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Thème

Distributions Non Paramétriques et Approximation de Modèles d'Attente

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Résumé

Dans cette thèse, nous étudions un nouveau modèle de file d'attente avec rappels où le serveur est sujet à des pannes et à des réparations. Dans une première partie, nous donnons une étude détaillée du point de vue file d'attente et du point de vue fiabilité. Dans le premier cas, nous étudions l'influence des pannes sur différentes caractéristiques du processus d'attente, en particulier (i) la condition nécessaire et suffisante de stabilité du système, (ii) la distribution conjointe du nombre de clients en orbite et de l'état du serveur, (iii) le nombre de rappels effectué par un client marqué, (iv) le temps d'attente, (v) la période d'activité. Du point de vue fiabilité, nous exprimons la loi du temps jusqu'à la première panne et la fonction de fiabilité elle-même en termes du volume d'activité de service.

Dans une seconde partie, nous nous intéressons à l'approximation par bornes (minoration/majoration) du modèle. Dans ce but, nous étudions les propriétés de monotonie et de comparabilité en utilisant la théorie générale des ordres stochastiques. Nous montrons la monotonie de l'opérateur de transition de la chaîne de Markov incluse par rapport aux ordres stochastique fort et convexe croissant. Nous obtenons, en régime stationnaire, des conditions de comparabilité de deux opérateurs de transition et du nombre de clients dans deux systèmes. Nous en déduisons des inégalités pour les caractéristiques moyennes de la période d'activité, du nombre de clients servis durant une période d'occupation, du nombre de périodes d'occupation de l'orbite et du temps d'attente. Nous obtenons aussi des inégalités pour les probabilités limites de l'état du serveur. Certaines bornes peuvent être obtenues pour des classes données de distributions non paramétriques (NBUE, HNBUE, L).

Abstract

In this thesis, we study a new unreliable retrial queue. In a first part, we give extensive analysis from queueing and reliability viewpoints. In the first case, we study the effect of breakdowns on the performance measures of the queueing system, in particular (i) the necessary and sufficient condition for the system to be stable, (ii) the joint distribution of the server state and the number of customers in the retrial group, (iii) the number of retrials made by a customer, (iv) the waiting time, (v) the busy period. From the reliability viewpoint, we analyse the time to the first failure of the server.

On the other hand, we investigate the monotonicity and comparability properties of the defined unreliable retrial queue using the general theory of stochastic ordering. We show the monotonicity of the embedded Markov chain relative to the strong stochastic ordering and increasing convex ordering. We obtain, in steady state, conditions of comparability of two transition operators and the number of customers in two systems. Inequalities are derived for the mean characteristics of the busy period, number of customers served during a busy period, number of orbit busy periods and waiting times. We also obtain inequalities for some probabilities of the steady state distribution of the server state. Bounds are obtained for given classes of non parametric distributions ($NBUE$, $HNBUE$, and $\mathcal{L}$ *).*

Contents

Introduction

Queueing theory is concerned with the mathematical modelling and analysis of systems that provide service to random demands [29, 33, 43, 46, 66, 68, 81, 103]. It is often assumed that customers who find all servers busy with no waiting position available are lost. However, the assumption about loss of customers is just a first order approximation to a real situation. Usually, such customers after a random time return to the system and try to get service again. The study of retrial queues is motivated by many applications, particularly in telephone networks [41-42, 84, 136], computer and telecommunication technology [67, 75, 112], call centers [4-5, 101], mobile telephone systems [13, 28, 98, 128], local computer networks [39, 72, 96], inventory systems [21, 85]. Because of the complexity of retrial queueing models, analytic results are generally difficult to obtain. Exact analytic solutions can be obtained only for special cases under restrictive assumptions: single sever and exponential distribution for interarrival time. Even in this case the closed-form expressions of performance measures are provided through transform methods which make the expressions cumbersome and the obtained results cannot be put into practice. However, there is a great number of numerical and approximation methods [20]. Qualitative properties of stochastic models constitute an important theoritical basis for approximation methods. Important qualitative properties and approximation methods are monotonicity and comparability which can be studied using the general theory of stochastic ordering .

Stochastic ordering is an area with a great number of applications as illustrated in the monographs of Muller and Stoyan [106], Shaked and Shanthikumar [115], Stoyan [123]. It has broad applications in queueing [1-3, 10, 26, 30, 34, 35-36, 45, 50, 51, 59,60, 61-62, 73, 82-83, 91, 106-107, 109, 115, 121-124, 137, 140] and it is used to:

(i) Study internal changes of performance due to parameter variations,

(ii) Compare distinct systems,

(iii) Approximate a system by a simpler one,

(iv) Obtain upper and lower bounds for the main performance measures of systems.

The approaches used to establish stochastic ordering results are coupling method, functional method and mapping method [52, 97, 106, 115-116, 123, 129]. In practice some constituent distributions belong to a given family of distributions. The classes of ageing distribution functions are interesting for comparison of "new" and "residual" life times. They arose in the context of reliability theory and are often used in queueing theory. From these distribution classes, bounds and inequalities can be obtained for the stationary distribution of the number of customers in the system and for the performance measures.

In the literature many results have been obtained on the stochastic comparison of standard queues but few papers are devoted to retrial queues [32, 78, 94-95, 110, 119-120]. Khalil and Falin [78] investigate some monotonicity properties, in steady state, of an $M/G/1$ retrial queue with exponential retrial times and linear retrial rate relative to strong stochastic ordering, increasing convex ordering and Laplace ordering. They show that the number of customers in steady state stochastically decreases when the arrival rate decreases with increasing of retrial rate and decreasing service time either stochastically or in the convex ordering. Inequalities are derived for the mean characteristics of the busy period and the number of customers served during a busy period. Boualem, Djellab and Aissani [32] derive stochastic comparison properties for an $M/G/1$ retrial queue with vacations and constant retrial policy. Sample path comparisons of queue length processes for a $G/G/1/K$ retrial queue are established in [94-95, 119]. Liang and Kulkarni [95] study the monotonicity properties of retrial queues in order to investigate how the retrial time distribution affects the behavior of the system. They assume retrial times to have phase type distributions and they show that systems with longer retrial times, with respect to the K-dominance, create more customers in the system and in the orbit. From these results, they derive monotonicity properties of several performance measures of interest. Liang [94] shows that if the hasard rate function of the retrial time distribution is decreasing, then stochastically longer service times or fewer servers will result in more customers in the system. Shin [119] considers several multi-server retrial queueing models with exponential retrial times such as $A^X/G/c/K$ retrial queue, two-node tandem retrial queue $A^X/G/c_1/K_1 \longrightarrow ./G/c_2/K_2$, MAP_1, $MAP_2/M/c$ retrial queue and $M/M/c/c$ retrial queue with negative arrivals. He shows that the number of customers in orbit and in the system as a whole are monotonically changed if the retrial rates in one system are bounded by the rates in the second one. The monotonicity results are applied to show the convergence of generalized truncated systems to the original one. Shin and Kim [120] consider Markovian retrial queues and deduce a stochastic order relation between two bivariate processes representing the number of customers in the service facility and one in the orbit by constructing the equivalent processes on a common probability space. Oukid and Aissani [110] obtain lower and upper bounds for the mean busy period of $GI/GI/1$ queue with breakdowns and FIFO queue. They consider an extension to the model of queue with classical retrial policy. The results are based on some reliability concepts.

In this thesis we are interested in a single retrial queue where the server is subject to breakdowns and repairs. We define a new model and study it from queueing and reliability viewpoints [127]. On the other hand, we investigate monotonicity and comparability properties of this model [126]. The organization of the thesis is as follows:

- In chapter one, we review tools for comparing random variables and stochastic processes via stochastic orders. We give definitions and characterizations of strong stochastic ordering, increasing convex ordering and Laplace ordering. We introduce some notions of ageing and their relation with stochastic orders. Monotonicity and comparability of stochastic processes are also examined;

- In chapter two, we give a review of studies in the area of unreliable retrial queues and introduce a new version which will be studied analytically in chapter three and by approximation through bounds using stochastic orders in chapter four;

- In chapter three, we study the effect of breakdowns on the performance measures of the queueing system, in particular (i) the necessary and sufficient condition for the system to be stable, (ii) the joint distribution of the server state and the number of customers in the retrial group, (iii) the waiting time, (iv) the busy period. From the reliability viewpoint, we analyse the time to the first failure of the server;

- Then in chapter four, we derive monotonicities of the major performance measures in terms of strong stochastic ordering and increasing convex ordering. The model is compared with a simpler counterpart of unreliable $M/M/1$ retrial queues where all distributions are exponential and hence bounds of performance measures are derived. We also discuss the conditions under which the comparison is made. The monotonicity properties are derived via the monotonicity of the embedded Markov chain. We present illustrative examples to show how close the bounds are to the exact expressions.

