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THÈME

Résultats d'existence pour certaines classes de problèmes aux limites à plusieurs points *

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Chapter 1

INTRODUCTION

The multi-point boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. For exemple, many problems in the theory of elastic stability can be handled as a multi-point problem (see [40], [41]). The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by IL'in and Moiseev [70]. Since then, many authors considered nonlinear multi-point boundary value problems (see [65], [31], [13], [66], [29]).

On the other hand, boundary value problems with p-Laplacan operator $(\phi_p(u'(t)))'$, where $(\phi_p(x)) = |x|^{p-2}\, x$, $(p > 1)$ have received a lot of attention in recent years. Many studies have been carried out to discuss the existence of solution or positive solution and multiple solutions for the boundary value problems (see [43], [34], [16], [23], [22], [27], [24]). As well known, the main difficulty that appears when passing from the case $p = 2$ to the case $p \neq 2$ is that, when $p = 2$, we can transform the differential equation into an equivalent integral operator for which a Green's function exists, so we can easily prove the equivalent integral operator in a cone, which is a very important result for discussing positive solution for boundary value problems in Banach spaces. However, for $p \neq 2$, it is difficult to find a Green's function in the equivalent integral operator since the $p-$Laplacian differential operator is nonlinear.

The operator $-\Delta_{p(t)}u = -\operatorname{div}\left(|\nabla u|^{p(t)-2}\,\nabla u\right)$ with $p(t) > 1$ is called the $p(t)-$Laplacian, which is a natural generalization of the $p-$Laplacian (where $p > 1$ is a constant). When $p(t)$ is not constant, the $p(t)-$Laplacian possesses more complicated nonlinearity than $p-$Laplacian, say, it is nonhomogeneous. For this reason, some of the above properties of the $p-$Laplacian problems may not hold for a general $p(t)-$Laplacian case.

In the first chapter, we present , some recent results obtained for the multi-point boundary value problems associated with the p-Laplacian and p(t)-Laplacian operator.

In the second chapter, we study existence and nonexistence of positive solutions to a singular three point boundary value problem (BVP for short)

$$\begin{cases} -\left(au'\right)'(t) = b(t)f(t,u(t)), \quad t \in (0,1), \\ u'(0) = 0, \ u(1) = \alpha u(\eta), \end{cases} \tag{1.1}$$

where $\mathbb{R}^+ = [0,+\infty)$, $\alpha, \eta \in (0,1)$, $a, b \in C((0,1), \mathbb{R}^+)$, $a > 0$ in $(0,1)$, $\dfrac{1}{a}$ is integrable on any compact subset of $(0,1]$, b does not vanish identically and is integrable on any compact subset of $[0,1)$ and $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. We give the criteria for the existence and nonexistence of positive solutions to BVP (1.1) which is determined by the comparison between the ratio $f(t,u)/u$ and a positive eigenvalue of the associated linear eigenvalue problem

$$\begin{cases} -(au')'(t) = \lambda b(t)u(t), \ t \in (0,1), \\ u'(0) = 0, \ u(1) = \alpha u(\eta). \end{cases} \tag{1.2}$$

In the third chapter, we investigate the existence of a positive solution to the $\phi-$Laplacian three point boundary value problem

$$\begin{cases} -\left(a\phi\left(u'\right)\right)'(x) = b\left(x\right)f(x,u(x)), \quad x \in \left(0,1\right), \\ u(0) = \alpha u\left(\eta\right), \ u'(1) = 0, \end{cases} \tag{1.3}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi\left(0\right) = 0$, $\alpha, \eta \in \left[0,1\right)$, $a, b \in C\left(\left[0,1\right], \left[0,+\infty\right)\right)$, $a > 0$ in $[0,1]$, b does not vanish identically, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. By using an homotopy property of the fixed point index, we obtain existence results. In the fourth chapter, we prove existence of a positive solutions to the p(t)-Laplacian three point boundary value problem (bvp for short)

$$\begin{cases} -(\phi(t,u'(t)))' = f(t,u(t)),\ t\in(0,1), \\ u(0)=\alpha u(\eta),\ u'(1)=0, \end{cases} \tag{1.4}$$

where $\mathbb{R}^+ = [0,+\infty)$, $\alpha,\eta\in(0,1)$, $\phi(t,x) = |x|^{p(t)-2}x$, $p\in C([0,1],(1,+\infty))$, and

$$f\in C\left([0,1]\times\mathbb{R}^+,\mathbb{R}^+\right).$$

By means of the auxiliary problem

$$\begin{cases} -u'(t) = f(t,Au(t)),\ t\in(0,1), \\ u(1)=0, \end{cases} \tag{1.5}$$

where Au is a positive solution to bvp (1.4) for all positive solution u to the auxiliary problem, we obtain existence results for the sublinear case and the superlinear case by using the homotopy property of the fixed point index. In the fifth chapter, we investigate the existence of a positive solution to the weighted $p(t)$-Laplacian three point boundary value problem

$$\begin{cases} -(a(t)\,\phi(t,u'(t)))' = f(t,u(t)),\ t\in(0,+\infty), \\ u(0)=\alpha u(\eta),\ u'(+\infty)=0, \end{cases} \tag{1.6}$$

where $\alpha\in(0,1)$, $\eta>0$, $\phi(t,x) = |x|^{p(t)-2}x$ for all $x\in\mathbb{R}^+ = [0,+\infty)$, $p:\mathbb{R}^+\to(1,+\infty)$ is a continuous bounded function and $a:(0,+\infty)\to(0,+\infty)$ is continuous.

To establish the main result of this paper we need to assume that:

$$\begin{cases} 1 < p^- := \inf_{t\in[0,\infty)} p(t) \le p^+ := \sup_{t\in[0,\infty)} p(t); \end{cases}$$

$$\begin{cases} \text{There exists } d\ge 0 \text{ such that} \\ \displaystyle\int_0^{+\infty} \tilde{a}(t)\,(1+t)^{-\frac{d}{p^+-1}}\,dt < +\infty, \quad \text{where } \tilde{a}(t) = 1/\psi(t,a(t)); \end{cases}$$

$$\begin{cases} f:(0,+\infty)\times\mathbb{R}^{+}\to\mathbb{R}^{+} \text{ is } L_d - \text{Carathéodory, that is:} \\ (i)\ \ f \ \text{ is a Carathéodory function,} \\ (ii)\ \text{ for all } \ r>0, \ \text{ there exists } h_r\in L_d\left(\mathbb{R}^{+}\right) \ \text{ such that} \\ \quad f\left(t,u\right)\leq h_r\left(t\right) \text{ for all } u\in[0,r] \text{ and at most } t>0, \text{ where} \\ \quad L_d\left(\mathbb{R}^{+}\right)=\left\{u:\mathbb{R}^{+}\to\mathbb{R}^{+} \text{ measurable and } \displaystyle\int_0^{+\infty}(1+t)^d\left|u(t)\right|dt<\infty\right\} \\ \quad \text{is equiped with the norm } \left|u\right|_d=\int_0^{+\infty}(1+t)^d\left|u(t)\right|dt. \end{cases}$$

By using the properties of the fixed point index, existence results of positive solutions are obtained.

In the sixth chapter, we obtain existence results of positive solutions for the third order boundary value problem

$$\begin{cases} -\left(pu''\right)'(t)=f(t,u(t)), \ \ t\in(0,1), \\ u(0)=au'(0)-b\lim\limits_{t\to 0}p(t)u''(t)=0, \\ cu'(1)+d\lim\limits_{x\to 1}p(t)u''(t)=0, \end{cases} \tag{1.7}$$

where $a,b,c,d\in\mathbb{R}^{+}$, $p\in C\left((0,1),(0,+\infty)\right)$, $f:(0,1)\times\mathbb{R}^{+}\to\mathbb{R}^{+}$ is a Carathéodory function ($f(\cdot,u)$ is measurable for u fixed and $f(t,\cdot)$ is continuous for almost $t\in(0,1)$) and $\mathbb{R}^{+}=[0,+\infty)$.

We assume in all this paper that

$$\int_0^1\frac{dt}{p(t)}<\infty$$

and

$$\Delta=ac\int_0^1\frac{dt}{p(t)}+ad+bc>0.$$

Under eingenvalue criteria, we obtain some abstract existence results related to index-jump property of positive linear compact operators, then to bvp (1.7).

Finally, a general conclusion will summarize all of this work.

Introduction

En raison de leurs applications physiques, l'étude des équations différentielles associées à des conditions aux limites à plusieurs points ont reçu beaucoup d'attention au cours des deux dernières décennies. Par exemple, de nombreux problèmes dans la théorie de la stabilité élastique non linéaire, peuvent être traités comme des problèmes aux limites à plusieurs points (Voir [40], [41]). Initiée par Il'in et Moiseev [70], l'étude des problèmes aux limites à plusieurs points a fait l'objet de nombreux travaux (Voir, [65], [31], [13], [66], [29]).

D'autre part, plusieurs études ont été menées afin de discuter de l'existence de solutions pour les problèmes aux limites avec un opérateur du type p-Laplacien $(\phi_p(u'(t)))'$, où $(\phi_p(x)) = |x|^{p-2}\,x$, $(p > 1)$. (Voir [43], [34], [16], [23], [22], [27], [24]). Comme on le sait, dans l'étude des problèmes non résonants, la principale difficulté qui apparaît , lors du passage du cas $p = 2$ au cas $p \neq 2$ est que, lorsque $p = 2$, on peut trouver la fonction de Green et l'opérateur intégral associé T et se ramener à l'étude de l'equation $Tu = u$ sur un espace de Banach sans beaucoups difficulté. Cependant, pour $p \neq 2$, il est difficile de trouver la fonction de Green pour l'opérateur intégral car l'opérateur differentiel $p-$Laplacien n'est pas linéair.

L'opérateur $p(t)-$Laplacien, $-\Delta_{p(t)}u = -\operatorname{div}\left(|\nabla u|^{p(t)-2}\,\nabla u\right)$ avec $p(t) > 1$ est une généralisation du $p-$Laplacien (Lorsque $p > 1$ est une constante). Les problèmes aux limites à plusieurs points avec un $p(t)-$Laplacien sont rarement étudiés. Lorsque $p(t)$ est non constante, $p(t)-$Laplacien rend le problème plus difficile, et l'opérateur intégral associé devient plus compliqué que celui associé au problème p-Laplacien, on dit qu'il est non homogène. Pour cette raison, certaines des propriétés ci-dessus des problèmes p-Laplacien ne s'applique plus pour le cas général p(t)-Laplacien. L'étude des problèmes $p(t)-$Laplacien fait l'objet de plusieurs contributions récentes (voir [73], [71], [72]).

Nous présentons dans le premier chapitre quelques résultats récents obtenus, pour les problèmes p(t)-Laplacien.

Dans le second chapitre, nous considérons le problème singulier à trois points

$$\begin{cases} -\left(au'\right)'(t) = b(t)f(t,u(t)), \quad t \in (0,1), \\ u'(0) = 0,\ u(1) = \alpha u(\eta), \end{cases} \tag{1.8}$$

où $\alpha, \eta \in (0,1),\ a, b \in C((0,1), \mathbb{R}^+),\ a > 0$ dans $(0,1)$, $\dfrac{1}{a}$ est intégrable sur tout sous intervalle compact de $(0,1],\ b$ est intégrable et non identiquement nulle sur tout sous intervalle compact de $[0,1)$ et $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ est continue. Nous donnons un critère pour l'existence et la non existence de solutions positives pour (1.8) qui est determiné par comparaison entre le rapport $f(t,u)/u$ et la valeur propre positive du problème linéaire au valeur propre associé

$$\begin{cases} -(au')'(t) = \lambda b(t)u(t),\ t \in (0,1), \\ u'(0) = 0,\ u(1) = \alpha u(\eta). \end{cases} \tag{1.9}$$

Dans le troisième chapitre, nous étudions le problème $\phi-$Laplacien à trois points

$$\begin{cases} -\left(a\phi\left(u'\right)\right)'(x) = b\left(x\right)f(x,u(x)), \quad x \in (0,1)\,, \\ u(0) = \alpha u\left(\eta\right),\ u'(1) = 0, \end{cases} \tag{1.10}$$

où $\phi : \mathbb{R} \to \mathbb{R}$ est un homéomorphisme strictement croissant avec $\phi\left(0\right) = 0,\ \alpha, \eta \in [0,1)\,,\ a, b \in C\left([0,1]\,,[0,+\infty)\right),\ a > 0$ dans $[0,1]$, b non identiquement nulle et $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ est continue. En utilisant la propriété de l'homotopie de l'indice de point fixe, les resultats d'existence de solutions positives sont obtenus. Dans le quatrième chapitre, nous considérons le problème p(t)-Laplacien à trois points

$$\begin{cases} -(\phi(t,u'(t)))' = f(t,u(t)),\ t \in (0,1), \\ u(0) = \alpha u(\eta),\ u'(1) = 0, \end{cases} \tag{1.11}$$

où $\alpha, \eta \in (0,1)$, $\phi(t,x) = |x|^{p(t)-2}\,x,\ p \in C\left([0,1]\,,(1,+\infty)\right)$ et $f \in C\left([0,1] \times \mathbb{R}^+, \mathbb{R}^+\right)$.

Par le biais du problème auxiliaire

$$\begin{cases} -u'(t) = f(t,Au(t)),\ t \in (0,1), \\ u(1) = 0, \end{cases} \tag{1.12}$$

où Au est une solution positive du problème (1.11) lorsque u est une solution du problème auxiliaire (1.12), nous obtenons les résultats d'existence de solutions positives pour le cas sous linéaire et surlinéaire par la méthode de l'indice de point fixe. Dans le cinquième chapitre, nous étudions le problème $p(t)$-Laplacien

$$\begin{cases} -(a(t)\,\phi(t,u'(t)))' = f(t,u(t)),\ t \in (0,+\infty), \\ u(0) = \alpha u(\eta),\ u'(+\infty) = 0, \end{cases} \tag{1.13}$$

où $\alpha \in (0,1)$, $\eta > 0$, $\phi(t,x) = |x|^{p(t)-2}x$ pour tout $x \in \mathbb{R}^+ = [0,+\infty)$, $p : \mathbb{R}^+ \to (1,+\infty)$ est une fonction continue et bornée et $a : (0,+\infty) \to (0,+\infty)$ est continue.

Nous supposons que:

$$\left\{ 1 < p^- := \inf_{t\in[0,\infty)} p(t) \leq p^+ := \sup_{t\in[0,\infty)} p(t); \right.$$

$$\begin{cases} \text{Il existe}\, d \geq 0 \,\text{tel que} \\ \displaystyle\int_0^{+\infty} \widetilde{a}\,(t)\,(1+t)^{-\frac{d}{p^+-1}}\,dt < +\infty, \quad \text{where } \widetilde{a}\,(t) = 1/\psi(t,a(t)); \end{cases}$$

$$\begin{cases} f : (0,+\infty)\times\mathbb{R}^+ \to \mathbb{R}^+ \text{ est } L_d - \text{Carathéodory, c'est à dire :} \\ (i) \ \ f \ \text{ est une fonction de Carathéodory,} \\ (ii) \ \text{Pour tout } \ r > 0, \ \text{ il existe } h_r \in L_d\,(\mathbb{R}^+) \ \text{ telle que} \\ \quad f\,(t,u) \leq h_r\,(t) \ \text{ pour tout } u \in [0,r] \ \text{ et p.p } t > 0, \ \text{où} \\ \quad L_d\left(\mathbb{R}^+\right) = \left\{u : \mathbb{R}^+ \to \mathbb{R}^+ \text{ mesurable et } \displaystyle\int_0^{+\infty} (1+t)^d\,|u(t)|\,dt < \infty\right\} \\ \quad \text{est muni de la norme } |u|_d = \int_0^{+\infty} (1+t)^d\,|u(t)|\,dt. \end{cases}$$

En utilisant le théorème de l'indice de point fixe [14], des résultats d'existence de solutions positives sont obtenus.

Dans le sixième chapitre, nous étudions le problème aux limites à trois points du troisième degré

$$\begin{cases} -\left(pu''\right)'(t) = f(t,u(t)), \ \ t \in (0,1)\,, \\ u(0) = au'(0) - b\lim\limits_{t\to 0} p(t)u''(t) = 0, \\ cu'(1) + d\lim\limits_{x\to 1} p(t)u''(t) = 0, \end{cases} \tag{1.14}$$

où $a, b, c, d \in \mathbb{R}^+$, $p \in C\left((0,1), (0,+\infty)\right)$, $f : (0,1) \times \mathbb{R}^+ \to \mathbb{R}^+$ est une fonction de Carathéodory ($f(\cdot, u)$ est mesurable pour u fixé et $f(t, \cdot)$ est continue pour presque tout $t \in (0,1)$).

Nous supposons que

$$\int_0^1 \frac{dt}{p(t)} < \infty$$

et

$$\Delta = ac \int_0^1 \frac{dt}{p(t)} + ad + bc > 0.$$

Sous une condition sur les valeurs propres, et en utilisant le théorème de Krein-Rutman, nous obtenons des resultats liés à la propriété de saut d'indice d'opérateurs compacts linéaires positives, pour l'existence de solutions pour une équation abstraite, donc pour le problème (1.14).

Enfin, une conclusion générale résumera l'ensemble des travaux de cette thèse.

Chapter 2

RECENT RESULTS FOR P-LAPLACIAN BVPS

The p-Laplacian problems with two-point, three-point and multi-point boundary conditions for ordinary differential equations have been studied extensively.

In 2005, D. Ma, Z. Du and W.Ge [15], proved the existence of at least one positive solutions for m-point boundary value problem (BVP) of the form

$$\begin{cases} (\phi_p(u'))' + a(t) f(t, u(t)) = 0, \ t \in (0,1), \\ u'(0) = \sum\limits_{i=1}^{m-2} a_i u'(\xi_i), \ u(1) = \sum\limits_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{2.1}$$

where $\phi_p(x) = |x|^{p-2} x$, $(p > 1)$, $0 < \sum\limits_{i=1}^{m-2} b_i < 1$, $1 < \sum\limits_{i=1}^{m-2} a_i < 1$, $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1$, $a(t) \in L^1(0,1)$, $f \in C([0,+\infty) \times [0,+\infty), \mathbb{R}^+)$. They have given sufficient conditions to obtain existence of positive solutions by using the monotone iterative technique in cones, they defined the sequence $(\omega_n)_n$ by $\omega_{n+1} = T\omega_n$ such that, by compacity of T, $(\omega_n)_n$ admits a sub-sequence converging to a fixed point ω^* of T, which is a solution of (2.1).

In [18], Xu has studied the following m-point boundary value problem (BVP)

$$\begin{cases} (\phi_p(u'))' + a(t) f(u(t)) = 0, \ t \in (0,1), \\ u'(0) = \sum\limits_{i=1}^{m-2} a_i u'(\xi_i), \\ u(1) = \sum\limits_{i=1}^{k} b_i u(\xi_i) - \sum\limits_{i=k+1}^{s} b_i u(\xi_i) - \sum\limits_{i=s+1}^{m-2} b_i u'(\xi_i), \end{cases} \tag{2.2}$$

where $1 < k < s < m-2$, a_i, $b_i > 0$ with $0 < \sum\limits_{i=1}^{k} b_i - \sum\limits_{i=k+1}^{s} b_i < 1$, $1 < \sum\limits_{i=1}^{m-2} a_i < 1$, $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1$, $a(t) \in C((0,1), [0,+\infty))$, $f \in C([0,+\infty), \mathbb{R}^+)$. They have shown the existence of one or two positive solutions if f is either superlinear or sublinear by applying fixed point theorems in cones.

Motivated by the results mentioned above, Hongkui Li, Fuyi Xu, Wenling Zhao and Hongling Geng [25], studied the existence of positive solutions of m-point boundary value problem with the nonlinear term f depending on the first-order derivative

$$\begin{cases} (\phi_p(u'))' + a(t)\, f(t, u(t), u'(t)) = 0, \ t \in (0,1), \\ u'(0) = \sum\limits_{i=1}^{m-2} a_i u'(\xi_i), \\ u(1) = \sum\limits_{i=1}^{k} b_i u(\xi_i) - \sum\limits_{i=k+1}^{s} b_i u(\xi_i) - \sum\limits_{i=s+1}^{m-2} b_i u'(\xi_i), \end{cases} \tag{2.3}$$

where $1 \le k \le s \le m-2$. They have obtained the existence of solutions by using the monotone iterative technique in cones. They made the following assumptions:

(H1) $a_i,\, b_i > 0$ with $0 < \sum\limits_{i=1}^{k} b_i - \sum\limits_{i=k+1}^{s} b_i < 1,\ 0 < \sum\limits_{i=1}^{m-2} a_i < 1,$

$$0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1,$$

(H2) $f \in C([0,1] \times [0,+\infty) \times \mathbb{R}, \mathbb{R}^+)$, $a(t) \in C([0,1], [0,+\infty))$ is not identically zero on any compact subinterval of $(0,1)$. In addition,

$$0 < \int_0^1 a(t)\, dt < +\infty.$$

Set

$$A = \phi_p^{-1}\left((l+1) \int_0^1 a(t)\, dt \right) . \max\{\Sigma, 1\},$$

with

$$l = \frac{\phi_p\left(\sum\limits_{i=1}^{m-2} a_i\right)}{1 - \phi_p\left(\sum\limits_{i=1}^{m-2} a_i\right)}, \ \Sigma = \frac{\left(1 + \sum\limits_{i=k+1}^{s} b_i + \sum\limits_{i=s+1}^{m-2} b_i\right)}{1 - \sum\limits_{i=1}^{k} b_i + \sum\limits_{i=k+1}^{s} b_i}.$$

Lemma 1 *For $u \in C^+([0,1])$, there exists a unique Au such that $\phi_p^{-1}(Au) = \sum\limits_{i=1}^{m-2} a_i C(\xi_i, u(\xi_i))$, where*

$$C(t, u(t)) = \phi_p^{-1}\left(Au - \int_0^t a(r) f(r, u(r), u'(r) dr \right).$$

For $C^{+}([0,1]) = \{u \in C([0,1]),\ u \geq 0\}$,

$$\begin{aligned} Tu &= -D\left(\sum_{i=1}^{i=k} b_i \int_{\xi_i}^{1} C(s,u(s))ds\right) - \sum_{i=k+1}^{i=s} b_i \int_{\xi_i}^{1} C(s,u(s))ds \\ &\quad + \sum_{i=s+1}^{i=m-1} b_i C(\xi_i, u(\xi_i)) - \int_{t}^{1} C(s,u(s))ds \end{aligned}$$

where

$$D = \frac{1}{1 - \sum\limits_{i=1}^{k} b_i + \sum\limits_{i=k+1}^{s} b_i}.$$

Lemma 2 *If $u \in C^{+}([0,1])$ is a fixed point of T then u is a solution of bvp (2.3).*

Their main result is following theorem.

Theorem 3 *Suppose conditions (H1) and (H2) hold and there exists a constant $a > 0$ such that*

(B1) $f(t, x_1, y_1) \leq f(t, x_2, y_2)$ for any $t \in [0,1]$, $0 \leq x_1 \leq x_2 \leq a$, $0 \leq |y_1| \leq |y_2| \leq a$;

(B2) $\max\limits_{t\in[0,1]} f(t,a,a) \leq \phi_p(\frac{a}{A})$;

(B3) $f(t,0,0)$ is not identically zero on $[0,1]$.

Then BVP (2.3) has two positive solutions ω^*, v^* such that $0 < \omega^* \leq a$, $0 < |(\omega^*)'| \leq a$, and

$$\lim_{n\to\infty} T^n \omega_0 = \omega^*,\ \lim_{n\to\infty} (T^n \omega_0)' = (\omega^*)'$$

and

$$\lim_{n\to\infty} T^n v_0 = v^*,\ \lim_{n\to\infty} (T^n v_0)' = (v^*)'$$

where

$$v_0 = 0 \text{ and } \omega_0(t) = a(\Sigma - t),\ \text{for } t \in [0,1].$$

In 2008, Hairong Lian and Weigao Ge [26], investigated the following four-point boundary value problem with the p-Laplacian

$$\begin{cases} (\varphi_p(u'(t)))' + f(t,u(t)) = 0, \ t \in (0,1), \\ u(0) - \alpha u(\xi) = 0, \ u(1) + \beta u'(\eta) = 0, \end{cases} \tag{2.4}$$

where $\varphi_p(x) = |x|^{p-2}\, x, (p>1),\ \alpha,\ \beta > 0,\ 0 < \xi < \eta < 1$ and $f \in C\left([0,1] \times \mathbb{R}^+, \mathbb{R}^+\right)$. By using the Krasonsel'skii's fixed point theorem for the operator $T = LF$ in the cone

$$K = \left\{u \in X^+ : \ u \text{ is concave, non decreasing on } [0,\xi] \text{ and nonincreasing on } [\xi,1]\right\}$$

of the Banach space $X = C\left([0,1]\right)$ equipped with the norm $\|u\| = \max\limits_{t\in[0,1]} |u(t)|$, where $Fu = f(.,u)$,

$$X^+ = \{u \in X : \ u(t) \geq 0, \ \text{for } t \in [0,1]\}$$

and

$$Lu(t) = \begin{cases} \frac{1}{\alpha}\varphi_q\left(\int\limits_{\xi}^{\sigma} u(r)\,dr\right) + \int\limits_{0}^{t} \varphi_q\left(\int\limits_{s}^{\sigma} u(r)\,dr\right) ds & t \in [0,\sigma] \\ \frac{1}{\beta}\varphi_q\left(\int\limits_{\sigma}^{\eta} u(r)\,dr\right) + \int\limits_{t}^{1} \varphi_q\left(\int\limits_{\sigma}^{s} u(r)\,dr\right) ds & t \in [\sigma,1] \end{cases}$$

with $\varphi_q = \varphi_p^{-1}$ and $q = \frac{p}{1-p}$. Let $\sigma \in [\xi,\eta]$ such that $u'(\sigma) = 0$. With some conditions for f, they have shown that for $u \in K$,

$$Tu(t) \geq w(t)\,\|u\|, \ t \in [0,1], \ \text{where } w(t) = \min\left\{\frac{1}{\eta}t, \frac{1}{1-\xi}(1-t)\right\}.$$

Set

$$f^a = \limsup_{u\to a} \max_{t\in[0,1]} \frac{f(t,u)}{\varphi_p(u)}, \ \ f_a = \liminf_{u\to a} \min_{t\in[0,1]} \frac{f(t,u)}{\varphi_p(u)},$$

$$\begin{aligned} \frac{1}{m} &= \max_{\sigma\in[\xi,\eta]} \frac{1}{2}\left(\alpha(\sigma-\xi)^{q-1} + \frac{1}{q}\sigma^q + \beta(\eta-\sigma)^{q-1} + \frac{1}{q}(1-\sigma)^q\right) \\ \frac{1}{M} &= \min_{\sigma\in[\xi,\eta]} \frac{1}{2}\left(\alpha(\sigma-\xi)^{q-1} + \frac{1}{q}\left(\sigma - \frac{1}{k}\right)^q + \beta(\eta-\sigma)^{q-1} + \frac{1}{q}\left(1 - \frac{1}{k} - \sigma\right)^q\right) \end{aligned}$$

where $k > 2$ is large enough such that $\eta, \xi \in \left[\frac{1}{k}, 1 - \frac{1}{k}\right]$.

They have considered the following conditions :

(H1) There exists a positive number R such that $M_\gamma \leq \Gamma N_\gamma$ for all $\gamma \in [0, R]$,

(H2) $M_\gamma \leq \Gamma N_\gamma$ for all $\gamma > 0$,

where

$$\begin{aligned} M_\gamma &= max\{f(t,u), t \in [0,\xi] \cup [\eta, 1], u \in [0, R]\}, \\ N_\gamma &= min\{f(t,u), t \in [\xi, \eta], u \in [w^* R, R]\}, w^* = min\{w(\xi), w(\eta)\}, \end{aligned}$$

$$\Gamma = \left(\min \left\{ \frac{\beta q \left(\eta - \xi\right)^{q-1} + \left(\eta - \xi\right)^q}{\xi^q}, \frac{\alpha q \left(\eta - \xi\right)^{q-1} + \left(\eta - \xi\right)^q}{\left(1 - \eta\right)^q} \right\} \right)^{p-1}.$$

The following existence and multiplicité results of positive solutions are given;

Theorem 4 *Let (H1) hold. Suppose there exist* $0 < r,\ \rho < R$ *such that either* $0 < r < \frac{m}{M}\rho < \rho < R$ *or* $0 < \rho < \tilde{w}r < r < R$ *and that the following conditions hold.*

(H3) $f(t,u) \leq (m\rho)^{p-1}$, *for all* $t \in [0,1], u \in [0,\rho]$.

