H_T + \frac{\alpha}{\gamma} \frac{M}{|M|} \times \partial_t M \tag{1.10}$$

appelée équation de Landau-Lifshitz-Gilbert (LLG) (voir [8]). L'équation ainsi formulée est intéressante dans la mesure où elle montre que le terme d'amortissement introduit un terme proportionnel à $\partial_t M$. Dans toute la suite, nous prendrons $|M| = 1$ ce qui sera le cas si on suppose qu'à l'instant $t = 0$, cette condition est satisfaite.

1.5 L'énergie magnétique volumique

Considérons un corps ferromagnétique occupant un domaine borné $\Omega \subset \mathbb{R}^3$. L'énergie magnétique est la somme de plusieurs contributions ([20, 28, 23, 1])

- *L'énergie d'anisotropie de volume* dont la forme est

$$\mathcal{E}_{av}(M) = \int_\Omega \psi(M) dx$$

où $\Phi : \mathbb{R}^3 \to \mathbb{R}^+$ est une fonction convexe qui dépend de la structure cristalline du matériau. Elle traduit le fait que l'aimantation tend à s'aligner selon certains axes appelés axes de facile aimantation. Une bonne approximation de ψ est

$$\psi(M) = \frac{1}{2}K_v(|M|^2 - (M \cdot U)^2)$$

où le vecteur unitaire U représente la direction de facile aimantation. Dans la suite, nous avons choisi de travailler avec la forme générale de $\psi(M)$.

- *L'énergie d'échange :* elle traduit la contribution de l'effet de la force électrostatique due au processus quantique d'interaction d'échange qui tend à aligner le moment magnétique moléculaire. Elle s'écrit

$$\mathcal{E}_{ech} = \frac{1}{2}\sum_{i,j}\int_\Omega A_{ij}\frac{\partial M}{\partial x_i}\frac{\partial M}{\partial x_j}dx$$

où A est un tenseur symétrique défini positif.

- *L'énergie du champ démagnétisant* définie par

$$\mathcal{E}_{dm}(M) = \frac{1}{2}\mu_0\int_{\mathbb{R}^3}|H|^2dx$$

appelée ainsi car le champ H induit s'oppose à l'aimantation. Dans la suite, nous considérerons les équations (1.7) de l'approximation magnétostatique des équations de Maxwell ce qui permet d'écrire $H = \nabla\varphi$ où φ est une fonction scalaire et

$$\mathcal{E}_{dm}(M) = \frac{1}{2}\mu_0\int_{\mathbb{R}^3}|\nabla\varphi|^2dx.$$

- *L'énergie Zeeman* qui s'exprime par

$$\mathcal{E}_Z(M) = -\int_\Omega H_s \cdot M dx$$

où $H_s = H_s(x)$ est un champ magnétique extérieur statique. Cette énergie traduit la tendance de l'aimantation à s'aligner sur le champ extérieur. Dans la suite, nous prendrons systématiquement $H_s = 0$.

L'énergie magnétique volumique est donc donnée par

$$\mathcal{E}_v(M) = \mathcal{E}_{av}(M) + \mathcal{E}_{ech}(M) + \mathcal{E}_{dm}(M) + \mathcal{E}_Z(M)$$

et à l'équilibre, elle est minimale. A chacun des termes contribuant à l'énergie, il correspond un terme du champ total H_T qui s'écrit par conséquent

$$H_T = -\nabla_M \psi(M) + \nabla \cdot (A \nabla M) + H_s + H$$

de sorte que $H_T = -\frac{\partial \mathbf{E}}{\partial M}$ où $\mathbf{E}$ représente la densité volumique de l'énergie.

1.6 L'énergie d'anisotropie de surface

Dans le cas où l'épaisseur du matériau ferromagnétique est très petite, on observe une forte aimantation de ses surfaces perpendiculairement à celles-ci [24]. Cet effet est décrit par l'énergie suivante appelée *énergie d'anisotropie de surface*

$$\mathcal{E}_{as} = \frac{K_s}{2} \int_{\partial\Omega} (|M|^2 - (M \cdot n)^2) d\xi$$

où n désigne la normale unitaire et sortante au bord $\partial\Omega$ de Ω. Il en découle une condition aux limites de la forme

$$M \times (A \frac{\partial M}{\partial n} - K_s (M \cdot n) n) = 0 \text{ sur } \partial\Omega$$

appelée condition de Rado-Weertmann (voir [26]).

Dans les matériaux ferromagnétiques, en plus de l'énergie d'anisotropie de surface, il intervient d'autres types d'énergie de surface comme nous allons le voir dans le paragraphe suivant

1.7 Matériaux ferromagnétiques en couches

Depuis quelques années, le domaine du micromagnétisme connaît un essor considérable en raison des développements dans le domaine informatique et des nouvelles technologies (et notamment la nanotechnologie).

Les structures magnétiques en couches ont bénéficié d'un intérêt tout particulier nourri par l'espoir que ce type de structures offre une possibilité d'obtenir des matériaux dotés de nouvelles propriétés magnétiques. Noter que les matériax magnétiques en couches sont utilisés pour le stockage de l'information et la réalisation de détecteurs.

Le couplage inter-couches est l'un des aspects de base des structures magnétiques en couches. Considérons une jonction F/N/F de deux couches ferromagnétiques parallèles (F) séparées par un métal non magnétique (N). Aux énergies définies précédemment vient s'ajouter une autre contribution due aux effets de surface. Considérons le cas où il n'y a pas d'anisotropie de surface, la magnétisation est alors stationnaire sur les surfaces externes de la structure. Cette condition s'écrit

$$M \times A\frac{\partial M}{\partial n} = 0$$

n étant la normale au bord unitaire et sortante. Cette condition s'applique aussi aux surfaces internes dans le cas d'un couplage magnétostatique seul. Cependant, les conditions aux limites doivent être modifiées en présence d'un couplage d'échange inter-couches (ie si la structure est de la forme F/N/F) dont l'énergie associée est

$$\begin{aligned}\mathcal{E}_J &= J_1\int(|M|^2 - M\cdot M')d\xi + \frac{J_2}{2}\int(|M|^2 - (M\cdot M')^2)d\xi \\ &= J_1\int(1 - M\cdot M')d\xi + \frac{J_2}{2}\int(1-(M\cdot M')^2)d\xi \qquad (1.11)\end{aligned}$$

où M et M' sont respectivement les magnétisations aux interfaces Σ et Σ', J_1 et J_2 sont des amplitudes. La condition aux limites sur Σ qui contient la contribution de l'échange volumique et le couplage d'échange inter-couches s'écrit

$$M \times (A\frac{\partial M}{\partial n} - J_1 M' - J_2(M\cdot M')M') = 0$$

Dans le cas où $J_2 = 0$ et $J_1 \neq 0$, elle est appelée la *condition aux limites de Hoffman.* Lorsque $J_1 = 0$ et $J_2 \neq 0$, elle est dite *condition d'échange inter-couches biquadratique.*

Dans le chapitre 2, nous focaliserons notre attention sur ce dernier cas, les effets des contributions de K_s et J_1 ayant déjà fait l'objet de nombreux travaux (voir [2, 11, 18] et les références citées dans ces articles) .

1.8 Effet d'un courant polarisé en spins sur la magnétisation

Le chapitre 3 sera consacré à l'étude de l'effet d'un courant polarisé en spins circulant perpendiculairement aux couches de la structure. Des expériences ont prouvé qu'un tel courant a pour effet de retourner l'orientation de la magnétisation dans les couches magnétiques et ce sans application d'un champ magnétique extérieur (voir [29, 4]). Ceci est dû à un transfert de spins aux couches magnétiques par les électrons de conduction de la couche d'espacement. Le transfert de spins produit une accumulation de spins autour de l'interface avec le métal non magnétique. En retour, l'accumulation de spins exerce un moment de torsion sur la magnétisation environnante. Plus précisément c'est la composante transverse (perpendiculaire à la magnétisation locale) de l'accumulation de spins qui produit cet effet-là, l'accumulation de spins longitudinale étant sans effet sur la magnétisation.

Pour décrire les équations de l'accumulation de spins et de la magnétisation, nous considérons une jonction F/N/F formée de deux couches ferromagnétiques parallèles au plan yOz séparées par un métal non magnétique. On injecte un courant électrique par une face externe de la structure et ce perpendiculairement aux couches ie parallèlement à l'axe Ox. Les courants électrique et magnétique j_e et j_m respectivement s'expriment alors par

$$j_e = C_0 E(x) - D_0 \frac{\partial n_0}{\partial x} - D \cdot \frac{\partial \mathbf{m}}{\partial x} \tag{1.12}$$

$$j_m = CE(x)C - \frac{\partial n_0}{\partial x} D - D_0 \frac{\partial \mathbf{m}}{\partial x} \tag{1.13}$$

où E est le champ électrique, $\mathbf{m}$ représente l'accumulation de spins et n_0 l'accumulation de charges. La constante C_0 et le vecteur $C = (C_1, C_2, C_3)$ sont liés à la conductivité alors que D_0 et $D = (D_1, D_2, D_3)$ sont relatifs à la diffusion. En introduisant les paramètres β et β' définis par $C = \beta C_0 M$, $D = \beta' D_0 M$ et en éliminant le champ électrique et la densité de charges des équations (1.12) et (1.12), on obtient l'équation

$$j_m = \beta j_e M - D_0 \left(\frac{\partial \mathbf{m}}{\partial x} - \beta\beta' (M \cdot \frac{\partial \mathbf{m}}{\partial x}) M\right) \tag{1.14}$$

dans laquelle le terme proportionnel à $\frac{\partial n_0}{\partial x}$ a été omis.

Considérons maintenant l'interaction entre l'accumulation de spins et la magnétisation locale qui s'exprime par l'interaction de contact

$$H_{int} = -J \mathbf{m}.M$$

L'équation de mouvement de l'accumulation de spins est donnée par

$$\frac{\partial \mathbf{m}}{\partial t} + \frac{\partial j_m}{\partial x} + \frac{J}{\hbar} \mathbf{m} \times M = -\frac{\mathbf{m}}{\tau_{sf}} \tag{1.15}$$

dans laquelle le terme $\frac{J}{\hbar}\mathbf{m} \times M$ représente le mouvement de précession de l'accumulation de spins autour de M. Cette équation s'écrit aussi

$$\frac{1}{D_0}\frac{\partial \mathbf{m}}{\partial t} = \frac{\partial^2 \mathbf{m}}{\partial x^2} - \beta\beta'(M \cdot \frac{\partial^2 \mathbf{m}}{\partial x^2})M - \frac{\mathbf{m}}{\lambda_{sf}^2} - \frac{\mathbf{m} \times M}{\lambda_J^2} \tag{1.16}$$

avec $\lambda_{sf} = \sqrt{D_0 \tau_{sf}}$ et $\lambda_J = \sqrt{\hbar D_0 / J}$. Quant à l'équation de mouvement de la magnétisation locale, elle est donnée par

$$\frac{\partial M}{\partial t} = -\gamma_0 M \times (H_T + J\mathbf{m}) + \alpha M \times \frac{\partial M}{\partial t} \tag{1.17}$$

dans laquelle le champ magnétique "total" H_T défini précédemment a été augmenté de $J\mathbf{m}$ résultant du couplage entre le moment local et l'accumulation de spins. Pour plus de détails, on pourra consulter [37].

Chapitre 2

Ferromagnets with biquadratic exchange coupling energy. Global existence of weak solutions

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A global existence theorem is proved for the Landau-Lifshitz-Gilbert equations with biquadratic exchange coupling energy acting on the interfaces of a material composed by two ferromagnetic layers separated by a nonmagnetic one. This energy is not convex. The magnetization M satisfies on the interfaces a coupled nonlinear Neumann boundary condition with cubic growth. We use several regularizations, in particular for the traces of the magnetization at the interfaces, to obtain global weak solutions of the problem with finite energy.

Key words. Ferromagnets, Landau-Lifschitz equations, biquadratic exchange coupling .

1991 AMS subject classifications. 73R05, 73K05, 47J35, 34G20, 35L10, 35K05, 46E35.

2.1 Biquadratic exchange coupling energy

The study of ferromagnetic layers had known a big expansion these last years because of their interest in the electronic industries. In this paper, We deal with a material constituted by two ferromagnetic layers separated by a nonmagnetic spacer. The adjacent interfaces of the ferromagnets are coupled via the so called biquadratic exchange energy see [13], [15], [19] for example. Until now, these effects have been neglected, in the literature we found the works [2] and [35] wherein the authors considered homogeneous boundary conditions while the case of bilinear exchange coupling energy was studied in [11] and [34].

Let $\widehat{\Omega} \subset \mathbb{R}^2$ be open and bounded, the domain occupied by the ferromagnetic material is denoted by $\Omega = \Omega^+ \cup \Omega^-$ where $\Omega^+ = \widehat{\Omega} \times (h, l)$, $\Omega^- = \widehat{\Omega} \times (-l, -h)$, and the nonmagnetic spacer occupies the domain $\Omega^0 = \widehat{\Omega} \times (-h, h)$. We denote by $\Gamma^\pm = \widehat{\Omega} \times \{z = \pm h\}$ the adjacent interfaces of the ferromagnetic material. The generic point of Ω will be denoted by $x = (\widehat{x}, z)$.

Let $M(t, x) \in S^2$ be the magnetization of Ω at the time $t \geq 0$ in the position x where S^2 denotes the unit sphere of $\mathbb{R}^3$. The biquadratic interlayer exchange coupling energy acting between the interfaces Γ^+ and Γ^- takes the form [15]

$$\mathcal{E}_{bq}(M) = \frac{K_{bq}}{2} \int_{\widehat{\Omega}} (1 - (M^+ \cdot M^-)^2) d\widehat{x} \tag{2.1}$$

where $M^\pm(t, \widehat{x}) = M(t, \widehat{x}, \pm h)$ and the biquadratic coupling parameter $K_{bq} > 0$ depends on the thickness of the ferromagnetic layers. Since by using the saturation condition $|M^\pm|^2 = 1$, we have $1 - (M^+ \cdot M^-)^2 = \frac{1}{4}|M^+ - M^-|^2 |M^+ + M^-|^2$ then the biquadratic energy penalizes the magnetization $M^\pm$ on the interfaces to become parallel ($M^+ = M^-$) or antiparallel ($M^+ = -M^-$). The biquadratic interlayer exchange constant K_{bq} is in fact a parameter smaller than the bilinear one J_{bl} appearing in the bilinear intelayer exchange energy $\mathcal{E}_{bl}(M) = \frac{J_{bl}}{2} \int_{\widehat{\Omega}} (1 - M^+ \cdot M^-) d\widehat{x}$ studied in [11] (see [15, 19]). The following study is made to complete the work given in [11].

Let us precise the model we shall discuss. In the ferromagnetic domain Ω the magnetization $M(t, x) \in S^2$ satisfies the Landau-Lifshitz-Gilbert (LLG) equations in $\mathbb{R}^+ \times \Omega$

$$\begin{cases} \partial_t M - \alpha M \times \partial_t M = -(1 + \alpha^2) M \times \mathcal{H}(M) \\ M(0, x) = M_0(x), \ |M_0(x)|^2 = 1 \text{ a.e.} \end{cases} \tag{2.2}$$

The effective magnetic field $\mathcal{H}(M)$ is given by

$$\mathcal{H}(M) = A\Delta M - \nabla_M \psi(M) + \nabla\varphi \tag{2.3}$$

where $A > 0$ is a fixed constant called the anisotropy exchange constant, $\nabla\varphi$ is the demagnetizing field given by the stray field equation (or magnetostatic equation)

$$\nabla \cdot (\nabla\varphi + \chi(\Omega)M) = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \tag{2.4}$$

$\chi(\Omega)$ being the characteristic function of Ω and $\nabla_M \psi(M)$ is the volume anisotropy field associated with a regular function $\psi \in C^2(\mathbb{R}^3)$ satisfying $\psi(X) \geq 0$ and $|D^2\psi(X)| \leq C$ for all vector $X \in \mathbb{R}^3$. The magnetization $M(t,x)$ satisfies the saturation condition

$$|M(t,x)|^2 = 1 \quad \text{a.e in} \quad \mathbb{R}^+ \times \Omega. \tag{2.5}$$

The boundary condition satisfied by M on $\mathbb{R}^+ \times (\partial\Omega \setminus (\Gamma^+ \cup \Gamma^-))$ is given by

$$M \times A\frac{\partial M}{\partial n} = 0 \tag{2.6}$$

while on $\mathbb{R}^+ \times \Gamma^\pm$ the biquadratic exchange coupling energy gives the boundary condition

$$M^\pm \times (\mp A(\partial_z M)^\pm + K_{bq}(|M^\mp|^2 M^\pm - (M^+ \cdot M^-)M^\mp)) = 0. \tag{2.7}$$

In this paper we shall discuss the global existence of weak solutions with finite energy of the problem (2.2)-(2.4)-(2.5)-(2.6)-(2.7). Without loss of generality we set $\alpha = 1$ and $A = 1$. Let us precise the notion of weak solution for our equations

Definition 2.1.1 *Let $M_0 \in \mathbb{H}^1(\Omega)$ be such that $|M_0(x)|^2 = 1$ a.e in Ω. We will say that (M,φ) is a global weak solution to the problem with finite energy if*
i) $M \in L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$, $\partial_t M \in L^2_{loc}(\mathbb{R}^+; \mathbb{L}^2(\Omega))$, $|M(t,x)|^2 = 1$ a.e
ii) $M(0,x) = M_0(x)$ in the trace sense
iii)$\nabla\varphi \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))$, $\nabla \cdot (\nabla\varphi + \chi(\Omega)M) = 0$
iv) for all $W \in \mathcal{D}(\mathbb{R}^+ \times \overline{\Omega})$ such that supp $W \subset [0,T] \times \overline{\Omega}$, it holds

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_Q \partial_t M \cdot M \times W \; dxdt + \dfrac{1}{2}\displaystyle\int_Q \; M \times \partial_t M \cdot M \times W \; dxdt \\ + \int_Q \nabla M \cdot M \times \nabla W \; dxdt - \displaystyle\int_Q (\nabla\varphi - \nabla_M\psi(M)) \cdot M \times W \; dxdt = \\ K_{bq} \displaystyle\int_{\widehat{Q}} (B^+(M^+,M^-) \cdot M^+ \times W^+ + B^-(M^+,M^-) \cdot M^- \times W^-) \; d\widehat{x}dt \end{cases}$$

where $Q = (0,T)\times\Omega$, $\widehat{Q} = (0,T)\times\widehat{\Omega}$ *and* $B^{\pm}(M^+, M^-) = |M^{\mp}|^2 M^{\pm} - (M^+ \cdot M^-) M^{\mp}$
iv) for all $t \geq 0$, *the energy inequality is satisfied*

$$\mathcal{E}(M(t)) + \frac{1}{2}\int_0^t |\partial_t M(s)|^2_{\mathbb{L}^2(\Omega)} ds \leq \mathcal{E}(M_0)$$

where

$$\mathcal{E}(M) = \frac{1}{2}\int_\Omega |\nabla M|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\varphi|^2 dx + \int_\Omega \psi(M) dx + \mathcal{E}_{bq}(M).$$

The main difficulty in this work relies in the nonlinear cubic boundary condition at the interfaces. Of course one can think that this difficulty can be solved by using a linearization procedure. Unfortunately this method does not lead to uniform estimates and so we are not able to pass to the limit. Indeed the energy estimate conducts to the evaluation of boundary terms of the form $\int_{\Gamma^\pm} F^n \partial_t M^{n+1} d\widehat{x}$ so the norm H^1 of the derivative $\partial_t M$ is needed while only its L^2-norm intervenes in the energy. This is also the reason for which the Galerkin method that is usually used, see [2], is inoperative in our case. Indeed to get an uniform bound for the energy we have to estimate boundary terms of the type $\int_{\widehat{\Omega}} (\mathcal{B}^+(M_n^+, M_n^-) \cdot \partial_t M_{n+1}^+ + \mathcal{B}^-(M_n^+, M_n^-) \cdot \partial_t M_{n+1}^-) d\widehat{x}$ which need an $\mathbb{L}^2$ bound of the trace of $\partial_t M_n^\pm$ and then a bound of $\partial_t M_n$ in $L^2(\mathbb{R}^+; \mathbb{H}^1(\Omega))$. In this work we are interested with weak solutions with finite energy then another approach to get them is used.

Our method deals directly with the nonlinear problem by using a monotonicity argument. For this, several approximations are necessary. First we need to penalize the saturation condition $|M| = 1$ (which is not convex), then a regularization and a further penalization on the boundary conditions are introduced. Finally, the obtained problem is solved by using an elliptic regularization of the equation in order to absorb the term $M \times \partial_t M$. All these transformations are done through small parameters ν, η and ε. Using a standard technique of abstract operators we prove existence of solutions $V^\varepsilon_{\eta,\nu}$ to the regularized problem then using the energy estimates, we pass to the limit in parameters ε, η and ν to show that the limit of $V^\varepsilon_{\eta,\nu}$ converges to a solution of our problem.

Let us precise in the next section the approximating procedure considered

2.2 The approximated models

The saturation condition $|M|^2 = 1$ allows to write $\partial_t M = -M \times (M \times \partial_t M)$. Substituting into the Landau-Lifshitz-Gilbert equation we obtain

$$M \times (\frac{M}{1+|M|} \times \partial_t M + \frac{1}{2}\partial_t M - \Delta M + \nabla_M \psi(M) - \nabla\varphi + pM) = 0 \tag{2.8}$$

where p is an arbitrarily scalar function depending eventually of $|M|$. For any test function ϕ with support in Q we get the weak formulation

$$\int_Q (\frac{M}{1+|M|} \times \partial_t M + \frac{1}{2}\partial_t M - \Delta M + \nabla_M \psi(M) - \nabla\varphi + pM) \cdot M \times \phi dxdt = 0$$

The idea is to obtain the solutions of our problem as a limit of approximated solutions of an intermediary problem which penalizes the saturation condition. The scalar function p plays the role of the penalization operator.

Let $\nu > 0$ and $\eta > 0$ be two fixed small parameters. The first one is related to the penalization of the saturation condition as well as to reduce the growth in the Neumann boundary condition. The second one is used to penalize the traces $M^{\pm}$ of M on the interfaces to remain bounded. We introduce the vector function U satisfying in $Q = (0,T) \times \Omega$ with $0 < T < \infty$, the intermediary problem (see [10])

$$\begin{cases} \frac{1}{2}\partial_t U - \Delta U = F_\nu(U) + G(U, \partial_t U) \text{ in } Q \\ U(0) = M_0, \ |M_0(x)|^2 = 1 \text{ in } \Omega \\ \dfrac{\partial U}{\partial n} = 0 \text{ on } (0,T) \times (\partial\Omega \setminus (\Gamma^+ \cup \Gamma^-)) \\ \mp(\partial_z U)^{\pm} + K_{bq}(B_\nu^{\pm}(U^+, U^-) + R_\eta(U^{\pm})) = 0 \text{ on } (0,T) \times \Gamma^{\pm} \end{cases} \tag{2.9}$$

with the stray equation

$$\nabla \cdot (\nabla\varphi + \chi(\Omega)U) = 0 \text{ in } Q^\infty = (0,T) \times \mathbb{R}^3. \tag{2.10}$$

We set

$$\begin{cases} G(U, \partial_t U) = -\frac{U}{1+|U|} \times \partial_t U \\ F_\nu(U) = -\nabla_U \psi(U) + \mathbb{D}(U) - \frac{1}{\nu}\nabla_U(\gamma(|U|)) \\ \gamma(y) = \frac{1}{2}(\sqrt{2} - \sqrt{1+y^2})^2 \ , \ y \in \mathbb{R} \end{cases} \tag{2.11}$$

$\mathbb{D}$ being the linear operator $U \mapsto \nabla\varphi$ where φ satisfies (2.10). The boundary operators $B_\nu^\pm(U^+,U^-)$ and $R_\eta(U^\pm)$ are defined by

$$\left\{\begin{array}{l} B_\nu^\pm(U^+,U^-) = \nabla_{U^\pm}\Phi_\nu(U^+,U^-), \quad \Phi_\nu(U^+,U^-) = \frac{1}{4\nu}\log(\phi_\nu(U^+,U^-)) \\ \phi_\nu(U^+,U^-) = 1 + 2\nu(|U^+|^2|U^-|^2 - (U^+\cdot U^-)^2) + \nu^2(|U^+ - M_o^+|^4 + |U^- - M_o^-|^4) \\ R_\eta(U^\pm) = \nabla_{U^\pm}\Theta_\eta(U^\pm), \quad \Theta_\eta(U^\pm) = \frac{1}{2\eta}(s(U^\pm) - \log(1+s(U^\pm))) \end{array}\right.$$

that is to say

$$\left\{\begin{array}{l} B_\nu^\pm(U^+,U^-) = \dfrac{|U^\mp|^2U^\pm - (U^+\cdot U^-)U^\mp + \nu|U^\pm - M_o^\pm|^2(U^\pm - M_o^\pm)}{\phi_\nu(U^+,U^-)} \\ R_\eta(U^\pm) = \dfrac{1}{\eta}\dfrac{s(U^\pm)U^\pm}{1+s(U^\pm)}, \;\; s(U^\pm) = max(|U^\pm|^2 - 1, 0) \end{array}\right.$$

Notice that $\Theta_\eta(U^\pm) = 0$ if and only if $|U^\pm| \leq 1$ as well as $\gamma(|U|) = 0$ if and only if $|U| = 1$ and if we multiply the equation (2.10) by φ and integrate on $\mathbb{R}^3$, we get

$$\int_\Omega \mathbb{D}(U).U dx = \int_\Omega \nabla\varphi.U dx = -|\nabla\varphi|^2_{\mathbb{L}^2(\mathbb{R}^3)} \tag{2.12}$$

which leads to the estimate

$$|\mathbb{D}(U)|_{\mathbb{L}^2(\mathbb{R}^3)} \leq |U|_{\mathbb{L}^2(\Omega)}. \tag{2.13}$$

Remark 2.2.1 *Let $W \in \mathcal{D}(\overline{Q})$, multiplying the intermediary problem* (2.9) *by $U\times W$ and observing that $\nabla_U(\gamma(|U|))\cdot U\times W = 0$ then integrating by parts we get the same weak formulation as problem* (2.2)-(2.4)-(2.5)-(2.6)-(2.7). *Hence the LLG equations appear as the projection of the intermediary problem in the plane orthogonal to M. Next, we observe that we can write $\frac{1}{2}\partial_t U + \frac{U}{1+|U|}\times\partial_t U = B(U)(\partial_t U)$ where $B(U)$ is an invertible 3×3 matrix . Hence the intermediary equation in* (2.9) *becomes $B(U)(\partial_t U) - \Delta U = F_\nu(U)$ which is a nonlinear heat equation with semilinear source term. Moreover setting $C(U) = B^{-1}(U)$ the equation takes the expression $\partial_t U - \nabla\cdot(C(U)\nabla U) + D_U(C(U))(\nabla U)(\nabla U) = C(U)(F_\nu(U))$ which is a quasilinear heat equation. In order to avoid the difficulties in solving the heat equation* (2.9) *we introduce the following singular perturbation with respect to $\partial_t U$.*

First of all we introduce the change of unknown function

$$U = e^{kt}(V_\nu^\eta + M_0) \tag{2.14}$$

for $k > 0$ fixed which will be chosen later . The constant $k > 0$ is used to obtain an $\mathbb{L}^2$ bound for V_ν^η. Hence, V_ν^η satisfies the intermediary problem

$$\begin{cases} \frac{1}{2}\partial_t V_\nu^\eta - \Delta V_\nu^\eta + \frac{k}{2} V_\nu^\eta = \mathbf{F}_\nu(t, V_\nu^\eta) + \mathbf{G}(t, V_\nu^\eta, \partial_t V_\nu^\eta) + \Delta M_0 - \frac{k}{2} M_0 \ \text{ in } \ Q \\ V_\nu^\eta(0) = 0 \ \text{ in } \ \Omega, \quad \frac{\partial(V_\nu^\eta + M_0)}{\partial n} = 0 \ \text{ on } \ (0,T) \times (\partial\Omega \setminus (\Gamma^+ \cup \Gamma^-)) \\ \mp\Big(\partial_z(V_\nu^\eta + M_0)\Big)^\pm + K_{bq}\mathbf{B}_\nu^\pm(t, V_\nu^{\eta,+}, V_\nu^{\eta,-}) + K_{bq}\mathbf{R}_\eta(t, V_\nu^{\eta,\pm}) = 0 \ \text{ on } \ (0,T) \times \Gamma^\pm \end{cases} \tag{2.15}$$

with

$$\begin{cases} \mathbf{F}_\nu(t, V) = e^{-kt} F_\nu(e^{kt}(V + M_0)) \\ \mathbf{B}_\nu^\pm(t, V^+, V^-) = e^{-kt} B_\nu^\pm(e^{kt}(V^+ + M_0^+), e^{kt}(V^- + M_0^-)) \\ \mathbf{R}_\eta(t, V^\pm) = e^{-kt} R_\eta(e^{kt}(V^\pm + M_0^\pm)) \\ \mathbf{G}(t, V, \partial_t V) = \frac{e^{kt}(V+M_0)}{1+e^{kt}|V+M_0|} \times \partial_t V \end{cases} \tag{2.16}$$

As pointed out in the previous remark, we introduce the following elliptic regularization of (2.15) with respect to the time variable. Let $\varepsilon > 0$ be a fixed small parameter we consider in Q the singular perturbation problem (see [21, 22])

$$\begin{cases} -\varepsilon^2 \partial_t^2 V_{\nu,\eta}^\varepsilon - \Delta V_{\nu,\eta}^\varepsilon + \frac{1}{2}\partial_t V_{\nu,\eta}^\varepsilon + \frac{k}{2} V_{\nu,\eta}^\varepsilon = \\ \qquad \mathbf{F}_\nu(t, V_{\nu,\eta}^\varepsilon) + \mathbf{G}(t, V_{\nu,\eta}^\varepsilon, \partial_t V_{\nu,\eta}^\varepsilon) + \Delta M_0 - \frac{k}{2} M_0 \ \text{ in } Q \\ V_{\nu,\eta}^\varepsilon(0) = 0, \ \partial_t V_{\nu,\eta}^\varepsilon(T) = 0 \text{ in } \ \Omega \\ \frac{\partial(V_{\nu,\eta}^\varepsilon + M_0)}{\partial n} = 0 \ \text{ on } \ (0,T) \times (\partial\Omega \setminus (\Gamma^+ \cup \Gamma^-)) \\ \mp\Big(\partial_z(V_{\nu,\eta}^\varepsilon + M_0)\Big)^\pm + K_{bq}\mathbf{B}_\nu^\pm(t, V_{\nu,\eta}^{\varepsilon,+}, V_{\nu,\eta}^{\varepsilon,-}) \\ +K_{bq}\mathbf{R}_\eta(t, V_{\nu,\eta}^{\varepsilon,\pm}) = 0 \ \text{ on } \ (0,T) \times \Gamma^\pm \end{cases} \tag{2.17}$$

Let us recall the crucial role played by the parameter η. It allows to show that the traces of the solutions on the interfaces are in the ball of radius 1. This condition is essential in passing to the limit in the nonlinear Neumann boundary condition.

In this paper we will use the following notations. If $D \subset \mathbb{R}^n$ is open and regular, $\mathbb{L}^p(D)$ will denote the vectorial Lebesgue space $(L^p(D))^3$, $|\cdot|$ and $(\cdot\,;\,\cdot)$ the norm and scalar product of $\mathbb{L}^2(D)$. The Hilbert spaces $\mathbb{H}^s(D)$ ($s = 1, 2$ or $\frac{1}{2}$) are the usual Sobolev spaces $(H^s(D))^3$. If D is a bounded domain of $\mathbb{R}^n$, $|D|$ will denote its Lebesgue measure. For $T > 0$, we set $Q = (0,T) \times \Omega$, $\widehat{Q} = (0,T) \times \widehat{\Omega}$ and $Q^\infty = (0,T) \times \mathbb{R}^3$. At the end let us announce that in the sequel C will represent various positive constants which are independent upon the different parameters.

Before going farther, let us note that $B_\nu^\pm$ and R_η are Lipschitz continuous on $\mathbb{R}^3$. We have the

Lemma 2.2.2 *Let $\eta > 0$ and $0 < \nu < 1$ then there exists $C > 0$ (which is independent of the parameters ν, η) such that for all $X, Y \in \mathbb{R}^3$ we have*

$$|R_\eta(X) - R_\eta(Y)| \leq 4\eta^{-1}|X - Y| \tag{2.18}$$

and

$$|B_\nu^\pm(X,Y) - B_\nu^\pm(X',Y')| \leq C\nu^{-1}(|X - X'| + |Y - Y'|). \tag{2.19}$$

Moreover R_η is monotone from $\mathbb{R}^3$ into $\mathbb{R}^3$:

$$(R_\eta(X) - R_\eta(Y)) \cdot (X - Y) \geq 0 \ \forall X, Y \in \mathbb{R}^3. \tag{2.20}$$

Proof. Computing the gradient of R_η, we get easily $|\nabla_X R_\eta(X)| \leq 4\eta^{-1}$ therefore R_η satisfies the Lipschitz property (2.18). Next for $X, Y \in \mathbb{R}^3$ let $B_{\nu,i}^\pm(X,Y) = N_i^\pm(X,Y)/D(X,Y)$ for $i = 1,2,3$ be the components of $B_\nu^\pm(X,Y)$. We have for all $i, j = 1,2,3$ and $X, Y \in \mathbb{R}^3$

$$\begin{cases} \partial_{X_j} N_i^+ = 2\nu(X - M_0^+)_i(X - M_0^+)_j + \nu|X - M_0^+|^2\delta_{i,j}) + |Y|^2\delta_{i,j} - Y_iY_j \\ \partial_{X_j} D = 4\nu(X_j|Y|^2 - (X\cdot Y)Y_j) + 4\nu^2|X - M_0^+|^2(X - M_0^+)_j \end{cases}$$

where $(\delta_{i,j})_{1\leq i,j\leq 3}$ denotes the identity matrix, thus $|\partial_{X_j} N_i^+| \leq 3\nu|X - M_0^+|^2 + 2|Y|^2$, $|\partial_{X_j} D| \leq 8\nu|X||Y|^2 + 4\nu^2|X - M_0^+|^3$. Since $D(X,Y) \geq 1 + \nu^2(|X - M_0^+|^4 + |Y - M_0^-|^4)$ and $|Y|^2 \leq 2|Y - M_0^-|^2 + 2|M_0^-|^2$, we have

$$\frac{|\partial_{X_j} N_i^+|}{D(X,Y)} \leq \frac{3\nu|X-M_0^+|^2}{1+\nu^2|X-M_0^+|^4} + \frac{2|Y|^2}{1+\nu^2|Y-M_0^-|^4} \leq C(1+\nu^{-1})$$

where $C > 0$ is a constant independent of ν. Furthermore, we have $|N_i^+ \partial_{X_j} D| \leq 16\nu|X|^2|Y|^4 + 16\nu^2|X-M_0^+|^3|X||Y|^2 + 4\nu^3|X-M_0^+|^6$ so $|N_i^+\partial_{X_j} D| \leq C\nu(|X-M_0^+|^2|Y-M_0^-|^4 + |X-M_0^+|^2 + |Y-M_0^-|^4+1) + C\nu^2(|X-M_0^+|^3 |Y-M_0^-|^2 + |X-M_0^+|^4 + |X-M_0^+|^3) + 4\nu^3|X-M_0^+|^6$ and since $D^2 \geq (1+\nu^2(|X-M_0^+|^4+|Y-M_0^-|^4))^2$ we get $2|X-M_0^+|^2|Y-M_0^-|^4 \leq |X-M_0^+|^4|Y-M_0^-|^4 + |Y-M_0^-|^4$ so

$$\begin{aligned}\frac{2\nu|X-M_0^+|^2|Y-M_0^-|^4}{D^2} &\leq \frac{\nu|X-M_0^+|^4|Y-M_0^-|^4}{1+2\nu^2|X-M_0^+|^4|Y-M_0^-|^4} + \frac{\nu|Y-M_0^-|^4}{1+2\nu^2|Y-M_0^-|^4} \\ &\leq C\nu^{-1}\end{aligned}$$

and

$$\frac{\nu(|X-M_0^+|^2+|Y-M_0^-|^4+1)}{D^2} \leq C(1+\nu^{-1}).$$

Similarly we have

$$\begin{aligned}&\frac{2\nu^2(|X-M_0^+|^4+|X-M_0^+|^3|Y-M_0^-|^2+|X-M_0^+|^3)+4\nu^3|X-M_0^+|^6}{D^2} \\ &\leq \frac{2\nu^2|X-M_0^+|^4}{1+2\nu^2|X-M_0^+|^4} + \frac{\nu^2|X-M_0^+|^2}{1+2\nu^2|X-M_0^+|^4} + \frac{\nu^2|X-M_0^+|^4|Y-M_0^-|^4}{1+2\nu^2|X-M_0^+|^4|Y-M_0^-|^4} \\ &+ \frac{\nu^2|X-M_0^+|^3}{1+\nu^4|X-M_0^+|^8} + \frac{4\nu^3|X-M_0^+|^6}{1+\nu^4|X-M_0^+|^8}\end{aligned}$$

and then

$$\begin{aligned}&\frac{2\nu^2(|X-M_0^+|^4+|X-M_0^+|^3|Y-M_0^-|^2+|X-M_0^+|^3)+4\nu^3|X-M_0^+|^6}{D^2} \\ &\leq C(1+\nu+\nu^{1/2}).\end{aligned}$$

Therefore if $0<\nu<1$, we have $|\nabla_X B_\nu^+(X,Y)| \leq C\nu^{-1}$ where $C>0$ is independent of ν. We get the same bound for $|\nabla_Y B_\nu^+(X,Y)|$, the same calculus are valid for B_ν^- so we deduce inequality (2.19). To prove the monotonicity of R_η we proceed as follows. For $X, Y \in \mathbb{R}^3$, we have

$$\eta(R_\eta(X)-R_\eta(Y))\cdot(X-Y)=\frac{s(X)}{1+s(X)}(|X|^2-X\cdot Y)+\frac{s(Y)}{1+s(Y)}(|Y|^2-X\cdot Y)$$

then

$$\eta(R_\eta(X)-R_\eta(Y))\cdot(X-Y)\geq(|X|-|Y|)(\frac{s(X)}{1+s(X)}|X|-\frac{s(Y)}{1+s(Y)}|Y|).$$

Writing

$$\frac{s(X)}{1+s(X)}|X|-\frac{s(Y)}{1+s(Y)}|Y|=(|X|-|Y|)\frac{s(X)}{1+s(X)}+|Y|(\frac{s(X)}{1+s(X)}-\frac{s(Y)}{1+s(Y)})$$

and using the inequality $(|X|-|Y|)(\frac{s(X)}{1+s(X)}-\frac{s(Y)}{1+s(Y)})\geq 0$ which holds for all $X,Y\in\mathbb{R}^3$, we get the whished result.