Chapter 1

Stochastic Orders and Non-Parametric Distributions

Stochastic orders are binary relations defined on sets of probability distributions, which formally describe intuitive ideas such as " being larger", " being more variable" or "being more dependent". Such stochastic order relations are now used in many areas of Probability and Statistics including Biology, Economics, Risks, Operations Research, Reliability and Queueing theory. For a comprehensive treatment see [106, 115, 123-124]. Stochastic orders lead to powerful approximation methods and bounds in situations where realistic stochastic models are too complex for rigourous treatment. Stochastic orders are also helpful in situations where fundamental model distributions are only known partially.

In this chapter we review tools for comparing random variables and stochastic processes. In section 1, we give definitions and characterizations of three most important orderings namely strong stochastic ordering ($\leq_{st}$), increasing convex ordering ($\leq_{icx}$) and Laplace ordering ($\leq_L$). Section 2 is devoted to ageing distributions which are interesting for comparing "new" and "residual" life times. Extremal elements (when they exist) for some classes of distribution functions are given in section 3. In section 4, we focus on the monotonicity and comparability of stochastic processes, in particular Markov processes. For details see [27, 31, 47, 73, 90, 102, 106, 113, 115-116, 123].

1.1 Univariate Stochastic Order

Random variables and their distributions can be ordered by comparing their properties, in some probabilistic sense. Many stochastic orderings are introduced and are used extensively in the studies of Probability and Statistics. Among the frequently discussed stochastic orderings: strong stochastic order, increasing convex order and Laplace order.

1.1.1 Usual (Strong) Stochastic Order

Definition and Characterizations

Let X and Y be two random variables with distribution functions F and G respectively.

Definition 1.1.1 *The random variable X is said to be smaller than the random variable Y with respect to usual stochastic order (written $X \leq_{st} Y$ or $F \leq_{st} G$) if*

$$F(t) \geq G(t) \text{ for all real } t$$

or equivalently, if

$$\overline{F}(t) \leq \overline{G}(t) \text{ for all real } t$$

where $\overline{F}(t) = 1 - F(t)$ is the survival function of X.

Interpretation: $X \leq_{st} Y$ means that X assumes small values with higher probability than Y does, and hence X assumes large values with smaller probability than Y does.

If X and Y are discrete random variables taking values in $\mathbb{N}$, with distribution $p_i = P(X = i)$ and $q_i = P(Y = i)$, $i \in \mathbb{N}$, respectively, then $X \leq_{st} Y$ if, and only if ,

$$\overline{p}_i \leq \overline{q}_i \text{ for all } i \in \mathbb{N}$$

where $\overline{p}_i = \sum_{j \geq i} p_j$.

The following theorem shows that $\leq_{st}$ is closely related to the pointwise comparison of random variables.

Theorem 1.1.1 *The following statements are equivalent*

(1) $X \leq_{st} Y$

(2) there exists two random variables $\widehat{X}$ *and* $\widehat{Y}$, *defined on the same probability space* $(\Omega, \mathcal{A}, P)$, *such that*

$$\widehat{X} =_{st} X$$

$$\widehat{Y} =_{st} Y$$

$$P[\widehat{X} \leq \widehat{Y}] = 1$$

In the following theorem, we give an important characterization in terms of expectations of increasing functions.

Theorem 1.1.2 *The following statements are equivalent*

(1) $X \leq_{st} Y$

(2) the inequality $E(f(X)) \leq E(f(Y))$ *holds for all increasing functions* f *for which both expectations exist.*

Theorem 1.1.3 *Let* X *and* Y *be random variables with finite expectations.*

(1) If $X \leq_{st} Y$ *then* $E(X) \leq E(Y)$

(2) If $X \leq_{st} Y$ *and* $E(X) = E(Y)$ *then* X *and* Y *have the same distribution.*

Remark 1.1.1

- *Strong stochastic order implies ordering of expectations;*
- *Different distributions with the same expectation cannot be ordered with respect to* $\leq_{st}$ *order.*

Part (1) of the previous theorem can be generalized to the comparison of higher moments.

Theorem 1.1.4 *If* $X \leq_{st} Y$, *then*

$$E(X^n) \leq E(Y^n) \quad \textit{for } n = 1, 3, 5, ...$$

whenever the expectations exist.

If, moreover, X *and* Y *are non-negative then*

$$E(X^n) \leq E(Y^n) \quad \textit{for all } n \in \mathbb{N}^*$$

Closure Properties

Theorem 1.1.5

(1) If $X \leq_{st} Y$ *and* g *is any increasing function, then*

$$g(X) \leq_{st} g(Y)$$

(2) Let $X_1, X_2, ..., X_n$ *be a set of independent random variables and let* $Y_1, Y_2, ..., Y_n$ *be another set of independent random variables. If* $X_i \leq_{st} Y_i$ *for* $i = 1, ..., n$*, then, for any increasing function* $g : \mathbb{R}^n \longrightarrow \mathbb{R}$*,*

$$g(X_1, X_2, ..., X_n) \leq_{st} g(Y_1, Y_2, ..., Y_n)$$

In particular,

$$\sum_{i=1}^{n} X_i \leq_{st} \sum_{i=1}^{n} Y_i$$

That is, the strong stochastic order is closed under convolutions.

(3) Let $\{X_i,\ i = 1, 2, ...\}$ *and* $\{Y_i,\ i = 1, 2, ...\}$ *be two sequences of random variables such that* $X_i \longrightarrow_{st} X$ *and* $Y_i \longrightarrow_{st} Y$ *as* $i \longrightarrow \infty$*, where "*$\longrightarrow_{st}$*" denotes convergence in distribution. If* $X_i \leq_{st} Y_i$*,* $i = 1, 2, ...,$ *then*

$$X \leq_{st} Y$$

That is , the strong stochastic order is closed with respect to weak convergence.

(4) Let X *,* Y *and* Θ *be random variables such that* $[X \mid \Theta = \theta] \leq_{st} [Y \mid \Theta = \theta]$ *for all* θ *in the support of* Θ*. Then:*

$$X \leq_{st} Y$$

The strong stochastic order is closed under mixtures.

Theorem 1.1.6 *Let* $\{X_i,\ i = 1, 2, ...\}$ *be a sequence of nonnegative independent random variables, and let* M *be a non-negative integer-valued random variable which is independent of the* X_i*'s. Let* $\{Y_i,\ i = 1, 2, ...\}$ *be another sequence of non-negative independent random variables, and let* N *be a non-negative integer–valued random variable which is independent of the* Y_i*'s. If* $X_i \leq_{st} Y_i$*,* $i = 1, 2, ...,$ *and if* $M \leq_{st} N$*, then*

$$\sum_{i=1}^{M} X_i \leq_{st} \sum_{i=1}^{N} Y_i$$

1.1.2 Increasing Convex Order

Definition and Equivalent Conditions

Definition 1.1.2 *Let X and Y be random variables with distribution functions F and G and with finite means.*

(1) X is less than Y in convex order (written $X \leq_{cx} Y$ or $F \leq_{cx} G$), if

$$E(f(X)) \leq E(f(Y))$$

for all convex functions f such that the expectations exist.

(2) X is less than Y in increasing convex order (written $X \leq_{icx} Y$ or $F \leq_{icx} G$), if

$$E(f(X)) \leq E(f(Y))$$

for all increasing convex functions f such that the expectations exist.

Theorem 1.1.7 *The following statements are equivalent:*

(1) $X \leq_{cx} Y$

(2) $X \leq_{icx} Y$ and $E(X) = E(Y)$

Corollary 1.1.1

(1) If $X \leq_{cx} Y$ then

$$E(X^n) \leq E(Y^n) \text{ and } E((X - E(X))^n) \leq E((Y - E(Y))^n)$$

for $n = 2, 4, 6, ...$
In particular, $X \leq_{cx} Y$ implies $var(X) \leq var(Y)$.

