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**Subject**

# Existence of positive solutions for the Sturm-Liouville problem with p-Laplacian

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**To the spirit of the parents**
**To my siblings**
**To my wife and my Sons.**

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## Abstract

This work is to prove the existence of three positive solutions for the problem Sturm-Liouville with p-Laplacian next

$$\left(\phi_p\left(u''\left(t\right)\right)\right)' + f\left(t, u\left(t\right)\right) = 0, \qquad t \in (0,1) \tag{1}$$

$$\alpha u\left(0\right) - \beta u'\left(0\right) = 0, \qquad \alpha, \beta \geq 0 \tag{2}$$

$$\gamma u\left(1\right) + \delta u'\left(1\right) = 0, \qquad \gamma, \delta \geq 0 \tag{3}$$

$$u''\left(0\right) = 0. \tag{4}$$

where $\phi_p\left(s\right) = \left|s\right|^{p-2} s \quad \forall p > 1, \;\; f \in C([0,1] \times [0,\infty); [0,\infty))$.

**Key words.** Existence of solutions, problem of third-order Sturm-Liouville, positive solution, Leggett-Williams fixed point theorem.

# Contents

# Introduction

Many mathematical models in nonlinear phenomena can give rise to problems for which only positive solutions make sens. The purpose of this work is to study the existence of at least three positive solutions for the third order Sturm-Liouville boundary value problem with p-Laplacien. This work has been proposed by C. Zhai and C. Guo [14] (2009).

These type of problems can arise in variety interesting applications such as in physics, biology or non-Newtonian mechanics. So in recent years, much authors were interested in the study of equations with p-Laplacien. For example, in non-Newtonian mechanics [6],[14], combustion theory [25], population biology [23],[24], nonlinear flow laws [9],[19],[20] and system of Monge-Kantorovitch partial differential equation [8].

Several authors have been more interested in the problems of the second order Sturm-Liouville p-Laplacian, see [12],[18],[22],[27],[28],[30] and to our knowledge few authors have studied the case of problems of third order Sturm-Liouville with p-Laplacian.

The main tool of this work is the use of fixed-point theorem of Leggett-Williams in a cone of the ordered Banach space which is a common thread in proving the existence of multiple positive solutions.

After rewrinting the initial value problem into an equivalent fixed point problem defined by an integral operator, Leggett-Williams used this fixed point theorem to prove the existence of three fixed points of this operator. For this, sufficient conditions on the nonhomogeneous term f and inequalities involving on the kernel of the integral operator are given to show the existence of three positive solutions of this problem. This theory has been widely used in recent years to show the existence of multiple positive solutions.

This paper is organized as follows: In Section 1, we give an overview of the basic results of the classical fixed point theorems. In Section 2, some preliminaries are established and

the main theorems are formulated and proved.

In Section 3, the proof of the existence of positive solutions of the third order Sturm-Liouville boundary value problem with p-Laplacien using the Leggett-Williams fixed point theorem is given.

Finally, in Section 4, we give an example to illustrate our results.

# Chapter 1

# Reminders

## Sommaire

## 1.1 Some Fixed Point Theorems

**Definition 1.1.1** *Let $X$ be a set and let $T : X \longrightarrow X$ be a function that maps $X$ into itself. (Such a function is often called an operator, a transformation, or a transform on $X$, and the notation $Tx$ is often used in place of $T(x)$). A fixed point of $T$ is an element $x \in X$ for which $T(x) = x$.*

**Remark 1.1.1** *Note that the definition of a fixed point requires no structure on either the set $X$ or the function $T$.*

### 1.1.1 Banach Fixed Point Theorem

**Definition 1.1.2** *Let $(X, d)$ be a metric space. A contraction of $X$ (also called a contraction mapping on $X$ ) is a function $f : X \longrightarrow X$ that satisfies*

$$\forall x, x^{'} \in X : d(f(x^{'}), f(x)) \leq \beta d\left(x^{'}, x\right)$$

*for some real number $\beta < 1$. Such a $\beta$ is called a contraction modulus of $f$.*

**Theorem 1.1.1** *Every contraction mapping is continuous.*

**Proof :** Let $T : X \longrightarrow X$ be a contraction on a metric space$(X, d)$, with modulus $\beta$, and let $\overline{x} \in X$. Let $\epsilon > 0$, and let $\delta = \epsilon$. Then $d\left(x, \overline{x}\right) < \delta \Longrightarrow d(Tx; T\bar{x}) \leq \beta\delta < \epsilon$ . Therefore $T$ is continuous at $\bar{x}$. Since $\bar{x}$ was arbitrary, $T$ is continuous on $X$. The above proof actually establishes that a contraction mapping is uniformly continuous.

**Definition 1.1.3** *Let $(X, d_x)$ and $(Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is uniformly continuous if for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\forall x; x^{'} \in X : d_x(x, x^{'}) < \delta \Longrightarrow d_Y(f(x), f(x^{'})) < \epsilon.$$

*Notice how this definition differs from the definition of continuity: uniform continuity requires that, for a given* $\epsilon$ *, a single will work across the entire domain of* $f$*.*

*Continuity allows that the* $\delta$ *may depend upon the point* $x$ *at which continuity of* $f$ *is being evaluated; uniform continuity requires that ( for a given* $\epsilon$ *) being within* $\delta$ *of any* $x \in X$ *guarantees that the image under* $f$ *is within* $\epsilon$ *of* $f(x)$*.*

**Theorem 1.1.2** *Every contraction mapping is uniformly continuous.*

**Theorem 1.1.3** *( Banach Fixed Point Theorem )Every contraction mapping on a complete metric space has a unique fixed point. (This is also called the Contraction Mapping Theorem.)*

***Proof :*** *Let* $T : X \longrightarrow X$ *be a contraction on the complete metric space* $(X, d)$*, and let be a contraction modulus of* $T$*. First we show that* $T$ *can have at most one fixed point.*

*Then we construct a sequence which converges and show that its limit is a fixed point of* $T$*.*

$(a)$ *Suppose* $x$ *and* $x^{'}$ *are fixed points of* $T$*. Then* $d(x, x^{'}) = d(Tx, Tx^{'}) \leq \beta d(x, x^{'})$*; since* $\beta < 1$*, this implies that* $d(x, x^{'}) = 0$*, i. e.,* $x = x^{'}$*.*

$(b)$ *Let* $x_0 \in X$*, and define a sequence* $\{x_n\}$ *as follows:*

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \; ... \; , x_n = Tx_{n-1} = T^nx_0, \; ...$$

*We first show that adjacent terms of* $\{x_n\}$ *grow arbitrarily close to one another | speci cally, that* $d(x_n, x_{n+1}) \leq \beta^n d(x_0, x_1)$ :

$$\begin{aligned} d(x_1, x_2) &\leq \beta d(x_0, x_1) \\ d(x_2, x_3) &\leq \beta d(x_1, x_2) \leq \beta^2 d(x_0, x_1) \\ &... \\ d(x_n, x_{n+1}) &\leq \beta d(x_{n-1}, x_n) \leq \beta^n d(x_0, x_1). \end{aligned}$$

*Next we show that if* $n < m$ *then* $d(x_n, x_m) < \beta^n \frac{1}{1-\beta} d(x_0, x_1)$ :

$$\begin{aligned}
d(x_n, x_{n+1}) &\leqq \beta^n d(x_0, x_1) \\
d(x_n, x_{n+2}) &\leqq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\
&\leqq \beta^n d(x_0, x_1) + \beta^{n+1} d(x_0, x_1) = \left(\beta^n + \beta^{n+1}\right) d(x_0, x_1) \\
&\dots \\
d(x_n, x_m) &\leqq (\beta^n + \beta^{n+1} + \dots + \beta^{m-1}) d(x_0, x_1) \\
&= \beta^n (1 + \beta + \beta^2 + \dots + \beta^{m-1-n}) d(x_0, x_1) \\
&< \beta^n (1 + \beta + \beta^2 + \dots) d(x_0, x_1) \\
&= \beta^n \frac{1}{1-\beta} d(x_0, x_1)
\end{aligned}$$

*Therefore* $\{x_n\}$ *is Cauchy: for* $\epsilon > 0$*, let* $N$ *be large enough that* $\beta^N \frac{1}{1-\beta} d(x_0, x_1) < \epsilon$*, which ensures that*

$$n, m > N \implies d(x_n, x_m) < \epsilon.$$

*Since the metric space* $(X, d)$ *is complete, the Cauchy sequence* $\{x_n\}$ *converges to a point* $x^* \in X$*. We show that* $x^*$ *is a fixed point of* $T$*: since* $x_n \longrightarrow x^*$ *and* $T$ *is continuous, we have* $Tx_n \longrightarrow Tx^*$ *i.e.,* $x_{n+1} \longrightarrow Tx^*$*. Since* $x_{n+1} \longrightarrow x^*$ *and* $x_{n+1} \longrightarrow Tx^*$ *, we have* $Tx^* = x^*$*.*

**Theorem 1.1.4** *If* $T$ *is a contraction on a complete metric space* $(X, d)$ *and* $\beta$ *is a contraction modulus of* $T$*, then for any* $x \in X$*,*

$$\forall n \in \mathbb{N}\colon d\left(T^n x, x^*\right) \leq \beta^n d\left(x, x^*\right),$$

*where* $x^*$ *is the unique fixed point of* $T$*.*

**Proof :**

$$\begin{aligned}
d\left(T^n x, x^*\right) &= d\left(TT^{n-1}x, Tx^*\right), \quad \text{because } x^* \text{ is a fixed point of } T \\
&\leq \beta d\left(T^{n-1}x, x^*\right), \ \text{because } T \ \text{ is a contraction} \\
&\leq \beta^2 d\left(T^{n-2}x, x^*\right) \\
&\leq \quad \dots \leq \beta^n d\left(T^0 x, x^*\right) \\
&= \beta^n d\left(x, x^*\right).
\end{aligned}$$

### 1.1.2 Brouwer Fixed Point Theorem

In the unit-interval example in the preceding section, the Banach Theorem seems somewhat limited. It seems intuitively clear that any continuous function mapping the unit interval into itself will have afixed point, but the Banach Theorem applies only to functions $f$ that satisfy $\left|f^{'}(x)\right| \leq \beta$ for some $\beta < 1$.

An elementary example of this is the function $f(x) = 1 - x$, which has an obvious fixed point at $x = \frac{1}{2}$, but whose derivative satisfies $\left|f^{'}(x)\right| = 1$ everywhere therefore $d(f(x), f(x^{'})) = d(x, x^{'})$ so $f$ is not a contraction and the Banach Fixed Point Theorem doesn't apply to $f$. The fixed point theorem due to Brouwer covers this case as well as a great many others that the Banach Theorem fails to cover because the relevant functions aren't contractions

**Definition 1.1.4** *Let $\mathbb{R}^n$ have its usual inner product $\langle ., . \rangle$ and let $\|.\|$ be the induced norm.*

Let $B^n := \{x \in \mathbb{R}^n : \|x\| < 1\}$ be the open unit ball, $\overline{B}^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ the closed unit ball, and $S^{n-1} := \{x : \|x\| = 1\}$ the unit sphere in $\mathbb{R}^n$.

**Theorem 1.1.5** *(Brouwer Fixed Point Theorem). Every continuous map $f : \overline{B}^n \to \overline{B}^n$ has a fixed point. That is there is an $x \in \overline{B}^n$ such that $f(x) = x$.*

*In* [18] *J. Milnor gave a proof of this result based on elementary multidimensional integral calculus.*

*In* [19] *C. A. Rogers simplied Milnor's proof.*

*Here we give an exposition of the Milnor-Rogers proof.*

**Lemma 1.1.1** *There is no $C^1$ map $f : \overline{B}^n \to S^{n-1}$ such that $f(x) = x$ for all $x \in S^{n-1}$.*

**Proof :** Assume, toward a contradiction, that such an $f : \overline{B}^n \to S^{n-1}$ exists.

For $t \in [0, 1]$ let

$$f_t(x) = (1 - t)x + tf(x) = x + tg(x)$$

where $g(x) = f(x) - x$. Note that for $x \in \overline{B}^n$ that

$$\|f_t(x)\| \le (1-t)\|x\| + t\|f(x)\| \le (1-t) + t = 1$$

and therefore $f_t : \overline{B}^n \to \overline{B}^n$.Also note that for all $x \in S^{n-1}$.

$$f_t(x) = (1-t)x + tf(x) = (1-t)x + tx = x$$

and thus $f_t$ fixes all points on $S^{n-1}$. As $f$ is $C^1$, the same is true for $g$ and therefore there is a constant $C$ such that for all $x_1, x_2 \in \overline{B}^n$

$$\|g(x_1) - g(x_2)\| \le C\|x_2 - x_1\|.$$

Now assume that there are distinct points $x_1$ and $x_2$ in $\overline{B}^n$ with $f_t(x_1) = f_t(x_2)$.

This implies $x_2 - x_1 = t(g(x_1) - g(x_2))$ and therefore

$$\|x_2 - x_1\| = t\|g(x_1) - g(x_2)\| \le Ct\|x_2 - x_1\|.$$

As $x_1 \neq x_2$ this implies $Ct \ge 1$. Thus when $t < 1/C$ the function $f_t : \overline{B}^n \to \overline{B}^n$ is injective. Let $G_t = f_t[B^n]$ be the image of the open unit ball under $f_t$ The derivative of $f_t$, viewed as a linear map $f_t^{'}(x) : \mathbb{R}^n \to \mathbb{R}^n$, is given by

$$f_t^{'}(x) = I + tg^{'}(x)$$

where $I$ is the identity map on $\mathbb{R}^n$. As $g$ is $C^1$ there is a $t_0$ such that $\det f_t^{'}(x) > 0$ for all $t \in [0, t_0]$. Then by the inverse function theorem $G_t$ is an open set for all $t \in [0, t_0]$.

