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Conformal geometry of pseudo-riemannian compact homogeneous manifolds

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Conformal geometry of pseudo-riemannian compact homogeneous manifolds

Abstract. In this thesis, we are studying homogeneous compact connected pseudo-riemannian manifolds under essential action of their conformal groups. We try a classification of these objects under the assumption that the non compact semi-simple part of the conformal group is (up to isomorphism) the Möbius group . We proved then that the manifold is conformally flat.

Keywords: Pseudo-riemannian conformal Geometry , Lichnerowicz conjecture, group actions, Einstein universe. .

الهندسة التوافقية للمنوعات النصف الريمانية المتراصة والمتجانسة

ملخص

في هذه الأطروحة، ندرس المنوعات النصف الريمانية المتراصة والمتجانسة تحت الفعل الأساسي لزمرتها المتوافقة. نحاول تصنيف هذه المنوعات على افتراض أن الجزء شبه البسيط غير المتراص من زمرتها المتوافقة متماثل محليا مع زمرة موبيوس . فبرهنا أن المنوعة سطحية توافقيا.

الكلمات المفتاحية : الهندسة النصف الريمانية التوافقية ، فراضية لتشنروفيتش ، أفعال الزمر ، فضاء أينشتاين.

Géométrie conforme des variétés pseudo-riemanniennes compactes homogènes

Résumé. Dans cette thèse, nous étudions les variétés pseudo-riemanniennes homogènes compactes et connexes sous l'action essentielle de leurs groupes conformes. Nous tentons une classification de ces objets sous l'hypothèse que la partie semi-simple non compacte du groupe conforme est (à isomorphisme près) le groupe de Möbius . Nous montrons alors que la variété est conformément plate.

Mots clés : Géométrie pseudo-riemannienne conforme, la conjecture de Lichnerowicz, action de groupe, univers d'Einstein.

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Contents

Chapter 1

Introduction

A pseudo-Riemannian manifold is a differentiable manifold M endowed with a pseudo-Riemannian metric g of signature (p, q). Two metrics g_1 and g_2 on M are said to be conformally equivalent if and only if $g_1 = \exp(f)g_2$ where f is C^∞ function. A conformal structure is then an equivalence class $[g]$ of a pseudo-Riemannian metric g and a conformal manifold is a manifold endowed with a pseudo-Riemannian conformal structure. A remarkable family of conformal manifolds is given by the conformally flat ones. These are pseudo-Riemannian conformal manifolds that are locally conformally diffeomorphic (i.e preserving the conformal structures) to the Minkowski space $\mathbb{R}^{p,q}$ i.e the vector space $\mathbb{R}^{p+q}$ endowed with the pseudo-Riemannian metric $-dx_0^2 - ... - dx_{p-1}^2 + dy_0^2 + ... + dy_{q-1}^2$.

The conformal group $\mathrm{Conf}(M, g)$ is the group of transformations that preserve the conformal structure $[g]$. It is said to be essential if there is no metric in the conformal class of g for which it acts isometrically. In the Riemannian case, the sphere $\mathbb{S}^n$ is a compact conformally flat manifold with an essential conformal group.

This notion of essentiality is very important in conformal geometry to distinguish it from pseudo-riemannian geometry. If the conformal group of a conformal manifold preserves a metric in this conformal class, then the conformal group acts as a subgroup of ISOM (M, g) and we are no longer in the field of conformal geometry but in the field of pseudo-riemannian geometry.

The Einstein universe $\mathrm{Ein}^{p,q}$ is the equivalent model of the standard conformal sphere in the pseudo-Riemannian setting. It admits a two-fold covering conformally equivalent to the product $\mathbb{S}^p \times \mathbb{S}^q$ endowed with the conformal class of $-g_{\mathbb{S}^p} \oplus g_{\mathbb{S}^q}$. It is conformally flat and its conformal group, which is in fact the pseudo-Riemannian Möbius group $\mathrm{O}(p+1, q+1)$, is essential. Actually the Einstein universe is the flat model of conformal pseudo-Riemannian geometry. This is essentially due to the fact that the Minkowski space embeds conformally as a dense open subset of the Einstein universe $\mathrm{Ein}^{p,q}$ and in addition to the Liouville theorem asserting that conformal local diffeomorphisms on $\mathrm{Ein}^{p,q}$ are unique restrictions of elements of $\mathrm{O}(p+1, q+1)$. Hence a manifold is conformally flat if and only if it admits an $(\mathrm{O}(p+1, q+1), \mathrm{Ein}^{p,q})$-structure.

In the sixties A. Lichnérowicz conjectured that among compact Riemannian manifolds, the sphere is the only essential conformal structure. This was generalised and proved independently by Obatta and Ferrand (see [24], [21]).

More precisely Ferrand proved in [21] the following theorem

Theorem 1.0.1 (Ferrand 1996)**.** *Let* $\mathrm{Conf}(M, g)$ *be the whole conformal group of a Riemannian manifold* M *with* $dim M = n \geq 2$*. If* M *is not conformally equivalent with* $\mathbb{S}^{\ltimes}$ *or* $\mathbb{R}^n$*, then* $\mathrm{Conf}(M, g)$ *is inessential, i.e. can be reduced to a group of isometries by a conformal change of metric.*

In the riemmanian setting, the conformal group is essential if and only if it is noncompact. This not true in the pseudo-Riemannian case. For example, there exist compact lorentzian manifolds with noncompact inessential conformal group (see the example in the introduction in Frances thesis [7]).

Historically, the pseudo-Riemannian Lichnérowicz conjecture, was raised by D'Ambra and Gromov [2]. Namely, if a compact pseudo-Riemannian conformal manifold is essential then it is conformally flat. This was disproved by Frances see [9], [11]. Neverthless the question is still open in the Lorentzian case. In [11], Frances and Melnick proved the Lichnerowicz conjecture for real analytic three dimensional Lorentzian manifolds. In the Lorentzian case, D'Ambra and Gromov conjectured in addition that conformal essential manifold are, up to finite cover, conformally equivalent to the Einstein space $\mathrm{Ein}^{1,n}$. This also was disproved by Frances in [8], where infinitely distinct (there is non conformal diffeomorphism betweeen them) essential conformally flat lorentz structure are constructed on the product of $\mathbb{S}^1$ by the connected sum of m copies of $\mathbb{S}^1 \times \mathbb{S}^{p+q-1}$. In particular, he constructed a 2-parameter family of non conformally flat (p,q) analytic pseudo-Riemannian essential structure on $\mathbb{S}^1 \times \mathbb{S}^{p+q-1}$

The work of Frances shows that the essentiality of the conformal group is not as restrictive as in the riemannian case and so it is not reasonable to consider the classification problem in the pseudo-riemannian case. A more restrictive situation consists in considering a manifold M with the conformal action of an essential group G that is algebraically interesting. This was adopted in the works of Zimmer, Bader, Zeghib, Frances and Pecastaing (see [31], [3], [12], [27], [28]). A direct consequence of [31] is that, when the group G is assumed to be simple then its real rank is less than $p+1$ (See [3]). In [3], [12] the maximal rank case is studied. They proved in particular that the simple essential group G is locally isomorphic to $SO(p+1,k)$, for some $p+1 \leq k \leq q+1$ and the manifold is conformally flat. Yet, the maximal real rank hypothesis restricts considerably the geometry. In the general case and even with the simpleness hypothesis, a wide variety of examples exists. In particular, all the examples constructed in [8] admit an essential conformal action of $\mathrm{PSL}(2,\mathbb{R})$. Pecastaing [27] proved that if the group G is locally isomorphic to $\mathrm{PSL}(2,\mathbb{R})$ then the manifold M is conformally flat. In [28], Pecastaing considered the minimal rank case. He detemines the smallest possible value of the index (p,q) of the pseudo riemannian metric. He proved in particular that when the index is optimal, the manifold M is conformally flat.

Several authors worked on the pseudo-Riemannian Lichnérowicz conjecture. In the general non homogeneous case, but with signature restrictions, we can quote Zimmer, Bader, Nevo, Frances, Zeghib, Melnick and Pecastaing (see [31], [3], [12], [27], [28], [26], [22]). Let us also quote [20] as a recent work in the Lorentz case.

We are investigating in this thesis, the homogeneous case where the non-compact semi-simple part of the conformal group is locally isomorphic to the Möbius group $\mathrm{SO}(1,n+1)$. More exactly, we prove the following classification result.

Theorem 1.0.2. *Let $(M,[g])$ be a conformal connected compact pseudo-Riemannian manifold. We suppose that there exists G a subgroup of the conformal group $\mathrm{Conf}(M,g)$ acting essentially and transitively on $(M,[g])$. We suppose moreover that the non-compact semi-simple part of G is locally isomorphic to the Möbius group $\mathrm{SO}(1,n+1)$. Then $(M,[g])$ is conformally flat. More precisely $(M,[g])$ is conformally equivalent to*

- *The conformal Riemannian $n-$sphere or;*
- *Up to a finite cover, the Einstein universe $\mathrm{Ein}^{1,1}$ (and $n=1$) or;*
- *Up to a double cover, the Einstein universe $\mathrm{Ein}^{3,3}$ (and $n=3$).*

Remark 1.0.3. *It turns out that, in the first and third cases, the acting group G is locally isomorphic to the Môbius group, that is, G is simple. In the second case, the universal cover $\tilde{G}$ is a subgroup of $\widetilde{\mathsf{SL}(2,\mathbb{R})} \times \widetilde{\mathsf{SL}(2,\mathbb{R})}$. It can in particular be $\widetilde{\mathsf{SL}(2,\mathbb{R})} \times \widetilde{\mathsf{SO}(2)}$.*

All the results mentionned in this thesis concern mostly the compact case (for the manifold). For the noncompact case, we mention the papers of Kühnel and Rademacher [18], [19] and Frances thesis [7].

Our thesis is divided into three chapters and two appendices. In the first chapter, we present some importants concepts of the theory we have used in our work. Among other things, we construct the flat conformal model of semi-Riemannian conformal geometry, which is the analogue of the sphere in Riemannian conformal geometry.

In chapter two, we look back at the remarkable results that have historically revolved around Lichnerowicz's conjecture.

Chapter three is devoted to our published work. In the two appendices, we present the machinery of Lie algebras and uniform subgroups used in our work.

Chapter 2

Some fundamental concepts and important facts about conformal pseudo-riemannian geometry

All objects we are dealing with (maps, manifolds, vector fields, ect...) are assumed to be smooth (C^∞).

A pseudo-Riemannian manifold is a differentiable manifold M endowed with a pseudo-Riemannian metric $< .,. >$ of signature (p, q), the dimension of M being equal to $p+q$. That is a symmetric nondegenerate (0,2) tensor field on M of constant index. This equivalent to say that for every $x \in M$, we have on the tangent space T_xM a nondegenerate symmetric bilinear form $< .,. >_x$ of signature (p, q) and that the map $x \in M \mapsto < X_x, Y_x >_x$ is smooth, for every vector fields X, Y on M. Two metrics $< .,. >_1$ and $< .,. >_2$ on M are said to be conformally equivalent if and only if $<>_1 = \exp(f) <>_2$ where f is smooth. More precisely

$$\forall x \in M, < X, Y >_1 = \exp(f(x)) < X, Y >_2, \forall X, Y \in T_xM$$

. We know that two quadratic forms are proportionnal if and only if they have the same nullcone. Hence, giving a conformal class of a metric on a pseudo-riemannian manifold is equivalent to giving a field of cones on that manifold.

A map f from a pseudo-Riemannian manifold $(M, < .\,,. >_M$ to another pseudo-Riemannian manifold $N, < .\,,. >_N$ is said to be conformal if $f^*(< .\,,. >_N)$ and $< .\,,. >_M$ are conformally equivalent. More precisely, there exists a smoth function f on M such that

$$\forall x \in M, \, , < d_x f(X), \, d_x f(Y) >_N = \exp(f(x)) < X, \, Y >_M, \, \forall X, \, Y \in T_x M$$

. A **conformal** structure is an equivalence class $[g]$ of a pseudo-Riemannian metric g and a conformal manifold is a manifold endowed with a pseudo-Riemannian conformal structure. The conformal group $\mathrm{Conf}(M, g)$ is the group of transformations that preserve the conformal structure $[g]$. It is said to be **essential** if there is no metric in the conformal class of g for which it acts isometrically. For example, we will see in the next chapter that, in the Riemannian case, the sphere $\mathbb{S}^n$ is a compact conformally flat manifold with an essential conformal group. According to a theorem in [17], $\mathrm{Conf}(M, g)$ is a Lie group for $\dim M \geq 3$.

All the notions we are reminding in this section can easily be rewritten for a finite dimensional vector space endowed with a nondegerate symmetric bilinear form. Let Q be the quadratic form in $\mathbb{R}^n$, $Q(x) = -x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_{p+q}^2$ $(p+q = n)$. To this quadratic form we associate the bilinear symmetric form $< x, y >= -\sum_{i=1}^{p} x_i y_i + \sum_{i=1}^{q} x_i y_i$. The couple (p, q) is called the **signature** of the quadratic form. We will note $\mathbb{R}^n = \mathbb{R}^{p,q}$. A vector v is called

- **timelike** if $Q(v) < 0$,
- **spacelike** if $Q(v) > 0$,
- **lightlike** (or **isotropic**) if $Q(v) = 0$.

Two vectors x and y are orthogonal if $< x, y >= 0$. We write $x \perp y$. The orthogonal of a subset A is $A^{\perp} = \{x \in \mathbb{R}^{p,q} \,|\, < x, y >= 0, \forall y \in \mathbb{R}^{p,q}\}$. The **null cône** of Q is the subset of lightlike vectors of $\mathbb{R}^{p,q}$:

$$\mathcal{C}^{p,q}(Q) = \{x \in \mathbb{R}^n \,|\, Q(x) = 0\}.$$

A subspace is called **degenerate** if it contains a vector orthogonal to it. It is called **totally isotropic** if it is included in $\mathcal{C}^{p,q}(Q)$ or equivalently it is included in its orthogonal.

The maximal dimension for the subspaces where the restriction of the bilinear form $< ., . >$ is positive definite is p. The maximal dimension for the subspaces where the restriction of the bilinear form $<>$ is negative definite is q. This is the **index** of $< ., . >$. The maximal dimension for an isotropic subspace is $\min(p, q)$.

The conformal group of $\mathbb{R}^{p,q}$, $p + q \geq 3$

It is well known that all elements of $(\mathbb{R} \times O(p, q)) \rtimes \mathbb{R}^{p+q}$ are conformal diffeomorphisms. In fact, a consequence of Liouville theorem (see below) says there is no others

$$Conf(\mathbb{R}^{p,q}) = (\mathbb{R} \times O(p, q)) \rtimes \mathbb{R}^{p+q}.$$

Topologically, $Conf(\mathbb{R}^{p,q}) \sim O(p) \times O(q) \times \mathbb{R}^{(p+1)(q+1)}$, $p+q \geq 3$ (a consequence of the polar decomposition).

If $p = q = 1$, $Conf(\mathbb{R}^{1,1})$ is given by $Diff(\mathbb{R}) \times Diff(\mathbb{R})$ augmented with $(x, y) \mapsto (y, x)$, $Diff(\mathbb{R})$ being the group of diffeomorphisms of $\mathbb{R}$.