(2) If X and Y are non-negative random variables, then $X \leq_{cx} Y$ implies $E(X^n) \leq E(Y^n)$ for all $n \in \mathbb{N}$.

(3) If $X \leq_{icx} Y$ then $E(X) \leq E(Y)$.

There is a simpler characterization of the increasing convex order. The next result shows that it is sufficient to consider only a small subclass of the set of all convex functions.

Theorem 1.1.8 *The following statements are equivalent:*

(1) $X \leq_{icx} Y$

(2) $E(X-t)_+ \leq E(Y-t)_+$ *for all real* t *or equivalently*

$$\int_t^{+\infty} \overline{F}(x)dx \leq \int_t^{+\infty} \overline{G}(x)dx$$

for all real t

If X and Y are discrete random variables taking on values $\mathbb{N}$ with distributions $p_i = P(X = i)$ and $q_i = P(Y = i)$, $i \in \mathbb{N}$,then $X \leq_{icx} Y$ if, and only if ,

$$\overline{\overline{p}}_i \leq \overline{\overline{q}}_i \text{ for all } i \in \mathbb{N}$$

where $\overline{\overline{p}}_i = \sum_{j \geq i} \overline{p}_j$.

Theorem 1.1.9 *The following statements are equivalent :*

(1) $X \leq_{icx} Y$

(2) there exists two random variables $\widehat{X}$ *and* $\widehat{Y}$, *defined on the same probability space* $(\Omega, \mathcal{A}, P)$, *such that*

$$\widehat{X} =_{st} X$$

$$\widehat{Y} =_{st} Y$$

$$E(\widehat{Y} \mid \widehat{X}) \geq \widehat{X} \quad \textit{almost surely}$$

and in addition the conditional law $(\widehat{Y} \mid \widehat{X} = x)$ *is stochastically increasing in* x, *i.e.*

$$(\widehat{Y} \mid \widehat{X} = x) \leq_{st} (\widehat{Y} \mid \widehat{X} = y), \quad x \leq y.$$

Closure Properties

Theorem 1.1.10

(1) If $X \leq_{icx} Y$ *and* g *is any increasing and convex function, then:*

$$g(X) \leq_{icx} g(Y)$$

(2) Let $X_1, X_2, ..., X_n$ a set of independent random variables and let $Y_1, Y_2, ..., Y_n$ another set of independent random variables. If $X_i \leq_{icx} Y_i$ for $i = 1, ..., n$, then:

$$g(X_1, X_2, ..., X_n) \leq_{icx} g(Y_1, Y_2, ..., Y_n)$$

for every increasing and componentwise convex function g.
In particular the increasing convex order is closed under convolutions

$$\sum_{i=1}^{n} X_i \leq_{icx} \sum_{i=1}^{n} Y_i$$

(3) Let $\{X_i,\ i = 1, 2, ...\}$ and $\{Y_i,\ i = 1, 2, ...\}$ be two sequences of random variables such that $X_i \longrightarrow_{st} X$ and $Y_i \longrightarrow_{st} Y$ as $i \longrightarrow \infty$. Assume that $E(X_+)$ and $E(Y_+)$ are finite and that $E(X_i)_+ \longrightarrow E(X_+)$, $E(Y_i)_+ \longrightarrow E(Y_+)$ as $i \longrightarrow \infty$. If $X_i \leq_{icx} Y_i$, $i = 1, 2, ...$, then

$$X \leq_{icx} Y$$

(4) Let X , Y and Θ be random variables such that $[X \mid \Theta = \theta] \leq_{icx} [Y \mid \Theta = \theta]$ for all θ in the support of Θ. Then

$$X \leq_{icx} Y$$

The increasing convex order is closed under mixtures.

Theorem 1.1.11 *Let $\{X_i,\ i = 1, 2, ...\}$ and $\{Y_i,\ i = 1, 2, ...\}$ each be a sequence of non-negative independent and identically distributed random variables such that $X_i \leq_{icx} Y_i$, $i = 1, 2, ...$ Let M and N be positive integer-valued random variables that are independent of the $\{X_i\}$ and the $\{Y_i\}$ sequences respectively, such that $M \leq_{icx} N$. Then:*

$$\sum_{i=1}^{M} X_i \leq_{icx} \sum_{i=1}^{N} Y_i$$

1.1.3 Laplace Transform Order

Definition and Characterizations

Definition 1.1.3 *Let X and Y be real random variables. X is said to be less than Y in Laplace order (written $X \leq_L Y$), if the Laplace Stieltjes transforms $L_X(s) = E(e^{-sX})$*

and $L_Y(s) = E(e^{-sY})$ *exist and satisfy*

$$L_X(s) \geq L_Y(s) \text{ for all } s \geq 0$$

If the random variables X and Y are of discrete type and (p_i), (q_i) are the corresponding distributions then $X \leq_L Y$ if and only if

$$\sum_{n\geq 0} z^n p_n \geq \sum_{n\geq 0} z^n q_n, \text{ for all } z \in [0,1]$$

Definition 1.1.4 *A real function f is called completely monotone if all its derivatives $f^{(n)}$ exist and satisfy $(-1)^n f^{(n)}(x) \geq 0$ for all x and for all $n \in \mathbb{N}$.*

Theorem 1.1.12 *$X \leq_L Y$ holds if and only if*

$$E(f(X)) \geq E(f(Y))$$

for all completely monotone functions f, provided that the expectations exist.
Or equivalently

$$E(f(X)) \leq E(f(Y))$$

for all functions f with a completely monotone derivative, for which the expectations exist.

Remark 1.1.2

(1) The function $f(x) = e^{-sx}, x \in \mathbb{R}$, is completely monotone.

(2) $f(x) = -e^{-sx}, x \in \mathbb{R}$, is a function with a completely monotone derivative.

Theorem 1.1.13 *Let X and Y be two non-negative random variables with survival functions $\overline{F}$ and $\overline{G}$, respectively. Then $X \leq_L Y$ if and only if*

$$\int_0^{+\infty} e^{-sx}\overline{F}(x)dx \leq \int_0^{+\infty} e^{-sx}\overline{G}(x)dx \text{ for all } s > 0$$

Remark 1.1.3 *If X is a positive random variable then*

(1) $\int_0^{+\infty} e^{-sx}\overline{F}(x)dx$ is Laplace transform of $\overline{F}$.

(2) $\int_0^{+\infty} e^{-sx}\overline{F}(x)dx = \dfrac{1 - L_X(s)}{s}$ *for all* $s > 0$.

(3) $\int_0^{+\infty} \overline{F}(x)dx = E(X)$

Theorem 1.1.14 *If* $X \leq_L Y$ *then* $E(X) \leq E(Y)$ *provided that the expectations exist.*

Closure Properties

Theorem 1.1.15

(1) If $X \leq_L Y$ *and* g *is a positive function with a completely monotone derivative, then:*

$$g(X) \leq_L g(Y)$$

(2) Let $X_1, X_2, ..., X_n$ *be a set of independent random variables and let* $Y_1, Y_2, ..., Y_n$ *be another set of independent random variables. If* $X_i \leq_L Y_i$ *for* $i = 1, ..., n$, *then*

$$g(X_1, X_2, ..., X_n) \leq_L g(Y_1, Y_2, ..., Y_n)$$

for all nonnegative function on $[0, +\infty[^n$ *such that* $\dfrac{d}{dx_i} g(x_1, ..., x_n)$ *is completely monotone in* x_i, $i = 1, ..., n$.
In particular the Laplace transform order is closed under convolutions, namely

$$\sum_{i=1}^{n} X_i \leq_L \sum_{i=1}^{n} Y_i$$

(3) Let $\{X_i,\ i = 1, 2, ...\}$ *and* $\{Y_i,\ i = 1, 2, ...\}$ *be two sequences of random variables such that* $X_i \longrightarrow_{st} X$ *and* $Y_i \longrightarrow_{st} Y$ *as* $i \longrightarrow \infty$. *If* $X_i \leq_L Y_i$, $i = 1, 2, ...$, *then*

$$X \leq_L Y$$

(4) Let X , Y *and* Θ *be random variables such that* $[X \mid \Theta = \theta] \leq_L [Y \mid \Theta = \theta]$ *for all* θ *in the support of* Θ. *Then:*

$$X \leq_L Y$$

The Laplace transform order is closed under mixtures.