This result leads to the following observation that will play an important role in this paper

Remark 2.2.3 *If $U^\pm\in\mathbb{H}^{\frac{1}{2}}((0,T)\times\widehat{\Omega})$, then $B_\nu^\pm(U^+,U^-)$ and $R_\eta(U^\pm)$ remain in $\mathbb{H}^{\frac{1}{2}}((0,T)\times\widehat{\Omega})$. Moreover if $M_0\in\mathbb{H}^1(\Omega)$, $|M_0(x)|^2=1$ on $\overline{\Omega}$ is such that its traces on $\Gamma^\pm$ satisfy either $M_0^+=M_0^-$ or $M_0^+=-M_0^-$ then $B_\nu^\pm(M_0^+,M_0^-)=0$.*

2.3 Solving the regularized problem (2.17)

Let us introduce the weak formulation of problem (2.17). We set $\mathbb{V}=\{V\in\mathbb{H}^1(Q); V(0)=0\}$ endowed with the usual norm of $\mathbb{H}^1(Q)$. $\langle\cdot;\cdot\rangle$ will denote the duality product between $\mathbb{V}$ and its dual $\mathbb{V}'$. We consider the operators defined from $\mathbb{V}$ to $\mathbb{V}'$ by

$$\langle\mathcal{A}_{\varepsilon,k}(V);W\rangle=\varepsilon^2(\partial_t V;\partial_t W)+(\nabla V;\nabla W)+\frac{1}{2}(\partial_t V;W)+\frac{k}{2}(V;W) \tag{2.21}$$

$$\langle\mathcal{F}_\nu(V);W\rangle=\int_Q \mathbf{F}_\nu(t,V)\cdot W\,dxdt \tag{2.22}$$

$$\langle \mathcal{G}(V); W\rangle = \int_Q \mathbf{G}(t, V, \partial_t V) \cdot W dx dt \tag{2.23}$$

$$\langle \mathcal{B}_\nu(V); W\rangle = K_{bq} \int_{\widehat{Q}} (\mathbf{B}_\nu^+(t, V^+, V^-) \cdot W^+ + \mathbf{B}_\nu^-(t, V^+, V^-) \cdot W^-) d\widehat{x} dt \tag{2.24}$$

$$\langle \mathcal{R}_\eta(V); W\rangle = K_{bq} \int_{\widehat{Q}} (\mathbf{R}_\eta(t, V^+) \cdot W^+ + \mathbf{R}_\eta(t, V^-) \cdot W^-) d\widehat{x} dt \tag{2.25}$$

for all $V, W \in \mathbb{V}$. Thus setting $L_{M_0}(W) = -(\nabla M_0; \nabla W) - \frac{k}{2}(M_0; W)$ for all $W \in \mathbb{V}$, the weak formulation of the problem (2.17) becomes : for all $W \in \mathbb{V}$

$$\langle (\mathcal{A}_{\varepsilon,k} + \mathcal{B}_\nu + \mathcal{R}_\eta - \mathcal{F}_\nu - \mathcal{G})(V^\varepsilon_{\eta,\nu}); W\rangle = L_{M_0}(W). \tag{2.26}$$

Hereafter we will precise some useful properties satisfied by the operators appearing in the weak formulation (2.26)

Lemma 2.3.1 *The following properties hold*
i) $\mathcal{A}_{\varepsilon,k}$ *is monotone continuous and coecive on* $\mathbb{V}$.
ii) *Operator* $\mathcal{R}_\eta$ *is monotone on* $\mathbb{V}$ *and satisfies for all* $U, V \in \mathbb{V}$ *the Lipschitz property*

$$|\mathcal{R}_\eta(U) - \mathcal{R}_\eta(V)|_{L^2(\widehat{Q})} \leq 4K_{bq}\eta^{-1}(|U^+ - V^+|_{L^2(\widehat{Q})} + |U^- - V^-|_{L^2(\widehat{Q})}) \tag{2.27}$$

iii) *The nonlinear operators* $\mathcal{B}_\nu$ *and* $\mathcal{F}_\nu$ *satisfy the Lipschitz properties*

$$|\mathcal{B}_\nu(U) - \mathcal{B}_\nu(V)|_{\mathbb{L}^2(\widehat{Q})} \leq CK_{bq}\nu^{-1}(|U^+ - V^+|_{L^2(\widehat{Q})} + |U^- - V^-|_{L^2(\widehat{Q})}) \ , \quad \forall\, U, V \in \mathbb{V} \tag{2.28}$$

$$|\mathcal{F}_\nu(U) - \mathcal{F}_\nu(V)|_{\mathbb{L}^2(Q)} \leq C\nu^{-1}|U - V|_{\mathbb{L}^2(Q)} \ , \quad \forall\, U, V \in \mathbb{L}^2(\Omega) \tag{2.29}$$

with $C > 0$ *independent of the fixed parameter* ν, $0 < \nu < 1$.

Proof. The first point is immediate and the second one is a consequence of the monotonicity property of R_η and (2.18). Finally according to (2.19), we obtain (2.28) and since $\nabla_U \psi(U)$ and $\nabla\gamma(|U|)$ are lipschitzian then using property (2.13) of the linear operator $\mathbb{D}$, we get (2.29).

Summarizing all these results, we obtain

Lemma 2.3.2 *For all ε, ν, $\eta > 0$ fixed, operator $\mathcal{L}^{\varepsilon}_{\nu,\eta} = \mathcal{A}_{\varepsilon,k} + \mathcal{B}_\nu + \mathcal{R}_\eta - \mathcal{F}_\nu$ is continuous. Moreover there exists $k_0 > 0$ (which depends of ν but not of ε and η) such that for $k > k_0$, $\mathcal{L}^{\varepsilon}_{\nu,\eta}$ is strongly monotone.*

Proof. For all $U, V \in \mathbb{V}$ it holds

$$\begin{cases} \langle \mathcal{L}^{\varepsilon}_{\nu,\eta}(U) - \mathcal{L}^{\varepsilon}_{\nu,\eta}(V); U - V\rangle \geq \varepsilon^2 |\partial_t U - \partial_t V|^2_{L^2(Q)} + \frac{1}{2}|\nabla U - \nabla V|^2_{L^2(Q)} \\ +\frac{1}{2}|U - V|^2_{L^2(Q)} + \frac{1}{4}|U(T) - V(T)|^2_{L^2(\Omega)}. \end{cases} \tag{2.30}$$

Indeed thanks to (2.28) and the classical estimate

$$|U^{\pm}|^2_{\mathbb{L}^2(\widehat{Q})} \leq \alpha |\nabla U|^2_{\mathbb{L}^2(Q)} + C_\alpha |U|^2_{\mathbb{L}^2(Q)}$$

which holds for all $U \in L^2(0, T; \mathbb{H}^1(\Omega))$ (where α is any positive constant and $C_\alpha > 0$ is such that $C_\alpha \to +\infty$ when $\alpha \to 0$), we obtain

$$|\langle \mathcal{B}_\nu(U) - \mathcal{B}_\nu(V); U - V\rangle| \leq C K_{bq} \alpha \nu^{-1} |\nabla U - \nabla V|^2_{L^2(Q)} + K_{bq} C_\alpha \nu^{-1} |U - V|^2_{L^2(Q)} \tag{2.31}$$

thus according to the monotonicity of $\mathcal{R}_\eta$ and the Lipschitz property of $\mathcal{F}_\nu$, we get

$$\begin{cases} \langle \mathcal{L}^{\varepsilon}_{\nu,\eta}(U) - \mathcal{L}^{\varepsilon}_{\nu,\eta}(V); U - V\rangle \geq \varepsilon^2 |\partial_t V - \partial_t V|^2_{L^2(Q)} + (1 - C K_{bq} \dfrac{\alpha}{\nu}) |\nabla U - \nabla V|^2_{L^2(Q)} \\ +(\dfrac{k}{2} - C\nu^{-1} - K_{bq} C_\alpha \nu^{-1}) |U - V|^2_{L^2(Q)} + \dfrac{1}{4}|U(T) - V(T)|^2_{L^2(\Omega)} \end{cases} \tag{2.32}$$

so for $\alpha = \nu(2CK_{bq})^{-1}$ if we choose $k > 2C\nu^{-1} + 2K_{bq}C_\alpha\nu^{-1} + 1$, we get the result.

Unfortunately, operator $-\mathcal{G}$ is not monotone nor lipschitzian. However, we will see that this perturbation of the monotone and coercive operator $\mathcal{L}^{\varepsilon}_{\nu,\eta}$ satisfies some properties which allow us to conclude that $\mathcal{L}^{\varepsilon}_{\nu,\eta} - \mathcal{G}$ is surjective. For this, we need the following result on the operators called operators of type M (see [21, 31] for example)

Lemma 2.3.3 *Let E and F be two operators defined on a reflexive Banach space $\mathbb{F}$ into its dual $\mathbb{F}'$. Under the following hypotheses*
i) *E is monotone continuous and coercive in $\mathbb{F}$*
ii) *F is bounded weakly continuous (that is from $\mathbb{F}$ weak into $\mathbb{F}'$ weak-$\star$) and the*

mapping $U \mapsto \langle F(U); U\rangle$ is weakly lower semi-continuous
iii) *$E + F$ is coercive on $\mathbb{F}$*
the operator $E + F$ is surjective.

We recall that an operator $N : \mathbb{F} \to \mathbb{R}$ is weakly lower semi-continuous if for $U_i \rightharpoonup U$ we have $\limsup N(U_i) \geq N(U)$. This lemma is a simplified version of a general result which involves operators of type M (see [31] for details). In our work we have $E = \mathcal{L}^{\varepsilon}_{\nu,\eta}$, $F = -\mathcal{G}$ and $\mathbb{F} = \mathbb{V}$. Let us prove the following

Lemma 2.3.4 *Operator $-\mathcal{G}$ satisfies the properties quoted in* **ii)** *of the previous lemma.*

Proof. First $\mathcal{G}$ is bounded from $\mathbb{V}$ into $\mathbb{V}'$ because for all $V \in \mathbb{V}$ we have $|\mathcal{G}(V)| \leq |\partial_t V|$. Assume $V_n \rightharpoonup V$ weakly in $\mathbb{V}$ then $V_n \to V$ strongly in $\mathbb{L}^2(Q)$ and $\partial_t V_n \rightharpoonup \partial_t V$ weakly in $\mathbb{L}^2(Q)$. Since $|\frac{e^{kt}(V_n+M_0)}{1+e^{kt}|V_n+M_0|}| \leq 1$ it follows that $\frac{e^{kt}(V_n+M_0)\cdot W}{1+e^{kt}|V_n+M_0|} \to \frac{e^{kt}(V+M_0)\cdot W}{1+e^{kt}|V+M_0|}$ strongly in $\mathbb{L}^2(Q)$ for all $W \in \mathbb{L}^2(Q)$ and then $\mathcal{G}(V_n) \rightharpoonup \mathcal{G}(V)$ weakly in $\mathbb{L}^2(Q)$ and finally weakly-$\star$ in $\mathbb{V}'$. Moreover since $(\mathcal{G}(V_n); V_n) = -(\mathcal{G}(V_n); M_0)$ thus from what precedes we deduce that $(\mathcal{G}(V_n); M_0) \to (\mathcal{G}(V); M_0) = -(\mathcal{G}(V); V)$.

Now we are able to prove the following existence theorem for problem (2.17)

Theorem 2.3.5 *Let $M_0 \in H^2(\Omega)$ be such that $|M_0(x)|^2 = 1$ in $\overline{\Omega}$, $\frac{\partial M_0}{\partial n} = 0$ on $\partial\Omega$ and either $M_0^+ = M_0^-$ or $M_0^+ = -M_0^-$ on $\widehat{\Omega}$ and let ε, $\eta > 0$, $0 < \nu < 1$ be fixed. Then there exists $k_0 > 0$ depending upon ν such that for $k > k_0$, the problem* (2.17) *admits a solution $V^{\varepsilon}_{\nu,\eta} \in \mathbb{V}$.*

Proof. Since $\mathcal{G}$ satisfies $|(\mathcal{G}(V); V)| \leq \int_Q |\partial_t V| dx$ and $\mathcal{L}^{\varepsilon}_{\nu,\eta}$ is coercive for $k > k_0$, then so is $\mathcal{L}^{\varepsilon}_{\nu,\eta} - \mathcal{G}$. Using lemmas 2.3.2 , 2.3.4 and 2.3.3 we conclude that for $k > k_0$, $\mathcal{L}^{\varepsilon}_{\nu,\eta} - \mathcal{G}$ is surjective from $\mathbb{V}$ into $\mathbb{V}'$. Therefore there exists $V^{\varepsilon}_{\nu,\eta} \in \mathbb{V}$ solving the equation

$$-\varepsilon^2 \partial_t^2 V^{\varepsilon}_{\nu,\eta} - \Delta V^{\varepsilon}_{\nu,\eta} + \frac{1}{2}\partial_t V^{\varepsilon}_{\nu,\eta} + \frac{k}{2} V^{\varepsilon}_{\nu,\eta} = \mathbf{F}_\nu(t, V^{\varepsilon}_{\nu,\eta}) + \mathbf{G}(t, V^{\varepsilon}_{\nu,\eta}, \partial_t V^{\varepsilon}_{\nu,\eta}) + \Delta M_0 - \frac{k}{2} M_0 \tag{2.33}$$

in $\mathbb{V}'$. Hence $V^{\varepsilon}_{\nu,\eta}(0) = 0$ and solves problem (2.17) in the weak sense. Moreover since $V^{\varepsilon}_{\nu,\eta}$ satisfies in the sense of distributions $-\varepsilon^2\partial_t^2 V^{\varepsilon}_{\nu,\eta} - \Delta V^{\varepsilon}_{\nu,\eta} + \frac{k}{2} V^{\varepsilon}_{\nu,\eta} \in L^2(Q)$ with the traces $\partial V^\varepsilon_{\nu,\eta}/\partial N \in H^{1/2}(\partial Q \setminus (\{0\} \times \Omega))$ (see remark 2.2.3) and $V^\varepsilon_{\nu,\eta}(0) = 0$ on Ω (here N denotes the outward unit normal to the boundary ∂Q of Q) then using the classical result of the regularity of solutions of elliptic problems with mixed boundary conditions we deduce that

$$V^\varepsilon_{\nu,\eta} \in \mathbb{H}^2(Q). \tag{2.34}$$

Indeed in view of the hypotheses given on M_0, we can transform our problem into a problem with zero boundary conditions according to the traces theorem given in [9]. Then the geometry of the domain $(0,T) \times \Omega^\pm$ leads by the well known reflection argument to the $\mathbb{H}^2$-regularity of the solutions.

Remark 2.3.6 *Notice that the coerciveness property of $\mathcal{L}^\varepsilon_{\nu,\eta} - \mathcal{G}$ does not allow to bound the solutions $V^\varepsilon_{\nu,\eta}$ independently of ε. The regularity $\mathbb{H}^2$ of the solutions $V^\varepsilon_{\nu,\eta}$ will be relevant to establish uniform estimates.*

2.4 Convergence as $\varepsilon \to 0$

We multiply (2.33) by $e^{-2kt}\partial_t V^\varepsilon_{\nu,\eta}$ and integrate on Q. To simplify notations, let us temporarily write $V^\varepsilon_{\nu,\eta} = V$, $U = e^{kt}(V + M_0)$, $\widetilde{U} = (U^+, U^-)$, $\widetilde{B}_\nu = (B^+_\nu, B^-_\nu)$, $\widetilde{R}_\eta(\widetilde{U}) = (R_\eta(U^+), R_\eta(U^-))$, $\widetilde{\Theta}_\eta(\widetilde{U}) = \Theta_\eta(U^+) + \Theta_\eta(U^-)$. First notice that

$$(\mathbf{G}(t, V, \partial_t V); e^{-2kt}\partial_t V) = 0 \tag{2.35}$$

and using integrations by parts, we get successively

$$(-\varepsilon^2\partial_t^2 V + \frac{1}{2}\partial_t V; e^{-2kt}\partial_t V) = \frac{\varepsilon^2}{2}|\partial_t V(0)|^2 + \frac{1}{2}(1 - 2\varepsilon^2 k)\int_0^T e^{-2kt}|\partial_t V|^2 dt \tag{2.36}$$

$$\frac{k}{2}(V + M_0; e^{-2kt}\partial_t V) = \frac{k}{4}e^{-4kT}|U(T)|^2 - \frac{k}{4}|M_0|^2 + \frac{k^2}{2}\int_0^T e^{-4kt}|U|^2 dt. \tag{2.37}$$

Then since $(-\Delta(V + M_0); e^{-2kt}\partial_t V) = (-\Delta U; e^{-4kt}(\partial_t U - kU))$, $\Phi_\nu(M_0^+, M_0^-) = 0$ and $\Theta_\eta(M_0^\pm) = 0$, we have

$$(-\Delta(V+M_0);e^{-2kt}\partial_t V)=\frac{1}{2}e^{-4kT}|\nabla U(T)|^2+K_{bq}e^{-4kT}\int_{\widehat{\Omega}}(\Phi_\nu+\widetilde{\Theta}_\eta)(\widetilde{U}(T))d\hat{x}$$
$$-\frac{1}{2}|\nabla M_0|^2+k\int_0^T e^{-4kt}|\nabla U|^2dt+4kK_{bq}\int_{\widehat{Q}}e^{-4kt}(\Phi_\nu+\widetilde{\Theta}_\eta)(\widetilde{U})d\hat{x}dt$$
$$-kK_{bq}\int_{\widehat{Q}}e^{-4kt}(\widetilde{B}_\nu+\widetilde{R}_\eta)(\widetilde{U})\cdot\widetilde{U}d\hat{x}dt.$$

Let us set for $U\in\mathbb{V}$

$$\tau_\eta(U^\pm)=4\Theta_\eta(U^\pm)-\frac{1}{\eta}\frac{s(U^\pm)}{1+s(U^\pm)}\ ,\ \widetilde{\tau}_\eta(\widetilde{U})=\tau_\eta(U^+)+\tau_\eta(U^-)\tag{2.38}$$

with $s(U^\pm)=max(|U^\pm|^2-1,0)$. We have $\tau_\eta(U^\pm)\geq 0$ and

$$(4\widetilde{\Theta}_\eta(\widetilde{U})-\widetilde{R}_\eta(\widetilde{U}))\cdot\widetilde{U}=\widetilde{\tau}_\eta(\widetilde{U})-\frac{1}{\eta}(\frac{s(U^+)}{1+s(U^+)}+\frac{s(U^-)}{1+s(U^-)})$$

so

$$k\int_{\widehat{Q}}e^{-4kt}(4\widetilde{\Theta}_\eta(\widetilde{U})-\widetilde{R}_\eta(\widetilde{U})\cdot\widetilde{U})d\hat{x}dt\geq k\int_{\widehat{Q}}e^{-4kt}\widetilde{\tau}_\eta(\widetilde{U})d\hat{x}dt-\frac{1}{2\eta}|\widehat{\Omega}|.\tag{2.39}$$

Furthermore as $\widetilde{B}_\nu(\widetilde{U})\cdot\widetilde{U}\leq C\nu^{-1}$ for $0<\nu<1$ (where $C>0$ is independent upon ν), we have

$$k\int_{\widehat{Q}}e^{-4kt}\widetilde{B}_\nu(\widetilde{U})\cdot\widetilde{U}d\hat{x}dt\leq C(4\nu)^{-1}|\widehat{\Omega}|\tag{2.40}$$

which together with (2.39) give

$$\begin{aligned}&(-\Delta(V+M_0);e^{-2kt}\partial_t V)\geq k\int_0^T e^{-4kt}|\nabla U|^2dt+\\&kK_{bq}\int_{\widehat{Q}}e^{-4kt}(4\Phi_\nu+\widetilde{\tau}_\eta)(\widetilde{U})d\hat{x}dt-\frac{1}{2}|\nabla M_0|^2-K_{bq}(\frac{C}{4\nu}+\frac{1}{2\eta})|\widehat{\Omega}|.\end{aligned}\tag{2.41}$$

In the same way we have $\gamma(|M_0|)=0$ so

$$(-\mathbf{F}_\nu(t,V);e^{-2kt}\partial_t V)=e^{-4kT}\int_\Omega(\psi(U(T))+\nu^{-1}\gamma(|U(T)|))dx-\int_\Omega\psi(M_0)dx$$
$$+4k\int_Q e^{-4kt}(\psi(U)+\nu^{-1}\gamma(|U|))dxdt+\frac{1}{2}e^{-4kT}\int_{\mathbb{R}^3}|\mathbb{D}(U(T))|^2dx$$
$$-\frac{1}{2}\int_{\mathbb{R}^3}|\mathbb{D}(M_0)|^2dx+2k\int_{Q\infty}e^{-4kt}|\mathbb{D}(U)|^2dxdt+k(F_\nu(U);e^{-4kt}U)$$

then writing $(\nabla_U\psi(U); e^{-4kt}U) = (\nabla_U\psi(U) - \nabla_U\psi(0); e^{-4kt}U) + (\nabla_U\psi(0); e^{-4kt}U)$ and using the Lipschitz property of $\nabla_U\psi$, we get

$$|(\nabla_U\psi(U); e^{-4kt}U)| \leq C\int_0^T e^{-4kt}(|U|^2 + |U|)dt \tag{2.42}$$

with a constant $C > 0$. Hence since $|\nabla_U\gamma(|U|)| \leq C|U|$ then using (2.13), we obtain for $0 < \nu < 1$

$$|(F_\nu(U); e^{-4kt}U)| \leq C\nu^{-1}\int_0^T e^{-4kt}|U|^2dt + C \tag{2.43}$$

which leads to

$$\begin{aligned}(-\mathbf{F}_\nu(t,V); e^{-2kt}\partial_t V) &\geq 4k\int_Q e^{-4kt}(\psi(U) + \nu^{-1}\gamma(|U|))dxdt - \int_\Omega \psi(M_0)dx \\ &+2k\int_{Q^\infty} e^{-4kt}|\mathbb{D}(U)|^2dxdt - \frac{1}{2}\int_{\mathbb{R}^3}|\mathbb{D}(M_0)|^2dx - kC\nu^{-1}\int_Q e^{-4kt}|U|^2dxdt - C.\end{aligned} \tag{2.44}$$

Combining the results obtained in (2.35), (2.36), (2.37), (2.41) and (2.44), we get

$$\begin{aligned}&\frac{1}{2}(1-2k\varepsilon^2)\int_0^T e^{-2kt}|\partial_t V|^2dt + k\int_0^T e^{-4kt}|\nabla U|^2dt + k(\frac{k}{2} - C\nu^{-1})\int_0^T e^{-4kt}|U|^2dt \\ &+kK_{bq}\int_{\widehat{Q}} e^{-4kt}(4\Phi_\nu + \widetilde{\tau}_\eta)(\widetilde{U})d\hat{x}dt + 4k\int_Q e^{-4kt}(\psi(U) + \nu^{-1}\gamma(|U|))dxdt \\ &+2k\int_{Q^\infty} e^{-4kt}|\mathbb{D}(U)|^2dxdt \leq C(k,\nu,\eta,M_0)\end{aligned} \tag{2.45}$$

where

$$\left\{\begin{aligned}&C(k,\nu,\eta,M_0) = \frac{k}{4}|M_0|^2 + \frac{1}{2}|\nabla M_0|^2 + \int_\Omega \psi(M_0)dx \\ &+\frac{1}{2}\int_{\mathbb{R}^3}|\mathbb{D}(M_0)|^2dx + C + K_{bq}(\frac{C}{4\nu} + \frac{1}{2\eta})|\widehat{\Omega}|.\end{aligned}\right. \tag{2.46}$$

Therefore ν and η being fixed, for $k > 2C\nu^{-1} + 1$ and $\varepsilon < (4k)^{-1/2}$, we get in particular

Lemma 2.4.1 *For every $0 < \nu < 1$ and $\eta > 0$, there exists $k_0 > 0$, $\varepsilon_0 > 0$ and a constant $C > 0$ which is independent of ε such that for $k > k_0$, $\varepsilon < \varepsilon_0$, the solutions $V^{\varepsilon}_{\nu,\eta}$ of the problem (2.17) satisfy*

$$\|V^{\varepsilon}_{\nu,\eta}\|_{\mathbb{V}} + |\mathbb{D}(V^{\varepsilon}_{\nu,\eta})|_{L^2(Q^\infty)} \leq C \tag{2.47}$$

Now we are able to pass to the limit in problem (2.17) when $\varepsilon \to 0$. The bounds given in lemma 2.4.1 imply that there exists a subsequence still denoted $V^{\varepsilon}_{\nu,\eta}$ and $V^{\eta}_{\nu} \in \mathbb{V}$ such that

$$\begin{cases} V^{\varepsilon}_{\nu,\eta} \rightharpoonup V^{\eta}_{\nu} \ \text{ in } \mathbb{H}^1(Q) \ \text{ weak} \\ V^{\varepsilon}_{\nu,\eta} \to V^{\eta}_{\nu} \ \text{ in } \mathbb{L}^2(Q) \text{ strong} \\ \partial_t V^{\varepsilon}_{\nu,\eta} \rightharpoonup \partial_t V^{\eta}_{\nu} \ \text{ in } \mathbb{L}^2(Q) \ \text{ weak} \\ V^{\varepsilon,\pm}_{\nu,\eta} \rightharpoonup V^{\eta,\pm}_{\nu} \ \text{ in } H^{1/2}(\widehat{Q}) \ \text{ weak.} \end{cases} \tag{2.48}$$

Since $\mathbb{D}$ is linear and $\nabla_U \psi$ and $\nabla_U \gamma$ are lipschitzian, we have

$$\begin{cases} \mathbb{D}(V^{\varepsilon}_{\nu,\eta}) \to \mathbb{D}(V^{\eta}_{\nu}) \ \text{ in } \mathbb{L}^2(Q) \text{ strong} \\ \nabla_U \psi(V^{\varepsilon}_{\nu,\eta}) \to \nabla_U \psi(V^{\eta}_{\nu}) \ \text{ in } \mathbb{L}^2(Q) \text{ strong} \\ \nabla_U \gamma(|V^{\varepsilon}_{\nu,\eta}|) \to \nabla_U \gamma(|V^{\eta}_{\nu}|) \ \text{ in } \mathbb{L}^2(Q) \text{ strong.} \end{cases} \tag{2.49}$$

For the boundary terms, we get the following

Lemma 2.4.2 *It holds that*

$$\begin{cases} V^{\varepsilon,\pm}_{\nu,\eta} \to V^{\eta,\pm}_{\nu} \ \textit{ in } \ L^2(\widehat{Q}) \ \textit{ strong} \\ \mathbf{B}^{\pm}_{\nu}(t, V^{\varepsilon,+}_{\nu,\eta}, V^{\varepsilon,-}_{\nu,\eta}) \to \mathbf{B}^{\pm}_{\nu}(t, V^{\eta,+}_{\nu}, V^{\eta,-}_{\nu}) \ \textit{ in } \ L^2(\widehat{Q}) \ \textit{ strong} \\ \mathbf{R}_{\eta}(t, V^{\varepsilon,\pm}_{\nu,\eta}) \to \mathbf{R}_{\eta}(t, V^{\eta,\pm}_{\nu}) \ \textit{in} \ L^2(\widehat{Q}) \ \textit{ strong .} \end{cases} \tag{2.50}$$

Proof. We get the strong convergence of the traces thanks to the compactness of the continuous imbedding $H^{1/2}(\widehat{Q}) \subset \mathbb{L}^2(\widehat{Q})$. Next the Lipschitz property of R_η and $B^{\pm}_{\nu}$ leads to the the strong convergence of $\mathbf{R}_{\eta}(t, V^{\varepsilon,\pm}_{\nu,\eta})$ and $\mathbf{B}^{\pm}_{\nu}(t, V^{\varepsilon,+}_{\nu,\eta}, V^{\varepsilon,-}_{\nu,\eta})$.

Passing to the limit in problem (2.17) when $\varepsilon \to 0$ with ν and η fixed, we get

Theorem 2.4.3 *Assume that M_0 satisfies the hypotheses of theorem 2.3.5 and let $\eta > 0$ and $0 < \nu < 1$ be fixed. Then for any $T > 0$ and $k > k_0$, (2.15) admits a solution $V^{\eta}_{\nu} \in \mathbb{V}$ obtained as the limit of the sequence $(V^{\varepsilon}_{\nu,\eta})_{\varepsilon}$ when $\varepsilon \to 0$.*

2.5 Convergence for $\eta \to 0$ and $\nu \to 0$

Let V_ν^η be the solution of (2.15) provided by theorem 2.3.2. We set

$$U_\nu^\eta = e^{kt}(V_\nu^\eta + M_0) \tag{2.51}$$

then $U_\nu^\eta \in \mathbb{H}^1(Q)$ and satisfies the intermediary problem (2.9). It follows that $\Delta U_\nu^\eta \in L^2(0,T;\mathbb{L}^2(\Omega))$, $\partial U_\nu^\eta/\partial n \in L^2(0,T;\mathbb{H}^{1/2}(\partial\Omega))$. Hence using the regularity property satisfied by the solution of a Laplace's equation we deduce that

$$U_\nu^\eta \in L^2(0,T;\mathbb{H}^2(\Omega)) \cap \mathbb{H}^1(Q). \tag{2.52}$$

Let us prove the following energy bound for U_ν^η

Lemma 2.5.1 *U_ν^η satisfies for $t \in (0,T)$ the energy equality*

$$\mathcal{E}_\nu^\eta(U_\nu^\eta(t)) + \frac{1}{2}\int_0^t |\partial_t U_\nu^\eta(s)|^2_{\mathbb{L}^2(\Omega)} ds = \mathcal{E}(M_0) \tag{2.53}$$

where

$$\begin{cases} \mathcal{E}_\nu^\eta(V) = \frac{1}{2}|\nabla V|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2}|\nabla\varphi|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \int_\Omega (\psi(V) + \nu^{-1}\gamma(|V|))dx \\ +K_{bq}\int_{\widehat{\Omega}} (\Phi_\nu(V^+,V^-) + \Theta_\eta(V^+) + \Theta_\eta(V^-))d\widehat{x} \end{cases} \tag{2.54}$$

$$\mathcal{E}(M_0) = \frac{1}{2}|\nabla M_0|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2}|\nabla\varphi_0|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \int_\Omega \psi(M_0)dx \tag{2.55}$$

and $\nabla\varphi = \mathbb{D}(V)$, $\nabla\varphi_0 = \mathbb{D}(M_0)$.

Moreover there exists $\nu_0 > 0$ and $C > 0$ which depends only upon the initial data M_0 such that for $0 < \nu < \nu_0$ and $\eta > 0$, we have

$$|U_\nu^\eta(t)|_{\mathbb{L}^2(\Omega)} \le C, \ \ t \in (0,T). \tag{2.56}$$

Proof. Recall that $\gamma(|M_0|) = 0$, $\Theta_\eta(M_0^\pm) = 0$ and $\Phi_\nu(M_0^+, M_0^-) = 0$ so if we multiply the equation (2.9) by $\partial_t U_\nu^\eta$ and integrate on $(0,t)\times\Omega$ we get the result stated in (2.53). Therefore since the function γ satisfies the inequality $s^2 \le 4\gamma(s)+3$, we obtain $|U_\nu^\eta(t)|^2_{\mathbb{L}^2(\Omega)} \le 4\nu\mathcal{E}(M_0) + 3|\Omega|$ which leads to (2.56).

Now we pass to the limit in (2.9) for $\eta \to 0$, ν being fixed. There exists a subsequence also denoted U_ν^η and $U_\nu \in L^\infty(0,T;\mathbb{H}^1(\Omega)) \cap \mathbb{H}^1(Q)$ such that when $\eta \to 0$ we have

$$\begin{cases} U_\nu^\eta \rightharpoonup U_\nu \ \text{ in } \ L^\infty(0,T;H^1(\Omega)) \ \text{ weak } -\star, \\ U_\nu^\eta \to U_\nu \ \text{ in } \ \mathbb{L}^2(Q) \ \text{ strong} \\ \partial_t U_\nu^\eta \rightharpoonup \partial_t U_\nu \ \text{ in } \ \mathbb{L}^2(Q) \ \text{ weak} \\ U_\nu^{\eta,\pm} \to U_\nu^\pm \ \text{ in } \ \mathbb{L}^2(\widehat{Q}) \ \text{ strong} \end{cases} \tag{2.57}$$

Moreover we have

$$\begin{cases} \mathbb{D}(U_\nu^\eta) \to \mathbb{D}(U_\nu) \ \text{ in } \ \mathbb{L}^2(Q) \ \text{ strong} \\ \nabla_U \psi(U_\nu^\eta) \to \nabla_U \psi(U_\nu) \ \text{ in } \mathbb{L}^2(Q) \text{ strong} \\ \nabla_U \gamma(|U_\nu^\eta|) \to \nabla_U \gamma(|U_\nu|) \ \text{ in } \mathbb{L}^2(Q) \text{ strong} \\ B_\nu^\pm(U_\nu^{\eta,+}, U_\nu^{\eta,-}) \to B_\nu^\pm(U_\nu^+, U_\nu^-) \ \text{ in } \ \mathbb{L}^2(\widehat{Q}) \ \text{ strong} \end{cases} \tag{2.58}$$

and we get

Lemma 2.5.2 *When $\eta \to 0$, it holds the following convergence*

$$\eta \Theta_\eta(U_\nu^{\eta,\pm}) \to 0 \text{ strongly in } L^\infty(0,T;L^1(\widehat{\Omega})). \tag{2.59}$$

Moreover U_ν is such that

$$|U_\nu^\pm(t,x)|^2 \leq 1 \text{ a.e. } (t,x) \in \widehat{Q} \tag{2.60}$$

and satisfies the energy inequality

$$\mathcal{E}^\nu(U_\nu(t)) + \frac{1}{2}\int_0^t |\partial_t U_\nu(s)|^2_{L^2(\Omega)} ds \leq \mathcal{E}(M_0) \tag{2.61}$$

where

$$\begin{cases} \mathcal{E}^\nu(U) = \frac{1}{2}|\nabla U|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2}|\mathbb{D}(U)|^2_{L^2(\mathbb{R}^3)} + \int_\Omega \psi(U)dx \\ +\nu^{-1}\int_\Omega \gamma(|U|)dx + K_{bq}\int_{\widehat{\Omega}} \Phi_\nu(U^+,U^-)d\widehat{x}. \end{cases} \tag{2.62}$$

Proof. The strong convergence of the traces leads to

$$2\eta\Theta_\eta(U_\nu^{\eta,\pm}) \to s(U_\nu^\pm) - \log(1 + s(U_\nu^\pm)) \ \text{ a.e. in } \widehat{Q} \tag{2.63}$$

but since $\eta|\Theta_\eta(U_\nu^{\eta,\pm})|_{L^\infty(0,T;L^1(\widehat{\Omega}))} \leq \eta\mathcal{E}(M_0)$, we get

$$\eta\Theta_\eta(U_\nu^{\eta,\pm}) \to 0 \text{ strongly in } L^\infty(0,T;L^1(\widehat{\Omega})).$$

Combining these results we conclude that $s(U_\nu^\pm) = 0$ a.e. in $\widehat{Q}$ which leads to (2.60). (2.61) is obtained by taking the limit in (2.53) when $\eta \to 0$.