The special case $p = 1$

(See B. O'Neill [25](chapter 5)for the details).

$\mathbb{R}^{1,q}$ is the Minkowski space. If we are working in a vector space with a nondegenerate bilinear symmetric form of signature of index one, we call it a **Lorentz space**.

Let (V, Q) a Lorents apce and $\mathscr{T}$ the subset of all timelike vectors in V. For $u \in \mathscr{T}$, we define the timecone of u by

$$\mathscr{C}(u) = \{u \in \mathscr{T} \mid < u, v >< 0\}$$

Let F a vector subspace of a Lorentz space. Then we have three possibilities mutually exclusive possibilities for F.

1. $< .,. >_F$ is positive definite: F is a **spacelike** vector subspace.
2. $< .,. >_F$ is nondegenerate: F is a **timelike** vector subspace.
3. $< .,. >_F$ is degenerate: F is a **lightlike** vector subspace.

We have the following important facts

1. F is timelike if and only if $F^{\perp}$ is spacelike.
2. F is lightlike if and only if $F^{\perp}$ is lightlike.
3. Orthogonal null vectors are collinear.
4. Orthogonal nonspacelike vectors are null hence collinear.
5. If $\dim F \geq 2$, F is timelike if and only if F contains two linearly independant null vectors if and only if F contains a timelike vector
6. If $\dim F \geq 2$, F is lightlike or degenerate if and only if F contains a null vector but contains no timelike vector if and only if the intersection of F and the nullcone is a one-dimensional subspace of V.

7. The Cauchy-Schwarz inequality holds in every spacelike subspace.

8. The inequality $| < v, w > | \geq |Q(v)||Q(w)|$ holds for any timelike vectors v , with equality if they are colinear.

9. If v and w are timelike vectors in the same timecone, we have

$$< v, w >= -|Q(v)|^{1/2}|Q(w)|^{1/2}cosh(\phi)$$

ϕ is called the hyperbolic angle between v and w.

10. If v and w are timelike vectors in the same timecone, we have

$$|Q(v)| + |Q(w)| \geq |Q(v + w)|.$$

11. A conformal mapping must preserve the causal type, angles between spacelike tangent vectors, and hyperbolic angles between tangent timelike vectors with the same time direction (in timelike manifolds).

12. Conversely, if a diffeomorphism between spacelike manifolds preserve angles between tangent vectors, then it is a conformal mapping.

2.1. The conformal group and the notion of essentiality

The notion of essentiality is very important in conformal geometry to distinguish it from pseudo-riemannian geometry. If the conformal group of a conformal manifold preserves a metric in the conformal class, then the conformal group acts (up to a conformal change of the metric) as a subgroup of ISOM(M, g) and we are no longer in the field of conformal geometry but in the field of pseudo-riemannian geometry. Moreover, in the context of riemannian geometry, the essentiality of the action (on compact manifolds) is equivalent to the non-compacity of the conformal group. Indeed, if $Conf(M)$ is compact, it preserves a volume form and by the proposition 4.3.3 (in the next chapter), it preserves a metric in the conformal class which implies inessentiality. Conversely, if it preserves a (riemannian) metric in the conformal class, it will be necessarily compact.

The pseudo-riemannian case contrasts with the riemannian one and the difficulty is bigger. Indeed, there exists compact pseudo-riemannian manifolds whose conformal group coincide with the group of isometries which is non-compact. Thus non-compacity does not necessarily imply essentiality in the pseudo-riemannian setting . (See for example the (lorentzian) example in Frances thesis, page 5, build with $PSL_2(\mathbb{R})$) .

2.2. The Einstein universe and Liouville theorem

The Einstein Universe is the first cosmological model based on the theory of general relativity discovered by Albert Einstein in 1915.

Einstein's universe is based on the cosmological principle proposed by Albert Einstein, namely the idea that the Universe is homogeneous and isotropic. It is a static solution of Einstein equation modelizing the univerese. For more details see [25] and [29] .

Moreover it's the right candidate in pseudo-Riemannian geometry to be the analog of the sphere as a compact flat conformal model in Riemannian geometry with an adequate Liouville theorem.

Indeed, just as conformal flat Riemannian manifolds are modeled on the conformal sphere as a $(G, X)-$structure, where X is the round sphere and G its conformal group , pseudo-riemannian flat manifolds will be modeled on Einstein Universe as a $(G, X)-$structure, where

X is the Einstein Universe and G its conformal group. All this thanks to the stereographic projection and Liouville's theorem.

Construction of Einstein Universe

We consider the epointed null cone ($n = p + q$)

$$\mathcal{N}^{p+1,q+1} = \left\{ x \in \mathbb{R}^{p+1,q+1,*} \,|\, Q^{p+1,q+1}(x) = 0 \right\}$$

(The sign * indicates that we have removed $0_{\mathbb{R}^{n+2}}$). The projectivization $P(\mathcal{N}^{p+1,q+1})$ which is the image by the canonical projection: $\pi : \mathbb{R}^{p+1,q+1} \longrightarrow \mathbb{R}P^{n+1}$ is a smooth submanifold of $\mathbb{R}P^{n+1}$ and inherits from the pseudo-Riemannian structure of $\mathbb{R}^{p+1,q+1}$ a type-(p, q) conformal class. This is obtained as follows.

Let $x \in \mathcal{N}^{p+1,q+1}$. The tangent space $\mathscr{T}_x\mathcal{N}^{p+1,q+1}$ of $\mathcal{N}^{p+1,q+1}$ at x is obviously $x^{\perp}$. Because $\mathbb{R}x \subset \mathscr{T}_x\mathcal{N}^{p+1,q+1}$, the restriction of $Q^{p+1,q+1}$ on $\mathcal{N}^{p+1,q+1}$ is degenerated. The restriction of π to $\mathcal{N}^{p+1,q+1}$ is a submersion from $\mathcal{N}^{p+1,q+1}$ onto its image by π. Now we have that $d_x\pi : \mathscr{T}_x\mathcal{N}^{p+1,q+1} \longrightarrow \mathscr{T}_{\pi(x)}\pi(\mathcal{N}^{p+1,q+1})$ is surjective with kernel equal to $\mathbb{R}x$ (because π is constant along $\mathbb{R}x$, we get $\ker d_x\pi$ contains $\mathbb{R}x$ but the two have the dimension one which gives the equality). We get then an isomorphism $\overline{d_x\pi} : \mathscr{T}_x\mathcal{N}^{p+1,q+1}/\ker d_x\pi \longrightarrow \pi(\mathcal{N}^{p+1,q+1})$.

We can define a metric $\tilde{Q}$ on $\pi(\mathcal{N}^{p+1,q+1})$ by transporting the degenerate metric $Q_{|\mathcal{N}^{p+1,q+1}}$ (it is a push-forward of $Q_{|\mathcal{N}^{p+1,q+1}}$)

$$\overline{Q}_{\pi(x)}(\overline{u}) = Q_x(u),$$

where, $u \in \mathscr{T}_x\mathcal{N}^{p+1,q+1}$ and $\overline{u} \in \mathscr{T}_{\pi(x)}\pi(\mathcal{N}^{p+1,q+1})$ are related by $\overline{u} = d_x\pi(u)$. This of course independent of the choice of u (because another choice u' verify $u' = u + z$, $z \in \ker d_x\pi = \mathbb{R}x$ and

$Q_x(u') = Q_x(u) + Q_x(z) + 2\,\langle u,\, z\rangle = Q_x(u)$ since $Q_x(z) = \langle u,\, z\rangle = 0$.) The metric obtained on $\pi(\mathcal{N}^{p+1,q+1})$ is non-degenerate and of signature (p,q). But this still depends on the choice of x. If we replace x by λx, $\lambda \in \mathbb{R}^*$, using $\lambda d_{\lambda x}\pi = d_x\pi$

(by differentiating $\pi \circ (x \mapsto \lambda x) = \pi$), we get

$$\overline{Q}_{\pi(x)}(\overline{u}) = \overline{Q}_{\pi(\lambda x)}(\overline{u}) = \overline{Q}_{\pi(\lambda x)}(d_x\pi(u)) = \overline{Q}_{\pi(\lambda x)}(d_{\lambda x}\pi(\lambda u))$$

$$= \overline{Q}_{\pi(x)}(d_{\lambda x}\pi(\lambda u)) = Q_x(\lambda u) = \lambda^2 Q_x(u).$$

Thus, we obtained on $\pi(\mathcal{N}^{p+1,q+1})$ a class of conformal non-degenerate metric of signature (p,q). We define, the Einstein universe of type (p, q), denoted $Ein^{p,q}$, the compact connected manifold $\pi(\mathcal{N}^{p+1,q+1})$ endowed with the precedent conformal class of metrics. This is a conformal manifold of dimension $p+q$. Moreover, the fact that the group $O(p+1,q+1)$ preserves the metric on $\mathbb{R}^{p+1,q+1}$ by means of which the conformal structure on $Ein^{p,q}$ is build, implies that $PO(p+1,q+1) = O(p+1,q+1)/\{\pm Id\}$ acts conformally and transitively on $Ein^{p,q}$. In fact we have the following **Liouville** theorem (see Frances thesis [7] for the proof)

Theorem 2.2.1. *Let $n = p+q \geq 3$, and U, V two connected open subsets of $Ein^{p,q}$ and $f : U \longrightarrow V$ a conformal map, then f is a restriction of a unique element of $PO(p+1,q+1)$.*

This implies that

$$Conf(Ein^{p,q}) = PO(p+1,q+1)$$

It should be noted that this result is a generalization of the riemannian case where we have the sphere instead the Einstein universe

The Einstein universe will play the same role in pseudo-riemannian geometry as the sphere in the riemannian setting. In the riemannian case $(p=0)$, $Ein^{0,q} = \mathbb{S}^q$ and

$$Conf(\mathbb{S}^q) = PO(1,q+1).$$

We have a conformal embedding of $\mathbb{R}^{p,q}$ into an open dense subset of $Ein^{p,q}$ given by (see [10], page 383)

$$(x_1, ..., x_n) \longrightarrow \left(-\frac{Q(x)}{2}, x_1, ..., x_n, 1\right)$$

where $Q(x) = 2x_0x_{p-1+q} + ... + 2x_{p-1}x_q + \sum_{i=p}^{q-1} x_i^2$. ($Q$ is a rewriting of the standard quadratic form of $\mathbb{R}^{p,q}$). This shows that $Ein^{p,q}$ is conformally flat. We can also see $Ein^{p,q}$ as a conformal compactification of the Minkowski space $\mathbb{R}^{p,q}$.

2.3. The special case of the dimension 2

Conformal pseudo-Riemannian structures in dimension 2 are not rigid, in particular the group of local conformal transformations has infinite dimension. Conformal actions on compact surfaces of Lie groups are however relevant even in this dimension. In the Riemannian case, the conformal group of a compact surface is in fact a Lie group, for example

$$\mathrm{Conf}(\mathbb{S}^2) = SO(1,3) = PSL(2,\mathbb{C})$$

. This is no longer the case of $Ein^{1,1}$. This latter surface is in fact the torus $\mathbb{S}^1 \times \mathbb{S}^1$, with the conformal structure given by the fact that the two factors are isotropic. So, $\mathrm{Conf}(Ein^{1,1})$ coincides with diffeomorphisms preserving the product structure. This is therefore $\mathrm{Diff}(\mathbb{S}^1) \times \mathrm{Diff}(\mathbb{S}^1)$ augmented with the involution $(x,y) \mapsto (y,x)$. On the other hand, from the quadric model of $Ein^{1,1}$, one gets the restricted conformal group $PSO(2,2)$ which is identified to $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$. Therefore $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ as well as $PSL(2,\mathbb{R}) \times SO(2)$ are two examples of Lie groups acting transitively and essentially on $Ein^{1,1}$. It turns out that $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ is a maximal Lie group in $\mathrm{Conf}(Ein^{1,1})$, that is there is no bigger Lie group contained in $\mathrm{Conf}(Ein^{1,1})$ which contains $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$.

Consider two finite covers of degrees m and n : $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$, and let $PSL^k(2,\mathbb{R})$ denote the k-cover of $PSL(2,\mathbb{R})$. Then $PSL^m(2,\mathbb{R}) \times PSL^n(2,\mathbb{R})$ acts conformally on $\mathbb{S}^1 \times \mathbb{S}^1$ endowed with the pull back of the conformal structure of $Ein^{1,1}$ via the covering of degree mn: $\mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow Ein^{1,1}$. Observe however that this conformal structure is also just given by the product structure, and so it is isomorphic to $Ein^{1,1}$. From all this, we infer that the Lie groups $PSL^m(2,\mathbb{R}) \times PSL^n(2,\mathbb{R})$ act faithfully transitively and essentially on $Ein^{1,1}$. In sum, on the same $Ein^{1,1}$, we have different actions of these Lie groups $PSL^m(2,\mathbb{R}) \times PSL^n(2,\mathbb{R})$, but they all finitely cover the $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$-action (so we can say they are all the same up a finite covers).

2.4. The conformal tensor

The Weyl tensor is a (1,3)-tensor field build with the Riemaann tensor. It has the special property that it is invariant under conformal changes to the metric. For the riemannian case, there is no need of Weyl tensor in dimension two as any 2-dimensional Riemannian manifold is conformally flat. This is a consequence of the existence of isothermal coordinates. In the lorentzian setting, this remains true: any two dimensional conformal manifolds is conformally flat. In coordinates, the Weyl tensor is given by

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(g_{ik}R_{jl} - g_{il}R_{jk} - R_{jk}g_{il} + R_{jl}g_{ik}) + \frac{R}{(n-1)(n-2)}(g_{ik}R_{jl} - g_{il}R_{jk})$$

The Weyl tensor is the traceless part of the Riemann tensor. Moreover, unlike the Riemann Tensor, the Weyl tensor is the same for two conformal metrics (see the book of Y. Choquet-Bruhat [6]). In the dimension 3, the Weyl tensor vanishes identically. We use instead the Cotton tensor given by:

$$C_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{4}(\nabla_j R g_{jk} - \nabla_k R g_{ij})$$

where R_{ij} is the Ricci tensor and R the scalar curvature.

Let's call the Weyl-tensor (in dimension greater or equal to 4) and the Cotton-tensor (in dimension three) the conformal tensor.

We have the folling theorem which characterizes the flatness of a conformal pseudo-riemannian manifold.