(H4) $f(t,u) > (Mr)^{p-1}$, *for all* $t \in [\frac{1}{k}, 1 - \frac{1}{k}], u \in [\tilde{w}\ r, r]$

Here $\tilde{w}\left(t\right) = \min\left\{w\left(\frac{1}{k}\right), w\left(1 - \frac{1}{k}\right)\right\}$. *Then the boundary value problem* (2.4) *has a positive solution* $u \in K$ *such that* $\|u\|$ *lies between* r *and* ρ.

Theorem 5 *Let (H2) hold. . Suppose that the following conditions hold.*

(H5) $f^0 < \varphi_p(m)$

(H6) $f_\infty > \varphi(\frac{M}{w})$.

Then the boundary value problem (2.4) *has a positive solution* $u \in K$.

Theorem 6 *Let (H2) hold. . Suppose that the following conditions hold.*

(H7) $f^\infty < \varphi_p(m)$

(H8) $f_0 > \varphi(\frac{M}{w})$.

Then the boundary value problem (2.4) *has a positive solution* $u \in K$.

In 2009, Chunmei Miao, Junfang Zhao and Weigao Ge [11], have given sufficient conditions for the existence of positive solutions of the four-point singular boundary value problem

$$\begin{cases} (\varphi_p(u'(t)))' + q(t)f(t,u(t),u'(t)) = 0, \ t \in (0,1), \\ u'(0) - \alpha u(\xi) = 0, \ u'(1) + \beta u(\eta) = 0, \end{cases} \tag{2.5}$$

where $\alpha, \beta > 0$, $0 < \xi < \eta < 1$, $q \in C([0,1])$, $q(t) > 0$ $t \in (0,1)$ and $f \in C([0,1] \times (0,+\infty) \times \mathbb{R}, (0,+\infty))$ may be singular at $u = 0$.

The following hypotheses are adopted ;

$(H1)$ $0 < \xi < \eta < 1$, $0 < \alpha \leq 1/\xi$, $0 < \beta \leq 1/(1-\eta)$, $q \in C[0,1], q(t) > 0, t \in (0,1)$;

$(H2)$ $f : [0,1] \times (0,+\infty) \times R \to (0,+\infty)$ is continuous, there are functions f_1, f_2 and h such that $0 < f(t,y,z) \leq h(z)[f_1(y) + f_2(y)]$ on $(0,1) \times (0,+\infty) \times R$

where f_1 is continuous, positive and nonincreasing on $(0,+\infty)$ and such that $\int_0^r f_1(t)\,dt < +\infty$, for all $r > 0$, f_2 is continuous, nonnegative and nondecreasing on $[0,+\infty)$ and h is continuous, positive and nondecreasing on $\mathbb{R}$;

$(H3)$ for given $H > 0$ and $L > 0$, there are a function $\psi_{H,L}$ and a constant $\gamma \in [0,1)$ such that $\psi_{H,L}$ is continuous on $[0,1]$, positive on $(0,1)$ and the inequality

$$f(t,y,z) > \psi_{H,L}(t)(\varphi_p(|z|))^\gamma$$

holds for $t \in [0,1]$, $y \in (0,H]$ and $z \in [-L,L]$;

$(H4)$ $I_1(x) = \int_0^x \left(\varphi_p^{-1}(u)\right)/h\left(\varphi_p^{-1}(u)\right) du < +\infty$, $x > 0$.

They have obtained the following Theorem:

Theorem 7 *Assume* $(H1) - (H4)$ *hold and*

$(H5)$ $\sup\limits_{c>0} \dfrac{c}{K\varphi_p^{-1}(j(c))} > 1,$ *where*

$$\begin{aligned} j(c) &= I_1^{-1}\left(|q|_0 f_2(M_0) M_0 + |q|_0 \int_0^c f_1(t)\,dt\right) \quad \textit{and} \\ K &= \max\left\{1+\frac{1}{\alpha}, 1+\frac{1}{\beta}\right\} \end{aligned}$$

then (2.5) *has a positive solution.*

The method used mainly depends on the theory of the Leray-Schauder degree in the Banach space $X = C^1([0,1])$ equipped with the norm $\|u\| = \max\{|u|_0, |u'|_0\}$, where $|u|_0 = \max\limits_{t\in[0,1]} u(t)$, of the completely continuous operator T defined by $Tu = LGu$, where $Gu(t) = b(t)f(t, u(t), u'(t))$ and

$$Lu(t) = \begin{cases} \frac{1}{\alpha}\varphi_p^{-1}\left(\int\limits_0^\sigma u(r)\,dr\right) + \int\limits_\xi^t \varphi_p^{-1}\left(\int\limits_s^\sigma u(r)\,dr\right) ds + \frac{A}{\alpha} & t\in[0,\sigma] \\ \frac{1}{\beta}\varphi_p^{-1}\left(\int\limits_\sigma^1 u(r)\,dr\right) + \int\limits_t^\eta \varphi_p^{-1}\left(\int\limits_\sigma^s u(r)\,dr\right) ds + \frac{A}{\alpha} & t\in[\sigma,1] \end{cases}$$

where $\sigma\in(0,1)$ such that

$$\begin{aligned} & \frac{1}{\alpha}\varphi_p^{-1}\left(\int_0^\sigma Gu(r)\,dr\right) + \int_\xi^\sigma \varphi_p^{-1}\left(\int_s^\sigma Gu(r)\,dr\right) ds \\ = {} & \frac{1}{\beta}\varphi_p^{-1}\left(\int_\sigma^1 Gu(r)\,dr\right) + \int_\sigma^\eta \varphi_p^{-1}\left(\int_\sigma^s Gu(r)\,dr\right) ds. \end{aligned}$$

They have obtained the fixed points of T, which are the positive solutions for a constructed nonsingular BVP

$$\begin{cases} (\varphi_p(u'(t)))' + q(t)f(t,u(t),u'(t)) = 0, \; t\in(0,1), \\ u'(0) - \alpha u(\xi) = -A, \; u'(1) + \beta u(\eta) = \frac{\beta}{\alpha}A, \end{cases} \tag{2.6}$$

for $A > 0$. Then by using the Arzelà-Ascoli theorem, they obtained positive solutions for the singular problem which is approximated by the family of solutions to the nonsingular BVP (2.6).

In 2010, Xiangfeng Li and Wanying Xu [77] studied the existence of two positive solutions for p-Laplace equation with sign-changing nonlinear terms

$$(\varphi_p(u'(t)))' + a(t)f(t, u(t), u'(t)) = 0, \ \ t \in (0,1), \tag{2.7}$$

subject to one of the following boundary conditions

$$u(0) - B_1 u'(\eta) = 0, u'(1) = 0 \tag{2.8}$$

or

$$u'(0) = 0, u(1) + B_2 u'(\eta) = 0 \tag{2.9}$$

where $\varphi_p(x) = |x|^{p-2} x$, $(p > 1)$, $\eta < \frac{1}{2}$ and under the following conditions,

(H1) $f \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$,

(H2) $a \in C([0,1], \mathbb{R}^+)$, $f(t,0,.) \geq 0$, $a(t).f(t,.,.)$ dont vanish identiquely in any subinterval of $(0,1)$ and $\int\limits_{\eta}^{1-\eta} a(t)dt > 0$.

(H3) B_1, B_2 are increasing, continuous and odd fuctions in $\mathbb{R}$ verifying :

$$\exists l, L \geq 0, \text{ such that } \ lx \leq B_i \leq Lx, x \geq 0, i = 1, 2.$$

By using the theorem of fixed point index for the operators $L, \hat{A}$ defined in the positive cone of the Banach space $E = C([0,1])$ (induced by the sup-norm) by

$$\begin{aligned} Lu(t) &= B_1\left(\varphi_q\left(\int\limits_{\eta}^{1} a(s)f(s,u(s),u'(s))ds\right)\right) + \int\limits_{0}^{t} \varphi_q\left(\int\limits_{s}^{1} a(r)f(r,u(r),u'(r))dr\right) ds, \\ \hat{A}u(t) &= B_2\left(\varphi_q\left(\int\limits_{0}^{\eta} a(s)f(s,u(s),u'(s))ds\right)\right) + \int\limits_{t}^{1} \varphi_q\left(\int\limits_{0}^{s} a(r)f(r,u(r),u'(r))dr\right) ds \end{aligned}$$

where $\varphi_q = \varphi_p^{-1}$, $q = \frac{p}{p-1}$, they have obtained the following results:

Theorem 8 *Assume that (H1), (H2) and (H3) hold, and suppose that there exist* $a, b, d,$ $0 < \frac{d}{\eta} < a < b < 1$. *Also assume that* f *satifies :*

(H4) $f(t,x,y) \geq 0$, $\forall\,(t,x,y) \in [\eta, 1-\eta] \times [d,b] \times \mathbb{R}$

(H5) $f(t,x,y) < \varphi_p\left(\frac{a}{M}\right)$, $\forall\,(t,x,y) \in [0,1] \times [0,a] \times \mathbb{R}$

(H6) $f(t,x,y) > \varphi_p\left(\frac{b}{N}\right)$, $\forall\,(t,x,y) \in [\eta, 1-\eta] \times [\eta b, b] \times \mathbb{R}$

where

$$\begin{aligned} M &= L.\varphi_q\left(\int_\eta^1 a(r)dr\right) + \int_0^1 \varphi_q\left(\int_s^1 a(r)dr\right), \\ N &= l.\varphi_q\left(\int_\eta^{1-\eta} a(r)dr\right) + \int_\eta^{1-\eta} \varphi_q\left(\int_s^{1-\eta} a(r)dr\right). \end{aligned}$$

Then BVP $(2.7)-(2.8)$ *has at least txo positive solutions* u_1, u_2 *such that* $0 < \|u_1\| < a < \|u_2\| < b$.

Theorem 9 *Assume that (H1), (H2) and (H3) hold, and suppose that there exist* $a, b, d,\ 0 < \frac{d}{\eta} < a < b < 1$. *Also assume that* f *satisfies :*

(H7) $f(t,x,y) \geq 0$, $\forall\,(t,x,y) \in [\eta, 1-\eta] \times [d,b] \times \mathbb{R}$

(H8) $f(t,x,y) < \varphi_p\left(a/\hat{M}\right)$, $\forall\,(t,x,y) \in [0,1] \times [0,a] \times \mathbb{R}$

(H9) $f(t,x,y) > \varphi_p\left(b/\hat{N}\right)$, $\forall\,(t,x,y) \in [\eta, 1-\eta] \times [\eta b, b] \times \mathbb{R}$

where

$$\begin{aligned} \hat{M} &= L.\varphi_q\left(\int_0^\eta a(r)dr\right) + \int_0^1 \varphi_q\left(\int_s^1 a(r)dr\right), \\ \hat{N} &= l.\varphi_q\left(\int_\eta^{1-\eta} a(r)dr\right) + \int_\eta^{1-\eta} \varphi_q\left(\int_\eta^{s} a(r)dr\right). \end{aligned}$$

Then BVP $(2.7)-(2.9)$ *has at least two positive solutions* u_1, u_2 *such that* $0 < \|u_1\| < a < \|u_2\| < b$.

In 2011, Donglong Bai and Hanying Feng [16] provided sufficient conditions for the existence of at least three positive solutions of the multipoint boundary value problem for the one-dimensional p-Laplacian

$$(\phi_p(u'(t)))' + q(t)f(t, u(t), u'(t)) = 0, t \in (0,1), \tag{2.10}$$

subject to the boundary conditions

$$u(0) = \sum_{i=1}^{i=m-2} a_i u(\xi_i), \ u'(1) = \beta u'(0) \tag{2.11}$$

where $\xi_i \in (0,1)$ with $\xi_1 < \xi_2 < ... < \xi_{m-2}$ with the following assumptions:
$(H1)$ $a_i \in [0,1)$ satisfies $\sum\limits_{i=1}^{i=m-2} a_i < 1$, $\beta \in (0,1)$;
$(H2)$ $f \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$;
$(H3)$ $q \in L^1(0,1)$ is nonegative on (0,1) and q is not identically zero on any subinterval of $(0,1)$. Furthermore, q satisfies $0 < \int\limits_0^1 q(t)dt < \infty$.

By using a fixed point theorem due to Avery and Peterson [64] in the Banach space $E = C^1([0,1])$ endowed with the norm $\|u\| = \max\{\max\limits_{t\in[0,1]} u(t), \max\limits_{t\in[0,1]} u'(t)\}$, they show the existence of three fixed points of the operator T defined in the cone

$$P = \{u \in E : u(t) \geq 0, u(0) = \sum_{i=1}^{i=m-2} a_i u(\xi_i), u'(1) = \beta u'(0), u \text{ is concave on } [0,1]\}$$

by $Tu = BFu(t)$ for $t \in [0,1]$, where $Fu(t) = f(t, u(t), u'(t))$ and

$$\begin{aligned} Bu(t) &= \int_0^t \left[\phi_p^{-1}\left(\int_s^1 q(r)u(r)dr\right) + \frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(r)u(r)dr\right] ds \\ &+ \frac{1}{\left(1 - \sum\limits_{i=1}^{i=m-2} a_i\right)} \sum_{i=1}^{i=m-2} a_i \int_0^{\xi_i} \left[\phi_p^{-1}\left(\int_s^1 q(r)u(r)dr\right) + \frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_0^1 q(r)u(r)dr\right] ds. \end{aligned}$$

For convenience they introduce following notation

$$K = \frac{4k^2}{(k\beta + \beta + 3k - 1)\left(1 - \sum\limits_{i=1}^{i=m-2} a_i\right) + (k^2 + k)\sum\limits_{i=1}^{i=m-2} a_i \left[2\xi_i - (1-\beta)\xi_i^2\right]}$$

with $k > \max\left\{\frac{1}{\xi_1}, \frac{1}{1-\xi_{m-2}}\right\}$ such that

$$0 < \int_{\frac{1}{k}}^{1-\frac{1}{k}} q(t)dt < \infty$$

$$\begin{aligned}
L &= \phi_p^{-1}\left(\frac{1}{1-\phi_p(\beta)}\int_0^1 q(r)dr\right), \\
M &= \int_{\frac{1}{k}}^{1-\frac{1}{k}}\left[\phi_p^{-1}\left(\int_s^{1-\frac{1}{k}} q(r)dr\right) + \frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_{\frac{1}{k}}^{1-\frac{1}{k}} q(r)dr\right]ds \\
&\quad + \frac{1}{\left(1-\sum\limits_{i=1}^{i=m-2} a_i\right)}\sum_{i=1}^{i=m-2} a_i \int_{\frac{1}{k}}^{\xi_i}\left[\phi_p^{-1}\left(\int_s^{1-\frac{1}{k}} q(r)dr\right) + \frac{\phi_p(\beta)}{1-\phi_p(\beta)}\int_{\frac{1}{k}}^{1-\frac{1}{k}} q(r)dr\right]ds \\
N &= B\theta(1),\ \text{with } \theta \equiv 1,\ \text{and } \bar{M} = 1 + \frac{\sum\limits_{i=1}^{i=m-2} a_i\xi_i}{1-\sum\limits_{i=1}^{i=m-2} a_i}.
\end{aligned}$$

They have obtained following result

Theorem 10 *Assume that (H1)–(H3) hold. Let* $0 < a < b \leq min\left\{\frac{1}{k}, \frac{\bar{M}}{1-k}\right\} d$, *and suppose that* f *satisfies the following conditions:*

(A1) $f(t,x,y) \leq \varphi_p(d/L)$, *for* $(t,x,y) \in [0,1] \times \left[0, \bar{M}d\right] \times [0,d]$

(A2) $f(t,x,y) \geq \varphi_p(kb/M)$, *for* $(t,x,y) \in \left[\frac{1}{k}, 1-\frac{1}{k}\right] \times [b,kb] \times [0,d]$

(A3) $f(t,x,y) < \varphi_p(a/N)$, *for* $(t,x,y) \in [0,1] \times [0,a] \times [0,d]$

Then the boundary-value problem $(2.10)-(2.11)$ *has at least three positive solutions* u_1, u_2, u_3 *such that* $\max\limits_{t\in[0,1]} |u_i'(t)| \leq d$, *for* $i = 1,2,3$, *and*

$$b < \min_{t\in\left[\frac{1}{k},1-\frac{1}{k}\right]} |u_1(t)|, \max_{t\in[0,1]} |u_1(t)| \leq \bar{M}d, a < \max_{t\in[0,1]} |u_2(t)| \leq kb$$

with

$$\min_{t\in\left[\frac{1}{k},1-\frac{1}{k}\right]} |u_2(t)| < b, \max_{t\in[0,1]} |u_1(t)| \leq \bar{M}d, a < \max_{t\in[0,1]} |u_3(t)| \leq a.$$

The operator $-\Delta_{p(t)}u = -\operatorname{div}\left(|\nabla u|^{p(t)-2}\nabla u\right)$ with $p(t) > 1$ is called the $p(t)-$Laplacian which is a natural generalization of the $p-$Laplacian. The p(t)-Laplacian can be applied to describe the physical phenomena with "pointwise different properties " which first arose from the nonlinear elasticity theory (see [82]). Differential equations involving the $p(t)$-Laplacian have received a great attention in recent years and many interesting results have been obtained (see for example [60], [57], [79], [20], [75], [78], [67] and [68]).

The $p(t)-$Laplacian possesses more complicated nonlinearity than $p-$Laplacian, say, it is inhomogeneous. For this reason, some of above properties of the $p-$Laplacian problems may not hold for a general $p(t)-$Laplacian case.

In 2008, Qihu Zhanga, Zheimei Qiu and Xiaopin Liu [56] considered the following weighted p(t) -Laplacian system:

$$\begin{cases} -\Delta_{p(t),w(t)}u + \delta f(t, u, (w(t))^{\frac{1}{p(t)-1}} u') = 0, \ t \in (0,1), \\ u(0) = \sum\limits_{i=1}^{m-2} \beta_i u(\eta_i) + e_0, \ u(1) = \sum\limits_{i=1}^{m-2} \alpha_i u(\xi_i) + e_1, \end{cases} \tag{2.12}$$

where $p \in C([0,1], \mathbb{R})$, $p(t) > 1$, $-\Delta_{p(t),w(t)}u = -(w(t)(|u'|^{p(t)-2} u'(t)))'$ is called the weighted $p(t)$ -Laplacian; $w \in C([0,1], \mathbb{R})$ satisfies $w(0) > 0$ for all $t \in (0,1)$ and $(w(t))^{\frac{-1}{p(t)-1}} \in L^1(0,1)$; $0 < \eta_1 < ... < \eta_{m-2} < 1$, $0 < \xi_1 < ... < \xi_{m-2} < 1$; $\alpha_i \geq 0$, $\beta_i \geq 0$ $(i = 1, ..., m-2)$, and $0 < \sum\limits_{i=1}^{m-2} \beta_i < 1$, $0 < \sum\limits_{i=1}^{m-2} \alpha_i < 1$, $e_0, e_1 \in \mathbb{R}^N$, $N > 0$; δ is a positive parameter.

Let $N \geq 1$, $I = [0,1]$, the function $f = (f_1, ..., f_N) : I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Caratheodory, i.e.

(i) For almost every $t \in I$ the function $f(t, ., .)$ is continuous;

(ii) For each $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, the function $f(., x, y)$ is measurable on I;

(iii) For each $R > 0$ there exists $\beta_R \in L^1(I, \mathbb{R})$ such that, for almost every $t \in I$, and every $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leq R$, $|y| \leq R$ one has;

$$|f(t,x,y)| \leq \beta_R(t).$$

Let $g : \mathbb{R}^n \to \mathbb{R}$. We say that g satisfies sub-$p^- - 1$ growth condition, if

$$\lim_{\|x\|_n \to +\infty} \frac{g(t,x)}{(\|x\|_n)^{q(t)-1}} = 0.$$

where $\|x\|_n = \sum_{k=1}^{k=n} |x_i|$, with $x_i \in \mathbb{R}$, for $x \in \mathbb{R}^n$.

In the following theorem, they applied Leray-Schauder's degree to prove existence of solutions for (2.12), when f satisfies sub-$p^- - 1$ growth condition.

Theorem 11 *If f satisfies sub-$p^- - 1$ growth condition, then for any fixed parameter δ, Problem* (2.12) *has at least one solution.*

In 2009, Qihu Zhanga, Zheimei Qiu and Xiaopin Liu [55] studied the existence of multiple solutions for weighted p(r)-Laplacian equation

$$-(w(r)(|u'|^{p(r)-2} u'(r)))' + f(r, u, (w(r))^{\frac{1}{p(r)-1}} u') = 0, \; r \in (T_1, T_2), \tag{2.13}$$

with

$$u(T_1) = 0 = u(T_2), \tag{2.14}$$

where $p; w \in C(I, \mathbb{R})$, $I = [T_1, T_2]$, $p(r) > 1$, $0 < w(r)$ for any $r \in (T_1, T_2)$ and $(w(r))^{\frac{1}{p(r)-1}} \in L^1(T_1, T_2)$, $-\Delta_{p(r)} u(r) = -(w(r)(|u'|^{p(r)-2} u'(r)))'$ is called weighted p(r)-Laplacian.

The function $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to be Carathéodory.

We set

$$C^1 = \left\{ \begin{array}{c} u \in C(I, \mathbb{R}), \; u' \in C((T_1, T_2), \mathbb{R}), \; \lim\limits_{t \to T_1^+} w(r)(|u'|^{p(r)-2} u'(r)) \\ = \lim\limits_{t \to T_2^-} w(r)(|u'|^{p(r)-2} u'(r)) \end{array} \right\}$$

the Banach space equipped with the norme $\|u\| = \|u\|_0 + \left\| (w(r))^{\frac{1}{p(r)-1}} u' \right\|_0$ where $\|u\|_0 = \sup_{t \in [T_1, T_2]} u(t)$.

We say a function u is a solution of (2.13), (2.14) , if $u \in C^1$ and $w(r)(|u'|^{p(r)-2}\, u'(r))$ is absolutely continuous and satisfies (2.13) and (2.14).

Functions $\alpha, \beta \in C^1$ are called subsolution and supersolution of (2.13) respectively, if

$w(r)(|\alpha'|^{p(r)-2}\, \alpha'(r))$ and $w(r)(|\beta'|^{p(r)-2}\, \beta'(r))$ are absolutely continuous and satisfy

$-\left(w(r)(|\alpha'|^{p(r)-2}\, \alpha'(r))\right)' + f(r, \alpha, (w(r))^{\frac{1}{p(r)-1}}\, \alpha') \leq 0$, a.e. in I,

$-\left(w(r)(|\beta'|^{p(r)-2}\, \beta'(r))\right)' + f(r, \beta, (w(r))^{\frac{1}{p(r)-1}}\, \beta') \geq 0$, a.e. in I.

Denote

$\Omega_0 = \{(t,x) \setminus t \in I,\ x \in [\alpha_1(t), \beta_2(t)]\}$,

$\Omega_1 = \{(t,x,y) \setminus\ (t,x) \in \Omega_0,\ y \in \mathbb{R}\}$,

where α_1 is a subsolution and β_2 supersolution of (2.13), such that $\alpha_1 \leq \beta_2$.

Assume that f satisfies the following conditions :

(H1) $|f(t,x,y)| \leq A_1(t,x)\, K_1(t,x,y) + A_2(t,x)\, K_2(t,x,y)$ for all $(t,x,y) \in \Omega_1$ where A_i are positive valued and continuous on Ω_0, K_i are positive valued and continuous on Ω_1 $(i = 1, 2)$.

(H2) There exist positive numbers M_1, M_2 such that

$K_1(t,x,y) \leq |y|\, h(|y|)$, $K_2(t,x,y) \leq M_1 h(|y|)\ \ \forall\, |y| \geq M_2$,

where $h \in C([1, +\infty), [1, +\infty))$ is increasing and satisfies $\int\limits_1^{+\infty} \dfrac{1}{h\left(|y|^{\frac{1}{p^- -1}}\right)} dy = \infty$, where $p^- = \min_{r \in I} p(r)$.

The main results are as follow

Theorem 12 *If f is Carathéodory and satisfies $(H1)$ and $(H2)$ on Ω_0 and Ω_1, α_1 and α_2 are subsolutions, β_1 and β_2 are supersolutions , which satisfy*

(i) $\alpha_i(T_j) \leq 0 \leq \beta_i(T_j)$, $i, j = 1, 2$,

(ii) $\alpha_1 \leq \alpha_2 \leq \beta_2$, $\alpha_1 \leq \beta_1 \leq \beta_2$,

(iii) $\alpha_2 \nleq \beta_1$,

then Problem (2.13), (2.14) *has at least three solutions* y_1, y_2, *and* y_3 *satisfying*

$$\alpha_1 \le y_1 \le \beta_1, \alpha_2 \le y_2 \le \beta_2,\ \alpha_1 \le y_3 \le \beta_2.$$

Theorem 13 *If f is Carathéodory and satisfies $(H1)$ and $(H2)$ on Ω_0 and Ω_1, α_1 and α_2 are subsolutions, β_1 and β_2 are supersolutions , which satisfy*

(i) $\alpha_i\left(T_j\right) < 0 < \beta_i\left(T_j\right),\ i,j = 1,2,$

(ii) $\alpha_1 \le \alpha_2 \le \beta_2,\ \alpha_1 \le \beta_1 \le \beta_2,$

(iii) $\alpha_2 \nleq \beta_1,$

(iv) If y is a solution of (2.13) *with $y \ge \alpha_2$ then $y > \alpha_2$ on* $\left(T_1, T_2\right),$

and

(v) If y is a solution of (2.13) *with $y \le \beta_1$ then $y < \beta_1$ on* $\left(T_1, T_2\right).$

then Problem (2.13), (2.14) *has at least three solutions y_1, y_2, and y_3 satisfying*

$$\alpha_1 \le y_1 \le \beta_1, \alpha_2 \le y_2 \le \beta_2,\ y_3 \nleq \beta_1 \text{ and } y_3 \ngeq \alpha_2.$$

Theorem 14 *If α_i are subsolutions and β_i are supersolutions $(i = 1, ..., N)$ of* (2.13), *which satisfy*

(i) $\alpha_i\left(T_j\right) \le 0 \le \beta_i\left(T_j\right),\ i,j = 1,...,N,$

(ii) $\alpha_i \le \beta_i \le \beta_{i+1}$ *and* $\alpha_i \le \alpha_{i+1} \le \beta_{i+1},\ i = 1,...,N$

(iii) $\alpha_{i+1} \nleq \beta_i,\ i = 1,...,N$

f is Carathéodory and satisfies $(H1)$ and $(H2)$ on Ω_0^* and Ω_1^* where

$\Omega_0^* = \left\{(t,x) \setminus t \in I,\ x \in \left[\alpha_1(t), \beta_N(t)\right]\right\},$

$\Omega_1^* = \left\{(t,x,y) \setminus\ (t,x) \in \Omega_0^*,\ y \in \mathbb{R}\right\}.$

Then Problem (2.13), (2.14) *has at least $2N-1$ solutions.*

In 2010, Guizhen Zhi, Liang Zhao, Guangxia Chen, Shujuan Wang and Qihu Zhang [21] considered the existence of solutions for the weighted $p(r)$ -Laplacian integro-differential system

$$-(w(r)(|u'|^{p(r)-2}u'(r)))' + f(r,u,(w(r))^{\frac{1}{p(r)-1}}u', S(u)) = 0,\ r \in (0,T),\ r \neq r_i \quad (2.15)$$

where $u:[0,1]\to\mathbb{R}^N$, with the following impulsive boundary value conditions:

$$\lim_{r\to r_i^+} u(r) - \lim_{r\to r_i^-} u(r) = A_i\left(\lim_{r\to r_i^-} u(r),\ \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)\right),\ i=1,...,k, \quad (2.16)$$

$$\begin{aligned} &\lim_{r\to r_i^+} (w(r))^{\frac{1}{p(r)-1}} u'(r) - \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r) && (2.17)\\ =\ & B_i\left(\lim_{r\to r_i^-} u(r),\ \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)\right),\ i=1,...,k, && (2.18)\end{aligned}$$

$$u(0)=u(T),\ \lim_{r\to 0^+} w(r)(|u'|^{p(r)-2}u'(r)) = \lim_{r\to T^-} w(r)(|u'|^{p(r)-2}u'(r)), \quad (2.19)$$

where $p;w\in C(J,\mathbb{R})$, $J=[0,T]$, $p(r)>1$, for any $r\in(0,T)$ and $(w(r))^{\frac{1}{p(r)-1}}\in L^1(0,T)$, $0<r_1<...<r_k<T$, $A_i, B_i\in C\left(\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R}^N\right)$; $S(u)(t)=\int_0^T \phi(s,t)\,u(s)\,ds$,

where $0\le\phi(.,.)\in C(J\times J,\mathbb{R})$. The function $f: J\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is assumed to be Carathéodory.