By possibly making $t_0$ smaller we also have that $f_t$ is injective for all $t \in [0, t_0]$.

We claim that $G_t = B^n$ for all $t \in [0, t_0]$. Assume that this is not the case. Then the boundary $\partial G_t$ will intersect the open ball $B^n$ at some point $y_0$. As $y_0 \in \partial G_t$ there is a sequence $x_\ell \in B^n$ such that

$$\lim_{\ell\to\infty} f_t(x_\ell) = y_0.$$

By the compactness of $\overline{B}^n$ we can pass to a subsequence and assume that $\lim\limits_{\ell\to\infty} x_\ell = x_0$ for some $x_0 \in \overline{B}^n$. Then, by the continuity of $f$ , we have

$$f_t(x_0) = y_0.$$

But, as $G_t$ is open and open sets are disjoints from their boundaries, $y_0$ is not in $G_t = f\left[B^n\right]$ thus

$$x_0 \in \overline{B}^n \setminus B^n = S^{n-1}.$$

But for $x_o \in S^{n-1}$ we have that

$$f_t\left(x_0\right) = x_0,$$

which implies that

$$y_0 = f_t\left(x_0\right) = x_0 \in S^{n-1},$$

which contradicts the assumption that $y_0$ is in $B^n$.Therefore for $t \in [0, t_0]$ the map $f_t : \overline{B}^n \to \overline{B}^n$ is a bijection.

Define a function $F : [0, 1] \to \mathbb{R}$ by

$$F\left(t\right) = \int_{\overline{B}^n} det\ f_t'\left(x\right) dx = \int_{\overline{B}^n} det\ \left(I + tg'\left(x\right)\right) dx$$

where $dx$ is the volume measure on $\mathbb{R}^n$. This is clearly a polynomial in $t$. And for $t \in [0, t_0]$ the function $f_t : \overline{B}^n \to \overline{B}^n$ is a bijection and so by the change of variable formula for multiple integrals $F\left(t\right)$ is just the volume of the image $f_t\left[\overline{B}^n\right] = \overline{B}^n$. That is

$$F\left(t\right) = \text{Volume }\left(\overline{B}^n\right) \qquad for\ t \in [0, t_0].$$

But a polynomial that is constant on an interval is constant everywhere.

Therefore $F\left(t\right)$ =Volume $\left(\overline{B}^n\right)$ for all $t \in [0, t_0]$ and in particular $F\left(1\right)$ =Volume $\left(\overline{B}^n\right) > 0$.

But

$$f_1\left(x\right) = f\left(x\right) \in S^{n-1}$$

for all $x$ and therefore

$$\left\langle f_1\left(x\right), f_1\left(x\right)\right\rangle = \left\|f_1\left(x\right)\right\|^2 = 1$$

for all $x$. Thus for any vector $\in \mathbb{R}^n$

$$2\left\langle f_1'\left(x\right)v, f_1\left(x\right)\right\rangle = \frac{d}{dt}\left\langle f_1\left(xt + tv\right), f_1\left(x + tv\right)\right\rangle |_{t=0} = \frac{d}{dt}1 |_{t=0} = 0.$$

This shows that the range of $f_1'(x)$ is contained in $f(x)^{\perp}$, the orthogonal compliment of $f(x)$. But then rank $f_1'(x) < n-1$ for all $x \in \overline{B}^n$ and therefore det $f_1'(x) = 0$ for all $x \in \overline{B}^n$. Whence

$$F(1) = \int_{\overline{B}^n} det\ f_1'(x)\, dx = 0.$$

This contradicts that $F(1) > 0$ and completes the proof.

**Proof :** of the Brouwer Fixed Point Theorem:

Let $f : \overline{B}^n \to \overline{B}^n$ be a continuous map.

Then by the Stone-Weierstrass theorem there is a sequence of $C^1$functions $p_\ell : \overline{B}^n \to R^n$ such the$\|f(x) - p_\ell\| < 1/\ell$ for all $x \in \overline{B}^n$.(In fact we can choose the $p_\ell$'s to be polynomials).

Then

$$\|p_\ell(x)\| \leq \|f(x)\| + \|p_\ell(x) - f(x)\| \leq 1 + 1/\ell.$$

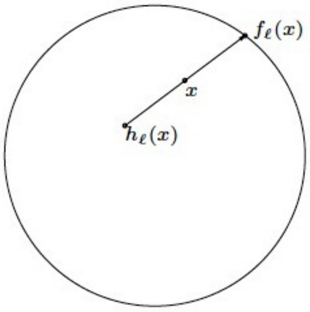
Therefore if

$$h_\ell = (1 + 1/\ell)^{-1} p_\ell$$

we have that $h_\ell : \overline{B}^n \to \overline{B}^n$ and $h_\ell \to f$ uniformly.

We claim that each $h_\ell$ has a fixed point in $\overline{B}^n$. For if not, let $f_\ell : \overline{B}^n \to S^{n-1}$ be map

$$f_\ell(x) = \text{point where the ray from } h_\ell(x) \text{ to } x \text{ meets } S^{n-1}.$$



If $h_\ell$ has no fixed point this map is $C^1$ and has $f_\ell(x) = x$ for all $x \in S^{n-1}$ contradicting Lemma (1.1.1).

Let $x_\ell$ be a fixed point of $h_\ell$, that is $h_\ell(x_\ell) = x_\ell$.

As $\overline{B}^n$ is compact we can pass to a subsquence and assume that $x_\ell \to x_0$ for some $x_0$ in $\overline{B}^n$. As $h_\ell \to f$ uniformly this implies

$$f(x_0) = \lim_{\ell\to\infty} h_\ell(x_\ell) = \lim_{\ell\to\infty} x_\ell = x_0.$$

That is $f$ has $x_0$ as a fixed point.

We get as a corollay, important enough to be called a theorem, a version of Lemma (1.1.1) where $f$ is not required to be $C^1$.

**Theorem 1.1.6** *There is no continuous map $f : \overline{B}^n \to S^{n-1}$with $f(x) = x$ for all $x \in S^{n-1}$.*

**Proof :** Assume that such an $f : \overline{B}^n \to S^{n-1}$existed.

Let $g : \overline{B}^n \to \overline{B}^n$ be given by $g(x) = -f(x)$.

Therefore $g$ also maps $\overline{B}^n$ into $S^{n-1}$.

Therefore if

$$x = g(x) \text{ we have } x \in S^{n-1}.$$

But for

$$x \in S^{n-1}, g(x) = -f(x) = -x \neq x.$$

Thus $g$ has no fixed point, contradicting Theorem (1.1.5).

### 1.1.3 Schauder Fixed Point Theorem

**Lemma 1.1.2** *Every bounded closed subset of a finite dimensional normed space is compact.*

**Theorem 1.1.7** *Let $K$ be a nonempty closed convex subset of a normed space.*

*Let $T$ be a continuous mapping of $K$ into a cumpact subset of $K$.*

*Then $T$ has fixed point in $K$.*

***Proof :*** *Let $\mathcal{E}$ be a Banach space and let $T(K) \subset A$, a compact subset of $K$. $A$ is contained in a closed convex bounded subset of $\mathcal{E}$.*

$$T(B \cap K) \subset T(K) \subset A \subset B$$

*so* $T(B \cap K)$ *is contained in a compact subset of* $B, K$ *and there is no loss of generality in supposing that* $K$ *is bounded.*

*If* $A_0$ *is a countable dense subset of the compact metric space* $A$*, then the set of all rational linear combinations of elements of* $A_0$ *is a countable dense subset of the closed linear subspace* $\mathcal{E}_0$ *spanned by* $A_0$ *and* $A_0 \subset \mathcal{E}_0$ *.*

*Then*

$$T(K \cap \mathcal{E}_0) \subset T(K) \subset A,$$

*a compact subset of* $\mathcal{E}_0$*, and* $K \cap \mathcal{E}_0$ *is closed and convex.*

*Hence without loss of generality we may assume that* $K$ *is a bounded closed convex subset of a separable normed space* $\mathcal{E}$ *with a strictly convex norm (Theorem Clarkson).*

*Given a positive integer* $n$*, there exists a*$\frac{1}{n}$*-net* $Tx_1, ..., Tx_m$ *say in* $TK$*, so that*

$$\min_{1 \leq k \leq n} \|Tx - Tx_k\| < \frac{1}{n}\,(x \in K) \tag{1.1.1}$$

*Let* $\mathcal{E}_n$ *denote the linear hull of*

$$Tx_1, ..., Tx_m . K_n = K \cap \mathcal{E}_n$$

*is a closed bounded subset of* $\mathcal{E}_n$ *and therefore compact (Lemma 1.1.2). Since the norm is strictly convex, the metric projection* $P_n$ *of* $\mathcal{E}$ *onto the convex compact subset* $K_n$ *exists.* $T_n = P_n T$ *is a continuous mapping of the non-empty convex compact subset* $K_n$ *into itself, and therefore by the Brouwer fixed point theorem, it has a fixed point* $u_n \varepsilon K_n$*,*

$$T_n u_n = u_n \tag{1.1.2}$$

*By (1.1.1), since*

$$Tx_k \in K_n (k = 1, 2, ..., m),$$

*we have*

$$\|Tx - T_n x\| < \frac{1}{n} \tag{1.1.3}$$

*The sequence* $\{Tu_n\}$*of* $TK$ *has a subsequence* $Tu_{n_k}$ *converging to a point* $v \in K$*. By (1.1.2) and (1.1.3),*

$$\|unk - v\| = \|T_{n_k} u_{n_k} - v\| \leq \|T_{n_k} u_{n_k} - Tu_{n_k}\| + \|Tu_{n_k} - v\| < \frac{1}{n} + \|Tu_{n_k} - v\| .$$

*Therefore*

$$\lim_{k\to\infty} u_{n_k} = u$$

*and by continuity of* $T$,

$$\lim_{k\to\infty} Tu_{n_k} = Tv \text{ or } Tv = v.$$

### 1.1.4 Krasnoselskii's Cone Fixed Point Theorem

**Definition 1.1.5** *Let* $\mathcal{E}$ *be a Banach space . A nonempty, closed set* $\mathcal{P} \subset \mathcal{E}$ *is a cone provided*

$(a)$ $\alpha u + \beta v \in \mathcal{P}$ *for all* $u, v \in \mathcal{P}$ *and all* $\alpha, \beta \geq 0$ *and*

$(b)$ $u, -u \in \mathcal{P}$ *implies* $u = 0$.

**Definition 1.1.6** *Let* $X$ *be any space and* $f$ *a map of* $X$, *or of a subset of* $X$, *into* $X$.

*A point* $x \in X$ *is called a fixed point for* $f$ *if* $x = f(x)$. *The set of all fixed points of* $f$ *is denoted by* $Fix\ f$.

**Exemple 1.1.1** *if* $\mathcal{E}$ *is a Banach space and* $K_\varrho = \{x \in \mathcal{E} \mid \|x\| \leq \varrho\}$*is the closed ball in* $\mathcal{E}$ *with center* $0$ *and radius* $\varrho$, *then* $r : \mathcal{E} \to K_\varrho$ *given by*

$$r(y) = \begin{cases} y & \text{for } \|y\| \leq \varrho, \\ \\ \varrho y \diagup \|y\| & \text{for } \|y\| > \varrho, \end{cases}$$

$Fix\ r = K_\varrho$

Let $X$ be a retract normed and $U$ open in $X$. Recall that by ,$\mathcal{K}\left(\overline{U}, X\right)$ we denote the set of all compact maps from $\overline{U}$ to $X$, and by $\mathcal{K}_{\partial U}\left(\overline{U}, X\right)$ the set of all maps $f \in \mathcal{K}\left(\overline{U}, X\right)$ that have no fixed points on $\partial U$. Observing that for $f \in \mathcal{K}_{\partial u}\left(\overline{U}, X\right)$ the map $f_{|U}$ is in $\mathcal{F}\left(\overline{U}, X\right)$.

**Definition 1.1.7** *Let* $X$ *be a Banach space,* $U$ *open in* $X$, *and* $f \in \mathcal{K}_{\partial u}\left(\overline{U}, X\right)$. *The fixed point index* $i(f, U)$ *of* $f$ *is given by*

$$i(f, U) = i\left(f_{|U}, U\right).$$

*The properties of the index for compact maps in* $\mathcal{F}\left(\overline{U},X\right)$ *immediately translate to properties of the index for maps in* $\mathcal{K}_{\partial u}\left(\overline{U},X\right)$, *and we obtain the following basic theorem:*

**Theorem 1.1.8** *Let* $X$ *be a Banach space,* $U\subset X$ *an arbitrary open subset, and* $\mathcal{K}_{\partial u}\left(\overline{U},X\right)$ *the set of all compact admissible maps from* $\overline{U}$ *to* $X$. *Then there exists an integer-valued fixed point index function* $f\rightarrow i\left(f\right)$ *for* $f\in\mathcal{K}_{\partial u}\left(\overline{U},X\right)$ *with the following properties:*

$(I)$ *(Normalization) If* $f\in\mathcal{K}_{\partial u}\left(\overline{U},X\right)$ *is a constant map* $u\rightarrow u_0$. *then* $i\left(f,U\right)=1 or\ 0$ *depending on whether or not* $u_o\in U$.