Theorem 2.4.1 (**Weyl–Schouten**). *A conformal pseudo-riemannian manifold of dimension greater or equal than three is conformally flat if its conformal tensor vanishes.*

Remark 2.4.2. *It is clear, that for a homogeneous manifold with a conformal structure, the manifold is conformally flat if its conformal tensor vanishes at some point.*

Chapter 3

Some historical and important results

In the sixties A. Lichnérowicz conjectured that among compact Riemannian manifolds, the sphere is the only essential conformal structure. This conjecture expresses a kind of rigidity in conformal geometry: not many varieties have a large conformal group. This conjecture was generalised and proved independently by Obatta and Ferrand (see [24], [21]). More precisely Ferrand proved in [21] the following theorem

Theorem 3.0.1 (Ferrand 1996)**.** *Let* $\mathrm{Conf}(M, g)$ *be the whole conformal group of a Riemannian manifold* M *with* $dim M = n \geq 2$*. If* M *is not conformally equivalent with the standard sphere* $\mathbb{S}^n$ *or* $\mathbb{R}^n$*, then* $\mathrm{Conf}(M, g)$ *is inessential, i.e. can be reduced to a group of isometries by a conformal change of metric.*

(In dimension 2, the theorem is a straightforward consequence of the uniformization theorem for Riemann surfaces.) The theorem says that up to conformal equivalence, the spheres and the euclidean spaces are the only riemannian manifolds with essential conformal groups. For example, if we take the flat torus, it is not difficult to prove that its conformal group coincide with its isometry group. To prove this theorem, she uses technique from the theory of global conformal invariants and quasiconformal analysis. It is not difficult to verify that for a compact riemannian manifold, the essentiality of the conformal group is equivalent to saying that it is not compact (see the precedent chapter).

Notice that the spheres and the euclidean spaces are conformally flat. In [2] (1990, section 6-7), D'Ambra and Gromov pointed out that, generically speaking, rigid geometric structures have no much symmetries, and therefore those that do have, must be sufficiently exceptional to be "classified". They conjectured that the flatness is necessary for the conformal group to act essentially. In particular, the only compact Lorentzian varieties for which the conformal group is essential are, up to finite quotients and coverings equivalent to the Einstein universe. This last fact is not true as proved by Frances [8] (2005) in the following theorem and the next.

Theorem 3.0.2. *For all $n \geq 3$ and $g \geq 1$, the manifold $\mathbb{S}^1 \times \underset{g}{\sharp}\mathbb{S}^{n-2}$ may be endowed with an infinite number of nonequivalent essential conformal lorentzian structures.*

(The symbol $\sharp$ designs the connected sum). The first examples illustrating the previous theorem will be obtained as Kleinian manifolds, that is quotients of open sets of $\mathbb{R} \times \mathbb{S}^{n-1}$ (the universal covering of $Ein^{1,n-1}$) by a discrete subgroup of $Conf(\mathbb{R} \times \mathbb{S}^{n-1})$. In the dimension three,

Theorem 3.0.3. *For all $g \geq 2$ and orientable surface Σ_g of genus g, the manifold $\mathbb{S}^1 \times \Sigma_g$ may be endowed with an infinite number of essential conformal flat lorentzian structures.*

In the same article, Frances points out that it is hypothetical to have a global classification of Lorentzian structures on compact varieties for which the conformal group is essential. Nevertheless, all the examples of essential structures he gives in this article are flat, which seems to strengthen the conjecture of D'Ambra and Gromov in the compact case. Let's call this conjecture by **the Pseudo-riemannian Lichnerowicz conjecture** (Unlike the riemannian case, the conjecture is false in the non compact case, see D. Alekseevski (1985) in [1] and W. Kühnel, W., H.B. Rademacher (1995) in [18]). But in an article (2015), the same autor in [9] proved the following theorem which defintively answers negatively to the pseudo-riemannian Lichnerowicz conjecture:

Theorem 3.0.4. *For every $p \geq 2$ and every $q \geq p$ one can construct on the product $\mathbb{S}^1 \times \mathbb{S}^{p+q-1}$ a 2-parameter family of distinct type-(p, q) analytic pseudo-Riemannian conformal structures, which are not conformally flat, and with an essential conformal group.*

In the same article, it points out that this theorem does not cover the lorentzian signature, so the conjecture remains open (at least at this time 2015) in this case. Let's call this conjecture (in the lorentzain case) : **The Lorentzian Lichnerowicz conjecture** which seems to be true assuming some restrictions as proved by Frances and Melnick (2021) in the following theorem

Theorem 3.0.5. *Let (M, g) be a 3-dimensional, compact, real analytic, lorentzian manifold with an essential action of the connectecd component of $Conf(M)$. Then (M, g) is conformally flat.*

The work of Frances shows that the essentiality of the conformal group is not as restrictive as in the riemannian case and so it is not reasonable to consider the classification problem in the pseudo-riemannian case. For that reason, it is reasonable to consider algebraic restriction on the conformal group to obtain some classification result.

Given a general compact pseudo-Riemannian manifold M of signature (p, q), thanks to a result from Zimmer [30] (1987), the real rank of the semisimple component of $Conf(M)$ is bounded by $\min\{p, q\} + 1$. This gives is a constraint for existence of conformal actions. What can be said if the rank of $Conf(M)$ is the maximal possible (we are quoting from

Zeghib and Frances article [12]). The answer is given by Bader and Nevo in [3] (2002) in the following theorem

Theorem 3.0.6. *Let G be a connected simple Lie group with finite center acting smoothly and conformally on a smooth compact pseudo-Riemannian manifold M of type (p, q) , $p \leq q$. Assume the (real) rank of G equals $p+1$. Then:*

1. *The group G is locally isomorphic to $SO^{\circ}(p+1, k+1)$, for some k such that $p \leq k \leq q$.*
2. *There exists a closed G-orbit, which is conformally equivalent to a finite cover of $Ein^{p,k}$.*
3. *In particular, if the G action is minimal, then M is conformally equivalent to a finite cover of $Ein^{p,k}$.*

In [12] (2005), Zeghib and Frances improved this result by relaxing the minimality condition and the finiteness of the center. They proved the following theorem

Theorem 3.0.7. *Let G be a connected simple Lie group acting smoothly and conformally on a smooth compact pseudo-Riemannian manifold M of type (p, q) , $2 \leq p \leq q$. Assume the rank of G equals $p+1$. Then:*

1. *The group G is locally isomorphic to $SO^{\circ}(p+1, k+1)$, for some k such that $p \leq k \leq q$.*
2. *Up to finite cover, M is conformally equivalent to $Ein^{p,k}$.*

In [26] (2017) Pecastaing in his contribution to deal with the Lorentzian Lichnerowicz conjecture studied the special case when a Lorentz manifolds of dimension at least three admits a conformal essential action of a Lie group locally isomorphic to $PSL(2, \mathbb{R})$.

Theorem 3.0.8. *Let $(M, [g])$, with dimension at lest 3, be a real-analytic compact connected Lorentz manifold admitting a faithful conformal essential action of a connected Lie group locally isomorphic to $PSL(2, \mathbb{R})$. Then, (M, g) is conformally flat.*

This result may seem restrictive since it is restricted to groups that are locally isomorphic to $PSL(2,\mathbb{R})$. But as the author points out, local copies of $PSL(2,\mathbb{R})$) sit in every non-compact simple Lie group. So the theorem has the following straight consequence

Corollary 3.0.9. *If a connected semi-simple Lie group acts faithfully, conformally and essentially on a real-analytic compact connected Lorentz manifold* $(M,[g])$ *of dimension at lest 3, then* $(M,[g])$ *is conformally flat.*

The author points out(at least at the time of publication of the article) that the question of conformal flatness for smooth compact Lorentz manifolds admitting an essential action of a Lie group locally isomorphic to $PSL(2,\mathbb{R})$ is still open.

Conclusion. Thus, we have just seen that the Pseudo-riemannian Lichnerowicz conjecture is not true in all generality. However, by imposing certain natural constraints on the conformal group or on the manifold, it may be true. In the next chapter, we will present our work by imposing an additional condition on the conformal group, in addition to the homogeneity of the manifold, and show that the conjecture remains true. We will even specify the geometry of the manifolds in question.

Chapter 4

Statement and proof of the main result

In this chapter, we prove our main result. But before getting into the subject, let us recall that the manifolds we'll be studying are homogeneous spaces, that is, manifolds on which a Lie group G acts transitively with smooth action. The manifold is then identified with the quotient G/H, where H is the stabilizer of a point called basepoint. The tangent space is then identified with $\mathfrak{g}/\mathfrak{h}$.

4.1. The main result

Theorem 4.1.1. *Let $(M,[g])$ be a conformal connected compact pseudo-Riemannian manifold. We suppose that there exists G a subgroup of the conformal group $\mathrm{Conf}(M,g)$ acting essentially and transitively on $(M,[g])$. We suppose moreover that the non-compact semi-simple part of G is locally isomorphic to the Möbius group $\mathrm{SO}(1,n+1)$. Then $(M,[g])$ is conformally flat. More precisely $(M,[g])$ is conformally equivalent to*

- *The conformal Riemannian $n-$sphere or;*
- *Up to a finite cover, the Einstein universe $\mathrm{Ein}^{1,1}$ (and $n=1$) or;*

- *Up to a double cover, the Einstein universe* $\mathrm{Ein}^{3,3}$ *(and* $n = 3$*).*

4.2. Preliminaries

4.2.1. Notations

Throughout this paper (M, g) will be a compact connected pseudo-Riemannian manifold of dimension n endowed with a transitive and essential action of the conformal group $G = \mathrm{Conf}(M, g)$. In general G is not connected but this does not prevent the connected component G^0 from acting transitively and essentially as proved in the following lemma.

Lemma 4.2.1. G^0 *acts transitively and essentially.*

Proof. The connected component G^0 acts transitively. Otherwise, M will be a disjoint union of G^0-orbits wich are open subsets of M.[1] This contradicts the fact that M is connected.

Let us show that it acts essentially. If not, it will preserve a metric g_0 in the conformal class of g. If $f \in G$, then f^*g_0 is another G^0-invariant metric since G^0 is normal in G. But any other G^0-invariant metric in the conformal class has the form αg_0, with α a G^0-invariant function and hence constant. It follows that $f^*g_0 = \alpha(f)g_0$, where $\alpha : G \longrightarrow \mathbb{R}^*$ is a homomorphism. In other words G acts by g_0-homotheties. Since M is compact, this implies $\alpha = \pm 1$, and hence G acts non- essentially. □

Thus, thanks to this lemma, we suppose without loss of generality that G is connected.

Fix a point x in M and denote by $H = \mathrm{Stab}(x)$ its stabilizer in G, $\mathfrak{g}$, $\mathfrak{h}$ the Lie algebras of G and H.

[1] G being locally connected, its connected component is open and its image G^0x will give an open orbit in M.

Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be a Levi decomposition of $\mathfrak{g}$, where $\mathfrak{s}$ is semi-simple and $\mathfrak{r}$ is the solvable radical of $\mathfrak{g}$. Denote by $\mathfrak{s}_{nc}$ the non-compact semi-simple factor of $\mathfrak{s}$, by $\mathfrak{s}_c$ the compact one. Let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{g}$. Let us denote respectively by S, S_{nc}, S_c, R and N the connected Lie sub-groups of G associated to $\mathfrak{s}$, $\mathfrak{s}_{nc}$, $\mathfrak{s}_c$, $\mathfrak{r}$ and $\mathfrak{n}$.

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{s}$ associated with a Cartan involution Θ.

Consider

$$\mathfrak{s} = \mathfrak{s}_0 \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{s}_\alpha = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{s}_\alpha$$

the root space decomposition of $\mathfrak{s}$, where Δ is the set of roots of $(\mathfrak{s}, \mathfrak{a})$.

Denote respectively by Δ^+, Δ^- the set of positive and negative roots of $\mathfrak{s}$ for some chosen notion of positivity on $\mathfrak{a}^*$. Then

$$\mathfrak{s} = \mathfrak{s}_- \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_+,$$

where

$$\mathfrak{s}_+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{s}_\alpha \text{ and } \mathfrak{s}_- = \bigoplus_{\alpha\in\Delta^-} \mathfrak{s}_\alpha.$$

For every $\alpha \in \mathfrak{a}^*$, consider $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}, \forall H \in \mathfrak{a} : ad_H(X) = \alpha(H)X\}$. We say that α is a weight if $\mathfrak{g}_\alpha \neq 0$. In this case $\mathfrak{g}_\alpha$ is its associated weight space. As $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$ (see [15, Theorem 13]) then for every $\alpha \neq 0$,

$$\mathfrak{g}_\alpha = \mathfrak{s}_\alpha \oplus \mathfrak{n}_\alpha,$$

where

$$\mathfrak{n}_\alpha = \{X \in \mathfrak{n}, \forall H \in \mathfrak{a} : ad_H(X) = \alpha(H)X\}.$$

We have then,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\neq 0} \mathfrak{g}_\alpha.$$

Finally we will denote respectively by A, S_+ the connected Lie subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{s}_+$.

4.3. General facts

We will prove some general results about the conformal group G of the homogeneous pseudoriemannian manifold M. We start with the following general fact:

Proposition 4.3.1. *We have that* $[\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{r}]$. *In particular the sub-algebra* $\mathfrak{s} \ltimes \mathfrak{n}$ *is an ideal of* $\mathfrak{g}$.

Proof. $\mathfrak{s}$ being semi-simple and $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$, we have that ad$\mathfrak{s}$ preserves $\mathfrak{n}$ and one of its supplementary in $\mathfrak{r}$, let say $\mathfrak{n}'$. We have then $[\mathfrak{s}, \mathfrak{r}] = [\mathfrak{s}, \mathfrak{n} \oplus \mathfrak{n}'] \subset [\mathfrak{s}, \mathfrak{n}] \oplus [\mathfrak{s}, \mathfrak{n}']$, but $[\mathfrak{s}, \mathfrak{n}'] \subset \mathfrak{n} \cap \mathfrak{n}' = \{0\}$ ($\mathfrak{s}$ acts trivially on $\mathfrak{n}'$), then $[\mathfrak{s}, \mathfrak{r}] \subset [\mathfrak{s}, \mathfrak{n}]$. The inclusion $[\mathfrak{s}, \mathfrak{n}] \subset [\mathfrak{s}, \mathfrak{r}]$ is obvious. Hence

$$[\mathfrak{s}, \mathfrak{g}] = [\mathfrak{s}, \mathfrak{s} \oplus \mathfrak{r}] \subset [\mathfrak{s}, \mathfrak{s}] \oplus [\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{s} \oplus [\mathfrak{s}, \mathfrak{n}] \subset \mathfrak{s} \ltimes \mathfrak{n}.$$

We deduce that $\mathfrak{s} \ltimes \mathfrak{n}$ is an ideal of $\mathfrak{g}$. □

Next we will prove:

Proposition 4.3.2. *The non-compact semi-simple factor S_{nc} of S is non trivial.*

Let us first start with the following simple observation:

Proposition 4.3.3. *If a conformal diffeomorphism f of (M, g) preserves a volume form ω on M, then it preserves a metric in the conformal class of g.*

Proof. Let f be a diffeomorphism preserving the conformal class $[g]$ and a volume form ω on M. Denote by ω_g the volume form defined on M by the metric g. On the one hand, there exists a C^∞ real function ϕ such that $\omega = e^{\phi}\omega_g$. Hence ω is the volume form defined by the metric $e^{\frac{2\phi}{n}}g$. On the other hand, we have $f^* e^{\frac{2\phi}{n}}g = e^{\psi}e^{\frac{2\phi}{n}}g$, for some C^∞ function ψ. Thus $f^*\omega = e^{\frac{n}{2}\psi}\omega$. But, f preserves the volume form ω, so $\psi = 0$ which means that f preserves the metric $e^{\frac{2\phi}{n}}g$. □

As a consequence we get:

Corollary 4.3.4. *The conformal group G preserves no volume form on M.*

Proof. It suffices to note that the function ϕ appearing in the proof of the proposition 4.3.3 is independent of the choice of the conformal diffeomorphism f. We then have a metric in the conformal class of g which is preserved by G. □

Let's return to the proof of the proposition 4.3.2. Assume that the non-compact semi-simple factor S_{nc} is trivial. Then by [30, Corollary 4.1.7] the group G is amenable. So it preserves a regular Borel measure μ on the compact manifold M. It is in particular a quasi-invariant measure with associated rho-function $\rho_1 = 1$ (in the sense of [4]). Let now ω_g be the volume form corresponding to the metric g. As the group G acts conformally and the action is C^∞, the measure ω_g is also quasi-invariant with C^∞ rho-function ρ_2 (see [4, Theorem B.1.4]). Again by [4, Theorem B.1.4], the measures μ and ω_g are equivalent and $\frac{d\mu}{d\omega_g} = \frac{1}{\rho_2}$. This shows that μ is a volume form. Then one use Corollary 4.3.4 to get the Proposition 4.3.2.