Theorem 1.1.16 *Let $\{X_i,\ i = 1, 2, ...\}$ and $\{Y_i,\ i = 1, 2, ...\}$ each be a sequence of non-negative independent and identically distributed random variables such that $X_i \leq_L Y_i$, $i = 1, 2, ...$ Let M and N be positive integer-valued random variables that are independent of the $\{X_i\}$ and the $\{Y_i\}$ sequences respectively, such that $M \leq_L N$. Then:*

$$\sum_{i=1}^{M} X_i \leq_L \sum_{i=1}^{N} Y_i$$

The orders $\leq_{st}$ and $\leq_{icx}$ are both stronger than $\leq_L$, namely:

$$\leq_{st} \Longrightarrow \leq_{icx} \Longrightarrow \leq_L$$

1.1.4 Smooth Generators of Integral Stochastic Orders

Many stochastic orders $\leq_s$ can be characterized by a given class $\mathcal{F}$ of real functions; $X \leq_s Y$ if $E(f(X)) \leq E(f(Y))$ for all $f \in \mathcal{F}$ provided that the expectations exist.
We call these stochastic orders "integral stochastic orders".

- If $\leq_s$ is the usual stochastic order then $\mathcal{F}$ is the class of increasing functions.
- If $\leq_s$ is the increasing convex order then $\mathcal{F}$ is the class of increasing convex functions.
- If $\leq_s$ is the Laplace order then $\mathcal{F}$ is the class of completely monotone functions.

There are several different classes of functions which characterize a given order [106]. For technical reasons, it is often helpful to know that it is sufficient to check $E(f(X)) \leq E(f(Y))$ for all $f \in \mathcal{F}$ which are sufficiently smooth. Indeed, differentiability of test functions is assumed. Denuit and Muller [47] show the following results for $\leq_{st}$ and $\leq_{icx}$ orders.

Theorem 1.1.17

(1) $X \leq_{st} Y$ if and only if

$$E(f(X)) \leq E(f(Y))$$

for all bounded differentiable increasing functions f.

(2) $X \leq_{icx} Y$ if and only if

$$E(f(X)) \leq E(f(Y))$$

for all bounded twice differentiable increasing convex functions f.

Denuit and Muller [47] show that provided a stochastic order is closed under convolution and weak convergence, its generator can be restricted to smooth functions.

Theorem 1.1.18 *Let $\leq_s$ be an integral stochastic order generated by a class $\mathcal{F}$ of continuous functions. If $\leq_s$ is closed under convolution then there is a generator $\mathcal{G}$ of this order relation which only consists of infinitely differentiable function.*

Corollary 1.1.2 *Let $\leq_s$ be an integral stochastic order generated by a class $\mathcal{F}$. If $\leq_s$ is closed under weak convergence and under convolution then it has a generator $\mathcal{G} \subset \mathcal{C}^{\infty}$.*

Note that an integral stochastic order is closed under weak convergence if, and only if, it has a generator consisting of bounded continuous functions [47].

1.2 Lifetime Distributions and Notions of Ageing

In reliability [27, 31, 90, 102], distributions of non-negative random variables that are called life distributions (ageing distributions or non parametric distributions) are usually classified by failure rate, mean residual lifetime and equilibrium distributions. Various classes of life distributions have been introduced and are often used in queueing theory.

Let X be a positive random variable with distribution F and mean m and denote by X_t a generic random variable with distribution function F_t given by

$$\overline{F}_t(x) = 1 - F_t(x) = \frac{\overline{F}(t+x)}{\overline{F}(t)},\ x \geq 0,\ t > 0 \text{ assuming } F(t) < 1$$

In reliability theory X is interpreted as the lifetime of a component, and $X_t = (X - t/X > t)$ as the residual lifetime of a component which has already survived the time t. In queueing theory, X may be the interarrival time, service time, repair time, retrial time and X_t is the residual interarrival time, remaining service time and so on.

Definition 1.2.1

(1) The distribution F is IFR (Increasing Failure Rate) if

$$\frac{\overline{F}(t+x)}{\overline{F}(t)} \text{ is decreasing in } t,\ \forall\ x \geq 0 \text{ for } \overline{F}(t) > 0;$$

(2) The distribution F is IFRA (Increasing Failure Rate in Average) if

$$\frac{-1}{t} \log \overline{F}(t) \text{ is increasing in } t \text{ for } \overline{F}(t) > 0;$$

(3) The distribution F is NBU (New Better than Used) if

$$\overline{F}(t+x) \leq \overline{F}(x)\overline{F}(t) \text{ for all } x,\ t \geq 0;$$

(4) The distribution F is NBUE (New Better than Used in Expectation) if

$$\int_x^{+\infty} \overline{F}(t)dt \leq m\overline{F}(x) \text{ for all } x \geq 0;$$

(5) The distribution F is HNBUE (Harmonically NBUE) if

$$\int_x^{+\infty} \overline{F}(t)dt \leq m \exp(\frac{-x}{m}) \text{ for all } x \geq 0;$$

(6) The distribution F belongs to the $\mathcal{L}$ class if

$$\int_0^{+\infty} e^{-sx} dF(x) \leq \frac{1}{ms+1} \text{ for all } s \geq 0$$

A dual class may be defined for each of the life distribution classes defined above by reversing the inequality or direction of monotonicity. We obtain, in this way, the DFR (Decreasing Failure Rate), $DFRA$, NWU (New Worse than Used), $NWUE$, $HNWUE$ and $\overline{\mathcal{L}}$ classes of distributions.

Such ageing properties can be characterized in terms of stochastic ordering as follows:

Theorem 1.2.1

(1) F is IFR (DFR) if and only if

$$F_t \leq_{st} (\geq_{st}) F_x \text{ for all } 0 \leq x \leq t;$$

(2) F is NBU (NWU) if and only if

$$F_x \leq_{st} (\geq_{st})\, F \text{ for all } x \geq 0;$$

(3) F is NBUE (NWUE) if and only if

$$F_e \leq_{st} (\geq_{st})\, F$$

where F_e is the equilibrium distribution; $F_e(x) = \dfrac{1}{m}\displaystyle\int_0^x \overline{F}(t)dt, x \geq 0$;

(4) F is HNBUE (HNWUE) if and only if

$$F \leq_{icx} (\geq_{icx})\, G$$

where G is the exponential distribution with the same mean as F;

(5) F is $\mathcal{L}$ ($\overline{\mathcal{L}}$) if and only if

$$F \geq_L (\leq_L)\, G$$

The ageing classes form an increasing sequence in the order of definition

$$\begin{aligned} IFR &\subset IFRA \subset NBU \subset NBUE \subset HNBUE \subset \mathcal{L} \\ DFR &\subset DFRA \subset NWU \subset NWUE \subset HNWUE \subset \overline{\mathcal{L}} \end{aligned}$$

The properties of interest concerning ageing classes are mainly on:

- Preservation or closure property of a given class under the reliability operations;
- Reliability bounds;
- Moment inequalities;
- Testing exponentiality against an ageing alternative;
- Arithmetic properties;
- The use in stochastic modelling.

1.3 Extremal Elements

There are many applications where it is interesting to have upper and lower bounds with respect to some stochastic order within a specific class of distributions.

Definition 1.3.1 *Let $\mathcal{M}$ be a set of distributions and $\preceq$ be a partial order on a class $\mathcal{D}$ of distributions with $\mathcal{M} \subset \mathcal{D}$, then*

(1) an element $F_{\max}$ $(F_{\min}) \in \mathcal{M}$ is called the maximum (minimum) if

$$F \preceq F_{\max} \, (F \succeq F_{\min}) \ \textit{for all } F \in \mathcal{M}$$

(2) The supremum (infimum) $F_{\sup}$ $(F_{\inf})$ is the smallest (greatest) distribution in $\mathcal{D}$ (with respect to $\preceq$), not necessarily in $\mathcal{M}$, for which

$$F \preceq F_{\sup} \, (F_{\inf} \preceq F) \quad \textit{for all } F \in \mathcal{M}$$

(3) If $F_{\sup}$ $(F_{\inf})$ exists and $F_{\sup} \in \mathcal{M}$ $(F_{\inf}) \in \mathcal{M}$, then $F_{\max} = F_{\sup} \, (F_{\min} = F_{\inf})$.