Denote

$$\begin{aligned} D_i &= \varphi\left(r_i,\lim_{r\to r_i^-}(w(r))^{\frac{1}{p(r)-1}}u'(r)+B_i\left(\lim_{r\to r_i^-}u(r),\lim_{r\to r_i^-}(w(r))^{\frac{1}{p(r)-1}}u'(r)\right)\right)\\ &\quad -\varphi\left(r_i,\lim_{r\to r_i^-}(w(r))^{\frac{1}{p(r)-1}}u'(r)\right)\end{aligned}$$

where $\varphi(r,y)=|y|^{p(r)-2}y$.

$PC^1=\left\{u\in PC|\ u'\in C\left(\mathring{J}_i\right),\ \lim_{r\to r_i^+}(w(r))^{\frac{1}{p(r)-1}}u'(r) \text{ and } \lim_{r\to r_{i+1}^-}(w(r))^{\frac{1}{p(r)-1}}u'(r) \text{ exist}\right\}$,

with $PC=\left\{u\in C(J_i)\mid u\left(r_i^+\right) \text{ exists } i=1,...,k\right\}$

where $u\left(r_i^+\right)=\lim_{r\to r_i^+}u(r)$, $J_0=[r_0,r_1]$, $J_i=]r_i,r_{i+1}]$.

Theorem 15 *Assume that Ω is an open bounded set in PC^1 such that the following conditions hold.*

(1°)

$$-(w(r)(|u'|^{p(r)-2}\,u'(r)))' + \lambda f(r, u, (w(r))^{\frac{1}{p(r)-1}}\,u', S\,(u)) = 0,\ \ r \in (0,T),\ \ r \neq r_i \tag{2.20}$$

$$\lim_{r\to r_i^+} u\,(r) - \lim_{r\to r_i^-} u\,(r) = \lambda A_i\left(\lim_{r\to r_i^-} u\,(r)\,,\ \lim_{r\to r_i^-}\ (w(r))^{\frac{1}{p(r)-1}}\,u'\,(r)\right),\ \ i = 1,...,k, \tag{2.21}$$

$$\lim_{r\to r_i^+}\ (w(r))^{\frac{1}{p(r)-1}}\,u'\,(r) - \lim_{r\to r_i^-}\ (w(r))^{\frac{1}{p(r)-1}}\,u'\,(r) \tag{2.22}$$

$$= \ \lambda D_i\left(\lim_{r\to r_i^-} u\,(r)\,,\ \lim_{r\to r_i^-}\ (w(r))^{\frac{1}{p(r)-1}}\,u'\,(r)\right),\ \ i = 1,...,k, \tag{2.23}$$

$$u\,(0) = u\,(T)\,,\ \ \lim_{r\to 0^+} w(r)(|u'|^{p(r)-2}\,u'(r)) = \lim_{r\to T^-} w(r)(|u'|^{p(r)-2}\,u'(r)), \tag{2.24}$$

has no solution on $\partial\Omega$.

(2°) *The equation*

$$\omega\,(l) = \frac{1}{T}\int_0^T f\,(r,l,0,S\,(l))\,dr + \frac{1}{T}\sum_{i=1}^{i=k} D_i\,(l,0) = 0, \tag{2.25}$$

has no solution on $\partial\Omega \cap \mathbb{R}^N$.

(2°) *The Brouwer degree* $d_B\left[\omega, \Omega\cap\mathbb{R}^N, 0\right] \neq 0$.

Then $(2.15)\,,\ (2.16)\,,\ (2.17)$ *and* (2.19) *has a solution on* $\bar{\Omega}$.

The methods in this theorem is based on the Leray-Schauder degree properties.

In 2011, Li Yin, Yunrui Guo, Guizhen Zhi and Qihu Zhang [42], considered the existence of solutions for the weighted p(r)-Laplacian system :

$$-(w(r)(|u'|^{p(r)-2}\,u'(r)))' + f(r, u, (w(r))^{\frac{1}{p(r)-1}}\,u') = 0,\ \ r \in (0,T),\ \ r \neq r_i \tag{2.26}$$

where $u : [0,1] \to \mathbb{R}^N$, with the following impulsive boundary value conditions:

$$\lim_{r\to r_i^+} u\,(r) - \lim_{r\to r_i^-} u\,(r) = A_i\left(\lim_{r\to r_i^-} u\,(r)\,,\ \lim_{r\to r_i^-}\ (w(r))^{\frac{1}{p(r)-1}}\,u'\,(r)\right),\ \ i = 1,...,k, \tag{2.27}$$

$$\lim_{r\to r_i^+} w(r)(|u'|^{p(r)-2}\,u'(r)) - \lim_{r\to r_i^-} w(r)(|u'|^{p(r)-2}\,u'(r)) \qquad (2.28)$$

$$= \quad B_i\left(\lim_{r\to r_i^-} u\left(r\right), \lim_{r\to r_i^-}\left(w(r)\right)^{\frac{1}{p(r)-1}} u'\left(r\right)\right),\ i=1,...,k, \qquad (2.29)$$

$$au\left(0\right) - b\lim_{r\to 0^+}\left(w(r)\right)^{\frac{1}{p(r)-1}} u'\left(r\right) = 0,\ cu(T) + d\lim_{r\to T^-} w(r)(|u'|^{p(r)-2}\,u'(r)) = 0, \quad (2.30)$$

where $p; w \in C\left(J,\mathbb{R}\right),\ J=\left[0,T\right],\ p\left(r\right)>1$, for any $r\in(0,T)$ and $\left(w(r)\right)^{\frac{1}{p(r)-1}} \in L^1(0,T),\ 0<r_1<...<r_k<T,\ A_i,\ B_i\in C\left(\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R}^N\right);\ a,b,c,d\in[0,+\infty)$, $ad+dc>0$.

The function $f : J\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is assumed to be Carathéodory.

For $\beta\in C\left(J,\mathbb{R}\right),\ \beta^- = \inf\left\{\beta,\ t\in J\right\}$ and $\beta^+ = \sup\left\{\beta,\ t\in J\right\}$.

Consider the space $X=\left\{x=\left(x_1,x_2\right),\ x_1,\in PC,\ x_2\in PC\right\}$ with the norm $\|x\| = \|x_1\|_0 + \|x_2\|_0$ where$\|u\|_0 = \sup_{t\in J} u(t)$.

$PC = \left\{u\in C\left(J_i\right) \mid u\left(r_i^+\right) \text{ exists } i=1,...,k\right\}$ with $u\left(r_i^+\right) = \lim_{r\to r_i^+} u\left(r\right),\ J_0 = \left[r_0,r_1\right]$ and $J_i = \left]r_i, r_{i+1}\right]$. Let

$$\begin{aligned} Sx &= \left(\varphi_{q(r)}\left(x_2/w\left(r\right)\right), f\left(r,x_1,\varphi_{q(r)}\left(x_2\right)\right), A_i, B_i, ax_1\left(0\right) - b\varphi_{q(0)}\left(x_2\left(0\right)\right), cx_1(T)+dx_2(T)\right), \\ Lx &= \left(x_1,x_2,\Delta x_1\left(r_i\right),\Delta x_2\left(r_i\right)\right) \end{aligned}$$

where $\varphi_{q(r)}\left(x\right) = |x|^{q(r)}.x$ with $q=\frac{p}{p-1}$, $\Delta x\left(r_i\right) = \lim_{r\to r^+} x\left(r\right) - \lim_{r\to r^-} x\left(r\right)$.

By using the properties of coincidence degree of (L,S) in the given open subset of X, they obtained the following results

Theorem 16 *Assume that the following conditions hold*

(1) $a>0$;

(2) $\lim_{|u|+|v|\to+\infty} f\left(r,u,v\right)/\left(|u|+|v|\right)^{\beta(r)-1} = 0$ *for* $r\in J$ *uniformly, where* $\beta\in C\left(J,\mathbb{R}\right)$, *and* $1<\beta^-<\beta^+<p^-$;

(3) $\sum\limits_{i=1}^{i=k} |A_i(u,v)| \leq C_1(|u|+|v|)^{\theta}$ *where* $(|u|+|v|)$ *is large enough, where* $0 < \theta < \frac{p^- -1}{p^+ -1}$;

(4) $\sum\limits_{i=1}^{i=k} |B_i(u,v)| \leq C_2(|u|+|v|)^{\varepsilon}$ *where* $(|u|+|v|)$ *is large enough, where* $0 < \varepsilon < \beta^+ - 1$.

Then, Problem $(2.26)-(2.30)$ *has at least one solution.*

Theorem 17 *Assume that the following conditions hold*

(1) $a = 0$;

(2) $\lim\limits_{|u|+|v|\to+\infty} f(r,u,v)/(|u|+|v|)^{\varepsilon} = 0$ *for* $r \in J$ *uniformly, where* $0 \leq \varepsilon < \min(1, p^- - 1)$;

(3) $\sum\limits_{i=1}^{i=k} |A_i(u,v)| \leq C_1(|u|+|v|)^{\theta}$ *where* $(|u|+|v|)$ *is large enough, where* $0 < \theta < 1$;

(4) $\sum\limits_{i=1}^{i=k} |B_i(u,v)| \leq C_2(|u|+|v|)^{\varepsilon}$ *where* $(|u|+|v|)$ *is large enough, where* $0 < \varepsilon < \beta^+ - 1$.

Then, Problem $(2.26)-(2.30)$ *has at least one solution.*

In 2013, Rong Dong, Yunrui Guo and Qihu Zhang [63], considered the existence of solutions for the weighted p(r)-Laplacian system

$$-\Delta_{p(t)}u + f(t,u,(w(t))^{\frac{1}{p(t)-1}}u', S(u), T(u)) = 0, \; t \in (0,1), \; t \neq t_i \tag{2.31}$$

where $u : [0,1] \to \mathbb{R}^N$, with the following impulsive boundary value conditions:

$$\lim_{r\to r_i^+} u(r) - \lim_{r\to r_i^-} u(r) = A_i\left(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)\right), \; i = 1,...,k, \tag{2.32}$$

$$\lim_{r\to r_i^+} w(r)(|u'|^{p(r)-2}u'(r)) - \lim_{r\to r_i^-} w(r)(|u'|^{p(r)-2}u'(r)) \tag{2.33}$$

$$= B_i\left(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{\frac{1}{p(r)-1}} u'(r)\right), \; i = 1,...,k, \tag{2.34}$$

$$u(0)=0,\ u(1)=\int_0^1 g(t)u(t)dt, \tag{2.35}$$

where $\Delta_{p(t)}u = (w(r)(|u'|^{p(r)-2}u'(r)))'$, $p; w \in C(J,\mathbb{R})$, $J=[0,1]$, $p(r)>1$, for any $r\in(0,1)$ and $(w(r))^{\frac{1}{p(r)-1}} \in L^1(0,T)$, $0<r_1<...<r_k<1$, A_i, $B_i \in C\left(\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R}^N\right)$, $\int_0^1 g(t)dt=\sigma$, $\sigma\in J$,

$S(u)(t) = \int_0^t k_*(t,s)u(s)ds$ and $S(u)(t) = \int_0^t h_*(t,s)u(s)ds$, $t\in[0,1]$ where k_*, $h_*\in C(J\times J,\mathbb{R})$.

The function $f : J\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Carathéodory. For $\beta\in C(J,\mathbb{R})$, $\beta^- = \inf\{\beta,\ t\in J\}$ and $\beta^+ = \sup\{\beta,\ t\in J\}$.

By homotopy transformation of the Leray-Schauder degree, they have given the following result

Theorem 18 *If f satisfies sub-$(p^- - 1)$ growth condition, we also assume that*

$$\begin{aligned}
\sum_{i=1}^{i=k}|A_i(u,v)| &\le C_1(1+|u|+|v|)^{\dfrac{q^+-1}{p^+-1}}, \\
\sum_{i=1}^{i=k}|B_i(u,v)| &\le C_2(1+|u|+|v|)^{q^+-1},\ \forall(u,v)\in\mathbb{R}^N\times\mathbb{R}^N,
\end{aligned}$$

then Problem $(2.31)-(2.35)$ *has at least one solution.*

Chapter 3

EIGENVALUE CRITERIA FOR EXISTENCE OF A POSITIVE SOLUTION TO A SINGULAR THREE POINT BVP

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3.1 Introduction

In this paper we study existence and nonexistence of positive solutions to the singular three point boundary value problem (BVP for short)

$$\begin{cases} -\left(au'\right)'(t) = b(t)f(t,u(t)), \quad t \in (0,1), \\ u'(0) = 0, \ u(1) = \alpha u(\eta), \end{cases} \tag{3.1}$$

where $\mathbb{R}^+ = [0, +\infty)$, $\alpha, \eta \in (0,1)$, $a, b \in C((0,1), \mathbb{R}^+)$, $a > 0$ in $(0,1)$, $\dfrac{1}{a}$ is integrable on any compact subset of $(0,1]$, b does not vanish identically and is integrable on any compact subset of $[0,1)$ and $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. By a *positive solution* to BVP (3.1), we mean a function $u \in C([0,1], \mathbb{R}^+) \cap C^1((0,1), \mathbb{R}^+)$ with $u(t) > 0$ on $[0,1]$ satisfying both the differential equation and the boundary conditions in (3.1).

Throughout this paper we assume that a and b satisfy the following conditions:

$$\lim_{t \to 0} \frac{1}{a(t)} \int_0^t b(s)ds = 0, \tag{3.2}$$

and

$$\int_0^1 \left(\frac{1}{a(s)} \int_0^s b(t)dt \right) ds < \infty. \tag{3.3}$$

The functions $a(t) = b(t) = t^{\alpha}(1-t)^{\beta}$ with $\alpha > -1$ and $\beta < 1$ are typical examples of functions satisfying hypotheses (3.2) and (3.3).

The study of multipoint BVP's was initiated by Il'in and Moiseev [70]. Since then, there has been much current attention focused on the study of multipoint BVP's, see [30], [76], [80], [12], [36], [37] and references cited therein.

In this paper, as in the papers [4] and [37], our criteria for the existence and nonexistence of positive solutions to BVP (3.1) are determined by the comparison between the ratio $f(t,u)/u$ and a positive eigenvalue of the associated linear eigenvalue problem

$$\begin{cases} -(au')'(t) = \lambda b(t)u(t), \ t \in (0,1), \\ u'(0) = 0, \ u(1) = \alpha u(\eta). \end{cases} \tag{3.4}$$

A special case of the BVP (3.1), when $\alpha = 0$, was studied recently in [2]. Here we extend the results to three point singular BVP. Since the weights a and b are singular, we cannot use the same arguments as in [4] and [37]. To overcome the difficulty caused by the singularity of a and b, as in [2], many new ideas are developed in this study.

3.2 Main results

Let E be the Banach space of all continuous functions defined on $[0,1]$ endowed with the sup-norm denoted by $\|\cdot\|$, K be the normal cone of all nonnegative functions in E and O be the open set in E which contains all positive functions.

Let $L : E \to E$ be the operator given for $u \in E$ by

$$Lu(t) = \int_t^1 \left(\frac{1}{a(s)} \int_0^s b(r)u(r)dr \right) ds + \frac{\alpha}{1-\alpha} \int_\eta^1 \left(\frac{1}{a(s)} \int_0^s b(r)u(r)dr \right) ds, \tag{3.5}$$

and $F : K \to K$ the operator defined for $u \in K$ by $Fu(t) = f(t, u(t))$ (which is well known that is continuous and maps bounded sets into bounded sets) and $T = LF$.

By Ascoli-Arzelá theorem, one can prove easily that hypothesis (3.3) implies that L is compact and so T is completely continuous. Moreover if $u \in K \setminus \{0\}$ is a fixed point of T then u is a positive solution to BVP (3.1).

Set

$$\Lambda^+ = \{\lambda \geq 0 : \text{ there exists } u \in K \setminus \{0\} \text{ such that } Lu \leq \lambda u\}$$

and

$$\Lambda^- = \{\lambda \geq 0 : \text{ there exists } u \in K \setminus \{0\} \text{ such that } Lu \geq \lambda u\}.$$

Lemma 19 *The operator L, defined by (3.5), admits a unique positive eigenvalue μ_1. Moreover we have*

$$\mu_1 = \inf \Lambda^+ = \sup \Lambda^-.$$

Proof. Existence and uniqueness of the positive eigenvalue follow from the well known Krein-Rutman theorem. It is easy to see that $L(K \setminus \{0\}) \subset O \subset \mathrm{int}K$ and L is strongly increasing. Therefore, we deduce from Theorem 19.3 in [39] or Theorem 7.C in [17], that $\mu_1 = r(L) > 0$ and μ_1 is the unique positive eigenvalue of L.

For the characterization $\mu_1 = r(L) = \inf \Lambda^+ = \sup \Lambda^-$, we will prove that $\mu_1 \in \Lambda^+ \cap \Lambda^-$.

Let $\lambda > 0$ and $u \in K \setminus \{0\}$ with $\|u\| = 1$ such that $Lu \geq \lambda u$. We have

$$u \leq L_\lambda^n u \text{ for all } n \in \mathbb{N}^*$$

where $L_\lambda = L/\lambda$. Hence the normality of K implies that

$$1 \leq \|L_\lambda^n u\|^{\frac{1}{n}} = \frac{\|L^n u\|^{\frac{1}{n}}}{\lambda} \leq \frac{\|L^n\|^{\frac{1}{n}}}{\lambda}.$$

Letting $n \to \infty$ we get

$$\lambda \leq \lim \|L^n\|^{\frac{1}{n}} = r(L) = \mu_1,$$

which means that Λ^- is bounded from above by μ_1 and $\lambda^- = \sup \Lambda^- \leq \mu_1$.

Let $\lambda^+ = \inf \Lambda^+$. We claim that $\lambda^+ \geq \lambda^-$. Indeed, if $\lambda^+ < \lambda^-$ then for arbitrary $\lambda \in (\lambda^+, \lambda^-)$ there exist $u, v \in K \setminus \{0\}$ such that $Lu \leq \lambda u$ and $Lv \geq \lambda v$.

Let $U = Lu$ and $V = Lv$. Since U and V belong to the open set O, there exists $t_0 > 0$ such that $U \geq V_0 = t_0 V$ and $(L_\lambda)([V_0, U]) \subset [V_0, U]$. We have that $\dfrac{L}{\lambda}$: $[V_0, U] \to [V_0, U]$ is compact and $[V_0, U] = \{u \in E : V_0 \leq u \leq U\}$ is a convex closed bounded set (the boundendness is due to the fact that the cone K is a normal cone). Schauder fixed point theorem implies that $\dfrac{L}{\lambda}$ has a fixed point ϕ lying in $[V_0, U]$. Moreover $\phi \neq 0$ since $V_0 \leq \phi \leq U$ and V_0, U belong to the open set O.

All this imply that $L\phi = \lambda\phi$ and λ is an eigenvalue of L. Since here $\lambda \in (\lambda^+, \lambda^-)$ is arbitrary, $(\lambda^+, \lambda^-) \subset \sigma(L)$ which contradicts to the fact that the spectrum $\sigma(L)$ of the compact operator L is discrete.

The above shows that $\lambda^+ \geq \lambda^-$. Also there is $\psi \in O$ such that $\mu_1\psi \leq L\psi = \mu_1\psi \leq \mu_1\psi$ (μ_1 is the positive eigenvalue of L given in the beginning of the proof) that is $\mu_1 \in \Lambda^+ \cap \Lambda^-$ and

$$\lambda^+ \leq \mu_1 \leq \lambda^- \leq \lambda^+$$

or

$$\lambda^+ = \mu_1 = \lambda^-.$$

This completes the proof of the lemma. ■

Theorem 20 *Assume that (3.2) and (3.3) hold. In addition we suppose that*

$$\int_0^1 \frac{dt}{a(t)} = \infty \qquad or \qquad \lim_{t \to 0} \frac{1}{a(t)} \neq 0. \tag{3.6}$$

Then $\lambda_1 = \mu_1^{-1}$ *is the unique positive eigenvalue of BVP (3.4). Moreover if one of the following conditions*

$$\inf\left\{\frac{f(t,u)}{u} : \ t \in [0,1) \ \text{ and } u > 0\right\} > \lambda_1 \tag{3.7}$$

or

$$\sup\left\{\frac{f(t,u)}{u} : \ t \in [0,1) \ \text{ and } u > 0\right\} < \lambda_1 \tag{3.8}$$

holds, then BVP (3.1) admits no positive solution.

Proof. We claim that for any $h \in E$, Lh is the unique solution to

$$\begin{cases} -(au')'(t) = b(t)h(t), \ t \in (0,1), \\ u'(0) = 0, \ u(1) = \alpha u(\eta). \end{cases} \tag{3.9}$$

Indeed, if u is a solution to BVP (3.9) then integrating the equation in (3.9) on $[\varepsilon, \zeta]$ with $\varepsilon, \zeta \in (0,1)$, we have

$$a(\zeta)u'(\zeta) - a(\varepsilon)u'(\varepsilon) = -\int_{\varepsilon}^{\zeta} b(s)h(s)ds.$$

This implies that $\lim_{\varepsilon \to 0} a(\varepsilon)u'(\varepsilon)$ exists and let it be equal to ℓ. Thus, we can write for all $t \in (0,1)$

$$u'(t) = \frac{\ell}{a(t)} - \frac{1}{a(t)} \int_0^t b(s)h(s)ds. \tag{3.10}$$

Condition (3.6) implies that $\ell = 0$. Indeed, if $\ell \neq 0$ and $\lim_{t \to 0} \frac{1}{a(t)} \neq 0$ we obtain from (3.10) and (3.2) the contradiction

$$0 = u'(0) = \lim_{t \to 0} \left(\frac{\ell}{a(t)} - \frac{1}{a(t)} \int_0^t b(s)h(s)ds \right) = \lim_{t \to 0} \frac{\ell}{a(t)} \neq 0,$$

and if $\ell \neq 0$ and $\int_0^1 \frac{dt}{a(t)} = +\infty$, integrating (3.10) on $[0,1]$, we have from (3.3) the contradiction

$$u(1) - u(0) = \int_0^1 \left(\frac{l}{a(t)} - \frac{1}{a(t)} \int_0^t b(s)h(s)ds \right) dt = \infty.$$

At this stage, integrating (3.10) on $[t,1]$ with $t \in [0,1)$ and taking in consideration the boundary condition $u(1) = \alpha u(\eta)$ we get

$$u(t) = \frac{\alpha}{1-\alpha} \int_{\eta}^{1} \left(\frac{1}{a(\zeta)} \int_0^{\zeta} b(r)h(r)dr \right) d\zeta + \int_t^1 \left(\frac{1}{a(\zeta)} \int_0^{\zeta} b(r)h(r)dr \right) d\zeta = Lh(t).$$

Now it follows from the above that 0 is not an eigenvalue of (3.4) and if $\lambda_1 > 0$ is an eigenvalue of (3.4) with an eigenvector $\phi \in K$, then $L\phi = \lambda_1^{-1}\phi$. Thus, the uniqueness of the positive eigenvalue of L implies that $\lambda_1^{-1} = \mu_1$.

Suppose that $\gamma = \inf\{f(t,u)/u : t \in [0,1) \text{ and } u > 0\} > \lambda_1$ (the case (3.8) can be checked similarly) and assume to the contrary that the BVP (3.1) has a positive solution u. Then from $u = LFu$ we have $u \geq \gamma Lu,\ \gamma^{-1} \in \Lambda^{+}$ and $\gamma^{-1} \geq \lambda^{+} = \mu_1 = \lambda_1^{-1}$ which contradicts $\gamma > \lambda_1$. This completes the proof of the theorem. ■

Remark 21 *We notice that if* $\displaystyle\int_0^1 \frac{dt}{a(t)} < \infty$ *and* $\displaystyle\lim_{t \to 0} \frac{1}{a(t)} = 0$, *then*

$$u(t) = \frac{\alpha}{1-\alpha} \int_\eta^1 \frac{ds}{a(s)} + \int_t^1 \frac{ds}{a(s)}$$

satisfies

$$\begin{cases} -(au')'(t) = 0,\ t \in (0,1), \\ u'(0) = 0,\ u(1) = \alpha u(\eta). \end{cases}$$

Remark 22 *Using Lemmas* **??** *and* **??**, *one can prove easily that for any* $R > 0$

1. $i(\gamma L, B(0,R) \cap K, K) = 0$ *if* $\gamma \lambda^{+} = \gamma/\lambda_1 > 1$ *and*

2. $i(\gamma L, B(0,R) \cap K, K) = 1$ *if* $\gamma \lambda^{-} = \gamma/\lambda_1 < 1$.

The statement of the existence result, given in the next theorem, needs the following notations. For $\nu = 0, +\infty$ let

$$f_\nu = \liminf_{u \to \nu} \left(\min_{t \in [0,1]} \frac{f(t,u)}{u} \right), \quad f^\nu = \limsup_{u \to \nu} \left(\max_{t \in [0,1]} \frac{f(t,u)}{u} \right).$$

Theorem 23 *Assume that (3.2) and (3.3) hold. In addition we suppose that one of the following conditions*

$$f^0 < \lambda_1 < f_\infty \leq f^\infty < \infty \tag{3.11}$$

or

$$f^\infty < \lambda_1 < f_0 \leq f^0 < \infty \tag{3.12}$$

holds. Then BVP (3.1) admits at least one positive solution.

Proof. We present the proof in the case where (3.11) holds, the other case can be checked similarly. The condition $f^0 < \lambda_1$ implies that for $\varepsilon > 0$ small enough there exists $\delta > 0$ such that for all $t \in [0,1]$ and $u \in [0,\delta]$, $f(t,u) \leq (\lambda_1 - \varepsilon)\, u$. Thus, if for $u \in K \cap \partial B(0,\delta)$, $Tu \geq u$ then $Lu \geq (\lambda_1 - \varepsilon)^{-1} u$ and $(\lambda_1 - \varepsilon)^{-1} \leq \lambda^- = \lambda_1^{-1}$ which is impossible. Hence we deduce from Lemma **??** that $i(T, K \cap B(0,\delta), K) = 1$.

Now, we have to prove the existence of $R > \delta$ such that

$$i(T, B(0,R) \cap K, K) = 0. \tag{3.13}$$

If (3.13) is true, additivity and solution properties of fixed point index implies that

$$\begin{aligned} i(T, \left(B(0,R) \setminus \overline{B}(0,\delta)\right) \cap K, K) &= i(T, B(0,R) \cap K, K) - i(T, B(0,\delta) \cap K, K) \\ &= -1, \end{aligned}$$

and the operator T admits a positive fixed point u with $\delta < \|u\| < R$ which is a positive solution to the BVP (3.1).