Let $W(t,x) \in \mathcal{D}(\overline{Q})$ be a test function. We multiply the equation (2.9) by $U_\nu^\eta \times W$ and integrate by parts. Observing that $\nabla_U \gamma(|U_\nu^\eta|) \cdot U_\nu^\eta \times W = 0$ and $R_\eta(U_\nu^{\eta,\pm}) \cdot U_\nu^{\eta,\pm} \times W^\pm = 0$, we get the weak formulation of (2.9)

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_Q \partial_t U_\nu^\eta \cdot U_\nu^\eta \times W \, dxdt + \int_Q \frac{U_\nu^\eta}{1+|U_\nu^\eta|} \times \partial_t U_\nu^\eta \cdot U_\nu^\eta \times W \, dxdt \\ + \displaystyle\int_Q \nabla U_\nu^\eta \cdot U_\nu^\eta \times \nabla W \, dxdt - \int_Q (\nabla \varphi_\nu^\eta - \nabla_U \psi(U_\nu^\eta)) \cdot U_\nu^\eta \times W \, dxdt \\ = K_{bq} \displaystyle\int_{\widehat{Q}} (B_\nu^+(U_\nu^{\eta,+}, U_\nu^{\eta,-}) \cdot U_\nu^{\eta,+} \times W^+ + B_\nu^-(U_\nu^{\eta,+}, U_\nu^{\eta,-}) \cdot U_\nu^{\eta,-} \times W^-) d\widehat{x}dt. \end{cases}$$

Using the strong convergence of U_ν^η and the weak convergence of $\partial_t U_\nu^\eta$ in $\mathbb{L}^2(Q)$, we can pass to the limit in each volume integral. Moreover the strong convergence in $\mathbb{L}^2(\widehat{Q})$ of the traces $U_\nu^{\eta,\pm}$ and $B_\nu^\pm(U_\nu^{\eta,+}, U_\nu^{\eta,-})$ allow to pass to the limit in the boundary terms. Hence the limit U_ν satisfies the weak formulation

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_Q \partial_t U_\nu \cdot U_\nu \times W dxdt + \int_Q \frac{U_\nu}{1+|U_\nu|} \times \partial_t U_\nu \cdot U_\nu \times W dxdt \\ + \displaystyle\int_Q \nabla U_\nu \cdot U_\nu \times \nabla W dxdt - \int_Q (\nabla \varphi_\nu - \nabla_U \psi(U_\nu)) \cdot U_\nu \times W dxdt \\ = K_{bq} \displaystyle\int_{\widehat{Q}} (B_\nu^+(U_\nu^+, U_\nu^-) \cdot U_\nu^+ \times W^+ + B_\nu^-(U_\nu^+, U_\nu^-) \cdot U_\nu^- \times W^-) d\widehat{x}dt. \end{cases} \tag{2.64}$$

Now we are able to prove our main theorem

Theorem 2.5.3 *Let $M_0 \in H^2(\Omega)$ be such that $|M_0(x)| = 1$ in $\overline{\Omega}$, $\frac{\partial M_0}{\partial n} = 0$ on $\partial\Omega$ and either $M_0^+ = M_0^-$ or $M_0^+ = -M_0^-$ on $\widehat{\Omega}$. There exists a weak solution of the Landau-Lifshitz equation with biquadratic interlayer exchange coupling energy satisfying $M \in L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$, $\partial_t M \in L^2_{loc}(\mathbb{R}^+; \mathbb{L}^2(\Omega))$, $|M(t,x)|^2 = 1$ a.e in $\mathbb{R}^+ \times \Omega$, $\nabla\varphi \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3))$ and the energy inequality*

$$\mathcal{E}(M(t)) + \frac{1}{2}\int_0^t |\partial_t M(s)|^2_{\mathbb{L}^2(\Omega)} ds \le \mathcal{E}(M_0) \tag{2.65}$$

where

$$\mathcal{E}(M) = \frac{1}{2}|\nabla M|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2}|\nabla\varphi|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \int_\Omega \psi(M)dx + K_{bq}\int_{\widehat{\Omega}} \Phi(M^+, M^-)d\widehat{x} \quad (2.66)$$

with $\Phi(M^+, M^-) = \frac{1}{2}(1 - (M^+ \cdot M^-)^2)$, $\nabla\varphi = \mathbb{D}(M)$, $\nabla\varphi_0 = \mathbb{D}(M_0)$ *and the initial energy is given by*

$$\mathcal{E}(M_0) = \frac{1}{2}|\nabla M_0|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2}|\nabla\varphi_0|^2_{L^2(\mathbb{R}^3)} + \int_\Omega \psi(M_0)dx \quad (2.67)$$

Proof. Let U_ν the limit of U_ν^η when $\eta \to 0$. Hence U_ν satisfies the energy inequality (2.61) and the trace estimate $|U_\nu^\pm(t, \widehat{x})|^2 \leq 1$ a.e. in $\mathbb{R}^+ \times \widehat{\Omega}$. Clearly the energy inequality implies the following convergence for a subsequence U_ν

$$\begin{cases} U_\nu \rightharpoonup M \text{ in } L^\infty(\mathbb{R}^+; H^1(\Omega)) \text{ weak} - \star, \\ \partial_t U_\nu \rightharpoonup \partial_t M \text{ in } L^2_{loc}(\mathbb{R}^+; L^2(\Omega)) \text{ weak}, \\ U_\nu \to M \text{ in } L^2_{loc}(\mathbb{R}^+; L^2(\Omega)) \text{ strong} \\ U_\nu^\pm \rightharpoonup M^\pm \text{ in } L^\infty(\mathbb{R}^+; \mathbb{H}^{1/2}(\widehat{\Omega})) \text{ weak} - \star \\ U_\nu^\pm \to M^\pm \text{ in } L^2_{loc}(\mathbb{R}^+; L^2(\widehat{\Omega})) \text{ strong} \end{cases} \quad (2.68)$$

for some $M \in L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega)) \cap \mathbb{H}^1(Q)$. Moreover we have

$$\begin{cases} \nabla\varphi_\nu \rightharpoonup \nabla\varphi \text{ in } L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3)) \text{ weak} - \star \\ \nabla\varphi_\nu \to \nabla\varphi \text{ in } L^2_{loc}(\mathbb{R}^+; L^2(\mathbb{R}^3)) \text{ strong} \\ \nabla_M\psi(U_\nu) \to \nabla_M\psi(M) \text{ in } L^2_{loc}(\mathbb{R}^+; L^2(\Omega)) \text{ strong.} \end{cases} \quad (2.69)$$

Since $\int_\Omega \gamma(|U_\nu|)dx \leq \nu\mathcal{E}(M_0)$, we have

$$\gamma(|U_\nu|) \to 0 \text{ strongly in } L^\infty(\mathbb{R}^+; L^1(\Omega))$$

and so a.e. in $\mathbb{R}^+ \times \Omega$. Hence combining this result with the strong convergence of U_ν we get $\gamma(|M|) = 0$ a.e. in $\mathbb{R}^+ \times \Omega$ that is $|M(t,x)|^2 = 1$ a.e. in $\mathbb{R}^+ \times \Omega$. Now we are interested by the convergence of the boundary terms $B_\nu^\pm(U_\nu^+, U_\nu^-) \cdot U_\nu^\pm \times W^\pm$ of the weak formulation (2.64). Since $U_\nu^\pm$ is bounded in $\mathbb{L}^\infty(\mathbb{R}^+ \times \widehat{\Omega})$ so is $B_\nu^\pm(U_\nu^+, U_\nu^-)$ so thanks to the Lebesgue dominated convergence theorem, we get the convergences

$$B_\nu^\pm(U_\nu^+, U_\nu^-) \to |M^\mp|^2 M^\pm - (M^+ \cdot M^-)M^\mp \text{ strongly in } L^2_{loc}(\mathbb{R}^+; \mathbb{L}^2(\widehat{\Omega})) \quad (2.70)$$

$$B_\nu^\pm(U_\nu^+, U_\nu^-) \cdot U_\nu^\pm \times W^\pm \to (|M^\mp|^2 M^\pm - (M^+ \cdot M^-)M^\mp) \cdot M^\pm \times W^\pm \tag{2.71}$$

for all $W \in \mathcal{D}(\mathbb{R}^+ \times \overline{\Omega})$. Hence M satisfies the weak formulation

$$\begin{cases} \frac{1}{2}\int_Q \partial_t M \cdot M \times W \; dxdt + \frac{1}{2}\int_Q \; M \times \partial_t M \cdot M \times W \; dxdt \\ + \int_Q \nabla M \cdot M \times \nabla W \; dxdt - \int_Q (\nabla\varphi - \nabla_M \psi(M)) \cdot M \times W \; dxdt = \\ K_{bq} \int_{\widehat{Q}} (B^+(M^+, M^-) \cdot M^+ \times W^+ + B^-(M^+, M^-) \cdot M^- \times W^-) \; d\widehat{x}dt \end{cases} \tag{2.72}$$

where $B^\pm(M^+, M^-) = |M^\mp|^2 M^\pm - (M^+ \cdot M^-)M^\pm$. This shows that M is a global weak solution of LLG equations. The energy estimate satisfied by M follows from the one satisfied by U_ν by passing to the limit when $\nu \to 0$. The proof of the theorem is now complete.

Chapitre 3

On a model of magnetization switching by spin-polarized current

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This paper is concerned with global existence of weak solutions to a model equations of magnetization reversal by spin-polarized current in a layer introduced in [37]. The local magnetization of the ferromagnet satisfies the usual Landau-Lifshitz equation which is coupled to the nonlinear heat equation satisfied by the spin accumulation field defined in all the layer. The coupling is due to the contact interaction energy. We use an hyperbolic regularization method with penalization of the saturation constraint satisfied by the local magnetization to prove global existence result, in any finite time interval, of weak solutions with finite energy. We present other models equations describing the magnetization switching by spin-polarized current and show that our method can be used to solve them.

Key words : Ferromagnets, spin-polarized current, spin accumulation, magnetization switching

Mathematics subject classification (MSC 2000) : 73R05, 73K05, 47J35

Short title : Magnetization switching

3.1 Introduction

The recent discovery that a spin-polarized electrical current can apply a large torque to a ferromagnet through direct transfer of spin angular momentum, offers the possibility of manipulating magnetic-device elements without applying any magnetic fields. Magnetic multilayers, devices which consist of alternating nanometer-scale-thick films of magnetic and nonmagnetic materials, have been the focus of scientific and technological interest since the discovery of their giant magnetoresistance. While the effect of the magnetic layers on the current has been widely studied, the inverse phenomenon, the effect of the current on the orientation of the magnetic layers, has received less attention. Recent theories [29, 4] predict that a large spin-polarized current passing perpendicularly through the layers can exert torques on the magnetic moments in the layers, due to direct transfer of spin angular momentum from the conduction electrons to the magnets. Convincing experiments have been now achieved and several theoretical approaches, extending the initial theory, have been recently developed for the interpretation of the existing experimental data. However the current density needed in the existing experiments is still relatively high, of the order of 10^7-10^9 A/cm^2, and a better understanding of the spin transfer mechanisms seems necessary to obtain a significant reduction of the current density. Another effect of the same type but probably requiring a smaller current density is the displacement of a domain wall by spin transfer from a spin-polarized current. It was found experimentally that the current-induced spin torque has many unique properties. Among them, the spin torque distorts domain structures, drives domain walls out of equilibrium, makes magnetization switching from one direction to another, and creates spin excitations. While several theoretical models have been put forward to formulate the spin torque, the exact form is still the subject of much debate. These theories suggest an interesting magnetization relaxation process caused by a spin-polarized current. A conventional physical picture consists of a magnetic junction with two ferromagnetic layers one of which (layer 1) is pinned and the other (layer 2) is free. Very thin spacer layer exists between the layers 1 and 2. The layer 2 has finite thickness and contacts at the second end with some nonmagnetic conductor (layer 3). The spin-polarized current may inject (or extract) nonequilibrium spins into layer 2. At a sufficiently large current density, the effective field leads to the magnetization reversal of the layer 2. In order to focus further on these phenomena which have the important consequence that they can act as an additional magnetic field on the magnetic background, one assumes that when the bilinear or biquadratic interlayer exchange coupling exists, their effects are negligible or one chooses the spacer layer thick enough to reduce the magnetic coupling between ferromagnets.

We refer respectively to [11] and [12] for some mathematical results on these effects.

The main model equation we consider in this work is introduced by Zhang, Levy and Fert [37]. The effective magnetic field $\mathcal{H}_e$ is given by $\mathcal{H}_e = \mathcal{H}_{LLG} + J\boldsymbol{m}$ where $\mathcal{H}_{LLG}$ is the classical effective magnetic field and $J\boldsymbol{m}$ corresponds to the contact interaction energy which is of the type $-J\int_\Omega \boldsymbol{m}\cdot\boldsymbol{M}\ dx$. The spin accumulation field $\boldsymbol{m}$ satisfies to a nonlinear heat equation coupled to the LLG equation. The main difference for the new magnetization equation relies in the fact that the energy of the system is not conserved. So, we consider the coupled system for $\boldsymbol{m}$ and the local magnetization $\boldsymbol{M}$ on the finite time interval $[0, T]$ for any $T > 0$ fixed.

Other models appear in the recent publications of physics to describe the magnetization reversal by current injection after the main contribution of Slonczewski [29] and Berger [4]. Some of them are [30], [3], [33] and [36] for a different approach. There is also other models, close to the previous ones, to describe the motion of domain wall by spin polarized current see [33]. In the last section of this work we recall these models equations and discuss their mathematical approaches.

Having briefly discussed the general context of this paper, let us go into its structure. In the second section, we set down the model equations and the initial-boundary conditions for spin accumulation and magnetization. The third section is dedicated to the main results. We investigate the global existence of weak solutions to the coupled problem for spin accumulation and local magnetization dynamics. We first give formal estimates and then by regularization procedures we prove global existence of weak solutions in the time interval $[0, T]$ for all fixed and finite T. In the fourth section, we discuss the case of stationary spin accumulation equation. We prove global existence of weak solutions by proceeding along the lines of the proof given in the third section. The last section of the paper is devoted to some other models appearing in the recent papers. We show that our method can be performed to establish global existence theory for weak solutions.

In the sequel we use the following notations. If D is an open and regular set of $\mathbb{R}^n$, $\mathbb{L}^2(\mathrm{D})$ will denote the vectorial Lebesgue space $(L^2(\mathrm{D}))^3$ with norm and scalar product denoted respectively by $|\cdot|_{\mathrm{D}}$ and $(\cdot;\cdot)_{\mathrm{D}}$. The Hilbert space $\mathbb{H}^1(\mathrm{D})$ is the usual Sobolev space $(H^1(\mathrm{D}))^3$. If D is a bounded domain of $\mathbb{R}^n$, $|\mathrm{D}|$ will denote its Lebesgue measure.

3.2 The model equations

The calculations presented in this paper combine phenomenological constitutive equations for the electric current and the spin current with a Landau-Lifshitz-Gilbert equation generalized to include spin-transfer torque. We adopt the model equations of magnetization switching by spin current proposed by [37], [27]. We also refer to [14]. Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be the canonical basis of $\mathbb{R}^3$ and let $\widehat{\Omega}$ be an open, bounded and regular domain of $\mathbb{R}^2$ with generic point $\widehat{x} = (y, z)$. We denote by x the abscissa of a point $X = (x, \widehat{x})$ of $\mathbb{R}^3$, $\Omega^0 = (-h, h) \times \widehat{\Omega}$ denotes the nonmagnetic spacer (N), $\Omega^1 = (h, l) \times \widehat{\Omega}$ and $\Omega^2 = (-L, -h) \times \widehat{\Omega}$ the ferromagnetic materials (F). We consider a magnetic junction of the type $F/N/F$ represented by the horizontal cylinder $\mathcal{O} = \Omega^1 \cup \Omega^0 \cup \Omega^2$. At the plane $x = -L$ an electric current j_e is injected in the direction $\boldsymbol{e}_1$ and flows in the domain $I = [-L, l]$. Then the background magnetization receives a spin torque. This torque leads to spin accumulation in N and in the regions of F close to the F/N and N/F interfaces. If one denotes by $\boldsymbol{m}(t, x)$ the spin accumulation vector in the domain Ω for $t \geq 0$ and $x \in I$, then the magnetization current $\boldsymbol{j}_m$, in the case where the local magnetization $\boldsymbol{M}$ is assumed to be uniform in $\Omega = \Omega^1 \cup \Omega^2$ (with respect to the variable X), is given by the expression

$$\boldsymbol{j}_m = \beta j_e \boldsymbol{M} - 2D_o(\partial_x \boldsymbol{m} - \beta\beta'(\boldsymbol{M} \cdot \partial_x \boldsymbol{m})\boldsymbol{M}). \tag{3.1}$$

Here, $\beta > 0$ and $\beta' > 0$ represent the spin polarization parameters, $D_o > 0$ the diffusion parameter. Theoretical analysis has been mostly confined to the case where the magnetization $\boldsymbol{M}$ of the layers is uniform. If $\boldsymbol{M}$ is not uniform in Ω then we can extend the expression (3.1) by setting

$$\boldsymbol{j}_m = \beta j_e \langle \boldsymbol{M} \rangle - 2D_o(\partial_x \boldsymbol{m} - \beta\beta'(\langle \boldsymbol{M} \rangle \cdot \partial_x \boldsymbol{m})\langle \boldsymbol{M} \rangle) \tag{3.2}$$

where $\langle \boldsymbol{M} \rangle$ is the mean value of $\boldsymbol{M}$ in $\widehat{\Omega}$, $\langle \boldsymbol{M} \rangle = |\widehat{\Omega}|^{-1} \int_{\widehat{\Omega}} \boldsymbol{M}(t, x, \widehat{x})\, \mathrm{d}\widehat{x}$. Hence, the spin accumulation is governed for $t \geq 0$ and $x \in I$, by the equation see [37], [27]

$$\frac{1}{d^2}\partial_t \boldsymbol{m} - \partial_x(\mathcal{A}(\langle \boldsymbol{M} \rangle)(\partial_x \boldsymbol{m})) + \frac{1}{\lambda_J^2} \boldsymbol{m} \times \langle \boldsymbol{M} \rangle + \frac{1}{\lambda_{\mathrm{sf}}^2} \boldsymbol{m} = -\frac{\beta}{d^2}\partial_x(j_e \langle \boldsymbol{M} \rangle) \tag{3.3}$$

where we defined the matrix $\mathcal{A}(\langle \boldsymbol{M} \rangle)$ by

$$\mathcal{A}(\langle \boldsymbol{M}\rangle)(\xi) = \xi - \beta\beta'(\langle \boldsymbol{M}\rangle \cdot \xi)\langle \boldsymbol{M}\rangle \tag{3.4}$$

for all $\xi \in \mathbb{R}^3$. The constants appearing in the equation are given by

$$d = \sqrt{2D_o},\ \lambda_{\rm sf} = d\sqrt{\tau_{\rm sf}},\ \lambda_J = d\sqrt{\hbar/J}, \tag{3.5}$$

where $\tau_{\rm sf}$ is the spin-flip relaxation time of the conduction electron, $J > 0$ is the exchange interaction constant and $\hbar$ is the Planck constant.
The equation (3.3) is completed by initial and boundary conditions. First, we assume that the spin accumulation $\boldsymbol{m}$ as well as the magnetization current $\boldsymbol{j}_m$ are continuous across the interfaces $x = -h$ and $x = h$ (notice that $\boldsymbol{M} = 0$ in Ω^0). We have the transmission boundary conditions

$$[\boldsymbol{m}(t,\cdot)]_{-h} = [\boldsymbol{m}(t,\cdot)]_h = 0,\ [\boldsymbol{j}_m(t,\cdot)]_{-h} = [\boldsymbol{j}_m(t,\cdot)]_h = 0 \tag{3.6}$$

where we use the notation $[u(t,\cdot)]_{x_0} = u(t, x_0 + 0) - u(t, x_0 - 0)$ for a function u. The electric current $j_e(t, x)$ is a given function defined for $t \geq 0$ in the interval $-L \leq x \leq l$ and is linked to the spin accumulation $\boldsymbol{m}$ by (3.2). The boundary conditions for $\boldsymbol{m}$ at the interfaces $x = -L$ and $x = l$ take the form

$$\Big(\mathcal{A}(\langle \boldsymbol{M}\rangle)(\partial_x \boldsymbol{m})(t,\cdot) - \frac{\beta}{d^2} j_e \langle \boldsymbol{M}\rangle(t,\cdot)\Big)_{x=-L,\ l} = 0. \tag{3.7}$$

The spin accumulation equation is coupled to the local magnetization $\boldsymbol{M}(t, x, \widehat{x})$ which satisfies the LLG equation in $\mathbb{R}^+ \times \Omega$ see [13], [5] for example

$$\partial_t \boldsymbol{M} - \alpha \boldsymbol{M} \times \partial_t \boldsymbol{M} = -\gamma \boldsymbol{M} \times (\mathcal{H}_{\rm eff} + J\boldsymbol{m}) \tag{3.8}$$

where $J\boldsymbol{m}$ is the field associated with the contact interaction between the local moment and the spin accumulation. The parameters $\gamma > 0$ and $\alpha > 0$ are respectively the gyromagnetic and the Gilbert damping parameters. The effective magnetic field $\mathcal{H}_{\rm eff}$ contains the contribution of the magnetostatic, the bulk anisotropy, the external fields and the magnetic field associated with the exchange energy

$$\mathcal{H}_{\rm eff}(\boldsymbol{M}) = \nabla \cdot (a\nabla \boldsymbol{M}) + \nabla\varphi - \nabla_{\boldsymbol{M}}\psi(\boldsymbol{M}) \tag{3.9}$$

where the exchange function $a(X)$ takes two positive values $a_1 > 0$ in Ω^1 and $a_2 > 0$ in Ω^2 and where $\varphi(t, X)$ satisfies in $\mathbb{R}^+ \times \mathbb{R}^3$ the stray field equation

$$\nabla \cdot (\nabla \varphi + \chi(\Omega)\boldsymbol{M}) = 0 \tag{3.10}$$

and $\nabla_{\boldsymbol{M}}\psi$ is the bulk anisotropy field associated with a regular function $\psi \in C^2(\mathbb{R}^3)$ satisfying $\psi(X) \geq 0$ and $|D^2\psi(X)| \leq C$ for all $X \in \mathbb{R}^3$.

To end this description of the model equations, let us rewrite the initial-boundary conditions satisfied by the field $(\boldsymbol{m}, \boldsymbol{M})$. We have for $\boldsymbol{m}$

$$\boldsymbol{m}(0,x) = \boldsymbol{m}_0(x) \ \text{ in } I, \ \boldsymbol{j}_m(t,-L) = \boldsymbol{j}_m(t,l) = 0 \ \text{ in } \mathbb{R}^+ \tag{3.11}$$

and for $\boldsymbol{M}$

$$\begin{cases} \boldsymbol{M}(0,X) = \boldsymbol{M}_0(X), \ |\boldsymbol{M}_0(X)|^2 = 1 \ \text{ a.e. in } \Omega \\ \boldsymbol{M}(t,h+0,\widehat{x}) = \boldsymbol{M}(t,-h-0,\widehat{x}) \ \text{ in } \mathbb{R}^+ \times \widehat{\Omega} \\ (\boldsymbol{M} \times a_1 \partial_x \boldsymbol{M})(t,h+0,\widehat{x}) = (\boldsymbol{M} \times a_2 \partial_x \boldsymbol{M})(t,-h-0,\widehat{x}) \ \text{ in } \mathbb{R}^+ \times \widehat{\Omega} \\ \boldsymbol{M} \times a(\nabla \cdot n)\boldsymbol{M} = 0 \text{ on } \mathbb{R}^+ \times (\partial\Omega \setminus (\Gamma^1 \cup \Gamma^2)) \end{cases} \tag{3.12}$$

where we set $\Gamma^1 = \{(h,\widehat{x}), \widehat{x} \in \widehat{\Omega}\}$, $\Gamma^2 = \{(-h,\widehat{x}), \widehat{x} \in \widehat{\Omega}\}$ and n is the unit outward normal to the boundary $\partial\Omega$ of Ω. Finally, the magnetization $\boldsymbol{M}(t,X)$ satisfies the saturation condition

$$|\boldsymbol{M}(t,X)|^2 = 1 \text{ a.e in } \mathbb{R}^+ \times \Omega.$$

As it is said in the introduction, we may use more complex boundary conditions coupling the interfaces $z = \pm h$ as the bilinear exchange one see [11] or the biquadratic exchange one see [12] for example.

In this section we discuss the model equations (3.3)-(3.11)-(3.8)-(3.12).

3.3 Global weak solutions to the problem

Our starting point is to give formally the a priori estimates satisfied by the solutions of the system (3.3)-(3.11)- (3.8)-(3.12).

3.3.1 Formal estimates

Any regular solution to the LLG equations (3.8) satisfies the relation

$$\alpha|\partial_t \boldsymbol{M}|^2 = \gamma(\mathcal{H}_{\text{eff}} + J\boldsymbol{m}) \cdot \partial_t \boldsymbol{M}$$

which leads after integration over Ω to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\boldsymbol{M}) + \frac{\alpha}{\gamma}|\partial_t \boldsymbol{M}|^2_\Omega = J\int_\Omega \boldsymbol{m}\cdot\partial_t\boldsymbol{M}\ \mathrm{d}X \tag{3.13}$$

where $\mathcal{E}(\boldsymbol{M})$ is the energy of the system given by

$$\mathcal{E}(\boldsymbol{M}) = \int_\Omega a(X)|\nabla \boldsymbol{M}|^2\ \mathrm{d}X + \int_{\mathbb{R}^3}|\nabla\varphi|^2\ \mathrm{d}X + 2\int_\Omega \psi(\boldsymbol{M})\ \mathrm{d}X. \tag{3.14}$$

By Cauchy-Schwarz inequality we get the estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\boldsymbol{M}) + \frac{\alpha}{2\gamma}|\partial_t \boldsymbol{M}|^2_\Omega \leq \frac{\gamma J^2|\widehat{\Omega}|}{2\alpha}|\boldsymbol{m}|^2_I. \tag{3.15}$$

Integration with respect to the time variable t leads to

$$\mathcal{E}(\boldsymbol{M}(t)) + \frac{\alpha}{\gamma}\int_0^t |\partial_t \boldsymbol{M}(s)|^2_\Omega\ \mathrm{d}s \leq \mathcal{E}(\boldsymbol{M}_0) + \frac{\gamma J^2|\widehat{\Omega}|}{\alpha}\int_0^t |\boldsymbol{m}(s)|^2_I\ \mathrm{d}s \tag{3.16}$$

where $\mathcal{E}(\boldsymbol{M}_0)$ is the initial energy defined by

$$\mathcal{E}(\boldsymbol{M}_0) = \int_\Omega a(X)|\nabla \boldsymbol{M}_0|^2\ \mathrm{d}X + \int_{\mathbb{R}^3}|\nabla\varphi_0|^2\ \mathrm{d}X + 2\int_\Omega \psi(\boldsymbol{M}_0)\ \mathrm{d}X \tag{3.17}$$

where φ_0 satisfies the stray field equation associated with $\boldsymbol{M}_0$.

Let us consider for $\boldsymbol{M}$ fixed the heat equation satisfied by $\boldsymbol{m}$. We have the energy estimate

$$\frac{1}{2d^2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{m}|^2_I + \frac{1}{\lambda^2_{\text{sf}}}|\boldsymbol{m}|^2_I + (\mathcal{A}(\langle \boldsymbol{M}\rangle)\partial_x\boldsymbol{m};\partial_x\boldsymbol{m})_{\mathbb{L}^2(I)} = \frac{\beta}{d^2}\int_I j_e\langle \boldsymbol{M}\rangle\cdot\partial_x\boldsymbol{m}\ \mathrm{d}x. \tag{3.18}$$

Assuming the parameters β and β' be such that

$$1 - \beta\beta' > 0 \tag{3.19}$$

then the matrix $\mathcal{A}(\langle \boldsymbol{M} \rangle)$ satisfies

$$\begin{cases} \mathcal{A}(\langle \boldsymbol{M} \rangle)(\xi) \cdot \xi \geq (1 - \beta\beta')|\xi|^2 \\ |\mathcal{A}(\langle \boldsymbol{M} \rangle)(\xi)| \leq (1 + \beta\beta')|\xi| \end{cases} \tag{3.20}$$

for all $\xi \in \mathbb{R}^3$. We deduce, by using the saturation condition $|\boldsymbol{M}(t,X)|^2 = 1$, the inequality

$$\begin{cases} \dfrac{1}{2d^2}|\boldsymbol{m}(t)|_I^2 + \dfrac{1}{\lambda_{\mathrm{sf}}^2}\displaystyle\int_0^t |\boldsymbol{m}(s)|_I^2 \,\mathrm{d}s + \dfrac{1-\beta\beta'}{2}\displaystyle\int_0^t |\partial_x \boldsymbol{m}|_I^2 \,\mathrm{d}s \leq \\ \dfrac{1}{2d^2}|\boldsymbol{m}_0|_I^2 + \dfrac{\beta^2}{2d^4(1-\beta\beta')}\displaystyle\int_0^t |j_e|_I^2 \,\mathrm{d}s. \end{cases} \tag{3.21}$$

Estimates (3.16) and (3.21) lead to a bound for the energy of the local magnetization $\boldsymbol{M}$ on the interval $[0,T]$ for T fixed and finite.

Remark 3.3.1 *Notice that the contact interaction energy satisfies the inequality*

$$|\int_\Omega \boldsymbol{m} \cdot \boldsymbol{M} \; dX| \leq |\widehat{\Omega}|^{1/2}(\int_I |\boldsymbol{m}|^2 \; dx)^{1/2}. \tag{3.22}$$

It follows that this energy is controlled by the contribution of the spin accumulation energy.

We shall prove a global existence result for weak solutions to problem (3.3)-(3.11)-(3.8)-(3.12). We set $Q = (0,T) \times \Omega$ and $\omega = (0,T) \times I$ for $T > 0$ fixed.

3.3.2 Regularization : the intermediary problem

We proceed along the lines of the method introduced in [10]. See also [12] for an use of the method. Let us consider a small parameter $\varepsilon > 0$. We introduce the vector function U^ε satisfying the intermediary problem

$$
\begin{cases}
\frac{\alpha}{\gamma}\partial_t U^\varepsilon - \nabla\cdot(a\nabla U^\varepsilon) + \frac{1}{\varepsilon^2}\nabla_U(p(|U^\varepsilon|)) = \\
\mathcal{L}(U^\varepsilon) + \mathcal{R}^\varepsilon(U^\varepsilon, \partial_t U^\varepsilon) + J\boldsymbol{m}^\varepsilon \text{ in } Q \\
U^\varepsilon(0, X) = \boldsymbol{M}_0(X) \text{ in } \Omega,\ |\boldsymbol{M}_0(X)|^2 = 1 \ \text{ a.e. in } \Omega \\
U^\varepsilon(t, h+0, \widehat{x}) = U^\varepsilon(t, -h-0, \widehat{x}) \text{ in } (0,T)\times\widehat{\Omega} \\
a_1\partial_x U^\varepsilon(t, h+0, \widehat{x}) = a_2\partial_x U^\varepsilon(t, -h-0, \widehat{x}) \text{ in } (0,T)\times\widehat{\Omega} \\
(a\nabla\cdot n)U^\varepsilon = 0 \text{ on } (0,T)\times(\partial\Omega\setminus(\Gamma^1\cup\Gamma^2)).
\end{cases}
\tag{3.23}
$$

The "spin accumulation" field $\boldsymbol{m}^\varepsilon$ satisfies the heat equation

$$
\begin{cases}
\frac{1}{d^2}\partial_t\boldsymbol{m}^\varepsilon - \partial_x(\mathcal{A}^\varepsilon(\langle U^\varepsilon\rangle)(\partial_x\boldsymbol{m}^\varepsilon)) + \frac{1}{\lambda_{\text{sf}}^2}\boldsymbol{m}^\varepsilon = \\
-\frac{1}{\lambda_J^2}\boldsymbol{m}^\varepsilon\times\mathcal{T}^\varepsilon(U^\varepsilon) - \frac{\beta}{d^2}\partial_x(j_e\mathcal{T}^\varepsilon(U^\varepsilon)) \ \text{ in } \omega \\
\boldsymbol{m}^\varepsilon(0,x) = \boldsymbol{m}_0(x) \ \text{ in } I \\
j^\varepsilon(U^\varepsilon, \boldsymbol{m}^\varepsilon)(t, -L) = j^\varepsilon(U^\varepsilon, \boldsymbol{m}^\varepsilon)(t, l) = 0 \ \text{ in } (0,T)
\end{cases}
\tag{3.24}
$$

where we set

$$
\begin{cases}
\mathcal{L}(U^\varepsilon) = \nabla\varphi^\varepsilon - \nabla_U\psi(U^\varepsilon),\ \mathcal{R}^\varepsilon(V,W) = \frac{1}{\gamma}\,\mathcal{S}^\varepsilon(V)\times W \\
\mathcal{S}^\varepsilon(\xi) = \xi/(\varepsilon+|\xi|),\ \forall\xi\in\mathbb{R}^3,\ \mathcal{T}^\varepsilon(V) = \langle V\rangle/(\varepsilon+\langle|V|\rangle) \\
p(r) = |\sqrt{1+r^2}-\sqrt{2}|^2, r\in\mathbb{R}^+ \\
\mathcal{A}^\varepsilon(\langle V\rangle)\xi = \xi - \beta\beta'(\mathcal{T}^\varepsilon(V)\cdot\xi)\mathcal{T}^\varepsilon(V),\ \forall\xi\in\mathbb{R}^3 \\
j^\varepsilon(V,\boldsymbol{m}) = \beta j_e\mathcal{T}^\varepsilon(V) - d^2\mathcal{A}^\varepsilon(\langle V\rangle)(\partial_x\boldsymbol{m}).
\end{cases}
\tag{3.25}
$$

Operator $\mathcal{L}$ is Lipschitz continuous from $\mathbb{L}^2(\Omega)$ into $\mathbb{L}^2(\Omega)$ by using the linearity of the stray field equation and the bound, $|D^2\psi(\xi)|\le C$ for all ξ, of ψ. We have for all $U,V\in\mathbb{L}^2(\Omega)$ the estimate

$$|\mathcal{L}(U)-\mathcal{L}(V)|_\Omega \le C|U-V|_\Omega \tag{3.26}$$

where $C>0$ represents various constants which are independent of ε. Moreover, operator $\mathcal{R}^\varepsilon(U^\varepsilon,\partial_t U^\varepsilon)$ satisfies locally the orthogonality property $\mathcal{R}^\varepsilon(U^\varepsilon,\partial_t U^\varepsilon)\cdot U^\varepsilon = \mathcal{R}^\varepsilon(U^\varepsilon,\partial_t U^\varepsilon)\cdot\partial_t U^\varepsilon = 0$. As we have for all $r\ge 0$

$$r^2 - 1 = p(r) + 2\sqrt{2}(\sqrt{r^2+1} - \sqrt{2}) \tag{3.27}$$

we can deduce the $\mathbb{L}^2(\Omega)$ bound for U knowing the $L^1(\Omega)$ bound of $p(|U|)$. The matrix operator $\mathcal{A}^\varepsilon$ satisfies for all $\xi \in \mathbb{R}^3$ the bounds

$$\mathcal{A}^\varepsilon(\langle V\rangle)(\xi)\cdot\xi \geq (1-\beta\beta')|\xi|^2,\ |\mathcal{A}^\varepsilon(\langle V\rangle)(\xi)| \leq (1+\beta\beta')|\xi| \tag{3.28}$$

and $\mathcal{T}^\varepsilon$ satisfies for all functions $U_1, U_2 \in \mathbb{L}^\infty(Q)$, the estimates

$$\begin{cases} |\mathcal{T}^\varepsilon(U_1)| \leq 1,\ |\mathcal{T}^\varepsilon(U_1) - \mathcal{T}^\varepsilon(U_2)|_\infty \leq C_\varepsilon |U_1 - U_2|_\infty \\ |\mathcal{T}^\varepsilon(U_1)\otimes\mathcal{T}^\varepsilon(U_1) - \mathcal{T}^\varepsilon(U_2)\otimes\mathcal{T}^\varepsilon(U_2)|_\infty \leq C_\varepsilon |U_1 - U_2|_\infty \end{cases} \tag{3.29}$$

where $C_\varepsilon > 0$ is a constant depending only of ε, $|\ .\ |_\infty$ denotes the norm of the two spaces $\mathbb{L}^\infty(Q)$ and $\mathbb{L}^\infty(\omega)$ while $V\otimes W$ is the matrix $(V\otimes W)_{i,\ j} = V_iW_j, 1 \leq i,j \leq 3$ for all vectors $V, W \in \mathbb{R}^3$. Moreover, for all $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}^3$, we may write $(\mathcal{A}^\varepsilon(\eta_1)\xi_1 - \mathcal{A}^\varepsilon(\eta_2)\xi_2)\cdot(\xi_1-\xi_2) = \mathcal{A}^\varepsilon(\eta_1)(\xi_1-\xi_2)\cdot(\xi_1-\xi_2) - \beta\beta'(\mathcal{T}^\varepsilon(\eta_1)\otimes\mathcal{T}^\varepsilon(\eta_1) - \mathcal{T}^\varepsilon(\eta_2)\otimes\mathcal{T}^\varepsilon(\eta_2))\xi_2\cdot(\xi_1-\xi_2)$ and then we get for all functions $U_1, U_2 \in \mathbb{L}^\infty(Q), V_1, V_2 \in \mathbb{L}^2(\omega)$

$$\begin{cases} (\mathcal{A}^\varepsilon(\langle U_1\rangle)V_1 - \mathcal{A}^\varepsilon(\langle U_2\rangle)V_2; V_1 - V_2)_\omega \geq \dfrac{(1-\beta\beta')}{2}|V_1-V_2|^2_\omega \\ \qquad\qquad -\dfrac{(\beta\beta')^2}{2(1-\beta\beta')}C_\varepsilon^2|U_1-U_2|^2_\infty|V_2|^2_\omega. \end{cases} \tag{3.30}$$

Let us give another property satisfied by operator $\mathcal{R}^\varepsilon$. Since $\mathcal{S}^\varepsilon$ is Lipschitz continuous with Lipschitz constant C_ε then for all $U_1, U_2 \in \mathbb{L}^\infty(Q), V_1, V_2 \in \mathbb{L}^2(Q)$ we have $\gamma(\mathcal{R}^\varepsilon(U_1,V_1) - \mathcal{R}^\varepsilon(U_2,V_2); V_1 - V_2)_Q = ((\mathcal{S}^\varepsilon(U_1) - \mathcal{S}^\varepsilon(U_2))\times V_2; V_1-V_2)_Q$ and then we get

$$|(\mathcal{R}^\varepsilon(U_1,V_1) - \mathcal{R}^\varepsilon(U_2,V_2); V_1-V_2)_Q| \leq \frac{C_\varepsilon}{\gamma}|U_1-U_2|_\infty|V_2|_Q|V_1-V_2|_Q. \tag{3.31}$$

3.3.3 Solving the intermediary problem (3.23)-(3.24)

Let us consider for $\nu > 0$ fixed the hyperbolic regularization of the intermediary problem (3.23)

$$\begin{cases} \nu^2\partial_t^2 U_\nu^\varepsilon - \nabla\cdot(a\nabla U_\nu^\varepsilon) + \mathcal{B}^\varepsilon(\rho^\nu \star \widetilde{U}_\nu^\varepsilon, \partial_t U_\nu^\varepsilon) = \\ -\dfrac{1}{\varepsilon^2}\nabla_U(p(|U_\nu^\varepsilon|)) + \mathcal{L}(U_\nu^\varepsilon) + \boldsymbol{J}\boldsymbol{m}_\nu^\varepsilon \ \text{ in } Q \\ U_\nu^\varepsilon(0,X) = \boldsymbol{M}_0(X) \ \text{ in } \Omega,\ |\boldsymbol{M}_0(X)|^2 = 1 \text{ a.e},\ \nu^2\partial_t U_\nu^\varepsilon(0,X) = 0 \ \text{ in } \Omega \end{cases} \tag{3.32}$$

with U_ν^ε satisfying the boundary condition of problem (3.23). We set

$$\mathcal{B}^\varepsilon(U,V) = \frac{\alpha}{\gamma}\, V - \mathcal{R}^\varepsilon(U,V) \tag{3.33}$$

$\widetilde{U}$ denotes the extension of U by 0 outside Q, $\star$ is the convolution product with respect to (t,X) and ρ^ν is a regularizing sequence. Notice that operator $\mathcal{B}^\varepsilon$ satisfies, locally for all U and V, the property

$$\mathcal{B}^\varepsilon(U,V)\cdot V = \frac{\alpha}{\gamma}|V|^2. \tag{3.34}$$

The spin accumulation field $\boldsymbol{m}_\nu^\varepsilon$ satisfies

$$\begin{cases} \dfrac{1}{d^2}\partial_t \boldsymbol{m}_\nu^\varepsilon - \partial_x(\mathcal{A}^\varepsilon(\langle\rho^\nu \star \widetilde{U}_\nu^\varepsilon\rangle)(\partial_x \boldsymbol{m}_\nu^\varepsilon)) + \dfrac{1}{\lambda_{\text{sf}}^2}\boldsymbol{m}_\nu^\varepsilon = \\ -\dfrac{1}{\lambda_J^2}\boldsymbol{m}_\nu^\varepsilon \times \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{U}_\nu^\varepsilon) - \dfrac{\beta}{d^2}\partial_x(j_e \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{U}_\nu^\varepsilon)) \ \text{ in } \omega \\ \boldsymbol{m}_\nu^\varepsilon(0,x) = \boldsymbol{m}_0(x) \ \text{ in } I \\ j_\nu^\varepsilon(U_\nu^\varepsilon, \boldsymbol{m}_\nu^\varepsilon)(t,-L) = j_\nu^\varepsilon(U_\nu^\varepsilon, \boldsymbol{m}_\nu^\varepsilon)(t,l) = 0 \ \text{ in } (0,T) \end{cases} \tag{3.35}$$

where $j_\nu^\varepsilon(U,\boldsymbol{m}) = \beta j_e \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{U}) - d^2\mathcal{A}^\varepsilon(\langle\rho^\nu \star \widetilde{U}\rangle)(\partial_x \boldsymbol{m})$.