In the general case the essentiality of the action ensure the non discretness of the stabilizer H.

Proposition 4.3.5. *The stabilizer H is not discrete.*

Proof. If it was not the case then H would be a uniform lattice in G. But as the action is essential, there is an element $h \in H$ that does not preserve the metric on $\mathfrak{g}/\mathfrak{h}$. So $|\det(\mathrm{Ad}_h)| \neq 1$ contradicting the unimodularity of G. $\square$

To finish this part let us prove the two following important Lemmas that will be used later:

Lemma 4.3.6. *Let $\pi : S_{nc} \longrightarrow \mathrm{GL}(V)$ be a linear representation of S_{nc} into a linear space V. Then, the compact orbits of S_{nc} are trivial.*

Proof. Assume that S_{nc} has a compact orbit $\mathcal{C} \subset V$. Then the convex envelope $\mathrm{Conv}(\mathcal{C} \cup -\mathcal{C})$ is an $S_{nc}-$invariant compact convex symmetric set with non empty interior. Thus the action of S_{nc} preserves the Minkowski gauge $\|.\|$ (which is in fact a norm) of $\mathrm{Conv}(\mathcal{C} \cup -\mathcal{C})$. But $\mathrm{Isom}\left(\mathrm{Conv}(\mathcal{C} \cup -\mathcal{C}), \|.\|\right)$ is compact. So the restriction of the representation π to $\mathrm{Conv}(\mathcal{C} \cup -\mathcal{C})$ gives rise to an homomorphism from a semi-simple group with no compact factor to a compact group and hence is trivial. $\square$

Lemma 4.3.7. *Let π_m be the representation given in example A.0.1 (appendix, page 51). We have*

$$\pi_m(Y)(V_m) \subset \pi_m\left(\mathbb{R}.H \oplus \mathbb{R}.X\right)(V_m)$$

Proof. We have V_m spanned by $\left\{z_1^k z_2^{m-k},\ 0 \leq k \leq m\right\}$. We have

$$\pi_m(Y)(z_1^k z_2^{m-k}) = -z_1\partial_{z_2}(z_1^k z_2^{m-k}) = -(m-k)z_1^{k+1}z_2^{m-k-1}$$

$$\pi_m(H)(z_1^k z_2^{m-k}) = (-z_1\partial_{z_1} + z_2\partial_{z_2})(z_1^k z_2^{m-k}) = -kz_1^k z_2^{m-k} + (m-k)z_1^k z_2^{m-k}$$

$$\pi_m(X)(z_1^k z_2^{m-k}) = -z_2\partial_{z_1}(z_1^k z_2^{m-k}) = -kz_1^{k-1}z_2^{m-k+1}$$

It is obvious that $\pi_m(Y)(z_1^k z_2^{m-k})$ can be obtained as a linear combination of $\pi_m(H)(z_1^k z_2^{m-k})$ and $\pi_m(X)(z_1^k z_2^{m-k})$.

□

Lemma 4.3.8. *A linear representation $\pi : \mathfrak{s}_{nc} \longrightarrow \mathrm{gl}(V)$ of $\mathfrak{s}_{nc}$ into a linear space V is completely determined by its restriction to $\mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_+$. More precisely,*

$$\pi_{\mathfrak{s}_{nc}}(V) = \mathrm{Vect}\left(\pi_{\mathfrak{a}\oplus\mathfrak{m}\oplus\mathfrak{s}_+}(V)\right).$$

Proof. It is in fact sufficient to show that

$$\pi_{\mathfrak{s}_-}(V) \subset \pi\left(\mathfrak{a} \oplus \mathfrak{s}_+\right)(V).$$

For that, fix $x \in \mathfrak{s}_{-\alpha} \subset \mathfrak{s}_-$ and let $a \in \mathfrak{a}$ such that $\mathbb{R}x \oplus \mathbb{R}a \oplus \mathbb{R}\Theta(x) \cong \mathfrak{sl}(2,\mathbb{R})$ (see for example [16, Proposition 6.52]). Thus the restriction of π to $\mathbb{R}x \oplus \mathbb{R}a \oplus \mathbb{R}\Theta(x)$ is isomorphic to a linear representation of $\mathfrak{sl}(2,\mathbb{R})$ into V. Using Weyl Theorem we can assume without loss of generality that this last is irreducible. But irreducible linear representations of $\mathfrak{sl}(2,\mathbb{R})$ into V are unique up to isomorphism (see for instance [14, Theorem 4.32]). Using lemma 4.3.7, (Y, H, X in the lemma 4.3.7 correspond respectively to $x, a, \Theta(x)$ in the lemma 4.3.8) it is then easy to check that they verify

$$\pi(x)(V) \subset \mathrm{Vect}\left(\pi_{\mathbb{R}a\oplus\mathbb{R}\Theta(x)}(V)\right)$$

This finishes the proof. □

4.4. Lie algebra formulation

4.4.1. Enlargement of the isotropy group

As the manifold G/H is compact, the isotropy subgroup H is a uniform subgroup of G. If H was discrete then it is a uniform lattice and in this case G would be unimodular. In the non discrete case, this imposes strong restrictions on the group H. When H and G are both complex algebraic it is equivalent to being parabolic i.e contains maximal solvable connected subgroup of H. In the real case, Borel and Tits [5] proved that an algebraic group H of a real linear algebraic group G is uniform if it contains a maximal connected triangular subgroup of G. Recall that a subgroup of G (respectively a sub-algebra of $\mathfrak{g}$) is said to be triangular if, in some real basis of $\mathfrak{g}$, its image under the adjoint representation is triangular.

Let $H^* = \mathrm{Ad}^{-1}\left(\overline{\mathrm{Ad}(H)}^{Zariski}\right)$ be the smallest algebraic Lie subgroup of G containing H. By [13, Corollary 5.1.1], the Lie algebra $\mathfrak{h}^*$ of H^* contains a maximal triangular sub-algebra of $\mathfrak{g}$. The sub-algebra $(\mathfrak{a} \oplus \mathfrak{s}_+) \ltimes \mathfrak{n}$ being triangular, we get the following fact:

Fact 4.4.1. *Up to conjugacy, the sub-algebra h^* contains $(\mathfrak{a} \oplus \mathfrak{s}_+) \ltimes \mathfrak{n}$.*

Consider the vector space $\mathrm{Sym}(\mathfrak{g})$ of bilinear symmetric forms on $\mathfrak{g}$. The group G acts naturally on $\mathrm{Sym}(\mathfrak{g})$ by $g.\Phi(X,Y) = \Phi(Ad_{g^{-1}}X, Ad_{g^{-1}}Y)$. Let $\langle .,. \rangle$ be the bilinear symmetric form on $\mathfrak{g}$ defined by

$$\langle X, Y \rangle = g\left(X^*(x), Y^*(x)\right),$$

where g is the pseudo-Riemannian metric, X^*, Y^* are the fundamental vector fields associated to X and Y and x is the point fixed previously. It is a degenerate symmetric form with kernel equal to $\mathfrak{h}$.

Let P be the subgroup of G preserving the conformal class of $\langle .,. \rangle$. It is an algebraic group containing H and normalizing the sub-algebra $\mathfrak{h}$. In particular, it contains H^*: the smallest algebraic group containing H. Using Fact 4.4.1 we get that up to conjugacy, the Lie algebra $\mathfrak{p}$ of P contains $(\mathfrak{a} \oplus \mathfrak{s}_+) \ltimes \mathfrak{n}$.

Proposition 4.4.2. *The Cartan sub-group A does not preserve the metric $\langle .,. \rangle$.*

Proof. First as $\mathfrak{h}$ is an ideal of $\mathfrak{p}$ then by taking quotient of both P and H by $H^{\mathfrak{r}}$, we can suppose that H is a uniform lattice of P and in particular that P is unimodular.

Assume that A preserves the metric $\langle .,. \rangle$. On the one hand, the groups S_+ and N preserve the conformal class of $\langle .,. \rangle$. On the other hand, they act on $\mathrm{Sym}(\mathfrak{g})$ by unipotent elements. So the groups A, S_+, and N preserve the metric $\langle .,. \rangle$. But by Iwasawa decomposition $(A \ltimes S_+)$ is co-compact in S. Thus the $S-$orbit of $\langle .,. \rangle$ is compact in $\mathrm{Sym}(\mathfrak{g})$ and hence trivial by Lemma 4.3.6. Therefore S and N are subgroups of P. This implies that for any $p \in P$, $\left|\det\left(\mathrm{Ad}_p\right)_{|\mathfrak{g}/\mathfrak{p}}\right| = 1$. Indeed, the action of G on $(\mathfrak{s}_c + \mathfrak{r})/\mathfrak{n}$ factors trough the product of the action of S_c on $\mathfrak{s}_c$ by the trivial action on $\mathfrak{r}/\mathfrak{n}$. As P contains S and N, its action on $\mathfrak{g}/\mathfrak{p}$ is a quotient of the action of S_c on $\mathfrak{s}_c$. But S_c is compact, thus it preserves some positive definite scalar product and hence the determinant $\left|\det\left(\mathrm{Ad}_p\right)_{|\mathfrak{g}/\mathfrak{p}}\right| = 1$.

Now let $h \in H$ such that Ad_h does not preserve $\langle .,. \rangle$. We have that

$$1 \neq \left|\det\left(\mathrm{Ad}_h\right)_{|\mathfrak{g}/\mathfrak{h}}\right| = \left|\det\left(\mathrm{Ad}_h\right)_{|\mathfrak{g}/\mathfrak{p}}\right| \left|\det\left(\mathrm{Ad}_h\right)_{|\mathfrak{p}/\mathfrak{h}}\right|$$

Finally we get $\left|\det\left(\mathrm{Ad}_h\right)_{|\mathfrak{p}/\mathfrak{h}}\right| \neq 1$ which contradicts the unimodularity of P. □

4.4.2. Distortion

The group P preserves the conformal class of $\langle .,. \rangle$. There exists thus an homomorphism $\delta : P \to \mathbb{R}$ such that: for every $p \in P$ and every $u, v \in \mathfrak{g}$,

$$\langle \mathrm{Ad}_p(u), \mathrm{Ad}_p(v) \rangle = e^{\delta(p)} \langle u, v \rangle = \left|\det\left(\mathrm{Ad}_p\right)_{|\mathfrak{g}/\mathfrak{h}}\right|^{\frac{2}{n}} \langle u, v \rangle \tag{4.1}$$

In particular if $p \in P$ preserves the metric then $\delta(p) = 0$ and

$$\langle \mathrm{Ad}_p(u), \mathrm{Ad}_p(v) \rangle = \langle u, v \rangle \tag{4.2}$$

Or equivalently

$$\langle \mathrm{ad}_p(u), v\rangle + \langle u, \mathrm{ad}_p(v)\rangle = 0 \tag{4.3}$$

It follows that if the action of $p \in P$ on $\mathfrak{g}$ is unipotent then $\delta(p) = 0$. Therefore, the homomorphism δ is trivial on S_- and N but not on A by Proposition 4.4.2. We continue to denote by δ the restriction of δ to A. We can see it alternatively as a linear form $\delta : \mathfrak{a} \to \mathbb{R}$, called **distortion**, verifying: for every $a \in \mathfrak{a}$ and every $u, v \in \mathfrak{g}$,

$$\langle \mathrm{ad}_a(u), v\rangle + \langle u, \mathrm{ad}_a(v)\rangle = \delta(a)\,\langle u, v\rangle \tag{4.4}$$

Definition 4.4.1. *Two weights spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are said to be paired if they are not $\langle .,.\rangle$ −orthogonal.*

Definition 4.4.2. *A weight α is a non-degenerate weight if $\mathfrak{g}_\alpha$ is not contained in $\mathfrak{h}$.*

Definition 4.4.3. *We say that a subalgebra $\mathfrak{g}'$ is a modification of $\mathfrak{g}$ if $\mathfrak{g}'$ projects surjectively on $\mathfrak{g}/\mathfrak{h}$ and in addition $\mathfrak{g}' \supset \mathfrak{s}_{nc}$. In this case $\mathfrak{g}'/\left(\mathfrak{g}' \cap \mathfrak{h}\right) = \mathfrak{g}/\mathfrak{h}$.*

Proposition 4.4.3. *If the weight space $\mathfrak{g}_0$ is degenerate then up to modification, $\mathfrak{g}$ is semi-simple and $M = G/H$ is conformally flat.*

Proof. On the one hand, $\mathfrak{g}_0 \subset \mathfrak{h}$ implies that $\mathfrak{a} \subset \mathfrak{h}$. As $\mathfrak{h}$ is an ideal of $\mathfrak{p}$, we get that $\mathfrak{s}_+ = [\mathfrak{s}_+, \mathfrak{a}] \subset \mathfrak{h}$. On the other hand, $\mathfrak{r} \subset \mathfrak{g}_0 + \mathfrak{n} \subset \mathfrak{g}_0 + [\mathfrak{n}, \mathfrak{a}] \subset \mathfrak{h}$. Thus, up to modification, we can assume that $\mathfrak{g}$ is semi-simple and that $\mathfrak{h}$ contains $\mathfrak{a} + \mathfrak{s}_+ + \mathfrak{m}$.

Now let α_{max} be the highest positive root and let $X \in \mathfrak{g}_{\alpha_{max}}$. Then $d_1 e^X : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ is trivial. Yet e^X is not trivial. We conclude using [10, Theorem 1.4]. □

A direct consequence of Equation 4.4, is that if $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are paired then $\alpha + \beta = \delta$. Indeed if $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are paired, there exist $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ such that $\langle x, y\rangle \neq 0$. By equation 4.4, we have for all $a \in \mathfrak{a}$

$$\langle \mathrm{ad}_a(x), y \rangle + \langle x, \mathrm{ad}_a(y) \rangle = \alpha(a) \langle x, y \rangle + \beta(a) \langle x, y \rangle = (\alpha(a) + \beta(a)) \langle x, y \rangle = \delta(a) \langle x, y \rangle$$

which gives

$$\alpha + \beta = \delta.$$

This shows that if α is a non-degenerate weight then $\delta - \alpha$ is also a non-degenerate weight. In particular if 0 is a non-degenerate weight, then $\mathfrak{g}_0$ and $\mathfrak{g}_\delta$ are paired and hence δ is a non-degenerate weight. In fact:

Proposition 4.4.4. *If* 0 *is a non-degenerate weight then* δ *is a root. Moreover* $\mathfrak{s}_\delta \not\subset \mathfrak{h}$.