To prove (3.13), let $\varepsilon, C_1, C_2 > 0$ such that $(f_\infty - \varepsilon) > \lambda_1$ and for all $(t,u) \in [0,1] \times \mathbb{R}^+$

$$(f_\infty - \varepsilon)u - C_1 \leq f(t,u) \leq (f^\infty + \varepsilon)u + C_2. \tag{3.14}$$

Consider the function $H : [0,1] \times K \to K$ defined by $H(t,u) = tTu + (1-t)(f_\infty - \varepsilon)Lu$ and let us prove the existence of $R > 0$ large enough such that for all $t \in [0,1]$ equation $H(t,u) = u$ has no solution in $\partial B(0,R) \cap K$. By the contrary suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial B(0,n) \cap K$ such that

$$u_n = t_n T u_n + (1 - t_n)(f_\infty - \varepsilon) L u_n.$$

Note that $v_n = \dfrac{u_n}{\|u_n\|} \in \partial B(0,1) \cap K$ and satisfies

$$v_n = t_n L\left(\frac{Fu_n}{\|u_n\|}\right) + (1 - t_n)(f_\infty - \varepsilon) L v_n.$$

It follows from (3.14) that

$$\left(f_\infty - \varepsilon\right) v_n - \frac{C_1}{\|u_n\|} \leq \frac{Fu_n}{\|u_n\|} \leq \left(f^\infty + \varepsilon\right) v_n + \frac{C_2}{\|u_n\|}.$$

Hence the normality of the cone K implies that $\dfrac{Fu_n}{\|u_n\|}$ is bounded. Then from the compactness of L we deduce the existence of a subsequence, denoted also (v_n), which converges to $v \in \partial B(0,1) \cap K$ satisfying $v \geq (f_\infty - \varepsilon) Lv$. That is $(f_\infty - \varepsilon)^{-1} \geq \lambda^+ = \lambda_1^{-1}$ which is impossible. So there exists $R > \delta$ such that equation $H(t,u) = u$ has no solution in $K \cap \partial B(0,R)$ and for a such $R > 0$, we deduce from homotopy property of fixed point index and remark 22 that

$$\begin{aligned} i(T, B(0,R) \cap K, K) &= i(H(1,\cdot), B(0,R) \cap K, K) \\ &= i(H(0,\cdot), B(0,R) \cap K, K) \\ &= i((f_\infty - \varepsilon) L, B(0,R) \cap K, K) \\ &= 0. \end{aligned}$$

This completes the proof of the theorem. ■

Chapter 4

EXISTENCE OF A POSITIVE SOLUTION TO A THREE POINT $\phi-$LAPLACIAN BVPS VIA HOMOTOPIC DEFORMATIONS

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4.1 Introduction

We are interested in this paper with the existence of a positive solution to the three point boundary value problem

$$\begin{cases} -\left(a\phi\left(u'\right)\right)'(x) = b\left(x\right)f(x,u(x)), \quad x \in (0,1)\,, \\ u(0) = \alpha u\left(\eta\right),\ u'(1) = 0, \end{cases} \tag{4.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi\left(0\right) = 0,\ \alpha, \eta \in [0,1)\,,\ a, b \in C\left([0,1]\,,[0,+\infty)\right),\ a > 0$ in $[0,1]$, b does not vanish identically, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

Because of their physical applications, the study of $\phi-$Laplacian second order differential equations subject to various boundary conditions have received a great deal of attention during the latter two decades; see [5]-[33], [69]-[54] and references therein. The differential operator in all of the cited papers, corresponds to the case where a is identically equal to 1. When seeking a positive solution when the nonlinearity positivity is guaranteed, authors are frequently led to using Krasnoselskii's compression and expansion of a cone principal to prove existence of a fixed point for some completely continuous operator $T : K \to K$ where K is a cone in some functional Banach space. For example, if we want to use Krasnoselskii's Theorem of norm compression and expansion of a cone, we may look for $0 < R_1,\ R_2$ such that $\|Tu\| \leq \|u\|$ for all $u \in K \cap \partial B(0, R_1)$ and $\|Tu\| \geq \|u\|$ for all $u \in K \cap \partial B(0, R_2)$, where $B(0, R)$ denotes the open ball centered at 0 and having radius R. The realization of the second inequality often requires a special cone left invariant by T; see the cone considered in [5] and [6] where a is identically equal to 1 and the cone K_p considered in Section 2 for the case $\phi = \phi_p$. But such a cone does not exist for general ϕ and a. To overcome this difficulty we use an homotopy deformation on the differential operator in (4.1), and we obtain existence results.

Throughout this chapter, ψ is the inverse function of ϕ.

In what follows, we let E be the Banach space of all continuous functions defined on $[0, 1]$ equipped with its sup-norm denoted $\|\cdot\|$; *i.e.*, for $u \in E$, $\|u\| = \sup\{|u(t)| : t \in [0, 1]\}$. K is the normal cone of nonnegative functions in E; *i.e.*, $K = \{u \in E : u(t) \geq 0, t \in [0, 1]\}$.

4.2 Related lemmas

Let $N : E \to E$ be defined for $u \in E$ by

$$Nu\,(x) = \frac{\alpha}{1-\alpha}\int_0^{\eta} \psi\left(\frac{1}{a\,(t)}\int_t^1 b(s)\phi\,(u(s))\,ds\right)dt + \int_0^{x} \psi\left(\frac{1}{a\,(t)}\int_t^1 b(s)\phi\,(u(s))\,ds\right)dt,$$

$F : K \to K$, the Nemytskii operator defined for $u \in K$ by

$$Fu\,(x) = \psi\,(f(x, u(x)))\,,$$

and $T = NF$.

When $\phi = \phi_p$ with $p > 1$, ψ, N and T are denoted, respectively, ψ_p, N_p and T_p.

It is easy to see that N is completely continuous (by the Ascoli-Arzela theorem), that F is bounded (maps bounded sets into bounded sets), and that u is a positive solution to (4.1) if and only if u is a nontrivial fixed point to the completely continuous operator $T = NF$.

For $p > 1$, the set $K_p = \left\{ u \in K : u(x) \geq \rho_p(x) \|u\| \text{ in } [0,1] \right\}$ is a cone in E where

$$\rho_p(x) = \frac{1}{\overline{\rho}} \int_0^x \frac{dt}{\psi_p\left(a(t)\right)} \text{ and } \overline{\rho} = \int_0^1 \frac{dt}{\psi_p\left(a(t)\right)}.$$

Lemma 24 *For all $p > 1$, $T_p(K) \subset K_p$.*

Proof. Let $u \in K$, $v = T_p u$ and set $w = v - \rho_p \|v\|$. We have that v is nondecreasing on $[0,1]$ and $\|v\| = v(1)$. Indeed, from $\left(a\phi_p\left(v'\right)\right)' = -b(t)f(t,u(t)) \leq 0$, we deduce that $a\phi_p\left(v'\right)$ is nonincreasing in $[0,1]$. Furthermore, it follows from $v'(1) = 0$ that $v' \geq 0$ in $[0,1]$ and v is nondecreasing on $[0,1]$, which in turn leads to $v(x) \geq v(0)$ on $[0,1]$. Assume that $v(0) < 0$. Then we get from $v(0) = \alpha v(\eta)$ that $\alpha \neq 0$ and $v(\eta) = \dfrac{1}{\alpha} v(0) < v(0)$, which contradicts v is nondecreasing. So, $v(x) \geq v(0) \geq 0$.

Now assume that for some $t_0 \in (0,1)$, $w(t_0) < 0$ and let $t_* \in (0,1)$ be such that

$$w(t_*) = \min_{t \in [0,1]} w(t) \text{ and } w'(t_*) = 0.$$

In this case, there exist $t_1, t_2 \in (0,1)$ such that

$$t_1 < t_* < t_2 \text{ and } w'(t_1) < w'(t_*) = 0 < w'(t_2),$$

that is,

$$v'(t_1) - \rho_p'(t_1) \|v\| < 0 < v'(t_2) - \rho_p'(t_2) \|v\|.$$

Since for all x, y, with $x \neq y$,

$$\left(\phi_p(x) - \phi_p(y)\right)(x - y) > 0,$$

we get

$$a(t_1)\left(\phi_p(v'(t_1)) - \phi_p(\rho_p'(t_1)\|v\|)\right) < 0 < a(t_2)\left(\phi_p(v'(t_2)) - \phi_p(\rho_p'(t_2))\|v\|\right),$$

which contradicts $\left(a\left(\phi_p(v') - \phi_p(\rho_p')\,\|v\|\right)\right)'(t) = -b\,(t)\, f(t, u(t)) \leq 0$. This completes the proof of the lemma ■

The proof of the next lemma is immediate, and so we omit the proof.

Lemma 25 *Let for all $p > 1$*

$$\begin{aligned} c(p) &= \frac{\alpha}{1-\alpha}\int_0^{\eta} \psi_p\left(\frac{1}{a(t)}\int_t^1 b(s)\phi_p\left(\rho_p\left(s\right)\right) ds\right) dt \\ &\quad + \int_0^1 \psi_p\left(\frac{1}{a(t)}\int_t^1 b(s)\phi_p\left(\rho_p\left(s\right)\right) ds\right) dt. \end{aligned}$$

Then for all $u \in K_p$

$$\|N_p u\| \geq c\,(p)\,\|u\|\,.$$

In the remainder of this section, we will present two results providing fixed point index calculations in the case where $\phi = \phi_p$. These are needed for the proofs of the main results of this paper. Set for $p > 1$

$$\gamma\,(p) = \int_{\frac{1}{2}}^1 \psi_p\left(\frac{1}{a(t)}\int_t^1 b(s)\phi_p\left(\rho_p\left(s\right)\right) ds\right) dt.$$

Lemma 26 *Assume that $\phi = \phi_p$ with $p > 1$ and*

$$\liminf_{x\to\infty}\left(\min_{t\in[0,1]} \frac{f(t,x)}{\phi_p\,(x)}\right) = l_\infty \text{ with } l_\infty \phi_p\left(\gamma\,(p)\right) > 1.$$

Then there exists $R_\infty\,(p) > 0$ such that $i\left(T_p, B\left(0, R\right)\cap K, K\right) = 0$ for all $R \geq R_\infty\,(p)$.

Proof. It follows, from the permanence property of the fixed point index and Lemma 24, that

$$i\left(T_p, B\left(0,R\right)\cap K, K\right) = i\left(T_p, B\left(0,R\right)\cap K_p, K_p\right).$$

Let $\epsilon > 0$ be such that $\left(l_\infty - \epsilon\right)\phi_p\left(\gamma\,(p)\right) > 1$. We deduce from the definition of l_∞ that there exists $r_\infty\,(p) > 0$ such that

$$f\,(t,u) \geq \left(l_\infty - \epsilon\right)\phi_p\,(u) \text{ for all } (t,u) \in [0,1]\times\left[r_\infty\,(p)\,, +\infty\right).$$

Thus, we have for all $u \in K_p \cap B(0,R)$, with $R > R_\infty(p) = \left(r_\infty(p)/\rho_p\left(\frac{1}{2}\right)\right)$,

$$\|Lu\| \geq Lu\left(\frac{1}{2}\right) \geq \int_0^{\frac{1}{2}} \psi_p\left(\frac{1}{a(t)}\int_t^1 b(s)f(s,u(s))ds\right)dt \geq \psi_p\left(l_\infty - \epsilon\right)\gamma(p)\|u\| > \|u\|$$

and by Lemma 87, $i\left(T_p, B(0,R)\cap K, K\right) = 0$. ■

Lemma 27 *Assume that* $\phi = \phi_p$ *with* $p > 1$, *and*

$$\liminf_{x\to 0}\left(\min_{t\in[0,1]}\frac{f(t,x)}{\phi_p(x)}\right) = l_0 \text{ with } l_0\phi_p(\gamma(p)) > 1.$$

Then there exists $R_0 > 0$ *such that* $i\left(T_p, B(0,R)\cap K, K\right) = 0$, *for all* $R \leq R_0$.

Proof. Let $\epsilon > 0$ be such that $(l_0 - \epsilon)\phi_p(\gamma(p)) > 1$. We deduce from the definition of l_0 that there exists $R_0(p) > 0$ such that

$$f(t,u) \geq (l_0 - \epsilon)\phi_p(u) \text{ for all } (t,u) \in [0,1]\times[0, R_0(p)].$$

As in the proof of Lemma 26, for all $u \in K_p \cap \partial B(0,R)$ with $0 < R < R_0(p)$, we have $\|Lu\| \geq \psi_p(l_0 - \epsilon)\gamma(p)\|u\| > \|u\|$ and so $i\left(T_p, B(0,R)\cap K, K\right) = i(T_p, B(0,R)\cap K_p, K_p) = 0$. ■

4.3 Main results

Throughout we assume the condition:

$$\begin{gathered} \textit{There exist } \alpha,\beta\in\mathbb{R} \text{ with } 0<\alpha<\beta \text{ such that} \\ t^\beta\phi(x) \leq \phi(tx) \leq t^\alpha\phi(x) \text{ for all } x \geq 0 \text{ and } t \in (0,1). \end{gathered} \tag{4.2}$$

We deduce immediately from (4.2)

$$t^{\frac{1}{\alpha}}\psi(x) \leq \psi(tx) \leq t^{\frac{1}{\beta}}\psi(x) \text{ for all } x \geq 0 \text{ and } t \in (0,1). \tag{4.3}$$

Let ψ^+,ψ^- be the functions defined on $[0,+\infty)$ by

$$\psi^+(x) = \begin{cases} x^{\frac{1}{\beta}} \text{ if } x \leq 1 \\ x^{\frac{1}{\alpha}} \text{ if } x \geq 1 \end{cases} \quad \psi^-(x) = \begin{cases} x^{\frac{1}{\alpha}} \text{ if } x \leq 1 \\ x^{\frac{1}{\beta}} \text{ if } x \geq 1. \end{cases}$$

It follows from (4.3) that, for all $t \geq 0$ and $x \geq 0$,

$$\psi^{-}(t)\,\psi(x) \leq \psi(tx) \leq \psi^{+}(t)\,\psi(x). \tag{4.4}$$

Set

$$f^{0} = \limsup_{u\to 0}\left(\max_{t\in[0,1]}\frac{\psi\left(f(t,u)\right)}{u}\right), \quad f^{\infty} = \limsup_{u\to+\infty}\left(\max_{t\in[0,1]}\frac{\psi\left(f(t,u)\right)}{u}\right),$$

$$\Gamma = \frac{\alpha}{1-\alpha}\int_0^{\eta}\psi^{+}\left(\frac{1}{a(t)}\int_t^1 b(s)ds\right)dt + \int_0^1 \psi^{+}\left(\frac{1}{a(t)}\int_t^1 b(s)ds\right)dt.$$

Theorem 28 *Assume that in addition to* (4.2), *the following conditions*

$$\Gamma f^{0} < 1,$$

$$\textit{there exists } p > 1 \textit{ such that } \lim_{x\to+\infty}\frac{\phi(x)}{\phi_p(x)} = 1, \tag{4.5}$$

$$\left(\phi_p\left(\gamma(p)\right)\right)^{-1} < \liminf_{x\to+\infty}\left(\min_{t\in[0,1]}\frac{f(t,x)}{\phi_p(x)}\right) = l_{\infty} \leq \limsup_{x\to+\infty}\left(\max_{\substack{t\in[0,1]\\ x\to 0}}\frac{f(t,x)}{\phi_p(x)}\right) = l^{\infty} < \infty,$$

are satisfied. Then Problem (4.1) *admits a positive solution.*

Proof. Let $\epsilon > 0$ be such that $(f^{0}+\epsilon)\,\Gamma < 1$. There exists $r_0 > 0$ such that

$$f(s,u) \leq \phi\left(\left(f^{0}+\epsilon\right)u\right) \text{ for all } (s,u) \in [0,1]\times[0,r_0].$$

Let $u \in K \cap \partial B(0,r)$ with $0 < r \leq r_0$. We have

$$\begin{aligned}
\|Tu\| &= Tu(1) \\
&\leq \frac{\alpha}{1-\alpha}\int_0^{\eta}\psi\left(\frac{1}{a(t)}\int_t^1 b(s)\phi\left(\left(f^{0}+\epsilon\right)u(s)\right)ds\right)dt \\
&\quad + \int_0^1\psi\left(\frac{1}{a(t)}\int_t^1 b(s)\phi\left(\left(f^{0}+\epsilon\right)u(s)\right)ds\right)dt \\
&\leq \Gamma\left(f^{0}+\epsilon\right)\|u\| \\
&< \|u\|.
\end{aligned}$$

So, by Lemma 86, $i\left(T, B(0,r)\cap K, K\right) = 1$ for all $r \in (0, r_0]$.

Now let us prove that there exists $r_\infty > R_\infty(p)$ such that $i\left(T, B\left(0, r\right) \cap K, K\right) = 0$. Let for $\theta \in [0,1]$, $\psi_\theta = \theta\psi + (1-\theta)\psi_p$, and consider the equation

$$u = T_\theta u, \tag{4.6}$$

where $T_\theta : K \to K$ is given for $u \in K$ by

$$\begin{aligned} T_\theta u\left(x\right) &= \frac{\alpha}{1-\alpha}\int_0^\eta \psi_\theta\left(\frac{1}{a\left(t\right)}\int_t^1 b(s)f(s,u(s))ds\right)dt \\ &\quad + \int_0^x \psi_\theta\left(\frac{1}{a\left(t\right)}\int_t^1 b(s)f(s,u(s))ds\right)dt. \end{aligned}$$

It is clear that T_θ is completely continuous, that $T_1 = T$ and $T_0 = T_p$.

In order to use the homotopy property of the fixed point index, let us prove that there exists $r_\infty > R_\infty(p)$ such that equation (4.6) has no solution in $\partial B(0, r_\infty) \cap K$. Assume to the contrary. Then there exists sequences $(\theta_n) \subset [0,1]$, $(r_n) \subset (R_\infty(p), +\infty)$ and $(u_n) \subset K$ with $\lim r_n = +\infty$, $u_n \in \partial B(0, r_n) \cap K$ such that

$$\frac{u_n}{\|u_n\|} = \frac{T_{\theta_n} u_n}{\|u_n\|}. \tag{4.7}$$

It is easy to see that hypothesis (4.5) implies $\lim_{x\to+\infty} \frac{\phi_\theta(x)}{\phi_p(x)} = 1$. Then $\lim_{x\to+\infty} \frac{\psi_\theta(x)}{\psi_p(x)} = 1$. Set $\delta_\theta = \psi_\theta - \psi_p$ and $\widetilde{T}_\theta = T_\theta - T_p$, where $\widetilde{T}_\theta : K \to E$ is given for $u \in K$ by

$$\begin{aligned} \widetilde{T}_\theta u\left(x\right) &= \frac{\alpha}{1-\alpha}\int_0^\eta \delta_\theta\left(\frac{1}{a\left(t\right)}\int_t^1 b(s)f(s,u(s))ds\right)dt \\ &\quad + \int_0^x \delta_\theta\left(\frac{1}{a\left(t\right)}\int_t^1 b(s)f(s,u(s))ds\right)dt. \end{aligned}$$

Then equation (4.7) becomes

$$\frac{u_n}{\|u_n\|} = N_p\left(\frac{Fu_n}{\|u_n\|}\right) + \frac{\widetilde{T}_{\theta_n} u_n}{\|u_n\|}. \tag{4.8}$$

At this stage, we claim that $\lim_{n\to\infty} \dfrac{\widetilde{T}_{\theta_n} u_n}{\|u_n\|} = 0$. Indeed, because of $l_\infty \le l^\infty < \infty$, there exists $c_1 > 0$ such that

$$\frac{Fu_n}{\|u_n\|} \le c_1.$$

Also, see that $\lim_{x\to+\infty}\left(\left|\delta_\theta\left(x\right)\right|/\psi_p\left(x\right)\right) = 0$ means that for arbitrary $\epsilon > 0$ there exists $c_\epsilon > 0$ such that for all $x > 0$

$$\left|\delta_\theta\left(x\right)\right| \leq \epsilon\psi_p\left(x\right) + c_\epsilon.$$

Thus, we have from the definition of T_θ that for all $x \in [0,1]$

$$\left|\frac{\widetilde{T}_\theta u_n(x)}{\|u_n\|}\right| \leq \frac{\epsilon}{1-\alpha}\int_0^1 \psi_p\left(\frac{1}{a(t)}\int_t^1 b(s)\frac{f(s,u_n(s))}{\phi_p\left(\|u_n\|\right)}ds\right)dt + \frac{c_\epsilon}{\|u_n\|}$$

which implies that

$$\limsup_{n\to\infty}\frac{\left\|\widetilde{T}_\theta u_n\right\|}{\|u_n\|} \leq \epsilon\frac{c_1}{1-\alpha}\int_0^1 \psi_p\left(\frac{1}{a(t)}\int_t^1 b(s)ds\right)dt$$

and since ϵ is arbitrary $\lim_{n\to\infty}\left(\widetilde{T}_{\theta_n}u_n/\left\|u_n\right\|\right) = 0$.

Set $v_n = \frac{u_n}{\|u_n\|}$ and $z_n = \frac{\widetilde{T}_{\theta_n}u_n}{\|u_n\|}$. From the compacteness of N_p and the boundedness of $\frac{Fu_n}{\phi_p(\|u_n\|)}$ it follows that there exists subsequences (θ_{n_k}) and (v_{n_k}) converging respectively to $\overline{\theta} \in [0,1]$ and $v \in \partial B\left(0,1\right)\cap K_p$ (see that $v_{n_k} - z_{n_k} = N_p\left(\frac{Fu_{n_k}}{\|u_{n_k}\|}\right) \in K_p$). Furthermore, it follows from $l_\infty > \left(\phi_p\left(\gamma(p)\right)\right)^{-1}$ that, for $\epsilon > 0$ with $\left(l_\infty - \epsilon\right)\phi_p\left(\gamma(p)\right) > 1$, there exists a constant $c_0 > 0$ such that for all $s \in [0,1]$ and $u \geq 0$,

$$f(s,u) \geq \left(l_\infty - \epsilon\right)\phi_p\left(u\right) - c_0\ . \tag{4.9}$$

Inserting (4.9) into (4.8), we obtain

$$v_{n_k} - z_{n_k} = N_p\left(\frac{Fu_{n_k}}{\|u_{n_k}\|}\right) \geq N_p\left(\psi_p\left(\left(l_\infty - \epsilon\right)\phi_p\left(v_{n_k}\right) - \frac{c_0}{\phi_p\left(\|u_{n_k}\|\right)}\right)\right).$$

Letting $n \to \infty$, we get $v \geq N_p\left(\psi_p\left(l_\infty - \epsilon\right)v\right)$, from which follows the contradiction,

$$1 = \|v\| \geq \left\|N_p\left(\psi_p\left(l_\infty - \epsilon\right)v\right)\right\| \geq c(p)\psi_p\left(l_\infty - \epsilon\right)\|v\| \geq \gamma(p)\psi_p\left(l_\infty - \epsilon\right) > 1.$$

Thus there exists $r_\infty > R_\infty(p)$ such that equation (4.6) admits no solution in $\partial B(0,r_\infty)\cap K$, we deduce from the homotopy property of the fixed point index and

Lemma 26, $i\left(T,B(0,r_\infty)\cap K,K\right)=i\left(T_p,B(0,r_\infty)\cap K,K\right)=0$. Finally, by excision and solution properties of the fixed point index, we deduce that $i\left(T,\left(B(0,r_\infty)\setminus\overline{B}(0,r)\right)\cap K,K\right)=-1$, where $r>0$ is small enough, and Problem (4.1) admits a positive solution u with $r<\|u\|<r_\infty$. ■

Theorem 29 *Assume that in addition to* (4.2), *the following conditions*

$$\Gamma f^\infty<1,$$

$$\text{there exists } p>1 \text{ such that } \lim_{x\to 0}\frac{\phi(x)}{\phi_p(x)}=1, \tag{4.10}$$

$$\left(\phi_p(\gamma(p))\right)^{-1}<\liminf_{x\to 0}\left(\min_{t\in[0,1]}\frac{f(t,x)}{\phi_p(x)}\right)=l_0\leq\limsup_{x\to 0}\left(\max_{\substack{t\in[0,1]\\ x\to 0}}\frac{f(t,x)}{\phi_p(x)}\right)=l^0<\infty,$$

are satisfied. Then Problem (4.1) *admits a positive solution.*

Proof. Let $\epsilon>0$ be such that $(f^\infty+\epsilon)\Gamma<1$. There exists $C_\epsilon>0$ such that

$$f(s,u)\leq\phi\left((f^\infty+\epsilon)u+C_\epsilon\right) \text{ for all } (s,x)\in[0,1]\times[0,+\infty).$$

We have for all $u\in K$,

$$\begin{aligned}\|Tu\| &= Tu(1)\\ &\leq \frac{\alpha}{1-\alpha}\int_0^\eta\psi\left(\frac{1}{a(t)}\int_t^1 b(s)\phi\left((f^\infty+\epsilon)u(s)+C_\epsilon\right)ds\right)dt\\ &\quad+\int_0^1\psi\left(\frac{1}{a(t)}\int_t^1 b(s)\phi\left((f^\infty+\epsilon)u(s)+C_\epsilon\right)ds\right)dt\\ &\leq \Gamma\left((f^\infty+\epsilon)\|u\|+C_\epsilon\right).\end{aligned}$$

So, we have for all $u\in K\cap B(0,r)$ with $r>\dfrac{C_\epsilon\Gamma}{1-\Gamma(f^\infty+\epsilon)}$, $\|Tu\|<\|u\|$ and by Lemma 86 $i\left(T,B(0,r)\cap K,K\right)=1$.

Arguing as in the proof of Theorem 28, we prove existence of $r_0>0$ small enough such that $i\left(T,B(0,r_0)\cap K,K\right)=0$, and by excision and solution properties of the fixed point index, we deduce that $i\left(T,\left(\overline{B}(0,r_\infty)\setminus B(0,r_0)\right)\cap K,K\right)=1$, and Problem (4.1) admits a positive solution u with $r_0<\|u\|<r_\infty$. ■

Remark 30 *Theorem 28 (resp. Theorem 29) holds true if* $\lim_{x\to+\infty} \frac{\phi(x)}{\phi_p(x)} = l > 0$ *(resp.* $\lim_{x\to+\infty} \frac{\phi(x)}{\phi_p(x)} = l > 0$*).*

Remark 31 $\phi(x) = \phi_{p_1}(x) + \phi_{p_2}(x)$, *where* $1 < p_1 < p_2$, *is a typical case where* (4.2) *and* (4.5) *or* (4.10) *are satisfied.*

Chapter 5

POSITIVE SOLUTIONS FOR A P(T)-LAPLACIAN THREE POINT BOUNDARY VALUE PROBLEM

Nadir Benkaci-Ali, Abdelhamid Benmezai and Johnny Henderson, appeared in Dynamical Systems and Applications (2013).

Key words: p(t)-Laplacian; Positive solution; Fixed point index theory.

5.1 Introduction

This paper deals with existence of a positive solutions to the p(t)-Laplacian three point boundary value problem (bvp for short)

$$\begin{cases} -(\phi(t,u'(t)))' = f(t,u(t)),\ t\in(0,1), \\ u(0)=\alpha u(\eta),\ u'(1)=0, \end{cases} \tag{5.1}$$

where $\mathbb{R}^+ = [0,+\infty)$, $\alpha,\eta\in(0,1)$, $\phi(t,x)=|x|^{p(t)-2}x$, $p\in C([0,1],(1,+\infty))$, and $f\in C([0,1]\times\mathbb{R}^+,\mathbb{R}^+)$.

By a positive solution to bvp (5.1) we mean a function $u\in C^1([0,1],\mathbb{R}^+)$ with $\phi(t,u'(t))\in C^1([0,1],\mathbb{R})$ and $u(t_0)>0$ for some $t_0\in[0,1[$ satisfying both the differentiel equation and the boundary conditions in bvp (5.1).

The operator $-(\phi(t,u'(t)))'$ is known as the monodimensional $p(t)$-Laplacian and becomes the p-Laplacian when $p(t)\equiv p\in(1,+\infty)$. Because of the physical interests (see for example [8], [19], [51] and references cited therein), differential equations involving the $p(t)$-Laplacian have received a great attention in recent years and many interesting results have been obtained (see for example [73], [71], [72], [74], [58], [59],

[60], [57], and [61]). See that the $p(t)$-Laplacian is inhomogeneous. This makes the study of differential equations involving $p(t)$-Laplacian more complicated. The difficulties encountered when studying various aspects of differential equations involing the p(t)-Laplacian are described in [74], [58], [60] and [57].

It is well known that in the case where p is constant and the positivity of the nonlinearity f is guaranteed, a positive solution of the bvp (5.1) is concave (see [5]). This fact makes the use of cone compression and expansion principal possible. However, in the case of general p, a positive solution to bvp (5.1) is not necessarily concave.

To overcome the difficulty caused by the loss of concavity, we have associated with the bvp (5.1), by means of an operator A depending on the function p, an auxiliary problem such that Au is a positive solution to the bvp (5.1) for all positive solution u to the auxiliary problem.

Thus, we present here an existence result for the sublinear case without many difficulties (see Theorem 28 and its proof). In the opposite case, an existence result for the superlinear case requires the use of a homotopical argument.