We fix ε and ν. In the sequel of this paragraph, we will show existence of solutions for the regularized problem (3.32)-(3.35) by applying a fixed point procedure. Let $V \in L^2(0,T;\mathbb{L}^2(\Omega))$ be fixed, we consider now the spin accumulation equation (3.35) associated with V

$$\begin{cases} \dfrac{1}{d^2}\partial_t \boldsymbol{m} - \partial_x(\mathcal{A}^\varepsilon(\langle\rho^\nu \star \widetilde{V}\rangle)(\partial_x \boldsymbol{m})) + \dfrac{1}{\lambda_{\text{sf}}^2}\boldsymbol{m} = \\ -\dfrac{1}{\lambda_J^2}\boldsymbol{m} \times \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{V}) - \dfrac{\beta}{d^2}\partial_x(j_e \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{V})) \ \text{ in } \omega \\ \boldsymbol{m}(0,x) = \boldsymbol{m}_0(x) \ \text{ in } I \\ j_\nu^\varepsilon(V, \boldsymbol{m})(t,-L) = j_\nu^\varepsilon(V, \boldsymbol{m})(t,l) = 0 \ \text{ in } (0,T). \end{cases} \tag{3.36}$$

The solution of (3.36) is such that $\boldsymbol{m} \in L^\infty(0,T;\mathbb{L}^2(I))$, $\partial_x \boldsymbol{m} \in L^2(0,T;\mathbb{L}^2(I))$ and $\partial_t \boldsymbol{m} \in L^2(0,T;\mathbb{H}^{-1}(I))$. Moreover, the following energy inequality holds

$$\left\{\begin{array}{l} \dfrac{1}{2d^2}|\boldsymbol{m}(t)|_I^2 + \dfrac{1}{\lambda_{\mathrm{sf}}^2}\displaystyle\int_0^t |\boldsymbol{m}(s)|_I^2 \,\mathrm{d}s + \dfrac{1-\beta\beta'}{2}\int_0^t |\partial_x \boldsymbol{m}(s)|_I^2 \,\mathrm{d}s \leq \\ \dfrac{1}{2d^2}|\boldsymbol{m}_0|_I^2 + \dfrac{\beta^2}{2d^4(1-\beta\beta')}\displaystyle\int_0^T |j_e|_I^2 \,\mathrm{d}s. \end{array}\right. \tag{3.37}$$

We define the map $\mathcal{Q} : V \in L^2(0,T;\mathbb{L}^2(\Omega)) \mapsto \boldsymbol{m} = \mathcal{Q}(V)$ solution of (3.36). We have

Lemma 3.3.2 *$\mathcal{Q}$ is Lipschitz continuous from $L^2(0,T;\mathbb{L}^2(\Omega))$ into $L^2(0,T;\mathbb{L}^2(I))$.*

Proof. Let $V_k \in L^2(0,T;\mathbb{L}^2(\Omega))$ and $\boldsymbol{m}_k = \mathcal{Q}(V_k)$ be the solution of (3.36) associated with V_k for $k = 1,2$. We set $\boldsymbol{m} = \boldsymbol{m}_1 - \boldsymbol{m}_2$ and $V = V_1 - V_2$. It follows that $\boldsymbol{m}$ satisfies the equation

$$\left\{\begin{array}{l} \dfrac{1}{d^2}\partial_t \boldsymbol{m} - \partial_x(\mathcal{A}^\varepsilon(\langle\rho^\nu \star \widetilde{V}_1\rangle)(\partial_x \boldsymbol{m}_1) - \mathcal{A}^\varepsilon(\langle\rho^\nu \star \widetilde{V}_2\rangle)(\partial_x \boldsymbol{m}_2)) + \dfrac{1}{\lambda_{\mathrm{sf}}^2}\boldsymbol{m} = \\ -\dfrac{1}{\lambda_J^2}\boldsymbol{m} \times \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{V}_1) - \dfrac{1}{\lambda_J^2}\boldsymbol{m}_2 \times (\mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{V}_1) - \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{V}_2)) \\ -\dfrac{\beta}{d^2}\partial_x(j_e(\mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{V}_1) - \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{V}_2))) \end{array}\right. \tag{3.38}$$

with homogeneous initial and boundary conditions. Multiplying by $\boldsymbol{m}$, integrating by parts and using the property satisfied by $\boldsymbol{\mathcal{A}}^\varepsilon$ we get

$$\left\{\begin{array}{l} \dfrac{1}{2d^2}|\boldsymbol{m}(t)|_I^2 + \dfrac{1}{2\lambda_{\mathrm{sf}}^2}\displaystyle\int_0^t |\boldsymbol{m}(s)|_I^2 \,\mathrm{d}s + \dfrac{1-\beta\beta'}{4}\int_0^t |\partial_x \boldsymbol{m}(s)|_I^2 \,\mathrm{d}s \leq \\ C_\varepsilon^2 |\rho^\nu \star \widetilde{V}|_\infty^2 \displaystyle\int_0^t (\dfrac{\lambda_{\mathrm{sf}}^2}{2\lambda_J^2}|\boldsymbol{m}_2|_I^2 + \dfrac{\beta^2|\beta'|^2}{2(1-\beta\beta')}|\partial_x \boldsymbol{m}_2|_I^2 + \dfrac{\beta^2}{(1-\beta\beta')d^4}|j_e|_I^2)\mathrm{d}s. \end{array}\right. \tag{3.39}$$

Finally arguing the fact that $|\rho^\nu \star \widetilde{V}|_\infty \leq C_\nu |V|_Q$ we deduce the continuity of the map $\mathcal{Q}$ from $L^2(0,T;\mathbb{L}^2(\Omega))$ into $L^\infty(0,T;\mathbb{L}^2(I)) \cap L^2(0,T;\mathbb{H}^1(I))$. The Lipschitz continuity in these spaces is then a consequence of (3.37).

Let us consider now for V fixed in $L^2(0,T;\mathbb{L}^2(\Omega))$ the equation

$$\begin{cases} \nu^2 \partial_t^2 U - \nabla \cdot (a \nabla U) + \mathcal{B}^\varepsilon(\rho^\nu \star \widetilde{V}, \partial_t U) = -\dfrac{1}{\varepsilon^2} \nabla_U (p(|U|)) + \\ \mathcal{L}(U) + J\mathcal{Q}(V) \quad \text{in } Q \\ U(0,X) = \boldsymbol{M}_0(X) \quad \text{in } \Omega, \ |\boldsymbol{M}_0(X)|^2 = 1 \text{ a.e}, \ \nu^2 \partial_t U(0,X) = 0 \quad \text{in } \Omega \end{cases} \tag{3.40}$$

with the same boundary conditions as in (3.23). Since the right-hand side of the equation is Lipschitz continuous (with respect to U) from $L^2(0,T;\mathbb{L}^2(\Omega))$ into $L^2(0,T;\mathbb{L}^2(\Omega))$ then the wave equation (3.40) admits a unique global weak solution U with finite energy see [6], [32]. It satisfies the estimate

$$\begin{cases} \nu^2 |\partial_t U(t)|_\Omega^2 + \mathcal{E}(U(t)) + \dfrac{2}{\varepsilon^2} \displaystyle\int_\Omega p(|U(t)|) \, \mathrm{d}X \\ + \dfrac{\alpha}{\gamma} \displaystyle\int_0^t |\partial_t U(s)|_\Omega^2 \, \mathrm{d}s \leq \mathcal{E}(\boldsymbol{M}_0) + \dfrac{J^2 |\widehat{\Omega}| \gamma}{\alpha} \displaystyle\int_0^t |\mathcal{Q}(V)(s)|_I^2 \, \mathrm{d}s. \end{cases} \tag{3.41}$$

Let $\mathcal{K}$ be the map defined by $\mathcal{K}(V) = U$, U solution of (3.40). We have the result

Lemma 3.3.3 *Operator $\mathcal{K} : L^2(0,T;\mathbb{L}^2(\Omega)) \to L^2(0,T;\mathbb{L}^2(\Omega))$ is continuous and compact.*

Proof. For ν and ε fixed, operator $\mathcal{K}$ is bounded from $L^2(0,T;\mathbb{L}^2(\Omega))$ into the Banach space $\mathcal{X} = L^\infty(0,T;\mathbb{H}^1(\Omega)) \cap H^1(0,T;\mathbb{L}^2(\Omega))$. By using Aubin's compactness lemma it follows that $\mathcal{X}$ is compactly embedded into $L^2(0,T;\mathbb{L}^2(\Omega))$ and then $\mathcal{K}$ is compact from $L^2(0,T;\mathbb{L}^2(\Omega))$ into itself. Let us prove that $\mathcal{K}$ is continuous. Let $V_k \in L^2(0,T;\mathbb{L}^2(\Omega))$ $k = 1,2$ and $U_k = \mathcal{K}(V_k)$ be the solution of (3.40). We set $V = V_1 - V_2$ and $U = U_1 - U_2$. Hence U satisfies the equation

$$\begin{cases} \nu^2 \partial_t^2 U - \nabla \cdot (a \nabla U) + \dfrac{\alpha}{\gamma} \, \partial_t U = \mathcal{R}^\varepsilon(\rho^\nu \star \widetilde{V}_1, \partial_t U_1) - \\ \mathcal{R}^\varepsilon(\rho^\nu \star \widetilde{V}_2, \partial_t U_2) - \dfrac{1}{\varepsilon^2} [\nabla_U (p(|U_1|)) - \nabla_U (p(|U_2|))] + \mathcal{L}(U_1) - \\ \mathcal{L}(U_2) + J(\mathcal{Q}(V_1) - \mathcal{Q}(V_2)) \end{cases} \tag{3.42}$$

with homogeneous initial condition and the same boundary conditions as in (3.23). Multiplying this equation by $\partial_t U$, integrating by parts, using the Lipchitz continuity of $\nabla_U(p(|U|))$ and $\nabla_U \psi$ and the property satisfied by $\mathcal{R}^\varepsilon$ we get

$$\begin{cases} \nu^2|\partial_t U(t)|^2_\Omega + |\sqrt{a}\ \nabla U(t)|^2_\Omega + |\nabla\varphi(t)|^2_\Omega + \dfrac{\alpha}{\gamma}\displaystyle\int_0^t |\partial_t U(s)|^2_\Omega \ \mathrm{d}s \leq \\ +\dfrac{3J^2|\widehat{\Omega}|\gamma}{\alpha}\displaystyle\int_0^t |\mathcal{Q}(V_1)-\mathcal{Q}(V_2)|^2_I \ \mathrm{d}s + \dfrac{3C_\varepsilon^2}{\alpha\gamma}|\rho^\nu \star \widetilde{V}|^2_\infty \displaystyle\int_0^T |\partial_t U_2(s)|^2_\Omega \ \mathrm{d}s \\ +\dfrac{3C_\varepsilon^2\gamma}{\alpha}\displaystyle\int_0^t |U(s)|^2_\Omega \ \mathrm{d}s \end{cases} \tag{3.43}$$

where φ is the solution of the stray equation associated with U. Next, using the inequality $|U(s)|^2 \leq \int_0^s |U(\alpha)|^2 \ \mathrm{d}\alpha \ + \int_0^t |\partial_t U(\alpha)|^2 \ \mathrm{d}\alpha$ for all $0 \leq s \leq t \leq T$ we get the estimate $|U(s)|^2 \leq (1+se^s)\int_0^t |\partial_t U(\alpha)|^2 \ \mathrm{d}\alpha$ and finally the previous energy estimate implies the following

$$\nu^2|\partial_t U(t)|^2_\Omega \leq C|\mathcal{Q}(V_1)-\mathcal{Q}(V_2)|^2_\omega + C_{\varepsilon,T}|\rho^\nu \star \widetilde{V}|^2_\infty + C_{\varepsilon,T}\int_0^t |\partial_t U(s)|^2_\Omega \ \mathrm{d}s. \tag{3.44}$$

Then using Gronwall inequality we obtain

$$|\partial_t U(t)|^2_\Omega \leq C_{\varepsilon,\nu,T}(|\mathcal{Q}(V_1)-\mathcal{Q}(V_2)|^2_\omega + |\rho^\nu \star \widetilde{V}|^2_\infty) \tag{3.45}$$

where C, $C_{\varepsilon,T}$ and $C_{\varepsilon,\nu,T}$ are various positive constants. Finally since we have $|\rho^\nu \star \widetilde{V}|_\infty \leq C_\nu |V|_{L^2(0,T;\mathbb{L}^2(\Omega))}$ and operator $\mathcal{Q}$ is Lipschitz continuous by the previous lemma we conclude that

$$\begin{cases} |\partial_t U(t)|^2_\Omega \leq C_{\varepsilon,\nu,T}\int_0^T |V(s)|^2_\Omega \ \mathrm{d}s \\ |U(t)|^2_\Omega \leq C_{\varepsilon,\nu,T}\int_0^T |V(s)|^2_\Omega \ \mathrm{d}s. \end{cases} \tag{3.46}$$

At the end we return to the energy estimate to deduce that

$$|\nabla U(t)|^2_\Omega \leq C_{\varepsilon,\nu,T}\int_0^T |V(s)|^2_\Omega \ \mathrm{d}s. \tag{3.47}$$

The continuity of operator $\mathcal{K}$ is then proved from $L^2(0,T;\mathbb{L}^2(\Omega))$ into $\mathcal{X}$ and consequently operator $\mathcal{K}$ is continuous and compact from $L^2(0,T;\mathbb{L}^2(\Omega))$ into itself.

Finally, by Schauder's fixed point theorem, there exists a fixed point $U_\nu^\varepsilon \in L^2(0,T;\mathbb{L}^2(\Omega))$ satisfying $\mathcal{K}(U_\nu^\varepsilon) = U_\nu^\varepsilon$. Moreover setting $\boldsymbol{m}_\nu^\varepsilon = \mathcal{Q}(U_\nu^\varepsilon)$ it follows that $U_\nu^\varepsilon \in \mathcal{X}$, $\boldsymbol{m}_\nu^\varepsilon \in L^\infty(0,T;\mathbb{L}^2(I)) \cap L^2(0,T;\mathbb{H}^1(I))$ and $(U_\nu^\varepsilon, \boldsymbol{m}_\nu^\varepsilon)$ is a weak solution of (3.32)-(3.35).

We proved the following result

Theorem 3.3.4 *Let $\nu > 0$ and $\varepsilon > 0$ be fixed. Let $\boldsymbol{m}_0 \in \mathbb{L}^2(I)$, $\boldsymbol{M}_0 \in \mathbb{H}^1(\Omega)$ such that $|\boldsymbol{M}_0(x)|^2 = 1$ a.e and $j_e \in L^2(\omega)$. Then there exists a weak solution $(\boldsymbol{m}_\nu^\varepsilon, U_\nu^\varepsilon) \in L^2(0,T;\mathbb{H}^1(I)) \times \mathbb{H}^1(Q)$ of the coupled problem (3.32)-(3.35). Moreover we have $\boldsymbol{m}_\nu^\varepsilon \in L^\infty(0,T;\mathbb{L}^2(I))$, $U_\nu^\varepsilon \in L^\infty(0,T;\mathbb{L}^2(\Omega))$ and the following energy inequalities hold*

$$\left\{ \begin{array}{l} \nu^2|\partial_t U_\nu^\varepsilon(t)|_\Omega^2 + \mathcal{E}(U_\nu^\varepsilon(t)) + \dfrac{2}{\varepsilon^2}\displaystyle\int_\Omega p(|U_\nu^\varepsilon(t)|)\ \mathrm{d}X \\ +\dfrac{\alpha}{\gamma}\displaystyle\int_0^t |\partial_t U_\nu^\varepsilon(s)|_\Omega^2\ \mathrm{d}s \le \mathcal{E}(\boldsymbol{M}_0) + \dfrac{J^2|\widehat{\Omega}|\gamma}{\alpha}\displaystyle\int_0^t |\boldsymbol{m}_\nu^\varepsilon(s)|_I^2\ \mathrm{d}s \end{array} \right. \tag{3.48}$$

and

$$\left\{ \begin{array}{l} \dfrac{1}{2d^2}|\boldsymbol{m}_\nu^\varepsilon(t)|_I^2 + \dfrac{1}{\lambda_{\mathrm{sf}}^2}\displaystyle\int_0^t |\boldsymbol{m}_\nu^\varepsilon(s)|_I^2\ \mathrm{d}s + \dfrac{1-\beta\beta'}{2}\displaystyle\int_0^t |\partial_x \boldsymbol{m}_\nu^\varepsilon(s)|_I^2\ \mathrm{d}s \le \\ \dfrac{1}{2d^2}|\boldsymbol{m}_0|_I^2 + \dfrac{\beta^2}{2d^4(1-\beta\beta')}\displaystyle\int_0^T |j_e|_I^2\ \mathrm{d}s \end{array} \right. \tag{3.49}$$

for all $t \in [0,T]$.

3.3.4 Passing to the limit for $\nu \to 0$

The estimates deduced from (3.48) and (3.49) allow to pass to the limit as $\nu \to 0$ in the coupled problem (3.32)-(3.35). We have

Lemma 3.3.5 *There exists $C > 0$ which is independent of ν and ε such that*

$$\left\{ \begin{array}{l} |U_\nu^\varepsilon|_{L^\infty(0,T;\mathbb{H}^1(\Omega))} + |\partial_t U_\nu^\varepsilon|_{L^2(0,T;\mathbb{L}^2(\Omega))} + |\nabla \varphi_\nu^\varepsilon|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \le C \\ |\boldsymbol{m}_\nu^\varepsilon|_{L^2(0,T;\mathbb{H}^1(I))} + |\partial_t \boldsymbol{m}_\nu^\varepsilon|_{L^2(0,T;\mathbb{H}^{-1}(I))} \le C. \end{array} \right. \tag{3.50}$$

The $\mathbb{L}^2$-bound of U_ν^ε is deduced from the inequality (associated with the function $p(r)$)

$$|U_\nu^\varepsilon(t)|_\Omega^2 \le |\Omega| + \int_\Omega p(|U_\nu^\varepsilon(t)|)\ \mathrm{d}X + 2\sqrt{2}\int_\Omega \sqrt{p(|U_\nu^\varepsilon(t)|)}\ \mathrm{d}X. \tag{3.51}$$

Therefore we can extract subsequences such that U_ν^ε converges towards a limit U^ε weakly in $\mathbb{H}^1(Q)$ and $\boldsymbol{m}_\nu^\varepsilon$ converges to a limit $\boldsymbol{m}^\varepsilon$ weakly in $L^2(0,T;\mathbb{H}^1(I))$. Moreover as a consequence of Aubin's compactness lemma we have the strong convergences

Lemma 3.3.6 *Let $\varepsilon > 0$ be fixed. Under the hypotheses of theorem 3.3.4 there exists subsequences U^ε_ν and $\boldsymbol{m}^\varepsilon_\nu$ such that*

$$\begin{cases} U^\varepsilon_\nu \to U^\varepsilon \text{ strongly in } L^2(0,T;\mathbb{L}^2(\Omega)) \\ \rho^\nu \star \widetilde{U}^\varepsilon_\nu \to \widetilde{U}^\varepsilon \text{ strongly in } \mathbb{L}^2(\mathbb{R}^4) \\ \boldsymbol{m}^\varepsilon_\nu \to \boldsymbol{m}^\varepsilon \text{ strongly in } L^2(0,T;\mathbb{L}^2(I)). \end{cases} \tag{3.52}$$

Since $\mathcal{S}^\varepsilon$ is Lipschitz continuous, we get that $\mathcal{S}^\varepsilon(\rho^\nu \star \widetilde{U}^\varepsilon_\nu) \to \mathcal{S}^\varepsilon(U^\varepsilon)$ strongly in $\mathbb{L}^2(Q)$ which allows the passage to the limit in (3.32). Therefore the equation (3.23) is satisfied in the sense of distributions and the boundary conditions are fulfilled. Moreover the property of $\mathcal{T}^\varepsilon$ leads to the weak convergence

$$\mathcal{A}^\varepsilon(\langle \rho^\nu \star \widetilde{U}^\varepsilon_\nu\rangle)(\partial_x \boldsymbol{m}^\varepsilon_\nu) \rightharpoonup \mathcal{A}^\varepsilon(\langle U^\varepsilon\rangle)(\partial_x \boldsymbol{m}^\varepsilon) \quad \text{in } \mathbb{L}^2(\omega) \tag{3.53}$$

as well as the strong convergences

$$\boldsymbol{m}^\varepsilon_\nu \times \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{U}^\varepsilon_\nu) \to \boldsymbol{m}^\varepsilon \times \mathcal{T}^\varepsilon(U^\varepsilon) \quad \text{in } \mathbb{L}^2(\omega) \tag{3.54}$$

$$j_e \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{U}^\varepsilon_\nu) \to j_e \mathcal{T}^\varepsilon(U^\varepsilon) \quad \text{in } \mathbb{L}^2(\omega). \tag{3.55}$$

Thus $\boldsymbol{m}^\varepsilon$ satisfies in the sense of distributions the equation

$$\begin{cases} \dfrac{1}{d^2}\partial_t \boldsymbol{m}^\varepsilon - \partial_x(\mathcal{A}^\varepsilon(\langle U^\varepsilon\rangle)(\partial_x \boldsymbol{m}^\varepsilon)) + \dfrac{1}{\lambda^2_{\mathrm{sf}}}\boldsymbol{m}^\varepsilon = \\ -\dfrac{1}{\lambda^2_J}\boldsymbol{m}^\varepsilon \times \mathcal{T}^\varepsilon(U^\varepsilon) - \dfrac{\beta}{d^2}\partial_x(j_e \mathcal{T}^\varepsilon(U^\varepsilon)). \end{cases} \tag{3.56}$$

Consequently $\dfrac{1}{d^2}\partial_t \boldsymbol{m}^\varepsilon - \partial_x(\mathcal{A}^\varepsilon(\langle U^\varepsilon\rangle)(\partial_x \boldsymbol{m}^\varepsilon) + \dfrac{\beta}{d^2}\partial_x(j_e \mathcal{T}^\varepsilon(U^\varepsilon)) \in \mathbb{L}^2(\omega)$ so as if we set $\mathcal{J}^\varepsilon = -\mathcal{A}^\varepsilon(\langle U^\varepsilon\rangle)(\partial_x \boldsymbol{m}^\varepsilon) + \dfrac{\beta}{d^2} j_e \mathcal{T}^\varepsilon(U^\varepsilon)$, we have $\left(d^{-2}\boldsymbol{m}^\varepsilon,\ \mathcal{J}^\varepsilon\right) \cdot N\Big|_{\partial\omega}$ which is well defined in $\mathbb{H}^{-1/2}(\partial\omega)$. Here N denotes the outward unit normal to the boundary $\partial\omega$ of ω. Let $\theta \in \mathbb{H}^{1/2}(\partial\omega)$ and $\eta \in \mathbb{H}^1(\omega)$ such that $\eta\Big|_{\partial\omega} = \theta$. Thus setting $\mathcal{J}^\varepsilon_\nu = -\mathcal{A}^\varepsilon(\langle \rho^\nu \star \widetilde{U}^\varepsilon_\nu\rangle)(\partial_x \boldsymbol{m}^\varepsilon_\nu) + \dfrac{\beta}{d^2} j_e \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{U}^\varepsilon_\nu)$ and $\mathcal{D}^\varepsilon_\nu = (d^{-2}(\boldsymbol{m}^\varepsilon_\nu - m^\varepsilon),\ \mathcal{J}^\varepsilon_\nu - \mathcal{J}^\varepsilon)$, we have

$$\int_{\partial\omega} \theta \mathcal{D}^\varepsilon_\nu \cdot\ N \ \ \mathrm{d}\sigma = \int_\omega \nabla\eta.\mathcal{D}^\varepsilon_\nu \mathrm{d}x\mathrm{d}t + \int_\omega \eta\nabla\cdot \mathcal{D}^\varepsilon_\nu \mathrm{d}x\mathrm{d}t$$

where $d\sigma$ is the Lebesgue measure on $\partial\omega$ and $\nabla = (\partial t, \partial x)$. We get

$$\begin{cases} \displaystyle\int_{\partial\omega} \theta \mathcal{D}^\varepsilon_\nu \cdot\ N \ \ \mathrm{d}\sigma = \int_\omega \nabla\eta.\mathcal{D}^\varepsilon_\nu \mathrm{d}x\mathrm{d}t - (1/\lambda^2_{\mathrm{sf}}) \int_\omega \eta(\boldsymbol{m}^\varepsilon_\nu - \boldsymbol{m}^\varepsilon)\mathrm{d}x\mathrm{d}t \\ \displaystyle -(1/\lambda^2_J) \int_\omega \eta(\boldsymbol{m}^\varepsilon_\nu \times \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{U}^\varepsilon_\nu) - \boldsymbol{m}^\varepsilon \times \mathcal{T}^\varepsilon(U^\varepsilon))\mathrm{d}x\mathrm{d}t \end{cases}$$

which leads to $\lim_{\nu\to 0}\int_{\partial\omega}\theta\mathcal{D}^\varepsilon_\nu\cdot N\ \mathrm{d}\sigma = 0$ that is

$$\begin{cases} \lim_{\nu\to 0}\Big(\frac{1}{d^2}\int_I \theta(T,x)(\boldsymbol{m}^\varepsilon_\nu - m^\varepsilon)(T,x)\mathrm{d}x - \frac{1}{d^2}\int_I \theta(0,x)(\boldsymbol{m}_0(x) - \boldsymbol{m}^\varepsilon(0,x))\mathrm{d}x \\ + \int_0^T \theta(t,l)(\mathcal{J}^\varepsilon_\nu - \mathcal{J}^\varepsilon)(t,l)\mathrm{d}t - \int_0^T \theta(t,-L)(\mathcal{J}^\varepsilon_\nu - \mathcal{J}^\varepsilon)(t,-L)\mathrm{d}t\Big) = 0 \end{cases}$$

Then for adequate test functions θ, we get the desired boundary conditions so m^ε satisfies (3.24). We proved the following result

Theorem 3.3.7 *Let $\varepsilon > 0$ be fixed. Let $\boldsymbol{m}_0 \in \mathbb{L}^2(I)$, $\boldsymbol{M}_0 \in \mathbb{H}^1(\Omega)$ such that $|\boldsymbol{M}_0(X)|^2 = 1$ a.e and $j_e \in L^2(\omega)$. Then there exists a weak solution $(\boldsymbol{m}^\varepsilon, U^\varepsilon) \in L^2(0,T;\mathbb{H}^1(I)) \times \mathbb{H}^1(Q)$ of the coupled problem (3.23)-(3.24). Moreover we have $\boldsymbol{m}^\varepsilon \in L^\infty(0,T;\mathbb{L}^2(I))$, $U^\varepsilon \in L^\infty(0,T;\mathbb{H}^1(\Omega))$ and the following energy inequalities hold*

$$\begin{cases} |\sqrt{a}\nabla U^\varepsilon(t)|^2_\Omega + \frac{2}{\varepsilon^2}\int_\Omega p(|U^\varepsilon(t)|)\ \mathrm{d}X \\ +|\nabla\varphi^\varepsilon(t)|^2_{\mathbb{R}^3} + 2\int_\Omega \psi(U^\varepsilon(t))\ \mathrm{d}X + \frac{\alpha}{\gamma}\int_0^t |\partial_t U^\varepsilon(s)|^2_\Omega\ \mathrm{d}s \leq \\ \mathcal{E}(\boldsymbol{M}_0) + \frac{J^2|\widehat{\Omega}|\gamma}{\alpha}\int_0^t |\boldsymbol{m}^\varepsilon(s)|^2_I\ \mathrm{d}s \end{cases} \tag{3.57}$$

and

$$\begin{cases} \frac{1}{2d^2}|\boldsymbol{m}^\varepsilon(t)|^2_I + \frac{1}{\lambda^2_{\mathrm{sf}}}\int_0^t |\boldsymbol{m}^\varepsilon(s)|^2_I\ \mathrm{d}s + \frac{1-\beta\beta'}{2}\int_0^t |\partial_x\boldsymbol{m}^\varepsilon(s)|^2_I\ \mathrm{d}s \leq \\ \frac{1}{2d^2}|\boldsymbol{m}_0|^2_I + \frac{\beta^2}{2d^4(1-\beta\beta')}\int_0^T |j_e|^2_I\ \mathrm{d}s \end{cases} \tag{3.58}$$

for $t \in [0,T]$.

3.3.5 Global solutions to (3.8)-(3.3)

We want to pass to the limit as $\varepsilon \to 0$ in the system (3.23)-(3.24). The estimate satisfied by U^ε given in theorem 3.3.7 is rewritten as

$$\begin{aligned} &\mathcal{E}^\varepsilon(U^\varepsilon(t)) + \frac{\alpha}{\gamma}\int_0^t |\partial_t U^\varepsilon|_\Omega\ \mathrm{d}s + \frac{2}{\varepsilon^2}\int_\Omega p(|U^\varepsilon(t)|)\ \mathrm{d}X \leq \\ &\mathcal{E}(\boldsymbol{M}_0) + C_0|\boldsymbol{m}_0|^2_I + C_1\int_0^T |j_e|^2_I\ \mathrm{d}s \end{aligned} \tag{3.59}$$

where $C_0 > 0$ and $C_1 > 0$ are two positive constants which are independent of ε and T. It follows that

$$\int_\Omega p(|U^\varepsilon(t)|)\ \mathrm{d}X \leq C\varepsilon^2. \tag{3.60}$$

Since we have $r^2 - 1 = p(r) + 2\sqrt{2}(\sqrt{1+r^2} - \sqrt{2})$ then we get

$$\int_\Omega ||U^\varepsilon|^2 - 1|\ \mathrm{d}X \leq C(\varepsilon^2 + |\Omega|^{1/2}\varepsilon) \tag{3.61}$$

so the sequence $(U^\varepsilon)_\varepsilon$ is bounded in $L^\infty(0,T;\mathbb{L}^2(\Omega))$. It follows that there exists a subsequence of $(U^\varepsilon)_\varepsilon$ and $\boldsymbol{M} \in \mathbb{L}^2(Q)$ such that we have

$$\begin{cases} U^\varepsilon \rightharpoonup \boldsymbol{M} \text{ weakly in } \mathbb{L}^2(Q) \\ |U^\varepsilon|^2 \to 1 \text{ strongly in } \mathbb{L}^1(Q). \end{cases} \tag{3.62}$$

The energy inequality leads also to

$$|\partial_t U^\varepsilon|_{L^2(0,T;\mathbb{L}^2(\Omega))} + |\nabla U^\varepsilon|_{L^\infty(0,T;\mathbb{L}^2(\Omega))} \leq C \tag{3.63}$$

which implies by using the classical compactness lemma the strong convergence

$$U^\varepsilon \to \boldsymbol{M} \text{ strongly in } L^2(0,T;\mathbb{L}^2(\Omega)) \tag{3.64}$$

so we have

$$|\boldsymbol{M}(t,X)|^2 = 1 \ \text{ a.e in } Q. \tag{3.65}$$

We pass to the limit in the weak formulation of the intermediary problem (3.23) satisfied by U^ε. We use test functions of the type $\phi \times U^\varepsilon$ where ϕ is a test function defined in Q satisfying $\phi(t, h+0, \widehat{x}) = \phi(t, -h-0, \widehat{x})$ in $(0,T) \times \widehat{\Omega}$. We have

$$\begin{cases} \dfrac{\alpha}{\gamma}\displaystyle\int_Q \partial_t U^\varepsilon \cdot \phi \times U^\varepsilon\ \mathrm{d}X\mathrm{d}t + \int_Q a(X)\nabla U^\varepsilon \cdot \nabla\phi \times U^\varepsilon\ \mathrm{d}X\mathrm{d}t - \\ \displaystyle\int_Q \mathcal{R}^\varepsilon(U^\varepsilon, \partial_t U^\varepsilon) \cdot \phi \times U^\varepsilon\ \mathrm{d}X\mathrm{d}t = \\ \displaystyle\int_Q \mathcal{L}(U^\varepsilon) \cdot \phi \times U^\varepsilon\ \mathrm{d}X\mathrm{d}t + J\int_Q \boldsymbol{m}^\varepsilon \cdot \phi \times U^\varepsilon\ \mathrm{d}X\mathrm{d}t. \end{cases} \tag{3.66}$$

We get the following result

Theorem 3.3.8 *Let $T > 0$ be fixed and $\boldsymbol{M}_0 \in \mathbb{H}^1(\Omega)$ be such that $|\boldsymbol{M}_0(X)|^2 = 1$ a.e, $\boldsymbol{m}_0 \in \mathbb{L}^2(I)$ and $j_e \in L^2(\omega)$. Then, there exists a global weak solution $(\boldsymbol{m}, \boldsymbol{M})$ of the coupled problem (3.8)-(3.12)-(3.3)-(3.11) such that $\boldsymbol{M} \in \mathbb{L}^\infty(Q) \cap \mathbb{H}^1(Q)$, $|\boldsymbol{M}(t,X)|^2 = 1$ a.e, $\boldsymbol{m} \in L^\infty(0,T;\mathbb{L}^2(I)) \cap L^2(0,T;\mathbb{H}^1(I))$ and satisfying the energy inequalities*

$$\begin{gathered}\mathcal{E}(\boldsymbol{M}(t)) + \frac{\alpha}{\gamma}\int_0^t |\partial_t \boldsymbol{M}(s)|_\Omega^2 \,\mathrm{d}s \leq \mathcal{E}(\boldsymbol{M}_0) + C_0|\boldsymbol{m}_0|_I^2 + C_1\int_0^T |j_e|_I^2 \,\mathrm{d}s \\ |\boldsymbol{m}(t)|_I^2 + C_2\int_0^t |\boldsymbol{m}(s)|_I^2 \,\mathrm{d}s + \frac{1-\beta\beta'}{2}\int_0^t |\partial_x \boldsymbol{m}(s)|_I^2 \,\mathrm{d}s \leq \\ C_0|\boldsymbol{m}_0|_I^2 + C_1\int_0^T |j_e|_I^2 \,\mathrm{d}s\end{gathered} \tag{3.67}$$

for all $t \in [0,T]$ where $C_k > 0$ for $k = 0,4$ are constants which are independent of $(\boldsymbol{m}, \boldsymbol{M})$ and T.

Proof. It is easy to see that the operator $M \mapsto \nabla\varphi$ with $\nabla\cdot(\nabla\varphi + \chi(\Omega)M) = 0$ in $\mathbb{R}^3$ is linear continuous from $\mathbb{L}^2$ into itself. Therefore we can pass to the limit in the stray field equation by using the strong convergence of U^ε to obtain that the strong limit $\nabla\varphi$ of $\nabla\varphi^\varepsilon$ in $\mathbb{L}^2(Q)$ satisfies the stray field equation (3.10). Moreover the Lipschitz continuity of $\nabla_U\psi$ allows passing to the limit in the bulk anisotropy field $\nabla_U\psi(U^\varepsilon)$ so that we have $\mathcal{L}(U^\varepsilon) \to \mathcal{L}(\boldsymbol{M})$ strongly in $L^2(0,T;\mathbb{L}^2(\Omega))$. Next, for all $\phi \in \mathbb{L}^2(Q)$ we have $\mathcal{S}^\varepsilon(U^\varepsilon) \times \phi \to \boldsymbol{M} \times \phi$ a.e and strongly in $L^2(0,T;\mathbb{L}^2(\Omega))$. Consequently $\mathcal{R}^\varepsilon(U^\varepsilon, \partial_t U^\varepsilon) \rightharpoonup \frac{1}{\gamma}\boldsymbol{M} \times \partial_t \boldsymbol{M}$ weakly in $L^2(0,T;\mathbb{L}^2(\Omega))$ and then $\int_Q \mathcal{R}^\varepsilon(U^\varepsilon, \partial_t U^\varepsilon) \cdot \phi \times U^\varepsilon \,\mathrm{d}X\mathrm{d}t \to \frac{1}{\gamma}\int_Q \boldsymbol{M} \times \partial_t \boldsymbol{M} \cdot \phi \times \boldsymbol{M} \,\mathrm{d}X\mathrm{d}t$. We pass to the limit in (3.66) to get

$$\left\{\begin{array}{l}\dfrac{\alpha}{\gamma}\displaystyle\int_Q \partial_t \boldsymbol{M} \cdot \phi \times \boldsymbol{M} \,\mathrm{d}X\mathrm{d}t + \int_Q a(X)\nabla\boldsymbol{M} \cdot \nabla\phi \times \boldsymbol{M} \,\mathrm{d}X\mathrm{d}t - \\ \dfrac{1}{\gamma}\displaystyle\int_Q \boldsymbol{M} \times \partial_t \boldsymbol{M} \cdot \phi \times \boldsymbol{M} \,\mathrm{d}X\mathrm{d}t = \\ \displaystyle\int_Q \mathcal{L}(\boldsymbol{M}) \cdot \phi \times \boldsymbol{M} \,\mathrm{d}X\mathrm{d}t + J\int_Q \boldsymbol{m} \cdot \phi \times \boldsymbol{M} \,\mathrm{d}X\mathrm{d}t\end{array}\right. \tag{3.68}$$

where $\boldsymbol{m}$ is the weak limit of $\boldsymbol{m}^\varepsilon$ in $L^2(0,T;\mathbb{L}^2(I))$. Hence $\boldsymbol{M}$ satisfies the equation (3.8) with the initial-boundary conditions (3.12). In addition of the energy inequality, $\boldsymbol{m}^\varepsilon$ is such that $(\partial_t \boldsymbol{m}^\varepsilon)_\varepsilon$ is uniformly bounded in $L^2(0,T;\mathbb{H}^{-1}(I))$ so $\boldsymbol{m}^\varepsilon$converges towards $\boldsymbol{m}$ weakly in $L^2(0,T;\mathbb{H}^1(I))$ and strongly in $L^2(0,T;\mathbb{L}^2(I))$. Let us consider the convergence of $\mathcal{A}^\varepsilon(\langle U^\varepsilon\rangle)(\partial_x \boldsymbol{m}^\varepsilon)$. First of all $\langle U^\varepsilon\rangle \to \langle \boldsymbol{M}\rangle$, $\langle |U^\varepsilon|\rangle \to 1$ a.e in ω. Therefore $\mathcal{T}^\varepsilon(U^\varepsilon) \to \langle \boldsymbol{M}\rangle$ and $\mathcal{T}^\varepsilon(U^\varepsilon)\otimes\mathcal{T}^\varepsilon(U^\varepsilon) \to \langle \boldsymbol{M}\rangle\otimes\langle \boldsymbol{M}\rangle$ a.e in ω so $\mathcal{T}^\varepsilon(U^\varepsilon)\otimes\mathcal{T}^\varepsilon(U^\varepsilon)\phi \to \langle \boldsymbol{M}\rangle\otimes\langle \boldsymbol{M}\rangle\phi$ strongly in $\mathbb{L}^2(\omega)$ for all $\phi\in\mathbb{L}^2(\omega)$. Consequently, we obtain

$$\mathcal{A}^\varepsilon(\langle U^\varepsilon\rangle)(\partial_x \boldsymbol{m}^\varepsilon) \rightharpoonup \mathcal{A}(\langle \boldsymbol{M}\rangle)(\partial_x \boldsymbol{m}) \quad \text{weakly in } \mathbb{L}^2(\omega).$$

A similar argument gives $\boldsymbol{m}^\varepsilon\times\mathcal{T}^\varepsilon(U^\varepsilon) \rightharpoonup \boldsymbol{m}\times\langle \boldsymbol{M}\rangle$ weakly in $\mathbb{L}^2(\omega)$ and $j_e\mathcal{T}^\varepsilon(U^\varepsilon) \to j_e\langle \boldsymbol{M}\rangle$ strongly in $\mathbb{L}^2(\omega)$ so taking the limit in (3.24), we get the equation (3.3). Proceeding along the lines of the precedent paragraph, we verify that $\boldsymbol{m}$ satisfies the initial and boundary conditions (3.11).
Finally we consider the energy inequalities satisfied by U^ε and $\boldsymbol{m}^\varepsilon$, we make use of the weak and strong convergences obtained and the lower semi-continuity of the norms to deduce the wished energy estimates on $\boldsymbol{M}$ and $\boldsymbol{m}$.