$(II)$ *(Additivity) If* $f\in\mathcal{K}_{\partial u}\left(\overline{U},X\right)$ *and Fix* $(f)\subset U_l\cup U_2\subset U$ *with* $U_l,U_2$ *open and disjoint, then*

$$i\left(f,U\right)=i\left(f,U_1\right)+i\left(f,U_2\right).$$

$(III)$ *(Homotopy) If* $h_t:\overline{U}\rightarrow X$ *is an admissible compact homotopy in* $\mathcal{K}_{\partial u}\left(\overline{U},X\right)$, *then*

$$i(h_0,U)=i(h_1,U).$$

$(IV)$ *(Existence) If* $ii\left(f,U\right)\neq 0$, *then*

$$Fix(f)\neq\emptyset.$$

$(V)$ *(Excision) If* $V$ *is an open subset of* $U$ *and if* $f\in\mathcal{K}_{\partial u}\left(\overline{U},X\right)$ *has no fixed points in* $U-V$, *then*

$$i\left(f,U\right)=i\left(f,V\right)$$

$(VI)$ *(Multiplicativity) If* $f_1\in\mathcal{K}_{\partial u}\left(\overline{U_1},X_1\right)$ *and* $f_2\in\mathcal{K}_{\partial u_2}\left(\overline{U_2},X_2\right)$, *then* $f_1\times f_2\in\mathcal{K}_{\partial u_1}\left(\overline{U_1}\times\overline{U_2},X_1\times X_2\right)$ *and*

$$i\left(f_1\times f_2,U_1\times U_2\right)=i\left(f_1,U_1\right)\cdot i\left(f_2,U_2\right)$$

$(VII)$ *(Commutativity) Let* $X,X^{'}$*be a Banach space, let* $U\subset X$, $U^{'}\subset X^{'}$ *be open, and let* $f:\overline{U}\rightarrow X^{'}, g:\overline{U}^{'}\rightarrow X$ *be continuous maps, at least one of them being compact. Define* $V=U\cap f^{-1}\left(U^{'}\right)$ *and* $V^{'}=U^{'}\cap g^{-1}\left(U\right)$. *then:*

$(i)$ *the maps*

$$gf:\overline{V}\rightarrow X,\qquad fg:\overline{V}^{'}\rightarrow X^{'}$$

*are compact,*

*(ii) if* $Fix\,(gf) \subset V$ *and* $Fix\,(fg) \subset V'$, *then*

$$i(gf, V) = i(fg, V').$$

*We list further frequently used properties of the index:*

$(VIII)$ *(Contraction) Let* $(X, A)$ *be a pair of Banach space with* $A$ *closed in* $X, U \subset X$ *open, and* $f \in \mathcal{K}_{\partial u}\left(\overline{U}, X\right)$ *with* $f\left(\overline{U}\right) \subset A$.

*Let* $\hat{f} = f_{\overline{u \cap A}} : \overline{U \cap A} \to A$ *be the contraction of* $f$. *Then*

$$\hat{f} \in \mathcal{K}_{\partial(u \cap A)}(\overline{U \cap A}, A) \text{ and } i(f, U) = i(\hat{f}, U \cap A).$$

$(IX)$ *(Localization) Let* $f \in \mathcal{K}_{\partial u}\left(\overline{U}, X\right)$ *and* $Fix\,(f) \subset U_1 \cup U_2 \subset U$ *with* $U_1, U_2$ *open and disjoint.*

*Suppose* $i\,(f, U) \neq 0$ *and* $i\,(f, U_1) = 0$. *Then*

$$Fix\,\left(f_{|U_2}\right) \neq \emptyset.$$

$(X)$*(Multiplicity) Let* $f \in \mathcal{K}_{\partial u}\left(\overline{U}, X\right)$ *and* $Fix\,(f) \subset U_1 \cup U_2 \subset U$ *with* $U_1, U_2$ *open and disjoint.*

*Suppose* $i\,(f, U) = 0$ *and* $i\,(f, U_1) \neq 0$. *Then*

$$Fix\,\left(f_{|U_1}\right) \neq \emptyset \text{ and } Fix\,\left(f_{|U_2}\right) \neq \emptyset.$$

( the proof of theorem can be found in [10] ).

**Definition 1.1.8** *Let* $\mathcal{E}$ *be a Banach space.* $C$ *a convex (not necessarily closed) subset of* $\mathcal{E}$, *and* $U \subset C$ *open with* $0 \in U$. *As before, we denote by* $\mathcal{K}_{\partial u}\left(\overline{U}, C\right)$ *the set of compact maps* $f : U \to C$ *that are fixed point free on* $\partial U$.

**Definition 1.1.9** *If* $\mathcal{P} \subset \mathcal{E}$ *is a cone and* $\varrho > 0$, *we let*

$$B_\varrho = \{x \in \mathcal{P} / \|x\| < \varrho\}, \quad S_\varrho = \{x \in \mathcal{P} / \|x\| = \varrho\}, \quad K_\varrho = S_\varrho \cup B_\varrho.$$

**Lemma 1.1.3** *Let* $\mathcal{E}$ *be a Banach space and* $|\ \ |$ *be any norm in* $\mathcal{E}$. *let* $f \in \mathcal{K}_{\partial u}\left(\overline{U}, C\right)$. *and assurne that one of the following conditions holds for all* $x \in \partial U$:

*(i)* $\mid f\,(x) \mid \leq \mid x \mid$,

*(ii)* $| f(x) | \leq | x - f(x) |$,

*(iii)* $| f(x) |^2 \leq | x |^2 + | x - f(x) |^2$,

*(iv)* $\langle x, f(x) \rangle \leq (x, x)$. *where* $\langle \cdot \rangle$ *is a scalar product in* $E$.

*Then*

$$i(f, U) = 1$$

( the proof of lemma can be found in [10] ).

**Lemma 1.1.4** *Let* $f \in \mathcal{K}_{S_\varrho}(K_\varrho, P)$ *satisfy* $\|f(x)\| \geq \|x\|$*for all* $x \in S_\varrho$. *Then:*

$(a)$ $i(f, B_\varrho) = 0$,

$(b)$ $f$ *has an eigenvector on* $S_\varrho$ *with characteristic value* $\mu \in (0, 1)$.

( the proof of lemma can be found in [10] ).

**Theorem 1.1.9** *(Krasnosel'skii). Let* $\mathcal{E}$ *be a Banach space,* $\mathcal{P} \subset \mathcal{E}$ *a proper cone, and assume that* $f : \mathcal{P} \to \mathcal{P}$ *is a completely continuous map such that for some numbers rand* $R$ *with* $0 < r < R$*, one of the following conditions is satisfied:*

$(a)$ $\|f(x)\| \leq \|x\|$ *for* $x \in S_r$*, and* $\|f(x)\| \geq \|x\|$ *for* $x \in S_R$.

$(b)$ $\|f(x)\| \geq \|x\|$ *for* $x \in S_r$*, and* $\|f(x)\| \leq \|x\|$ *for* $x \in S_R$.

*Then* $f$ *has a fixed point* $x$ *with* $r \leq \|x\| \leq R$.

**Proof :** We may assume that $f$ has no fixed points on $S_r$ and $S_R$. We make the following observations:

$(i)$ if $(a)$ holds, then $i(f, B_R) = 0$ by (Lemma 1.1.4) and $i(f, B_r) = 1$ by (Lemma 1.1.3),

$(ii)$ if $(b)$ holds, then $i(f, B_R) = 1$ by (Lemma 1.1.3) and $i(f, B_r) = 0$ by (Lemma 1.1.4).

## 1.2 Lebesgue and Arzelà–Ascoli Theorems

### 1.2.1 The Lebesgue Dominated Convergence Theorem

**Definition 1.2.1** *Let*

$$F = \{f_i : X \to Y, i \in I\}$$

*be a family of functions indexed by $I$, where $X$ is an arbitrary set and $Y$ is the metric space with metric $d$.*

*We call $F$ uniformly bounded if there exists an element $a$ from $Y$ and a real number $M$ such that*

$$d\left(f_i\left(x\right), a\right) \leq M \quad \forall i \in I,\ \forall x \in X.$$

**Theorem 1.2.1** *If $\{f_n\}_{n\geq 0}$ is a sequence of measurable functions and $f_n \to f$ a.e, on a set $S$ and there is an integrable*

*function $g$ on $S$ such that $|f_n| < g$ for all $n$, then $f$ is integrable and $\int\limits_S f_n dx \to \int\limits_S f dx$*

**Proof :** Notice that all fn are integrable since $|f_n| < g$ and g is integrable.

Since $f_n \to f$, $f$ is measurable, and f is integrable since $|f| < g$.

Let $\epsilon > 0$ and let $T$ be a finite measure subset of $S$ such that $g$ is bounded on $T$ and $\int\limits_{S-T} g dx < \epsilon$.

Clearly $\{f_n\}$ is uniformly bounded (by a bound for g) on $T$,

so

$$\int\limits_T f_n dx \to \int\limits_T f dx$$

by the bounded convergence theorem. Since $|f_n| < g$ and $|f| < g$,

$$\left|\int\limits_{S-T} f_n dx\right| < \epsilon\ , \quad \left|\int\limits_{S-T} f dx\right| < \epsilon$$

Therefore, if we choose $N$ so that $\left|\int\limits_T f_n dx - \int\limits_T f dx\right| < \epsilon$ for $n > N$,

then for $n > N$

$$\left|\int\limits_S f_n dx - \int\limits_S f dx\right| \leq \left|\int\limits_T f_n dx - \int\limits_T f dx\right| + \left|\int\limits_{S-T} f_n dx - \int\limits_{S-T} f dx\right| < 3\epsilon$$

The dominated convergence theorem says in essence that pointwise convergence of $\{f_n\}$ implies convergence of the integrals, provided only that the graphs of the $f_n$ all lie in some fixed finite area.

If the $f_n$ are positive and their graphs are not all in some fixed finite area, then the integrals $\int f_n dx$ may get too big, but they are always at

least as big as $\int f dx$ in the limit.

### 1.2.2 Arzelà–Ascoli Theorem

**Definition 1.2.2** *Consider a subset* $A \subset C\left[0,1\right]$. *We call* $A$ *equicontinuous if,*

$$\forall \epsilon > 0, \quad \exists \, \delta > 0$$

*such that,*

$$\forall f \in A, \forall x, y \in \left[0,1\right], \left|x-y\right| < \delta \implies \left|f(x) - f(y)\right| < \epsilon.$$

*This is very similar to the definition for continuity of a specific function* $f$ *in* $A$.

*The only difference is that, for a given* $\epsilon$, *we must choose* $\delta > 0$ *which works for all possible functions* $f$ *in* $A$. *That is, we must choose the before we are allowed to look at the function, making this a property of a set rather than a function, whereas in proving continuity, we choose our* $\delta$ *for a specific function.*

**Remark 1.2.1** *Since we are trying to show that equi continuity is the additional property which compact sets in* $C\left[0,1\right]$ *contain, our example of a closed, bounded but not compact subset of* $C\left[0,1\right]$, *should fail to be equi continuous.*

*We will prove that the unit ball in* $C\left[0,1\right]$ *is not equi continous.*

*Recall that the unit ball contained the sequence* $\left\{x^n\right\}_{n\geq 1}$.

**Theorem 1.2.2** *For* $A \subset C\left[0,1\right]$, $A$ *is compact if and only if* $A$ *is closed, bounded, and equicontinuous.*

**Proof :** $A$ compact $\implies$ $A$ closed, bounded, and equicontinuous.

We have already shown, that $A$ must be closed and bounded. It remains to show that $A$ is equicontinuous.

To do so, we assume first that $A$ is compact but not equicontinuous.

Since it is not equicontinuous, we note that $\exists \epsilon > 0$ such that $\forall \delta \exists x, y \in \left[0,1\right]$ and $f \in A$ such that

$$\left|x-y\right| < \delta,$$

but

$$\left|f(x) - f(y)\right| > \epsilon.$$

In particular, we can create a chain of $\delta$'s, which we label by $\delta_n = \frac{1}{n}$, such that $\exists x_n, y_{\ n} \in [0,1]$ and $f_n \in A$

such that

$$|x_n, y_n| < \delta_n = \frac{1}{n},$$

but

$$|f_n\left(x_n\right), f_n\left(y_n\right)| > \epsilon.$$

This of course defines at least one sequence of functions in $A$.

We choose one sequence of functions with this property, and we note that by the above this sequence cannot possibly be equicontinuous.

Also, all subsequences$f_{n(k)}$ of this sequence would clearly also have the property that for the $n\left(k\right)$'th function in the sequence, $\exists x_{n(k)}, y_{\ n(k)} \in [0,1]$ such that

$$\left|x_{n(k)}, y_{n(k)}\right| < \delta$$

but

$$\left|f_{n(k)}\left(x_{n(k)}\right), f_{n(k)}\left(y_{n(k)}\right)\right| > \epsilon,$$

by the above. And so,it is also true that no subsequence can be equicontinuous.

However, we have shown already that all convergent sequences must in fact be equicontinuous.

And so, under the assumption that $A$ is not equicontinuous, we have demonstrated the existence of a sequence in $A$ with no subsequence that converges.

This is a contradiction with the assumption that $A$ is compact, and so we conclude that $A$ must be equicontinuous.

$A$ closed, bounded, and equicontinuous $\implies$ $A$ compact.

We begin by considering an arbitrary sequence $\{f_n\}_{n\geq 1}$ in A.

We must show that it contains a convergent subsequence. Unfortunately, it is not very clear how to do this.

Intuitively, we may look at the interval $[0,1]$, and find a subsequence which converges pointwise at one point, $x_0$. We could then find a subsequence of that subsequence which converged at a second point, $x_1$, and so on.

This would work if $[0,1]$ had only finitely many points. Unfortunately, the interval has uncountably many points, and so this strategy must be modified.

The first modification is to use a diagonal argument, familiar from previous arguments in analysis, to extend convergence of a subsequence from a finite number of points to a countable set of points. We will then use the equicontinuity property to extend convergence at a well-chosen countable set of points to uniform convergence over the entire interval $[0,1]$. For now, we continue the proof.

We let $x_1, x_2, .., x_n, ..$ be an enumeration of the rational points of $[0,1]$.

This is possible since, as we have shown, the rationals are a countable set.

We note that $\{f_n\}_{n\geq 1}$, evaluated at x1, forms an infinite sequence of real numbers.