5.2 Preliminaries

Throughout this paper, $\psi(t,\cdot)$ denotes the inverse function of $\phi(t,\cdot)$ and we have

$$\psi(t,x) = |x|^{q(t)-2}\, x \text{ where } q(t) = \frac{p(t)}{p(t)-1}.$$

The real numbers, p^-, p^+ are defined by

$$p^- = \min_{t\in[0,1]} p(t), \quad p^+ = \max_{t\in[0,1]} p(t).$$

E will denote the Banach space of all continuous functions defined on $[0,1]$ equipped with the sup-norm denoted by $\|\cdot\|$, and E^+, K and P are cones of E defined by

$$E^+ = \{u \in E : u \geq 0 \; in \; [0,1]\},$$

$$K = \left\{ u \in E^+ : u \text{ is nonincreasing on } [0,1] \right\}$$

and

$$P = \{ u \in K,\ u(t) \geq (1-t) \|u\| \} .$$

The linear operator $L : E \to E$ is given for $u \in E$ by

$$Lu(t) = \int_t^1 u(s) ds$$

and $F, A_p, T : E^+ \to E^+$ are the operators defined for $u \in E^+$ by

$$Fu(t) = f(t, A_p u(t)),$$

$$A_p u(t) = \frac{\alpha}{1-\alpha} \int_0^\eta \psi\left(s, u(s)\right) ds + \int_0^t \psi\left(s, u(s)\right) ds$$

and

$$T = LFA_p.$$

Lemma 32 *The operator A_p is continuous.*

Proof. Let $(u_n) \subset E^+$ be a sequence converging to $u \in E^+$. There exists $M > 0$ such that $(u_n) \subset \overline{B}(0, M)$. From the uniform continuity of ψ on the compact $[0,1] \times [-M, M]$ we deduce that for $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that, for all $s \in [0,1]$ and all $x, y \in [-M, M]$, $|x - y| \leq \delta_\epsilon$ implies $|\psi(s, x) - \psi(s, y)| \leq \epsilon$.

Thus for n large enough, we have for all $t \in [0,1]$

$$|A_p u_n(t) - A_p u(t)| \leq \int_0^t |\psi(s, u_n(s)) - \psi(s, u(s))| \, ds \leq \epsilon.$$

This means that $\lim A_p u_n = A_p u$ and A_p is continuous ■

Lemma 33 *We have that*

1. *T is completly continuous,*

2. *$T(E^+) \subset K$ and*

3. If $u \in E^+$ is a fixed point of T then $v = A_p u$ is a positive solution to BVP (5.1).

Proof.

1. Since all the operators $L,\ F$ and A_p are continuous, $T = LFA_p$ is continuous. Moreover the continuity of the functions ψ and f make the operators F and A_p bounded (map bounded sets into bounded sets) and Ascoli-Arzela Theorem makes L a compact operator. Thus, $T = LFA_p$ is completely continuous.

2. Let $u \in E^+$ and $v = Tu$. We have $v'(t) = -f(t, A_p u(t)) \leq 0$ for all $t \in (0,1)$. This, together with $v(1) = 0$, leads to v is a nonnegative and nonincreasing function on $[0,1]$. Namely $v \in K$.

3. If $u \in E^+$ is a fixed point of T, then u satisfies the auxiliary problem
$$\begin{cases} -u'(t) = f(t, A_p u(t)) \ t \in (0,1), \\ u(1) = 0. \end{cases}$$
Let $v = A_p u$. We have for all $t \in (0,1)$,
$$\phi(t, v'(t)) = \phi(t, \psi(t, u(t))) = u(t). \tag{5.2}$$
Then for all $t \in (0,1)$,
$$-(\phi(t, v'(t)))' = f(t, v(t)).$$
Also we have from (5.2), $v'(1) = u(1) = 0$, and from
$$\begin{aligned} v(0) &= A_p u(0) = \frac{\alpha}{1-\alpha}\int_0^\eta (u(s))^{\frac{1}{p(s)-1}}\,ds, \\ v(\eta) &= \frac{1}{1-\alpha}\int_0^\eta (u(s))^{\frac{1}{p(s)-1}}\,ds, \end{aligned}$$
we have $v(0) = \alpha v(\eta)$. Thus $v = A_p u$ is a positive solution to bvp (5.1).

This completes the proof of the lemma ■

5.3 Main results

The statements of the main results, we introduce the following notations.

$$\begin{aligned} \Lambda_1 &= \left(\tfrac{1-\alpha}{1-\alpha(1-\eta)}\right)^{p^- -1}, & \Lambda_2 &= \tfrac{1}{1-\eta}\left(\tfrac{1-\alpha}{\eta}\right)^{p^- -1}, \\ \Lambda_3 &= \left(\tfrac{1-\alpha}{1-\alpha(1-\eta)}\right)^{p^+ -1}, & \Lambda_4 &= \tfrac{1}{1-\eta}\left(\tfrac{1-\alpha}{\eta}\right)^{p^+ -1}, \\ f_0 &= \liminf_{u\to 0}\left(\min_{t\in[\eta,1]}\frac{f(t,u)}{u^{p^- -1}}\right), & f^\infty &= \limsup_{u\to +\infty}\left(\max_{t\in[0,1]}\frac{f(t,u)}{u^{p^- -1}}\right), \\ f^0 &= \limsup_{u\to 0}\left(\max_{t\in[0,1]}\frac{f(t,u)}{u^{p^+ -1}}\right), & f_\infty &= \liminf_{u\to +\infty}\left(\min_{t\in[\eta,1]}\frac{f(t,u)}{u^{p^+ -1}}\right). \end{aligned}$$

Let

$$\psi^+(x) = \begin{cases} x^{\frac{1}{p^+ -1}} & \text{if } x \le 1, \\ x^{\frac{1}{p^- -1}} & \text{if } x \ge 1, \end{cases} \qquad \psi^-(x) = \begin{cases} x^{\frac{1}{p^- -1}} & \text{if } x \le 1, \\ x^{\frac{1}{p^+ -1}} & \text{if } x \ge 1. \end{cases}$$

Then we have for all $x \ge 0$

$$\psi^-(x) \le \psi(t,x) \le \psi^+(x). \tag{5.3}$$

Thus, the nonincreasing property of the functions in the cone K leads to

$$A_p u(\eta) \ge \frac{\eta}{1-\alpha}\psi^-(u(\eta)), \tag{5.4}$$

$$\|A_p u\| = A_p u(1) \le A_p u(\eta) + \psi^+(u(\eta)), \tag{5.5}$$

and

$$\|A_p u\| = A_p u(1) \le \frac{1-\alpha(1-\eta)}{1-\alpha}\psi^+(u(0)), \tag{5.6}$$

for all $u \in K$.

Theorem 34 (the sublinear case) *Assume that*

$$f^\infty < \Lambda_1 \le \Lambda_2 < f_0. \tag{5.7}$$

Then BVP (5.1) *admits at least one positive solution.*

Proof. Let $\epsilon > 0$ such that $\Lambda_1 - \epsilon > f^{\infty}$. There exists $C > 0$ such that for all $t \in [0, 1]$ and $u \geq 0$

$$f(t,u) \leq (f^{\infty} + \epsilon)\, u^{p^- - 1} + C. \tag{5.8}$$

Let $R_{\infty} > \max\left(C\Lambda_1/\Lambda_1 - (f^{\infty} + \epsilon), 1\right)$ and $u \in \partial B(0, R_{\infty}) \cap K$. We have from (5.8) and (5.6)

$$\begin{aligned} Tu(0) &= \int_0^1 f(s, A_p u(s))ds \leq \int_0^1 (f^{\infty} + \epsilon)\,(A_p u(s))^{p^- - 1}\, ds + C \\ &\leq (f^{\infty} + \epsilon) \left(\frac{1 - \alpha\,(1 - \eta)}{1 - \alpha} \psi^{+}\,(u(0))\right)^{p^- - 1} + C \\ &= \frac{(f^{\infty} + \epsilon)}{\Lambda_1} u\,(0) + C < u\,(0)\,. \end{aligned}$$

Then $Tu - u \notin K$. This means that for all $u \in \partial B(0, R_{\infty}) \cap K$, $Tu \ngeq u$. So we have from Lemma **??**, $i(T, K \cap B(0, R_{\infty}), K) = 1$.

Let $\epsilon > 0$ small enough so that $f_0 > (\Lambda_2 + \epsilon)$. There exists $\delta > 0$ such that for all $t \in [\eta, 1]$ and $u \in [0, \delta]$

$$f(t,u) \geq (f_0 - \epsilon)\, u^{p^- - 1}. \tag{5.9}$$

Let $\delta_1 = \min\left(1, \delta, \Lambda_3 \delta^{p^+ - 1}\right)$ and $u \in K \cap \partial B(0, \delta_1)$. We have from (5.6,) $\|A_p u\| \leq \delta$ and from (5.9),

$$\begin{aligned} Tu(\eta) &= \int_{\eta}^1 f(s, A_p u(s))ds \\ &\geq \int_{\eta}^1 (f_0 - \epsilon)\,(A_p u(s))^{p^- - 1}\, ds \\ &\geq (f_0 - \epsilon)\,(1 - \eta)\,(A_p u(\eta))^{p^- - 1}\,. \end{aligned}$$

Then from (5.4) follows

$$Tu(\eta) \geq (f_0 - \epsilon)\, \Lambda_2^{-1} u\,(\eta) > u\,(\eta)\,.$$

Then $u-Tu\notin K$. This means that for all $u\in K\cap\partial B(0,\delta_1)$, $Tu\nleq u$. So, we deduce from Lemma **??** that $i(T,K\cap B(0,\delta_1),K)=0$.

Finally, we have from additivity and solution properties of the fixed point index

$$i(T,(B(0,R_\infty)\setminus B(0,\delta_1))\cap K,K)=1$$

and the operator T admits a positive fixed point u with $\delta_1<\|u\|<R_\infty$. Thus by Lemma 33, A_pu is a positive solution to bvp (5.1) ■

In what follows, we let for $u\geq 0$, $g(u)=\gamma u^{p^+-1}$ with $\gamma>\Lambda_4$, and $T_0=LGA_p$ where $G:E^+\to E^+$ is defined for $u\in E^+$ by $Gu\left(t\right)=g\left(u\left(t\right)\right)$. The following lemma provides a fixed point computation for the operator T_0.

Lemma 35 *For all* $R>R^0=\max\left(\frac{1}{1-\eta},\Lambda_4\right)$,

$$i(T_0,B(0,R)\cap K,K)=0.$$

Proof. First, we need to prove that $T_0\left(E^+\right)\subset P$. Let $u\in E^+$ and $v=T_0u$. We have that $v\in K$ and $v'=-g\left(A_pu\right)$ is nonincreasing on $[0,1]$. This means that v is concave and reaches its maximum value at $t=0$. Then from the concavity of v, we deduce that for all $t\in[0,1]$

$$v(t)\geq(1-t)\,v(0)=(1-t)\left\|v\right\|. \tag{5.10}$$

Now, let $R>R_0$ and $u\in\partial B(0,R)\cap P$. We have from (5.10) and (5.4), for all $t\in[0,\eta]$,

$$u\left(t\right)\geq u\left(\eta\right)\geq\left(1-\eta\right)R>1,$$

and for all $t\in[\eta,1]$,

$$A_pu\left(t\right)\geq A_pu\left(\eta\right)\geq\frac{\eta}{1-\alpha}\left(u\left(\eta\right)\right)^{\frac{1}{p^+-1}}\geq\left(\frac{R}{\Lambda_4}\right)^{\frac{1}{p^+-1}}>1.$$

Thus, we obtain from (5.4) ,

$$\begin{aligned} T_0u(\eta) &> \int_\eta^1 \Lambda_4(A_pu(s))^{p^+-1}ds \\ &\geq \Lambda_4 (1-\eta) (A_pu(\eta))^{p^+-1} \\ &\geq \Lambda_4 (1-\eta) \left(\frac{\eta}{1-\alpha}\right)^{p^+-1} u(\eta) = u(\eta). \end{aligned}$$

This means that for all $u \in \partial B(0,R) \cap P,\ T_0u \nleq u$. So by Lemma **??**, $i(T_0, B(0,R) \cap P, P) = 0$.

Then from the permanence property of the fixed point index, we get $i(T_0, B(0,R) \cap K, K) = 0$. ■

Theorem 36 (the superlinear case) *Assume that*

$$f^0 < \Lambda_3 \leq \Lambda_4 < f_\infty. \tag{5.11}$$

Then BVP (5.1) *admits a positive solution.*

Proof. Let $\epsilon > 0$ such that $f^0 + \epsilon < \Lambda_3$. There exists $\delta > 0$ such that, for all $t \in [0,1]$ and $u \in [0,\delta]$,

$$f(t,u) \leq \left(f^0 + \epsilon\right) u^{p^+-1}.$$

Let $\delta_2 = \min\left(1, \delta, \Lambda_3 \delta^{p^+-1}\right)$ and $u \in \partial B(0,\delta_2) \cap K$. We have from (5.6)

$$\|A_pu\| = A_pu\,(1) \leq \frac{1-\alpha\,(1-\eta)}{1-\alpha} \delta_2^{\frac{1}{p^+-1}} \leq \delta.$$

The above estimates together with (5.6) lead to

$$\begin{aligned} Tu(0) &= \int_0^1 f(s, A_pu(s))ds \leq \int_0^1 \left(f^0 + \epsilon\right) (A_pu(s))^{p^+-1} ds \\ &< \Lambda_3 \Lambda_3^{-1} u(0) = u(0). \end{aligned}$$

This means that $Tu \ngeq u$ for all $u \in \partial B(0,\delta_2) \cap K$, so, we deduce from Lemma **??** that $i(T, K \cap B(0, r_0), K) = 1$.

In order to conclude by the additivity and solution properties of the fixed point index as in the proof of Theorem 34, we have to prove the existence of an $R^{\infty} > \delta_2$ such that

$$i(T, K \cap B(0, R^{\infty}), K) = 0.$$

Consider for $\theta \in [0,1]$, the fixed point equation

$$u = (1-\theta)\, Tu + \theta T_0 u. \tag{5.12}$$

We claim that there exists $R^{\infty} > \delta_2$ such that equation (5.12) has no solution in $K \cap \partial B\,(0, R^{\infty})$. Indeed, to the contrary, assume that for all integer $n \geq 1$ there exist $\theta_n \in [0,1]$ and $u_n \in K \cap \partial B(0,n)$ such that

$$u_n = (1-\theta_n)\, Tu_n + \theta_n T_0 u_n. \tag{5.13}$$

We have that $\lim_{n\to\infty} \|Au_n\| = \lim_{n\to\infty} A_p u_n\,(1) = +\infty$. Indeed, if there exists a subsequence (u_{n_k}) such that $\|A_p u_{n_k}\|$ is bounded by a constant $M > 0$ and

$$M_1 = \sup\left\{f(s,x) + g(x) : s \in [0,1],\ x \in [0,M]\right\},$$

then we have the contradiction

$$\|u_{n_k}\| = u_{n_k}(0) = (1-\theta_{n_k}) \int_0^1 f(s, A_p u_{n_k}(s))ds + \theta_{n_k} \int_0^1 g(A_p u_{n_k}(s))ds \leq M_1.$$

The reminder of the proof involves showing that $\lim\limits_{n\to\infty} Au_n(\eta) = +\infty$. We distinguish two cases:

- If $\lim\limits_{n\to\infty} u_n(\eta) = +\infty$, then we have from (5.4), $\lim_{n\to\infty} Au_n(\eta) = +\infty$.

- If there exists a subseqence (u_{n_k}) such that $(u_{n_k}(\eta))$ is bounded and

$$m = \sup_{k\in\mathbb{N}} (u_{n_k}(\eta)),$$

then we have from (5.5) the contradiction

$$\|Au_{n_k}\| = Au_{n_k}(1) \leq Au_{n_k}(\eta) + \psi^+(m).$$

Let $\epsilon > 0$ be such that $f_\infty, \gamma > (\Lambda_4 + \epsilon)$. There exists $R_2 > 0$ such that, for all $t \in [\eta, 1]$ and $u > R_2$,

$$f(t,u) \geq (f_\infty - \epsilon) u^{p^+-1}.$$

Let $R^\infty > \max(R_2, R^0)$. We have from (5.13) and (5.4) for n large, the contradiction

$$\begin{aligned}
u_n(\eta) &= \int_\eta^1 ((1-\theta_n) f(s, A_p u_n(s)) + \theta_n g(A_p u_n(s)))\, ds \\
&> \int_\eta^1 (\Lambda_4)(A_p u_n(s))^{p^+-1} ds \\
&\geq (1-\eta)(\Lambda_4)(A_p u_n(\eta))^{p^+-1} \\
&\geq (\Lambda_4)\Lambda_4^{-1} u_n(\eta) \\
&\geq u_n(\eta).
\end{aligned}$$

The claim is proved, and by the homotopy property of the fixed point index, we have

$$i(T, B(0,R)\cap K, K) = i(T_0, B(0,R)\cap K, K) = 0.$$

This completes the proof ■

Remark 37 *Theorem 34 and Theorem 36 cover, respectively, the cases*

$$\lim_{u\to 0} \frac{f(t,u)}{\phi(t,u)} = +\infty \text{ and } \lim_{u\to+\infty} \frac{f(t,u)}{\phi(t,u)} = 0 \text{ uniformly on } [0,1],$$

and

$$\lim_{u\to 0} \frac{f(t,u)}{\phi(t,u)} = 0 \text{ and } \lim_{u\to+\infty} \frac{f(t,u)}{\phi(t,u)} = +\infty \text{ uniformly on } [0,1].$$

Thus, if $f(t,u) = u^{q(t)-1}$, *then bvp* (5.1) *admits a positive solution in both the cases* $1 < q(t) < p(t)$ *and* $1 < p(t) < q(t)$.

Example 38 *Consider the bvp* (5.1) *with*

$$p(t) = \frac{3+t}{1+t},\ f(t,u) = \frac{Au + Bu^2}{1+u} \text{ and } \alpha = \eta = \frac{1}{2}$$

where A, B *are positive real numbers.*

By simple computations we get that $p^+ = 3$, $p^- = 2$, *and*

$$\Lambda_1 = \frac{2}{3},\ \Lambda_2 = 2,\ f_0 = A,\ f^\infty = B.$$

We deduce from Theorem 34 that BVP (5.1) *admits a positive solution if* $B < \frac{2}{3} < 2 < A$.

Example 39 *Consider the bvp* (5.1) *with* $f(t,u) = Au^\sigma \exp(u)$ *where* σ *and* A *are positive real numbers. By simple computations we get that*

$$f_\infty = +\infty \text{ and } f^0 = \begin{cases} 0 & \text{if } \sigma > p^+ - 1 \\ A & \text{if } \sigma = p^+ - 1 \\ +\infty & \text{if } \sigma < p^+ - 1. \end{cases}$$

We deduce from Theorem 36 that BVP (5.1) *admits a positive solution if* $\sigma > p^+ - 1$ *or* $\sigma = p^+ - 1$ *and* $A < \left(\frac{1-\alpha}{1-\alpha(1-\eta)}\right)^{p^+-1}$.

Example 40 *Consider the bvp* (5.1) *with* $f(t,u) = u^\theta$ *where* θ *is a positive real number. We have*

$$f_0 = \begin{cases} 0 & \text{if } \theta > p^- - 1 \\ 1 & \text{if } \theta = p^- - 1 \\ +\infty & \text{if } \theta < p^- - 1 \end{cases} \qquad f^\infty = \begin{cases} +\infty & \text{if } \theta > p^- - 1 \\ 1 & \text{if } \theta = p^- - 1 \\ 0 & \text{if } \theta < p^- - 1 \end{cases}$$

$$f^0 = \begin{cases} 0 & \text{if } \theta > p^+ - 1 \\ 1 & \text{if } \theta = p^+ - 1 \\ +\infty & \text{if } \theta < p^+ - 1 \end{cases} \qquad f_\infty = \begin{cases} +\infty & \text{if } \theta > p^+ - 1 \\ 1 & \text{if } \theta = p^+ - 1 \\ 0 & \text{if } \theta < p^+ - 1. \end{cases}$$

So we deduce from Theorem 34 and Theorem 36 that BVP (5.1) *admits a positive solution if* $\theta \in (0, p^- - 1) \cup (p^+ - 1, +\infty)$.

Chapter 6

POSITIVE SOLUTION FOR A P(T)-LAPLACIAN THREE POINT BOUNDARY VALUE PROBLEM ON THE HALF LINE

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6.1 Introduction

This paper deals with the existence of a positive solution to the $p(t)$-Laplacian three point boundary value problem (bvp for short)

$$\begin{cases} -(a\,(t)\,\phi(t,u'(t)))' = f(t,u(t)), \ t \in (0,+\infty), \\ u(0) = \alpha u(\eta), \ u'(+\infty) = 0, \end{cases} \tag{6.1}$$

where $\alpha \in (0,1)$, $\eta > 0$, $\phi(t,x) = |x|^{p(t)-2}\,x$ for all $x \in \mathbb{R}^+ = [0,+\infty)$, $p : \mathbb{R}^+ \to (1,+\infty)$ is a continuous bounded function, $a : (0,+\infty) \to (0,+\infty)$ is continuous and $f : (0,+\infty) \times \mathbb{R}^+ \to \mathbb{R}^+$ is a Carathéodory function, that is:

(i) $f(t,\cdot)$ is continuous for almost $t > 0$,

(ii) $f\,(\cdot,u)$ is mesurable for all $u \geq 0$.

Throughout this paper, $\psi\,(t,\cdot)$ is the inverse function of $\phi\,(t,\cdot)$ and it is given by

$$\psi(t,x) = |x|^{q(t)-2}\,x \quad \text{where} \quad q\,(t) = \frac{p(t)}{p(t)-1}.$$

To establish the main result of this paper we need to assume that:

$$\left\{ 1 < p^{-} := \inf_{t\in[0,\infty)} p(t) \leq p^{+} := \sup_{t\in[0,\infty)} p(t); \right. \tag{6.2}$$

$$\left\{ \begin{array}{l} \text{There exists}\, d \geq 0 \,\text{such that} \\ \displaystyle\int_0^{+\infty} \widetilde{a}\,(t)\,(1+t)^{-\frac{d}{p^{+}-1}}\,dt < +\infty, \quad \text{where } \widetilde{a}\,(t) = 1/\psi(t,a(t)); \end{array} \right. \tag{6.3}$$

$$\left\{ \begin{array}{l} f \text{ is } L_d - \text{Carathéodory, that is:} \\ (i) \;\; f \;\; \text{is a Carathéodory function,} \\ (ii) \;\; \text{for all} \;\; r > 0, \;\; \text{there exists } h_r \in L_d\left(\mathbb{R}^{+}\right) \;\; \text{such that} \\ \quad f\,(t,u) \leq h_r\,(t) \;\text{ for all } u \in [0,r] \;\text{ and almost } t > 0, \text{ where} \\ \quad L_d\left(\mathbb{R}^{+}\right) = \left\{ u : \mathbb{R}^{+} \to \mathbb{R}^{+} \text{ measurable and } \displaystyle\int_0^{+\infty} (1+t)^{d}\,|u(t)|\,dt < \infty \right\} \\ \quad \text{is equiped with the norm } |u|_d = \int_0^{+\infty} (1+t)^{d}\,|u(t)|\,dt. \end{array} \right. \tag{6.4}$$

By a positive solution to BVP (6.1) we mean a function $u \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ with $a(t)\phi(t,u'(t)) \in C(\mathbb{R}^+,\mathbb{R})$ and $u\,(t_0) > 0$ for some $t_0 \in [0,+\infty)$ satisfying both the differential equation and the boundary conditions in BVP (6.1).

The operator $-(a(t)\phi(t,u'(t)))'$ is known as the monodimensional $p(t)$-Laplacian and becomes the p-Laplacian when $p(t) \equiv p \in (1,+\infty)$. Because of the physical interests (see for example [8], [19], [51] and references cited therein), differential equations involving the $p(t)$-Laplacian have received a great attention in recent years and many interesting results have been obtained (see for example [73], [71], [72], [74], [58], [59], [60], [57], and [61]). Note that the $p(t)$-Laplacian is inhomogeneous. This makes the study of differential equations involving $p(t)$-Laplacian more complicated. The difficulties encountered when studying various aspects of differential equations involving the $p(t)$-Laplacian are described in [74], [58], [60] and [57].

In our case we met the following difficulty. Integrating directly, the problem of finding a positive solution to the problem (6.1) is equivalent to the problem of seeking a fixed point for the operator $Tu(t) = \int_0^t \psi\left(s, \frac{1}{a(s)} \int_s^{+\infty} f(\tau, u(\tau))\, d\tau\right) ds$, where u lies in a cone of a suitable Banach space. Unfortunately and in contrast to those provided in [53], [5] and [52] (in particular see the cone K_p in [52]) the above operator T does not provide a framework for applying the above principle because we don't know a cone which is invariant by T on which we can use a cone compression and expansion theorem.

To overcome the difficulty caused by the loss of concavity, we have associated with the bvp (6.1), by means of an operator A depending on the function p, an auxiliary problem such that Au is a positive solution to the bvp (6.1) for all positive solution u to the auxiliary problem.

Thus, we have obtained an existence result for the sublinear case without many difficulties (see Theorem 28 and its proof). In contrast, existence result for the superlinear case requires the use of homotopical argument.

6.2 Preliminaries

We need also to introduce some Banach spaces and cones. Let for $d \geq 0$, E_d be the Banach space of continuous functions u defined on $\mathbb{R}^+$ such that $\lim_{t\to+\infty} (1+t)^d u(t) = u_\infty \in \mathbb{R}$ equipped with the norm $\|u\|_d = \sup_{t\geq 0} (1+t)^d |u(t)|$.

Note that if $d = 0$ then E_0 is the Banach space of all continuous functions defined on $\mathbb{R}^+$, having a finite limit at $+\infty$ and $\|u\|_0 = \sup_{t\geq 0} |u(t)|$ and so $L_0(\mathbb{R}^+) = L^1(\mathbb{R}^+)$.

In the sequel we let E_d^+ and $L_d^+(\mathbb{R}^+)$ be respectively the cones of nonnegative functions of E_d and $L_d(\mathbb{R}^+)$, and

$$K_d = \left\{u \in E_d^+ : u \text{ is nonincreasing on } \mathbb{R}^+\right\}.$$

In order to prove compactness of some operators we will make use of the following lemma.

Lemma 41 *([9]) Let M be a subset of E_d. Then M is relatively compact in E if the following conditions hold:*

(i) M is bounded in E_d;

(ii) The functions belonging to $\left\{u:\ u(t) = (1+t)^d x(t),\ x \in M\right\}$ are equicontinuous on any compact interval of $\mathbb{R}^+$;

(iii) The functions from $\left\{u:\ u(t) = (1+t)^d x(t),\ x \in M\right\}$ are equiconvergent.

Now let us introduce some operators. Let the operator $T = LFA$ where $L : L_d(\mathbb{R}^+) \to E_d,\ F : E_0^+ \to L_d^+(\mathbb{R}^+)$ and $A : E_d^+ \to E^+$ are defined by

$$\begin{aligned}
Lu(t) &= \int_t^{+\infty} u(\tau)d\tau \quad \text{for all } u \in L_d\left(\mathbb{R}^+\right), \\
Fu(t) &= f(t, u(t)) \text{ for all } u \in E_0^+ \quad \text{and} \\
Au(t) &= \frac{\alpha}{1-\alpha}\int_0^{\eta} \tilde{a}(s)\,\psi(s, u(s))\,ds + \int_0^t \tilde{a}(s)\,\psi(s, u(s))\,ds \text{ for all } u \in E_d^+.
\end{aligned}$$

We need to present some properties related to the functions ϕ and ψ which will be very helpful in the sequel.