3.4 The stationary spin accumulation equation

The time scale appearing in the spin magnetization is shorter than the time scale of the LLG equation (see [37], [27] for a discussion) then we may consider stationary solutions of the spin accumulation equation

$$\begin{cases} \partial_x(\mathcal{A}(\langle \boldsymbol{M}\rangle)\partial_x \boldsymbol{m}) - \dfrac{1}{\lambda_{\text{sf}}^2}\boldsymbol{m} - \dfrac{1}{\lambda_J^2}\boldsymbol{m}\times\langle \boldsymbol{M}\rangle = \dfrac{\beta}{d^2}\partial_x(j_e\langle \boldsymbol{M}\rangle) \\ \boldsymbol{j}_m(t,-L) = \boldsymbol{j}_m(t,l) = 0 \end{cases} \tag{3.69}$$

where j_e is assumed to be independent of time. We are interested with the global solutions to the system (3.8)-(3.12)-(3.69). We should verify that the energy associated with $\boldsymbol{m}$ allows to deduce an uniform bound for the energy of $\boldsymbol{M}$. For all $t\geq 0$, we have

$$\frac{(1-\beta\beta')}{2}|\partial_x \boldsymbol{m}|_I^2 + \frac{1}{\lambda_{\text{sf}}^2}|\boldsymbol{m}|_I^2 \leq \frac{\beta^2}{2d^4(1-\beta\beta')}|j_e|_I^2. \tag{3.70}$$

Next, it holds that

$$J|\int_0^t\int_\Omega \boldsymbol{m}\cdot\partial_t\boldsymbol{M}\ \mathrm{d}X\mathrm{d}s| \leq \frac{\alpha}{2\gamma}\int_0^t\int_\Omega|\partial_t\boldsymbol{M}|^2\ \mathrm{d}X\mathrm{d}s + \frac{J^2\gamma|\widehat{\Omega}|}{2\alpha}\frac{\beta^2\lambda_{\text{sf}}^2}{2d^4(1-\beta\beta')}|j_e|_I^2\ t.$$

Hence our method allows to prove a global existence result for weak solutions to the system (3.8)-(3.12)-(3.69). To do that we introduce the regularized problem

$$\begin{cases} \nu^2\partial_t^2 U_\nu^\varepsilon - \nabla\cdot(a(X)\nabla U_\nu^\varepsilon) + \mathcal{B}^\varepsilon(\rho^\nu \star \widetilde{U}_\nu^\varepsilon, \partial_t U_\nu^\varepsilon) = \\ -\frac{1}{\varepsilon^2}\nabla_U(p(|U_\nu^\varepsilon|)) + \mathcal{L}(U_\nu^\varepsilon) + J\boldsymbol{m}_\nu^\varepsilon \\ U_\nu^\varepsilon(0,X) = \boldsymbol{M}_0(X),\ |\boldsymbol{M}_0(X)|^2 = 1 \text{ a.e},\ \nu^2\partial_t U_\nu^\varepsilon(0,X) = 0 \text{ in } \Omega \end{cases} \tag{3.71}$$

coupled to the elliptic equation

$$\begin{cases} -\partial_x(\mathcal{A}^\varepsilon(\langle\rho^\nu \star \widetilde{U}_\nu^\varepsilon\rangle)(\partial_x \boldsymbol{m}_\nu^\varepsilon)) + \frac{1}{\lambda_{\text{sf}}^2}\boldsymbol{m}_\nu^\varepsilon = \\ -\frac{1}{\lambda_J^2}\boldsymbol{m}_\nu^\varepsilon \times \mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{U}_\nu^\varepsilon) - \frac{\beta}{d^2}\partial_x(j_e\mathcal{T}^\varepsilon(\rho^\nu \star \widetilde{U}_\nu^\varepsilon)) \end{cases} \tag{3.72}$$

with the boundary conditions defined previously. Proceeding along the lines of the proof given in section 3.3 we can prove the same global existence result of weak solutions $(\boldsymbol{m}, \boldsymbol{M})$ on the finite time interval $[0,T]$ for all $T>0$ fixed. More precisely, we have

Proposition 3.4.1 *Let $T>0$ be fixed, $\boldsymbol{M}_0 \in \mathbb{H}^1(\Omega)$ be such that $|\boldsymbol{M}_0(X)|^2 = 1$ a.e. and $j_e \in L^2(I)$. There exists a global weak solution $(\boldsymbol{m}, \boldsymbol{M})$ of the coupled problem* (3.8)-(3.12)-*(3.69) such that $\boldsymbol{M} \in \mathbb{L}^\infty(Q) \cap \mathbb{H}^1(Q)$, $|\boldsymbol{M}(t,X)|^2 = 1$ a.e, $\boldsymbol{m} \in \mathbb{H}^1(I)$ and satisfying the energy inequalities*

$$\begin{aligned} \mathcal{E}(\boldsymbol{M}(t)) + \frac{\alpha}{\gamma}\int_0^t |\partial_t \boldsymbol{M}(s)|_\Omega^2 \,\mathrm{d}s \le \mathcal{E}(\boldsymbol{M}_0) + C_1 T |j_e|_I^2 \\ \frac{1}{\lambda_{sf}^2}|\boldsymbol{m}|_I^2 + \frac{1-\beta\beta'}{2}|\partial_x \boldsymbol{m}|_I^2 \le C_2 |j_e|_I^2 \end{aligned} \tag{3.73}$$

for all $t \in [0,T]$ where $C_k > 0$ for $k=1,2$ are constants which are independent of $(\boldsymbol{m}, \boldsymbol{M})$ and T.

3.5 Concluding remarks

In the case where the local magnetization is uniform in the ferromagnet layer, the torque $\boldsymbol{m}_\perp$ acting on the switching of the magnetization takes the form see [37] and [27]

$$J\boldsymbol{m}_\perp = b\boldsymbol{M}^2 + a\mathbf{M}^2 \times \boldsymbol{M}^1 \tag{3.74}$$

where the transverse spin accumulation $\boldsymbol{m}_\perp$ is the component of $\boldsymbol{m}$ which is orthogonal to $\boldsymbol{M}^1$, the local magnetization of the thin layer Ω^1, a and b are two constants. We denote by $\boldsymbol{M}^2$ the local magnetization of the thick layer Ω^2. It is assumed that in the thick ferromagnet, the local magnetization $\mathbf{M}^2$ polarizes the current and is uniform and pinned. Notice that the longitudinal component $\boldsymbol{m}_{\|}$ (parallel to $\boldsymbol{M}^1$) of the spin accumulation has no effect in the product $\boldsymbol{M}^1 \times \boldsymbol{m}$. Hence, the LLG equation becomes

$$\partial_t \boldsymbol{M} - \alpha \boldsymbol{M} \times \partial_t \boldsymbol{M} = -\gamma \boldsymbol{M} \times (\mathcal{H}_e + b\mathbf{V} + a\mathbf{V} \times \mathrm{M}) \tag{3.75}$$

where we set $\boldsymbol{M} = \boldsymbol{M}^1$ and $\boldsymbol{V} = \boldsymbol{M}^2$. Generally, the following form $\boldsymbol{V} = \cos(\theta)\boldsymbol{e}_3 - \sin(\theta)\boldsymbol{e}_1$ is used where $\theta \in [\pi/2, \pi]$ is the angle between $\boldsymbol{V}$ and $\mathbf{e}_3$ the final position of $\boldsymbol{M}$. In this model the effective magnetization field does not contain the exchange magnetic field $\nabla \cdot (a\nabla \boldsymbol{M})$ and then the LLG equation becomes a set of ordinary differential equations.

To describe the switching of the local magnetization across a ferromagnetic domain wall, the following model is used in [33]. Let $\Omega = (-l, l) \times \widehat{\Omega}$ be a bounded and regular horizontal cylinder of $\mathbb{R}^3$ representing the ferromagnet. The cross section $\widehat{\Omega}$ is included in the plane $(\mathbf{e}_2, \mathbf{e}_3)$. The local magnetization satisfies in Ω the equation

$$\partial_t \boldsymbol{M} - \alpha \boldsymbol{M} \times \partial_t \boldsymbol{M} = -\gamma \boldsymbol{M} \times \mathcal{H}_e - u\partial_x \boldsymbol{M} \tag{3.76}$$

where x denotes the coordinate along which the current flows, $X = (x, \widehat{x}) \in \Omega$ with $\widehat{x} = (y, z)$. The effective magnetic field $\mathcal{H}_e$ contains exchange energy, volume anisotropy energy and magnetostatic. The initial-boundary conditions are

$$\boldsymbol{M}(0, X) = \boldsymbol{M}_0(X) \ \text{ in } \Omega \ , \quad \boldsymbol{M} \times (a\partial_n \boldsymbol{M}) = 0 \ \text{ on } (0, T) \times \partial\Omega. \tag{3.77}$$

Multiplying equation (3.76) by $\boldsymbol{M}$ we get the transport $\partial_t |\boldsymbol{M}|^2 + u\partial_x |\boldsymbol{M}|^2 = 0$ which implies the saturation condition $|\boldsymbol{M}(t, x, \widehat{x})|^2 = 1$ a.e if $|\boldsymbol{M}(t, -l, \widehat{x})|^2 = |\boldsymbol{M}(0, x, \widehat{x})|^2 = 1$ (we assume that $u > 0$). Therefore we may write $u\partial_x \boldsymbol{M} = -u\boldsymbol{M} \times (\boldsymbol{M} \times \partial_x \boldsymbol{M})$ so equation (3.76) has the equivalent form

$$\partial_t \boldsymbol{M} - \alpha \boldsymbol{M} \times \partial_t \boldsymbol{M} = -\gamma \boldsymbol{M} \times (\mathcal{H}_e - \frac{u}{\gamma} \boldsymbol{M} \times \partial_x \boldsymbol{M}). \tag{3.78}$$

A model equations of the same type is used in [3] to describe the dynamic of the local magnetization in the presence of spin-polarized current in a layer of the type F/N/F. The modified LLG equation is given in the ferromagnet layer Ω by

$$\partial_t \boldsymbol{M} - \alpha \boldsymbol{M} \times \partial_t \boldsymbol{M} = -\gamma \boldsymbol{M} \times (\mathcal{H}_e + \beta \boldsymbol{M} \times (\boldsymbol{j} \cdot \nabla) \boldsymbol{M}) \tag{3.79}$$

where $\beta > 0$ is a constant and $\boldsymbol{j}$ the current . Initial-boundary conditions and transmission boundary conditions at the interfaces F/N and N/F are assumed to be verified. In this case the equation can be rewritten as

$$\partial_t \boldsymbol{M} - \alpha \boldsymbol{M} \times \partial_t \boldsymbol{M} = -\gamma \boldsymbol{M} \times \mathcal{H}_e + \gamma\beta (\boldsymbol{j} \cdot \nabla) \boldsymbol{M}. \tag{3.80}$$

To solve this problem, we proceed as in section 3.3 and introduce the regularized-penalized equation

$$\begin{cases} \nu^2 \partial_t^2 U_\nu^\varepsilon - \nabla \cdot (a(X) \nabla U_\nu^\varepsilon) + \mathcal{B}^\varepsilon(\rho^\nu \star \widetilde{U}_\nu^\varepsilon, \partial_t U_\nu^\varepsilon) = \\ -\frac{1}{\varepsilon^2} \nabla_U (p(|U_\nu^\varepsilon|)) + \mathcal{L}(U_\nu^\varepsilon) + \beta (\boldsymbol{j} \cdot \nabla) U_\nu^\varepsilon \\ U_\nu^\varepsilon(0, X) = \boldsymbol{M}_0(X), \ |\boldsymbol{M}_0(X)|^2 = 1 \text{ a.e}, \ \nu^2 \partial_t U_\nu^\varepsilon(0, X) = 0. \end{cases} \tag{3.81}$$

The following energy estimate holds

$$\begin{cases} \nu^2 |\partial_t U_\nu^\varepsilon(t)|_\Omega^2 + \mathcal{E}(U_\nu^\varepsilon(t)) + \frac{2}{\varepsilon^2} \int_\Omega p(|U_\nu^\varepsilon(t)|) \, \mathrm{d}X \\ +\frac{\alpha}{\gamma} \int_0^t |\partial_t U_\nu^\varepsilon(s)|_\Omega^2 \, \mathrm{d}s \leq \mathcal{E}(\boldsymbol{M}_0) + \frac{\gamma\beta^2}{\alpha} |j|_\infty^2 \int_0^t |\nabla U_\nu^\varepsilon(s)|_\Omega^2 \, \mathrm{d}s. \end{cases} \tag{3.82}$$

For $V \in L^2(0, T; \mathbb{L}^2(\Omega))$, we introduce $U = \mathcal{R}(V)$ the solution of the wave equation

$$\begin{cases} \nu^2 \partial_t^2 U - \nabla \cdot (a(X) \nabla U) + \mathcal{B}^\varepsilon(\rho^\nu \star \widetilde{V}, \partial_t U) = \\ -\frac{1}{\varepsilon^2} \nabla_U (p(|U|)) + \mathcal{L}(U) + \beta (\boldsymbol{j} \cdot \nabla) U \\ U(0, X) = \boldsymbol{M}_0(X), \ |\boldsymbol{M}_0(X)|^2 = 1 \text{ a.e}, \ \nu^2 \partial_t U(0, X) = 0. \end{cases} \tag{3.83}$$

Proceeding along the lines of the proof of our main theorem, we prove that $\mathcal{R}$ is continuous and compact from $L^2(0, T; \mathbb{L}^2(\Omega))$ into itself. Then as a consequence of

Schauder fixed point theorem, we get a solution U_ν^ε of the intermediary wave equation. We pass to the limit as $\nu \to 0$ then as $\varepsilon \to 0$ and finally prove, as in section 3.3, a global existence result of weak solutions.

To end this review, let us discuss the model equations considered in [36] to describe the magnetization switching by spin-polarized current. It is given by

$$\partial_t \boldsymbol{M} = -\gamma \boldsymbol{M} \times \mathcal{H}_e - \alpha \boldsymbol{M} \times (\boldsymbol{M} \times \mathcal{H}_e) - f_{||}(t)\boldsymbol{M} \tag{3.84}$$

where the time dependent function $f_{||}$ depends of the injected current see [36]. One observes that the saturation condition is not fulfilled by $\boldsymbol{M}$ and we have

$$|\boldsymbol{M}(t,X)|^2 = |\boldsymbol{M}_0(X)|^2 e^{2F(t)},\ F(t) = -\int_0^t f_{||}(s)ds. \tag{3.85}$$

Assuming that the magnetic field $\mathcal{H}_e(\boldsymbol{M})$ depends linearly of $\boldsymbol{M}$ and setting $\mathbf{U} = \boldsymbol{M}e^{-F(t)}$ we deduce that $\mathbf{U}$ satisfies the classical form of the LLG equation

$$\partial_t \mathbf{U} = -\gamma(t)\mathbf{U} \times \mathcal{H}_e(\mathbf{U}) - \alpha(t)\mathbf{U} \times (\mathbf{U} \times \mathcal{H}_e(\mathbf{U})) \tag{3.86}$$

where

$$\gamma(t) = \gamma e^{F(t)},\ \alpha(t) = \alpha e^{2F(t)}. \tag{3.87}$$

One has $|\mathbf{U}(t)|^2 = 1$ and as usually we can transform this equation in the LLG form. We have $\mathbf{U} \times \partial_t \mathbf{U} = -\gamma(t)\mathbf{U} \times (\mathbf{U} \times \mathcal{H}_e(\mathbf{U})) + \alpha(t)\mathbf{U} \times \mathcal{H}_e(\mathbf{U})$. Consequently $-\alpha(t)\mathbf{U} \times (\mathbf{U} \times \mathcal{H}_e(\mathbf{U})) = (\alpha(t)/\gamma(t))\mathbf{U} \times \partial_t \mathbf{U} - (\alpha^2(t)/\gamma(t))\mathbf{U} \times \mathcal{H}_e(\mathbf{U})$. We get

$$\partial_t \mathbf{U} - \frac{\alpha(t)}{\gamma(t)}\mathbf{U} \times \partial_t \mathbf{U} = -\frac{\alpha^2(t) + \gamma^2(t)}{\gamma(t)}\mathbf{U} \times \mathcal{H}_e(\mathbf{U}) \tag{3.88}$$

and $\alpha(t)/\gamma(t) = (\alpha/\gamma)e^{F(t)}$, $(\alpha^2(t) + \gamma^2(t))/\gamma(t) = ((\alpha^2 e^{2F(t)} + \gamma^2)/\gamma)e^{F(t)}$ or

$$\partial_t \mathbf{U} + \frac{\gamma(t)}{\alpha(t)}\mathbf{U} \times \partial_t \mathbf{U} = \frac{\alpha^2(t) + \gamma^2(t)}{\alpha(t)}\mathcal{H}_e(\mathbf{U}). \tag{3.89}$$

The energy estimate follows from the equality $|\partial_t \mathbf{U}|^2 + ((\alpha^2(t) + \gamma^2(t))/\alpha(t))\mathcal{H}_e(\mathbf{U}) \cdot \partial_t \mathbf{U} = 0$ which gives the energy equality

$$\mathcal{E}(\mathbf{U}(t)) + 2\int_0^t \frac{\alpha(s)}{\alpha^2(s) + \gamma^2(s)}|\partial_t \mathbf{U}(s)|^2 ds = \mathcal{E}(\boldsymbol{M}_0) \tag{3.90}$$

where $\mathcal{E}$ is the classical energy in the ferromagnet. Notice that we have for all s

$$\frac{\alpha(s)}{\alpha^2(s)+\gamma^2(s)} = \frac{\alpha}{\alpha^2 e^{2F(s)}+\gamma^2} \geq \frac{\alpha}{\alpha^2 e^{2MT}+\gamma^2} \tag{3.91}$$

for all s assuming $|f_{||}(t)| \leq M$ for all $t \in [0,T]$. Hence under this hypothesis, the global existence theory of weak solutions can be performed by using the method developed here.

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Deuxième partie

Retournement de domaine dans une tige piézoélectrique mince

Chapitre 1

Introduction à la piézoélectricité

Un matériau piézoélectrique (du grec "piezein", presser) est une substance qui se polarise sous l'effet d'une contrainte (effet piézoélectrique direct) et inversement, lorsqu'il est mis en présence d'un champ électrique, il se déforme (effet piézoélectrique indirect). Ce phénomène a été découvert par les frères Pierre et Jacques Curie sur le quartz et le sel de Rochelle en 1880. Electriquement neutre extérieurement, une déformation exercée sur un cristal provoque un déplacement des barycentres des charges positives et négatives dans la structure du cristal, ce qui est à l'origine de l'apparition d'un dipôle élémentaire qui se traduit par la polarisation des surfaces du cristal et donc l'apparition d'un champ électrique. Inversement lorsqu'un champ électrique externe est appliqué au cristal, les ions des cellules sont déplacés par des forces électrostatiques, ce qui engendre une déformation mécanique du cristal. Cet effet peut être observé sur de nombreux matériaux tels que le quartz, la berlinite, le nitrate de lithium et les céramiques piézoélectriques. On notera que seuls les cristaux ne possédant pas de centre de symétrie sont piézoélectriques et que la plupart des matériaux piézoélectriques sont aussi ferroélectriques.

Pouvant convertir un effort mécanique en tension électrique et réciproquement, les matériaux piézoélectriques permettent d'obtenir des vibrations mécaniques ou électriques, à l'origine de beaucoup d'applications dans les systèmes de télécommunication, les générateurs ou capteurs d'ultrason, les commutateurs, les transducteurs, les résonateurs dans les oscillateurs électroniques, etc... Actuellement Le PZT ou $Pb(Zr,Ti)O_3$ représente le matériau piézoélectrique le plus utilisé dans les applications.

1.1 Les tenseurs en piézoélectricité

Pendant longtemps, la description des propriétés des matériaux s'est faite dans le cadre de l'approximation de Hooke reliant linéairement effort et déformation. La relation tensorielle la plus simple entre le moment électrique $\overrightarrow{P}$ et le tenseur des contraintes T est la suivante (on utilise la convention de sommation sur les indices répétés)

$$P_i = d_{ijk}T_{jk}$$

d étant le tenseur (d'ordre 3) de piézoélectricité. Ce même tenseur permet de relier le champ électrique E au tenseur des déformations S

$$S_{ij} = d_{ijk}E_k.$$

Cependant les effets électrique et mécanique apparaissent simultanément de sorte qu'il n'est pas possible d'exprimer par exemple la déformation uniquement en fonction du champ électrique, l'addition d'un terme purement mécanique est nécessaire dans l'approche linéaire.

L'étude de la piézoélectricité repose étroitement sur l'analyse des propriétés élastiques et diélectriques des matériaux. Le tenseur des contraintes T est défini par

$$T_{ij} = \frac{\Delta F_i}{\Delta s_j}$$

où ΔF représente la force qui s'applique sur une surface élémentaire Δs. Sous l'action des forces extérieures, les points du solide se déplacent, la déformation qui en résulte est définie par le tenseur symétrique (d'ordre 2)

$$S_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

où u représente le déplacement et x la position. Dans un corps purement élastique, la contrainte et la déformation sont reliées par la relation

$$T_{ij} = C_{ijkl}S_{kl} \tag{1.1}$$

le tenseur (d'ordre 4) C étant le tenseur de rigidité élastique. D'autre part, un solide diélectrique est polarisé en présence d'un champ électrique E. Cette polarisation s'exprime par

$$P_i = \varepsilon_0 \ \chi_{ij}E_j$$

où χ est le tenseur des susceptibilités électriques et ε_0 est la permittivité du vide. Le solide piézoélectrique est polarisé par le champ électrique appliqué et par la tension électrique résultant de la déformation du cristal. La polarisation totale du solide est donc la somme de deux termes, le premier est proportionnel au champ électrique appliqué et le second est proportionnel à la déformation

$$P_i = \varepsilon_0 \chi_{ij} \; E_j + d_{ijk} T_{jk}.$$

De la même manière, l'induction électrique D sera la somme de deux termes

$$D_i = \varepsilon_{ij} E_j + d_{ijk} T_{jk} \tag{1.2}$$

où ε est le tenseur de permittivité. Cette superposition des effets mécanique et électrique impose aussi un terme supplémentaire à l'équation de la contrainte (1.1).

1.2 Equations en piézoélectricité linéaire

L'utilisation des fonctions thermodynamiques appropriées permet de généraliser les équations précédentes aussi bien dans le cas linéaire que dans le cas non linéaire. Ainsi d'après le premier principe de la thermodynamique, si par unité de volume le cristal reçoit une quantité de chaleur dQ et est soumis à un travail dW dû à des forces extérieures, l'accroissement dU de l'énergie interne est donné par

$$dU = dQ + dW.$$

Le travail qu'il faut fournir pour modifier l'état du cristal comprend un terme mécanique $dW_m = T_{ij} dS_{ij}$ et un terme électrique $dW_e = E_i dD_i$. Pour une transformation adiabatique, $dQ = 0$, on obtient

$$dU = T_{ij} dS_{ij} + E_i dD_i.$$

U est une fonction de la déformation et de l'induction électrique. Cette relation donne lieu à [13], [12]

$$T_{ij} = \frac{\partial U}{\partial S_{ij}}, \;\; E_i = \frac{\partial U}{\partial D_i}.$$

Pour faire apparaître les variables S et E, on introduit le potentiel thermodynamique (enthalpie électrique)

$$\psi = U - E_i D_i \tag{1.3}$$

$$d\psi = T_{ij} dS_{ij} - D_i dE_i.$$

Nous avons alors

$$T_{ij} = \frac{\partial \psi}{\partial S_{ij}}, \ D_i = -\frac{\partial \psi}{\partial E_i}$$

et

$$e_{ijk} = -\frac{\partial T_{ij}}{\partial E_k} = \frac{\partial D_k}{\partial S_{ij}} = -\frac{\partial^2 \psi}{\partial E_k \partial S_{ij}}$$

représente le tenseur (d'ordre 3) de piézoélectricité. Le tenseur de rigidité élastique s'écrit

$$C_{ijkl} = \frac{\partial T_{ij}}{\partial S_{kl}}$$

et la permittivité s'exprime par

$$\varepsilon_{ij} = \frac{\partial D_i}{\partial E_j}.$$

Noter que les tenseurs obéissent aux règles de symétrie suivantes

$$\begin{cases} C_{ijkl} = C_{ijlk} = C_{klij} \\ e_{kij} = e_{kji} \\ d_{kij} = d_{kji} \\ \varepsilon_{ij} = \varepsilon_{ji}. \end{cases}$$

Les équations constitutives (en fonction des variables S et E) sont données par

$$\begin{cases} T_{ij} = C_{ijkl} S_{kl} - e_{kij} E_k \\ D_i = e_{ikl} S_{kl} + \varepsilon_{ik} E_k. \end{cases} \tag{1.4}$$

Avec le couple de variables (T, E), on peut aboutir aux relations suivantes

$$\begin{cases} S_{ij} = \mathcal{C}_{ijkl} T_{kl} - d_{kij} E_k \\ D_i = d_{ikl} T_{kl} + \varepsilon_{ik} E_k. \end{cases}$$

Ceci peut-être vu comme une généralisation de la loi de Hooke en élasticité et les équations constitutives en électrostatique. De plus, la loi fondamentale de mécanique ainsi que l'approximation électrostatique des équations de Maxwell en absence de charge volumique, conduisent à [9]

$$\begin{cases} -\dfrac{\partial T_{ij}}{\partial x_j} + \rho \dfrac{\partial^2 u_i}{\partial t^2} = 0 \\ -\dfrac{\partial D_i}{\partial x_i} = 0 \\ E_i = -\dfrac{\partial \Phi}{\partial x_i} \end{cases} \tag{1.5}$$

où ρ représente la densité de masse et Φ le potentiel électrique. Les relations (1.4) et (1.5) constituent les équations de la piézoélectricité linéaire. Quand les signaux appliqués sont très petits, la réponse piézoélectrique demeure raisonnablement linéaire et réversible. A noter que la théorie linéaire de la piézoélectricité est utilisée pour la description des ondes élastiques et la résolution de problèmes concernant les oscillations à faibles amplitudes. Cette théorie n'est pas exacte pour des contraintes fortes ou des champs électriques intenses (voir [14] et les références qui y sont citées).

1.3 Le retournement de domaine en piézoélectricité

L'effet piézoélectrique tel que décrit précédemment est très faible. Par exemple, même avec le PZT qui déploie le plus grand effet piézoélectrique conventionnel, la déformation est de seulement 0.01%. Dans un matériau, une région de constante polarisation est appelée un domaine ferroélectrique et les domaines avec des polarisations électriques différentes existent dans les matériaux piézoélectriques sans qu'ils soient macroscopiquement polaires. La polarisation entre les domaines dépend de la symétrie du cristal et peut former des angles de 180°, 90°, etc... Quand un champ électrique est appliqué au matériau, il se produit un retournement de domaine et la direction de polarisation tend à s'aligner le long de la direction du champ appliqué.

Globalement, le retournement de domaine conduit à un retournement macroscopique du vecteur de polarisation et dans certains cas, la magnitude de la déformation associée à ce processus peut atteindre 1 à 5%. Pour décrire ce phénomène, suivant [7],[6], on introduit en plus des champs et tenseurs définis précédemment qui décrivent la structure macroscopique, le nombre effectif $\mathbf{n}$ de dipôles alignés défini comme la projection du moment du dipôle sur la direction du champ électrique (ici on ne tient compte que des retournements de domaines réversibles) ainsi que les champs suivants pour décrire la micro-structure

$$\mathbf{T}: \text{ la micro-contrainte et } \mathbf{h}: \text{ la micro-force interactive.}$$

On suppose que l'énergie ψ dépend de $\mathbf{n}$ de la façon suivante

$$\psi = \psi_1(S, E, \mathbf{n}) + \frac{1}{2}|\nabla \mathbf{n}|^2$$

où $\psi_1 = \mathcal{H}(S, E) + \mathbf{n} \cdot \mathcal{F}(S, E) + \frac{1}{2}\mathcal{K}(S, E)|\mathbf{n}|^2$, $\mathcal{H}(S, E)$ étant l'enthalpie définie par

$$\mathcal{H}(S, E) = \frac{1}{2}C_{ijkl}S_{kl}S_{ij} - e_{ikl}S_{kl}E_i - \frac{1}{2}\varepsilon_{ij}E_iE_j$$

et on suppose que $\mathcal{F}$ est une fonction quadratique de ses arguments. On calcule les expressions du champ des contraintes $T_{ij} = \dfrac{\partial \psi_1}{\partial S_{ij}}$ et du champ de déplacements électriques $D_i = -\dfrac{\partial \psi_1}{\partial E_i}$ en négligeant les termes en $|\mathbf{n}|^2$. La micro-contrainte est donnée par $\mathbf{T}_{ij} = \dfrac{\partial \mathbf{n}_i}{\partial x_j}$, quant à la micro-force $\mathbf{h}$, elle est la somme de deux contributions : une partie dissipative donnée par $\mathbf{h}_i^d = -\gamma \dfrac{\partial \mathbf{n}_i}{\partial t}$, $\gamma > 0$ et une partie non dissipative donnée par $\mathbf{h}_i^{nd} = -\dfrac{\partial \psi_1}{\partial \mathbf{n}_i}$ et nous avons les lois de conservation mécaniques et électrostatique

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} - \nabla \cdot T = 0 \\ \dfrac{\partial \mathbf{n}}{\partial t} - \nabla \cdot \mathbf{T} - \mathbf{h}^{nd} = 0 \\ -\nabla \cdot D = 0 \end{cases} \tag{1.6}$$

dans le cas où aucune force ni charge volumiques ne sont appliquées.

1.4 Tige piézoélectrique mince avec retournement de domaine

Soit $\mathcal{P} = \mathcal{D} \times I$ un cylindre de génératrices parallèles à la direction $\mathbf{k}$, $\mathcal{D} \subset \mathbb{R}^2$ borné et $I =]0, L[$ (on se place dans un repère cartésien $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ orthonormé et on note x_i, $i = 1, 2, 3$ les composantes de x). F. Davì dans [7] a introduit la notion phénoménologique de minceur de la tige $\mathcal{P}$ en faisant les hypothèses suivantes :
i/ le nombre effectif de dipôles alignés dépend uniquement de la variable x_3 et est parallèle à $\mathbf{k}$

$$\mathbf{n} = n(x_3, t)\mathbf{k}, \nabla \mathbf{n} = \partial_{x_3} n(x_3, t)\ \mathbf{k} \otimes \mathbf{k}$$

ii/ le potentiel électrique dépend linéairement de $x' = (x_1, x_2)$

$$\phi(x, t) = \varphi(x_3, t) + x' \cdot \mathbf{\Phi}(x_3, t)$$

ce qui donne

$$E(x, t) = -\Phi_1(x_3, t)\mathbf{i} - \Phi_2(x_3, t)\mathbf{j} - (\partial_{x_3}\varphi(x_3, t) + x' \cdot \partial_{x_3}\Phi(x_3, t))\mathbf{k}$$

iii/ la section transversale de $\mathcal{P}$ est rigide dans son plan et reste orthogonale à l'axe de déformation. Autrement dit, seuls les déplacements de la forme $S = s\mathbf{k} \otimes \mathbf{k}$ sont admissibles. Les seuls matériaux compatibles avec cette hypothèse sont ceux qui admettent l'axe Ox_3 comme axe de symétrie.

Ces hypothèses conduisent à des déplacements et déformations de la forme

$$u(x,t) = w_1(x_3,t)\mathbf{i} + w_2(x_3,t)\mathbf{j} + (v(x_3,t) - x' \cdot \partial_{x_3} w(x_3,t))\mathbf{k}$$

$$S(x,t) = (\partial_{x_3} v(x_3,t) - x' \cdot \partial^2_{x_3} w(x_3,t))\mathbf{k} \otimes \mathbf{k}.$$

En intégrant les équations (1.6) sur la section du cylindre $\mathcal{D}$, on obtient des équations à une variable $x_3 \in]0, L[$ [7]. Ces équations dans le cas particulier où $u = v(x_3,t)\mathbf{k}$ et $E = -\partial_{x_3}\varphi(x_3,t)\mathbf{k}$, sont définies par Davi dans [6] comme les équations de la piézoélectricité avec retournement de domaine en une dimension d'espace et elles constituent le modèle que nous avons considéré dans notre travail.

Chapitre 2

Local solutions to a model of piezoelectric materials

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A local existence theorem is proved for a nonlinear coupled system modelling the electromechanical motion of a one-dimensional piezoelectric body with domain switching. The system is composed by a heat equation describing the behavior of the number of electric dipoles and by a wave equation governing the dynamic of the electric displacement. The main coupling in the system appears in the time dependent velocity of the waves depending on the number of electric dipoles. The proof of the result relies on a time decay estimate satisfied by the number of electric dipoles and an uniform estimate of the solution of the regularized wave equation.

Key words. piezoelectric materials, heat equation, wave equation .
2000 AMS subject classifications. 74F15, 74K10, 47J35, 34G20,35K05,46E35.

2.1 Introduction and main result

We are dealing with a one-dimensional model to describe the dynamic of piezoelectric bodies with switchable domain introduced by Daví in [6] and [7]. The model equations proposed is the following. In $(0,1)$, we have $\partial_x(C(f)\partial_x u + H\partial_x\varphi) = \rho\partial_t^2 u$, $K\partial_x\varphi - H\partial_x u = 0$ and $\sigma^2\partial_x^2 f + Mf - D\partial_x u = \gamma^2\partial_t f$ where $K > 0$ is the permittivity constant, $H > 0$ the piezoelectric stress, and $C(f) = C + Df$ where $C > 0$ is the elasticity constant and $D > 0$, $M > 0$, σ^2 and $\rho > 0$ are material parameters. The number of aligned dipoles $f(t,x)$ is defined as the projection of the dipole moments onto the direction of the electric field. The displacement field is denoted by $u(t,x)$ and the electric displacement is given by $\partial_x u$. The above equations are equivalent to the following ones where we assumed $M = 0$ and we equated the other constants to one

$$\begin{cases} \partial_t f - \partial_x^2 f = -\partial_x u \ \text{ in } (0,T)\times\mathbb{R} \\ f(0,x) = f_0(x) \ \text{ in } \mathbb{R} \\ \partial_t^2 u - \partial_x[(1+f)\partial_x u] = 0 \ \text{ in } (0,T)\times\mathbb{R} \\ u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x) \ \text{ in } \mathbb{R} \end{cases} \tag{2.1}$$

where $t \geq 0$ is the time variable, $x \in \mathbb{R}$ is the space one. The main observation is that the coupling of both equations makes the velocity of the wave equation time dependent and depends of the solution of the heat equation. Moreover, the model equation do not have any natural conserved quantities as for example the energy of the system. This difficulty together with the fact that the wave operator is nonlinearly time dependent do not allow to give a general result.

Theorem 2.1.1 *Under the following hypotheses,*

$$\begin{cases} f_0 \in W^{1,\infty}(\mathbb{R})\,,\ f_0^{(2)}(x) \ \textit{exists a.e. in } \mathbb{R} \\ u_0 \in H^1(\mathbb{R}), u_1 \in L^2(\mathbb{R}), f_1 \equiv f_0^{(2)} - u_0' \in L^\infty(\mathbb{R}), \\ f_0(x) \geq F_0 > 0\ ,\ f_1(x) \leq -F_0 \ \textit{a.e. in } \mathbb{R} \end{cases} \tag{2.2}$$

there exists $T > 0$ depending only on the data f_0, u_0, u_1, F_0 such that (2.1) has a solution (f,u) such that $f \in W^{1,\infty}((0,T)\times\mathbb{R})$ and $u \in L^\infty(0,T;H^1(\mathbb{R})) \cap W^{1,\infty}(0,T;L^2(\mathbb{R}))$ satisfying the equations (2.1) in the sense of distributions, the initial conditions in $L^2(\mathbb{R})$ and

$$f \geq 0\,,\ \partial_t f \leq 0 \ \textit{a.e. in } (0,T)\times\mathbb{R}.$$

Remark 2.1.2 *Let us give an example of initial data satisfying our hypothesis. We set $u_0 = a\exp(-x^2)$, then $u_0 \in H^1(\mathbb{R})$ and let f_0 be a $2w$-periodic function with $0 < w < 1$ fixed, defined by $f_0(x) = 1 - x^2$ for $-w \leq x \leq w$. We have*

$f_0 \in W^{1,\infty}(\mathbb{R}), f_0(x) \geq 1 - w^2 > 0, f_0^{(2)}(x) - u_0'(x) = -2 + 2ax\exp(-x^2)$ *a.e. so that we have* $f_0^{(2)}(x) - u_0'(x) \leq 2(-1 + \frac{1}{\sqrt{2}}\exp(-\frac{1}{2})) < 0$ *a.e if we have chosen* a *to be such that* $0 < a < \sqrt{2}\exp(\frac{1}{2})$.

Remark 2.1.3 *Solving explicitly the heat equation, il follows that* u *satisfies the following non-local non-linear wave equation with memory*

$$\begin{aligned} &\partial_t^2 u - \partial_x((1 + (H(t,.) * f_0)(x))\partial_x) \\ &+\partial_x\Big(\Big(\int_0^t H(t-s, x-.) * \partial_x u(s,.)ds\Big)\partial_x u\Big) = 0 \end{aligned} \tag{2.3}$$

Remark 2.1.4 *Let us explain formally the method we use to obtain existence of local solutions. The elastic energy* $E(t) = (|\partial_t u|_2^2 + |(1+f)^{1/2}\partial_x u|_2^2)^{1/2}$ *deduced from the wave equation satisfies*

$$\frac{dE^2(t)}{dt} = \int_{\mathbb{R}} \partial_t f |\partial_x u|^2 \, dx$$

then $E^2(t)$ *decreases if we have* $\partial_t f(t,x) \leq 0$ *a.e. The second condition is* $f(t,x) \geq 0$ *a.e. to insure the non-negativity of the energy. Since the sign of* $\partial_x u$ *is not fixed then the heat equation implies that* $f(t,x) \geq 0$ *for a short time depending on* $E^2(0)$ *assuming the initial data to be such that* $f_0 \geq F_0 > 0$. *The estimate* $\partial_t f(t,x) \leq 0$ *is satisfied also for a short time depending on* $E(0)$ *by assuming that* $\partial_t f(0,x) \leq -F_0$ *a.e. To solve the wave equation satisfied by* u, *it is convenient to have* $\partial_{tx}^2 u$ *in* L^2. *We have added to* $E^2(t)$ *the artificial viscosity energy* $\varepsilon \int_{(0,T)\times\mathbb{R}} |\partial_{tx}^2|^2 \, dxdt$ *which allows to control the regularity of* u.