Since $A$ is closed and bounded, each $\{f_n\}$ must also be bounded, and so our sequence of real numbers, $\{f_n(x_1)\}$, is also bounded.

By the Bolzano-Weierstrass Theorem, then, there exists a subsequence of our sequence of real numbers which converges. This is equivalent to stating that there is a subsequence of $\{f_n\}_{n\geq 1}$ which converges pointwise at $x_1$.

For notational convenience, we label this sequence $f_{n_1(k)}$, where $n_1(k)$ is a strictly increasing function from the positive integers to the positive integers.

With exactly the same argument, we can create a subsequence of that subsequence which converges at $x_2$, which we label $f_{n_2(k)}$. Since $f_{n_1(k)}$ converges at $x_1$ and $f_{n_2(k)}$ is a subsequence of $f_{n_1(k)}$fn1(k), $f_{n_2(k)}$ must also converge at $x_1$.

We can continue this chain of subsequences, and so obtain a sequence, for each positive integer m, a subsequence $f_{n_m(k)}$ which converges at the rational points $x_1, x_2, .., x_m$, created in such a way that $f_{n_m(k)}$ is a subsequence of $f_{n_{(m-1)}(k)}$.

Thus, for any particular finite number of rational points in $[0,1]$, we can find a subsequence which converges at those rational points.

As pointed out earlier, this will not be enough to find a subsequence which converges on the entire interval. However, we have not yet used the hypothesis of equicontinuity.

Before doing that, we define a sequence $\{g_n\}_{n\geq 1}$, by making the $n$'th function in the sequence equal to the $n$'th function in the sequence $f_{n_m(k)}$.

That is, the $n$'th function in $g$ is equal to the $n$'th function of the $n$'th subsequence of the $f_n$'s.

We note that, for any given rational point $x_i$, $g_n$ is a subsequence of $f_{n_i(k)}$ for all $n > i$, and so $g_n$ converges at $x_i$. Thus, this sequence in fact converges at every single rational point on $[0, 1]$. Since the elements of $\{g_n\}$ are all taken from subsequences of $\{f_n\}$, we note that it is also a subsequence of $\{f_n\}$. At this point, it remains to show that $g_n$ converges everywhere on $[0, 1]$, and also that it converges uniformly.

First, we will show that it is a Cauchy sequence. We consider an arbitrary $x \in [0, 1]$. We note immediately that, by the triangle inequality,:

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x) - g_m(x_i)|$$

for any point $x_i$ in $[0, 1]$.

Here, for the first time, we use equicontinuity.

We can choose a such that $|x - x_i| < \delta$ implies both that

$$|g_n(x) - g_n(x_i)| < \frac{\epsilon}{3}$$

and that

$$|g_m(x) - g_m(x_i)| < \frac{\epsilon}{3}.$$

This $\delta$, we recall, is completely independent of $m$ and $n$, and it is also completely independent of $x, x_i$.

We note that the rational points are dense in the reals, and so we choose now $x_i$ to be a rational point satisfying $|x - x_i| < \delta$.

As for the middle term, $g_n$ converges at $x_i$, and so $g_n$ evaluated at $x_i$ forms a Cauchy sequence, after we have already chosen $x_i$.

Thus, $\exists N > 0$ such that $m, n > N$ forces the middle term to be less than $\frac{\epsilon}{3}$ . And so, we have shown that $g_n(x)$ is itself a Cauchy sequence, that is, it converges pointwise everywhere on $[0, 1]$.

We need to show now that this convergence is uniform.

That is, the convergence is essentially independent of $x$.

The proof above of pointwise convergence depended on $x$.

Fortunately, it can actually be modified to prove not only pointwise convergence, but uniform convergence. Once again to start, we also $\epsilon > 0$ to be given. Since $A$ is equicontinuous, we can choose a $\delta$ independent of $n$ and $x$ such that

$$|x - x_i| < \delta$$

implies that

$$|g_n(x) - g_n(x_i)| < \frac{\epsilon}{3} \text{ for all } n.$$

Now, we partition the interval into intervals of length $\frac{\delta}{2}$ . We can now choose exactly one rational point in each such interval. We are now looking at a finite number of rational points, only. Since g converges at each rational point, for each rational point $x_j$ which we are looking at,

there exists an $N_j$ such that $m, n > N_j$ implies that

$$|g_m(x_j) - g_n(x_j)| < \frac{\epsilon}{3}.$$

Now, we define $N$ to be the maximum over all the $N_j$ 's, which exists, since we are taking the maximum over a finite set.

Having done this preparatory work, we will now show that in fact the convergence is uniform.

For $\epsilon, \delta, N$ and the set of rational points $x_j$ with their associated partition as above, we continue.

Note that for any $x \in [0, 1]$, we can choose one of our special rational points, $x_j$ , that is within $\delta$ of $x$. Choosing this rational point, and forcing $m, n$ to be strictly greater than $N$, we obtain:

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x) - g_m(x_i)|$$

but

$$|g_n(x) - g_n(x_j)| < \frac{\epsilon}{3}, |g_m(x) - g_m(x_j)| < \frac{\epsilon}{3}$$

since

$$|x - xj| < \delta.$$

We also see that

$$\forall x_j \in [0,1]\,, |g_m(x_j) - g_m(x_j)| < \frac{\epsilon}{3},$$

since we have already imposed the restriction $m, n > N > N_j$ .

And so

$$|g_n(x) - g_m(x)| < \epsilon,$$

as we wished to show. Since $A$ is closed and this sequence converges, it must of course converge to a function in $A$.

Thus, from the assumptions that $A$ is closed, bounded and equicontinuous, we have demonstrated for a general sequence the existence of a convergent subsequence with limit point in $A$.

# Chapter 2

# Leggett-Williams Fixed Point Theorem

**Sommaire**

## 2.1 Preliminaries

**Definition 2.1.1** *Let $\mathcal{E}$ be a real Banach space and $P$ be a nonempty, convex closed set in $\mathcal{E}$. We say that $P$ is a cone if it satisfies the following properties:*

*(i) $\lambda u \in P$ for $u \in P$, $\lambda \geq 0$,*

*(ii) $u, -u \in P$ implies $u = \theta$ ($\theta$ denotes the null element of $\mathcal{E}$).*

*If $P \subset \mathcal{E}$ is a cone, we denote the order induced by $P$ on $\mathcal{E}$ by $\leq$ . For $u, v \in P$,we write $u \leq v$ if $v - u \in P$.*

**Definition 2.1.2** *The map $\varphi$ is said to be a nonnegative continuous concave functional on $P$ provided that $\varphi : P \longrightarrow [0, \infty)$ is continuous and*

$$\varphi(tx + (1-t)y) \geq t\varphi(x) + (1-t)\varphi(y)$$

*for all $x, y \in P$ and $0 \leq t \leq 1$.*

**Definition 2.1.3** *Let $0 < a < b$ be given and let $\varphi$ be a nonnegative continuous concave functional on the cone P. Define the convex sets $P_r$ , $\bar{P}_r$ and $P(\varphi, a, b)$ by*

$$P_r = \{y \in P : \|y\| < r\},$$

$$\bar{P}_r = \{y \in P : \|y\| \leq r\}, \ 0 < r < \infty$$

$$P(\varphi, a, b) = \{y \in P : a \leq \varphi(y), \|y\| \leq b\},$$

*and*

$$P_\infty = P,$$

## 2.2 Leggett-Williams Theorem

Let the maps $A : P_c \to P$ satisfying the following property:

($*$) A has a continuous extension $A_1 : P \to P$ such that: *range* $A_1$ = *range* $A$, and $A_1$ has no fixed points in $P \setminus P_c$ .

**Definition 2.2.1** *Let* $\mathcal{E}$ *be a real Banach space. A subset* $K$ *of* $\mathcal{E}$ *is said to be a retract of* $\mathcal{E}$ *if there exists a continuous map* $P : \mathcal{E} \to K$ *such that* $P_x = x$ *for all* $x \in K$.

**Remark 2.2.1** *Any closed convex subset* $C$ *of* $E$ *is a retract of* $E$. *That is, there is a continuous mapping* $R : E \to C$ *such that* $Rx = x$ *for each* $x \in C$.

**Lemma 2.2.1** *Let* $Q$ *be a retract of a Banach space* $\mathcal{E}$. *For every open subset* $U$ *of* $Q$ *and every completely continuous map* $A \colon \bar{U} \longrightarrow Q$ *which has no fixed points on* $\partial U$ *boundary of* $U$, *there exists an integer* $i\left(A, U, Q\right)$ *satisfying*:

$(i)$ *if* $A \colon \bar{U} \longrightarrow U$ *is a constant map, then*

$$i\left(A, U, Q\right) = 1;$$

$(ii)$ *if* $U_1$ *and* $U_2$ *are disjoint open subsets of* $U$ *such that* $A$ *has no fixed points on* $\bar{U} \setminus \left(U_1 \cup U_2\right)$ *then*

$$i\left(A, U, Q\right) = i\left(A, U_1, Q\right) + i\left(A, U_2, Q\right),$$

*where*

$$i\left(A, U_k, Q\right) = i\left(A\backslash\bar{U}_k, U_k, Q\right), k = 1, 2;$$

$(iii)$ *if* $I$ *is a compact interval in* $\mathbb{R}$ *and* $h \colon I \times \bar{U} \longrightarrow Q$ *is a continuous map with relatively compact range such that* $h\left(\lambda, x\right) \neq x$ *for* $\left(\lambda, x\right) \in I \times \partial U$,*then* $i\left(h\left(\lambda, .\right), U, Q\right)$ *is well -defined and independent of* $\lambda$,

$(iv)$ *if* $i\left(A, U, Q\right) \neq 0$, *then* $A$ *has at least one fixed point in* $U$,

$(v)$ *if* $Q_1$ *is a retract of* $Q$ *and* $A(\bar{U}) \subset Q_1$, *then*

$$i\left(A, U, Q\right) = i\left(A, U \cap Q_1, Q_1\right);$$

*where*

$$i\left(A, U \cap Q_1, Q_1\right) = i\left(A\backslash\overline{U \cap Q_1}, U \cap Q_1, Q_1\right),$$

$(vi)$ *if* $V$ *is open in* $U$ *and* $A$ *has no fixed points in* $\bar{U} \setminus V$, *then*

$$i\left(A, U, Q\right) = i\left(A, V, Q\right).$$

*Our first result gives sufficient conditions for an operator* $A \colon P_c \longrightarrow P$ *to have at least one nonzero fixed point.* (*The proof can be found in* [17])

**Theorem 2.2.1** *suppose* $A\colon P_c \longrightarrow P$ *is completely continuous ( i.e.,* $T$ *is continuous and compact ) and suppose there exist a concave positive functional* $\alpha$ *with* $\alpha(x) \leq \|x\| \ \ (x \in P)$ *and numbers* $b > a > 0$ $(b \leq c)$ *satisfying the following conditions :*

(1) $\{x \in S(\alpha, a, b) : \alpha(x) > a\} \neq \phi,$ *and* $\alpha(Ax) > a$ *if* $x \in S(\alpha, a, b);$

(2) $Ax \in P_c$ *if* $x \in S(\alpha, a, c);$

(3) $\alpha(Ax) > a$ *for all* $x \in S(\alpha, a, c)$ *with* $\|Ax\| > b.$

*Then* $A$ *has a fixed point* $x$ *in* $S(\alpha, a, c).$

***Proof :*** *Set*

$$U = \{x \in S(\alpha, a, c) : \alpha(x) > a\}.$$

*Then* $U$ *is the interior of* $S(\alpha, a, c)$ *in* $P_c$ *. Suppose that* $x \in \partial U$ *is a fixed point of* $A$*.*

*Then* $\alpha(x) = a$ *and either* $x \in S(\alpha, a, b)$ *or* $\|x\| > b$*; but if* $x \in S(\alpha, a, b)$*, then*

$$\alpha(x) = \alpha(Ax) > a,$$

*and if* $\|x\| > b$*, then*

$$\|Ax\| > b \text{ and } \alpha(x) = \alpha(Ax) > a.$$

*Hence* $A$ *has no fixed points in* $\partial U$*,and there exists an integer* $i(A, U, P_c)$ *satisfying properties* $(i) - (vi)$ *of lemma* (2.2.1). *Choose* $x_0 \in S(\alpha, a, b)$ *such that* $\alpha(x_0) > a$*, and define the map* $h\colon [0, 1] \times \bar{U} \longrightarrow P_c$ *by*

$$h(t, x) = (1 - t) Ax + tx_0.$$

*Clearly* $h$ *is continuous and* $h\left([0, 1] \times \bar{U}\right)$ *is relatively compact. Suppose there exists* $(t, x) \in [0, 1] \times \partial U$ *such that* $h(t, x) = x$*. Then* $\alpha(x) = a$*. if* $\|Ax\| > b$ *, then by condition* (3) $\alpha(Ax) > a$*, so that*

$$\alpha(x) = \alpha(h(t, x)) = \alpha((1 - t) Ax + tx_0) \geq (1 - t) \alpha(Ax) + t\alpha(x_0) > a,$$

*a contradiction. On the other hand, if* $\|Ax\| \leq b$*, then*

$$\|x\| = \|(1 - t) Ax + tx_0\| \leq (1 - t) \|Ax\| + t \|x_0\| \leq b,$$

*so that* $x \in S(\alpha, a, b)$. *Hence, by condition* (1), $\alpha(Ax) > a$ *and again we arrive at the contradiction*

$$\alpha(x) = \alpha(1-t)Ax + tx_0 > a.$$

*It follows that for each* $(t, x) \in [0, 1] \times \partial U$, $h(t, x) \neq x$. *Therefore by* $(i)$ *and* $(iii)$ *of Lemma* (2.2.1),

$$i(A, U, P_c) = i(x_0, U, P_c) = 1.$$

*Hence by* $(iv)$ *of Lemma* (2.2.1), $A$ *has a fixed point in* $U$.