We note that ϕ and ψ are multiplicative i.e. for all $t, x, y \geq 0$

$$\phi(t, xy) = \phi(t, x)\,\phi(t, y) \text{ and } \psi(t, xy) = \psi(t, x)\,\psi(t, y).$$

Let ψ^-, ψ^+ be the functions defined by

$$\psi^+(x) = \begin{cases} x^{\frac{1}{p^+-1}} \text{ if } x \leq 1, \\ x^{\frac{1}{p^--1}} \text{ if } x \geq 1 \end{cases} \quad \text{and} \quad \psi^-(x) = \begin{cases} x^{\frac{1}{p^--1}} \text{ if } x \leq 1, \\ x^{\frac{1}{p^+-1}} \text{ if } x \geq 1. \end{cases}$$

Thus we have for all $x, t \geq 0$

$$\psi^-(x) \leq \psi(t, x) \leq \psi^+(x). \tag{6.5}$$

The nonincreasing property of a function $u \in K_d$ leads to

$$Au(\eta) \geq (1-\alpha)^{-1} \left(\int_0^{\eta} \tilde{a}(s)\, ds \right) \psi^{-}(u(\eta)), \tag{6.6}$$

$$\|Au\|_0 = \sup_{t \geq 0} Au(t) = Au(+\infty) = Au(\eta) + \int_{\eta}^{+\infty} \tilde{a}(s)\, \psi(s, u(s))\, ds, \tag{6.7}$$

and for all $t \geq \eta$

$$Au(t) \leq \mu(t)\, \psi^{+}(\|u\|_d) \leq \mu_\infty \psi^{+}(\|u\|_d) \tag{6.8}$$

where

$$\mu(t) = (1-\alpha)^{-1} \int_0^t \tilde{a}(s)\, \psi\left(s, (1+s)^{-d}\right) ds,$$

and

$$\mu_\infty = \mu(+\infty) = (1-\alpha)^{-1} \int_0^{+\infty} \tilde{a}(s)\, \psi\left(s, (1+s)^{-d}\right) ds.$$

Because of (6.8) the operator A is well defined and using the fact that for all $u \in L_d(\mathbb{R}^{+})$

$$\left| \lim_{t \to +\infty} (1+t)^d \int_t^{+\infty} u(s) ds \right| \leq \lim_{t \to +\infty} \int_t^{+\infty} \left(1+s^d\right) |u(s)|\, ds = 0,$$

the opearator L is well defined.

It is easy to prove the following lemma.

Lemma 42 *Assume that (6.2)-(6.4) hold. If $u \in E_d^{+}$ is a fixed point of T then Au is a positive solution of BVP (6.1).*

Proof. Let $u \in E_d^{+}$ and $v = Au$, we have

$$v(0) = Au(0) = \frac{\alpha}{1-\alpha} \int_0^{\eta} \tilde{a}(s)\, \psi(s, u(s))\, ds$$

and

$$v(\eta) = Au(\eta) = \frac{1}{1-\alpha} \int_0^{\eta} \tilde{a}(s)\, \psi(s, u(s))\, ds$$

leading to $v(0) = \alpha v(\eta)$. Also we have that

$$\begin{aligned} v'(t) &= \widetilde{a}(t)\,\psi(t,u(t)) = \widetilde{a}(t)\,\psi\left(t,(1+t)^{-d}\right)\psi\left(t,(1+t)^{d}u(t)\right) \\ &\leq \widetilde{a}(t)\,(1+t)^{\frac{-d}{p^{+}-1}}\,\psi^{+}\left(\|u\|_{d}\right). \end{aligned}$$

So, from hypothesis (6.3) follows that $\lim_{t\to+\infty} v'(t) = 0$.

Finally, if u is a fixed point of T then u satisfies

$$-u'(t) = f(t, Au(t)) = f(t, v(t))$$

leading to

$$-\left(a(t)\,\phi(t, v'(t))\right)' = f(t, v(t)).$$

■

Lemma 43 *Assume that (6.2) and (6.3) hold, then the operator $A : E_d^{+} \to E_0^{+}$ is continuous.*

Proof. Let $(u_n) \subset E_d^{+}$ be a sequence converging to $u \in E_d^{+}$. There exists $R > 0$ such that for all integer $n \geq 1$, $\|u_n\|_d \leq R$.

We have that

$$|Au_n(t) - Au(t)| \leq (1-\alpha)^{-1}\int_0^{+\infty} g_n(s)ds$$

where

$$g_n(s) = \widetilde{a}(s)\,|\psi(s, u_n(s)) - \psi(s, u(s))|.$$

The continuity of ψ and $\widetilde{a}$ imply that $\lim\limits_{n\to\infty} g_n(s) = 0$ for all $s \geq 0$. Also we have that

$$g_n(s) \leq 2\widetilde{a}(s)\,(1+s)^{-\frac{d}{p^{+}-1}}\psi^{+}(R)$$

and

$$\int_0^{+\infty} 2\widetilde{a}(s)\,(1+s)^{-\frac{d}{p^{+}-1}}\psi^{+}(R)ds < \infty.$$

By Lebesgue Dominated Convergence Theorem, we deduce that $\lim\limits_{n\to\infty} Au_n = Au$ in E_0 and A is continuous. ■

Lemma 44 *Assume that (6.2)-(6.4) hold, then T is completely continuous.*

Proof. The continuity of T follows from that of A, F and L.

Let B be a subset of E_d bounded by a positive constant R and let us prove by means of Lemma 41 that $T(B)$ is relatively compact in E_d.

First, we have from (6.8) and hypothesis (6.4) that for all $u \in B$

$$\|Tu\|_d = \sup_{t \geq 0} (1+t)^d \, Tu(t) \leq \int_0^{+\infty} (1+s)^d \, h_\delta(s) \, ds < +\infty,$$

where $\delta = \mu \psi^+ (R)$.

Then, for $t_1, t_2 \in [\xi, \zeta]$ with $t_1 < t_2$ and $v = Tu$, $u \in B$, we have

$$\begin{aligned}
& \left|(1+t_1)^d v(t_1) - (1+t_2)^d v(t_2)\right| \\
\leq \; & \left((1+t_2)^d - (1+t_1)^d\right) \int_{t_2}^{+\infty} f(s, Au(s)) \, ds + (1+t_1)^d \int_{t_1}^{t_2} f(s, Au(s)) \, ds \\
\leq \; & (t_2 - t_1) \, c_{\xi,\zeta} \int_0^{+\infty} h_\delta(s) \, ds + (1+\zeta)^d \int_{t_1}^{t_2} h_\delta(s) \, ds,
\end{aligned}$$

where

$$c_{\xi,\zeta} = \max_{s_1, s_2 \in [\xi,\zeta]} \left| \frac{(1+s_2)^d - (1+s_1)^d}{s_2 - s_1} \right|.$$

This means that the functions are equicontinuous on $[\xi, \zeta]$.

Next, note that for all $u \in B$

$$\lim_{t \to +\infty} (1+t)^d \, Tu(t) \leq \lim_{t \to +\infty} \int_t^{+\infty} (1+s)^d \, h_\delta(s) \, ds = 0,$$

resulting that the functions belonging to $\left\{u : \; u(t) = (1+t)^d x(t), \; x \in T(B)\right\}$ are equiconvergent at $+\infty$. ■

6.3 Main results

6.3.1 The sublinear case

The statement of the main result of this subsection needs to suppose that f satisfies the following conditions:

$$\begin{cases} \text{There exist two nonnegative functions } h_1, h_2 \in L_d(\mathbb{R}^+) \text{ with} \\ \mu_\infty^{p^- -1} |h_1|_d < 1 \text{ such that } \ f(t,x) \leq h_1(t)x^{p^- -1} + h_2(t) \\ \text{ for all } t, x \geq 0. \end{cases} \tag{6.9}$$

$$\begin{cases} \text{There exist } \theta > \eta,\ \delta > 0 \text{ and a function } h_\delta \in L_d([\eta,\theta]) \text{ with} \\ \Lambda_1 \int_\eta^\theta h_\delta(s)\,ds > 1, \text{ such that } f(t,u) \geq h_\delta(t)\,u^{p^- -1} \text{ for all } u \in [0,\delta] \\ \text{and } t \in [\eta,\theta], \text{where } \Lambda_1 = \left((1-\alpha)^{-1} \displaystyle\int_0^\eta \widetilde{a}(s)\,ds \right)^{p^- -1}. \end{cases} \tag{6.10}$$

Theorem 45 *Assume that (6.2), (6.3), (6.4), (6.9) and (6.10) hold. Then BVP (6.1) admits at least one positive solution.*

Proof. Let $r = \min\left(1, (\mu(\theta)\,\delta)^{p^+ -1}\right)$, we have for all $u \in K \cap \partial B(0,r)$ and $t \in [0,\theta]$ that

$$Au(t) \leq \mu(\theta)\,\psi^+(\|u\|_d) = \delta.$$

Let $u \in K \cap \partial B(0,r)$ and $v = Tu$. We have

$$\begin{aligned} Tu(\eta) &= \int_\eta^{+\infty} f(s, Au(s))ds \geq \int_\eta^\theta f(s, Au(s))ds \\ &\geq \int_\eta^\theta h_\delta(s)\,(Au(s))^{p^- -1}\,ds. \end{aligned} \tag{6.11}$$

This, together with (6.6) leads to

$$Tu(\eta) \geq u(\eta)\,\Lambda_1 \int_\eta^\theta h_\delta(s)\,ds > u(\eta).$$

This means that $Tu \nleq u$ for all $u \in \partial B(0,r) \cap K$ and we have by Lemma **??** $i(T, K_d \cap B(0,r), K_d) = 0$.

Now, let $u\in K\cap\partial B(0,R_{\infty})$ with $R_{\infty}>\max\left\{1,\left(\left|h_2\right|_d/\left(1-\mu^{p^- -1}(\infty)\left|h_1\right|_d\right)\right)\right\}$. We have from (6.9) that for all $t\geq 0$

$$Tu(t)\leq\int_t^{+\infty}\Big(h_1(s)\left(Au(s)\right)^{p^- -1}+h_2(s)\Big)\,ds,$$

from which

$$(1+t)^d\,Tu(t)\leq\int_t^{+\infty}\left(1+s^d\right)\Big(h_1(s)\left(Au(s)\right)^{p^- -1}+h_2(s)\Big)\,ds.$$

This together with (6.8) lead to

$$\begin{aligned}\left\|Tu\right\|_d &\leq \int_t^{+\infty}(1+s)^d\,h_1(s)\left(\mu(\infty)\,\psi^+\left(\left\|u\right\|_d\right)\right)^{p^- -1}ds+\int_t^{+\infty}(1+s)^d\,h_2(s)ds\\ &\leq \mu_{\infty}^{p^- -1}\left|h_1\right|_d\left\|u\right\|_d+\left|h_2\right|_d<\left\|u\right\|_d.\end{aligned}$$

So we have from Lemma **??** that $i(T,K_d\cap B\left(0,R_{\infty}\right),K_d)=1$. This means that $\left\|Tu\right\|_d\leq\left\|u\right\|_d$ for all $u\in\partial B(0,R_{\infty})\cap K$. Finally, the additivity and solution properties of the fixed point index imply that

$$i(T,\big(B(0,R_{\infty})\setminus\overline{B}(0,r)\big)\cap K,K)=i(T,K\cap B(0,R_{\infty}),K)-i(T,K\cap B(0,r),K)=1$$

and the operator T admits a positive fixed point u with $\delta<\|u\|<R_{\infty}$ and by Lemma 42 Au is a positive solution to bvp (6.1). ■

6.3.2 The superlinear case

In this subsection we let

$$\Lambda_2=\left((1-\alpha)^{-1}\int_0^{\eta}\widetilde{a}\,(s)\,ds\right)^{p^+ -1}$$

and $g\in C\left(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+\right)$, $\theta>\eta$ and $q\in L^1\left([\eta,\theta]\right)$ be given and satisfying $g(t,u)\geq q\,(t)\,u^{p^+ -1}$ and $\Lambda_2\int_{\eta}^{\theta}q\,(s)\,ds>1$. Let $T_0=LGA\Gamma$ where $G,\Gamma:E_0^+\to E_0^+$ are defined for $u\in E_0^+$ by $Gu\,(t)=g\,(u\,(t))$ and $\Gamma u\,(t)=\max\{u(t),1\}$. The following lemma provides a fixed point computation for the operator T_0.

Lemma 46 *Assume that (6.2)-(6.4) hold with $d = 0$. Then for all $R > 0$*

$$i(T_0, B(0, R) \cap K, K) = 0.$$

Proof. We have from (6.6) that for all $u \in K \cap \partial B\,(0, R)$

$$\begin{aligned} T_0 u(\eta) &= \int_\eta^{+\infty} g\,(t, A\Gamma u(s))ds \\ &\geq \int_\eta^\theta q\,(s)\,(A\Gamma u(s))^{p^+-1} ds \\ &\geq \int_\eta^\theta q\,(s)\,ds\,(A\Gamma u(\eta))^{p^+-1} \\ &\geq \int_\eta^\theta q\,(s)\,ds\,(Au(\eta))^{p^+-1} \\ &\geq u(\eta)\Lambda_2 \int_\eta^\theta q\,(s)\,ds > u(\eta). \end{aligned}$$

This means that for all $u \in \partial B(0, R) \cap K,\ T_0 u \nleq u$. So, by Lemma **??**, $i(T_0, B(0, R) \cap K, K) = 0$. ∎

To establish the existence result for the superlinear case, we will assume that there exist $R > \delta > 0,\ \theta > \eta$ and two functions $h_1,\ h_\delta \in L_0\,(\mathbb{R}^+) = L^1\,(\mathbb{R}^+)$ with $\Lambda_2 \int_\eta^\theta h_1\,(s)\,ds > 1$ and $|h_\delta|_0\,\mu_\infty^{p^+-1} < 1$ such that

$$f\,(t, u) \geq h_1\,(t)\,u^{p^+-1} \text{ for all } t \in [\eta, \theta]\,, u \geq R \tag{6.12}$$

and

$$f\,(t, u) \leq h_\delta\,(t)\,u^{p^+-1} \text{ for all } t \geq 0 \text{ and } u \leq \delta. \tag{6.13}$$

Theorem 47 *Assume that (6.2), (6.3), (6.4) with $d = 0$, (6.12) and (6.13) hold. Then BVP (6.1) admits a positive solution.*

Proof. Let $r = \min\left(1, (\delta/\mu_\infty)^{p^+-1}\right)$ and $u \in \partial B(0, r) \cap K$. We have from (6.8)

$$\|Au\|_0 \leq \mu_\infty r^{p^+-1} \leq \delta.$$

Thus for all $u \in \partial B(0,r) \cap K$

$$\begin{aligned}\|Tu\|_0 &= \sup_{t\geq 0} Tu(t) \leq \int_0^{+\infty} f(s, Au(s))ds \\ &\leq \int_0^{+\infty} h_\delta(s)\,(Au(s))^{p^+-1}\,ds \\ &\leq \mu_\infty^{p^+-1} \int_0^{+\infty} h_\delta(s)\,ds\,\|u\|_0 < \|u\|_0\,.\end{aligned}$$

This means that $\|Tu\|_0 < \|u\|_0$ for all $u \in \partial B(0,r) \cap K$, so, we deduce from Lemma **??** that $i(T, K \cap B(0,r), K) = 1$. To conclude, by the additivity and solution properties of the fixed point index as in the proof of Theorem 45, we have to prove the existence of an $R^\infty > r$ such that

$$i(T, K \cap B(0, R^\infty), K) = 0.$$

Consider for $\nu \in [0,1]$, the fixed point equation

$$u = (1-\nu)\,Tu + \nu T_0 u. \tag{6.14}$$

We claim that:

there exists $R^\infty > r$ such that equation (6.14) has no solution in $K \cap \partial B\,(0, R^\infty)$.

Indeed, to the contrary, assume that for all integer $n \geq 1$ there exist $\nu_n \in [0,1]$ and $u_n \in K \cap \partial B(0,n)$ such that

$$u_n = (1-\nu_n)\,Tu_n + \nu_n T_0 u_n. \tag{6.15}$$

We have that $\lim_{n\to\infty} \|Au_n\| = \lim_{n\to\infty} Au_n(+\infty) = +\infty$, since, if there exists a subsequence (u_{n_k}) such that $\|Au_{n_k}\|$ is bounded by a constant $M > 0$ and

$$M_1 = \sup\left\{f(s,x) + g(x) : s \in [0,1],\ x \in [0,M]\right\},$$

then we have the contradiction

$$\|u_{n_k}\| = u_{n_k}(0) = (1-\nu_{n_k}) \int_0^{+\infty} f(s, Au_{n_k}(s))ds + \nu_{n_k} \int_0^{+\infty} g(Au_{n_k}(s))ds \leq M_1.$$

The reminder of the proof amounts to showing that $\lim\limits_{n\to\infty} Au_n(\eta) = +\infty$. We distinguish two cases:

- If $\lim_{n\to\infty} u_n(\eta) = +\infty$, then we have from (6.6), $\lim_{n\to\infty} Au_n(\eta) = +\infty$.

- If there exists a subsequence (u_{n_k}) such that $(u_{n_k}(\eta))$ is bounded and

$$m = \sup_{k\in\mathbb{N}} (u_{n_k}(\eta)),$$

then we have from (6.7) the contradiction

$$\|Au_{n_k}\|_0 = Au_{n_k}(+\infty) \leq Au_{n_k}(\eta) + \mu_\infty \psi^+(m) < +\infty.$$

We have from (6.15) and (6.6) for n large, the contradiction

$$\begin{aligned} u_n(\eta) &= \int_\eta^{+\infty} ((1-\nu_n) f(s, Au_n(s)) + \nu_n g(A\Gamma u_n(s)))\, ds \\ &> \int_\eta^{\theta} (h_1(s) + q(s)) (Au_n(s))^{p^+-1}\, ds \\ &\geq \int_\eta^{\theta} (h_1(s) + q(s)) (Au_n(\eta))^{p^+-1}\, ds \\ &\geq \Lambda_2 u_n(\eta) \int_\eta^{\theta} (h_1(s) + q(s))\, ds > u_n(\eta). \end{aligned}$$

Thus the claim is proved, and by the homotopy property of the fixed point index, we have

$$i(T, B(0, R^\infty) \cap K, K) = i(T_0, B(0, R^\infty) \cap K, K) = 0.$$

This completes the proof ■

6.3.3 The separable variables case

Consider the bvp

$$\begin{cases} -(a(t)\,\phi(t, u'(t)))' = b(t)\,\chi(u(t)),\ t \in (0, +\infty), \\ u(0) = \alpha u(\eta),\ u'(+\infty) = 0, \end{cases} \tag{6.16}$$

where a, ϕ, α, η are those of the introduction, $b \in L_d(\mathbb{R}^+)$ is nonnegative and $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

Let $\theta > \eta$ and set

$$\chi_0 = \liminf_{u\to 0} \frac{\chi(u)}{u^{p^- -1}}, \quad \chi^\infty = \limsup_{u\to +\infty} \frac{\chi(u)}{u^{p^- -1}}$$

$$\chi^0 = \limsup_{u\to 0} \frac{\chi(u)}{u^{p^+ -1}}, \quad \chi_\infty = \liminf_{u\to +\infty} \frac{\chi(u)}{u^{p^+ -1}}.$$

We deduce from Theorems 45 and 47 respectively the following corollaries 48 and 49.

Corollary 48 *Assume that (6.2), (6.3), (6.4) and*

$$\begin{cases} \text{there exists } \theta > \eta \text{ such that} \\ \mu_\infty^{p^- -1} \chi^\infty \left|b\right|_d < 1 < \chi_0 \Lambda_1 \int_\eta^\theta b(s)ds \end{cases}$$

hold. Then the BVP (6.16) admits a positive solution.

Proof. Let $\epsilon > 0$ be such that $(\chi_0 - \epsilon)\Lambda_1 \int_\eta^\theta b(s)ds > 1$, and there exists $\delta > 0$ such that $\chi(u) \geq (\chi_0 - \epsilon) u^{p^- -1}$ for all $u \in [0,\delta]$. Thus for all $u \in [0,\delta]$ and all $t \in [\eta,\theta]$, $b(t)\chi(u) \geq (\chi_0 - \epsilon) b(t) u^{p^- -1}$ and Hypothesis (6.10) is satisfied with $h_\delta = (\chi_0 - \epsilon) b$.

Let $\varepsilon > 0$ be such that $\mu_\infty^{p^- -1} (\chi^\infty + \varepsilon) \left|b\right|_d < 1$, there exists $C > 0$ such that $\chi(u) \leq (\chi^\infty + \varepsilon) u^{p^- -1} + C$ for all $u \geq 0$.

So, we have that the hypothesis (6.9) is satisfied with $h_1 = (\chi^\infty + \varepsilon) b$ and $h_2 = Cb$.

The assertion follows from Theorem 45. ■

Corollary 49 *Assume that (6.2), (6.3), (6.4) and*

$$\begin{cases} \text{there exists } \theta > \eta \text{ such that} \\ \mu_\infty^{p^+ -1} \chi^0 \left|b\right|_d < 1 < \chi_\infty \Lambda_2 \int_\eta^\theta b(s)ds \end{cases}$$

hold. Then the BVP (6.16) admits a positive solution.

Proof. Let $\epsilon > 0$ be such that $(\chi_\infty - \epsilon)\Lambda_2 \int_\eta^\theta b(s)ds > 1$, and there exists $R > 0$ such that $\chi(u) \geq (\chi_\infty - \epsilon) u^{p^- -1}$ for all $u \geq R$. Thus for all $u \geq R$ and all $t \in$

$[\eta,\theta]$, $b(t)\chi(u) \geq (\chi_0-\epsilon)b(t)u^{p^--1}$ and hypothesis (6.12) is satisfied with $h_1 = (\chi_\infty-\epsilon)b$.

Let $\varepsilon > 0$ be such that $\mu_\infty^{p^+-1}(\chi^0+\varepsilon)|b|_d < 1$, and there exists $\delta > 0$ such that $\chi(u) \leq (\chi^0+\varepsilon)u^{p^--1}$ for all $u \in [0,\delta]$.

So, we have that Hypothesis (6.13) is satisfied with $h_\delta = (\chi^0+\varepsilon)b$.

The assertion follows from Theorem 47. ■

6.4 Examples

Example 50 *Theorems 45 and 47 cover, respectively, the cases*

$$\lim_{u\to 0}\frac{f(t,u)}{u^{p^--1}} = +\infty \quad \textit{and} \quad \lim_{u\to+\infty}\frac{f(t,u)}{u^{p^--1}} = 0 \textit{ uniformly on } [0,1],$$

and

$$\lim_{u\to 0}\frac{f(t,u)}{u^{p^+-1}} = 0 \quad \textit{and} \quad \lim_{u\to+\infty}\frac{f(t,u)}{u^{p^+-1}} = +\infty \textit{ uniformly on } [0,1].$$

In particular, they cover respectively the cases

$$f(t,u) = u^{m(t)} \textit{ with } \sup_{t\geq 0} m(t) < p^- - 1$$

and

$$f(t,u) = u^{m(t)} \textit{ with } p^+ - 1 < \inf_{t\geq 0} m(t).$$

Example 51 *Consider the BVP (6.1) with*

$$\begin{aligned} a(t) &= 1+t,\ p(t) = \frac{3+2t}{1+t},\ f(t,u) = b(t)\chi(u) \textit{ where } b(t) = (1+t)^{-\frac{3}{2}}, \\ \chi(u) &= \frac{Au+Bu^2}{1+u}\ A,B>0 \textit{ and } \alpha = \eta = \frac{1}{2}. \end{aligned}$$

By simple computations we get that

$$p^+ = 3,\ p^- = 2,\ \chi_0 = A,\ \chi^\infty = B,$$
$$\frac{1}{1+t} \leq \tilde{a}(t) = \psi\left(t, \frac{1}{1+t}\right) \leq \frac{1}{\sqrt{1+t}} \textit{ and } \ln(9/4) \leq \Lambda_1 \leq 1.$$

Also choosing $d = \dfrac{3}{2}$, *we have*

$$\mu_\infty \le 8, \ \ |b|_{\frac{3}{2}} = 2,$$

$$\lim_{\theta\to+\infty} \int_\eta^\theta b(s)ds = \lim_{\theta\to+\infty} 2\left(\frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{\sqrt{1+\theta}}\right) = \frac{2\sqrt{2}}{\sqrt{3}}.$$

We deduce from Corollary 48 that BVP (6.1) admits a positive solution if

$$B < \frac{1}{16} < \sqrt{\frac{3}{2}} \frac{1}{4\ln\left(\frac{3}{2}\right)} < A.$$

Example 52 *Consider the BVP (6.1) with*

$$\begin{aligned} a(t) &= (1+t)^3, \ p(t) = \frac{3+2t}{1+t}, \ f(t,u) = b(t)\chi(u) \\ b(t) &= \frac{1}{1+t^2}, \ \chi(u) = u^\sigma \exp(u), \ \sigma < 2. \end{aligned}$$

We have that

$$\chi^0 = 0 \ \textit{and} \ \chi_\infty = +\infty.$$

We deduce from Corollary 49 that BVP (6.1) admits a positive solution.

Chapter 7

EIGENVALUE CRITERIA FOR EXISTENCE OF POSITIVE SOLUTION TO SINGULAR THIRD-ORDER BVPS VIA THE INDEX-JUMP PROPERTY

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7.1 Introduction

This paper deals with existence of positive solutions to the third order boundary value problem (bvp for short)

$$\begin{cases} -\left(pu''\right)'(t) = f(t,u(t)), \;\; t \in (0,1), \\ u(0) = au'(0) - b\lim\limits_{t\to 0} p(t)u''(t) = 0, \\ cu'(1) + d\lim\limits_{x\to 1} p(t)u''(t) = 0, \end{cases} \tag{7.1}$$

where $a,b,c,d \in \mathbb{R}^+$, $p \in C\left((0,1), (0,+\infty)\right)$, $f : (0,1) \times \mathbb{R}^+ \to \mathbb{R}^+$ is a Carathéodory function ($f(\cdot,u)$ is measurable for u fixed and $f(t,\cdot)$ is continuous for almost $t \in (0,1)$) and $\mathbb{R}^+ = [0,+\infty)$.

Throughout this chapter, we assume that

$$\int_0^1 \frac{dt}{p(t)} < \infty \tag{7.2}$$

and

$$\Delta = ac\int_0^1 \frac{dt}{p(t)} + ad + bc > 0. \tag{7.3}$$

The second order version of Problem (7.1) have been studied in [1]. More precisely, the authors of [1] have proved that the bvp

$$\begin{cases} -\frac{1}{p(t)}\left(pu'\right)'(t) = q(t)f(t,u(t)), \quad t \in (0,1), \\ au(0) - b\lim\limits_{t\to 0} p(t)u'(t) = 0, \\ cu(1) + d\lim\limits_{t\to 1} p(t)u'(t) = 0, \end{cases} \tag{7.4}$$

where a, b, c and d are nonnegative real numbers such that $\Delta > 0$, $p \in C\left((0,1),(0,+\infty)\right)$ satisfies (7.2), $q \in C\left((0,1),[0,+\infty)\right)$ does not vanish identically on $(0,1)$ and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, admits a positive solution under one of the following eigenvalue criterion

$$f^0 < \lambda_1 < f_\infty, \tag{7.5}$$

and

$$f^\infty < \lambda_1 < f_0, \tag{7.6}$$

with

$$f_\nu = \liminf_{u\to\nu}\left(\min_{t\in[0,1]}\frac{f(t,u)}{u}\right), \quad f^\nu = \limsup_{u\to\nu}\left(\max_{t\in[0,1]}\frac{f(t,u)}{u}\right),$$

for $\nu = 0, +\infty$ and λ_1 is the smallest positive eigenvalue of

$$\begin{cases} -\frac{1}{p(t)}\left(pu'\right)'(t) = \lambda q(t)u(t), \quad t \in (0,1), \\ au(0) - b\lim\limits_{t\to 0} p(t)u'(t) = 0, \\ cu(1) + d\lim\limits_{t\to 1} p(t)u'(t) = 0. \end{cases}$$

As mentioned in [1], the existence conditions (7.5) and (7.6) are in some sense optimal. All the results in [1] are proved by means of fixed point index calculations obtained as follows. Consider the familly of bvps

$$\begin{cases} -\frac{1}{p(t)}\left(pu'\right)'(t) = \lambda q(t)f(t,u(t)), \quad t \in (0,1), \\ au(0) - b\lim\limits_{t\to 0} p(t)u'(t) = 0, \\ cu(1) + d\lim\limits_{t\to 1} p(t)u'(t) = 0, \end{cases}$$

where λ is a positive real parameter and let ϕ_1 be the positive eigenfunction associated with λ_1. It is proved in [1] that

$$\int_0^1 -(pu')'\phi_1 = \int_0^1 pq\phi_1\lambda f(t,u), \quad \int_0^1 -(p\phi_1')'u = \int_0^1 pq\phi_1\lambda_1 u \tag{7.7}$$

and

$$\int_0^1 -(pu')'\phi_1 + (p\phi_1')'u = 0, \tag{7.8}$$

leading to

$$\int_0^1 pq\phi_1 (\lambda f(t,u) - \lambda_1 u) = 0, \tag{7.9}$$

which combined with hypothesis (7.5) or (7.6) give us that the fixed point index of some operator relatively to some cone is equal to 0 or 1.