Remark 2.1.5 *Let us discuss the stability of the system for the weak-*$\star$ *convergence. Let* (f^n, u^n) *be a sequence of solutions of the problem such that* $0 \leq f^n(t,x) \leq C$ *and* $0 \leq E^n(t) \leq C$ *uniformly. Let* f *be the weak-*$\star$ *limit in* $L^\infty((0,T)\times\mathbb{R})$ *of* (f^n) *and* u *such that* $\partial_t u^n \rightharpoonup \partial_t u$, $\partial_x u^n \rightharpoonup \partial_x u$ *weakly-*$\star$ *in* $L^\infty(0,T;L^2(\mathbb{R}))$. *Using the integral representation of the solutions* f^n *and the regularizing effect of the heat equation, we easily deduce that* $f^n(t,x) \to f(t,x)$ *a.e.* $(t,\ x)$ *and then* (f,u) *is a solution of the problem. If* f^n *is just assumed to be a non-negative uniformly bounded sequence then, in general, the stability of the wave equation fail without assuming an uniform bound for* $\partial_t f^n$ *in* $L^\infty((0,T)\times\mathbb{R})$.

The content of the paper is the following. In the second section, we recall the L^∞-estimates satisfied by the solution of the heat equation and its first partial derivatives. In the third section, we consider the wave equation with given time dependent velocity. To solve it, we introduce an artificial viscosity energy in the model namely $\varepsilon \int_0^T \int_{\mathbb{R}} |\partial_{tx}^2 u|^2 \, dxds$ where $\varepsilon > 0$ and consider the associated approximated problem. We establish existence and uniqueness of the solution and the decay of its energy. Then, in the fourth section, we introduce the approximate nonlinear problem associated with (2.1). The proof of the existence of a solution for this problem is based on an iteration method and on uniform estimates which allow to obtain the convergence of the iterative scheme. The last paragraph is devoted to the proof of Theorem 2.1.1.

In the sequel, we use the following notations. We denote by $|.|_p$ the norms of the Lebesgues spaces either $L^p(\mathbb{R})$ or $L^p((0,T)\times\mathbb{R})$ and by $\|.\|$ the norm of the space $L^\infty(0,T;L^2(\mathbb{R}))$. We set ∂_t, ∂_x the partial derivatives with respect to t and x, $\partial_t^2, \partial_{tx}^2, \partial_x^2$ the second partial derivatives. Next, for a function $f(x)$ with $x \in \mathbb{R}$, we set f', $f^{(2)}$ for the first and the second derivative of the function.

2.2 The heat equation

In this section we discuss the heat equation in order to show that for some initial data the associated solution f satisfies $f(t,x) \geq 0$ and $\partial_t f(t,x) \leq 0$. The first result shows that the velocity of the wave equation verifies $1 + f(t,x) \geq 1$ and the second one implies that the dissipated energy satisfies $\int_{\mathbb{R}} \partial_t f |\partial_x u|^2 \, dx \leq 0$. This inequality gives the decay of the energy. We shall prove a maximum principle for f and its derivatives (see [1] for example). Let $T > 0$ be fixed, we consider the heat equation

$$\begin{cases} \partial_t f - \partial_x^2 f = -\partial_x v \ \text{ in } (0,T)\times\mathbb{R} \\ \\ f(0,x) = f_0(x) \ \text{ in } \mathbb{R} \end{cases} \tag{2.4}$$

where v is a fixed function. The following result holds

Proposition 2.2.1 *Let v be such that $v \in L^\infty(0,T;H^1(\mathbb{R})) \cap C([0,T];L^2(\mathbb{R}))$, $\partial_{tx}^2 v \in L^2((0,T)\times\mathbb{R})$ and $v(0,.) = u_0$. Then, under the hypotheses of Theorem 2.1.1, there* *exists a unique solution $f \in W^{1,\infty}((0,T)\times\mathbb{R})$ of problem (2.4). Moreover, there exists $C_1, C_2 > 0$ which are independent of the data such that the following estimates hold*

$$f(t,x) \geq F_0 - C_1 T^{\frac{3}{4}}\|\partial_x v\| \text{ a.e. in } (0,T)\times\mathbb{R} \tag{2.5}$$

$$\partial_t f(t,x) \leq -F_0 + C_2 T^{\frac{1}{4}}\|\partial_t v\| \text{ a.e. in } (0,T)\times\mathbb{R} \tag{2.6}$$

$$|f|_\infty \leq |f_0|_\infty + C_1 T^{\frac{3}{4}}\|\partial_x v\| \tag{2.7}$$

$$|\partial_x f|_\infty \leq |f_0'|_\infty + C_2 T^{\frac{1}{4}}\|\partial_x v\| \tag{2.8}$$

$$|\partial_t f|_\infty \leq |f_1|_\infty + C_2 T^{\frac{1}{4}}\|\partial_t v\|. \tag{2.9}$$

Proof. The solution of (2.4) is given by

$$f(t,x) = H(t,\cdot)\star f_0(\cdot)(x) - \int_0^t H(t-s,\cdot)\star\partial_x v(s,\cdot)(x)ds \tag{2.10}$$

where $H(t,x) = \dfrac{1}{\sqrt{4\pi t}}\exp(-\dfrac{x^2}{4t})$ is the heat kernel. Using Cauchy- Schwartz inequality, we easily deduce that

$$F_0 - \int_0^t (8\pi(t-s))^{-1/4}|\partial_x v(s,\cdot)|ds \leq f(t,x) \tag{2.11}$$

and

$$f(t,x) \leq |f_0|_\infty + \int_0^t (8\pi(t-s))^{-1/4}|\partial_x v(s,\cdot)|ds \tag{2.12}$$

which implies (2.5) and (2.7). Taking in (2.10) the derivative of f with respect to the x variable, we get similarly (2.8). To prove (2.6) and (2.9) we proceed as follows. The time derivative $\partial_t f$ is a solution of the problem $\partial_t(\partial_t f) - \partial_x^2(\partial_t f) = -\partial_{tx}^2 v$ in $(0,T)\times\mathbb{R}$, $\partial_t f(0,x) = f_1(x)$ in $\mathbb{R}$. Then it is given by $\partial_t f(t,x) = H(t)\star f_1(x) - \int_0^t \partial_x H(t-s,\cdot)\star\partial_t v(s,\cdot)(x)ds$. Proceeding as previously, we get the stated result .

2.3 The wave equation with artificial viscosity

Let $g(t,x)$ be a fixed function. We consider the Cauchy problem for the wave equation with artificial viscosity

$$\begin{cases} \partial_t^2 u - \partial_x[(1+g)\partial_x u] - \varepsilon\partial_x^2[\partial_t u] = 0 \ \text{ in } (0,T)\times\mathbb{R} \\ \\ u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x) \ \text{ in } \mathbb{R}. \end{cases} \tag{2.13}$$

For $T>0$ and $\varepsilon>0$ fixed, we shall prove the following result

Proposition 2.3.1 *Let u_0, u_1, f_0, f_1 as in Theorem 2.1.1 and let $g \in W^{1,\infty}(0,T;$ $L^\infty(\mathbb{R}))$ be such that $g(0,x) = f_0(x)$, $g \geq 0$, $\partial_t g \leq 0$ a.e. in $(0,T)\times\mathbb{R}$. Then problem (2.13) has a unique solution $u \in C([0,T];H^1(\mathbb{R}))$ such that $u \in C^1([0,T];L^2(\mathbb{R})) \cap H^1(0,T;H^1(\mathbb{R}))$, $\partial_t^2 u \in L^2(0,T;H^{-1}(\mathbb{R}))$. Moreover, for every $t \in [0,T]$, the following energy inequality holds*

$$E^2(t) + 2\varepsilon\int_0^t |\partial_{tx}^2 u(s)|_2^2 \leq E_0^2 \tag{2.14}$$

where $E^2(t) = |\partial_t u(t)|_2^2 + |\sqrt{1+g}\ \partial_x u(t)|_2^2$ and $E_0 = E(0)$.

Our existence proof is based on a theorem of Lions [15] (Chap. III) that we recall hereafter. For a detailed study of a wave equation with time dependent velocity, we refer the reader to Reference [11] for example in the case of regular coefficients and to Reference [3] if not. We get in Reference [2] other results related to time-dependent velocity of the wave equation. The wave equation with such an artificial viscosity had been studied in [10, 19]. Note that (2.13) is equivalent to a Cauchy problem associated to a first order differential equation of the form $\frac{du(t)}{dt} = A(t)u(t)$ and we can solve it using the method developed in [4, 5, 18].

Theorem 2.3.2 *Let $\mathcal{H}$ be a Hilbert space provided with a norm $|\cdot|$ and an inner product $(\cdot,\cdot)$. Let $\mathcal{V}$ be a subspace of $\mathcal{H}$ provided with a prehilbertian norm $\|\cdot\|$ such that the injection $\mathcal{V}\subset\mathcal{H}$ is continuous. We consider a bilinear form B*

$$B : \mathcal{H}\times\mathcal{V} \ni (U,\phi) \to B(U,\phi) \in \mathbb{R}$$

such that $B(\cdot,\phi)$ is continuous on $\mathcal{H}$, for any fixed $\phi\in\mathcal{V}$ and

$$|B(\phi, \phi)| \geq \alpha \|\phi\|^2, \forall \phi \in \mathcal{V}, \text{ with } \alpha > 0.$$

Then, given a linear form L in $\mathcal{V}'$, there exists a solution U in $\mathcal{V}$ of the problem

$$B(U, \phi) = L(\phi) \ , \forall \phi \in \mathcal{V}.$$

The following result will be also useful in the sequel

Lemma 2.3.3 *For $k > 0$ and under hypotheses of Proposition 2.3.1, the operator $(-\partial_x[(1+g)\partial_x] + kI)$ is an isomorphism from $L^2(0, T, H^1(\mathbb{R}))$ into $L^2(0, T, H^{-1}(\mathbb{R}))$ and from $H^1(0, T, H^1(\mathbb{R}))$ into $H^1(0, T, H^{-1}(\mathbb{R}))$.*

Proof. Let $V = L^2(0, T, H^1(\mathbb{R}))$ and V' its dual. Thanks to the properties of g, the bilinear form

$$b(u, v) = \int_0^T \int_{\mathbb{R}} \Big((1 + g)(\partial_x u)(\partial_x v) + kuv\Big) dxdt \tag{2.15}$$

is continuous and coercive on $V \times V$. So for all $f \in V'$, there exists a unique solution to the variational equation

$$b(u, v) = \langle f, v \rangle_{V', V} \ , \ \forall v \in V. \tag{2.16}$$

Hence u is the unique solution in V of the equation

$$-\partial_x[(1 + g)\partial_x u] + ku = f. \tag{2.17}$$

Now if $f \in H^1(0, T, H^{-1}(\mathbb{R}))$, then taking the time derivative in (2.17), we get

$$-\partial_x[(1 + g)\partial_x(\partial_t u)] + k(\partial_t u) = \partial_t f + \partial_x[(\partial_t g)\partial_x u] \tag{2.18}$$

and since the right hand side of (2.18) is in V', we conclude thanks to the first step that $\partial_t u \in V$.

Now we come back to the proof of proposition 2.3.1.

Proof. Let u be a solution of (2.13). We set $\partial_t u = v, (\tilde{u}, \tilde{v}) = \exp(-kt)(u, v)$ where $k > 0$ is a constant. Hence (2.13) is equivalent to the system

$$\begin{cases} \partial_t \tilde{u} + k\tilde{u} - \tilde{v} = 0 \ \text{ in } (0,T)\times\mathbb{R} \\ \partial_t \tilde{v} - \partial_x[(1+g)\partial_x \tilde{u}] - \varepsilon\partial_x^2 \tilde{v} + k\tilde{v} = 0 \text{ in } (0,T)\times\mathbb{R} \\ \tilde{u}(0,x) = u_0(x), \tilde{v}(0,x) = u_1(x) \ \text{ in } \mathbb{R} \end{cases} \tag{2.19}$$

Let $\mathcal{H}$ be the Hilbert space $\mathcal{H} = L^2(0,T;H^1(\mathbb{R})) \times L^2(0,T;H^1(\mathbb{R}))$ provided with the norm $||.||_{\mathcal{H}}$ defined for all $U = (u,v) \in \mathcal{H}$ by

$$||U||^2_{\mathcal{H}} = \int_0^T (|||u(t)|||^2_{H^1(\mathbb{R})} + \|v(t)\|^2_{H^1(\mathbb{R})})dt \tag{2.20}$$

where $\|.\|_{H^1(\mathbb{R})}$ is the usual norm in $H^1(\mathbb{R})$ and

$$|||u(t)|||^2_{H^1(\mathbb{R})} = \int_{\mathbb{R}} [(1+g)|\partial_x u|^2 + k|u|^2]dx. \tag{2.21}$$

As $g \in L^\infty((0,T)\times\mathbb{R})$, $g \geq 0$ a.e., this norm is equivalent to the natural one.
Next we define the space of infinitely differentiable functions with compact support in $[0,T[\times\mathbb{R}$, $\mathcal{V} = (\mathcal{D}([0,T[\times\mathbb{R}))^2$ provided with the norm

$$|\phi|^2_{\mathcal{V}} = \|\phi\|^2_{\mathcal{H}} + \frac{1}{2}(|||\varphi(0)|||^2_{H^1(\mathbb{R})} + |\psi(0)|^2),\ \forall\phi = (\varphi,\psi) \in \mathcal{V} \tag{2.22}$$

which makes the injection $\mathcal{V} \subset \mathcal{H}$ continuous.

Now we introduce a bilinear form B on $\mathcal{H}\times\mathcal{V}$ and a linear form L on $\mathcal{V}$ defined by

$$\begin{aligned} B(U,\phi) &= \int_0^T\int_{\mathbb{R}} \Big(-(1+g)\partial_x u \partial^2_{tx}\varphi - ku\partial_t\varphi - (\partial_t g)(\partial_x u)(\partial_x\varphi)\Big)dxdt \quad (2.23) \\ &+ \int_0^T\int_{\mathbb{R}} \Big((1+g)(\partial_x(ku-v))(\partial_x\varphi) + k(ku-v)\varphi\Big)dxdt \\ &+ \int_0^T\int_{\mathbb{R}} \Big(-v\partial_t\psi + (1+g)(\partial_x u)(\partial_x\psi) + \varepsilon(\partial_x v)(\partial_x\psi) + kv\psi\Big)dxdt \end{aligned}$$

$$L(\phi) = \int_{\mathbb{R}} [(1+g(0,x))\partial_x u_0(x)\partial_x\varphi(0,x) + ku_0(x)\varphi(0,x) + u_1(x)\psi(0,x)]dx. \tag{2.24}$$

where $U = (u, v), \phi = (\varphi, \psi)$. Then clearly $B(\cdot, \phi)$ is continuous on $\mathcal{H}$, for any fixed $\phi \in \mathcal{V}$ and L is continuous on $\mathcal{V}$. Let us prove the corciveness of B on the space $\mathcal{V}$. First set for $(U, \phi) \in \mathcal{H} \times \mathcal{V}$

$$B_1(U,\phi) = \int_0^T \int_{\mathbb{R}} \Big(-(1+g)\partial_x u \partial^2_{tx}\varphi - ku\partial_t\varphi - (\partial_t g)(\partial_x u)(\partial_x \varphi) - v\partial_t \psi \Big) dxdt \quad (2.25)$$

then for all $(U, \phi) \in \mathcal{V} \times \mathcal{V}$, we have

$$\begin{aligned} B_1(U,\phi) = -B_1(\phi, U) - \int_0^T \int_{\mathbb{R}} (\partial_t g)(\partial_x u)(\partial_x \varphi) dxdt \\ + \int_{\mathbb{R}} [(1+g(0,x))\partial_x u(0,x)\partial_x\varphi(0,x) + ku(0,x)\varphi(0,x) + v(0,x)\psi(0,x)]dx \end{aligned} \quad (2.26)$$

so for $\phi \in \mathcal{V}$, we get

$$\begin{aligned} B_1(\phi,\phi) &= -\frac{1}{2}\int_0^T \int_{\mathbb{R}} (\partial_t g)|\partial_x\varphi|^2 dxdt \\ &+ \frac{1}{2}\int_{\mathbb{R}} [(1+g(0,x))|\partial_x\varphi(0,x)|^2 + k|\varphi(0,x)|^2 + |\psi(0,x)|^2]dx \\ &\geq \frac{1}{2}(\||\varphi(0)\||^2_{H^1(\mathbb{R})} + |\psi(0)|^2) \end{aligned} \quad (2.27)$$

because $\partial_t g \leq 0\, a.e.$ Now we set $B_2(U,\phi) = B(U,\phi) - B_1(U,\phi)$. As

$$\begin{aligned} B_2(\phi,\phi) &= k\int_0^T \int_{\mathbb{R}} ((1+g)|\partial_x\varphi|^2 + k|\varphi|^2) dxdt - k\int_0^T \int_{\mathbb{R}} \varphi\psi dxdt \\ &+ \varepsilon \int_0^T \int_{\mathbb{R}} |\partial_x\psi|^2 dxdt + k\int_0^T \int_{\mathbb{R}} |\psi|^2 dxdt \\ &\geq \frac{k}{2}\int_0^T \int_{\mathbb{R}} (1+g)|\partial_x\varphi|^2 dxdt + (k^2 - \frac{k}{2})\int_0^T \int_{\mathbb{R}} |\varphi|^2 dxdt \\ &+ \varepsilon \int_0^T \int_{\mathbb{R}} |\partial_x\psi|^2 dxdt + \frac{k}{2}\int_0^T \int_{\mathbb{R}} |\psi|^2 dxdt \end{aligned} \quad (2.28)$$

for all $\phi \in \mathcal{V}$, then adding (2.27) and (2.28) we get the coerciveness estimate

$$B(\phi,\phi) \geq \alpha \|\phi\|^2_{\mathcal{V}}, \forall\, \phi \in \mathcal{V} \quad (2.29)$$

with $\alpha = min(\varepsilon, \frac{k}{2}, 1) > 0$. Thus thanks to the Theorem 2.3.2, there exists a solution $\tilde{U}$ in $\mathcal{H}$ to the variational equation

$$B(\tilde{U},\phi) = L(\phi),\ \forall \phi \in \mathcal{V}. \quad (2.30)$$

Set $\tilde{U} = (\tilde{u}, \tilde{v})$, then using (2.30), we get

$$-\partial_x[(1+g)\partial_x(\partial_t\tilde{u} + k\tilde{u} - \tilde{v})] + k(\partial_t\tilde{u} + k\tilde{u} - \tilde{v}) = 0 \tag{2.31}$$

and

$$\partial_t\tilde{v} - \partial_x[(1+g)\partial_x\tilde{u}] - \varepsilon\partial_x^2\tilde{v} + k\tilde{v} = 0 \tag{2.32}$$

in the sense of distributions. In particular (2.31) gives

$$\partial_t\Big[-\partial_x[(1+g)\partial_x\tilde{u}] + k\tilde{u}\Big] = \partial_x[(1+g)\partial_x(k\tilde{u} - \tilde{v})] - k(k\tilde{u} - \tilde{v}) - \partial_x[(\partial_t g)\partial_x\tilde{u}]$$

so $\partial_t\Big[-\partial_x[(1+g)\partial_x\tilde{u}] + k\tilde{u}\Big] \in L^2(0,T,H^{-1}(\mathbb{R}))$, then $-\partial_x[(1+g)\partial_x u] + ku \in H^1(0,T,H^{-1}(\mathbb{R}))$. Therefore using Lemma 2.3.3, we get $\tilde{u} \in H^1(0,T,H^1(\mathbb{R}))$. Coming back to (2.31), we conclude that

$$\partial_t\tilde{u} + k\tilde{u} - \tilde{v} = 0 \ \text{ in } L^2(0,T,H^1(\mathbb{R})). \tag{2.33}$$

On the other hand (2.32) leads to $\partial_t\tilde{v} \in L^2(0,T,H^{-1}(\mathbb{R}))$. Thus $\partial_t(\tilde{u},\tilde{v})$ belongs to $L^2(0,T,H^1(\mathbb{R})) \times L^2(0,T,H^{-1}(\mathbb{R}))$ and $(\tilde{u},\tilde{v}) \in \mathcal{C}([0,T]; H^1(\mathbb{R}) \times L^2(\mathbb{R}))$, so the traces $\tilde{u}(0,.), \tilde{v}(0,.)$ are well defined in $H^1(\mathbb{R})$ and $L^2(\mathbb{R})$ respectively (see Reference [16]). Taking the duality product of (2.31) by a function $\varphi \in \mathcal{D}([0,T[\times\mathbb{R})$, the duality product of (2.32) by a function $\psi \in \mathcal{D}([0,T[\times\mathbb{R})$, adding the obtained terms and using (2.30) we show that

$$\tilde{u}(0,x) = u_0(x), \tilde{v}(0,x) = u_1(x) \text{ a.e. in } \mathbb{R} \tag{2.34}$$

so $(\tilde{u},\tilde{v})$ solves the problem (2.19) and $u = \exp(kt)\tilde{u}$ is a solution of (2.13) satisfying the properties of Proposition 2.3.1.

Now, to prove the energy estimate (2.14), we multiply the equation (2.13) by $\partial_t u$ and integrate by parts with respect to x , we get

$$\frac{1}{2}\frac{d}{dt}|\partial_t u|_2^2 + \frac{1}{2}\int_{\mathbb{R}}(1+g)\,\partial_t\left[(\partial_x u)^2\right]\,dx + \varepsilon|\partial_{tx}^2 u(t)|_2^2 = 0$$

which implies

$$\frac{d}{dt}\left[|\partial_t u|_2^2 + |\sqrt{1+g}\,\partial_x u|_2^2\right] + 2\varepsilon|\partial_{tx}^2 u(t)|_2^2 = \int_{\mathbb{R}}(\partial_t g)|\partial_x u|^2\,dx \le 0 \tag{2.35}$$

and leads to (2.14).

For the uniqueness, since (2.13) is linear, it is enough to show that if $u_0 = 0$, $u_1 = 0$, then $u = 0$ and this is an immediate consequence of the energy inequality (2.14).

2.4 The coupling problem with artificial viscosity

We consider the following approximation of the problem (2.1)

$$\begin{cases} \partial_t f - \partial_x^2 f = -\partial_x u \ \text{ in } (0,T)\times\mathbb{R} \\ f(0,x) = f_0(x) \ \text{ in } \mathbb{R} \\ \partial_t^2 u - \partial_x[(1+f)\partial_x u] - \varepsilon\partial_x^2[\partial_t u] = 0 \ \text{ in } (0,T)\times\mathbb{R} \\ u(0,x) = u_0(x), \partial_t u(0,x) = u_1(x) \ \text{ in } \mathbb{R} \end{cases} \tag{2.36}$$

where $\varepsilon > 0$ is fixed.Then we have

Theorem 2.4.1 *Under the hypothesis of Theorem 2.1.1, there exists $T > 0$ which is independent of ε satisfying*

$$F_0 - C_1 T^{\frac{3}{4}} E_0 \ \geq 0, \ F_0 - C_2 T^{\frac{1}{4}} E_0 \ \geq \ 0 \tag{2.37}$$

where $E_0 = \Big(|u_1|_2^2 + |\sqrt{1+f_0}\, u_0'|_2^2\Big)^{\frac{1}{2}}$ and C_1, C_2 are the constants given in Proposition 2.2.1 such that problem (2.36) *admits a unique weak solution* $(f^\varepsilon, u^\varepsilon)$ *defined on the time interval [0,T]. This solution satisfies*

$$\begin{aligned} & u^\varepsilon \in C([0,T]; H^1(\mathbb{R})) \cap C^1([0,T]; L^2(\mathbb{R})) \cap H^1(0,T; H^1(\mathbb{R})\ ) \\ & f^\varepsilon \in W^{1,\infty}((0,T)\times\mathbb{R}) \\ & f^\varepsilon \geq 0, \ \partial_t f^\varepsilon \leq 0, \quad a.e. \ in \ (0,T)\times\mathbb{R}. \end{aligned} \tag{2.38}$$

Moreover, the following uniform bounds with respect to ε hold

$$|f^\varepsilon|_\infty \leq |f_0|_\infty + F_0, \ |\partial_x f^\varepsilon|_\infty \leq |f_0'|_\infty + F_0, \ |\partial_t f^\varepsilon|_\infty \leq |f_1|_\infty + F_0 \tag{2.39}$$

$$\|u^\varepsilon\|^2 \ \leq \ \exp T\Big(|u_0|_2^2 + E_0^2 T\Big) \tag{2.40}$$

$$\|\partial_t u^\varepsilon\|^2 + \|\partial_x u^\varepsilon\|^2 \ \leq \ |u_1|_2^2 + |\sqrt{1+f_0}\, u_0'|_2^2. \tag{2.41}$$

The proof will be done in several steps (see References [20, 17] for example). We first prove the existence of solution by using the following iterative scheme.

The iterative scheme. We set $u^{(0)} = 0$ and $f^{(0)} = 0$ and define for all $n \in \mathbb{N}$, f^{n+1} as the solution of the linear heat equation

$$\partial_t f^{n+1} - \partial_x^2 f^{n+1} = -\partial_x u^n \text{ in } (0,T) \times \mathbb{R},\ f^{n+1}(0,x) = f_0(x) \text{ in } \mathbb{R} \tag{2.42}$$

then we define u^{n+1} as the solution of the linear wave equation

$$\begin{cases} \partial_t^2 u^{n+1} - \partial_x[(1+f^{n+1})\partial_x u^{n+1}] - \varepsilon \partial_x^2[\partial_t u^{n+1}] = 0 \text{ in } (0,T) \times \mathbb{R} \\ u^{n+1}(0,x) = u_0(x), \partial_t u(0,x) = u_1(x) \text{ in } \mathbb{R}. \end{cases} \tag{2.43}$$

The following results give uniform estimates for the sequence $(f^n, u^n)_n$ of solutions.

Proposition 2.4.2 *Under hypotheses of Theorem 2.1.1, let $T > 0$ be the time existence defined by* (2.37)*. Then the sequence of solutions $(f^n, u^n)_n$ is well defined on [0,T], $f^n(t,x) \geq 0$, $\partial_t f^n(t,x) \leq 0$ a.e. in $(0,T) \times \mathbb{R}$ for all $n \in \mathbb{N}$ and the uniform bounds with respect to n and ε are satisfied*

$$|f^n|_\infty \leq |f_0|_\infty + F_0,\ |\partial_x f^n|_\infty \leq |f_0'|_\infty + F_0,\ |\partial_t f^n|_\infty \leq |f_1|_\infty + F_0 \tag{2.44}$$

$$\|u^n\|^2 \leq \exp T\Big(|u_0|_2^2 + E_0^2 T\Big),\ \|\partial_t u^n\|^2 + \|\partial_x u^n\|^2 + 2\varepsilon \int_0^T |\partial_{tx}^2 u^n(s)|^2\, ds \leq E_0^2 \tag{2.45}$$

Proof. The proof is done by induction on n. Suppose the results of Proposition 2.4.2 hold for (f^n, u^n). Using the results of Proposition 2.2.1, we deduce that f^{n+1} is well defined and thanks to (2.45) at order n, we obtain $f^{n+1}(t,x) \geq F_0 - C_1 T^{\frac{3}{4}} \|\partial_x u^n\| \geq F_0 - C_1 T^{\frac{3}{4}} E_0 \geq 0$ and $\partial_t f^{n+1}(t,x) \leq -F_0 + C_2 T^{\frac{1}{4}} \|\partial_t u^n\| \leq -F_0 + C_2 T^{\frac{1}{4}} E_0 \leq 0$ a.e. in $(0,T) \times \mathbb{R}$ then we have $f^{n+1}(t,x) \geq 0$, $\partial_t f^{n+1}(t,x) \leq 0$ *a.e.* in $(0,T) \times \mathbb{R}$.
Using again Proposition 2.2.1, thanks to (2.37) we get $|f^{n+1}|_\infty \leq |f_0|_\infty + C_1 T^{\frac{3}{4}} \|\partial_x u^n\|$ and then $|f^{n+1}|_\infty \leq |f_0|_\infty + C_1 T^{\frac{3}{4}} E_0 \leq |f_0|_\infty + F_0$. Similarly, we have $|\partial_x f^{n+1}|_\infty \leq |f_0'|_\infty + C_2 T^{\frac{1}{4}} \|\partial_x u^n\|$ and thus we have $|\partial_x f^{n+1}|_\infty \leq |f_0'|_\infty + C_2 T^{\frac{1}{4}} E_0 \leq |f_0'|_\infty + F_0$ and $|\partial_t f^{n+1}|_\infty \leq |f_1|_\infty + C_2 T^{\frac{1}{4}} \|\partial_t u^n\| \leq |f_1|_\infty + C_2 T^{\frac{1}{4}} E_0 \leq |f_1|_\infty + F_0$ and finally we get (2.44) at order $(n+1)$.

From Proposition 2.3.1 it follows that u^{n+1} is well defined and we get easily (2.45) at order $(n+1)$ by using (2.14). Since $(u^{n+1})^2(t,x) = u_0^2(x)+2\int_0^t u^{n+1}(s,x)\partial_t u^{n+1}(s,x)\,ds$, we deduce the estimate

$$|u^{n+1}(t)|_2^2 \leq |u_0|_2^2 + \int_0^t |u^{n+1}(s)|_2^2\,ds + \int_0^t |\partial_t u^{n+1}(s)|_2^2\,ds \tag{2.46}$$

so by using (2.45) and Gronwall's inequality we obtain (2.40).

Convergence of the iterative scheme. We introduce the Banach space $\mathcal{E}$ defined by $\mathcal{E} = \mathcal{C}([0,T];H^1(\mathbb{R})) \cap \mathcal{C}^1([0,T];L^2(\mathbb{R})) \cap H^1(0,T;H^1(\mathbb{R}))$. We shall show that $(f^n)_n$ and $(u^n)_n$ are Cauchy sequences in $W^{1,\infty}((0,T)\times\mathbb{R})$ and $\mathcal{E}$ respectively. For this purpose, we set $g^n = f^{n+1} - f^n$, $v^n = u^{n+1} - u^n$ for all $n \in \mathbb{N}$ then g^n satisfies the heat equation

$$\partial_t g^n - \partial_x^2 g^n = -\partial_x v^{n-1} \ \text{ in } (0,T)\times\mathbb{R},\ g^n(0,x) = 0 \ \text{ in } \mathbb{R} \tag{2.47}$$

and v^n the wave equation with source term

$$\begin{cases} \partial_t^2 v^n - \partial_x[(1+f^{n+1})\partial_x v^n] - \varepsilon\partial_x^2[\partial_t v^n] = \partial_x[g^n\partial_x u^n] \text{ in } (0,T)\times\mathbb{R} \\ v^n(0,x) = 0, \partial_t v^n(0,x) = 0 \text{ in } \mathbb{R}. \end{cases} \tag{2.48}$$

Multiplying (2.48) by $\partial_t v^n$ and integrating by parts with respect to space variable, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[|\partial_t v^n(t)|_2^2 + |\sqrt{1+f^{n+1}}\,\partial_x v^n(t)|_2^2] + \varepsilon|\partial_{tx}^2 v^n(t)|_2^2 \\ &= \frac{1}{2}\int_{\mathbb{R}} \partial_t f^{n+1}|\partial_x v^n|^2\,dx - \int_{\mathbb{R}} g^n(\partial_x u^n)(\partial_{tx}^2 v^n)\,dx. \end{aligned}$$

Using the property satisfied by $\partial_t f^{n+1}$ and Young's inequality we get the estimate

$$\begin{aligned} &|\partial_t v^n(t)|_2^2 + |\sqrt{1+f^{n+1}}\,\partial_x v^n(t)|_2^2 + 2\varepsilon\textstyle\int_0^t |\partial_{tx}^2 v^n(s)|_2^2\,ds \\ &\leq \frac{1}{\varepsilon}\int_0^t |g^n\partial_x u^n(s)|_2^2\,ds + \varepsilon\int_0^t |\partial_{tx}^2 v^n(s)|_2^2\,ds. \end{aligned}$$

Hence, by (2.45), it holds that

$$|\partial_t v^n(t)|_2^2 + |\sqrt{1+f^{n+1}}\,\partial_x v^n(t)|_2^2 + \varepsilon\int_0^t |\partial_{tx}^2 v^n(s)|_2^2\,ds \leq \frac{1}{\varepsilon}E_0^2\int_0^t |g^n(s)|_\infty^2\,ds.$$

Let us consider the heat equation (2.47). Proceeding as in the first section we get the inequality $|g^n(t)|_\infty \leq \Big[\int_0^t |H(t-s)|_2^2\, ds\Big]^{\frac{1}{2}} \Big[\int_0^t |\partial_x v^{n-1}(s)|_2^2\, ds\Big]^{\frac{1}{2}}$ which shows that

$$|g^n(t)|_\infty \ \leq \ C_3 t^{\frac{1}{4}} \Big[\int_0^t |\partial_x v^{n-1}(s)|_2^2\, ds\Big]^{\frac{1}{2}} \tag{2.49}$$

where $C_3 > 0$ is a constant which is independent of the data, ε and n. Since we have $f^{n+1} \geq 0$, then we get the estimate

$$|\partial_t v^n(t)|_2^2 + |\partial_x v^n(t)|_2^2 + \varepsilon \int_0^t |\partial_{tx}^2 v^n(s)|_2^2\, ds \leq \frac{C_3^2 T^{3/2}}{\varepsilon} E_0^2 \int_0^t |\partial_x v^{n-1}(s)|_2^2\, ds \tag{2.50}$$

for all $n \in \mathbb{N}^*$. In particular we have

$$|\partial_x v^n(t)|_2^2 \leq \frac{C_3^2 T^{\frac{3}{2}}}{\varepsilon} E_0^2 \int_0^t |\partial_x v^{n-1}(s)|_2^2\, ds. \tag{2.51}$$

We set $V^n(t) = |\partial_x v^n(t)|_2^2$ and $C_\varepsilon = \frac{C_3^2 T^{3/2}}{\varepsilon} E_0^2$. Then (2.51) takes the form

$$V^n(t) \leq C_\varepsilon \int_0^t V^{n-1}(s)\, ds \ \ \forall t \in [0,T]\,, \forall n \in \mathbb{N}^* \tag{2.52}$$

which implies that $V^n(t) \leq C_\varepsilon^n \frac{t^{n-1}}{(n-1)!} \int_0^t V^0(s)\, ds$ for all $t \in [0,T]$ and for all $n \in \mathbb{N}^*$. Thus we obtain

$$\|\partial_x u^{n+1} - \partial_x u^n\|^2 \leq 2 C_\varepsilon^n \frac{T^n}{(n-1)!} E_0^2\ , \ \forall n \in \mathbb{N}.$$

We deduce that $(\partial_x u^n)_n$ is a Cauchy sequence in $\mathcal{C}([0,T]; L^2(\mathbb{R}))$, then coming back to (2.50), we see that $(\partial_t u^n)_n$ and $(\partial_{tx}^2 u^n)_n$ are also Cauchy sequences in $\mathcal{C}([0,T]; L^2(\mathbb{R}))$ and $L^2((0,T)\times\mathbb{R})$ respectively. Proceeding as for the proof of (2.45), we can show that

$$\|v^n\|^2 \leq \exp T \|\partial_t v^n\|^2. \tag{2.53}$$

We conclude that $(u^n)_n$ is a Cauchy sequence in $\mathcal{E}$. Moreover, thanks to (2.49), we see that $(f^n)_n$ is a Cauchy sequence in $L^\infty((0,T)\times\mathbb{R})$. Then, proceeding along the lines of the proof of Proposition 2.2.1, we obtain

$$|\partial_x g^n|_\infty \leq C_2 T^{\frac{1}{4}} \|\partial_x v^{n-1}\|,\ |\partial_t g^n|_\infty \leq C_2 T^{\frac{1}{4}} \|\partial_t v^{n-1}\|$$

and therefore$(f^n)_n$ is a Cauchy sequence in $W^{1,\infty}((0,T)\times\mathbb{R})$.

It follows that the sequences $(u^n)_n$ and $(f^n)_n$ converge strongly, respectively in $\mathcal{E}$ and $W^{1,\infty}((0,T)\times\mathbb{R})$ towards u^ε and f^ε. Letting $n\to+\infty$ in (2.42) and (2.43), we conclude that $(f^\varepsilon,u^\varepsilon)$ is a solution of the coupling problem with viscosity (2.36) and satisfies the inequalities (2.38)-(2.41).

Uniqueness of the solution. Let $(f^\varepsilon,u^\varepsilon)$ and $(h^\varepsilon,w^\varepsilon)$ be two solutions of (2.36), we set $g^\varepsilon=f^\varepsilon-h^\varepsilon$ and $v^\varepsilon=u^\varepsilon-w^\varepsilon$. Then, proceeding as in the proof of (2.51), (2.53) and (2.49), we obtain the following inequalities

$$
\begin{aligned}
|\partial_x v^\varepsilon(t)|_2^2 &\le \frac{C_3^2T^{3/2}}{\varepsilon}E_0^2\int_0^t|\partial_x v^\varepsilon(s)|_2^2\,ds &\quad (2.54)\\
\|v^\varepsilon\|^2 &\le \exp T\|\partial_x v^\varepsilon\|^2 &\quad (2.55)\\
|g^\varepsilon(t)|_\infty^2 &\le C_3^2T^{1/2}\int_0^t|\partial_x v^\varepsilon(s)|_2^2\,ds. &\quad (2.56)
\end{aligned}
$$

So, by Gronwall's inequality, (2.54) gives $|\partial_x v^\varepsilon(t)|_2^2=0$ a.e. in $(0,T)$ then using (2.55) and (2.56), we obtain $v^\varepsilon(t,x)=0$, $g^\varepsilon(t,x)=0$ a.e. in $(0,T)\times\mathbb{R}$ which ends the proof of Theorem 2.4.1.