Remark 1.3.1 *If $\mathcal{M}$ is the set of all distributions on the real line, then extremal elements do not exist for any of the orderings considered so far.*

We give below the extremal elements (when they exist) for some classes of distribution functions [106, 123].

• For the class $\mathcal{M}_m$ of distribution functions with mean m

i) θ_0 is its $\leq_L$ −infimum

ii) θ_m is its $\leq_L$ −maximum

iii) θ_m is its $\leq_{icx}$ −minimum

• For the class $\mathcal{M}^{[a,b]}$ of distributions on $[a, b]$

i) θ_a is the minimum with respect to $\leq_{st}$ and $\leq_{icx}$

ii) θ_b is the maximum with respect to $\leq_{st}$ and $\leq_{icx}$

• For the class $\mathcal{M}_m^{NBUE}$ of all distribution functions on $[0, +\infty[$ with mean m that are $NBUE$

i) θ_m is its $\leq_{icx}$ −minimum

ii) $Exp(\frac{1}{m})$ is its $\leq_{icx}$ −maximum

iii) $Exp(\frac{1}{m})$ is its $\leq_{L}$ −minimum

iv) θ_m is its $\leq_L$ −maximum

Note that θ_c is Dirac distribution at c, and $Exp(c)$ is exponential distribution with mean $\frac{1}{c}$.

1.4 Monotonicity and Comparability of Stochastic Processes

1.4.1 Definitions

We consider stochastic processes $\{X_t,\ t \in T\}$ with the state space $(S, \mathcal{S})$. Usually, (X_n) is written in the discrete case. Typically, S is the real line or a subset of it or S is a very general set; for example a polish space (complete separable metric space) with a closed partial order. On the other hand $\mathcal{S}$ is a σ-algebra of subsets of S; for example the borel σ-algebra. We denote by $\preceq$ a partial order on the space $P_{\mathcal{S}}$ of all probability distributions on $\mathcal{S}$. It is a general partial order, but in most cases it will be an integral stochastic order.

Definition 1.4.1 *(Monotonicity) A stochastic process $\{X_t,\ t \in T\}$ is said to be increasing (decreasing) with respect to $\preceq$ if*

$$X_s \preceq (\succeq)\, X_t \tag{1.4.1}$$

for all $s,\ t$ in T with $s < t$.

Remark 1.4.1 *Sometimes instead of " increasing" the term monotone is used.*

In the case of discrete time, (1.4.1) holds if and only if

$$X_n \preceq (\succeq)\, X_{n+1} \tag{1.4.2}$$

for $n = 0, 1, 2, ...$

Definition 1.4.2 *(Comparability) Let (X_t) and (Y_t) be stochastic processes with the state space $(S, \mathcal{S})$. Then, the process (X_t) is smaller than (Y_t) with respect to $\preceq$, symbolically $(X_t) \preceq (Y_t)$ if for all t*

$$X_t \preceq Y_t \tag{1.4.3}$$

1.4.2 Comparability and Monotonicity of Markov Processes

Let $\mathbf{T}$, $\mathbf{T}^{'}$ and $\mathbf{T}^{''}$ be operators on $P_{\mathcal{S}}$ with general state space $(S, \mathcal{S})$, i.e. mappings from $P_{\mathcal{S}}$ into itself, and let $\preceq$ be a partial order on $P_{\mathcal{S}}$.

Definition 1.4.3

(1) An operator $\mathbf{T}$ is said to be $\preceq$-monotone if

$$\mathbf{T}P^{'} \preceq \mathbf{T}P^{''}$$

holds for all $P^{'}$ and $P^{''} \in P_{\mathcal{S}}$ with $P^{'} \preceq P^{''}$.

(2) The operator $\mathbf{T}^{'}$ is said to be smaller than $\mathbf{T}^{''}$ if

$$\mathbf{T}^{'}P \preceq \mathbf{T}^{''}P$$

for all $P \in P_{\mathcal{S}}$; in symbols $\mathbf{T}^{'} \preceq \mathbf{T}^{''}$.

For applications to discrete time Markov processes, the comparability of state distributions $P_n^{'}$ and $P_n^{''}$ defined by

$$P_n^{'} = \mathbf{T}^{'n} P_0^{'} \text{ and } P_n^{''} = \mathbf{T}^{''n} P_0^{''} \text{ for } n = 1, 2, ...$$

is of interest for different initial distributions $P_0^{'}$ and $P_0^{''}$ and operators $\mathbf{T}^{'}$ and $\mathbf{T}^{''}$, where $\mathbf{T}^{'n} P = \mathbf{T}^{'}\left(\mathbf{T}^{'n-1}P\right)$.

Theorem 1.4.1 *Let $\mathbf{T}^{'}$ and $\mathbf{T}^{''}$ be operators on $P_{\mathcal{S}}$ and $P_0^{'}$, $P_0^{''} \in P_{\mathcal{S}}$. Then*

$$P_0^{'} \preceq P_0^{''} \tag{1.4.4}$$

implies

$$P_n^{'} \preceq P_n^{''} \text{ for all } n = 1, 2, ... \tag{1.4.5}$$

if there is a $\preceq$-monotone operator $\mathbf{T}$ *on* $P_{\mathcal{S}}$ *for which*

$$\mathbf{T}^{'} \preceq \mathbf{T} \preceq \mathbf{T}^{''} \tag{1.4.6}$$

In other notation theorem 1.4.1 says: If $\left(X_n^{'}\right)$ and $\left(X_n^{''}\right)$ are homogeneous Markov chain with transition operators $\mathbf{T}^{'}$ and $\mathbf{T}^{''}$ for which an operator $\mathbf{T}$ exists such that (1.4.6) is true, then

$$X_0^{'} \preceq X_0^{''}$$

implies

$$X_n^{'} \preceq X_n^{''} \text{ for all } n$$

If the transition operator $\mathbf{T}$ of the Markov chain (X_n) is $\preceq$-monotone, then $X_0 \preceq X_1$ implies $X_n \preceq X_{n+1}$ for all n.

Homogenous Markov chains (X_n) on an arbitrary state space $(S, \mathcal{S})$ can be described by their transition kernel $Q(x, B)$, namely

$$Q(x, B) = P(X_{n+1} \in B \mid X_n = x) \text{ for } x \in S \text{ and } B \in \mathcal{S}$$

or, in the case of real-valued processes, by their transition distribution function

$$F(y \mid x) = P(X_{n+1} \leq y \mid X_n = x)$$

Stoyan [123], (see also [106]), gives criterion of monotonicity and comparability for $\leq_{st}$ and $\leq_{icx}$ orders.

Theorem 1.4.2

(1) An operator $\mathbf{T}$ *is monotone with respect to* $\leq_{st}$ *order if and only if*

$$\textit{for all } y,\ F(y \mid x) \textit{ is decreasing in } x.$$

or equivalently

$$\overline{F}(y \mid x) = 1 - F(y \mid x) \textit{ is increasing in } x$$

(2) An operator $\mathbf{T}$ *is monotone with respect to* $\leq_{icx}$ *order if and only if*

$$\text{for all } y, \int_{y}^{+\infty} \overline{F}(t \mid x)\, dt \text{ is increasing and convex in } x$$

In the case of a Markov chain with discrete state space $S = \{1, 2, ...\}$ monotonicity can be expressed in terms of the transition probabilities $P_{ij} = P(X_{n+1} = j \mid X_n = i)$.

Theorem 1.4.3

(1) An operator $\mathbf{T}$ *is monotone with respect to* $\leq_{st}$ *order if and only if*

$$\overline{P}_{n-1m} \leq \overline{P}_{nm} \text{ for all } n, m$$

where $\overline{P}_{nm} = \sum_{l \geq m} P_{nl}$.

(2) An operator $\mathbf{T}$ *is monotone with respect to* $\leq_{icx}$ *order if and only if*

$$2\overline{\overline{P}}_{nm} \leq \overline{\overline{P}}_{n-1m} + \overline{\overline{P}}_{n+1m} \text{ for all } n, m$$

where $\overline{\overline{P}}_{nm} = \sum_{l \geq m} \overline{P}_{nl}$.