We have from (7.7) and (7.8) that

$$\int_0^1 (pu')'\phi_1 = \int_0^1 (p\phi_1')'u.$$

Roughtly speaking, this means that the second order differential operator $-\frac{d}{dt}\left(p\frac{d}{dt}\cdot\right)$ is self-adjoint. Also note that there is some compatibility between the bilinear form in (7.7)-(7.9) and the cone of nonnegative functions.

In the same spirit as in [53], [1], [2], [4], [37] and references therein, we will obtain in this chapter existence results of positive solutions to bvp (7.1) under eingenvalue criteria. These results can not be obtained with the same technics of [1], since the third order differential operator $-\frac{d}{dt}\left(p\frac{d^2}{dt^2}\cdot\right)$ is not self-adjoint. To overcome this difficulty, we will develop in Section 2 some ideas taken from [3], [53], [1] and [2] to obtain abstract existence results related to index-jump property of positive linear compact operators.

7.2 Abstract Background

For sake of completeness let us recall some basic facts needed in this paper. Let E be a real Banach space endowed with norm $\|\cdot\|$, $L(E)$ is the set of all linear continuous mapping from E into E. For $L \in L(E)$, $r(L) = \lim_{n\to\infty} \|L^n\|^{\frac{1}{n}}$ denotes the spectral radius of L.

In all the following, $Q^+(E)$ denotes the subset of $L(E)$ consisting of all compact positive operators.

Definition 53 *Let $L \in Q^+(E)$. L is said to have the index-jump property at $\mu > 0$ if for all $\gamma, R > 0$ we have*

$$i(\gamma L, B(0,R) \cap K, K) = \begin{cases} 1, & if\ \gamma\mu < 1, \\ 0, & if\ \gamma\mu > 1. \end{cases}$$

Remark 54 *If $L \in Q^+(E)$ has the index-jump property at μ, then μ is a unique positive eigenvalue of L. Indeed, if for all $u \in K \setminus \{0\}$, $Lu \neq \mu u$, then $i\left(\frac{L}{\mu}, K_1, K\right)$ is well defined and we have for any sequences $(\alpha_n) \subset (0, 1/\mu)$ and $(\beta_n) \subset (1/\mu, +\infty)$ with $\lim \alpha_n = \lim \beta_n = 1/\mu$, the contradiction*

$$0 = \lim i(\beta_n L, K_R, K) = i(L/r(L), K_R, K) = \lim i(\alpha_n L, K_R, K) = 1.$$

Remark 55 *It follows from Remark 54 that, if $L \in Q^+(E)$ has the index-jump property at μ, then*

$$\inf \Lambda_L^+ \leq \mu \leq \sup \Lambda_L^-,$$

where

$$\begin{aligned} \Lambda_L^+ &= \{\lambda \geq 0 : there\ exists\ u \in K \setminus \{0\}\ \ such\ that\ \ Lu \leq \lambda u\}\ \ and \\ \Lambda_L^- &= \{\lambda \geq 0 : there\ exists\ u \in K \setminus \{0\}\ \ such\ that\ \ Lu \geq \lambda u\}. \end{aligned}$$

Definition 56 *Let $L \in Q^+(E)$. L is said to have the strongly index-jump property at $\mu > 0$ if*

$$\mu = \inf \Lambda_L^+ = \sup \Lambda_L^-.$$

Clearly, we have:

Proposition 57 *If $L \in L(E)$ has the strongly index-jump property at μ, then L has the index-jump property at μ.*

Proof. Let $R > 0$, $\gamma \in (0, 1/\mu)$ and $u \in \partial K_R$ such that $\gamma L u \geq u$. This implies that $1/\gamma \in \Lambda_L^-$ and $1/\gamma \leq \sup \Lambda_L^- = \mu$ which contradicts $\gamma \in (0, 1/\mu)$. So, by Lemma **??**, $i(\gamma L, K_R, K) = 1$.

Let $R > 0$, $\gamma > 1/\mu$ and $u \in \partial K_R$ such that $\gamma L u \leq u$. This implies that $1/\gamma \in \Lambda_L^+$ and $1/\gamma \geq \inf \Lambda_L^+ = \mu$ which contradicts $\gamma > 1/\mu$. So, by Lemma **??**, $i(\gamma L, K_R, K) = 0$. This completes the proof. ■

Remark 58 *Let $L \in Q^+(E)$ having the strongly index-jump property at $\mu > 0$. We have from Remark 54 that $\mu \leq r(L)$. Thus uniqueness of μ combined with Theorem 19.2 in [39] (Proposition 7.26 in [17]) or Theorem 19.3 in [39] (Theorem 7.C in [17]) leads to that, if K is total or L is strongly positive, then $\mu \leq r(L)$.*

In the following, we present a class of operators of $Q^+(E)$ having the strongly index-jump property and consequently the index-jump property.

Proposition 59 *Let $L \in Q^+(E)$ be strongly positive. If K is normal, or L is $K-$normal, then L has the strongly index-jump property at $r(L)$.*

Proof. It follows from the Krein-Rutman Theorem (Theorem 7.C in [17] or Theorem 19.3 in [39]) that $r(L) > 0$ and $r(L)$ is the unique positive eigenvalue of L. Moreover $r(L)$ is simple and it is associated with an eigenvector in int$K$. Note that $r(L) \in \Lambda_L^+ \cap \Lambda_L^-$ and $\inf \Lambda_L^+ \leq r(L) \leq \sup \Lambda_L^-$. Set $\lambda^+ = \inf \Lambda_L^+$ and $\lambda^- = \sup \Lambda_L^-$, we want to prove that $r(L) = \lambda^+ = \lambda^-$. To this aim let $\lambda > 0$ and $u \in K \setminus \{0\}$ be such that $Lu \geq \lambda u$. We have for all integer $n \geq 1$,

$$L^n u \geq \lambda^n u.$$

Since L is $K-$normal (the case K normal can be checked similarly) we have for all integer n,

$$\|L^n\| \geq \left\|L^{n+1}u\right\| \geq \lambda^n \|Lu\|,$$

from which follows

$$\|L^n\|^{1/n} \geq \lambda \|Lu\|^{1/n}.$$

Letting $n \to \infty$, we obtain

$$r(L) = \lim_{n\to\infty} \|L^n\|^{1/n} \geq \lambda.$$

Since λ is arbitrary in the above calculations and $r(L) \in \Lambda_L^-$ we deduce that $r(L) = \lambda^-$.

Now, assume that $\lambda^+ < \lambda^-$ and let $\lambda \in (\lambda^+, \lambda^-)$. For a such λ, there exist $u, v \in K \setminus \{0\}$ such that $Lu \leq \lambda u$ and $Lv \geq \lambda v$. Set $U = Lu$ and $V = Lv$ and see that, since L is strongly positive, $U, V \in \operatorname{int}(K)$ and there exists $\delta > 0$, small enough, such that $U \geq V_\delta = \delta V$.

If L is $K-$normal, we have that $C = \overline{L([V_\delta, U])}$ (in the case K normal we take $C = [V_\delta, U]$) is a closed bounded convex set in E. Moreover we have that the compact operator $T = L/\lambda$ maps C into itself. So, Schauder's fixed point theorem guarantees existence of a fixed point $w \in C$ of T and λ is a positive eigenvalue of L. This shows that $(\lambda^-, \lambda^+) \subset \sigma(L)$ which contradicts to the fact that L is compact. We conclude from all the above that $\lambda^+ \leq r(L) \leq \lambda^- \leq \lambda^+$ and then $r(L) = \lambda^- = \lambda^+$. This completes the proof. $\blacksquare$

Now set

$$\mathrm{SIJ}(E) = \left\{L \in Q^+(E) : L \text{ has the strongly index-jump property at } r(L)\right\}$$

and

$$\mathrm{IJ}(E) = \left\{L \in \overline{SIJ(E)} : r(L) > 0\right\}.$$

We need in the following, to recall a fundamental result proved by Nussbaum in [62] as quoted in [2] and [37].

Lemma 60 *Let (L_n) be a sequence of compact linear operators on a Banach space X and suppose that $L_n \to L$ in operator norm as $n \to \infty$. Then $r(L_n) \to r(L)$.*

Proposition 61 *If $L \in IJ(E)$, then L has the index-jump property at $r(L)$.*

Proof. Let $\gamma, R > 0$ with $\gamma \neq 1/r(L)$. Then we have $\gamma L_n \to \gamma L$ in the operator norm. So, if $\gamma < 1/r(L)$ then, for n large enough, $\gamma < 1/r(L_n)$ and

$$i(\gamma L, B(0,R) \cap K, K) = i(\gamma L_n, B(0,R) \cap K, K) = 1,$$

and, if $\gamma > 1/r(L)$ then, for n large enough, $\gamma > 1/r(L_n)$ and

$$i(\gamma L, B(0,R) \cap K, K) = i(\gamma L_n, B(0,R) \cap K, K) = 0.$$

The proposition is proved ■

Definition 62 *Let $L \in Q^+(E)$. L is said to have the $\sigma-$ index-jump property if $r(L) > 0$ and there exists a convex subset C of $SIJ(E)$ such that $L \in \overline{C}$.*

Remark 63 *We deduce from Proposition 61 that, if $L \in Q^+(E)$ has the $\sigma-$index-jump property at $r(L)$, then $L \in IJ(E)$ and so, L has the index-jump property at $r(L)$.*

We present now an existence principle for positive solutions (solutions in $K \setminus \{0\}$) to the abstract equation

$$u = LFu \tag{7.10}$$

where $L \in Q^+(E)$ has the $\sigma-$index-jump property at $r(L)$ and $F : K \to K$ is continuous and bounded.

Theorem 64 *Assume that K is a normal cone in E, $L \in Q^+(E)$ has the $\sigma-$index-jump property and there exist three positive constants α, β, γ and three functions $G_i : K \to K, i = 1, 2, 3$ such that*

$$\alpha r(L) < 1 < \beta r(L),$$

and for all $u \in K$,

$$Fu \leq \alpha u + G_1(u),$$

$$\beta u - G_2(u) \leq F(u) \leq \gamma u + G_3(u).$$

Then, if one of the following situations

$$G_1(u) = \circ\left(\|u\|\right) \textit{ near } 0 \textit{ and } G_i(u) = \circ\left(\|u\|\right) \textit{ near } \infty \textit{ for } i = 2, 3, \tag{7.11}$$

and

$$G_1(u) = \circ\left(\|u\|\right) \textit{ near } \infty \textit{ and } G_i(u) = \circ\left(\|u\|\right) \textit{ near } 0 \textit{ for } i = 2, 3, \tag{7.12}$$

holds true, equation (7.10) *admits a positive solution.*

Proof. We present the proof in the case where (7.11) holds, the other case is checked similarly. We have to prove the existence of $0 < r < R$ such that

$$i(LF, B(0,r) \cap K, K) = 1 \quad \text{and} \quad i(LF, B(0,R) \cap K, K) = 0.$$

Then the additivity and the solution properties of fixed point index will imply that

$$i(LF, \left(B(0,R) \setminus \overline{B}(0,r)\right) \cap K, K) = i(LF, K_R, K) - i(LF, K_r, K) = -1,$$

and equation (7.10) admits a positive solution u, with $r < \|u\| < R$.

Now, consider the function $H_1 : [0,1] \times K \to K$ defined by $H_1(t,u) = tLFu + (1-t)\alpha Lu$ and let us prove the existence of $r > 0$, small enough, such that for all $t \in [0,1]$ the equation $H_1(t,u) = u$ has no solution in $\partial B(0,r) \cap K$. In the contrary, suppose that for all integer $n \geq 1$, there exist $t_n \in [0,1]$ and $u_n \in \partial B\left(0, \frac{1}{n}\right) \cap K$ such that

$$u_n = t_n LFu_n + (1-t_n)\alpha Lu_n.$$

Note that $v_n = \dfrac{u_n}{\|u_n\|} \in \partial B(0,1) \cap K$ and satisfies

$$v_n = t_n L\left(\frac{Fu_n}{\|u_n\|}\right) + (1-t_n)\alpha Lv_n.$$

The normality of the cone K, combined with the inequalities

$$\frac{Fu_n}{\|u_n\|} \leq \alpha v_n + \frac{G_1 u_n}{\|u_n\|}$$

and the fact that $G_1(u_n) = \circ(\|u_n\|)$ at 0 shows that $\frac{Fu_n}{\|u_n\|}$ is bounded. From the compactness of L we conclude the existence of a subsequence, denoted also (v_n), that converges to $v \in \partial B(0,1) \cap K$, satisfying $v \leq \alpha L v$. Namely, $\frac{1}{\alpha} \leq r(L)$ which contradicts $\alpha r(L) < 1$.

For a such $r > 0$, we deduce from the homotopy property of fixed point index that

$$\begin{aligned} i(LF, K_r, K) &= i(H_1(1,\cdot), K_r, K) \\ &= i(H_1(0,\cdot), K_r, K) \\ &= i(\alpha L, K_r, K) \\ &= 1. \end{aligned}$$

At this stage let $(L_n) \subset \mathrm{SIJ}(E)$ be a sequence of operators converging to L such that, for all $t \in [0,1]$ and all integers n, m, $(tL_n + (1-t)L_m) \in \mathrm{SIJ}(E)$. We deduce from Lemma 60 that there exists an integer n_∞ such that, for all $n, m \geq n_\infty$ and all $t \in [0,1]$, $\beta r(tL_n + (1-t)L_m) > 1$. We need to prove that there exists $R_1 > 0$, such that for all $R > R_1$ and all integers $n, m \geq n_\infty$,

$$i(L_n F, B(0,R) \cap K, K) = i(L_m F, B(0,R) \cap K, K).$$

Consider the function $H_\infty : [0,1] \times K \to K$ defined by $H_\infty(t,u) = tL_n Fu + (1-t)L_m Fu$ and let us prove the existence of $R_1 > 0$, large enough, such that for all $t \in [0,1]$ and all $R > R_1$, the equation $H_\infty(t,u) = u$ has no solution in $\partial B(0,R) \cap K$. In the contrary, suppose that there is a sequence (R_j), with $\lim R_j = \infty$, such that for all integer $j \geq 1$, there exist $t_j \in [0,1]$ and $u_j \in \partial B(0,R_j) \cap K$ such that

$$u_j = t_j L_n F u_j + (1-t_j) L_m F u_j.$$

Note that $v_j = \dfrac{u_j}{\|u_j\|} \in \partial B(0,1) \cap K$ and satisfies

$$v_j = t_j L_n\left(\frac{Fu_j}{\|u_j\|}\right) + (1-t_j)L_m\left(\frac{Fu_j}{\|u_j\|}\right).$$

The normality of the cone K, combined with inequalities

$$\beta v_n - \frac{G_2 u_j}{\|u_j\|} \le \frac{Fu_j}{\|u_j\|} \le \gamma v_n + \frac{G_3 u_j}{\|u_j\|}$$

and the fact that $G_i(u_j) = \circ\left(\|u_j\|\right)$ at ∞ for $i = 2,3$ leads to $\dfrac{Fu_j}{\|u_j\|}$ is bounded. From the compactness of L_n and L_m we obtain the existence of a subsequence, denoted also (v_j), which converges to $v \in \partial B(0,1) \cap K$ and $\bar{t} \in [0,1]$ such that $v \ge \beta\left(\bar{t}L_n + \left(1-\bar{t}\right)L_m\right)(v)$. This, with $\left(\bar{t}L_n + \left(1-\bar{t}\right)L_m\right) \in SIJ(E)$ leads to $\dfrac{1}{\beta} \ge r(\bar{t}L_n + \left(1-\bar{t}\right)L_m)$, which contradicts to $\beta r\left(\bar{t}L_n + \left(1-\bar{t}\right)L_m\right) > 1$.

Let $n_* \ge n_\infty$ be fixed. With the same arguments as above we prove that there exists $R_* > 0$ such that for all $R > R_*$, the equation $H_*(t,u) = u$ has no solution in $K \cap \partial B(0,R)$ where $H_* : [0,1] \times K \to K$ is defined by $H_*(t,u) = tL_{n_*}Fu + (1-t)\beta L_{n_*}u$. Then for all $R > R_*$

$$\begin{aligned} i(L_{n_*}F, K_R, K) &= i(H_*(1,\cdot), K_R, K) \\ &= i(H_*(0,\cdot), K_R, K) \\ &= i(\beta L_{n_*}, K_R, K) \\ &= 0. \end{aligned}$$

Thus, we have for $R > \max(R_*, R_1)$ and all $n \ge n_\infty$

$$\begin{aligned} i(LF, K_R, K) &= \lim i(L_nF, K_R, K) \\ &= i(\beta L_{n_*}, K_R, K) \\ &= 0. \end{aligned}$$

This completes the proof. ■

Now consider the abstract equation

$$u = Tu \tag{7.13}$$

where $T : K \to K$ is a completely continuous mapping. Using similar arguments as those used in the previous proof we obtain the following existence result.

Theorem 65 *Assume that there exist $L^0, L^\infty \in Q^+(E)$ each having the index jump property respectively at $r(L^0)$ and $r(L^\infty)$, and there exist two continuous functions $G^0, G^\infty : K \to E$ with $G^0(u) = \circ(\|u\|)$ near 0 and $G^\infty(u) = \circ(\|u\|)$ near ∞, such that*

$$T = L^0 + G^0 = L^\infty + G^\infty.$$

Then equation (7.13), admits at least one positive solution, if one of the following conditions

$$r(L^\infty) < 1 < r\left(L^0\right) \tag{7.14}$$

and

$$r\left(L^0\right) < 1 < r(L^\infty) \tag{7.15}$$

is satisfied.

7.3 Auxiliary lemmas

We have from hypotheses (7.2) and (7.3) that the functions $\Phi_{ab}(t) = b + a\displaystyle\int_0^t \frac{d\tau}{p(\tau)}$ and $\Psi_{cd}(t) = d + c\displaystyle\int_t^1 \frac{d\tau}{p(\tau)}$ are well defined on $[0,1]$ and the Green function asociated with the bvp

$$\begin{cases} -(pu')'(t) = 0, \quad t \in (0,1), \\ au(0) - b\lim\limits_{t\to 0} p(t)u'(t) = 0, \\ cu(1) + d\lim\limits_{t\to 1} p(t)u'(t) = 0, \end{cases}$$

is given by

$$G(t,s) = \frac{1}{\Delta}\begin{cases} \Phi_{ab}(s)\Psi_{cd}(t), & 0 \le s \le t \le 1, \\ \Phi_{ab}(t)\Psi_{cd}(s), & 0 \le t \le s \le 1. \end{cases}$$

We will use in this work the following spaces: $E = C([0,1], \mathbb{R})$ equiped with the sup-norm, that is for $u \in E$, $\|u\|_\infty = \sup\{|u(t)|, \ t \in [0,1]\}$, K is the cone of all nonnegative functions in E and P is the cone defined by

$$P = \{u \in K, \ u(t) \ge \rho(t)\|u\|\}$$

where for all $t \in [0,1]$,

$$\begin{aligned} \rho(t) &= \int_0^t \rho_*(s)ds \quad \text{and} \\ \rho_*(s) &= \begin{cases} \frac{1}{\Delta}\min\left(c\Phi_{ab}(s), a\Psi_{cd}(s)\right), \ \text{if } ac \neq 0, \\ \dfrac{\Psi_{cd}(s)}{\Psi_{cd}(0)}, \ \text{if } a = 0, \\ \dfrac{\Phi_{ab}(s)}{\Phi_{ab}(1)}, \ \text{if } c = 0. \end{cases} \end{aligned}$$

Note that the condition $ac\int_0^1 \frac{dt}{p(t)} + ad + bc > 0$ implies that the situation $a = c = 0$ does not occur and makes $\rho > 0$ in $(0,1)$.

Let

$$X = \left\{ \begin{array}{c} u \in C^1([0,1], \mathbb{R}) \cap C^2((0,1), \mathbb{R}) : \ \lim\limits_{t\to 0} p(t)u''(t) \text{ and} \\ \\ \lim\limits_{t\to 1} p(t)u''(t) \text{ exist and are finite }, \\ \\ u(0) = au'(0) - b\lim\limits_{t\to 0} p(t)u''(t) = 0 = cu'(1) + d\lim\limits_{t\to 1} p(t)u''(t) = 0 \end{array} \right\}$$

equiped with the norm $\|u\|_X = \|u\|_\infty + \|u'\|_\infty + \|pu''\|_\infty$,

$$L_G^1 = \left\{u : (0,1) \to \mathbb{R} \text{ mesurable}, \ \int_0^1 G(t,t)|u(t)|\,dt < \infty\right\}$$

equiped with the norm $\|u\|_G = \int_0^1 G(t,t)\,|u(t)|\,dt$ and let K_G be the cone of nonnegative functions in L^1_G.

The proof of the following lemma is based on elementary facts of analysis, so it is omitted.

Lemma 66 *$(X, \|\cdot\|_X)$ is a Banach space.*

We need to introduce the following subset of X,

$$N = \left\{ \begin{array}{c} u \in X : \; u' > 0 \text{ in } (0,1), \\ b\,|u'(0)| + a\left|\lim_{t\to 0} p(t)u''(t)\right| > 0 \text{ and } d\,|u'(1)| + c\left|\lim_{t\to 1} p(t)u''(t)\right| > 0 \end{array} \right\}.$$

Lemma 67 *N is an open set in X.*

Proof. We have $N^c = F_1 \cup F_2$ where

$$\begin{aligned} F_1 &= \{u \in X, \text{ there exists } t \in (0,1) \text{ such that } u'(t) \le 0\} \text{ and} \\ F_2 &= \left\{u \in X,\; u'(0) = \lim_{t\to 0} p(t)u''(t) = 0 \text{ or } u'(1) = \lim_{t\to 1} p(t)u''(t) = 0\right\}. \end{aligned}$$

It is clear that F_2 is a closed set, so, we have to show that $\overline{F}_1 \subset F_1 \cup F_2$. To this aim let $(u_n) \subset F_1$ with $\lim u_n = u$ (in the Banach space X) and $u' > 0$ in $(0,1)$ and let $(x_n) \subset (0,1)$ be such that $u_n'(x_n) \le 0$, and $\lim x_n = \overline{x} \in \{0,1\}$.

Let $\overline{x} = 0$ (the case $\overline{x} = 1$ can be checked similarly). The positivity of u' on $(0,1)$ implies that $u'(0) \ge 0$ and $\lim_{t\to 0} p(t)u''(t) \ge 0$. Thus we have

$$0 \le u'(0) = \lim u_n'(x_n) \le 0,$$

that is, $u'(0) = 0$. Moreover, for each integer n there exists $y_n \in (0, x_n)$ such that

$$0 \ge \frac{u_n'(x_n)}{x_n} = u_n''(y_n).$$

This leads to

$$0 \le \lim p(y_n)\, u_n''(y_n) \le 0.$$

We conclude from the above discussion that $u'(0) = \lim_{t\to 0} p(t)u''(t) = 0$ and $u \in F_2$. This completes the proof. ■

Now consider the bvp

$$\begin{cases} -(pu'')'(t) = v(t), \ t \in (0,1), \\ u(0) = au'(0) - b \lim_{t\to 0} p(t)u''(t) = 0, \\ cu'(1) + d \lim_{t\to 1} p(t)u''(t) = 0, \end{cases} \tag{7.16}$$

where $v \in L^1_G$ and let $\pounds : L^1_G \to X$ be the linear operator defined for $u \in L^1_G$ by

$$\pounds u(t) = \int_0^t \left(\int_0^1 G(s,\tau)u(\tau)d\tau \right) ds.$$

It is easy to prove the following:

Lemma 68 *Assume that* (7.2) *and* (7.3) *hold. Then for all* $v \in L^1_G$, $u = \pounds v$ *is the unique solution of the bvp* (7.16).

Let for all $q \in K_G$, $\pounds_q : E \to X$ be the operator given for $u \in E$ by

$$\pounds_q u(t) = \int_0^t \left(\int_0^1 G(s,\tau)q(\tau)u(\tau)d\tau \right) ds \tag{7.17}$$

and $L_q = i \circ \pounds_q$, where i is the compact embedding of X into E.

Lemma 69 *Assume that* (7.2) *and* (7.3) *hold. Then, for all* $q \in K_G$, *the operator* $\pounds_q$ *is completely continuous and maps* K *into* P.

Proof. The compactness of $\pounds_q$ is due to the fact that i is compact.

Let $u \in K$, $\omega = \pounds_q u$ and $v = \omega'$. Since

$$\omega(t) - \rho(t) \|\omega\|_\infty = \int_0^t (v(s) - \rho_*(s) \|\omega\|_\infty) ds$$

and $\|\omega\|_\infty \le \|v\|_\infty$, we have to prove that, for all $t \in [0,1]$,

$$v(t) \ge \rho_*(t) \|v\|_\infty .$$

To this aim, let $t_0 \in [0,1]$ be such that $v(t_0) = \|v\|_\infty > 0$ (the case $v = 0$ is obvious). We distinguish two cases:

Case 1. $t_0 \in \{0,1\}$. If $t_0 = 0$ (the case $t_0 = 1$ can be checked similarly) we have in this situation $a = 0$, $b \neq 0$ and $c > 0$ then $\lim_{t\to 0} p(t)v'(t) = 0$; indeed if $a > 0$ it follows from

$$0 < av(0) = b \lim_{t\to 0} p(t)v'(t)$$

that $v' > 0$ in a right neighborhood of 0 and this contradicts $v(0) = \max_{t\in[0,1]} v(t)$.

Setting $w = v - \dfrac{v(0)\Psi_{cd}}{\Psi_{cd}(0)}$ we have

$$w(0) = 0, \tag{7.18}$$

$$(pw')' = -qu \leq 0 \text{ in } (0,1), \tag{7.19}$$

$$cw(0) + d \lim_{t\to 1} p(t)w'(t) = 0 \tag{7.20}$$

and

$$\lim_{t\to 0} p(t)w'(t) = \frac{cv(0)}{\Psi_{cd}(0)} > 0. \tag{7.21}$$

We claim that $w(1) \geq 0$; indeed if $w(1) < 0$ it follows from (7.20) that $\lim_{t\to 1} p(t)w'(t) > 0$ which together with (7.21) and (7.18) imply that there exist $t_1, t_2, t_3 \in (0,1)$ such that $t_2 < t_1 < t_3$ and $p(t_2)w'(t_2) < p(t_1)w'(t_1) = 0 < p(t_3)w'(t_3)$ and contradicts (7.19). Now if for $z_0 \in (0,1)$ $w(z_0) < 0$ and $w'(z_0) = 0$, there exist $z_1, z_2 \in (0,1)$ with $z_1 < z_0 < z_2$ and $w'(z_1) < w'(z_0) = 0 < w'(z_2)$ and this contradicts (7.19).

This means that $w \geq 0$ in $[0,1]$ and $v(t) \geq \dfrac{\Psi_{cd}(t)}{\Psi_{cd}(0)} v(0) \geq \rho_*(t) \|v\|_\infty$ for all $t \in [0,1]$.

Case 2. $t_0 \in (0,1)$. In this situation $a \neq 0$; indeed, if $a = 0$ then $\lim_{t\to 0} p(t)v'(t) = 0$. Thus we deduce from $(pv')' = -qu \leq 0$ that $v' \leq 0$ in $(0,1)$ which contradicts $t_0 > 0$. We obtain $v'(t_0) = 0$. From the formula

$$v(t) = \frac{\Psi_{cd}(t)}{\Delta} \int_0^t \Phi_{ab}(s)q(s)u(s)ds + \frac{\Phi_{ab}(t)}{\Delta} \int_t^1 \Psi_{cd}(s)q(s)u(s)ds \tag{7.22}$$

we have

$$p(t)v'(t) = -\frac{c}{\Delta}\int_0^t \Phi_{ab}(s)q(s)u(s)ds + \frac{a}{\Delta}\int_t^1 \Psi_{cd}(s)q(s)u(s)ds. \tag{7.23}$$

Taking $t = t_0$ in (7.23) we get

$$\int_{t_0}^1 \Psi_{cd}(s)q(s)u(s)ds = \frac{c}{a}\int_0^{t_0} \Phi_{ab}(s)q(s)u(s)ds. \tag{7.24}$$

Taking $t = t_0$ in (7.22) we get from (7.24)

$$v(t_0) = \frac{1}{a}\int_0^{t_0} \Phi_{ab}(s)q(s)u(s)ds > 0.$$

Then from (7.24) follows that $c \neq 0$ and

$$\begin{aligned} v(t_0) &= \frac{1}{a}\int_0^{t_0} \Phi_{ab}(s)q(s)u(s)ds \\ &= \frac{1}{c}\int_{t_0}^1 \Psi_{cd}(s)q(s)u(s)ds. \end{aligned} \tag{7.25}$$

At the end we obtain from (7.22) and (7.25) that, for $t \geq t_0$,

$$v(t) \geq \frac{a}{\Delta}\Psi_{cd}(t)\, v(t_0)$$

and for $t \leq t_0$,

$$v(t) \geq \frac{c}{\Delta}\Phi_{ab}(t)\, v(t_0),$$

that is, for all $t \in [0,1]$,

$$v(t) \geq \rho_*(t)\, \|v\|_\infty .$$

This completes the proof. ■

Lemma 70 *Assume that* (7.2) *and* (7.3) *hold and let* $q \in K_G \setminus \{0\}$. *We have:*

i) *If* q *does not vanish identically on any subinterval of* $(0,1)$, *then* $L_q \in SIJ(E)$.

ii) L_q *has the* $\sigma-$*index-jump property at* $r(L_q)$.