**Remark 2.2.2** *The condition:*

$\alpha(Ax) > a$ *for all* $x \in S(\alpha, a, b)$ *with* $\|Ax\| > b$ *will be satisfied if either of the following conditions holds:*

$$(i)\ \ \alpha(Ax) \geq \frac{a}{b}\|Ax\|, \qquad\qquad x \in S(\alpha, a, b);$$

$$(ii)\ \ \|Ax\| - \alpha(Ax) \leq b - a, \qquad\qquad x \in S(\alpha, a, b).$$

**Theorem 2.2.2** *Let* $T : \overline{P}_c \longrightarrow P$ *be a completely continuous operator and let* $\varphi$ *be a nonnegative continuous concave functional on* $P$ *such that* $\varphi(y) \leq \|y\|$ *for all* $y \in \overline{P}_c$. *Suppose that there exist* $0 < a < b < c$ *such that*

(*a"*) $\{y \in P(\varphi, b, c) : \varphi(y) > b\} \neq \emptyset$, *and* $\varphi(Ty) > b$ *for* $y \in P(\varphi, b, c)$;

(*b"*) $\|Ty\| < a$ *for* $\|y\| \leq a$;

(*c"*) $\varphi(Ty) > \dfrac{b}{c}\|Ty\|$ *for* $y \in \overline{P}_c$*with* $\|Ty\| > c$.

*Then* $T$ *has at least two fixed points* $y_1, y_2$ *in* $\overline{P}_c$ *satisfying*

$$\|y_1\| < a, \|y_2\| > a \text{ and } \varphi(y_2) < b.$$

**Proof :** Let $T_1$ be the extension of $T$ described in property $(*)$, and choose $r \geq c$ such that

$$T_1(P_r) \subset P_r.$$

Note that conditions (1) and (2) of Theorem (2.2.1) hold for $T_1$.

If $y \in P(\varphi, b, r)$ and $\|T_1 y\| > b$, then

$$T_1 y = Tx \text{ for some } x \in \bar{P}_c \text{ and } \varphi(T_1 y) = \varphi(T_1 x) > b \, (\text{since } \|Tx\| > c)\,.$$

Hence condition (3) of Theorem (2.2.1) is satisfied for $P_r$ and $T_1$ , and $T_1$ has at least three fixed points in $P_r$ .

Since $T_1$ has no fixed points in $P \setminus \bar{P}_c$ , these fixed points lie in $\bar{P}_c$ and therefore are fixed points of $T$ .

It is possible to obtain two fixed points of $T$ even if $T$ dose not satisfy property $(*)$.

In this case condition (3) must be replaced by stronger conditions of the type in Remark (2.2.1).

**Theorem 2.2.3** ***(Leggett-Williams).*** *Let* $T : \bar{P}_c \longrightarrow \bar{P}_c$ *be a completely continuous operator and let* $\varphi$ *be a nonnegative continuous concave functional on* $P$ *such that*

$\varphi(y) \le \|y\|$ for all $y \in \bar{P}_c$. Suppose that there exist $0 < a < b < d \le c$ such that

$$(a')\{y \in P(\varphi, b, d) : \varphi(y) > b\} \neq \emptyset \text{ and } \varphi(Ty) > b \text{ for } y \in P(\varphi, b, d);$$

$$(b')\ \|Ty\| < a \text{ for } \|y\| \le a;$$

$$(c')\varphi(Ty) > b \text{ for } y \in P(\varphi, b, c) \text{ with } \|Ty\| > d.$$

Then $T$ has at least three fixed points $y_1, y_2, y_3$ in $\bar{P}_c$ satisfying

$$\|y_1\| < a,\ \varphi(y_2) > b,\ \|y_3\| > a \text{ and } \varphi(y_3) < b.$$

**Proof :** Let $U_1 = \left\{ y \in \bar{P}_c \colon \|y\| < a \right\}$ and $U_2 = \{y \in P(\varphi, b, c) : \varphi(y) > b\}$. Then $U_1$ and $U_2$ are convex open sets in $\bar{P}_c$ and $T$ has no fixed points on $\partial U_1 \cup \partial U_2 = \partial(U_1 \cup U_2)$.

By $(ii)$ of lemma (2.2.1),

$$i\left(T, \bar{P}_c, \bar{P}_c\right) = i\left(T, U_1 \cup U_2, \bar{P}_c\right) + i\left(T, \bar{P}_c \setminus \left(\overline{U_1 \cup U_2}\right), \bar{P}_c\right)$$

and

$$i\left(T, U_1 \cup U_2, \overset{-}{P}_c\right) = i\left(T, U_1, \overset{-}{P}_c\right) + i\left(T, U_{2,}\overset{-}{P}_c\right),$$

so that

$$i\left(T, \overset{-}{P}_c \setminus \left(\overline{U_1 \cup U_2}\right), \overset{-}{P}_c\right) = i\left(T, \overset{-}{P}_c, \overset{-}{P}_c\right) - \sum_{j=1}^{2} i\left(T, U_j, \overset{-}{P}_c\right).$$

suppose $V$ is a convex open subset of $\overset{-}{P}_c$ such that $T\colon V \longrightarrow V$ and $T$ has no fixed points on $\partial V$.

By property $(v)$ of lemma (2.2.1) ,

$$i\left(T, V, \overset{-}{P}_c\right) = i\left(T, V, \overset{-}{V}\right),$$

since $\overset{-}{V}$ is a retract of $\overset{-}{P}_c$. Fixe $y_0$ in $V$ and define $h\colon\ [0,1] \times \overset{-}{V} \longrightarrow \overset{-}{V}$ by

$$h(t, y) = (1-t)\,Ty + ty_0.$$

Now if $h(t, y) = y$ for some $y \in \partial V$, then $t = 0$, since otherwise $h(t, y) \in V$. But then $Ty = y$ for some $y \in \partial V$, which is assumed not to be the case. Then by $(i)$ and $(iii)$ of lemma (2.2.1),

$$i\left(T, V, \overset{-}{V}\right) = i\left(y_0, V, \overset{-}{V}\right) = 1.$$

Now $T(U_1) \subset U_1$ and $T\left(\overset{-}{P}_c\right) \subset \overset{-}{P}_c$ , so that

$$i\left(T, U_1, \overset{-}{P}_c\right) = 1 = i\left(T, \overset{-}{P}_c, \overset{-}{P}_c\right).$$

(note that $T$ has no fixed points on the boundary of $U_1$ in $\overset{-}{P}_c$, and that the boundary of $\overset{-}{P}_c$ in $\overset{-}{P}_c$ is empty.) Also, it follows from the proof of Theorem (2.2.1) that $i(T, U_2, P_c) = 1$.

Therefore

$$i\left(T, \overline{P}_c \setminus \left(\overline{U_1 \cup U_2}\right), \overline{P}_c\right) = 1 - 2 = -1.$$

By property $(iv)$ of Lemma (2.2.1) , $T$ has a fixed point in $\overset{-}{P}_c \setminus \left(\overline{U_1 \cup U_2}\right)$ .By Schauder's theorem, $T$ has a fixed point in $U_1$, and by Theorem (2.2.1) , $T$ has a fixed point in $U_2$. Therefore $T$ has at least three fixed points in $\overset{-}{P}_c$.

# Chapter 3

# STURM-LIOUVILLE BOUNDARY-VALUE PROBLEM

## Sommaire

## 3.1 Sturm-Liouville

In this section ,we would like to proof that under suitable conditions, the third order Sturm-Liouville boundary value problem, with $p-Laplacian$ , has three positive solutions.

We consider the third-order Sturm-Liouville boundary value problem,with $p-Laplacian$

$$\left(\phi_p\left(u''(t)\right)\right)'+f\left(t,u(t)\right)=0, \qquad t\in(0,1) \tag{3.1.1}$$

$$\alpha u(0)-\beta u'(0)=0, \qquad \alpha,\beta\geq 0 \tag{3.1.2}$$

$$\gamma u(1)+\delta u'(1)=0, \qquad \gamma,\delta\geq 0 \tag{3.1.3}$$

$$u''(0)=0. \tag{3.1.4}$$

$$\text{where :}\quad \phi_p(s) \quad = \quad |s|^{p-2}s, \qquad p>1$$

and assume the following assumptions

- $$\rho=\gamma\beta+\alpha\gamma+\alpha\delta \quad >0, \qquad (A1)$$
- $$0<\sigma:=\min\left\{\frac{4\delta+\gamma}{4(\delta+\gamma)},\frac{\alpha+4\beta}{4(\alpha+\beta)}\right\}<1 \qquad (A2)$$
- $$f\in C([0,1]\times[0,\infty);[0,\infty)). \qquad (A3)$$

We want to prove that under conditions $(A1)$ , $(A2)$ and $(A3)$ , by using the Leggett -williams theorem, the problem $(3.1.1)-(3.1.4)$ has three positive solutions.

Before starting the main results, we need some definitions and lemmas.

**Definition 3.1.1** *A positive solution of the boundary value problem* (3.1.1) − (3.1.4) *is a function* $u(t) \in C^2[0,1]$,*which is positive on* $0 < t < 1$ *and satisfies the problem* (3.1.1) − (3.1.4) $\cdot$

We start by giving some properties of the function $\phi$

**Lemma 3.1.1** *if*

$$\phi_p(s) = |s|^{p-2} s \ , \quad p > 1$$

*then*

$$\phi_p^{-1}(s) = \phi_q(s) \quad for \quad \frac{1}{p} + \frac{1}{q} = 1, \quad i.e. \quad (p-1)(q-1) = 1$$

**Proof :** We have

$$\phi_p(s) = |s|^{p-2} s = |s|^{p-1} \cdot \frac{s}{|s|},$$

then

$$\left|\phi_p(s)\right| = |s|^{p-1}$$

and

$$\frac{\phi_p(s)}{\left|\phi_p(s)\right|} = \frac{s}{|s|},$$

so

$$\phi_p(s) = s^{p-1} \quad if \quad s \geq 0.$$

When

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad \text{i.e.} \qquad (p-1)(q-1) = 1,$$

then

$$\begin{aligned} \left(\phi_q \circ \phi_p\right)(s) &= \phi_q\left(\phi_p(s)\right) = \left|\phi_p(s)\right|^{q-1} \frac{\phi_p(s)}{\left|\phi_p(s)\right|} \\ &= \left(|s|^{p-1}\right)^{q-1} \frac{s}{|s|} = \frac{|s|^{(p-1)(q-1)} s}{|s|} = s \end{aligned}$$

therefore

$$\phi_q \circ \phi_p = \phi_p \circ \phi_q = 1,$$

and then

$$\phi_q^{-1} = \phi_p \qquad \bullet$$

In the next we give the Green function of the differential equation

$$u''(t) = 0, \qquad t \in [0,1]$$

**Lemma 3.1.2** *The Green function of the differential equation*

$$u''(t) = 0, \qquad t \in [0,1]$$

*with respect to the boundary value conditions* $(3.1.2), (3.1.3)$, *is given by*

$$G(t\ ,s) = \begin{cases} \frac{1}{\rho}(\gamma+\delta-\gamma t)(\beta+\alpha s) & if \quad 0 \leq s \leq t \leq 1 \\ \\ \frac{1}{\rho}(\gamma+\delta-\gamma s)(\beta+\alpha t) & if \quad 0 \leq t \leq s \leq 1 \end{cases}$$

$$where \qquad \rho = \gamma\beta + \alpha\gamma + \alpha\delta$$

**Proof :** Let $L$ be a linear differential operator defined by

$$LU = u'' \ \ in \ \ (0,1)$$

and let $D$ be the boundary condition operator defined by

$$DU = \begin{cases} \alpha u(0) - \beta u'(0) \\ \\ \gamma u(1) + \delta u'(1) \end{cases}$$

We seek the Green function $G(t\ ,s)$ of the problem

$$\begin{cases} LU = 0 \quad \text{ in } \quad (0,1) \\ \\ DU = 0 \end{cases}$$

Which satisfies the following conditions

1$-$ $G\left(t\ ,s\right)$ is continuous in $t$ and $s$

2$-$ for $t\neq s$ $LG\left(t\ ,s\right)=0$

3$-$ $DG$ satisfies the boundary conditions

4$-$ The jump derivative $G'\left(s_{+0}\ ,s\right)-G'\left(s_{-0}\ ,s\right)=1$ i.e.

$$\lim_{\varepsilon\longrightarrow 0^{+}} G'\left(s\ +\varepsilon ,s\right)-G'\left(s-\varepsilon ,s\right)=1$$

The Green function for the linear operator is defined as the solution to

$$G''\left(t,s\right)=\delta\left(t-s\right)$$

if $t\neq s$ then the general solution is

$$G\left(t,s\right)=\begin{cases} a_1t+b_1 & t<s \\ a_2t+b_2 & t>s \end{cases}$$

for $t<s$ the boundary condition at $t=0$ implies

$$\alpha b_1-\beta a_1=0,\quad \text{so}\quad a_1=\frac{\alpha b_1}{\beta}\cdot$$

We skip the equation $G\left(1,s\right)=0$ because $t\neq 1$ if $t<s$ and $s\neq 1$.

So, for $t<s$

$$G\left(t,s\right)=\frac{\alpha b_1}{\beta}t+b_1=\frac{b_1}{\beta}\left(\alpha t+\beta\right).$$

for $t>s$, the boundary condition at $t=1$ implies

$$\begin{aligned} \gamma G\left(1,s\right)+\delta G\prime\left(1,s\right) &= \gamma\left(a_2+b_2\right)+\delta a_2 \\ &= 0 \end{aligned}$$

so that

$$a_2=-\frac{\gamma b_2}{\delta+\gamma}\cdot$$

For the same reasons, we skip the equation $G\left(0,s\right)=0$, then for $t>s$

$$G\left(t,s\right)=-\frac{\gamma b_2}{\delta+\gamma}t+b_2=\frac{b_2}{\delta+\gamma}\left(-\gamma t+\delta+\gamma\right)$$

Thus we have

$$G(t,s) = \begin{cases} \dfrac{b_1}{\beta}(\alpha t + \beta) & t < s \\ \\ \dfrac{b_2}{\gamma+\delta}(-\gamma t + \delta + \gamma) & t > s \end{cases}$$

Where $b_1$ and $b_2$ depend on $s$ .