Proof. First we will prove that the subalgebras $\mathfrak{a}$ and $\mathfrak{n}_\delta$ are $\langle ., . \rangle$ −orthogonal. Let $a \in \mathfrak{a}$ such that $\delta(a) \neq 0$. Using Equation 4.4 for a, $u = a$ and $v \in \mathfrak{n}_\delta$, we get, $\langle a, \mathrm{ad}_a(v) \rangle = \delta(a) \langle a, v \rangle$. But v preserves $\langle ., . \rangle$, thus by Equation 4.3, $\delta(a) \langle a, v \rangle = 0$. Hence $\langle a, v \rangle = 0$, for every $v \in \mathfrak{n}_\delta$. We conclude by continuity.

Now if δ was not a root then $\mathfrak{s}_\delta = 0$ and $\mathfrak{g}_\delta = \mathfrak{n}_\delta$. Thus $\mathfrak{a}$ and $\mathfrak{g}_\delta$ are orthogonal. Which implies that $\mathfrak{a} \subset \mathfrak{h}$. But $\mathfrak{h}$ is an ideal of $\mathfrak{p}$, so $\mathfrak{g}_\delta = [\mathfrak{g}_\delta, \mathfrak{a}] \subset \mathfrak{h}$. This contradicts the fact that $\mathfrak{g}_\delta$ is paired with $\mathfrak{g}_0$.

To finish we need to prove that $\mathfrak{s}_\delta \not\subset \mathfrak{h}$. If this was not the case then $\mathfrak{a}$ would be orthogonal to $\mathfrak{g}_\delta$. Hence $\mathfrak{g}_\delta \subset \mathfrak{h}$ which contradicts again the fact that $\mathfrak{g}_\delta$ is paired with $\mathfrak{g}_0$.

□

4.4.3. The isotropy group is big

From now and until the end we will suppose that the non-compact semi-simple part S_{nc} of G is locally isomorphic to the Möbius group $\mathrm{SO}(1, n+1)$. In this case the Cartan Lie algebra

$\mathfrak{a}$ is one dimensional and we have $\mathfrak{s}_{nc} = \mathfrak{s}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_{\alpha}$, where α is a positive root, $\mathfrak{a} \cong \mathbb{R}$, $\mathfrak{m} \cong \mathfrak{so}(n)$, and $\mathfrak{s}_{-\alpha} \cong \mathfrak{s}_{\alpha} \cong \mathbb{R}^n$. Moreover, $\mathfrak{g}_{\pm\alpha} = \mathfrak{s}_{\pm\alpha} \oplus \mathfrak{n}_{\pm\alpha}$, $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_c \oplus \mathfrak{r}_0$, $\mathfrak{g}_{\beta} = \mathfrak{n}_{\beta}$ for every $\beta \neq 0, \pm\alpha$ and $\mathfrak{r} = \mathfrak{r}_0 \oplus \bigoplus_{\beta\neq 0} \mathfrak{n}_{\beta}$.

In section 4.4.1 we saw that the isotropy group H is contained in the algebraic group P which turn out to be big i.e to contain the connected Lie groups A, S_{α} and N. Our next result shows that the group H itself is big:

Proposition 4.4.5. *The Lie algebra* $\mathfrak{h}$ *contains* $\mathfrak{a} \oplus \mathfrak{s}_{\alpha} \oplus \bigoplus_{\beta\neq 0} \mathfrak{n}_{\beta}$.

Proof. We have that $\mathfrak{a} \subset \mathfrak{h}$. Indeed, if 0 is a degenerate weight then we are done. If not, then δ is a root and $\mathfrak{a} \subset \mathfrak{g}_0$ is orthogonal to every $\mathfrak{g}_{\beta}$ with $\beta \neq \delta$. From the proof of Proposition 4.4.4 we know that $\mathfrak{a}$ and $\mathfrak{n}_{\delta}$ are orthogonal. Thus it remains to show that $\mathfrak{a}$ and $\mathfrak{s}_{\delta}$ are orthogonal. For that, let $x \in \mathfrak{s}_{\delta}$ then $\Theta(x) \in \mathfrak{s}_{-\delta}$ and $[x, \Theta(x)] \neq 0$ in $\mathfrak{a}$. Now using Equation 4.3 and the fact that one of x or $\Theta(x)$ preserve $\langle .,. \rangle$, we get $\langle \mathrm{ad}_x(\Theta(x)), x \rangle = 0$. But $\mathfrak{a}$ is one dimensional so it is orthogonal to $\mathfrak{s}_{\delta}$.

To end this proof, we have that $\mathfrak{h}$ is an ideal of $\mathfrak{p}$ and so

$$\mathfrak{a} \oplus \mathfrak{s}_{\alpha} \oplus \bigoplus_{\beta\neq 0} \mathfrak{n}_{\beta} = \mathfrak{a} \oplus \left[\mathfrak{a}, \mathfrak{s}_{\alpha} \oplus \bigoplus_{\beta\neq 0} \mathfrak{n}_{\beta}\right] \subset \mathfrak{h} \oplus [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{h}.$$

□

As a consequence we get:

Corollary 4.4.6. *If* 0 *is a non-degenerate weight, then* $\mathfrak{s}_{-\alpha}$ *is paired with* $\mathfrak{g}_0$ *and* $\delta = -\alpha$.

Indeed, there exists $x \in \mathfrak{s}_{-\alpha}$ and $y \in \mathfrak{g}_0$ such that $\langle x, y \rangle \neq 0$. Admitting this for the moment we have for all $a \in \mathfrak{a}$

$$\langle \mathrm{ad}_a(x), y \rangle + \langle x, \mathrm{ad}_a(y) \rangle = -\alpha(a) \langle x, y \rangle = \delta(a) \langle x, y \rangle$$

which gives $-\alpha = \delta$. Now, if there is no $x \in \mathfrak{s}_{-\alpha}$ and $y \in \mathfrak{g}_0$ such that $\langle x, y\rangle \neq 0$, we have $\langle \mathfrak{s}_{-\alpha}, \mathfrak{g}_0\rangle = 0$, we have also $\langle \mathfrak{g}_0, \mathfrak{g}_0\rangle = 0$ (if not, $\mathfrak{g}_0$ will be paired with itself, which gives us $0+0 = \delta$ which is impossible), and $\left\langle \mathfrak{g}_0, \mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \bigoplus_{\beta\neq 0} \mathfrak{n}_\beta \right\rangle = 0$ because $\mathfrak{a}\oplus\mathfrak{s}_\alpha\oplus\bigoplus_{\beta\neq 0} \mathfrak{n}_\beta \subset \mathfrak{h}$. Then, all this implies that $\langle \mathfrak{g}_0, \mathfrak{g}\rangle = 0$ (remember that $\mathfrak{g} = \mathfrak{s}_{-\alpha} \oplus \underbrace{\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{s}_c \oplus \mathfrak{r}_0}_{\mathfrak{g}_0} \oplus \mathfrak{s}_\alpha \oplus \bigoplus_{\beta\neq 0} \mathfrak{n}_\beta$) which gives $\mathfrak{g}_0 \subset \mathfrak{h}$. This contradicts the fact 0 is a non-degenerate weight.

4.4.4. A suitable modification of $\mathfrak{g}$

We will show that $\mathfrak{g}$ admits a suitable modification $\mathfrak{g}'$. This allows us to considerably simplify the proofs in the next section. More precisely, we have:

Proposition 4.4.7. *The solvable radical decomposes as a direct sum $\mathfrak{r} = \mathfrak{r}_1 \oplus \mathfrak{r}_2$, where $\mathfrak{r}_1$ is a subalgebra commuting with the semi-simple factor $\mathfrak{s}$ and $\mathfrak{r}_2$ is an $\mathfrak{s}-$invariant linear subspace contained in $\mathfrak{h}$. In particular $\mathfrak{g}' = \mathfrak{s} \oplus \mathfrak{r}_1$ is a modification of $\mathfrak{g}$.*

To prove Proposition 4.4.7, we need the following lemma:

Lemma 4.4.8. *We have $[\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{h}$.*

Proof of Lemma 4.4.8. First we prove that $[\mathfrak{n}, \mathfrak{g}_0] \subset \mathfrak{h}$. For this, note that by the Jacobi identity and the fact that $\mathfrak{n}$ is an ideal of $\mathfrak{g}$, we have $\left[\bigoplus_{\beta\neq 0} \mathfrak{n}_\beta, \mathfrak{g}_0\right] = \bigoplus_{\beta\neq 0} \mathfrak{n}_\beta$ which in turn is a subset of $\mathfrak{h}$ by Proposition 4.4.5. Thus one need to prove that $[\mathfrak{n}_0, \mathfrak{g}_0] \subset \mathfrak{h}$. We know that $\mathfrak{n}$ preserve the metric $\langle ., .\rangle$. So using Equation 4.3 for $p \in \mathfrak{n}_0$, $u \in \mathfrak{g}_0$ and $v \in \mathfrak{g}_\delta$ gives us: $\langle \mathrm{ad}_p(u), v\rangle + \langle u, \mathrm{ad}_p(v)\rangle = 0$. But once again by Jacobi identity, the fact that $\mathfrak{n}$ is an ideal of $\mathfrak{g}$ and Proposition 4.4.5 we have $\mathrm{ad}_p(v) \in \mathfrak{g}_\delta \cap \mathfrak{n} = \mathfrak{n}_\delta \subset \mathfrak{h}$. So $\langle \mathrm{ad}_p(u), v\rangle = 0$, which means that $[\mathfrak{n}_0, \mathfrak{g}_0]$ is orthogonal to $\mathfrak{g}_\delta$. Using the fact that $[\mathfrak{n}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$ and that $\mathfrak{g}_0$ is orthogonal to every $\mathfrak{g}_\beta$ for $\beta \neq \delta$ we get that $[\mathfrak{n}_0, \mathfrak{g}_0] \subset \mathfrak{h}$.

Next we have that $\mathfrak{s}_c \subset \mathfrak{g}_0$ thus $[\mathfrak{s}_c, \mathfrak{n}] \subset [\mathfrak{g}_0, \mathfrak{n}] \subset \mathfrak{h}$.

Finally we finish by proving that $[\mathfrak{s}_{nc}, \mathfrak{n}] \subset \mathfrak{h}$. On the one hand we have,

$$[\mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{m}, \mathfrak{n}] \subset [\mathfrak{a} \oplus \mathfrak{s}_\alpha, \mathfrak{n}] + [\mathfrak{g}_0, \mathfrak{n}] \subset \mathfrak{h} + \mathfrak{h} \subset \mathfrak{h}.$$

On the other hand, as $\mathfrak{s}_{nc}$ is semi-simple we have by Lemma 4.3.8 that

$$[\mathfrak{s}_{nc}, \mathfrak{n}] \subset \mathrm{Vect}\left([\mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_\alpha, \mathfrak{n}]\right) \subset \mathfrak{h}$$

.

Proof of Proposition 4.4.7. The subalgebra $[\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{r}]$ is $\mathfrak{s}-$invariant, so it admits an $\mathfrak{s}-$invariant supplementary subspace $\mathfrak{r}_1'$ in $\mathfrak{r}$. But $\mathfrak{s}$ acts trivially on $\mathfrak{r}/[\mathfrak{s}, \mathfrak{n}]$ and thus it acts trivially on $\mathfrak{r}_1'$. We take $\mathfrak{r}_1$ to be the $\mathfrak{s}-$invariant subalgebra generated by $\mathfrak{r}_1'$ (in fact the action of $\mathfrak{s}$ on $\mathfrak{r}_1$ is trivial).

It is clear that $\mathfrak{r}_1$ is a direct sum of $\mathfrak{r}_1'$ and $\mathfrak{r}_1''$: an $\mathfrak{s}-$invariant subspace of $[\mathfrak{s}, \mathfrak{n}]$. Consider $\mathfrak{r}_2$ to be the supplementary of $\mathfrak{r}_1''$ in $[\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{r}]$. It is $\mathfrak{s}-$invariant and by Lemma 4.4.8 we have $\mathfrak{r}_2 \subset \mathfrak{h}$.

4.5. The Möbius conformal group: a classification theorem

This section is devoted to prove Theorem 4.1.1. We distinguish two situations: when $\mathfrak{m}$ is contained in $\mathfrak{h}$ and when it is not. In this last one, we first consider the case where only the non-compact semi-simple part S_{nc} is non trivial. Then deduce from it the general case. From now and until the end we will assume, up to modification, that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}_1$.

4.5.1. The Frances-Melnick case

We suppose that the sub-algebra $\mathfrak{m}$ is contained in $\mathfrak{h}$. Then we have the following proposition:

Proposition 4.5.1. *M is conformally equivalent to the standard sphere $\mathbb{S}^n$ or the Einstein universe* $\mathrm{Ein}^{1,1}$.

Proof. Assume first that $\mathfrak{g}_0$ is contained in $\mathfrak{h}$. Then by Proposition 4.4.3, M is conformally flat and after modification, $\mathfrak{r} = 0$. Moreover, $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{s}_{-\alpha}$. This is because $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_c$ and $[\mathfrak{m}, X] = \mathfrak{s}_{-\alpha}$ for every $X \neq 0$ in $\mathfrak{s}_{-\alpha}$. Thus M is conformally equivalent to $\mathrm{SO}(1, n+1)/\mathrm{CO}(n) \ltimes \mathbb{R}^n \cong \mathbb{S}^n$.

Now suppose that $\mathfrak{g}_0$ is not in $\mathfrak{h}$. In this case $\mathfrak{g}_{-\alpha} = \mathfrak{g}_\delta$ is paired with $\mathfrak{g}_0$. But $\mathfrak{a}$, $\mathfrak{m}$ and $\mathfrak{n}_\delta$ are contained in $\mathfrak{h}$ so $\mathfrak{s}_{-\alpha}$ is paired with $\mathfrak{s}_c \oplus (\mathfrak{r}_0 \cap \mathfrak{r}_1)$. Note that $\mathfrak{m}$ acts on $\mathfrak{s}_{-\alpha} \oplus (\mathfrak{s}_c \oplus (\mathfrak{r}_0 \cap \mathfrak{r}_1))$ by preserving the pairing (in fact the action of $\mathfrak{m}$ preserves the metric $\langle ., . \rangle$). On the contrary for $n \geq 2$, $\mathfrak{m} \cong \mathfrak{so}(n)$ acts trivially on $\mathfrak{r}_0 \cap \mathfrak{r}_1$ and transitively on $\mathfrak{s}_{-\alpha} - \{0\}$, so $n = 1$. As the metric is of type (p, q), we conclude that the projection of $\mathfrak{s}_c \oplus (\mathfrak{r}_0 \cap \mathfrak{r}_1)$ on $\mathfrak{g}/\mathfrak{h}$ is $\cong \mathbb{R}$. Thus, after modification $\mathfrak{g} = \mathfrak{so}(1, 2) \oplus \mathbb{R} = \mathfrak{u}(1, 1)$, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{s}_\alpha = \mathbb{R} \oplus \mathbb{R}$ and hence M is, up to cover, conformally equivalent to $\mathrm{Ein}^{1,1}$.