2.5 End of the proof of Theorem 2.1.1

Let $(f^\varepsilon,u^\varepsilon)$ be the solution of (2.36) provided by Theorem 2.4.1. Using the estimates therein, we deduce that $(f^\varepsilon,u^\varepsilon)_\varepsilon$ is uniformly bounded in the Banach space

$$\mathcal{F}=W^{1,\infty}((0,T)\times\mathbb{R})\times\Big(L^\infty(0,T;H^1(\mathbb{R}))\cap W^{1,\infty}(0,T;L^2(\mathbb{R}))\Big).$$

Therefore we can extract a subsequence, still denoted $(f^\varepsilon,u^\varepsilon)_\varepsilon$ such that when ε goes to 0, $(f^\varepsilon,u^\varepsilon)_\varepsilon$ converges weakly-$\star$ in $\mathcal{F}$ towards a limit (f,u). Thanks to the compactness of the imbedding of $W^{1,\infty}(0,T)\times\mathbb{R})$ into $L^\infty_{loc}((0,T)\times\mathbb{R})$, we can pass to the limit in (2.36) and we obtain

$$
\begin{aligned}
&\partial_t f-\partial_x^2 f=-\partial_x u \ \text{ in } \mathcal{D}'((0,T)\times\mathbb{R})\\
&\partial_t^2 u-\partial_x[(1+f)\partial_x u]=0 \ \text{ in } \mathcal{D}'((0,T)\times\mathbb{R})
\end{aligned}
\qquad (2.57)
$$

Moreover as $W^{1,\infty}((0,T)\times\mathbb{R})\times W^{1,\infty}(0,T;L^2(\mathbb{R}))$ is continuously embedded in $C([0,T];L^\infty(\mathbb{R}))\times C([0,T];L^2(\mathbb{R}))$, we deduce that

$$f(0,x)=f_0(x) \text{ in } L^\infty(\mathbb{R}),\ u(0,x)=u_0(x) \text{ in } L^2(\mathbb{R}). \tag{2.58}$$

On the other hand, since $f^\varepsilon \to f$ a.e. in $(0,T)\times\mathbb{R}$ then thanks to (2.38), we get

$$f \geq 0 \text{ a.e. in } (0,T)\times\mathbb{R}. \tag{2.59}$$

Now let $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}), \varphi \geq 0$. Since we have

$$0 \ \geq \ \int_0^T\int_{\mathbb{R}} \partial_t f^\varepsilon \varphi(t,x)\,dxdt \to \int_0^T\int_{\mathbb{R}} \partial_t f \varphi(t,x)\,dxdt,$$

then we deduce that $\partial_t f \leq 0$ in $\mathcal{D}'((0,T)\times\mathbb{R})$ and since $\partial_t f \in L^\infty((0,T)\times\mathbb{R})$, we conclude that

$$\partial_t f \leq 0 \text{ a.e. in } (0,T)\times\mathbb{R}. \tag{2.60}$$

Finally (2.57) implies $\partial_t^2 u \in L^2(0,T;H^{-1}(\mathbb{R}))$. So $\partial_t u \in H^1(0,T;H^{-1}(\mathbb{R}))$ and $\partial_t u(0,.)$ is well defined in $L^2(\mathbb{R})$. Thus multiplying the second pde of (2.36) by $\phi \in \mathcal{D}([0,T[\times\mathbb{R})$ and integrating by parts, we get

$$\int_0^T\int_{\mathbb{R}}\Big[u^\varepsilon\partial_t^2\phi+(1+f^\varepsilon)(\partial_x u^\varepsilon)(\partial_x\phi)+\varepsilon(\partial_{tx}^2 u^\varepsilon)(\partial_x\phi)\Big]\,dxdt = \int_{\mathbb{R}} u_1(x)\phi(0,x)\,dx-\int_{\mathbb{R}} u_0(x)\partial_t\phi(0,x)\,dx,$$

which gives when ε goes to 0

$$\int_0^T\int_{\mathbb{R}}\Big[u\partial_t^2\phi+(1+f)(\partial_x u)(\partial_x\phi)\Big]\,dxdt = \int_{\mathbb{R}} u_1(x)\phi(0,x)\,dx-\int_{\mathbb{R}} u_0(x)\partial_t\phi(0,x)\,dx.$$

Integrating by parts this last equality and making use of (2.57), we arrive at

$$\int_{\mathbb{R}}[\partial_t u(0,x)-u_1(x)]\phi(0,x)\,dx-\int_{\mathbb{R}}[u(0,x)-u_0(x)]\partial_t\phi(0,x)\,dx=0$$

which gives thanks to (2.58)

$$\int_{\mathbb{R}}[\partial_t u(0,x)-u_1(x)]\phi(0,x)\,dx=0\ \ ,\forall\phi\in\mathcal{D}([0,T[\times\mathbb{R}).$$

Therefore, we have $\partial_t u(0,x)=u_1(x)$ in $\mathcal{D}'(\mathbb{R})$ and then in $L^2(\mathbb{R})$. Hence Theorem 2.1.1 is proved.

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Troisième partie

Le problème aux limites de Vlasov-Poisson-Fokker-Planck

Chapitre 1

Introduction à la physique des plasmas

Un plasma est un gaz (électriquement neutre) formé d'ions et d'électrons qui coexistent éventuellement avec des atomes neutres mais, le terme plasma introduit par le physicien I. Langmuir, désignait initialement un gaz entièrement ionisé. Le plasma compose les 99% de l'univers d'après les astrophysiciens ; il est présent dans les étoiles et dans l'ionosphère où il joue un rôle important pour les télécommunications en réfléchissant les ondes radio. Sur terre, il n'existe pas à l'état naturel car l'ionisation d'un gaz requiert des températures très élevées (de l'ordre de 10 000°C) par contre, il est formé en laboratoire pour diverses applications de haute technologie telles que l'optique et la microélectronique mais surtout dans le développement de la fusion thermonucléaire.

1.1 Description de l'évolution d'un plasma

Plusieurs modèles sont utilisés pour décrire l'évolution d'un plasma, l'idéal étant de décrire l'évolution de chaque particule composant le plasma. Dans la pratique toutefois, cela s'avère inapplicable en raison du nombre élevé de particules présentes dans un plasma, on a alors recours à des modèles approchés.

Les modèles fluides sont utilisés pour un plasma proche de l'état d'équilibre thermodynamique mais ils ne peuvent pas rendre compte de certaines instabilités qui relèvent essentiellement de particularités de trajectoires des particules, ces dernières ne pouvant être décrites que par des équations cinétiques.

Les modèles cinétiques reposent sur une description statistique du plasma en considérant la fonction de distribution des particules dans l'espace des phases position-vitesse. Habituellement, on définit une fonction de distribution f_α pour chaque type de particules présentes dans le plasma. Le nombre probable de particules α qui au temps t occupent la position x à dx près, animées d'une vitesse v à dv près est alors donné par $f_\alpha(t,x,v)dxdv$.

Ces modèles sont de type Boltzmann lorsque les interactions binaires entre particules proches dominent et de type Vlasov quand les interactions sont régies par le champ électique moyen que les particules engendrent.

1.1.1 Les équations de Vlasov-Poison

Dans la suite, on considèrera que "le plasma est formé d'un seul type de particules" et on omettra l'indice α qui faisait référence au type de la particule. Soit $f(t,x,v)$ la fonction de distribution des particules, la densité de particules n et leur nombre total N sont donnés par

$$n(t,x) = \int f(t,x,v)dv, \quad N = \int\int f(t,x,v)dxdv.$$

La densité massique ρ et la vitesse macroscopique u sont telles que

$$\rho(t,x) = mn(t,x), \quad n(t,x)u(t,x) = \int vf(t,x,v)dv$$

où m représente la masse d'une particule de l'espèce considérée, alors que la densité de charge et la densité de courant sont données respectivement par

$$\sigma(t,x) = qn(t,x)$$

$$j(t,x) = qn(t,x)u(t,x).$$

où q représente la charge de la particule considérée.

L'approximation du champ moyen consiste à négliger les collisions. On peut alors considérer que toutes les particules du plasma se déplacent sous l'action de deux champs $E(t,x)$ et $B(t,x)$. Les particules obéissent alors aux équations de mouvement

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = \frac{q}{m}(E + v \times B). \tag{1.1}$$

Dans ces conditions, la fonction de distribution $f(t,x,v)$ vérifie l'équation de Liouville qui exprime la conservation du nombre de particules au cours du mouvement

$$\frac{d}{dt}(f(t,x,v)dxdv) = 0$$

ou encore

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx}{dt} \cdot \nabla_x f + \frac{dv}{dt} \cdot \nabla_v f = 0$$

car le mouvement étant hamiltonien, il laisse invariant l'élément $dxdv$. En utilisant les relations (1.1), on obtient

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \frac{q}{m}(E + v \times B) \cdot \nabla_v f = 0. \tag{1.2}$$

Les champs moyens E et B sont liés par les équations de Maxwell à la densité de charge σ et à la densité de courant j par

$$\begin{cases} \varepsilon_0 \dfrac{\partial E}{\partial t} = \dfrac{1}{\mu_0} \nabla \times B - j, & \varepsilon_0 \nabla \cdot E = \sigma \\ \dfrac{\partial B}{\partial t} = -\nabla \times E, & \nabla \cdot B = 0 \end{cases} \tag{1.3}$$

où ε_0 et μ_0 désignent respectivement la permittivité électrique et la perméabilité magnétique dans le vide. Les équations (1.2) et (1.3) forment un système complet qui permet de décrire l'évolution du plasma à partir de l'état initial défini par la donnée à $t = 0$ de la fonction de distribution f_0 et des champs E_0 et B_0 vérifiant $\nabla \cdot B_0 = 0$ et $\varepsilon_0 \nabla \cdot E_0 = \sigma_0$. Le couplage des deux équations précédentes est appelé le système de Vlasov-Maxwell. En approximation électrostatique, le champ électrique $E = -\nabla\varphi$ où φ est le potentiel électrique qui satisfait à l'équation de Poisson

$$-\varepsilon_0 \Delta \varphi = \sigma \tag{1.4}$$

et le système formé par les équations (1.2) et (1.4) est appelé le système de Vlasov-Poisson.

Une conséquence importante des équations précédentes est la conservation de l'énergie. En supposant que le plasma est localisé dans l'espace c.à.d. $f \to 0$ quand $|x| \to \infty$ ou qu'il soit confiné dans un volume fermé avec des conditions aux limites appropriées, alors en désignant par $\mathcal{E}_c = \frac{1}{2} m \int |v|^2 f(t,x,v) dxdv$ l'énergie cinétique, il vient de l'équation de Liouville (1.2) que

$$\frac{d\mathcal{E}_c}{dt} = \int E \cdot j dx$$

ensuite d'après les équations de Maxwell (si on s'intéresse au système complet), on a

$$\frac{1}{2}\frac{d}{dt}\int(\varepsilon_0|E|^2+\mu_0^{-1}|B|^2)dx=-\int E\cdot j\,dx$$

d'où on déduit la conservation de l'énergie totale exprimée par

$$\frac{d}{dt}\Big(\mathcal{E}_c+\frac{1}{2}\int(\varepsilon_0|E|^2+\mu_0^{-1}|B|^2)dx\Big)=0.$$

1.1.2 Le système de Vlasov-Poisson-Fokker-Planck

L'effet des collisions sur le mouvement des particules quoique faible en général, peut avoir des conséquences importantes. Leur importance pour la fusion contrôlée vient du fait que les collisions imposent des limites au confinement d'un plasma (chaud et peu dense) dans un champ magnétique. Hormis la conductibilité thermique, l'effet le plus important des collisions se traduit par la diffusion perpendiculairement au champ magnétique.

Pour décrire cet effet, considérons un plasma homogène dans l'espace et observons le mouvement d'une particule témoin. Notons $P(\Delta t, v, \Delta v)$ la probabilité pour que la vitesse de la particule subisse une variation Δv du fait des collisions pendant le temps Δt, à partir de la vitesse initiale v. La fonction de distribution $f(t,v)$ à l'instant t se déduit de celle à l'instant $t-\Delta t$ en considérant toutes les variations possibles Δv pendant Δt à partir de $v-\Delta v$; elle est donnée par

$$f(t,v)=\int f(t-\Delta t, v-\Delta v)P(\Delta t, v-\Delta v, \Delta v)d(\Delta v).$$

Les relations suivantes $\int P(\Delta t, v, \Delta v)d(\Delta v)=1$, $\int \Delta v_i P(\Delta t, v, \Delta v)d(\Delta v)=\langle \Delta v_i\rangle$ et $\int \Delta v_i\Delta v_j P(\Delta t, v, \Delta v)d(\Delta v)=\langle\Delta v_i\Delta v_j\rangle$ sont satisfaites avec $\langle\Delta v_i\rangle$ et $\langle\Delta v_i\Delta v_j\rangle$ représentant respectivement la moyenne statistique des variations Δv_i des composantes v_i de la vitesse et la moyenne quadratique. On peut montrer que $\langle\Delta v_i\rangle$ et $\langle\Delta v_i\Delta v_j\rangle$ sont proportionnelles à Δt, plus précisément

$$\langle\Delta v_i\rangle=\frac{q}{m}\Delta t\ \langle E_i\rangle,\quad \langle\Delta v_i\Delta v_j\rangle=C\frac{q^2}{m^2}\Delta t\ \delta_{ij}$$

où δ est le symbole de Kronecker et C une constante. En admettant que $f(t,v)$ et $P(t,v,\Delta v)$ varient peu lorsque v varie de Δv, on peut utiliser un développement limité à l'ordre 2 de $f(t-\Delta t, v-\Delta v)P(\Delta t, v-\Delta v, \Delta v)$ au point v qui conduit à l'approximation suivante de $f(t,v)$

$$f(t,v) = f(t-\Delta t, v) - \frac{\partial}{\partial v_i}(f(t-\Delta t, v)\langle\Delta v_i\rangle) + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}(f(t-\Delta t, v)\langle\Delta v_i \Delta v_j\rangle).$$

En divisant cette équation par Δt, on obtient lorsque $\Delta t \to 0$ (on pose $\frac{q}{m} = 1$ pour simplifier) l'équation suivante

$$\frac{\partial f}{\partial t} + E \cdot \nabla_v f - \frac{C}{2}\Delta_v f = 0 \tag{1.5}$$

valable rappelons-le, dans le cas d'un plasma homogène en espace. Dans le cas général, on a

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + E \cdot \nabla_v f - \frac{C}{2}\Delta_v f = 0 \tag{1.6}$$

qui couplée avec (1.4) forme le système de Vlasov-Poisson-Fokker-Planck.

Pour finir notons que cet effet des collisions sur les particules d'un plasma tend à établir une distribution de vitesses isotrope et stationnaire c.à.d. une distribution maxwellienne (en $\exp(-|v|^2)$). Pour plus de détails, on pourra consulter [8, 12, 23, 25, 28].

Chapitre 2

Boundary value problem of Vlasov-Poisson-Fokker-Planck system : Behavior of solutions in presence of a small coefficient of diffusion

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In this paper, we prove global existence of a weak solution for the boundary value problem of the Vlasov-Poisson-Fokker-Planck (VPFP) system in 1 and 2 dimensions. The method involves a detailed analysis of the linear Vlasov-Fokker-Planck equation which leads to uniform estimates of the solutions of (VPFP) system with respect to the diffusion coefficient. As a consequence we show that these solutions converge to those of Vlasov-Poisson system when the diffusion coefficient goes to 0.

Key words. nonlinear problem, semilinear hyperbolic systems.
AMS (MOS) classification : 35Q20, 35D05, 82A45, 85A05

2.1 Introduction and main result

We are dealing with the boundary problem of the Vlasov-Poisson-Fokker-Planck system (VPFP)

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_v f - \sigma \Delta_v f = 0, \ (t,x,v) \in \mathbb{R}^+ \times \omega \times \mathbb{R}^d \\ \Delta_x \Phi(t,x) = \rho(t,x), \ (t,x) \in \mathbb{R}^+ \times \omega, \ \Phi(t,x) = 0, \ (t,x) \in \mathbb{R}^+ \times \partial\omega \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\, dv, \ (t,x) \in \mathbb{R}^+ \times \omega \\ f(0,x,v) = f_0(x,v), \ (x,v) \in \omega \times \mathbb{R}^d \\ f(t,x,v) = g(t,x,v), \ (t,x,v) \in \mathbb{R}^+ \times \Gamma_- \end{cases} \tag{2.1}$$

where $f(t,x,v), \Phi(t,x)$ and $\rho(t,x)$ are respectively the distribution function of the particles in a plasma, the self-consistent potential and the electric charge. The position x belongs to a bounded regular domain ω of $\mathbb{R}^d$, $d \geq 1$, $v \in \mathbb{R}^d$ is the velocity, $t \geq 0$ is the time and $\sigma > 0$ is the diffusion coefficient, $\Gamma_- = \big\{(x,v) \in \partial\omega \times \mathbb{R}^d, v \cdot \nu(x) < 0\big\}$, $\Gamma_+ = \big\{(x,v) \in \partial\omega \times \mathbb{R}^d, v \cdot \nu(x) > 0\big\}$ where $\partial\omega$ is the boundary of ω and $\nu(x)$ is the outward unit normal onto $\partial\omega$.
The boundary condition is an absorbing one and corresponds to the injection of particles at the boundary according to the distribution function g, but one can also consider a reflecting-type condition. A lot of results on the Vlasov equations are scattered throughout the literature and more of them had been collected in the excellent references [14, 7]. The problem of (VPFP) has been studied in the whole space by [5, 10, 26] whereas for the problem in a bounded domain an existence result of weak solution is given in [6] considering absorbing or reflecting boundary conditions.
When $\sigma = 0$, we get the so called Vlasov-Poisson system (VP)

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_v f = 0, \ (t,x,v) \in \mathbb{R}^+ \times \omega \times \mathbb{R}^d \\ \Delta_x \Phi(t,x) = \rho(t,x), \ (t,x) \in \mathbb{R}^+ \times \omega, \ \Phi(t,x) = 0, \ (t,x) \in \mathbb{R}^+ \times \partial\omega \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\, dv, \ (t,x) \in \mathbb{R}^+ \times \omega \\ f(0,x,v) = f_0(x,v), \ (x,v) \in \omega \times \mathbb{R}^d \\ f(t,x,v) = g(t,x,v), \ (t,x,v) \in \mathbb{R}^+ \times \Gamma_- \end{cases} \tag{2.2}$$

which has been studied by several authors essentially in the free space. They investigate existence of classical, weak or renormalized solutions. We cite [2, 3, 13, 18, 19, 20, 21, 22, 27, 30, 31] and the references therein. The problem on bounded domains didn't make the object of an abundant literature. The stationary problem had been analyzed in [15] while the time dependent one had been treated in [15]. The authors proved existence of weak solutions using the method of characteristics and an elliptic regularization of the Poisson's equation in order to apply a fixed point theorem.
The aim of this work is to prove existence of a weak solution of the (VPFP) system. This result had been obtained in [6] using an approximation of the linear Vlasov-Fokker-Planck (VFP) equation (say when Φ is known) by elliptic-parabolic-type equation to obtain L^1 and L^∞ estimates. Here we propose a direct approach which relies on the decay of the solution when the velocity goes to infinity to get these estimates. This paper is organized as follows. In the second section, we establish existence and uniqueness of a solution of the linear (VFP) system then we give maximum principles in L^1 and L^∞ for this solution. In the third section, we introduce an iterative scheme which converges thanks to the results obtained in the second section and the uniform estimates on $|(1+|v|^2)^{\frac{\gamma}{2}} f^n|_{L^\infty(\Omega)}$. Finally, the fourth section is devoted to the behavior of the solution when $\sigma \to 0$ and we show that at the limit we get a solution of the (VP) system.

Throughout this paper, we will use the following notations. For $T > 0$, we set

$$\Omega = \omega \times \mathbb{R}^d, \quad \Omega_T = (0,T) \times \Omega, \quad \omega_T = (0,T) \times \omega, \quad \Gamma = \Gamma_+ \cup \Gamma_-$$

$$X = L^2(\omega_T, H^1(\mathbb{R}^d_v)),\ X' = L^2(\omega_T, H^{-1}(\mathbb{R}^d_v)), Y = \{u \in X, \partial_t u + v \cdot \nabla_x u \in X'\}.$$

X and Y will be provided respectively with the norms

$$|u|^2_X = |u|^2_{L^2(\Omega_T)} + |\nabla_v u|^2_{L^2(\Omega_T)}\ ,\ |u|^2_Y = |u|^2_X + |\partial_t u + v \cdot \nabla_x u|^2_{X'}.$$

Finally, we denote by $L^p(\Gamma)$ the set of functions g such that $|g|^p$ are integrable on Γ with respect to the measure $d\Gamma = |v \cdot \nu(x)| ds dv$, ds being the Lebesgue measure on $\partial\omega$ and for $\alpha > 0$, $\mathbb{L}^2_\alpha(\Gamma) = \left\{g \in L^2(\Gamma), (1+|v|^2)^{\frac{\alpha}{2}} g \in L^2(\Gamma)\right\}$, $|g|^2_{\mathbb{L}^2_\alpha(\Gamma)} = \displaystyle\int_\Gamma (1+|v|^2)^\alpha g^2 d\Gamma$.

The main result of this paper is the following

Theorem 2.1.1 *Let $d \leq 2$, $\gamma > d$, f_0 and g satisfying the hypothesis*

$$f_0 \in L^\infty(\Omega) \cap \mathbb{L}^2_\gamma(\Omega),\ g \in L^\infty_{loc}(\mathbb{R}^+; L^\infty(\Gamma_-)) \cap {}^2_{loc}(\mathbb{R}^+; \mathbb{L}^2_\gamma(\Gamma_-)),\ f_0 \geq 0,\ g \geq 0 \text{ a.e.}$$

then the problem (2.1) admits a weak solution (f^σ, Φ^σ) defined on $\mathbb{R}^+$ such that for all $T > 0$, $f^\sigma \in L^\infty(0,T; L^\infty(\Omega) \cap \mathbb{L}^2_\gamma(\Omega))$, $\Phi^\sigma \in L^\infty(0,T; H^2(\omega)) \cap C([0,T];\ H^1(\omega))$ and

$$\begin{cases} \partial_t f^\sigma + v \cdot \nabla_x f^\sigma + \nabla_x \Phi^\sigma \cdot \nabla_v f - \sigma \Delta_v f^\sigma = 0, \text{ in } \mathcal{D}'(\Omega_T) \\ f^\sigma(0,x,v) = f_0(x,v) \text{ a.e. in } \Omega \\ f^\sigma(t,x,v) = g(t,x,v) \text{ a.e. in } (0,T) \times \Gamma_- \end{cases} \tag{2.3}$$

$$\Delta_x \Phi^\sigma = \rho^\sigma \text{ in } \omega_T,\ \Phi^\sigma = 0 \text{ a.e. in } (0,T) \times \partial\omega \tag{2.4}$$

where $\rho^\sigma(t,x) = \int_{\mathbb{R}^d} f^\sigma(t,x,v)\,dv$. Moreover $f^\sigma \geq 0$ a.e. in Ω_T, $\nabla_v f^\sigma \in L^2(0,T; \mathbb{L}^2_\gamma(\Omega))$ and for all $\sigma_0 > 0$, $\sigma \in]0, \sigma_0]$, $t \in [0,T]$, it holds the following estimates

$$\begin{cases} |f^\sigma(t)|_{L^\infty(\Omega) \cap \mathbb{L}^2_\gamma(\Omega)} + |(1+|v|^2)^{\frac{\gamma}{2}} f^\sigma(t)|_{L^\infty(\Omega)} + \sigma^{\frac{1}{2}} |\nabla_v f^\sigma(t)|_{\mathbb{L}^2_\gamma(\Omega)} \leq C \\ |\rho^\sigma(t)|_{L^1(\omega) \cap L^\infty(\omega)} + |\Phi^\sigma(t)|_{H^2(\omega)} + |\nabla_x \Phi^\sigma(t)|_{L^\infty(\omega)} + |\partial_t \Phi^\sigma(t)|_{H^1(\omega)} \leq C \end{cases}$$

where $C > 0$ depends only on $d, \gamma, \sigma_0, T, f_0$ and g.

The proof of the theorem will follow from sections 2.2 and 2.3. As a consequence, when we let $\sigma \to 0$ we get

Corollary 2.1.2 *Under the hypothesis of Theorem 2.1.1, the initial-boundary value problem (2.2) admits a weak solution (f, Φ) defined on $\mathbb{R}^+$ such for every $T > 0$, $f \in L^\infty(0,T; L^\infty(\Omega) \cap \mathbb{L}^2_\gamma(\Omega))$, $\Phi \in \mathbf{C}(0,T; H^1(\omega)) \cap L^\infty(0,T; H^2(\omega))$,*

$$\begin{cases} \displaystyle\int_{\Omega_T} f\Big(-\partial_t \varphi - v \cdot \nabla_x \varphi - E \cdot \nabla_v \varphi \Big)\, dxdvdt \ = \\ \displaystyle\int_\Omega f_0 \varphi(0,x,v)\, dxdv + \int_0^T \int_{\Gamma_-} g\varphi d\Gamma dt. \end{cases} \tag{2.5}$$

for all test function $\varphi \in \mathcal{D}([0,T[\times\overline{\Omega})$, $\varphi = 0$ on $(0,T) \times \Gamma_+$ and

$$\Delta_x \Phi(t,x) = \rho(t,x), (t,x) \in \omega_T, \Phi(t,x) = 0, (t,x) \in (0,T) \times \partial\omega$$

where $E = \nabla_x \Phi$ and $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)dv$. Moreover E, $\rho \in L^\infty(\omega_T)$ and $f \geq 0$ a.e. in Ω_T.

2.2 The linear Vlasov-Fokker-Planck problem

Let $T > 0$, we consider the problem (VFP)

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + a \cdot \nabla_v f - \sigma \Delta_v f = U, \ (t,x,v) \in \Omega_T \\ f(0,x,v) = f_0(x,v), \ (x,v) \in \Omega \\ f(t,x,v) = g(t,x,v), \ (t,x,v) \in (0,T) \times \Gamma_- \end{cases} \tag{2.6}$$

which governs the distribution of particles in the domain Ω in the presence of a known electric field a. Here U is a source term in $L^2(\Omega_T)$ and the initial condition f_0 and the boundary data g are given in $L^2(\Omega)$ and $L^2((0,T) \times \Gamma_-))$ respectively. In order to solve the problem of existence, we use a variational method based on a theorem of J.L. Lions (see [24]), then we give a meaning to the traces of the solution in Lemma 2.2.3 using a result of density that we recall in Lemma 2.2.2 (see [11] for more details). The existence result for this problem is the following

Proposition 2.2.1 *If $f_0 \in L^2(\Omega)$, $U \in L^2(\Omega_T)$, $a \in L^\infty(\Omega_T)$, $\nabla_v \cdot a \in L^\infty(\Omega_T)$, $g \in L^2((0,T) \times \Gamma_-)$, $\sigma > 0$ then the problem (2.6) has a unique solution f in the subclass of functions of Y satisfying the equation in $D'(\Omega_T)$ and the initial and boundary conditions in $L^2(\Omega)$ and $L^2((0,T) \times \Gamma_-)$ respectively.*

For the proof of this proposition we need these lemmas

Lemma 2.2.2 *(see [11]) Let $\widetilde{Y}$ the set of functions u belonging to $C^\infty(\overline{\omega}_T; H^1(\mathbb{R}^d))$ satisfying Supp $u \subset \overline{\Omega}_T$ and $u = 0$ in the neighborhood of $\{0,T\} \times \Gamma$ and $(0,T) \times \Gamma_0$ where $\Gamma_0 = \big\{(x,v) \in \partial\omega \times \mathbb{R}^d, v.\nu(x) = 0\big\}$. Then $\widetilde{Y}$ is dense in Y.*

Lemma 2.2.3 *(see appendix) The mapping $\mathcal{T} : \widetilde{Y} \to L^2(\Omega) \times L^2(\Omega) \times L^2((0,T) \times \Gamma)$ defined by $\mathcal{T}(u) = (u(0,x,v), u(T,x,v), u(t)_{|\Gamma})$ can be continuously extended to Y. Moreover for u and $\widetilde{u}$ in Y, we have*

$$\left\langle \frac{\partial u}{\partial t} + v \cdot \nabla_x u, \widetilde{u} \right\rangle_{X',X} + \left\langle \frac{\partial \widetilde{u}}{\partial t} + v \cdot \nabla_x \widetilde{u}, u \right\rangle_{X',X} =$$

$$\int_\Omega (u\widetilde{u})(T) - (u\widetilde{u})(0) dx dv + \int_0^T \int_\Gamma u\widetilde{u} \ (v \cdot \nu(x)) ds dv dt. \tag{2.7}$$

Proof. Let $\lambda > \frac{1}{2}|\nabla_v \cdot a|_{L^\infty(\Omega_T)}$, then $u(t,x,v) = e^{-\lambda t} f(t,x,v)$ solves the problem

$$\begin{cases} \partial_t u + v \cdot \nabla_x u + a \cdot \nabla_v u - \sigma \Delta_v u + \lambda u = \widetilde{U},\ (t,x,v) \in \Omega_T \\ u(0,x,v) = f_0(x,v),\ (x,v) \in \Omega \\ u(t,x,v) = \widetilde{g}(t,x,v),\ (t,x,v) \in (0,T) \times \Gamma_- \end{cases} \tag{2.8}$$

where $\widetilde{U} = e^{-\lambda t} U$ and $\widetilde{g} = e^{-\lambda t} g$. In order to get the existence result we use a variational method, so we introduce the space $Z = \left\{\varphi \in D([0,T[\times\overline{\Omega}); \varphi = 0 \ on\ \Gamma_+\right\}$ endowed with the norm

$$|\varphi|_Z^2 = |\varphi|_X^2 + \frac{1}{2}\int_\Omega \varphi^2(0,x,v)dxdv + \frac{1}{2}\int_0^T \int_{\Gamma_-} \varphi^2(t,x,v)d\Gamma dt. \tag{2.9}$$

It is obvious that the injection of Z into X is continuous. Then we consider the bilinear form B on $X \times Z$ and the linear form L on Z defined respectively by

$$B(u,\varphi) = \int_{\Omega_T} u(-\frac{\partial \varphi}{\partial t} - v \cdot \nabla_x \varphi + \lambda \varphi)dxdvdt + \int_{\Omega_T} \nabla_v u.(a\varphi + \sigma \nabla_v \varphi)dxdvdt \tag{2.10}$$

$$L(\varphi) = \int_{\Omega_T} \widetilde{U}\varphi dxdvdt + \int_\Omega f_0(x,v)\varphi(0,x,v)dxdv + \int_0^T \int_{\Gamma_-} \widetilde{g}(t,x,v)\varphi(t,x,v)d\Gamma dt. \tag{2.11}$$

One can easily show that $B(\varphi,\varphi) \geq \min\Big((\lambda - \frac{1}{2}|\nabla \cdot a|_{L^\infty(\Omega_T)}),\, \sigma,\, 1\Big)\ |\varphi|_Z^2,\ \forall \varphi \in Z$, hence thanks to a theorem of Lions (see appendix), there exists $u \in X$ such that

$$B(u,\varphi) = L(\varphi), \forall \varphi \in Z. \tag{2.12}$$

Therefore u satisfies the equation of (2.8) in the sense of distributions and since $\partial_t u + v \cdot \nabla_x u = \widetilde{U} - a \cdot \nabla_v u + \sigma \Delta_v u - \lambda u \in X'$ then u belongs to Y. According to lemma 2.2.3, $u(0,.,.)$ is well defined in $L^2(\Omega)$, moreover writing

$$\left\langle \frac{\partial u}{\partial t} + v \cdot \nabla_x u, \varphi \right\rangle_{X',X} + \langle a \cdot \nabla_v u - \sigma \Delta_v u, \varphi\rangle_{X',X} + \lambda \langle u, \varphi\rangle = \left\langle \widetilde{U}, \varphi \right\rangle_{X',X} \tag{2.13}$$

and using (2.7) and (2.12), we get

$$\int_\Omega [u(0,x,v) - f_0(x,v)]\varphi(0,x,v)dxdv = 0,\ \forall \varphi \in Z, Supp\, \varphi \subset [0,T[\times\Omega \tag{2.14}$$

consequently $u(0,x,v) = f_0(x,v)$ in $L^2(\Omega)$. We proceed in a similar way to prove that $u(t,x,v) = \widetilde{g}(t,x,v)$ in $L^2((0,T) \times \Gamma_-)$, therefore $u \in Y$ and satisfies (2.8) which ends the proof of the existence of a solution. It remains to prove the uniqueness of the solution of (2.8) in Y. Let $u \in Y$ a weak solution of (2.8) corresponding to $f_0 \equiv 0$, $\widetilde{g} \equiv 0$ and $\widetilde{U} \equiv 0$. As

$$\left\langle \frac{\partial u}{\partial t} + v \cdot \nabla_x u, u \right\rangle_{X',X} + \langle a.\nabla_v u, u\rangle_{X',X} + \sigma \left|\nabla_v u\right|^2_{L^2(\Omega_T)} + \lambda \left|u\right|^2_{L^2(\Omega_T)} = 0 \quad (2.15)$$

then thanks to (2.7) and the nullity of the boundary conditions, we get

$$\left\{ \begin{array}{l} \frac{1}{2}|u(T)|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^T \int_{\Gamma_+} u^2 \, d\Gamma dt - \frac{1}{2}\int_{\Omega_T} (\nabla_v \cdot a) u^2 dx dv dt + \\ \sigma |\nabla_v u|^2_{L^2(\Omega_T)} + \lambda |u|^2_{L^2(\Omega_T)} = 0 \end{array} \right. \quad (2.16)$$

so $(\lambda - \frac{1}{2}|\nabla_v \cdot a|_{L^\infty(\Omega_T)})|u|^2_{L^2(\Omega_T)} \leq 0$ and since $\lambda > \frac{1}{2}|\nabla_v \cdot a|_{L^\infty(\Omega_T)}$ then $|u|_{L^2(\Omega_T)} = 0$ and the proof of proposition 2.2.1 is complete.

In the remainder of this paragraph, we shall prove the positivity of the solution provided by proposition 2.2.1 and some regularity results in L^∞, L^1 and the space Y. Later we analyze the corresponding densities of charge, current and electric field. Let us begin with a Green formula that will be useful to show that f is nonnegative.

Lemma 2.2.4 *(see appendix)*
Let $u \in Y$ then $u^+ = max(u, 0) \in X$, $u^- = max(-u, 0) \in X$ and

$$\left\langle \frac{\partial u}{\partial t} + v \cdot \nabla_x u, u^- \right\rangle_{X',X} = \frac{1}{2}\int_\Omega \left|u^-(0,x,v)\right|^2 dx dv - \frac{1}{2}\int_\Omega \left|u^-(T,x,v)\right|^2 dx dv$$

$$-\frac{1}{2}\int_0^T \int_\Gamma \left|u^-(t,x,v)\right|^2 \; (v.\nu(x)) ds dv dt \quad (2.17)$$

Proposition 2.2.5 *The solution f of (2.6) satisfies*
$(i) f_0 \geq 0,\ U \geq 0,\ g \geq 0$ a.e. $\Rightarrow f \geq 0$ a.e.
$(ii) f_0 \in L^\infty(\Omega),\ g \in L^\infty((0,T) \times \Gamma_-)$, $U \in L^1(0,T;\ L^\infty(\Omega)) \Rightarrow f \in L^\infty(\Omega_T)$ and

$$|f(t)|_{L^\infty(\Omega)} \leq |f_0|_{L^\infty(\Omega)} + |g|_{L^\infty((0,T)\times\Gamma_-)} + \int_0^T |U(s)|_{L^\infty(\Omega)} ds,\ t \in (0,T) \text{ a.e.} \quad (2.18)$$

Proof. (ii) can be obtained from (i) replacing f by $f - |f_0|_{L^\infty(\Omega)} - |g|_{L^\infty((0,T)\times\Gamma_-)} - \int_0^T |U(s)|_{L^\infty(\Omega)} ds$. To prove (i), let $u(t,x,v) = e^{-\lambda t} f(t,x,v)$ the solution of (2.8). We have

$$\left\langle \widetilde{U}, u^- \right\rangle_{X',X} = \left\langle \frac{\partial u}{\partial t} + v.\nabla_x u, u^- \right\rangle_{X',X} + \left(a.\nabla_v u + \sigma \nabla_v u + \lambda u, u^-\right)_{L^2(\Omega_T)}.$$

Observing that $(g^+, g^-)_{L^2} = 0,\ \forall g \in L^2$ and using lemma 2.2.3 and lemma 2.2.4, we obtain

$$\left\langle \widetilde{U}, u^- \right\rangle_{X',X} = \frac{1}{2}\int_\Omega \left|u^-(0)\right|^2 dxdv - \frac{1}{2}\int_\Omega |u^-(T)|^2 dxdv$$

$$-\frac{1}{2}\int_0^T\int_\Gamma |u^-|^2 (v.\nu(x)) dsdvdt - \left(a.\nabla_v u^-, u^-\right)_{L^2(\Omega_T)} - \sigma|\nabla_v u^-|^2_{L^2(\Omega_T)} - \lambda|u^-|^2_{L^2(\Omega_T)}.$$

Since the data are positive, we get

$$\begin{aligned}\left\langle \widetilde{U}, u^- \right\rangle_{X',X} &= -\frac{1}{2}\int_\Omega \left|u^-(T)\right|^2 dxdv - \frac{1}{2}\int_0^T\int_{\Gamma_+} |u^-|^2 d\Gamma dt \\ &+ \frac{1}{2}\int_{\Omega_T} (\nabla_v \cdot a)|u^-|^2 dxdvdt - \sigma|\nabla_v u^-|^2_{L^2(\Omega_T)} - \lambda|u^-|^2_{L^2(\Omega_T)} \\ &\leq \left(\frac{1}{2}|\nabla_v \cdot a|_{L^\infty(\Omega_T)} - \lambda\right)|u^-|^2_{L^2(\Omega_T)}.\end{aligned}$$

Finally since $\left\langle \widetilde{U}, u^- \right\rangle_{X',X} \geq 0$ and $\frac{1}{2}|\nabla_v.a|_{L^\infty(\Omega_T)} - \lambda < 0$, we get immediately $|u^-|_{L^2(\Omega_T)} = 0$.

Proposition 2.2.6 *Let $\gamma > d$, $f_0 \in \mathbb{L}^2_\gamma(\Omega)$, $g \in L^2(0,T;\mathbb{L}^2_\gamma(\Gamma_-))$, $a \in L^\infty(\Omega_T)$ $U = 0$ in Ω_T then the solution f of (2.6) is such that $(1+|v|^2)^{\gamma/2} f \in Y$. Moreover it holds that*

$$|f|^2_{L^\infty(0,T;\,\mathbb{L}^2_\gamma(\Omega))} + \sigma|\nabla_v f|^2_{\mathbb{L}^2_\gamma(\Omega_T)} \leq \alpha \exp(\beta T) \tag{2.19}$$

where $\alpha = 2|f_0|^2_{\mathbb{L}^2_\gamma(\Omega)} + 2|g|_{\mathbb{L}^2_\gamma((0,T)\times\Gamma_-)}$ and $\beta = 2\sigma\gamma(\gamma+5) + 2\gamma|a|_{L^\infty(\Omega_T)}$.