Ensuring continuity of the Green function at $t = s$ implies

$$\frac{b_1}{\beta}(\alpha s + \beta) = \frac{b_2}{\gamma+\delta}(-\gamma s + \delta + \gamma)$$

which gives the following equation

$$\left(\frac{b_1\alpha}{\beta} + \frac{b_2\gamma}{\gamma+\delta}\right) s + b_1 - b_2 = 0 \cdot \tag{3.1.5}$$

Since $G$ satisfies the assumption $(4)$ , then

$$\begin{cases} \dfrac{b_1\alpha}{\beta} + \dfrac{b_2\gamma}{\gamma+\delta} = -1 & (3.1.6) \\ b_1 - b_2 = s \cdot & (3.1.7) \end{cases}$$

Equation (3.1.6) gives

$$(\delta+\gamma)\,\alpha b_1 + \gamma\beta b_2 = -\beta(\gamma+\delta)$$

using (3.1.7) and the assumption $(A_1)$ , we obtain

$$b_1 = -\frac{\beta}{\rho}(-\gamma s + \gamma + \delta)$$

and replacing $b_2$ by

$$b_2 = b_1 - s$$

we obtain the Green function for this problem :

$$G(t\ ,s) = \begin{cases} \frac{1}{\rho}(\gamma+\delta-\gamma t)(\beta+\alpha s) & if \quad 0 \le s \le t \le 1 \\ \\ \frac{1}{\rho}(\gamma+\delta-\gamma s)(\beta+\alpha t) & if \quad 0 \le t \le s \le 1 \end{cases} \qquad \bullet$$

The Green function satisfies the following properties:

**Lemma 3.1.3** *For the Green function defined in lemma* ( 3.1.2)*, we have the following*

results :

$$\frac{G(t,s)}{G(s,s)} \leq 1 \qquad for \quad t \in [0,1], \quad s \in [0,1]$$

$$\frac{G(t,s)}{G(s,s)} \geq \sigma \qquad for \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \quad s \in [0,1]$$

**Proof :** let

$$\varphi(t) = \gamma + \delta - \gamma t \qquad \text{and} \qquad \psi(t) = \beta + \alpha t \qquad \text{for} \quad t \in [0,1],$$

then we rewrite the Green function $G$ as:

$$G(t,s) = \begin{cases} \frac{1}{\rho}\, \varphi(t)\ \psi(s), & 0 \leq s \leq t \leq 1 \\ \\ \frac{1}{\rho}\, \varphi(s)\ \psi(t), & 0 \leq t \leq s \leq 1 \end{cases}$$

and

$$G(s,s) = \frac{1}{\rho}\, \varphi(s)\, \psi(s)$$

we have:

$$\varphi(s) = \gamma(1-s) + \delta \geq 0 \qquad \text{since} \ \ \gamma \geq 0, \quad \delta \geq 0 \quad \text{and} \ \ s \leq 1,$$

and

$$\psi(s) = \beta + \alpha s \geq 0 \qquad \text{since} \ \ \beta \geq 0, \quad \alpha \geq 0 \quad \text{and} \ \ s \geq 0 \quad \text{and} \ \ \rho > 0.$$

For $s \leq 1$, we have:

$$s \leq 1 + \frac{\delta}{\gamma} \quad \text{because} \ \ \rho > 0 \ \text{ and then } \ \varphi(s) > 0.$$

For $s \geq 0$, we have:

$$s > -\frac{\beta}{\alpha} \ \text{ and then } \ \psi(s) > 0$$

Thus:

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \dfrac{\varphi(t)}{\varphi(s)}, & 0 \leq s \leq t \leq 1 \\ \\ \dfrac{\psi(t)}{\psi(s)}, & 0 \leq t \leq s \leq 1 \end{cases}$$

and

$$\frac{G(t,s)}{G(s,s)} - 1 = \begin{cases} \dfrac{\varphi(t)}{\varphi(s)} - 1 = \dfrac{\gamma \cdot (s-t)}{\gamma + \delta - \gamma s} \leq 0 & 0 \leq s \leq t \leq 1 \\ \\ \dfrac{\psi(t)}{\psi(s)} - 1 = \dfrac{\alpha \cdot (t-s)}{\beta + \alpha s} \leq 0 & 0 \leq t \leq s \leq 1 \end{cases}$$

hence

$$\frac{G(t,s)}{G(s,s)} \leq 1$$

We next prove that

$$\frac{G(t,s)}{G(s,s)} \geq \sigma \quad \text{for} \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right], \qquad s \in [0,1] \quad \text{Where } \sigma \text{ is given by } (A2)$$

From the lemma $(3.1.2)$, we have :

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \dfrac{\gamma + \delta - \gamma t}{\gamma + \delta - \gamma s}, & \frac{1}{4} \leq s \leq t \leq \frac{3}{4} \\ \\ \dfrac{\beta + \alpha t}{\beta + \alpha s}, & \frac{1}{4} \leq t \leq s \leq \frac{3}{4} \end{cases}$$

Since $\gamma \geq 0$, then

$$\gamma + \delta - \gamma s \leq \gamma + \delta \qquad \text{for} \quad s \geq 0$$

and so,

$$\frac{1}{\gamma + \delta - \gamma s} \geq \frac{1}{\gamma + \delta},$$

and

$$t \leq \tfrac{3}{4}, \qquad \text{implies} \qquad -\gamma t \geq \tfrac{-3}{4}\gamma$$

so that

$$\gamma+\delta-\gamma t \geq \gamma+\delta-\frac{3\gamma}{4}=\frac{4\delta+\gamma}{4}$$

and hence

$$\frac{\gamma+\delta-\gamma t}{\gamma+\delta-\gamma s} \geq \frac{4\delta+\gamma}{4\left(\gamma+\delta\right)}.$$

If $t \geq \frac{1}{4}$ then

$$\beta+\alpha t \geq \beta+\frac{\alpha}{4}=\frac{4\beta+\alpha}{4}$$

and $s \leq 1$ implies

$$\beta+\alpha s \leq \beta+\alpha$$

hence

$$\frac{1}{\beta+\alpha s} \geq \frac{1}{\beta+\alpha}$$

and

$$\frac{\beta+\alpha t}{\beta+\alpha s} \geq \frac{4\beta+\alpha}{4\left(\beta+\alpha\right)}.$$

Thus

$$\frac{G\left(t,s\right)}{G\left(s,s\right)} \geq \sigma \quad \text{for } t \in \left[\frac{1}{4},\frac{3}{4}\right] \quad \text{and} \quad s \in [0,1] \qquad \bullet$$

We next assume that using the above results , and the assumptions $(A1)-(A3)$, the next theorem gives the form of the solutions of the problem (3.1.1)−(3.1.4) :

**Theorem 3.1.1** *let* $C^{+}[0,1]=\{u\ :\ [0,1]\longrightarrow[0,+\infty[,\quad u\in C\ \ [0,1]\}$

Where $C\ [0,1]$ be endowed with the maximum norm $\|u\|=\max\limits_{0\leq t\leq 1}\left|u\left(t\right)\right|$, then

$$u\left(t\right)==\int_{0}^{1}G\left(t,v\right)F\left(v\right)dv$$

where

$$F\left(v\right)=\phi_{q}\left(\int_{0}^{v}f\ \ \left(s,u\left(s\right)\right)ds\right)$$

is a solution of the problem (3.1.1) − (3.1.4) and $G$ is the Green function defined by lemma $(3.1.2)$.

**Proof :** let

$$u(t)=\int_0^1 G(t,v)\,F(v)\,dv,$$

from $(A3)$ and lemma $(3.1.2)$, we easily get $u(t)\geq 0,\quad t\in[0,1]$.

$$\begin{aligned}
u(t) &= \int_0^t G(t,v)\,F(v)\,dv+\int_t^1 G(t,v)\,F(v)\,dv \\
&= \frac{1}{\rho}(\gamma+\delta-\gamma t)\int_0^t(\beta+\alpha v)\,F(v)\,dv \\
&+ \frac{1}{\rho}(\beta+\alpha t)\int_t^1(\gamma+\delta-\gamma v)\,F(v)\,dv, \\
u'(t) &= \frac{1}{\rho}(\gamma+\delta-\gamma t)(\beta+\alpha t)\,F(t)-\frac{\gamma}{\rho}\int_0^t(\beta+\gamma v)\,F(v)\,dv \\
&- \frac{1}{\rho}(\beta+\alpha t)(\gamma+\delta-\gamma t)\,F(t)+\frac{\alpha}{\rho}\int_t^1(\gamma+\delta-\gamma v)\,F(v)\,dv \\
&= \frac{-\gamma}{\rho}\int_o^t(\beta+\alpha v)\,F(v)\,dv+\frac{\alpha}{\rho}\int_t^1(\gamma+\delta-\gamma v)\,F(v)\,dv, \\
u''(t) &= \frac{-\gamma}{\rho}(\beta+\alpha t)\,F(t)-\frac{\alpha}{\rho}(\gamma+\delta-\gamma t)\,F(t)\quad=-F(t) \\
&= -\phi_q\left(\int_0^t f\;(s,u(s))\,ds\right),
\end{aligned}$$

so that

$$\phi_q^{-1}(u''(t))=\phi_p(u''(t))=-\int_0^t f(s,u(s))\,ds$$

$$\left(\phi_p(u''(t))\right)'=-f(t,u(t))\cdot$$

For the boundary conditions, we have:

$$\alpha u(0)-\beta u'(0)=\alpha\int_0^1 G(0,\upsilon)F(\upsilon)\,d\upsilon-\frac{\beta\alpha}{\rho}\int_0^1(\gamma+\delta-\gamma\upsilon)F(\upsilon)\,d\upsilon=0\cdot$$

Since

$$\int_0^1 G(0,\upsilon)F(\upsilon)\,d\upsilon=\frac{\beta}{\rho}\int_0^1(\gamma+\delta-\gamma\upsilon)F(\upsilon)\,d\upsilon,$$

and

$$\gamma u(1)+\delta u'(1)=\gamma\int_0^1 G(1,\upsilon)F(\upsilon)\,d\upsilon-\frac{\gamma\delta}{\rho}\int_0^1(\beta+\alpha\upsilon)F(\upsilon)\,d\upsilon=0$$

because

$$\int_0^1 G(1,\upsilon)F(\upsilon)\,d\upsilon=\frac{\delta}{\rho}\int_0^1(\beta+\alpha\upsilon)F(\upsilon)\,d\upsilon$$

forther

$$u''(0)=-F(0)=0$$

Thus, we complete the proof .

In order to show that solutions of the boundary value problem $(3.1.1)-(3.1.4)$, are fixed points of $a$ continuous operator $T$, let us define:

$$T\colon C^{+}[0,1]\longrightarrow C^{+}[0,1]$$

such that:

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,\upsilon)\,\phi_q\left(\int_o^{\upsilon} f\ \ (s,u(s))\,ds\right)d\upsilon\\ &= \int_0^1 G(t,\upsilon)F(\upsilon)\,d\upsilon \qquad \forall u\in C^{+}[0,1]\end{aligned}$$

**Theorem 3.1.2** *The operator* $T\colon C^+[0,1] \longrightarrow C^+[0,1]$ *is completely continuous; i.e.,* $T$ *is continuous and compact.*

**Proof :** first from theorem (3.1.1) $(Tu)(t) \geq 0, \ \ t \in [0,1] \ \ for \ \ u \in C^+[0,1]$

then

$$T\colon C^+[0,1] \longrightarrow C^+[0,1].$$

next, we show that $T\colon C^+[0,1] \longrightarrow C^+[0,1]$ is continuous.

Suppose

$$\{u_n\} \subset C^+[0,1], \ u_n \longrightarrow \bar{u}\,(n \longrightarrow +\infty).$$

Then

$$\bar{u} \in C^+[0,1]$$

and there exists a constant $M_0 > 0$ such that $\|u_n\| \leq M_0, \ \ \|\bar{u}\| \leq M_0$.

Let

$$M_1 = max\{f(t,u)|t \in [0,1], \ u \in [0,M_0]\}.$$

Then for $t \in [0,1]$, we have:

$$\begin{aligned}
|Tu_n(t) - T\bar{u}(t)| &\leq \int_0^1 G(t,\upsilon)\left|\phi_q\left(\int_0^\upsilon f(s,u_n(s))\,ds\right) - \phi_q\left(\int_0^\upsilon f(s,\bar{u}(s))\,ds\right)\right| d\upsilon \\
&\leq \int_0^1 G(\upsilon,\upsilon)\left|\phi_q\left(\int_0^\upsilon f(s,u_n(s))\,ds\right) - \phi_q\left(\int_0^\upsilon f(s,\bar{u}(s))\,ds\right)\right| d\upsilon \\
&\leq \int_0^1 G(\upsilon,\upsilon)\left[\left|\phi_q\left(\int_0^\upsilon f(s,u_n(s))\,ds\right)\right| + \left|\phi_q\left(\int_0^\upsilon f(s,\bar{u}(s))\,ds\right)\right|\right] d\upsilon \\
&\leq \int_0^1 G(\upsilon,\upsilon)\left[\left|\phi_q\left(\int_0^\upsilon M_1 ds\right)\right| + \left|\phi_q\left(\int_0^\upsilon M_1 ds\right)\right|\right] d\upsilon \\
&\leq \int_0^1 G(\upsilon,\upsilon)\left[\left|\phi_q(M_1\upsilon)\right| + \left|\phi_q(M_1\upsilon)\right|\right] d\upsilon \quad \text{but} \ \ \upsilon \leq 1 \\
&\leq \int_0^1 2\phi_q(M_1)\,G(\upsilon,\upsilon)\,d\upsilon
\end{aligned}$$

Note that by hypothesis $f(t,u)$ is continuous.