Theorem 1.4.4

(1) The transition operators $\mathbf{T}^{'}$ *and* $\mathbf{T}^{''}$ *satisfy* $\mathbf{T}^{'} \leq_{st} \mathbf{T}^{''}$ *if and only if*

$$F_1(y \mid x) \geq F_2(y \mid x) \text{ for all } x \text{ and } y$$

(2) The transition operators $\mathbf{T}^{'}$ *and* $\mathbf{T}^{''}$ *satisfy* $\mathbf{T}^{'} \leq_{icx} \mathbf{T}^{''}$ *if and only if*

$$\int_{y}^{+\infty} \overline{F_1}(t \mid x)\, dt \leq \int_{y}^{+\infty} \overline{F_2}(t \mid x)\, dt \text{ for all } x \text{ and } y$$

where $F_1(y \mid x)$ *and* $F_2(y \mid x)$ *are the transition distribution functions of* $\mathbf{T}^{'}$ *and* $\mathbf{T}^{''}$ *respectively.*

In the case of Markov chains with discrete state space $S = \{1, 2, ...\}$, we have:

Theorem 1.4.5

(1) $\mathbf{T}' \leq_{st} \mathbf{T}''$ *if and only if*

$$\overline{P_{nm}^{(1)}} \leq \overline{P_{nm}^{(2)}} \text{ for all } n, m$$

(2) $\mathbf{T}' \leq_{icx} \mathbf{T}''$ *if and only if*

$$\overline{\overline{P_{nm}^{(1)}}} \leq \overline{\overline{P_{nm}^{(2)}}} \text{ for all } n, m$$

where $\left(P_{nm}^{(1)}\right)$ *and* $\left(P_{nm}^{(2)}\right)$ *are the transition probabilities associated to* $\mathbf{T}'$ *and* $\mathbf{T}''$.

Chapter 2

Unreliable M/G/1 Retrial Queue with Geometric Loss and Random Reserved Time

During the last years, many authors considered and analysed queueing systems with retrial in which blocked customer enters orbit and repeats his/her call after a random amount of time. These models have been used in telephone switching systems, telecommunication networks, computer networks, computer and communication systems. The main features of the theory of retrial queues can be found in [17, 20, 37, 56, 89]. In complex models, the repeated attempts are combined with a variety of queueing phenomena leading to a large number of variants and generalizations of the main retrial queues. Among these variants, we mention retrial queues with: batch arrivals, finite population, multiclass queues, negative arrivals and disasters, impatient customers, priorities, breakdowns, feedback, vacation, generalized retrial policies.

Most queueing systems with repeated attempts assume that each customer in the retrial group seeks service independently of each other after a random time exponentially distributed. This discipline for access to the server from the retrial group is called classical retrial policy. However, the retrial rate may be independent of the number of customers in the orbit (if any). This discipline was introduced by Fayolle [58] and it is known as constant retrial policy. Farahmand [57] calls this model a retrial queue with first come first served (FCFS) orbit. For example, if the customers in orbit are queued in FCFS discipline, only the customer at the head of the queue is allowed to access the server. Gómez-Corral [65] studied the case of general retrial times. Artalejo and Gomez [19] introduced the versatile retrial policy which incorporates simultaneously the classical linear retrial policy and the constant one.

Customers can be either patient or impatient. For impatient customers, we distinguish two cases of abandonment: balking and renegeing. Balking of arriving customers describes their probabilistic decision to leave the system or to join the orbit. A waiting customer possibly reneges after a random time if his service has not begun.

Queues with servers subject to breakdowns and repairs are often encountered in practical applications. The interruptions of the service due to breakdowns and repairs lead to the growth of the waiting times for the customers and present an additional way for an arriving customer to enter the retrial orbit. Customers whose service is interrupted by a failure may have the option of remaining at the server until the repair is complete, leaving the system entirely or returning to the orbit. The breakdowns may be active (preemptive) or passive according to whether the failures occur in a working or idle period of the server. Besides, failures can take place after a random amount of service time or just before starting the service. If failures are preemptive, there are three different types of interruptions [63]:

(i) Preemptive-resume: After the repair is completed,the customer service proceeds as if there was not interruption,

(ii) Preemptive-repeat identical: After the repair is completed, the entire service is repeated. The new service time is exactly the same as the old one,

(iii) Preemptive-repeat different: After the repair is completed, the service is repeated. The new service time is statstically identical to and independent of the other service times.

Since the papers of Aissani [6] and Kulkarni and Choi [88] many studies on the unreliable retrial queues have been carried out from queueing and reliability viewpoints.

In section 1, we give a review of studies in the area of unreliable retrial queues. In section 2, we introduce a new version of retrial queues with server breakdowns and two types of impatient customers.

2.1 Literature Review: Retrial Queues with Server Breakdowns

Kulkarni and Choi [88] consider an $M/G/1$ retrial queue with classical retrial policy where the server is subject to active and idle breakdowns. The life time of the server is exponential. They consider two different models. In model I, the customer whose service is interrupted has to either leave the system or rejoin the retrial group. In model II, the customer whose service is interrupted stays at the server and restarts the service when repair is completed. For model II, they derive the stability condition and study the limiting behavior of the system by using the tools of Markov regenerative processes.

Aissani [7] considers an unreliable $M/G/1$ retrial queue with bulk arrivals, active and passive breakdowns, and classical retrial policy. The breakdown flow is described by a renewal process. The occurence of a breakdown can generate various situations (i) the service begins at first, (ii) the service begins at the point it was interrupted, (iii) the interrupted unit enters to orbit, (iv) the interrupted unit leaves the system. The author studies the distribution of the number of customers in the system by an adequate Markovization and obtains a genaralization of the Pollazeck-Khintchin formula.

Aissani [8] considers a version of the unreliable $M/G/1$ retrial queue with the problem of redundancy. He gives the condition of existence of a stationary regime, the steady state distribution of the system and obtains the generating function of the number of customers in orbit.

Artalejo [15] obtains sufficient conditions for ergodicity of Markovian multiserver queues with retrials and breakdowns. He presents a recursive algorithm to compute the steady state probabilities for the unreliable $M/G/1$ retrial queue. The algorithm is based on a regenerative approach.

Yang and Li [139] study a retrial queueing model with the server subject to starting failures where the repeated customers are persistent. They present, under the classical retrial policy, the necessary and sufficient condition for the system to be stable and derive analytical results for the queue length distribution as well as some performance measures of the system in steady state. They show that the general stochastic decomposition law for $M/G/1$ vacation models also holds for their system.

Aissani [9] analyses an unreliable single retrial queue with two type of customers (persistent and impatient customers) where the server is subject to passive and active breakdowns. He takes into consideration the corrective and preventive actions of maintenance. He obtains the generating function of the number of orbiting customers in the system. The heavy loading case is also discussed.

Aissani and Artalejo [12] consider a single server retrial queueing system subject to active and independent breakdowns where the retrial times are exponentially distributed and the service times are general. They introduce the concept of fundamental server period and an auxiliary queueing system with breakdowns and option for leaving the system. They obtain simplified expressions for the partial generating functions of the server state and the number of customers in the retrial group, a recursive scheme for computing the limiting probabilities and closed form formulae for the second order partial moments. Some stochastic decomposition results are also investigated.

Wang and Cao [131] analyse an $M/G/1$ retrial queue under classical retrial policy where the server is subject to Poisson active breakdowns. The customer whose service is interrupted remains in the service in order to conclude his remaining service after repair. Performance measures of the system and some reliability indexes are obtained.

Djellab [48]-[49] considers the $M/G/1$ retrial queue subjected to Poisson breakdowns. She uses its stochastic decomposition property to approximate the model performance in the case of general retrial times. The customers whose service is interrupted are obliged either to leave the system or to join the orbit.

Krishna Kumar, Pavai and Vijayakumar [86] discuss an $M/G/1$ retrial queue with Bernoulli feedback, FCFS orbit and general retrial times where the server is subject to starting failure. They derive the necessary and sufficient condition for the stability of the system and they obtain various performance measures. The general decomposition law is shown to hold for their model as well.

Aissani [11] considers retrial queueing systems with batch arrivals, server breakdowns and vacations. He studies (i) the effect of retrials, vacations and breakdowns on the performance measures of queueing service systems, (ii) some optimal control problems of vacation and retrial policies.

Almási, Roszik and Sztrik [14], propose the use of MOSEL tool to study homogeneous finite-source retrial queue with unreliable servers. One year later, Sztrik, Almási and Roszik [125] consider the case of heterogeneous finite-source retrial queues.