□

4.5.2. The non-compact semi-simple case

Here we suppose that $\mathfrak{m}$ is not contained in $\mathfrak{h}$, the compact semi-simple part $\mathfrak{s}_c$ and the radical solvable part $\mathfrak{r}_1$ are both trivial. We will show:

Proposition 4.5.2. *The pseudo-Riemannian manifold M is conformally equivalent to* $\mathrm{Ein}^{3,3}$

By corollary 4.4.6, δ is a negative root. In particular $\delta = -\alpha$ and $\mathfrak{g}_{-\alpha}$ is paired with $\mathfrak{g}_0$. In addition, $\mathfrak{g} = \mathfrak{s}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_\alpha$ and $\mathfrak{a} \oplus \mathfrak{s}_\alpha \subset \mathfrak{h}$. We have:

Proposition 4.5.3. *The root space $\mathfrak{s}_\delta$ does not intersect $\mathfrak{h}$. In particular the metric is of type (n, n).*

Proof. If it was not the case then let $0 \neq X \in \mathfrak{s}_\delta \cap \mathfrak{h}$. We have $[[X, \mathfrak{s}_{-\delta}], X] = \mathfrak{s}_\delta = \mathfrak{s}_{-\alpha}$. Let $Y \in \mathfrak{s}_{-\alpha}$, there exists $Z \in \mathfrak{s}_\alpha$ $(\subset \mathfrak{h})$ such that $Y = [[X, Z], X] \in \mathfrak{h}$ so $\mathfrak{s}_\delta \subset \mathfrak{h}$. This contradicts the fact that $\mathfrak{g}_\delta$ is paired with $\mathfrak{g}_0$.

Let's explain why the metric is of type (n, n). We have $\mathfrak{g} = \mathfrak{s}_{-\alpha} \oplus V \oplus \mathfrak{h}$ where $V \subset \mathfrak{g}_o$ and $V \cap \mathfrak{h} = \{0\}$. The subspace $\mathfrak{s}_{-\alpha}$ is totally isotropic (it is not paired with itself because $-\alpha - \alpha \neq -\alpha = \delta$). The same is true for V ($V \subset \mathfrak{g}_o$ and $\mathfrak{g}_o$ is not paired with itself because $0 + 0 \neq -\alpha = \delta$). Then $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{s}_{-\alpha} \oplus V$. If the metric is of type (p, q), $p \leq q$, we will have $\dim \mathfrak{s}_{-\alpha} \leq p$, $\dim V \leq p$. On the other hand, $p + q = \dim \mathfrak{s}_{-\alpha} + \dim V \leq 2p$ which gives us $p = q$. $\square$

Consider the bracket $[.,.] : \mathfrak{s}_\alpha \times \mathfrak{s}_{-\alpha} \longrightarrow \mathfrak{a} \oplus \mathfrak{m}$. Denote by $. \wedge . : \mathfrak{s}_\alpha \times \mathfrak{s}_{-\alpha} \longrightarrow \mathfrak{m}$ and $. \vee . : \mathfrak{s}_\alpha \times \mathfrak{s}_{-\alpha} \longrightarrow \mathfrak{a}$ its projections on $\mathfrak{m}$ and $\mathfrak{a}$ respectively. Direct computations give us:

Lemma 4.5.4.

1. *$\forall X \in \mathfrak{s}_\alpha$, $\forall x \in \mathfrak{s}_{-\alpha}$: $X \vee x = -\Theta(x) \vee \Theta(X)$.*

2. *$\forall X \in \mathfrak{s}_\alpha$, $\forall x \in \mathfrak{s}_{-\alpha}$, $\forall y \in \mathfrak{s}_{-\alpha}$: $[X \wedge x, y] = [X \wedge y, x] - [\Theta(x) \wedge y, \Theta(X)]$*

Proof. 1. Remmember that $[\mathfrak{s}_\alpha, \mathfrak{s}_{-\alpha}] \subset \mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$. Let $X \in \mathfrak{s}_\alpha$ and $x \in \mathfrak{s}_{-\alpha}$. We have

$$[X, x] = X \wedge x + X \vee x$$

$$[\Theta(x), \Theta(X)] = \Theta(x) \wedge \Theta(X) + \Theta(x) \vee \Theta(X)$$

We have also

$$\begin{aligned}\Theta([X,x]) &= -[\Theta(x),\Theta(X)]\\ &= \Theta(X\wedge x)+\Theta(X\vee x).\end{aligned}$$

Uniqueness implies that $\Theta(X\vee x)=\Theta(x)\vee\Theta(X)$, but $X\vee x\in\mathfrak{a}\subset\mathfrak{p}=\{X\,|\,\Theta(X)=-X\}$ thus $X\vee x=\Theta(x)\vee\Theta(X)$.

2. By Jacobi identity, this identity is equivalent to

$$[X\vee x,y]=[X\vee y,x]+[[\Theta(x),y],\Theta(X)]-[\Theta(x)\vee y,\Theta(X)]$$

But $[X\vee x,y]=-\alpha(X\vee x)y$, $[X\vee y,x]=-\alpha(X\vee y)x$ and $[\Theta(x)\vee y,\Theta(X)]=-\alpha(\Theta(x)\vee y)\Theta(X)$. Thus the identity is equivalent to

$$\alpha(X\vee y)x-\alpha(X\vee x)y-\alpha(\Theta(x)\vee y)\Theta(X)=[[\Theta(x),y],\Theta(X)]$$

whose proof is straightforward by calculation.

$\square$

The Cartan involution identifies $\mathfrak{s}_\alpha$ and $\mathfrak{s}_{-\alpha}$, which when identified with $\mathbb{R}^n$, $\mathfrak{m}$ acts on them as $\mathfrak{so}(n)$. Indeed we have

$$\mathfrak{s}_{-\alpha}=\left\{\begin{bmatrix}0 & 0_{1\times n} & 0\\ x & 0_{n\times n} & 0\\ 0 & -x^t & 0\end{bmatrix},\ x\in\mathbb{R}^n\right\},$$

$$\mathfrak{a}=\left\{\begin{bmatrix}a & 0_{1\times n} & 0\\ 0_{n\times 1} & 0_{n\times n} & 0\\ 0 & 0_{1\times n} & -a\end{bmatrix},\ a\in\mathbb{R}\right\},$$

$$\mathfrak{m}=\left\{\begin{bmatrix}0 & 0_{1\times n} & 0\\ 0_{n\times 1} & M & 0\\ 0 & 0_{1\times n} & 0\end{bmatrix},\ M\in\mathfrak{so}(n)\right\},$$

$$\mathfrak{s}_\alpha = \left\{ \begin{bmatrix} 0 & -x^t & 0 \\ 0 & 0_{n\times n} & x \\ 0 & 0_{1\times n} & 0 \end{bmatrix}, x \in \mathbb{R}^n \right\},$$

We see that $\mathfrak{s}_\alpha \cong \Theta(\mathfrak{s}_\alpha) = \mathfrak{s}_{-\alpha} = \mathbb{R}^n$. If $A \in \mathfrak{m}$ and $X \in \mathfrak{s}_\alpha$, we have

$$[A, X] = AX - XA = \begin{bmatrix} 0 & -(Mx)^t & 0 \\ 0_{n\times 1} & 0_{n\times n} & Mx \\ 0 & 0_{1\times n} & 0 \end{bmatrix}$$

which shows that the action of $\mathfrak{m}$ on $\mathfrak{s}_\alpha$ is equivalent to the equation of $\mathfrak{so}(n)$ on $\mathbb{R}^n$.

In this case, the map $.\vee.$ can be seen as a bilinear symmetric map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$, and when composed with α gives rise to an $\mathfrak{m}-$invariant scalar product $\langle ., . \rangle_0$ on $\mathbb{R}^n$. Moreover, by Lemma 4.5.4, for every $x, X \in \mathbb{R}^n$, $X \wedge x$ is the antisymmetric endomorphism of $\mathbb{R}^n$ defined by $X \wedge x(y) = \langle X, y \rangle_0 x - \langle x, y \rangle_0 X$.

Let $x, X \in \mathbb{R}^n$ and consider P the plane generated by x, X. Then $X \wedge x$ when seen as element of $\mathfrak{m} \cong \mathfrak{so}(n)$ is the infinitesimal generator of a one parameter group acting trivially on the orthogonal $P^\perp$ of P with respect to the scalar product $\langle ., . \rangle_0$. Hence $X \wedge x \in \mathfrak{so}(P)$. More generally:

Proposition 4.5.5. *Let E be a linear subspace of $\mathbb{R}^n$ and let $x \in E$. Consider $\mathfrak{c}$ the Lie subalgebra of $\mathfrak{so}(n)$ generated by $\{X \wedge x / X \in E\}$. Then $\mathfrak{c}$ equals the Lie algebra linearly generated by $\{X \wedge X' / X, X' \in E\}$, which in turn equals $\mathfrak{so}(E)$, the Lie algebra of orthogonal transformations preserving E and acting trivially on its orthogonal (with respect to $\langle ., . \rangle_0$).*

Proof. First we have $\mathfrak{c}(E) \subset E$ and hence $\mathfrak{c} \subset \mathfrak{so}(E)$. It is then sufficient to prove that $\mathfrak{c}$ and $\mathfrak{so}(E)$ have the same dimensions. For that let $\{x, X_2, ..., X_k\}$ be a basis of E. Note that $\{X_2 \wedge x, ..., X_k \wedge x, [X_i \wedge x, X_j \wedge x], \text{ for } 2 \leq i < j \leq k\}$ are linearly independent. Thus $\mathfrak{c} = \mathfrak{so}(E)$. $\square$

For every $x \neq 0 \in \mathfrak{s}_{-\alpha}$ consider: $Z_x = \{X \neq 0 \in \mathfrak{s}_\alpha,$ such that $[X, x] \in \mathfrak{h}\}$. Denote $\overline{\Theta(Z_x)}$ the projection of $\Theta(Z_x)$ in $\mathfrak{g}/\mathfrak{h}$. Then:

Proposition 4.5.6. *The family* $\left\{\overline{\Theta(Z_x)}\backslash\{0\}\right\}_{x\in\mathfrak{s}_{-\alpha}}$ *form a partition of* $\mathfrak{g}/\mathfrak{h}$. *More precisely:*

$$\overline{\Theta(Z_x)} \cap \overline{\Theta(Z_y)} = \{0\} \Longleftrightarrow x \notin \Theta(Z_y) \Longleftrightarrow y \notin \Theta(Z_x).$$

Proof. By Proposition 4.5.5 we have,

$$[Z_x, \Theta(Z_x)] = \mathfrak{a} \oplus \mathrm{alg}\left(\{X \wedge x / X \in Z_x\}\right) \subset \mathfrak{h}.$$

This implies that

$$\Theta(Z_x) = \Theta(Z_y) \Longleftrightarrow x \in \Theta(Z_y) \Longleftrightarrow y \in \Theta(Z_x).$$

Hence, the projections $\left\{\overline{\Theta(Z_x)}\backslash\{0\}\right\}_{x\in\mathfrak{s}_{-\alpha}}$ form a partition of $\mathfrak{g}/\mathfrak{h}$. □

Next we prove:

Proposition 4.5.7. *The pseudo-Riemannian manifold M is conformally flat.*

Proof. We need to prove that the Weyl tensor W (or the Cotton tensor C if the dimension of M is 3) vanishes. Actually we will just make use of their conformal invariance property. Namely: if f is a conformal transformation of M then,

$$d_x f\, \mathrm{W}(X, Y, Z) = \mathrm{W}(d_x f(X), d_x f(Y), d_x f(Z)) \tag{4.5}$$

We denote by $\bar{x}$ the projection in $\mathfrak{g}/\mathfrak{h}$ of an element $x \in \mathfrak{g}$. A direct application of Equation 4.5 gives us:

1. $\mathrm{W}(\bar{x}, \bar{y}, \bar{z}) = 0$ for every $x, y, z \in \mathfrak{s}_{-\alpha}$;

2. $\mathrm{W}(\bar{x}, \bar{y}, \overline{m}) = 0$ for every $x, y \in \mathfrak{s}_{-\alpha}$ and every $m \in \mathfrak{m}$;

3. $[X, \mathrm{W}(\bar{x}, \overline{m}_1, \overline{m}_2)] = \mathrm{W}([X, \bar{x}], \overline{m}_1, \overline{m}_2)$ for every $X \in \mathfrak{s}_\alpha$, $x \in \mathfrak{s}_{-\alpha}$ and every $m_1, m_2 \in \mathfrak{m}$.

Let $x \in \mathfrak{s}_{-\alpha}$, $m_1, m_2 \in \mathfrak{m}$. Then, from Equation 4.5 we obtain:

$$[\Theta(x), \mathrm{W}(\bar{x}, \overline{m}_1, \overline{m}_2)] = \mathrm{W}([\Theta(x), \bar{x}], \overline{m}_1, \overline{m}_2) = 0.$$

In other words

$$\mathrm{W}(\bar{x}, \overline{m}_1, \overline{m}_2) \in \overline{\Theta(Z_x)}.$$

Now let $x, y \in \mathfrak{s}_{-\alpha}$, $X \in \mathfrak{s}_\alpha$ and $m \in \mathfrak{m}$. Then again Equation 4.5 gives us:

$$\mathrm{W}(\bar{x}, [X, \bar{y}], \overline{m}) + \mathrm{W}([X, \bar{x}], \bar{y}, \overline{m}) = 0.$$

But $\mathrm{W}(\bar{x}, [X, \bar{y}], \overline{m}) \in \overline{\Theta(Z_x)}$ and $\mathrm{W}([X, \bar{x}], \bar{y}, \overline{m}) \in \overline{\Theta(Z_y)}$. Thus, Proposition 4.5.6 gives us:

1. If $y \notin \Theta(Z_x)$ then $\mathrm{W}(\bar{x}, [X, \bar{y}], \overline{m}) = 0$;
2. In the case $y \in \Theta(Z_x)$ and $X \in Z_x$, we have $\mathrm{W}(\bar{x}, [X, \bar{y}], \overline{m}) = 0$
3. If $y \in \Theta(Z_x)$ and $X \notin Z_x$. Then because $\Theta(X) \notin \Theta(Z_x)$ we have:

$$\mathrm{W}(\bar{x}, [X, \bar{y}], \overline{m}) = \mathrm{W}(\bar{x}, \left[\Theta(y), \overline{\Theta(X)}\right], \overline{m}) = 0.$$

So as a conclusion we get $\mathrm{W} = 0$. □

We finish this section by proving Proposition 4.5.2:

Proof of Proposition 4.5.2. First note that if $n = 1$ then $\mathfrak{m} = 0$. Thus we assume $n \geq 2$. So far we have seen that $M = \mathrm{SO}(1, n+1)/H$ is a conformally flat pseudo-Riemannian manifold of signature (n, n). Since the Lie algebra $\mathfrak{h}$ contains $\mathfrak{a} + \mathfrak{s}_\alpha$, the group H° is cocompact in

$\mathrm{SO}^\circ(1, n+1)$. Therefore $\mathrm{SO}^\circ(1, n+1)/H^\circ$ is connected and compact, with a connected isotropy and hence simply connected. As M is connected, it covers $\mathrm{SO}^\circ(1, n+1)/H^\circ$ and thus equals it.