We know that $\phi_q\left(\int_0^v f\left(s,u\right)ds\right)$ is continuous in $u$ on $[0,+\infty)$ and

$$\left(\int_0^v f\ \ (s,u)\,ds\right)\geq 0$$

then

$$\phi_q\left(\int_0^v f\ \ (s,u)\,ds\right)$$

is continuous in $u$ on $[0,+\infty)$.

Then for each $\varepsilon>0$, there exists $\delta_1>0$, such that

$$|u_1-u_2|<\delta_1$$

and knowing that

$$G\left(v,v\right)\neq 0$$

then

$$\left|\phi_q\left(\int_0^v f\ \ (s,u_1\left(s\right))\,ds\right)-\phi_q\left(\int_0^v f\ \ (s,u_2\left(s\right))\,ds\right)\right|<\frac{\varepsilon}{G\left(v,v\right)}$$

In view of $u_n(s)\longrightarrow\bar{u}(s)$, as $n\longrightarrow+\infty$, there exists $n\in N$, $N>0$,

for $n>N$ with

$$|u_n(s)-\bar{u}(s)|<\delta_1$$

we have:

$$\left|\phi_q\left(\int_0^v f\ \ (s,u_n\left(s\right))\,ds\right)-\phi_q\left(\int_0^v f\ \ (s,\bar{u}\left(s\right))\,ds\right)\right|<\frac{\varepsilon}{G\left(v,v\right)}$$

Thus for $\varepsilon>0$, there exists $N>0$, such that when $n>N$

$$G\left(v,v\right)\left|\phi_{q}\left(\int_{0}^{v}f\ \left(s,u_{n}\left(s\right)\right)ds\right)-\phi_{q}\left(\int_{0}^{v}f\ \left(s,\bar{u}\left(s\right)\right)ds\right)\right|<\varepsilon\quad\text{a.e.}\quad\left[0,1\right].$$

An application of Lebesgue's dominated convergence theorem implies

$$|Tu_n(t)-T\bar{u}(t)|\longrightarrow 0\ (as\ n\longrightarrow+\infty),\ t\in[0,1].$$

So the operator $T:C^{+}\left[0,1\right]\longrightarrow C^{+}\left[0,1\right]$ is continuous.

Next we prove that $T$ is compact.

Let $\Omega\subset C^{+}\left[0,1\right]$ be a bounded set, then there exists $R>0$ such that

$$\Omega\subset\{u\in C^{+}\left[0,1\right]:\ \|u\|\leq R\}.$$

Let

$$M=max\{f(t,u)\ :\ t\in[0,1],\ u\in\Omega\}.$$

For any $u\in\Omega$, we have:

$$\begin{aligned}\left|\left(Tu\right)\left(t\right)\right| &= \left|\int_{0}^{1}G\left(t,v\right)\phi_{q}\left(\int_{0}^{v}f\left(s,u\left(s\right)\right)ds\right)dv\right|\\ &\leq \int_{0}^{1}G\left(v,v\right).\phi_{q}\left(\int_{0}^{v}Mds\right)dv\\ &\leq \int_{0}^{1}G\left(v,v\right).\phi_{q}\left(M\ v\right)dv,\\ &\leq \int_{0}^{1}G\left(v,v\right)\phi_{q}\left(M\right)dv\qquad\text{for}\quad v\leq 1\cdot\end{aligned}$$

which implies that $T\ (\Omega)$ is uniformly bounded.

Furthermore, for any $u\in\Omega$ and $t\in[0,1]$,we hav:

$$\begin{aligned}\left(Tu\right)'\left(t\right) &= \left[\int_{0}^{t}\frac{1}{\rho}\left(\gamma+\delta-\gamma t\right)\left(\beta+\alpha v\right)\phi_{q}\left(\int_{0}^{v}f\ \left(s,u\left(s\right)\right)ds\right)\right]'\\ &+ \left[\int_{t}^{1}\frac{1}{\rho}\left(\beta+\alpha t\right)\left(\gamma+\delta-\gamma v\right)\phi_{q}\left(\int_{0}^{v}f\ \left(s,u\left(s\right)\right)ds\right)\right]'\end{aligned}$$

$$\begin{aligned}
(Tu)'(t) &= -\frac{\gamma}{\rho}\int_0^t (\beta+\alpha v)\,\phi_q\left(\int_0^v f(s,u(s))\,ds\right)dv \\
&+ \frac{\alpha}{\rho}\int_t^1 (\gamma+\delta-\gamma v)\,\phi_q\left(\int_0^v f(s,u(s))\,ds\right)dv,
\end{aligned}$$

so

$$\begin{aligned}
\left|(Tu)'(t)\right| &\leq \phi_q(M)\left[\left|-\frac{\gamma}{\rho}\int_0^t (\beta+\alpha v)\,dv + \frac{\alpha}{\rho}\int_t^1 (\gamma+\delta-\gamma v)\,dv\right|\right] \\
&\leq \phi_q(M)\left[\frac{\gamma}{\rho}\int_0^t (\beta+\alpha v)\,dv + \frac{\alpha}{\rho}\int_t^1 (\gamma+\delta-\gamma v)\,dv\right] \\
&\leq \phi_q(M)\left[\left(\frac{\gamma\beta-\gamma\alpha-\delta\alpha}{\rho}\right)t + \frac{\gamma\alpha}{\rho}t^2 + \frac{\alpha\gamma+2\delta\alpha}{2\rho}\right] \\
&\leq \phi_q(M)\left[\left(\frac{\gamma\beta-\delta\alpha}{\rho}\right)t + \frac{\alpha\gamma+2\delta\alpha}{2\rho}\right] \qquad \text{for } t^2 \leq t \\
&\leq \phi_q(M)\,\frac{2\gamma\beta-2\delta\alpha+\alpha\gamma+2\delta\alpha}{2\rho} \qquad \text{for } t \leq 1 \\
&= \phi_q(M)\,\frac{2\gamma\beta+\alpha\gamma}{2\rho} \\
&\leq \phi_q(M)\,\frac{2\gamma\beta+2\alpha\gamma+2\delta\alpha}{2\rho} \\
&\leq \phi_q(M)\,\frac{2\rho}{2\rho} \\
&= \phi_q(M)\cdot 1 = \phi_q(M).
\end{aligned}$$

Hence $\left\|T(u)'\right\| \leq \phi_q(M)$. So we can easily prove that $T(\Omega)$ is equicontinuous.

The Arzela-Ascoli Theorem guarantees that $T(\Omega)$ is relatively compact and therefore that $T$ is compact. since $T$ is completely continuous then by Theorem (3.1.1), any fixed point of $T$ i.e. ( $Tu = u$) is a solution of the problem $(3.1.1)-(3.1.4)$.

### 3.1.1 The main results

In this section we present the main results which are given by the following two theorems:

we denote

$$\zeta(a) = max\{f(t,u)\colon 0 \leq t \leq 1,\ 0 \leq u \leq a\}, \tag{3.1.8}$$

$$\psi(b) = \min\left\{f(t,u) : \frac{1}{4} \leq t \leq \frac{3}{4},\ b \leq u \leq \frac{b}{\sigma^2}\right\} \tag{3.1.9}$$

**Theorem 3.1.3** *Assume* $(A1)-(A3)$, *and that there exist constants* $(0 < a < b)$ *such that*

$$\zeta(a) < (ma)^{p-1}$$

$$\psi(b) \geq (\ell b)^{p-1}$$

Then the boundary value problem $(3.1.1)-(3.1.4)$ has at least two positive solutions $u_1, u_2$ satisfying

$$\|u_1\| < a,$$

$$\underset{t\in\left[\frac{1}{4},\frac{3}{4}\right]}{min}\, u_2(t) < b,$$

and

$$\|u_2\| > a,$$

where

$$m = \left(\int_0^1 G(s,s)\,ds\right)^{-1} = \frac{6\rho}{\alpha\gamma + 3\alpha\delta + 3\beta\gamma + 6\beta\delta}, \tag{3.1.10}$$

$$\ell = \frac{2}{\sigma 4^{1-q}}\left(\int_{1/4}^{3/4} G\left(\frac{1}{2},s\right)ds\right)^{-1} = \frac{2}{\sigma 4^{1-q}} \cdot \frac{32\rho}{3\alpha\gamma + 7\alpha\delta + 7\beta\gamma + 16\beta\delta} \tag{3.1.11}$$

**Theorem 3.1.4** *Assume* $(A1)-(A3)$ *and that there exist constants* $a, b, c$ *such that*

$0 < a < b < \sigma^2 c$ implies

$$\zeta(a) < (ma)^{p-1} \tag{3.1.12}$$

$$\psi(b) \geq (\ell b)^{p-1} \tag{3.1.13}$$

$$\zeta(c) \leq (mc)^{p-1} \tag{3.1.14}$$

Then the boundary value problem $(3.1.1)-(3.1.4)$ has at least three positive solutions $u_1, u_2$ and $u_3$ with

$$\|u_1\| < a, \tag{3.1.15}$$

$$\underset{t\in\left[\frac{1}{4},\frac{3}{4}\right]}{min} u_2(t) > b \tag{3.1.16}$$

and

$$\|u_3\| > a, \tag{3.1.17}$$

$$\underset{t\in\left[\frac{1}{4},\frac{3}{4}\right]}{min} u_3(t) < b, \tag{3.1.18}$$

where $\sigma$ is given as in $(A2)$ and $m, \ell$ are given as in Theorem $(3.1.3)$.

The proof of these theorems are based upon the legget–williams fixed point theorem.

Before starting the main results, let us define the Banach space :

$$\mathcal{E} = C^+[0,1]$$

which is endowed with the *maximum norm*

$$\|y\| = \max_{t\in[0,1]} |y(t)| .$$

The cone $P$ :

$$P \subset \mathcal{E} \quad : \quad P = \left\{ u \in C^{+}[0,1] , \quad \min_{t\,\in\left[\frac{1}{4},\frac{3}{4}\right]} u(t) \geq \sigma \|u\| \right\}$$

Where $\sigma$ is given by the assumption $(A2)$, and the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in [0,1]$.

**Proposition 3.1.1** *$P$ is a convex, closed cone such that* $T(P) \subset P$

**Proof :** (1) we first prone that $P$ is a cone

$$\text{Let} \quad \lambda \geq 0 \qquad \text{for any} \quad u \in P \ : \ \lambda u \in P$$

because

$$u \in P : \quad u \in C^{+}[0,1] , \quad \min_{t\,\in\left[\frac{1}{4},\frac{3}{4}\right]} u(t) \geq \sigma \|u\|$$

$$\lambda \geq 0 \implies \lambda u \geq 0 \ \text{ and } \ u \in C^{+}[0,1] ,$$

$$\lambda \min_{t\,\in\left[\frac{1}{4},\frac{3}{4}\right]} u(t) \geq \sigma \cdot \lambda \cdot \|u\|$$

$$\min_{t\,\in\left[\frac{1}{4},\frac{3}{4}\right]} (\lambda u(t)) \geq \sigma \|\lambda u\|$$

Now we show that if

$$u, -u \ \in P \qquad \Longrightarrow \qquad u = 0$$

We have:

$$u \in P = \left\{ u \in C^{+}[0,1] , \quad \min_{t\,\in\left[\frac{1}{4},\frac{3}{4}\right]} u(t) \geq \sigma \|u\| \right\}$$

then

$$-u \in P = \left\{ -u \in C^{+}[0,1], \quad \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} -u(t) \geq \sigma \|-u\| \right\}$$

Suppose

$$u, -u \ \in P, \quad \text{then if} \quad u \neq 0$$

$$u \in C^{+}[0,1], \quad -u \in C^{+}[0,1],$$

so

$$u \geq 0, \ \ -u \geq 0 \qquad \text{and then} \quad u = 0$$

(2) $P$ is convex, i.e

$$\forall \lambda \in [0,1], \quad \forall \ u_{1,}u_2 \ \in P, \ \lambda u_1 + (1-\lambda) u_2 \in P.$$

We have:

$$\lambda u_1 + (1-\lambda) u_2 \geq 0, \qquad \lambda u_1 + (1-\lambda) u_2 \in C^{+}[0,1],$$

since

$$u_1 \in C^{+}[0,1], \ \ u_2 \in C^{+}[0,1]$$

$$\begin{aligned}
\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} (\lambda u_1 + (1-\lambda) u_2) &\geq \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \lambda (u_1) + \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} (1-\lambda) u_2 \\
&\geq \lambda \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} (u_1) + (1-\lambda) \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} (u_2) \\
&\geq \lambda\sigma \|u_1\| + (1-\lambda) \sigma \|u_2\| \\
&\geq \sigma \|\lambda u_1\| + \sigma \|(1-\lambda) u_2\| \\
&\geq \sigma (\|\lambda u_1\| + \|(1-\lambda) u_2\|) \\
&\geq \sigma \|\lambda u_1 + (1-\lambda) u_2\|
\end{aligned}$$

then $P$ is convexe

(3) $P$ is closed : let $u_n \in P$ such that $\underset{n \longrightarrow +\infty}{u_n \longrightarrow} \bar{u}$ we must show that $\bar{u} \in P$.

$$u_n \in P = \left\{ u_n \in C^+ [0,1] \, , \, \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_n (t) \geq \sigma \left\| u_n \right\| \right\}$$

$$u_n \in C^+ [0,1] \, , \quad \bar{u} \in C^+ [0,1] \, ,$$

$$\lim_{n \longrightarrow +\infty} \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_n (t) \geq \lim_{n \longrightarrow +\infty} \sigma \left\| u_n \right\|$$

$$\lim_{n \longrightarrow +\infty} \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_n (t) \geq \sigma \left\| \bar{u} \right\|$$

$$\exists \; t_1 \in \left[\frac{1}{4}, \frac{3}{4}\right] , \quad \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_n (t) = u_n (t_1)$$

$$\lim_{n \longrightarrow +\infty} \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_n (t) = \lim_{n \longrightarrow +\infty} u_n (t_1) = \bar{u} (t_1) = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} \bar{u} (t) \, .$$

Then

$$\min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} \bar{u} (t) \geq \sigma \left\| \bar{u} \right\|$$

then $P$ is closed

Let us Now state some properties of the operator $T$.