Proof. We set

$$X_\gamma = \left\{u \in \mathbb{L}^2_\gamma(\Omega_T); \nabla_v u \in \mathbb{L}^2_\gamma(\Omega_T)\right\},\ Z = \left\{\varphi \in D([0,T[\times\overline{\Omega}); \varphi = 0 \text{ on } \Gamma_+\right\}$$

$$|u|^2_{X_\gamma} = |u|^2_{\mathbb{L}^2_\gamma(\Omega_T)} + |\nabla_v u|^2_{\mathbb{L}_\gamma(\Omega_T)},\ |\varphi|^2_Z = |\varphi|^2_{X_\gamma} + \frac{1}{2}|\varphi(0)|^2_{\mathbb{L}^2_\gamma(\Omega)} + \frac{1}{2}\int_0^T |\varphi|^2_{\mathbb{L}^2_\gamma(\Gamma_-)} dt.$$

Let $\lambda > 0$, we define a bilinear form b on $X_\gamma \times Z$ and a linear form l on Z by

$$b(u,\varphi) = \int_{\Omega_T} u(\partial_t\varphi - v.\nabla_x\varphi + \lambda\varphi)(1+|v|^2)^\gamma dxdvdt$$
$$+\int_{\Omega_T} \nabla_v u \cdot (a\varphi + \sigma\nabla_v\varphi + 2\sigma\gamma v(1+|v|^2)^{-1}\varphi)(1+|v|^2)^\gamma dxdvdt$$

$$l(\varphi)=\int_{\Omega} f_0(x,v)\varphi(0,x,v)(1+|v|^2)^{\gamma}dxdv+\int_0^T\int_{\Gamma_-} g(t,x,v)\varphi(t,x,v)(1+|v|^2)^{\gamma}d\Gamma dt.$$

Thanks to these definitions, Lions' theorem applies (if λ is large enough), so there exists $u_1\in X_\gamma$ satisfying $b(u_1,\varphi)=l(\varphi)\,,\forall\varphi\in Z$, therefore $\partial_t u_1+v\cdot\nabla_x u_1+a\cdot\nabla_v u_1-\sigma\Delta_v u_1+\lambda u_1=0$ in $D'(\Omega_T)$. Since $X_\gamma\subset X$, we get from this last equality that $\partial_t u_1+v\cdot\nabla_x u_1\in X'$. Thus $u_1\in Y$ and we can easily show that u_1 is a weak solution of (2.8). We deduce from the uniqueness of the solution in Y that $u=u_1$, so the solution f of (2.6) is such that $(1+|v|^2)^{\gamma/2}\,f\in X$. Now setting $F=(1+|v|^2)^{\gamma/2}\,f$, we get

$$\partial_t F+v\cdot\nabla_x F+(a+2\sigma\gamma v(1+|v|^2)^{-1})\cdot\nabla_v F-\sigma\Delta_v F=R_1+R_2$$

where $R_1=\gamma v\cdot a(1+|v|^2)^{\gamma/2-1}f,\ R_2(\sigma\gamma(\gamma+2)\frac{|v|^2}{1+|v|^2}-\sigma\gamma d)(1+|v|^2)^{\gamma/2-1}f$ and since R_1 and R_2 belong to $L^2(\Omega_T)$, then $\partial_t F+v\cdot\nabla_x F\in X'$ hence $(1+|v|^2)^{\gamma/2}f\in Y$. As

$$\begin{cases}\langle\partial_t F+v.\nabla_x F+(a+2\sigma\gamma v(1+|v|^2)^{-1})\cdot\nabla_v F,F\rangle_{X',X}\\ +\sigma\,|\nabla_v F|^2_{L^2(\Omega_T)}=(R_1+R_2,F)_{L^2(\Omega_T)}\end{cases}$$

therefore applying (2.7), we obtain

$$\begin{cases}\displaystyle\int_{\Omega}(1+|v|^2)^{\gamma}\Big[|f(T,x,v)|^2-|f(0,x,v)|^2\Big]dxdv+2\sigma|\nabla_v[(1+|v|^2)^{\frac{\gamma}{2}}f]|^2_{L^2(\Omega_T)}+\\ \displaystyle\int_0^T\int_{\Gamma}(1+|v|^2)^{\gamma}f^2v.\nu(x)dsdvdt+\int_{\Omega_T}(a+2\sigma\gamma v(1+|v|^2)^{-1}).\nabla_v[(1+|v|^2)^{\gamma}f^2]\\ \displaystyle=2\gamma\int_{\Omega_T}\Big[v.a+((\sigma\gamma+2\sigma)|v|^2)(1+|v|^2)^{-1}-\sigma d)\Big](1+|v|^2)^{\gamma-1}f^2dxdvdt.\end{cases}$$

Hence

$$\begin{aligned}&|f(T)|^2_{\mathbb{L}^2_\gamma(\Omega)}+2\sigma|\nabla_v[(1+|v|^2)^{\gamma/2}f]|^2_{L^2(\Omega_T)}\le|f_0|^2_{\mathbb{L}^2_\gamma(\Omega)}+|g|^2_{\mathbb{L}^2_\gamma((0,T)\times\Gamma_-)}\\ &+2\gamma\int_0^T|a|_{L^\infty(\Omega_T)}|f(t)|^2_{\mathbb{L}^2_\gamma(\Omega)}dt+2\sigma\gamma(\gamma+5)\int_0^T|f(t)|^2_{\mathbb{L}^2_\gamma(\Omega)}dt.\end{aligned}\tag{2.20}$$

So using Gronwall inequality, we get

$$|f(T)|^2_{\mathbb{L}^2_\gamma(\Omega)}\le\frac{\alpha}{2}\exp(\beta T)\tag{2.21}$$

which conducts to (2.19).

Proposition 2.2.7 *Under the hypothesis of proposition 2.2.1, if a is independent of v , we have*
$f_0 \in L^1(\Omega),\ U \in L^1(\Omega_T),\ g \in L^1((0,T)\times\Gamma_-) \Rightarrow f \in L^\infty(0,T;L^1(\Omega))$. *Moreover*

$$|f(t)|_{L^1(\Omega)} \leq |f_0|_{L^1(\Omega)} + \int_0^T (|g(\tau)|_{L^1(\Gamma_-)} + |U(\tau)|_{L^1(\Omega)})d\tau,\ \text{a.e. } t \in (0,T). \quad (2.22)$$

Proof. Proceeding as in [10] we will prove the proposition for f^+, the same argument works for f^- which leads to the result. The idea is based on the construction of a sequence satisfying $\varphi_\varepsilon(f) \to f^+$ a.e. in Ω_T as $\varepsilon \to 0$.
Let $\psi_\varepsilon \in C^\infty(\mathbb{R})$ such that $\psi_\varepsilon(u) = 0$ if $u \leq 0$, $\psi_\varepsilon(u) = 1$ if $u \geq \varepsilon$ and ψ_ε increases in $[0,\varepsilon]$, then the needed sequence is $\varphi_\varepsilon(u) = \int_{-\infty}^u \psi_\varepsilon(s)ds$. Since φ_ε and ψ_ε are regular, $\varphi_\varepsilon(f)$ and $\psi_\varepsilon\ (f)$ belong to Y and we can write

$$\langle \partial_t f + v\cdot\nabla_x f, \psi_\varepsilon(f)\rangle_{X',X} + \langle a\cdot\nabla_v f - \sigma\Delta_v f, \psi_\varepsilon(f)\rangle_{X',X} = \langle U, \psi_\varepsilon(f)\rangle_{X',X} \quad (2.23)$$

$$\langle a\cdot\nabla_v f, \psi_\varepsilon(f)\rangle_{X',X} = -\int_{\Omega_T} (\nabla\cdot a)\,\varphi_\varepsilon(f)dxdvdt = 0 \quad (2.24)$$

$$-\sigma\,\langle \Delta_v f, \psi_\varepsilon(f)\rangle_{X',X} = \sigma\left(\nabla_v f, \psi_\varepsilon^{'}(f)\nabla_v f\right)_{L^2(\Omega_T)} = \sigma\int_{\Omega_T} \psi_\varepsilon^{'}(f)\,|\nabla_v f|^2\,dxdvdt. \quad (2.25)$$

Now, let us prove that

$$\left\{\begin{array}{l} \langle \partial_t f + v\cdot\nabla_x f, \psi_\varepsilon(f)\rangle_{X',X} \geq \\ \displaystyle\int_\Omega \Big(\varphi_\varepsilon(f)(T,x,v) - \varphi_\varepsilon(f)(0,x,v)\Big)dxdv - \int_0^T\int_{\Gamma_-} \varphi_\varepsilon(f)d\Gamma dt. \end{array}\right. \quad (2.26)$$

Applying lemma 2.2.2, there exists a sequence $(f^n)_n$ in $\widetilde{Y}$ converging to f in Y so

$$\left\{\begin{array}{l} \langle \partial_t f^n + v\cdot\nabla_x f^n, \psi_\varepsilon(f^n)\rangle_{X',X} = \displaystyle\int_{\Omega_T} \partial_t\left[\varphi_\varepsilon(f^n)\right] + v.\nabla_x\left[\varphi_\varepsilon(f^n)\right]dxdvdt \\ = \displaystyle\int_\Omega \varphi_\varepsilon(f^n)(T,x,v)dxdv - \int_\Omega \varphi_\varepsilon(f^n)(0,x,v)dxdv + \int_0^T\int_\Gamma \varphi_\varepsilon(f^n)v.\nu(x)dsdvdt \\ \geq \displaystyle\int_\Omega \varphi_\varepsilon(f^n)(T,x,v)dxdv - \int_\Omega \varphi_\varepsilon(f^n)(0,x,v)dxdv - \int_0^T\int_{\Gamma_-} \varphi_\varepsilon(f^n)(t,x,v)d\Gamma dt. \end{array}\right. \quad (2.27)$$

We can easily pass to the limit in the first side of this inequality ; for the second one, we show that there exists a positive constant c_ε, $\left(c_\varepsilon = \frac{1}{2}|\psi_\varepsilon^{'}|_{L^\infty(\mathbb{R})}\right)$ such that

$|\varphi_\varepsilon(u)| \leq c_\varepsilon |u|^2$, $\forall u \in \mathbb{R}$ hence (2.26) holds. Finally combining (2.23), (2.24), (2.25) and (2.26) we get

$$\int_\Omega \varphi_\varepsilon(f)(T)dxdv - \int_\Omega \varphi_\varepsilon(f)(0)dxdv - \int_0^T \int_{\Gamma_-} \varphi_\varepsilon(f)(t,x,v)d\Gamma dt - \langle U, \psi_\varepsilon(f)\rangle_{X',X} \leq 0$$

therefore, as $0 \leq \psi_\varepsilon(u) \leq 1$ and $\varphi_\varepsilon(u) \leq |u|$, $\forall u \in \mathbb{R}$, we obtain

$$\int_\Omega \varphi_\varepsilon(f)(T,x,v)dxdv \leq |f_0|_{L^1(\Omega)} + \int_0^T \Big(|g(\tau)|_{L^1(\Gamma_-)} + |U(\tau)|_{L^1(\Omega)}\Big) d\tau. \tag{2.28}$$

Hence thanks to Fatou's lemma, we deduce that $f^+(T,.,.) \in L^1(\Omega), \forall T > 0$ and satisfies

$$\left|f^+(T)\right|_{L^1(\Omega)} \leq |f_0|_{L^1(\Omega)} + \int_0^T \Big(|g(\tau)|_{L^1(\Gamma_-)} + |U(\tau)|_{L^1(\Omega)}\Big) d\tau. \tag{2.29}$$

The next proposition deals with the densities of charge ρ, current j and the potential Φ corresponding to the solution f built in proposition 2.2.1. We recall that

$$\begin{aligned} &\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)dv,\ j(t,x) = \int_{\mathbb{R}^d} vf(t,x,v)dv, \\ &\Delta_x\Phi(t,x) = \rho(t,x) \ \text{ in } \omega_T,\ \Phi(t,x) = 0 \ \text{ on } (0,T)\times\partial\omega. \end{aligned} \tag{2.30}$$

Proposition 2.2.8 *Let γ as in theorem 2.1.1 then if f is a positive function on Ω_T such that $f \in L^\infty\Big(0,T;\ L^\infty(\Omega) \cap \mathbb{L}^2_\gamma(\Omega)\Big)$, we have $\rho \in L^\infty\Big(0,T;\ L^\infty(\omega) \cap L^1(\omega)\Big)$, $j \in L^\infty(0,T;L^2(\omega))$, $\Phi \in L^\infty(0,T;H^2(\omega))$. Moreover for every $t \in (0,T)$*

$$|\rho(t)|_{L^1(\omega)} = |f(t)|_{L^1(\Omega)} \tag{2.31}$$

$$|\rho(t)|_{L^\infty(\omega)} \leq K_1 |f(t)|_{L^\infty(\Omega)}^{\frac{\gamma-d}{\gamma}} \left|(1+|v|^2)^{\frac{\gamma}{2}} f(t)\right|_{L^\infty(\Omega)}^{\frac{d}{\gamma}} \tag{2.32}$$

$$|E(t)|_{L^\infty(\omega)} \leq K_2 |\rho(t)|_{L^1(\omega)}^{\frac{1}{d}} |\rho(t)|_{L^\infty(\omega)}^{\frac{d-1}{d}} \tag{2.33}$$

$$|j(t)|_{L^2(\omega)} \leq K_3 |f(t)|^2_{\mathbb{L}^2_\gamma(\Omega)} \tag{2.34}$$

$$|\Phi(t)|_{H^2(\omega)} \leq K_4 |\rho(t)|_{L^2(\Omega)} \tag{2.35}$$

where $E = \nabla_x\Phi$ and K_i denote positive constants depending only on d and γ.

Proof. (2.31) and (2.34) are evident, (2.35) is a well known result, (2.32) can be found in [10] while for (2.33) we refer the reader to [16].

2.3 The full problem of (VPFP) system

Coming back to the (VPFP) system (2.1), we shall prove (in three steps) the existence of a solution obtained like a limit of a sequence of solutions of the linear (VFP) equations. This sequence will be defined according to the results of section 2 and we establish that it satisfies as well as the corresponding densities of charge, current and electric field uniform estimates allowing the passage to the limit.

Step 1. Let $T > 0$, we introduce the functional sequences $(f^n)_n$, $(\rho^n)_n$, $(j^n)_n$ and $(\Phi^n)_n$ defined as follows. First we set at order 0, $f^0 \equiv 0$, $\rho^0 \equiv 0$, $j^0 \equiv 0$, $\Phi^0 \equiv 0$, then assuming these functions known at order n, we define f^{n+1} thanks to proposition 2.2.1 as the solution of the linear (VFP) problem

$$\begin{cases} \partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} + \nabla_x \Phi^n \cdot \nabla_v f^{n+1} - \sigma \Delta_v f^{n+1} = 0, \ (t,x,v) \in \Omega_T \\ f^{n+1}(0,x,v) = f_0(x,v), (x,v) \in \Omega \\ f^{n+1}(t,x,v) = g(t,x,v), (t,x,v) \in (0,T) \times \Gamma_- \end{cases} \tag{2.36}$$

then setting

$$\begin{cases} \rho^{n+1}(t,x) = \displaystyle\int_{\mathbb{R}^d} f^{n+1}(t,x,v)dv \\ j^{n+1}(t,x) = \displaystyle\int_{\mathbb{R}^d} v f^{n+1}(t,x,v)dv \end{cases} \tag{2.37}$$

we define Φ^{n+1} as the solution of the Dirichlet problem

$$\Delta_x \Phi^{n+1}(t,x) = \rho^{n+1}(t,x), \ (t,x) \in \omega_T, \ \ \Phi^{n+1}(t,x) = 0, \ (t,x) \in (0,T) \times \partial\omega. \tag{2.38}$$

We set also

$$E^n(t,x) = \nabla_x \Phi^n(t,x), \forall n \in \mathbb{N}. \tag{2.39}$$

Step 2. We establish uniform estimates on $(f^n)_n$,$(\rho^n)_n$,$(j^n)_n$ and $(\Phi^n)_n$ summarized in this proposition

Proposition 2.3.1 *Under the hypothesis of Theorem 2.1.1, the sequences $(f^n)_n$, $(\rho^n)_n$, $(j^n)_n$ and $(\Phi^n)_n$ are well defined. Moreover for all $T > 0$, $\sigma_0 > 0$, $n \in \mathbb{N}$, $\sigma \in]0, \sigma_0]$, we have*

$$f^n \geq 0 \ a.e. \ in \ \Omega_T \ and \ |f^n(t)|_{L^\infty(\Omega)} \leq C_0 \tag{2.40}$$

$$|f^n(t)|_{L^1(\Omega)} = |\rho^n(t)|_{L^1(\omega)} \leq C_1 \tag{2.41}$$

$$\left|(1+|v|^2)^{\frac{\gamma}{2}} f^n(t)\right|_{L^\infty(\Omega)} \leq C_2 \tag{2.42}$$

$$|\rho^n(t)|_{L^\infty(\omega)} + |E^n(t)|_{L^\infty(\omega)} \leq C_3 \tag{2.43}$$

$$|f^n(t)|_{\mathbb{L}^2_\gamma(\Omega)} + \sigma^{1/2} \, |\nabla_v f^n|_{\mathbb{L}^2_\gamma(\Omega_T)} + |j^n(t)|_{L^2(\omega)} \leq C_4 \tag{2.44}$$

$$|\Phi^n(t)|_{H^2(\omega)} \leq C_5 \tag{2.45}$$

$$|\partial_t \Phi^n(t)|_{H^1(\omega)} \leq C_6 \tag{2.46}$$

for a.e. $t \in (0,T)$, where the constants C_i depend only on the data d, f_0, g, γ, σ_0 and T.

Proof. Let us prove the points (2.40) to (2.46) by induction. Suppose that the proposition is true at order n, then it follows from proposition 2.2.1 that f^{n+1} is well defined and according to propositions 2.2.5 and 2.2.7, $f^{n+1} \geq 0$ a.e. in Ω_T and (2.40) and (2.41) are satisfied with

$$C_0 = |f_0|_{L^\infty(\Omega)} + |g|_{L^\infty((0,T)\times\Gamma_-)}, \; C_1 = |f_0|_{L^1(\Omega)} + \int_0^T |g(\tau)|_{L^1(\Gamma_-)} d\tau. \tag{2.47}$$

Proof of (2.42). Thanks to proposition 2.2.6, $F^{n+1} = (1+|v|^2)^{\gamma/2} f^{n+1} \in Y$ and solves the problem

$$\begin{cases} \partial_t F^{n+1} + v.\nabla_x F^{n+1} + (E^n + 2\sigma\gamma v(1+|v|^2)^{-1}) \cdot \nabla_v F^{n+1} - \sigma \Delta_v F^{n+1} = \\ R_1^{n+1} + R_2^{n+1} \\ F^{n+1}(0,x,v) = (1+|v|^2)^{\frac{\gamma}{2}} f_0(x,v) \;, \quad (x,v) \in \Omega \\ F^{n+1}(t,x,v) = (1+|v|^2)^{\frac{\gamma}{2}} g(t,x,v) \;, \quad (t,x,v) \in (0,T) \times \Gamma_-. \end{cases} \tag{2.48}$$

Here we used the notations introduced in proof of proposition 2.2.6. Hence, by (2.18)

$$\begin{cases} |F^{n+1}(t)|_{L^\infty(\Omega)} \leq |(1+|v|^2)^{\frac{\gamma}{2}} f_0|_{\mathbb{L}^\infty(\Omega)} + |(1+|v|^2)^{\frac{\gamma}{2}} g|_{\mathbb{L}^\infty((0,T)\times\Gamma_-)} + \\ 2\gamma \displaystyle\int_0^t |E^n(s)|_{L^\infty(\omega)} \left|(1+|v|^2)^{\frac{\gamma-1}{2}} f^{n+1}(t)\right|_{L^\infty(\Omega)} dt \\ +2\sigma\gamma(\gamma+5) \displaystyle\int_0^t |F^{n+1}(s)|_{\mathbb{L}^\infty(\Omega)} ds. \end{cases} \tag{2.49}$$

As $\left|(1+|v|^2)^{\frac{\gamma-1}{2}} f^{n+1}(t)\right|_{L^\infty(\Omega)} \leq C(\gamma)|f^{n+1}(t)|_{L^\infty(\Omega)}^{\frac{1}{\gamma}}\left|(1+|v|^2)^{\frac{\gamma}{2}} f^{n+1}(t)\right|_{L^\infty(\Omega)}^{1-\frac{1}{\gamma}}$ (see [13]), we obtain thanks to the interpolation inequalities (2.33) and (2.32)

$$|E^n(t)|_{L^\infty(\omega)}\left|(1+|v|^2)^{\frac{\gamma-1}{2}} f^{n+1}(t)\right|_{L^\infty(\Omega)} \leq C(C_0,d,\gamma)|F^n(t)|_{L^\infty(\Omega)}^{\frac{d-1}{\gamma}}|F^{n+1}(t)|_{L^\infty(\Omega)}^{1-\frac{1}{\gamma}}$$

thus setting $G_n(t) = Max(1, |F^n(t)|_{L^\infty(\Omega)})$, (2.49) gives

$$G_{n+1}(t) \leq A_1 + A_2 \int_0^t G_{n+1}(t)dt + A_3 \int_0^t (G_n(s))^{\frac{d-1}{\gamma}}(G_{n+1}(s))^{1-\frac{1}{\gamma}} ds \qquad (2.50)$$

where $A_i, i = 1,2,3$ are positive constants depending only on f_0, g, C_0, γ, d and T. Replacing A_1 by a greater constant which we will also denote A_1 , we can write again (2.50) with a strict inequality

$$G_{n+1}(t) < A_1 + A_2 \int_0^t G_{n+1}(s)ds + A_3 \int_0^t (G_n(s))^{\frac{d-1}{\gamma}}(G_{n+1}(s))^{1-\frac{1}{\gamma}} ds. \qquad (2.51)$$

Now consider the differential equation

$$\dot{\alpha}(t) = (A_2 + A_3)\alpha(t)^{\frac{d-2}{\gamma}+1},\ \alpha(0) = A_1 \qquad (2.52)$$

which has a global solution since $d \leq 2$ then we prove by induction on n that

$$G_n(t) \leq \alpha(t),\ \forall n \in \mathbb{N}. \qquad (2.53)$$

So (2.42) holds with $C_2 = \alpha(T)$.

Proof of (2.43) to (2.45). It comes from (2.32), (2.40) and (2.42) that

$$|\rho^n(t)|_{L^\infty(\omega)} \leq B_1 \qquad (2.54)$$

where $B_1 = K_1 C_0^{\frac{\gamma-d}{\gamma}} C_2^{\frac{d}{\gamma}}$, then using (2.33), (2.41) and (2.54), we get

$$|E^n(t)|_{L^\infty(\omega)} \leq B_2 \qquad (2.55)$$

with $B_2 = K_2 C_1^{\frac{1}{d}} B_1^{\frac{d-1}{\gamma}}$. Then combining (2.19), (2.55) and (2.34) we get (2.44). Moreover we get (2.45) thanks to (2.35), (2.41) and (2.43).

Proof of (2.46). We have (see [4]) $\partial_t \rho^n + \nabla \cdot j^n = 0$ in $D'(\omega_T)$ then thanks to (2.44) $\partial_t \rho^n$ is uniformly bounded in $L^\infty(0,T;H^{-1}(\omega))$ and since $\partial_t \Phi^n$ satisfies

$\Delta_x(\partial_t\Phi^n) = \partial_t\rho^n, \partial_t\Phi^n_{|\partial\omega} = 0$, $\partial_t\Phi^n$ is uniformly bounded in $L^\infty(0,T;H^1(\omega))$.

Step 3. We prove the convergence of the iterative scheme.
In the following, the constants C_i are those involving in proposition 2.3.1. It follows from (2.42), (2.40) and (2.41) that one can extract a subsequence of $(f^n)_n$ which converges towards a function f^σ in $L^\infty(0,T;\mathbb{L}^2_\gamma(\Omega))$ weak$-\star$, $f^\sigma \in L^\infty(\Omega_T) \cap L^\infty(0,T;L^1(\Omega))$ and satisfies for $t \in (0,T)$ and $\sigma \in]0,\sigma_0]$

$$f^\sigma \geq 0 \text{ a.e. in } \Omega_T \text{ and } |f^\sigma(t)|_{L^\infty(\Omega)} \leq C_0, \tag{2.56}$$

$$|f^\sigma(t)|_{L^1(\Omega)} \leq C_1 \tag{2.57}$$

$$|f^\sigma(t)|_{L^2_\gamma(\Omega)} \leq C_4. \tag{2.58}$$

Furthermore according to (2.43), we can extract a subsequence of $(\rho^n)_n$ also denoted $(\rho^n)_n$ which converges in $L^\infty(\omega_T)$ weak$-\star$ towards a function ρ^σ satisfying

$$|\rho^\sigma(t)|_{L^\infty(\omega)} \leq C_3, \quad \text{a.e. } t \in (0,T)\ ,\ \forall\sigma \in]0,\sigma_0] \tag{2.59}$$

and in virtue of (2.45), (2.46) and the Aubin's compactness theorem we can extract a subsequence of $(\Phi^n)_n$ also denoted $(\Phi^n)_n$ which converges strongly towards a function Φ^σ in $L^\infty(0,T;H^1(\omega))$. Moreover $\Phi^\sigma \in L^\infty(0,T;H^2(\omega)) \cap W^{1,\infty}(0,T;H^1(\omega))$ is the solution of the equation

$$\Delta_x\Phi^\sigma = \rho^\sigma, (t,x) \in \omega_T \quad , \Phi^\sigma(t,x) = 0, (t,x) \in (0,T)\times\partial\omega \tag{2.60}$$

and it holds for almost every $t \in (0,T)$ and $\sigma \in]0,\sigma_0]$ that

$$|\Phi^\sigma(t)|_{H^2(\omega)} \leq C_5 \tag{2.61}$$

$$|\partial_t\Phi^\sigma(t)|_{L^2(\omega)} \leq C_6. \tag{2.62}$$

Now let $\varphi \in D([0,T[\times\overline{\Omega})$ such that $\varphi = 0$ on Γ_+. Using (2.36) we get

$$\left\{ \begin{array}{l} \displaystyle\int_{\Omega_T} f^{n+1}(-\partial_t\varphi - v.\nabla_x\varphi - \sigma\Delta_v\varphi) - f^{n+1}\nabla_x\Phi^n.\nabla_v\varphi dxdvdt = \\ \displaystyle\int_\Omega f_0\varphi(0)dxdv + \int_0^T\int_{\Gamma_-} g\varphi d\Gamma dt. \end{array} \right. \tag{2.63}$$

Thus using the previous results, we obtain by passing to the limit

$$\left\{ \begin{array}{l} \displaystyle\int_{\Omega_T} f^\sigma(-\partial_t\varphi - v.\nabla_x\varphi - \sigma\Delta_v\varphi) - f^\sigma\nabla_x\Phi^\sigma.\nabla_v\varphi dxdvdt = \\ \displaystyle\int_{\Omega} f_0\varphi(0)dxdv + \int_0^T\int_{\Gamma_-} g\varphi d\Gamma dt. \end{array} \right. \tag{2.64}$$

Consequently

$$\partial_t f^\sigma + v\cdot\nabla_x f^\sigma + \nabla_x\Phi^\sigma\cdot\nabla_v f^\sigma - \sigma\Delta_v f^\sigma = 0 \text{ in } D'(\Omega_T). \tag{2.65}$$

In virtue of (2.44), $\nabla_v f^\sigma \in L^2(\Omega_T)$ then $f^\sigma \in X$. Furthermore, since $\nabla_x\Phi^\sigma \in L^\infty(\omega_T)$, we deduce from (2.65) that $\partial_t f^\sigma + v\cdot\nabla_x f^\sigma \in X'$ thus $f^\sigma \in Y$. The initial and boundary conditions

$$\left\{ \begin{array}{l} f^\sigma(0,x,v) = f_0(x,v) \text{ in } L^2(\Omega) \\ \\ f^\sigma(t,x,v) = g(t,x,v) \text{ in } L^2((0,T)\times\Gamma\_) \end{array} \right. \tag{2.66}$$

are fulfilled thanks to (2.64) and lemma 2.2.3. Now in order to prove that

$$\rho^\sigma(t,x) = \int_{\mathbb{R}^d} f^\sigma(t,x,v)dv \tag{2.67}$$

we set $\widetilde{\rho}(t,x) = \displaystyle\int_{\mathbb{R}^d} f^\sigma(t,x,v)dv$. Thanks to (2.43), ρ^n and $\widetilde{\rho}$ belong to $L^\infty(0,T;L^2(\omega))$ so for all $\psi \in L^1(0,T;L^2(\omega))$ we can write

$$\langle \rho^n - \widetilde{\rho}, \psi\rangle = \int_{\Omega_T} (f^n - f^\sigma)(1+|v|^2)^{\gamma/2}(1+|v|^2)^{-\gamma/2}\psi(t,x)dxdvdt. \tag{2.68}$$

As the function $(t,x,v) \to (1+|v|^2)^{-\gamma/2}\psi(t,x)$ belongs to $L^1(0,T;L^2(\Omega))$ and $((1+|v|^2)^{\gamma/2}f^n)_n$ converges in $L^\infty(0,T;L^2(\Omega))$ weak$-^\star$ towards $(1+|v|^2)^{\gamma/2}f^\sigma$ then $\langle \rho^n - \widetilde{\rho}, \psi\rangle \to 0$ as $n\to\infty$ which leads to (2.67). Finally, we see that according to (2.44) , we have

$$\sigma^{1/2}|(1+|v|^2)^{\gamma/2}\nabla_v f^\sigma|_{L^2(\Omega_T)} \le C_4 \tag{2.69}$$

thus (f^σ, Φ^σ) is a weak solution of (VPFP) system satisfying the properties of theorem 2.1.1.

2.4 Proof of Corollary 2.1.2

According to (2.58), (2.59), (2.61) and (2.62), we can extract subsequences of (f^σ), (ρ^σ), (Φ^σ) such that when σ goes to 0, $f^\sigma \rightharpoonup f$ in $L^\infty(0,T;\mathbb{L}^2_\gamma(\Omega))$ weak$-\star$, $\rho^\sigma \rightharpoonup \rho$ in $L^\infty(\omega_T)$ weak$-\star$, $\Phi^\sigma \to \Phi$ in $L^\infty(0,T;H^1(\omega))$ strongly. Moreover $f \in L^\infty(\Omega_T) \cap L^\infty(0,T;L^1(\Omega))$, $\Phi \in L^\infty(0,T;H^2(\omega)) \cap W^{1,\infty}(0,T;H^1(\omega))$ and the estimates (2.58),(2.59),(2.61) and (2.62) are still satisfied by f, ρ and Φ . Let $\varphi \in D([0,T[\times\overline{\Omega})$, $\varphi = 0$ over Γ_+. As $\nabla_x\Phi \in L^\infty(\omega_T)$, then taking the limit in (2.64), we obtain

$$-\int_{\Omega_T} f(\partial_t\varphi+v\cdot\nabla_x\varphi+\nabla_x\Phi\cdot\nabla_v\varphi)dxdvdt = \int_\Omega f_0\varphi(0)dxdv+\int_0^T\int_{\Gamma_-} g\varphi d\Gamma dt \quad (2.70)$$

Furthermore, we show as previously that $\rho(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)dv$ a.e. in ω_T , $\Delta_x\Phi = \rho$ a.e. in ω_T, $\Phi(t,x) = 0$ a.e. in $(0,T)\times\partial\omega$ thus (f,Φ) is a weak solution of (2.2) which ends the proof of corollary 2.1.2.

Remark 2.4.1 *Our method failed in dimension 3 of space because it gives a local in time existence. This is due to the equation (2.52) which admits in this case a local solution blowing up at finished time. The authors in [1], [4] and [6] overcame this difficulty by using both energy and entropy estimates.*

2.5 Appendix

• **Theorem of Lions** (see [8]) Let H be a Hilbert space provided with the norm $|\cdot|$ and the inner product $(.,.)$ and let V be a subspace provided with the norm $||\cdot||$ such that the injection of V into H is continuous. We consider the bilinear form B on $H\times V$ satisfying
(i) For every $\varphi \in V$, $B(\cdot,\varphi)$ is continuous on H.
(ii) $\exists\alpha > 0; \forall\varphi \in V, B(\varphi,\varphi) \geq \alpha||\varphi||^2_V$.
Then, for every continuous form L on V, there exists u belonging to H such that

$$B(u,\varphi) = L(\varphi),\ \forall\varphi \in V.$$

• **Proof of Lemma 2.2.3**. Let $u \in \widetilde{Y}$, using a partition of unity we can write $u = u_1+u_2+u_3+u_4+u_5$ with $u_i \in \widetilde{Y}$ and *Supp* $u_1 \subset [0,T[\times\ \Omega$, *Supp* $u_2 \subset]0,T]\times\ \Omega$, *Supp* $u_3 \subset]0,T[\times \overline{\Omega}$, $u_3 = 0$ in the neighborhood of $]0,T[\times \Gamma_+$, *Supp* $u_4 \subset]0,T[\times \overline{\Omega}$, $u_4 = 0$ in the neighborhood of $]0,T[\times \Gamma_-$, *Supp* $u_5 \subset]0,T[\times \Omega$. We have

$$\int_\Omega |u_1(0,x,v)|^2\,dxdv = -\int_0^T \Big(\int_\Omega \partial_t |u_1(0,x,v)|^2 dxdv\Big)dt = -2\int_{\Omega_T} u_1\partial_t u_1 dxdvdt$$

hence

$$\left\{\begin{array}{l} \displaystyle\int_\Omega |u_1(0,x,v)|^2 dxdv = -2\int_{\Omega_T} u_1(\partial_t u_1 + v\cdot\nabla_x u_1)dxdvdt \\ \le 2|\partial_t u_1 + v\cdot\nabla_x u_1|_{X'}|u_1|_X \le \|u_1\|_Y^2 . \end{array}\right. \tag{2.71}$$

We obtain as above $\displaystyle\int_\Omega |u_2(T,x,v)|^2 dxdv \le |u_2|_Y^2$. Furthermore we have

$$\left\{\begin{array}{l} \displaystyle\int_0^T\int_{\partial\omega\times\mathbb{R}^d} |u_3(t,x,v)|^2|v.\nu(x)|dsdvdt = -\int_0^T\int_{\Gamma_-} |u_3(t,x,v)|^2(v.\nu(x))dsdvdt \\ \displaystyle = \int_{\Omega_T} v\cdot\nabla_x u_3^2 dxdvdt = 2\int_{\Omega_T} v\cdot(u_3\nabla_x u_3)dxdvdt \\ \displaystyle = 2\int_{\Omega_T} u_3(\partial_t u_3 + v\cdot\nabla_x u_3)dxdvdt \le |u_3|_Y^2. \end{array}\right.$$

The same inequality applies for u_4. Therefore, the application $\mathcal{T}$ is linear continuous on $\widetilde{Y}$ provided with the norm of Y. By virtue of lemma 2.2.2, we can extend it by continuity to Y. Now for u and $\widetilde{u} \in \widetilde{Y}$, it holds the identity

$$\left\{\begin{array}{l} \displaystyle \big\langle \tfrac{\partial u}{\partial t} + v\cdot\nabla_x u, \widetilde{u}\big\rangle_{X',X} + \big\langle \tfrac{\partial \widetilde{u}}{\partial t} + v\cdot\nabla_x \widetilde{u}, u\big\rangle_{X',X} = \int_{\Omega_T} (\partial_t(u\widetilde{u}) + v\cdot\nabla_x(u\widetilde{u}))dxdvdt \\ \displaystyle = \int_\Omega (u(T,x,v)\widetilde{u}(T,x,v) - u(0,x,v)\widetilde{u}(0,x,v))dxdv + \int_0^T\int_\Gamma u\widetilde{u}\; v.\nu(x)dsdvdt \end{array}\right. \tag{2.72}$$

so we can extend it on Y thanks to the continuity of the trace application $\mathcal{T}$ of lemma 2.2.3.

• **Proof of lemma 2.2.4**. (i) is obvious. For the proof of (ii), we need the following result

Lemma 2.5.1 *(see [29]) Let $V \subset H \subset V'$ be Hilbert spaces. We assume that the application $u \to u^-$ is continuous from V to V. If $u \in L^2(0,T;V)\cap C(0,T;H)$ and $\partial_t u \in L^2(0,T;V')$ then*

$$\int_0^T \left\langle \frac{du}{dt}, u^-\right\rangle_{V',V} dt = \frac{1}{2}\left(\left|u^-(0)\right|_H^2 - \left|u^-(T)\right|_H^2\right). \tag{2.73}$$

We will prove (ii) for $u \in \widetilde{Y}$ and the result will be extended to $u \in Y$ by density thanks to the continuity of the trace application T. Let $u \in \widetilde{Y}$, we have

$$\left\langle \frac{\partial u}{\partial t}, u^- \right\rangle_{X',X} = \int_0^T \left\langle \frac{\partial u}{\partial t}, u^- \right\rangle_{L^2(\Omega)} dt \tag{2.74}$$

then applying lemma 2.5.1 with $V = H = L^2(\Omega)$, we obtain

$$\left\langle \frac{\partial u}{\partial t}, u^- \right\rangle_{X',X} = \frac{1}{2} \left(\int_\Omega \left| u^-(0,x,v) \right|^2 dxdv - \int_\Omega \left| u^-(T,x,v) \right|^2 dxdv \right). \tag{2.75}$$

Now for the calculus of $\langle v \cdot \nabla_x u, u^- \rangle_{X',X}$, since $u \in \widetilde{Y}$ then $u \in H^1(\Omega_T)$, $u^+, u^- \in H^1(\Omega_T)$ and $\partial_{x_i}(u^+) = 0$ if $u \leq 0$ a.e., $\partial_{x_i}(u^-) = 0$ if $u \geq 0$ a.e. Thus $\displaystyle\int_{\Omega_T} (v \cdot \nabla_x u^+) u^- dxdvdt = 0$ and

$$\left\langle v \cdot \nabla_x u, u^- \right\rangle_{X',X} = \int_{\Omega_T} (v \cdot \nabla_x u) u^- dxdvdt = -\frac{1}{2} \int_{\Omega_T} v \cdot \nabla_x \left| u^- \right|^2 dxdvdt \tag{2.76}$$

hence, using Green's formula, we get

$$\left\langle v \cdot \nabla_x u, u^- \right\rangle_{X,X'} = -\frac{1}{2} \int_0^T \int_\Gamma \left| u^- \right|^2 v \cdot \nu(x) dsdvdt \tag{2.77}$$

We conclude the proof of lemma by combining (2.75) and (2.77).

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Bibliographie générale

En plus des références données à la fin de chacune des parties composant cette thèse, nous avons rassemblé quelques travaux que nous avons répertoriés en deux volets, l'un consacré au ferromagnétisme et l'autre à la piézoélectricité

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