On the one hand, the Einstein universe $\mathrm{Ein}^{n,n}$ is simply connected. Thus M is identified to $\mathrm{Ein}^{n,n}$. So $\mathrm{SO}(1, n+1)$ acts transitively on $\mathrm{Ein}^{n,n}$ with isotropy H. By Montgomery Theorem [23, Theorem A] any maximal compact subgroup in $\mathrm{SO}(1, n+1)$, e.g. $K_2 = \mathrm{SO}(n+1)$, acts transitively on $\mathbb{S}^n \times \mathbb{S}^n$ the two fold cover of $\mathrm{Ein}^{n,n}$.

On the other hand, the conformal group of $\mathrm{Ein}^{n,n}$ is $\mathrm{SO}(n+1, n+1)$. A maximal compact subgroup of it is $K_1 = \mathrm{SO}(n+1) \times \mathrm{SO}(n+1)$. Up to conjugacy , we can assume $K_2 \subset K_1$. Therefore, $K_2 = \mathrm{SO}(n+1)$ acts via a homomorphism

$$\rho = (\rho_1, \rho_2) : \mathrm{SO}(n+1) \to \mathrm{SO}(n+1) \times \mathrm{SO}(n+1)$$

. If $\mathrm{SO}(n+1)$ is simple, then:

- either ρ_1 or ρ_2 is trivial and the other one is bijective, in which case $\rho(\mathrm{SO}(n+1))$ does not act transitively on $\mathbb{S}^n \times \mathbb{S}^n$,

- or both are bijective, and $\rho(\mathrm{SO}(n+1)$ is up to conjugacy in $\mathrm{SO}(n+1) \times \mathrm{SO}(n+1)$ the diagonal $\{(g, g)/g \in \mathrm{SO}(n+1)\}$. The latter, too, does not act transitively on $\mathbb{S}^n \times \mathbb{S}^n$.

Hence $\mathrm{SO}(n+1)$ must be non-simple which implies $n = 1$ or $n = 3$. But $n = 1$ was excluded, and then remains exactly the case $n = 3$, for which M is conformally equivalent to $\mathrm{Ein}^{3,3}$.

4.5.3. The general case

In this section we will show Theorem 4.1.1 in the general case. We suppose that $\mathfrak{g} = \mathfrak{s}_{-\alpha} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{s}_c \oplus \mathfrak{r}_1$. Let us denote by $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{h}$ so that $\mathfrak{so}(1, n+1) \cap \mathfrak{h} = \mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{m}_0$.

A priori the subalgebra $\mathfrak{m}_0$ could be of any dimension in $\mathfrak{m}$. Nevertheless the hypothesis $\mathfrak{m} \not\subset \mathfrak{h}$ restricts drastically the possibilities. So we have:

Proposition 4.5.8. *The subalgebra $\mathfrak{m}_0$ has codimension n in $\mathfrak{m}$.*

Proof. If $n = 2$ then $\mathfrak{m} = \mathfrak{so}(2)$. Hence $[p, \mathfrak{s}_\alpha] = \mathfrak{a} \oplus \mathfrak{m}$ for any non null $p \in \mathfrak{s}_{-\alpha}$. Recall that $\mathfrak{s}_{-\alpha}$ preserves the metric so by applying Equation 4.3 for $p = v \in \mathfrak{s}_{-\alpha}$, $u \in \mathfrak{s}_\alpha$ we get $\langle \mathfrak{s}_{-\alpha}, \mathfrak{m} \rangle = 0$. Thus $\mathfrak{m} \subset \mathfrak{h}$ which contradicts our hypothesis.

Assume that $n \geq 3$ and suppose that $\mathfrak{m}_0$ has codimension less then $n-1$. Denote by M_0 the connected subgroup of $\mathrm{SO}(n)$ corresponding to $\mathfrak{m}_0$.

If the action of M_0 on $\mathfrak{s}_{-\alpha} \cong \mathbb{R}^n$ is reducible then M_0 preserves the splitting $\mathbb{R}^d \times \mathbb{R}^{n-d}$ and hence is contained in $\mathrm{SO}(d) \times \mathrm{SO}(n-d)$. Thus M_0 has codimension bigger than the codimension of $\mathrm{SO}(d) \times \mathrm{SO}(n-d)$ which in turn achieves its minimum if $d = 1$ or $n - d = 1$ and hence $M_0 = \mathrm{SO}(n-1)$. One can identify $\mathfrak{m}_0$ with $\mathfrak{so}(E)$ for some $n-1$ dimensional linear subspace E of $\mathfrak{s}_{-\alpha}$. Let then $e \in \mathfrak{s}_{-\alpha}$ such that $\mathfrak{s}_{-\alpha} = \mathbb{R}e \oplus E$. Fix a non zero element $x \in \Theta(E)$, we have $\langle \mathrm{ad}_x(e), X \rangle + \langle e, \mathrm{ad}_x X \rangle = 0$ for every $X \in \mathfrak{s}_{-\alpha}$ and so in particular $\langle e, \mathrm{ad}_x e \rangle = 0$. In addition by Proposition 4.5.5, $[E, \Theta(E)] = \mathfrak{a} \oplus \mathfrak{m}_0 \subset \mathfrak{h}$ thus $\langle \mathrm{ad}_x e, X \rangle = 0$ for every $X \in E$ and hence $\mathrm{ad}_x e$ is orthogonal to $\mathfrak{s}_{-\alpha}$. This implies that $x \wedge e \in \mathfrak{h} \cap \mathfrak{m} = \mathfrak{m}_0 = \mathfrak{so}(E)$ which contradicts the fact that $x \wedge e$ is the infinitesimal rotation of the plane $\mathbb{R}e \oplus \mathbb{R}x$.

The last case to consider is when M_0 acts irreducibly. Let $m \in \mathfrak{m}_0$, $X \in \mathfrak{s}_{-\alpha}$ and $y \in \mathfrak{s}_c \oplus \mathfrak{r}_1$ then

$$\langle \mathrm{ad}_m(X), y \rangle + \langle X, \mathrm{ad}_m y \rangle = 0.$$

But $\mathrm{ad}_m y = 0$ and hence $\mathfrak{s}_c \oplus \mathfrak{r}_1$ is orthogonal to $[\mathfrak{m}_0, \mathfrak{s}_{-\alpha}]$ which is equal to $\mathfrak{s}_{-\alpha}$ by irreducibility. Thus $\mathfrak{s}_c \oplus \mathfrak{r}_1 \subset \mathfrak{h}$ and we are in the non-compact semi-simple case. Therefore $n = 3$ and

$\mathfrak{m} \cong \mathfrak{so}(3)$. Non trivial Sub-algebras of $\mathfrak{so}(3)$ have dimension one and are reducible. So the only left possibility is $\mathfrak{m}_0 = \mathfrak{m} \cong \mathfrak{so}(3)$ which show that $\mathfrak{m} \subset \mathfrak{h}$ and this is a contradiction.

The last case to consider is when M_0 acts irreducibly. Let $m \in \mathfrak{m}_0$, $X \in \mathfrak{s}_{-\alpha}$ and $y \in \mathfrak{s}_c \oplus \mathfrak{r}_1$ then

$$\langle \mathrm{ad}_m(X), y\rangle + \langle X, \mathrm{ad}_m\, y\rangle = 0.$$

But $\mathrm{ad}_m\, y = 0$ and hence $\mathfrak{s}_c \oplus \mathfrak{r}_1$ is orthogonal to $[\mathfrak{m}_0, \mathfrak{s}_{-\alpha}]$ which is equal to $\mathfrak{s}_{-\alpha}$ by irreducibility. Thus $\mathfrak{s}_c \oplus \mathfrak{r}_1 \subset \mathfrak{h}$ and we are in the non-compact semi-simple case. Therefore $n = 3$ and $\mathfrak{m} \cong \mathfrak{so}(3)$. Non trivial Sub-algebras of $\mathfrak{so}(3)$ have dimension one and are reducible. So the only left possibility is $\mathfrak{m}_0 = \mathfrak{m} \cong \mathfrak{so}(3)$ which show that $\mathfrak{m} \subset \mathfrak{h}$ and this is a contradiction. $\square$

End of Proof of Theorem 4.1.1.

By Proposition 4.5.8, $\mathfrak{m}_0$ is of codimension n in $\mathfrak{m}$. But $\mathfrak{s}_{-\alpha}$ is paired with $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{s}_c \oplus (\mathfrak{r}_1 \cap \mathfrak{r}_0)$. Thus $\mathfrak{s}_c \oplus (\mathfrak{r}_1 \cap \mathfrak{r}_0) \subset \mathfrak{h}$ and we are also in the non-compact semi-simple case. Therefore $n = 3$ and M is conformally equivalent to $\mathrm{Ein}^{3,3}$.

Chapter 5

Conclusion

Our work deals with the so called pseudo-Riemannian Lichnerowicz Conjecture. We wanted to understand all compact pseudo- Riemannian manifolds whose conformal group does not preserve any metric in the conformal class. We focused on the case where the non-compact semi- simple part is the Möbius group $O(1,n)$ (for a certain value of n) and we assumed that the manifold is compact homogeneous.

The general non homogeneous case, but with signature restrictions, was amply studied by Zimmer, Bader, Nevo, Frances, Zeghib, Melnick and Pecastaing (see [31], [3], [12], [27], [28], [26], [22]). Let us also quote [20] as a recent work in the Lorentz case. In contrast, our work deals with arbitrary signature but the restrictions we used are: the transivity of the action and the specification of the noncompact part of the semisimple part of the conformal group as being locally isomorphic to the Möbius group. We hope that in the near future, we can get rid at least of the second restriction and to assume instead the noncompact semi-simple part of the group G has real rank 1 Moreover we think that under certain hypothesis, this is possible whith compact complex riemannian manifolds admitting a conformal riemannian holomorphic structure with a transitive and essential action of a complex semisimple Lie group.

Appendix A

Some facts about Lie algebras

This section is devoted to recall some basic notions and fondamental results known in the literature regarding Lie algebras and Lie groups we used in our work. All the details are found in [16], [14] and [15].

All Lie algebras used in our work are finite-dimensional over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$).

Definition A.0.1 (**Lie algebra, subalgebra of a Lie algebra, ideal, Lie algebra homomorphism**).

1. *A Lie algebra $\mathfrak{g}$ is a vector space over a field with a product $[\,,\,]$ (called **Lie bracket**) satisfying*
   - *The product is linear in each variable and for all X, Y, Z in $\mathfrak{g}$ we have*
   - $[X, X] = 0$ *or equivalently* $[X, Y] = -[Y, X]$,
   - $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *(the **Jacobi Identity**).*
2. *A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ which is stable under the Lie bracket.*

3. *A subspace* $\mathfrak{k}$ *of a Lie algebra* $\mathfrak{g}$ *is an ideal in* $\mathfrak{g}$ *if* $[\mathfrak{k},\mathfrak{g}] \subset \mathfrak{k}$. *It is clear that an ideal is a subalgebra.*

4. *A linear map:* $\pi : \mathfrak{g} \longrightarrow \mathfrak{h}$ *between two Lie algebras is* ***a homomorphism of Lie algebras*** *if*

$$\pi\left([X,Y]\right) = \left[\pi(X),\pi(Y)\right], \forall X,\, Y \in \mathfrak{g}.$$

It is clear that $\ker\pi$ *is an ideal of* $\mathfrak{g}$.

Definition A.0.2 (**Centre, Centralizer, normalizer**).

1. *The center of a Lie algebra is* $Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X,Y] = 0, \forall Y \in \mathfrak{g}\}$.

2. *Let* $\mathfrak{s}$ *be a subset of a Lie algebra* $\mathfrak{s}$. *The centralizer of* $\mathfrak{s}$ *is* $Z_{\mathfrak{g}}(\mathfrak{s}) = \{X \in \mathfrak{g} \mid [X,Y] = 0, \forall Y \in \mathfrak{s}\}$.

3. *Let* $\mathfrak{s}$ *be a subset of a Lie algebra* $\mathfrak{s}$. *The normalizer of* $\mathfrak{s}$ *is* $N_{\mathfrak{g}}(\mathfrak{s}) = \{X \in \mathfrak{g} \mid [X,Y] \in \mathfrak{s}, \forall Y \in \mathfrak{s}\}$.

It is clear that the center of a Lie algebra is an ideal and that the centralizer and normalizer are subalgebras.

Definition A.0.3 (**Solvable Lie algebra**). *Let* $\mathfrak{g}$ *be a Lie algebra. We define*

$$\mathfrak{g}^{\circ} = \mathfrak{g},\, \mathfrak{g}^{1} = [\mathfrak{g},\mathfrak{g}]\,,\, \mathfrak{g}^{i+1} = \left[\mathfrak{g}^{i},\mathfrak{g}^{i}\right].$$

The decreasing sequence

$$\mathfrak{g} = \mathfrak{g}^{\circ} \supseteq \mathfrak{g}^{1} \cdots \supseteq \mathfrak{g}^{\hat{\imath}} \cdots$$

is called the commutator series of $\mathfrak{g}$.

We say that $\mathfrak{g}$ *is solvable if* $\mathfrak{g}^{i} = 0$ *for some* i.

It is clear that each $\mathfrak{g}^{i}$ is an ideal of $\mathfrak{g}$. It is also clear that a nonzero solvable Lie algebra $\mathfrak{g}$ has a nonzero abelian ideal which is given by the last nonzero $\mathfrak{g}^{i}$.

Every Lie algebra $\mathfrak{g}$ contains a (unique) maximal solvable ideal. It is called the radical of the Lie algebra (see [16].) and it is noted by rad$\mathfrak{g}$.

Definition A.0.4 (**Nilpotent Lie algebra**). *Let* $\mathfrak{g}$ *be a Lie algebra. We define*

$$\mathfrak{g}_\circ = \mathfrak{g},\ \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]\ ,\ \mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i]\ .$$

The decreasing sequence

$$\mathfrak{g} = \mathfrak{g}_\circ \supseteq \mathfrak{g}_1 \cdots \supseteq \mathfrak{g}_{\hat{\imath}} \cdots$$

is called the lower central series of $\mathfrak{g}$.

We say that $\mathfrak{g}$ *is nilpotent if* $\mathfrak{g}_i = 0$ *for some* i.

It is clear that each $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$. It is also clear that a nonzero nilpotent Lie algebra $\mathfrak{g}$ has nonzero center, the last nonzero $\mathfrak{g}_i$ being in the center. We have also that for all i

$$\mathfrak{g}^i \subset \mathfrak{g}_i,$$

hence every nilpotent algebra is solvable.

Every Lie algebra contains a (unique) maximal nilpotent ideal (which is contained in tne radical). It is called the radical nilpotent of the Lie algebra.