Let

$$\begin{aligned} \|Tu\| &= \max \left\{ (Tu) (t) : t \in [0,1] \right\} \\ &= \max \left\{ \int_0^1 G \; (t, v) \, F \; (v) \, dv : t \in [0,1] \right\} \end{aligned} \tag{3.1.19}$$

so it is easy to see

$$\|Tu\| \leq \int_0^1 G(\upsilon,\upsilon) F(\upsilon)\, d\upsilon \quad \text{for all} \quad \upsilon \in [0,1] \qquad \bullet$$

Let

$$D_a = \{(t,\upsilon) : t \in [0,1], \upsilon \in [0,a]\} \tag{3.1.20}$$

and let

$$u \in C^+[0,1] \qquad \text{such that} \qquad 0 \leq u(t) \leq a \quad \Longrightarrow \|u\| \leq a$$

**Lemma 3.1.4**

$$if \;\; \|u\| \leq a, \quad Then \quad \|Tu\| < a$$

**Proof :**

$$f(s,u(s)) \leq \max \; \{f \;\; (t,\upsilon) : \; (t,\upsilon) \in D_a\}$$

so

$$f(s,u(s)) \leq \zeta(a).$$

and by hypothesis

$$\zeta(a) < (ma)^{p-1}$$

Where

$$m^{-1} = \int_0^1 G(\upsilon,\upsilon)\, d\upsilon$$

Therefore

$$\int_0^\upsilon f(s,u(s))\, ds < \int_0^\upsilon (ma)^{p-1}\, ds = (ma)^{p-1}\upsilon < (ma)^{p-1} \qquad \forall \upsilon \in [0,1]$$

Then

$$\phi_q\left(\int_0^1 f \;\; (s,u(s))\, ds\right) < \left((ma)^{p-1}\right)^{q-1} = (ma)^{(p-1)(q-1)} = ma$$

Finaly

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t,\upsilon)\,F(\upsilon)\,d\upsilon < ma\int_0^1 G(t,\upsilon)\,d\upsilon \\ &\leq ma\int_0^1 G(\upsilon,\upsilon)\,d\upsilon = a \quad \cdot \end{aligned}$$

Let

$$0 < \sigma < 1, \quad b > 0$$

and

$$D_b = \left\{(t,\upsilon) : t \in \left[\frac{1}{4},\frac{3}{4}\right], \quad \upsilon \in \left[b,\frac{b}{\sigma^2}\right]\right\} \tag{3.1.21}$$

Such that

$$G(\upsilon,\upsilon) \leq \sigma G(t,\upsilon) \qquad \forall \ (t,\upsilon) \in D_b$$

Let

$$\varphi\colon C^+[0,1] \longrightarrow [0,+\infty)$$

be the nonnegative continuous functional defined by

$$\varphi(u) = \min\left\{u(t) : t \in \left[\frac{1}{4},\frac{3}{4}\right]\right\} \tag{3.1.22}$$

Evidently, for

$$u \in C^+[0,1] \qquad\qquad \varphi(u) \leq \|u\| \quad \cdot$$

**Lemma 3.1.5**

$$\varphi(T(u)) \geq \sigma\|T(u)\|$$

**Proof :**

$$\begin{aligned} \varphi(T(u)) &= \min\left\{\int_0^1 G(t,\upsilon)\,F(\upsilon)\,d\upsilon\colon t \in \left[\frac{1}{4},\frac{3}{4}\right]\right\} \\ &\geq \int_0^1 \sigma G(\upsilon,\upsilon)\,F(\upsilon)\,d\upsilon \\ &= \sigma\int_0^1 G(\upsilon,\upsilon)\,F(\upsilon)\,d\upsilon \\ &\geq \sigma\|Tu\| \qquad \bullet \end{aligned}$$

**Lemma 3.1.6**

$$\|Tu\| > b \quad for \quad b \leq u(t) \leq \frac{b}{\sigma^2}, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right]$$

**Proof :** $f(s, u(s)) \geq \min\{f(t, \upsilon) : (t, \upsilon) \in D_b\} = (\ell b)^{p-1}, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right]$
and since

$$\int_{1/4}^{3/4} G\left(\frac{1}{2}, \upsilon\right) d\upsilon = \frac{3\alpha\gamma + 7\alpha\delta + 7\beta\gamma + 16\beta\delta}{32\rho}$$

then from (3.1.11)

$$\int_{1/4}^{3/4} G\left(\frac{1}{2}, \upsilon\right) d\upsilon = \frac{2(4^{q-1})}{\ell\sigma} \quad .$$

Now, we have:

$$\int_0^\upsilon f(s, u(s))\, ds \geq \int_0^\upsilon (\ell b)^{p-1}\, ds = (\ell b)^{p-1} \int_0^\upsilon ds \geq (\ell b)^{p-1} \int_0^{1/4} ds = \frac{(\ell b)^{p-1}}{4},$$

$$F(\upsilon) = \phi_q\left(\int_0^\upsilon f(s, u(s))\, ds\right) \geq \left(\frac{(\ell b)^{p-1}}{4}\right)^{q-1} = \frac{(\ell b)^{(p-1)(q-1)}}{4^{q-1}} = \frac{\ell b}{4^{q-1}},$$

so:

$$\begin{aligned} \|Tu\| &\geq (Tu)\left(\frac{1}{2}\right) = \int_0^1 G\left(\frac{1}{2}, \upsilon\right) F(\upsilon)\, d\upsilon \\ &\geq \int_{1/4}^{3/4} G\left(\frac{1}{2}, \upsilon\right) F(\upsilon)\, d\upsilon \\ &\geq \frac{\ell b}{4^{q-1}} \int_{1/4}^{3/4} G\left(\frac{1}{2}, \upsilon\right) d\upsilon = \frac{2b}{\sigma} > b \qquad \bullet \end{aligned}$$

**Lemma 3.1.7** *if* $\|u\| \leq \frac{b}{\sigma^2}$ *and* $\|Tu\| > \frac{b}{\sigma^2}$ *then* $\varphi(u) > b$

**Proof :** we have that if $\|u\| \leq \frac{b}{\sigma^2}$ , *then* $\|Tu\| > \frac{b}{\sigma^2}$

so,

$$\varphi(u) \geq \sigma \|Tu\| > \sigma^2 \|Tu\| > \sigma^2 \frac{b}{\sigma^2} > b$$

by Theorem (2.2.3) (Leggett-Williams [10]).

**Proof :** Of the Theorem (3.1.4) First: we have $0 < a < b < \dfrac{b}{\sigma^2}$ and from Theorem (3.1.2), and Lemma (3.1.5) we know that $T\colon P \longrightarrow P$ is completely continuous.

it is easy to see that $T\colon \bar{P}_{\frac{b}{\sigma^2}} \longrightarrow P$ is completely continuous.

If we choose $u(t) = \dfrac{b}{\sigma^2}$

$$\left[P\left(\varphi, b, \frac{b}{\sigma^2}\right) = \left\{u \in P : \quad b \leq \varphi(u) \quad ; \quad \|u\| \leq \frac{b}{\sigma^2}\right\}\right]$$

then from Lemma (3.1.6) and Lemma (3.1.7) $u \in P\left(\varphi, b, \dfrac{b}{\sigma^2}\right)$

so

$$\left\{u \in P\left(\varphi, b, \frac{b}{\sigma^2}\right) : \varphi(u) > b\right\} \neq \emptyset.$$

Thus for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$ and from Lemma (3.1.5), and Lemma (3.1.6),

$$\varphi(Tu) > b \ \ \forall u \in P\left(\varphi, b, \frac{b}{\sigma^2}\right)$$

Therefore, condition $(a')$ of theorem (2.2.3), is satisfied.

Now if $u \in \bar{P}_a$, then from Lemma (3.1.4), $\|Tu\| < a.$ for $u \in \bar{P}_a$

This shows that $T\colon \bar{P}_a \longrightarrow P_a$.

and then $(b')$ of theorem (2.2.3) is satisfied.

Finally, assuming $u \in \bar{P}_{\frac{b}{\sigma^2}}$ with $\|Tu\| > \dfrac{b}{\sigma^2}$ then from Lemma (3.1.5)

$$\begin{aligned} \varphi(Tu) &\geq \sigma \|Tu\| \\ \varphi(Tu) &\geq \sigma^2 \|Tu\| = b \big/ \tfrac{b}{\sigma^2} \|Tu\|, \end{aligned}$$

so condition $(c')$ is satisfied.

Thus using theorem (2.2.3), $T$ has at least three fixed points.

That is to say, the boundary value problem (3.1.1) − (3.1.4) has at least three positive solutions $u_1, u_2$ and $u_3$ with $\|u_1\| < a, \quad \min\limits_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_2(t) > b, \quad \|u_3\| > a$

and

$$\min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_3(t) < b.$$

## 3.2 AN EXAMPLE

### Sommaire

Now we consider an example to illustrate our results.

Consider the third-order Sturm-Liouville boundary value problem, with p-Laplacian,

$$(\phi_3(u''(t)))' + [\varphi(t)\,h(u(t))]^2 = 0, \qquad t \in (0,1) \tag{3.2.1}$$

$$u(0) - u'(0) = 0, \tag{3.2.2}$$

$$u(1) = 0, \tag{3.2.3}$$

$$u''(0) = 0. \tag{3.2.4}$$

$Where\ \varphi(t) = 4t,\ t \in [0,1]\ and$

$$h(u) = \begin{cases} u/2\,, & 0 \le u \le 3/512\,; \\ \dfrac{1021}{4}u - \dfrac{3057}{2048}, & \dfrac{3}{512} \le u \le \dfrac{5}{512}; \\ 1, & \dfrac{5}{512} \le u \le \dfrac{5}{32}; \\ \dfrac{8}{59}u + \dfrac{231}{236}, & \dfrac{5}{32} \le u \le 2; \\ 5u/8\,, & u \ge 2. \end{cases}$$

In this example, we note that $p = 3, \qquad \alpha = \beta = \gamma = 1, \delta = 0$.After a simple calculation, we get $q = 3/2, \qquad \rho = 2$, $\sigma = 1/4 < 1$, $G(s,s) = \frac{1}{2}(1 - s^2)$ and

$$m = \frac{6\rho}{\alpha\gamma + 3\alpha + 3\beta\gamma + 6\beta\delta} = 3, \qquad \ell = \frac{2}{\sigma 4^{1-q}} \cdot \frac{32\rho}{3\alpha\gamma + 7\alpha\delta + 7\beta\gamma + 16\beta\delta} = \frac{512}{5}.$$

We choose $a = \dfrac{3}{512}$, $b = \dfrac{5}{512}$, $c = 2$ Evidently, $a < b < \sigma^2 c$ and

$(i)$ for $t \in [0,1], \qquad 0 \le u \le \dfrac{3}{512}$,

$$\text{we have} \quad f(t,u) = [\varphi(t)\, h(u)]^2 \le \left[4 \times \frac{1}{2} \times 3/512\right]^2 < (ma)^2 \ .$$

$(ii)$ for $t \in \left[\frac{1}{4}, \frac{3}{4}\right], \qquad \frac{5}{512} \le u \le \frac{b}{\sigma^2} = \frac{5}{32}$,

$$\text{we have} \quad f(t,u) = [\varphi(t)\, h(u)]^2 \ge \left[4 \times \frac{1}{4} \times 1\right]^2 = (\iota b)^2 .$$

$(iii)$ for $t \in [0,1], \qquad 0 \le u \le 2$,

$$\text{we have} \quad f(t,u) = [\varphi(t)\, h(u)]^2 \le \left[4 \times 1 \times \left(\frac{8}{59} \times 2 + \frac{231}{236}\right)\right]^2 \le (mc)^2 .$$

$$\text{Thuse} \qquad \zeta(a) < (ma)^2, \qquad \psi(b) \ge (\iota b)^2, \qquad \zeta(c) \le (mc)^2 .$$

Hence, all the conditions of Theorem (3.1.4) are satisfied. An application of Theorem (3.1.4) implies that (4.1.1) – (4.1.4) has at least three positive solutions $u_1, u_2$ and $u_3$ with

$$\|u_1\| < \frac{3}{512},$$

$$\min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_2(t) > \frac{5}{512},$$

$$\|u_3\| > \frac{3}{512}$$

and

$$\min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_3(t) < \frac{5}{512}.$$

# Conclusion

In this work, the existence of at least three positive solutions to the third order Sturm-Liouville boundary value problem, with p-Laplacian was established.

The differential equation was converted to an integral operator associated with the boundary value problem.

Growth conditions are imposed on the nonlinear term of the problem and inequalities involving an associated Green's function to the boundary value problem are used in order to apply the Leggett -Williams Fixed point theorem to cones in ordered Banach space.

The Leggett -Williams Fixed point Theorem is based upon the relationships between the operator, the norm defined on the Banach space and a continuous convave functional.

Using the Legget-Williams theorem to this operator, the existence of three positive fixed points was established. In this study, the main tool is a fixed point theorem of a completely continuous operator defined on a cone of an ordered Banach space.

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