Proof. **i)** First, we see that 0 is not an eigenvalue of L_q; indeed if 0 is an eigenvalue of L_q associated with an eigenvector η, then $\pounds(q\eta) = 0$. It follows then, from Lemma 68, that $q\eta = 0$. Since q does not vanish identically on any subinterval of $(0,1)$ we get the contradiction $\eta = 0$.

Let $\pounds_q^X : X \to X$ be the restriction of $\pounds_q$ to X and $K_X = K \cap X$. Since $G > 0$ in $(0,1) \times (0,1)$ and q does not vanish identically on any subinterval of $(0,1)$, we have for all $u \in K_X \setminus \{0\}$, $v = L_q^X u$ satisfies

$$v'(t) = \int_0^1 G(t,\tau)\, q(\tau)\, u(\tau)\, d\tau > 0 \quad \text{for all} \quad t \in (0,1),$$

$$b\,|v'(0)| + a\left|\lim_{t\to 0} p(t)v''(t)\right| = \frac{b^2+a^2}{\Delta}\int_0^1 \Psi_{cd}(s)\, q(s)u(s)ds > 0,$$

$$d\,|v'(1)| + c\left|\lim_{t\to 1} p(t)v''(t)\right| = \frac{d^2+c^2}{\Delta}\int_0^1 \Phi_{ab}(s)\, q(s)u(s)ds > 0,$$

that is, $v = L_q^X u \in N \subset \operatorname{int}_X(K_X)$. Thus L_q^X is strongly positive and we have from the Krein-Rutman theorem that $r(L_q^X) = \lambda_1(q) > 0$ is the unique positive eigenvalue of $\pounds_q^X$.

Let $u_1, u_2 \in K_X$ with $u_1 \le u_2$ and set $U_1 = \pounds_q^X u_1$, $U_2 = \pounds_q^X u_2$. We have

$$U_1'(s) = \int_0^1 G(s,t)q(t)u_1(t)dt \le \int_0^1 G(s,t)q(t)u_2(t)dt = U_2'(s),$$

$$\begin{aligned}\left|\lim_{s\to 0} p(s)U_1''(s)\right| &= \left|\frac{a}{\Delta}\int_0^1 \Psi_{cd}(t)q(t)u_1(t)dt\right| \\ &\le \left|\frac{a}{\Delta}\int_0^1 \Psi_{cd}(t)q(t)u_2(t)dt\right| \\ &= \left|\lim_{s\to 0} p(s)U_2''(s)\right|\end{aligned}$$

and

$$\begin{aligned}\left|\lim_{s\to 1} p(s)U_1''(s)\right| &= \left|\frac{c}{\Delta}\int_0^1 \Phi_{ab}(t)q(t)u_1(t)dt\right| \\ &\le \left|\frac{a}{\Delta}\int_0^1 \Phi_{ab}(t)q(t)u_2(t)dt\right| \\ &= \left|\lim_{s\to 1} p(s)U_2''(s)\right|.\end{aligned}$$

Since for $i = 1, 2$

$$\|pU_i''\|_\infty = \max\left(\left|\lim_{s\to 0} p(s)U_i''(s)\right|, \left|\lim_{s\to 1} p(s)U_i''(s)\right|\right),$$

we get $\|pU_1''\|_\infty \le \|pU_2''\|_\infty$ and then $\|U_1\|_X \le \|U_2\|_X$ and $\pounds_q^X$ is K_X-normal. So by Proposition 59, $\pounds_q^X \in \mathrm{SIJ}(X)$ and $r(\pounds_q^X) = \inf \Lambda^+_{\pounds_q^X} = \sup \Lambda^-_{\pounds_q^X}$, where

$$\Lambda^+_{\pounds_q^X} = \left\{\lambda \ge 0 : \text{there exists } u \in K_X \setminus \{0\} \text{ such that } \pounds_q^X u \le \lambda u\right\}$$

and

$$\Lambda^-_{\pounds_q^X} = \left\{\lambda \ge 0 : \text{there exists } u \in K_X \setminus \{0\} \text{ such that } \pounds_q^X u \ge \lambda u\right\}.$$

Now, it is clear that for $\nu = +, -$, $\Lambda^\nu_{\pounds_q^X} \subset \Lambda^\nu_{L_q}$. Thus, let $\lambda > 0$ and $u \in K \setminus \{0\}$ be such that $L_q u \le \lambda u$ (resp. $L_q u \ge \lambda u$). Since $\ker(L_q) \cap K = \{0\}$, $U = L_q u = \pounds_q^X u \in K_X \setminus \{0\}$ and satisfies $\pounds_q^X U = L_q U \le \lambda U$ (resp. $\pounds_q^X U = L_q U \ge \lambda U$) that is $\lambda \in \Lambda^+_{\pounds_q^X}$ (resp. $\lambda \in \Lambda^-_{\pounds_q^X}$). So, for $\nu = +, -$, $\Lambda^\nu_{\pounds_q^X} = \Lambda^\nu_{L_q}$ leading to i) of the Lemma.

ii) Let $c = \|L_q \rho\|$. We have from Lemma 69

$$L\rho(t) \ge c\rho(t), \tag{7.26}$$

from which follows

$$L_q^n \rho \ge c^n \rho, \text{ for all integer } n \ge 1,$$

and from the normality of K

$$\left\|L_q^n\right\|^{1/n} \ge \left\|L_q^n \rho\right\|^{1/n} \ge c\,\|\rho\|_\infty > 0, \text{ for all integer } n \ge 1.$$

Letting $n \to \infty$ we get

$$r(L) = \lim_{n\to\infty} \|L^n\|^{1/n} = c > 0.$$

Since K is a total cone in E, we deduce from Theorem 19.2 in [39] that $r(L)$ is a positive eigenvalue of L_q.

In the end, let $C = \{L_{q+\epsilon} : \epsilon \in (0,1)\} \subset \mathrm{SIJ}(E)$ and note that $L_q = \lim L_{q_n}$, where $q_n = q + (1/n)$, $L_{q_n} \in C$ and for all $t \in [0,1]$ and all integers $n, m \ge 1$, $(tL_{q_n} + (1-t)\, L_{q_m}) \in C$. This means that L_q has the $\sigma-$index-jump property. This completes the proof. $\blacksquare$

7.4 Existence results for third-order BVPs

In the beginning of this section we focus our attention on the linear eigenvalue problem

$$\begin{cases} -\left(pu''\right)'(t) = \mu q(t)u(t), \quad t \in (0,1), \\ u(0) = au'(0) - b \lim\limits_{t\to 0} p(t)u''(t) = 0, \\ cu'(1) + d \lim\limits_{t\to 1} p(t)u''(t) = 0, \end{cases} \tag{7.27}$$

where $q \in K_G \setminus \{0\}$ and μ is a real parameter.

Definition 71 *We say that μ is a positive eigenvalue of (7.27) if $\mu > 0$ and there exists $\phi \in K \setminus \{0\}$ such that the pair (μ, ϕ) satisfies all equations in (7.27).*

We deduce from Lemma 70 and Remark 54 the following:

Corollary 72 *Assume that (7.2) and (7.3) hold. Then the eigenvalue problem (7.27) admits a smallest eigenvalue $\mu_1(q) > 0$ which is a positive eigenvalue of Problem (7.27) associated with an eigenvector ϕ_1. Moreover if q does not vanish identically on any subinterval of $(0,1)$, then $\mu_1(q)$ is the unique positive eingenvalue of Problem (7.27).*

Consider the bvp

$$\begin{cases} -\left(pu''\right)'(t) = m(t)g(t,u(t)), \ t \in (0,1), \\ u(0) = au'(0) - b \lim\limits_{t\to 0} p(t)u''(t) = 0, \\ cu'(1) + d \lim\limits_{t\to 1} p(t)u''(t) = 0 \end{cases} \tag{7.28}$$

where $m \in K_G \setminus \{0\}$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

In the following, we denote for $\nu = 0, +\infty$

$$g_\nu = \liminf_{u\to\nu} \left(\min_{t\in[0,1]} \frac{g(t,u)}{u} \right), \quad g^\nu = \limsup_{u\to\nu} \left(\max_{t\in[0,1]} \frac{g(t,u)}{u} \right).$$

Theorem 73 *Assume that (7.2), (7.3) and one of the following situations*

$$g^0 < \mu_1(m) < g_\infty \le g^\infty < \infty, \tag{7.29}$$

and

$$g^\infty < \mu_1(m) < g_0 \le g^0 < \infty \tag{7.30}$$

hold. Then Problem (7.28) admits at least a positive solution.

Proof. First we have from Lemma 68 that u is a positive solution of (7.28) if and only if $u = L_m F u$ where $F : K \to K$ is defined for $u \in E$ by $Fu(t) = g(t, u(t))$.

Note that hypothesis (7.29) (the other case is checked similarly) implies that there exists $\varepsilon > 0$, small enough, and positive constants C_1, C_2 such that

$$F(u) \le (\mu_1(m) - \varepsilon) u + F_0(u), \text{ for all } u \in K,$$

$$(\mu_1(m) + \varepsilon) u - C_1 \le F(u) \le (g^\infty + \varepsilon) u + C_2, \text{ for all } u \in K,$$

where $F_0 u(t) = \max\{g(t, u(t) - g^0 u(t), 0\}$.

Thus the claim of Theorem 73 follows from Lemma 70 and Theorem 64 ■

Theorem 74 *Assume that (7.2), (7.3) and one of the following situations*

$$g^0 < \mu_1(m) < g_\infty = \infty, \tag{7.31}$$

and

$$g^\infty < \mu_1(m) < g_0 = \infty \tag{7.32}$$

hold. Then Problem (7.1) admits at least a positive solution.

Proof. Let F be that of proof of Theorem 73 and assume that hypothesis (7.31) holds (the other case is checked similarly). There exists $\varepsilon > 0$, small enough, such that

$$F(u) \le (\mu_1 - \varepsilon) u + F_0(u), \text{ for all } u \in K,$$

where $F_0 u(t) = \max\{g(t, u(t) - g^0 u(t), 0\}$.

As in the proof of Theorem 64, there exists $r>0$ such that

$$i\left(L_mF,K_r,K\right)=1.$$

Since $g_\infty=\infty$, for $M=1+\dfrac{1}{A}$ with $A=\int_0^{1/2}\left(\int_0^1 G\left(s,t\right)q\left(t\right)\rho\left(t\right)dt\right)ds$ there exists a positive constant C such that

$$g\left(t,u\right)\geq Mu-C,\ \text{ for all } u\geq 0.$$

Thus for all $u\in K\cap\partial B(0,R_\infty)$ with $R_\infty>C/A$ we have

$$\begin{aligned}
L_mFu(1/2) &= \int_0^{1/2}\left(\int_0^1 G\left(s,t\right)q\left(t\right)\rho\left(t\right)g(t,u(t))dt\right)ds\\
&\geq \int_0^{1/2}\left(\int_0^1 G\left(s,t\right)q\left(t\right)\rho\left(t\right)\left(Mu\left(t\right)-C\right)dt\right)ds\\
&\geq \int_0^{1/2}\left(\int_0^1 G\left(s,t\right)q\left(t\right)\rho\left(t\right)\left(M\rho\left(t\right)\left\|u\right\|_\infty-C\right)dt\right)ds\\
&= MA\left\|u\right\|_\infty-C>\left\|u\right\|_\infty\geq u\left(1/2\right).
\end{aligned}$$

This means that $L_mFu\nleq u$ for all $u\in K\cap\partial B(0,R_\infty)$. So we have from Lemma **??**,

$$i\left(L_mF,K_{R_\infty},K\right)=0.$$

Then from the additivity of the fixed point index

$$i(L_mF,\left(B(0,R_\infty)\setminus\overline{B}(0,r)\right)\cap K,K)=i(L_mF,BK_{R_\infty},K)-i(L_mF,K_r,K)=-1$$

and L_mF has a positive fixed point, which is a positive solution to Problem (7.1) ■

Now consider the bvp

$$\begin{cases}
-\left(pu''\right)'\left(t\right)=h(t,u(t)),\ \ t\in\left(0,1\right),\\
u(0)=au'(0)-b\lim\limits_{t\to 0}p(t)u''(t)=0,\\
cu'(1)+d\lim\limits_{t\to 1}p(t)u''(t)=0,
\end{cases}\tag{7.33}$$

where the function $h:(0,1)\times\mathbb{R}^+\to\mathbb{R}^+$ is L^1_G-Carathéodory function, that is

- $h(\cdot, u)$ is a mesurable function for all $u \geq 0$,
- $h(t, \cdot)$ is continous for at most $t \in (0,1)$,
- for all $r > 0$ there exists $\psi_r \in L^1_G$ such that for $u \in [0, r]$

$$|h(t,u)| \leq \psi_r(t) \text{ for almost } t \in (0,1).$$

We assume in the following that there exist $\alpha, \beta \in K_G \setminus \{0\}$ such that

$$\lim_{u \to 0} \frac{h(t,u)}{u} = \alpha(t) \text{ and } \lim_{u \to +\infty} \frac{h(t,u)}{u} = \beta(t) \text{ in } L^1_G. \tag{7.34}$$

Theorem 75 *Assume that (7.2), (7.3), (7.34) and one of the following situations*

$$\mu_1(\alpha) < 1 < \mu_1(\beta), \tag{7.35}$$

and

$$\mu_1(\beta) < 1 < \mu_1(\alpha) \tag{7.36}$$

hold. Then Problem (7.33) *admits a positive solution.*

Proof. First we have from Lemma 68, that u is a positive solution of (7.33) if and only if $u = \pounds Fu$ where $F : K_G \to K$ is defined for $u \in E$ by $Fu(t) = h(t, u(t))$.

We have

$$\pounds F = L_\alpha + \pounds F_\alpha = L_\beta + \pounds F_\beta$$

where $F_\alpha(u) = h(t,u) - \alpha u$ and $F_\beta(u) = h(t,u) - \beta u$.

We have from (7.34) that

$$\pounds F_\alpha(u) = \circ(\|u\|) \text{ at } 0 \text{ and } \pounds F_\beta(u) = \circ(\|u\|) \text{ at } \infty.$$

Thus the assertion of Theorem 75 follows immediatly from Theorem 65, since $r(L_\alpha) = 1/\mu_1(\alpha)$ and $r(L_\beta) = 1/\mu_1(\beta)$. ∎

Example 76 *Consider the boundary value problem*

$$\begin{cases} -\left(t^{\alpha}u''\right)' = t^{\beta}\chi\left(u\right), & t \in (0,1)\,, \\ u(0) = u'\left(0\right) = u'\left(1\right) = 0, \end{cases} \tag{7.37}$$

where $\alpha \in (-1,1)\,,\ \beta > -2$ *and* $\chi : \mathbb{R}^{+} \to \mathbb{R}^{+}$ *is continuous.*

Straightforward computations lead to

$$\Phi_{1,0}\left(t\right) = \frac{1}{1-\alpha}t^{1-\alpha}, \quad \Psi_{1,0}\left(t\right) = \frac{1}{1-\alpha}\left(1-t^{1-\alpha}\right),$$

$$G\left(t,s\right) = \begin{cases} s^{1-\alpha}\left(1-t^{1-\alpha}\right), & 0 \le s \le t \le 1, \\ t^{1-\alpha}\left(1-s^{1-\alpha}\right), & 0 \le t \le s \le 1 \end{cases}$$

and

$$\rho\left(t\right) = \frac{1}{2-\alpha}\min\left(t^{2-\alpha},\left(1-t^{2-\alpha}\right)\right).$$

It is easy to see that $\alpha \in (-1,1)$ and $\beta \in (-2,+\infty)$ implies

$$\int_0^1 G\left(s,s\right)s^{\alpha+\beta}ds < \infty.$$

Now the smallest eigenvalue λ_1 and its normalized eigenvector ϕ_1 satisfy

$$\phi_1\left(t\right) = \lambda_1 \int_0^t \int_0^1 G\left(s,\tau\right)\tau^{\beta}\phi_1\left(\tau\right)d\tau ds.$$

Since we can not compute λ_1 we will give estimates of its value.We have

$$\begin{aligned} 1 &= \max_{t\in[0,1]}\phi_1\left(t\right) = \phi_1\left(1\right) = \lambda_1\int_0^1\int_0^1 G\left(s,\tau\right)\tau^{\beta}\phi_1\left(\tau\right)d\tau ds \\ &\le \lambda_1\int_0^1\int_0^1 G\left(s,\tau\right)\tau^{\beta}d\tau ds. \end{aligned}$$

This leads to

$$\lambda_1 \ge \left(\int_0^1\int_0^1 G\left(s,\tau\right)\tau^{\beta}d\tau ds\right)^{-1} = \Gamma_1.$$

Also because of $\phi_1(t) \geq \rho(t)$, we have

$$\begin{aligned} 1 &= \max_{t\in[0,1]} \phi_1(t) = \phi_1(1) = \lambda_1 \int_0^1 \int_0^1 G(s,\tau)\,\tau^\beta \phi_1(\tau)\,d\tau ds \\ &\geq \lambda_1 \int_0^1 \int_0^1 G(s,\tau)\,\tau^\beta \rho(\tau)\,d\tau ds \end{aligned}$$

leading to

$$\lambda_1 \leq \left(\int_0^1 \int_0^1 G(s,\tau)\,\tau^\beta \rho(\tau)\,d\tau ds \right)^{-1} = \Gamma_2.$$

Let for $\nu = 0, +\infty$

$$\chi_\nu = \liminf_{u\to\nu} \left(\frac{\chi(u)}{u} \right), \quad \chi^\nu = \limsup_{u\to\nu} \left(\frac{\chi(u)}{u} \right)$$

we deduce from Theorem 73 and Theorem 74 that if $\chi^\infty < \Gamma_1 \leq \Gamma_2 < \chi_0$ or $\chi^0 < \Gamma_1 \leq \Gamma_2 < \chi_\infty$ then Problem (7.37) has at least one positive solution.

Example 77 *Consider the boundary value problem*

$$\begin{cases} -(t^\alpha u'')' = t^{\beta_0} \dfrac{A_0 u}{u+1} + t^{\beta_\infty} \dfrac{A_\infty u^2}{u+1}, \quad t \in (0,1), \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{7.38}$$

where $\alpha \in (-1,1)$, $A_0, A_\infty > 0$ *and* $\beta_0, \beta_\infty > -2$.

It is easy to see that $\alpha \in (-1,1)$ and $\beta_0, \beta_\infty \in (-2,+\infty)$ implies that for $i = 0, \infty$

$$\int_0^1 G(s,s)\, s^{\alpha+\beta_i} ds < \infty.$$

Let for $i = 0, \infty$, λ_1^i be the smallest eigenvalue of

$$\begin{cases} -(t^\alpha u'')' = \lambda A_i t^{\beta_i} u, \quad t \in (0,1), \\ u(0) = u'(0) = u'(1) = 0. \end{cases}$$

Computing as in the above example we obtain for $i = 0, \infty$

$$\frac{\Gamma_1^i}{A_i} \leq \lambda_1^i \leq \frac{\Gamma_2^i}{A_i}$$

where

$$\begin{aligned}\Gamma_1^i &= \left(\int_0^1 \int_0^1 G(s,\tau)\,\tau^{\beta_i} d\tau ds\right)^{-1} \textit{ and} \\ \Gamma_2^i &= \left(\int_0^1 \int_0^1 G(s,\tau)\,\tau^{\beta_i} \rho(\tau)\, d\tau ds\right)^{-1}.\end{aligned}$$

We deduce from Theorem 75 that if $A_0 < \Gamma_1^0, \Gamma_2^\infty < A_\infty$ or $A_\infty < \Gamma_1^\infty, \Gamma_2^0 < A_0$ then Problem (7.38) has at least one positive solution.

Chapter 8

CONCLUSION

Dans ce travail, nous avons donné des résultats d'existence de solutions positives non nulles pour certaines classes de problèmes aux limites à plusieurs points associés à des équations différentielles non linéaires. Dans chaque cas on s'est ramené à l'étude de l'existence de point fixe de l'opérateur associé, en utilisant les propriétés de l'indice du point fixe, et la comparaison entre la non linéarité et la première valeur propre du problème auxiliaire, plusieurs résultats d'existences on été fournis.

Cependant, beaucoup de types de problèmes feront l'objet de notre recherche et qui sont peu abordés dans la littérature, tels que les problèmes singuliers aux limites à plusieurs points avec un opérateur p(t)-Laplacien d'ordre supérieur.

Appendice

Let us recall some elements related to the functional space and fixed point index theory. Let E be a real Banach space.

Definition 78 *A nonempty closed convex subset K of E is said to be an* ordered cone *if $K \cap (-K) = \{0\}$ and $(tK) \subset K$ for all $t \geq 0$.*

Definition 79 *Let $X = C(I, \mathbb{R}^m)$ be the Banach space equipped with the norm $\|\|$, M a subset of X and I a subset of $\mathbb{R}^n$, $n \geq 0$. Then M is equicontinuous on any compact subset of I if*
$\forall J \subset I$ compact,
For $\epsilon > 0$, there exist $\delta > 0$, such that for $t, t' \in J$, if $\|t - t'\|_{\mathbb{R}^n} < \delta$ then $\|u(t) - u(t')\|_{\mathbb{R}^m} < \epsilon$, for all $u \in M$.

Lemma 80 *([14]) Let I be a compact subset of $\mathbb{R}$ and M a subset of the Banach space $X = C(I)$. Then M is relatively compact in X if the following conditions hold:*

(i) M is bounded in X;

(ii) The functions belonging to M are equicontinuous .

Let for $d \geq 0$, E_d be the Banach space of continuous functions u defined on $\mathbb{R}^+$ such that $\lim_{t\to+\infty} (1+t)^d u(t)$ exist in $\mathbb{R}$ equipped with the norm $\|u\|_d = \sup_{t\geq 0} (1+t)^d |u(t)|$.

Note that if $d = 0$ then E_0 is the Banach space of all continuous functions defined on $\mathbb{R}^+$, having a finite limit at $+\infty$ and $\|u\|_0 = \sup_{t\geq 0} |u(t)|$.

Lemma 81 *([9]) Let E_d be a real Banach space endowed with norm $\|\cdot\|$, M a subset of E_d. Then M is relatively compact in E_d if the following conditions hold:*

(i) M is bounded in E_d;

(ii) The functions belonging to $\left\{u: \ u(t) = (1+t)^d x(t), \ x \in M\right\}$ *are equicontinuous on any compact interval of* $\mathbb{R}^+$*;*

(iii) The functions from $\left\{u: \ u(t) = (1+t)^d x(t), \ x \in M\right\}$ *are equiconvergent.*

Let E be a real Banach space endowed with norm $\|\cdot\|$, K a cone of E and Ω an open subset of E. For $R>0$, $B(0,R)$ denotes the open ball of radius R centered at 0 in E.

Definition 82 *$T:\Omega \to E$ is completely continuous if it is continuous and compact (i.e: TB is relatively compact in E for all bounded subset B of Ω).*

Definition 83 *Let U be an open bounded subset of K such that $U \subset B(0,R)$, $R>0$. For any completly continuous mapping $f:\bar{U} \to K$ with $f(x) \neq x$ for all $x \in \partial U$, and retraction $r: E \to K$ (i.e. r continuous and $r(x) = x$ for all $x \in K$), the integer given by*

$$i(f,U,K) = \deg(I - f \circ r, B(0,R) \cap r^{-1}(U), 0)$$

where deg *is the Leray-Schauder degree, is well defined and is called fixed point index.*

Fixed point index properties:

1. Normality : $i(f,U,K) = 1$ if $f(x) = x_0 \in \bar{U}$ for all $x \in \bar{U}$.

2. Homotopy invariance : Let $H:[0,1]\times\bar{U} \to K$ be a completely continuous mapping such that $H(t,x) \neq x$ for all $(t,x) \in [0,1]\times\partial\bar{U}$. The integer $i(H(t,.),U,K)$ is independant of t.

3. Additivity :

$$i(f,U,K) = i(f,U_1,K) + i(f,U_2,K)$$

whenever U_1 and U_2 are two disjoint open subsets of U such that f has no fixed point in $\bar{U}\diagdown(U_1 \cup U_2)$.

4. Permanence : If K' is a retractof K with $f(\bar{U}) \subset K'$ then

$$i(f, U, K) = i(f, U \cap K', K').$$

5. Solution property : If $i(f, U, K) \neq 0$ then f admits a fixed point in U.

Let $T : \bar{B}(0, R) \cap K \to K$ be a compact mapping.

For any $x, y \in E$, by the equivalence $x \leq y \Leftrightarrow y - x \in K$, the convex cone K induces an order relation $\leq$ in E.

The following two lemmas can be found in [14]. They provide fixed point index computations.

Lemma 84 *If* $Tu \ngeq u$ *for all* $u \in \partial B(0, R) \cap K,$ *then* $i(T, B(0, R) \cap K, K) = 1.$

Lemma 85 *If* $Tu \nleq u$ *for all* $u \in \partial B(0, R) \cap K,$ *then* $i(T, B(0, R) \cap K, K) = 0.$

Lemma 86 *If* $\|Tx\| < \|x\|$ *for all* $x \in \partial B(0, R) \cap K,$ *then*

$$i\,(T, B(0, R) \cap K, K) = 1.$$

Lemma 87 *If* $\|Tx\| > \|x\|$ *for all* $x \in \partial B(0, R) \cap K,$ *then*

$$i\,(T, B(0, R) \cap K, K) = 0.$$

For sake of completeness let us recall some basic facts needed in the following chapters. Let $L(E)$ be the set of all linear continuous mapping from E into E. For $L \in L(E)$, $r(L) = \lim_{n\to\infty} \|L^n\|^{\frac{1}{n}}$ denotes the spectral radius of L.

Let K be a cone in E, $L \in L(E)$ is said to be increasing if $L(K) \subset K$ and it is said to be strongly increasing if int$K \neq \emptyset$ and $L(K \setminus \{0\}) \subset \text{int}(K)$.

A cone K is said to be:

- normal if there exists a positive constant N such that for all $u, v \in K,\ u \leq v$ implies $\|u\| \leq N \|v\|$,

- solid if $\operatorname{int}(K) \neq \emptyset$,

- total if $\overline{K-K} = E$.

Let K be an ordered cone in E, $L \in L(E)$ is said to be:

- positive if $L(K) \subset K$,

- strongly positive if K is solid and $L(K \setminus \{0\}) \subset \operatorname{int}(K)$,

- $K-$normal if for all $u, v \in K$,

$$u \leq v \text{ implies } \|Lu\| \leq \|Lv\|.$$

In all the following, $Q^{+}(E)$ denotes the subset of $L(E)$ consisting of all compact positive linear operators.

The following Krein-Rutman Theorems, can be founded in Theorem 7.C in [17] or Theorem 19.3 in [39].

Theorem 88 (Krein-Rutman)
Let $L \in Q^{+}(E)$. If K is total and $r(T) > 0$, then $r(T)$ is an eigenvalue of L with positive eigenvector.

Theorem 89 (Krein-Rutman)
Let $L \in Q^{+}(E)$ be strongly positive. If K is normal, or L is $K-normal$, then $r(L) > 0$ and $r(L)$ is the unique positive eigenvalue of L.

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