Definition A.0.5. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra. Then:*

1) $\mathfrak{g}$ *is simple if* $\mathfrak{g}$ *is nonabelian and has no proper nonzero ideals,*

2) $\mathfrak{g}$ *is semisimple if* $\mathfrak{g}$ *has nonzero abelian ideals, or equivalently* $\mathfrak{g}$ *has no nonzero solvable ideals., i.e.* $rad(\mathfrak{g}) = 0$ *where* $rad(\mathfrak{g})$ *is the solvable radical of* $\mathfrak{g}$.

Remark A.0.1. *In the definition of a simple Lie algebra we can replace the nonabelian condition by the fact that* $\dim \mathfrak{g}$ *is* ≥ 2

Remark A.0.2. *If* $\mathfrak{g}$ *is simple then* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Definition A.0.6. *Let $\mathfrak{g}$ be a Lie algebra. The symmetric bilinear map*

$$k : \mathfrak{g} \times \mathfrak{g} \longmapsto \mathbb{K}, \ (X, Y) \longmapsto Tr(adXadY)$$

where adX is the linear transformation from $\mathfrak{g}$ to $\mathfrak{g}$ defined by $adX(Y) = [X, \ Y]$,

is called the killing form of $\mathfrak{g}$.

Theorem A.0.3 (**Cartan**)**.** *Let $\mathfrak{g}$ be a real Lie algebra then $\mathfrak{g}$ is semisimple if and only if its killing form is nondegenerate.*

The following theorem will be of crucial importance in our work (see [16, Theorem B.2, page 660]).

Theorem A.0.4 (**Levi decomposition**)**.** *Let $\mathfrak{g}$ be a real Lie algebra, then there exists a semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{g}$ is the semidirect product*

$$\mathfrak{g} = \mathfrak{s} \oplus_{\pi} (\mathrm{rad}\mathfrak{g})$$

for a suitable homomorphism $\pi : s \longrightarrow \mathrm{Der}_{\mathbb{R}}(\mathrm{rad}\mathfrak{g})$.

Definition A.0.7 (**Cartan decomposition**)**.** *Let $\mathfrak{g}$ be a Lie algebra. A Cartan decomposition of $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ such that:*

1- $[l, \ l] \subset \mathfrak{l}$, $[\mathfrak{l}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{l}$

2- The killing form k of $\mathfrak{g}$ is negative definite on $\mathfrak{l}$ and positive definite on $\mathfrak{p}$.

Definition A.0.8. *Let θ be an involution of $\mathfrak{g}$, that is an automorphism with square equal to the identity. We say that θ is a Cartan involution of $\mathfrak{g}$ if the symmetric bilinear form*

$$k_{\theta}(X, Y) = -k(X, \theta(Y)),$$

where k is the killing form of $\mathfrak{g}$, is positive definite.

A Cartan decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ determines a Cartan involution by the formula $\theta = Id$ on $\mathfrak{l}$ and $\theta = -Id$ on $\mathfrak{p}$. Conversely, let θ be a Cartan involution of $\mathfrak{g}$. Then we have $\mathfrak{l} \oplus \mathfrak{p}$, where:

$$\mathfrak{l} = \{X \in \mathcal{G}, \theta(X) = X\}, \, \mathfrak{p} = \{X \in \mathcal{G}, \theta(X) = -X\},$$

It is then easy to show that this decomposition is orthogonal with respect to the killing form of $\mathfrak{g}$ and that $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}$, $[\mathfrak{l}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{l}$. Since k_θ is positive definite, k is negative definite on $\mathfrak{l}$ and positive definite on $\mathfrak{l}$.

The following proposition affirms the existence of Cartan involution in the semisimple case.

Proposition A.0.5. *If $\mathfrak{g}$ is a real semisimple Lie algebra, then $\mathfrak{g}$ has a Cartan involution and any two Cartan involutions of $\mathfrak{g}$ are conjugate via an inner automorphism of $\mathfrak{g}$.*

Definition A.0.9 (Representation of a Lie algebra). *Let $\mathfrak{g}$ a real or complex Lie algebra and V a finite-dimensional complex vector space. Let $\rho : \mathfrak{g} \longrightarrow gl(V)$. We say that ρ is a finite-dimensional complex representation of $\mathfrak{g}$ into $gl(V)$ if it is a Lie algebra homorphism.*

Definition A.0.10 (Invariant subspace. Irreducible representation of a Lie algebra). *Let $\rho : \mathfrak{g} \longrightarrow gl(V)$ a finite-dimensional complex representation of $\mathfrak{g}$. An invariant subspace for the representation ρ is a vector subspace W of V verifying $\pi(\mathfrak{g})W \subset W$. We say that the representation ρ on a nonzero vector space V is **irreducible** if the only invariant subspaces are $\{0_V\}$ and V.*

Definition A.0.11 (Equivalent representations). *We say that two representations $\rho_1 : \mathfrak{g} \longrightarrow gl(V_1)$ and $\rho_2 : \mathfrak{g} \longrightarrow gl(V_2)$ of a Lie algebra $\mathfrak{g}$ are equivalent if there exists an isomorphism ϕ from V_1 onto V_2 such that for all $X \in \mathfrak{g}$, $\phi \circ \rho_1(X) = \rho_2(X) \circ \phi$.*

We have the following proposition (see [14], proposition 4.6 page 93) that we used implicitely in the proof of lemma 4.3.8

Proposition A.0.6. *Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then every finite-dimensional complex representation π of $\mathfrak{g}$ has a unique extension to a complex linear representation of $\mathfrak{g}_{\mathbb{C}}$ given by*

$$\pi(X + iY) = \pi(X) + i\pi(Y), \forall\, X,\, Y \in \mathfrak{g}.$$

Furthermore, π is irreducible as a representation of $\mathfrak{g}$ if and only if it is irreducible as a representation of $\mathfrak{g}_{\mathbb{C}}$.

Example A.0.1 (A representation of $\mathfrak{sl}(2, \mathbb{C})$, see [14], page 97)**.** *Consider the space V_m of homogeneous polynomials in two complex variables z_1, z_2 with degree m. We have* $\dim V_m = m + 1$. *We have then the following representation $\pi_m : \mathfrak{sl}(2, \mathbb{C}) \longrightarrow \mathfrak{gl}(V_m)$ given by*

$$\pi_m(H) = -z_1\partial_{z_1} + z_2\partial_{z_2}$$

$$\pi_m(X) = -z_2\partial_{z_1}$$

$$\pi_m(Y) = -z_1\partial_{z_2}.$$

where $\{H, \,, X\,, Y\}$ is the standard basis of $\mathfrak{sl}(2, \mathbb{C})$ given by

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We have the following important proposition

Proposition A.0.7. *The representation π_m is irreducible.*

The next proposition is of much great importance for the proof of lemma 4.3.8. It says that in some sense π_m is the only irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$.

Proposition A.0.8 (see [14], page 102)**.** *For each integer $m \geq 0$, there exists up to equivalence a unique irreducible complex-linear representation π of $\mathfrak{sl}(2, \mathbb{C})$*

In other words, any irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ is equivalent to π_m.

Root-space decomposition

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ be a Cartan decomposition. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. It is called a Cartan subalgebra of $\mathfrak{g}$. For $\lambda \in \mathfrak{a}^*$ we have

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \ / \ [H,\ X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\lambda$ is nontrivial and $\lambda \neq 0$, we call λ a restricted root of $\mathfrak{g}$ or a root of $(\mathfrak{g},\ a)$ and $\mathfrak{g}_\lambda$ is called a restricted root space. The set of restricted roots of $\mathfrak{g}$ is denoted by Δ.

The next theorem plays an important role in our work.

Theorem A.0.9. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ with respect to the Cartan involution θ. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Then, we have:*

a) $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda$ *(orthogonal direct sum),*

b) $[\mathfrak{g}_\lambda,\ \mathfrak{g}_\gamma] \subset \mathfrak{g}_{\lambda+\gamma}$ *and* $\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$.

c) $\mathfrak{go} = \mathfrak{a} \oplus \mathfrak{m}$, *where* $\mathfrak{m} = \mathrm{Z}_{\mathfrak{l}}(\mathfrak{a})$ *(the centralizer of* $\mathfrak{a}$ *in* $\mathfrak{l}$*).*

The decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda$ is called the restricted-root space decomposition of $\mathfrak{g}$.

Example 1. ***Root space decomposition of*** $\mathfrak{sl}(n,\ \mathbb{R})$

Let $G = SL(n,\ \mathbb{R})$ be the set of matrices whose determinant is equal to 1, its Lie algebra is $\mathfrak{g} = \mathfrak{sl}(n,\ \mathbb{R})$ of all real matrices with traces equal to 0. Let k be the killing form of $\mathfrak{sl}(n,\mathbb{R})$ and $\theta :\ \mathfrak{sl}(n,\mathbb{R}) \mapsto \mathfrak{sl}(n,\mathbb{R})$ be the map defined by $\theta(X) = -X^t$. Then,

$$\theta[X,Y] = [\theta(X),\theta(Y)]$$

and

$$k_\theta(X,Y) = -k(X,\theta(Y)) = k(Y,X)$$

$k(X,\, X) \geq 0$, *then* k_θ *is a Cartan involution and*

$$\mathfrak{sl}(n,\mathbb{R}) = l \oplus p$$

where $\mathfrak{l}$ *is the space of skew symmetric matrices of* $\mathfrak{sl}(n,\, \mathbb{R})$ *and* $\mathfrak{p}$ *is the space of symmetric matrices of* $\mathfrak{sl}(n,\, \mathbb{R})$. *The space* $\mathfrak{a}$ *of diagonal matrices withe trace 0 is a maximal abelian subspace of* $\mathfrak{p}$. *Let* $f_i \in \mathfrak{a}^*$ *such that* $f_i(diag\ d_j) = d_i$. *Then the set of restricted roots of* $\mathfrak{sl}(n,\, \mathbb{R})$ *is the set of functions* $f_{ij} = f_i - f_j$ *with* $i \neq j$ *and*

$$\mathfrak{g}_{f_{ij}} = \{X \in \mathfrak{sl}(n,\mathbb{R}) /\, [A,X] = f_{ij}(A).X\}$$

is the space of all matrices (X_{lk}) *such that* $X_{lk} = 0$ *for* $(l,\, k) \neq (i,\, j)$. *The dimension of* $\mathfrak{g}_{f_{ij}}$ *is 1.*

For $n = 2$, *we have* $\dim a = 1$ *and the set of restricted roots consist of* $\{f_{12},\, -f_{12}\}$ *and the dimension of* $\mathfrak{g}_{f_{12}}$ *is* 1.

Restricted Roots of $\mathfrak{o}(p,q)$

Let $G = O(p,q)$ *the set of* $n \times n$ *matrices* M *where* $n = p+q$ *verifying*

$$M^{tr} J_{p,q} M = J_{p,q}$$

$J_{p,q}$ *being the diagonal matrix whose first p elements are all equal to -1 and the last q elements to 1. Let* $\mathfrak{so}(p,q)$ *be its Lie algebra, then* $M \in \mathfrak{so}(p,q)$ *if* M *has the form*

$$M = \begin{bmatrix} A & B \\ B^t & D \end{bmatrix}$$

where A is $p \times p$ skew symmetric matrix and D is $q \times q$ skew symmetric matrix. Then the Cartan involution is given by $\theta = -M^t$. Then we find

$$\mathfrak{l} = \{M \in \mathfrak{so}(p,q), B = 0\}$$

and

$$\mathfrak{l} = \{M \in \mathfrak{so}(p,q), A = D = 0\}$$

Let $\mathfrak{a}$ be the maximal abelian subalgebra of $\mathfrak{p}$ of matrices of the form

$$M = \begin{bmatrix} 0 \\ C \end{bmatrix}$$

where

$$C = \begin{bmatrix} 0 & 0 & a_q \\ 0 & a_i & 0 \\ a_1 & 0 & 0 \end{bmatrix}$$

Let $\alpha_i \in \mathfrak{a}^*$ defined by $\alpha_i(M) = a_i$, then the functions α_i, $-\alpha_i$, $\alpha_i + \alpha_j$, $\alpha_i - \alpha_j$ $i \neq j$ are the restricted roots of $\mathfrak{so}(p,q)$.

The particular case $\mathfrak{g} = \mathfrak{o}(1, n+1)$

Here we find that $\mathfrak{g}_\alpha$ consists of matrices given by

$$\begin{bmatrix} 0 & -x_1 & ... & -x_n & 0 \\ 0 & ... & ... & ... & -x_1 \\ ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... \\ 0 & ... & ... & ... & -x_n \\ 0 & ... & ... & ... & 0 \end{bmatrix}$$

and for $\mathfrak{g}_{-\alpha}$ the matrices are given by

$$\begin{bmatrix} 0 & 0 & ... & ... & 0 \\ x_1 & 0 & ... & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ x_n & 0 & ... & ... & 0 \\ 0 & -x_1 & ... & -x_n & 0 \end{bmatrix}$$

The subalgebras and are obviously isomorphic to $\mathbb{R}^n$.

The matrices of the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{o}(1, n+1)$ are of the form

$$\begin{bmatrix} a & 0 & ... & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & -a \end{bmatrix}$$

$\mathfrak{a}$ is obviously isomorphic to $\mathbb{R}$.

Finally, the matrices of $\mathfrak{m}$ are of the form

$$\begin{bmatrix} 0 & ... & 0 \\ 0 & M & 0 \\ 0 & ... & 0 \end{bmatrix}$$

where M is in $\mathfrak{so}(\mathfrak{n})$ which shows that $\mathfrak{m}$ is isomorphic to $\mathfrak{so}(\mathfrak{n})$.

The root α acts on $\mathfrak{a}$ as follows

$$\alpha(X) = a$$

for every X in $\mathfrak{a}$, where X is given by

$$\begin{bmatrix} a & 0 & ... & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & -a \end{bmatrix}$$

We have also that

$$\mathfrak{s}_0 = \mathfrak{a} \oplus \mathfrak{m}.$$

Appendix B

Algebraic uniform subgroups.

A subgroup H of a Lie group G is called uniform (or cocompact) if there exists a compact subset Q of G such that $G = QH$. A Lie subgroup H of G is uniform if and only if G/H is compact.

A subgroup H of a complex linear algebraic group G is said to be parabolic if it contains a max- imal connected solvable subgroup of the group G.

An algebraic subgroup H of a complex linear algebraic group is uniform if and only if H is a parabolic subgroup. The analogous statement in the real case has the following form (Borel and Tits 1965, see [5]): an algebraic subgroup H of a real linear algebraic group G is uniform if and only if H con- tains a maximal connected subgroup of the group G, the elements of which can be, in some base, simultaneously represented by triangular matrices.

Following V. V. Gorbatsevich and A. L. Onishchik [13], we say that a subgroup H of a Lie group G is called Ad-algebraic if $H = Ad^{-1}(H*)$, where $H*$ is a certain (real) algebraic subgroup of $GL(\mathfrak{g})$. Clearly, an Ad-algebraic subgroup of G is a Lie subgroup. If G is a real algebraic group then any Ad-algebraic subgroup of it is an algebraic subgroup in the usual sense, containing the centre of G. The intersection of any family of subgroups of a Lie group G is an Ad- algebraic subgroup. From this it follows that for every subgroup H of a Lie group G there exists the smallest Ad-algebraic subgroup H containing H.

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