Wu, Brill, Hlynka and Wang [138] consider two retrial orbits in their M/G/1 system. The first one with an FCFS discipline, and the second is reserved specifically for customers preempted by a server failure. Repair times and retrials from first orbit are generally distributed while retrials from second orbit are exponentially distributed. Customers in second orbit remain persistent since they have already completed some amount of service.

Atencia, Fortes, Moreno and Sánchez [24] analyse a retrial queue with active breakdowns where the interrupted customers have the option of joining the orbit or remaining in the server for the repair in order to conclude his remaining service. Two disciplines to access to the server from the orbit are considered: classical retrial policy and constant retrial policy. For each system various characteristic quantities are found.

Gharbi and Ioualalen [64] give a detailed analysis of finite-source retrial systems with multiple servers subject to random breakdowns and repairs using generalized stochastic Petri nets model. They consider different breakdowns disciplines. The main steady state performance and reliability indices are derived and several numerical calculations were performed to show the effect of servers number, retrial, failure,and repair rates on the performability measures of the system.

Li and Wang [93] study an $M/G/1$ retrial queue with two phase service, feedback and FCFS orbit where the server is subject to starting failures and customers are allowed to balk and renege at particular times. All customers demand the first "essential" service, whereas only some of them demand the second " multi-optional" service. The steady state solutions for some queueing and reliability measures of the system are obtained. Wang and Li [132] investigate a repairable $M/G/1$ retrial queue with Bernoulli vacation, setup times and two-phase service.

Li, Ying and Zhao [92] consider a BMAP/G/1 retrial queue with general repair time. They develop two algorithms, the first one to compute the stationary performance measures of an M/G/1 continous-time level dependent Markov chain, and the second to calculate the mean of the first passage time with regard to this M/G/1.

Sherman and Kharoufeh [117] analyze an unreliable M/M/1 retrial queue with infinite-capacity orbit and normal queue. Customers join the retrial orbit if and only if they are interrupted by a server breakdowns. In their analysis , they assume that the total number of customers in the system is a Markovian process which is not true. Falin [54] suggests an alternative method for analysis of the Markov process which describes the functioning of the system and finds the distribution of the server state, the number of customers in the queue and the number of customers in the retrial group in steady state. Sherman, Kharoufeh and Abramson [118] consider the same model with general service time.

Brian Crawford [44] considers the performance evaluation of an Markovian $M/M/2$ retrial queue for which both servers are subject to active and idle breakdowns. Customers may abandon service requests if they are blocked from service upon arrival, or if their service is interrupted by a server failure. Customers choosing to remain in the system enter a retrial orbit for a random amount of time before attempting to access an available server. The author proposes a method for approximating the steady state probabilities.

Mokaddis, Metwally and Zaki [104] study the $M/G/1$ retrial queue with Bernoulli feedback, single vacation and FCFS orbit where the server is subject to starting failures and the retrial time is governed by an arbitrary distribution. Various performance measures are derived.

Atencia, Bouza and Moreno [23] consider an $M^{[X]}/G/1$ retrial queue subject to breakdowns where the retrial time is exponential and independent of the number of customers applying for service. The customer just being served before server failure waits for the server to complete his remaining service.The stochastic decomposition property and the asymptotic behaviour under high rate of retrials are discussed. They also give a recursive scheme to compute the distribution of the number of served customers during the k-busy period and the ordinary period.

Kim, Klimenok and Orlovsky [79] consider a multi-server $BMAP/PH/N$ retrial queue with unreliable servers and classical retrial policy. Input of breakdowns is described by the Markovian arrival process MAP, repair times have PH distribution. Customer whose service was interrupted goes to the orbit or leaves the system. They outline an algorithm for calculating the stationary distribution, derive formulas for important performance measures and present illustrative numerical examples.

Jain and Bhargava [69] analyse an unreliable server bulk arrival retrial queue with two class non-preemptive priority subscribers. The two types of subscribers arrive according to poisson flow. The subscribers in each class arrive to the system in batches; the batch sizes follow the geometric law. The server is subject to active breakdowns and the service is restarted after repair. The same authors, consider in [70] an unreliable $M/G/1$ queueing system with Bernoulli feedback, modified vacation, repeated attempts, discouragement and k-phase repair. First phase repair is essential whereas other (k-1) phases are optional. The time until the repairman starts the first phase repair is called setup time. The authors obtain the probability generating function of steady state queue size. Some queueing as well as reliability indices are also derived. The effects of various system parameters on the system performance indices are examined numerically.

Jain and Mishra [71] study the reliability analysis of an unreliable server retrial queueing system with bulk arrivals and state dependent rates where the server is subject to active breakdowns. Steady state and transient state analysis for the prediction of reliability indices is presented.

Wang [130] focuses on the unreliable $M_1, M_2/G_1, G_2/1$ retrial queues with two different types of primary customers. In case of blocking, the first type of customers can be queued whereas the second type of customers must leave the service area but return after some random period of time to try their luck again. The server is subject to active breakdowns and it has a service-type dependent, exponentially distributed lifetime as well as a service-type dependent, generally distributed repair time. The author obtains a steady state solution for queueing and reliability measures of interest.

Boualem, Djellab and Aissani [32] consider an $M/G/1$ retrial queue with vacations and constant retrial policy. They derive several stochastic comparison properties in the sense of some stochastic orders.

Choudhury and Deka [40] consider an $M^X/G/1$ retrial queue with an additional second phase of optional service and unreliable server where breakdowns are preemptive. Further concept of Bernoulli admission mechanism is introduced in the model.

Oukid and Aissani [110] obtain lower bound and new upper bound for the mean busy period of $GI/GI/1$ queue with breakdowns and FIFO queue. They consider an extension to the model of queue with retrial. The results are based on some reliability concepts.

On the other hand, a few number of papers have recently appeared on discrete-time retrial queue with unreliable server. We refer to [25, 105, 133-135].

Atencia and Moreno [25] study a discrete time $Geo/G/1$ retrial queue where the server is subject to starting failures. They analyse the embedded Markov chain and present some performance measures of the system in steady state. Then, they develop a procedure for calculating the distributions of the orbit and system size as well as the marginal distributions of the orbit size when the server is idle, busy or down.

Moreno [105] studies a discrete time $Geo/G/1$ retrial queue, in which the server is subject to failures and the server lifetime is general. She studies the stability condition of the system and gives the distributions of the number of customers in the orbit and in the system together with various performance measures. The stochastic decomposition law of the system size is shown to be hold for her model. She also gives bounds for the proximity between the steady state distributions for her queueing system and its corresponding without retrials.

Wang and Zhao [135] consider a discrete time $Geo/G/1$ retrial queue with FCFS orbit and general retrial time where the server is subject to starting failures and the interrupted customer must join the orbit after failure. They obtain the ergodicity condition of the embedded Markov Chain and the performance measures of the system. Stochastic decomposition laws are also given. In [134] they consider the same model in which all the arriving customers require a first "essential" service while some of them ask for second "optional" service.

Wang and Zhang [133] treat a discrete time single server retrial queue with geometrical arrivals of positive and negative customers in which the server is subject to breakdowns and repairs. They analyse the Markov chain underlying the queueing system and obtain its ergodicity condition. The generating functions of the number of customers in the orbit and in the system are also obtained along with the marginal distributions of the orbit size when the server is idle, busy or down.

In this thesis we analyse an M/G/1 retrial queue with general retrial times where the server is subject to Poisson and active breakdowns with repairs. The primary customers who find the server busy, failed or reserved are allowed to balk or are queued in orbit in accordance with an FCFS discipline. Retrial attempt of the customer at the head of the retrial queue begins only when the server becomes empty. Each time that a customer is blocked on his attempt he/she will either leave the system or stay in orbit for later retrial. The customer whose service is interrupted can stay at the server waiting for the repair or enters a service orbit. After the repair is completed, the server resumes service immediately if the customer in service has remained in the service position. Otherwise the server is required to search for the customer in the service orbit. The random time until the customer in service orbit resumes service is called the reserved time that we suppose is generally distributed.

2.2 The Mathematical Model

We consider a single retrial queue with breakdowns and repairs. The following assumptions describe the mathematical model [127]:

• **The arrival process:** Primary customers arrive in a Poisson process with rate λ.

• **The service times:** Successive service times are independent with common probability distribution function $B(x)$, density function $b(x)$, Laplace Stieltjes transform $L_B(s)$ and first two moments β_1, β_2.

• **The life time and the repair time:** The server is subject to active breakdowns, i.e., the server can fail when it is rending service. It fails after an exponential amount time with mean $\frac{1}{\mu}$. When the server fails, it is repaired immediately and the time required to repair is a random variable with general distribution $C(x)$, density function $c(x)$, Laplace Stieltjes transform $L_C(s)$ and first two moments γ_1, γ_2.

• **The service discipline and the retrial rule:** If the server is free the primary customer is served immediately and leaves the system after service completion, otherwise he will decide either to leave the system without service with probability $1-p$ or to enter the retrial group (called orbit) with probability p. We assume that only the customer at the head of the orbit is allowed to access the server and that the retrial begins only when the server is idle. If the server is busy upon retrial, the customer can either return to the orbit with probability q or leave the system with complementary probability $1-q$. The successive retrial times are governed by an arbitrary probability distribution function $A(x)$, with corresponding density function $a(x)$ and Laplace-Stieltjes transform $L_A(s)$.

• **The repair discipline and the reserved time:** The customer whose service is interrupted either remains in the service position with probability r or enters a service orbit with probability $1-r$. If the customer in service enters the service orbit upon server failure, after repair the server is required to search for customer in service orbit. We refer to the time until the customer in service orbit resumes service as the "reserved time" that we suppose is generally distributed with distribution function $D(x)$, density function $d(x)$, Laplace Stieltjes transform $L_D(s)$ and first two moments η_1, η_2. The server is not allowed to accept new customers until the customer in service leaves the system. The server is said to be blocked if the server is busy, under repair or reserved. Service for a customer resumes after the repair time and reserved time.

At any service completion, the server becomes idle. The length of the idle period of the server is determined by the competition between an exponential law of rate λ (a primary customer) and the general retrial time distribution (the retrial customer at the head of the orbit) which determines the next customer who accesses the server.

Inter-arrival times, retrial times, service times, repair times and reserved times are assumed to be mutually independent. The time until failure is independent of the other times.

The functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$ and $\eta(x)$ are the conditional completion rates for repeated attempts, for service, for repair and for reserved time, respectively; i.e.,

$$\alpha(x) = \frac{a(x)}{1 - A(x)}, \beta(x) = \frac{b(x)}{1 - B(x)}, \gamma(x) = \frac{c(x)}{1 - C(x)}, \eta(x) = \frac{d(x)}{1 - D(x)}$$

The state of the system at time t can be described by the Markov process

$$\{X(t), t \geq 0\} = \{(J(t), J^*(t), Q(t), \xi_0(t), \xi_1(t), \xi_2(t), \xi_3(t))\,; t \geq 0\}$$

We denote

- $J(t)$ the server state at time t;
- $J^*(t)$ the state of the customer in service after failure at time t;
- $Q(t)$ the number of customers in the retrial queue at time t;
- $\xi_0(t)$ the elapsed retrial time when $J(t) = 0$ and $Q(t) > 0$;
- $\xi_1(t)$ the elapsed service time if $J(t) = 1$ or $J(t) = 2$ or $J(t) = 3$;
- $\xi_2(t)$ the elapsed repair time when $J(t) = 2, J^*\,(t) = 0$ or 1;
- and $\xi_3(t)$ the elapsed reserved time if $J(t) = 3$.

  where

$$J(t) = \begin{cases} 0, & \text{if the server is idle} \\ 1, & \text{if the server is busy} \\ 2, & \text{if the server is under repair} \\ 3, & \text{if the server is reserved} \end{cases}$$

$$J^*\,(t) = \begin{cases} 0, & \text{if the customer in service remains in service position after server failure} \\ 1, & \text{if the customer in service enters a service orbit after server failure} \end{cases}$$

Special cases of our model can be deduced by setting appropriate parameters as follows:

- If $A(x) = 1 - e^{-\theta x}$ and $\theta \to \infty$, then this retrial queue becomes a classical $M/G/1$ queue with repairable server and balking;
- If $\mu = 0$, this retrial reduces to the $M/G/1$ retrial queue with impatient customers;
- If $p = q = r = 1$, $\mu \neq 0$, $D(x) = 1 - e^{-\eta x}$, $\eta \to \infty$, this retrial queue reduces to the $M/G/1$ retrial queue with persistent customers and repairable server where the customer whose service is interrupted remains in service position.

By assuming exponential service times, repair times, reserved times and retrial times with means β_1, γ_1, η_1 and α_1 respectively, the supplementary variables $\xi_0(t),\ \xi_1(t),\ \xi_2(t)$ and $\xi_3(t)$ are omitted and the state of the system is described by the Markov process $\{X(t), t \geq 0\} = \{(J(t), J^*(t), Q(t))\,; t \geq 0\}$. The possible transitions are as follows:

State	Transitions	Rates	Events
(0,n)	(1,n)	λ	Arriving of a primary customer
	(1,n-1)	$1/\alpha_1$	Arriving of a repeated customer
(1,n)	(1,n+1)	λp	Entering of a customer into the orbit
	(0,n)	$1/\beta_1$	Service completion
	(2,0,n)	$r\mu$	Failure and the customer in service remains in service position
	(2,1,n)	(1-r)μ	Failure and the customer in service enters into the service orbit
(2,0,n)	(2,0,n+1)	λp	Entering of a customer into the orbit
	(1,n)	$1/\gamma_1$	Repair completion
(2,1,n)	(2,1,n+1)	λp	Entering of a customer into the orbit
	(3,n)	$1/\gamma_1$	Repair completion
(3,n)	(3,n+1)	λp	Entering of a customer into the orbit
	(1,n)	$1/\eta_1$	Reserved time completion

Chapter 3

Analytical Study

The system performance can be described in terms of its main characteristics, such as limit distribution, number of customers in the system, waiting time, busy period, number of customers served , loss probability in the case of impatient customers, and so on.

In this chapter we analyse the unreliable retrial queue described in section 2, chapter 2, from queueing and reliability viewpoints. In section 1, we define the generalized service time. The necessary and sufficient condition for the stability of the system is obtained in section 2, using an embedded Markov chain. By supplementary variables method, we study in section 3 the steady-state distribution of the system. We describe it in terms of generating functions and then we derive some performance measures. Section 4 is devoted to retrials. In particular, we give the distribution of the number of retrials made by a customer, loss probabilities, and the distribution of the number of primary customers that get service before the customer at the head of the orbit. Waiting time and busy period are discussed in section 5 and section 6 respectively. In section 7, we analyse the time to the first failure of the server.

3.1 Generalized Service Time

To calculate the queueing indices, we need to define the generalized service time (also called blocking time or basic server sojourn time) for unreliable systems.

The generalized service time S_n^* of the $n-th$ customer is the length of time since the $n-th$ customer begins to be served until the service is completed. Note that S_n^* includes the service time and some possible repair times and reserved times. We denote by S^* the generic random variable corresponding to the sequence (S_n^*) of independent , identically distributed random variables with common distribution $B^*(x)$ and Laplace Stieltjes transform $L_{B^*}(s)$.

Lemma 3.1.1 [127] *The distribution function, Laplace Stieltjes transform and expected value of S^* are :*

$$B^*\left(x\right)=\sum_{n\geq 0}\sum_{k=0}^{n}\int_0^x\binom{n}{k}e^{-\mu y}\frac{\left(\mu y\right)^n}{n!}\left(1-r\right)^{n-k}r^kC_{n,k}^{(2)}\left(x-y\right)dB(y) \tag{3.1.1}$$

$$L_{B^*}\left(s\right)=L_B\left(s+\mu-\mu\left\{r+\left(1-r\right)L_D\left(s\right)\right\}L_C\left(s\right)\right) \tag{3.1.2}$$

$$E\left(S^*\right)=\beta_1\left\{1+\mu\left[\left(1-r\right)\eta_1+\gamma_1\right]\right\} \tag{3.1.3}$$

respectively, where $C_{n,k}^{(2)}(x)$ represents the two fold convolution of $C^{(n)}(x)$ (which is n fold convolution of $C(x)$) and $D^{(n-k)}(x)$ (which is $n-k$ fold convolution of $D(x)$).

Proof. Let $N\left(S\right)$, $N_0\left(S\right)$, $N_1\left(S\right)$ be the number of failures during the interval since the customer in service begins to be served until the service is completed, the number of failures during the same period where the customer in service remains in service position after failure and the number of failures where the customer in service enters the service orbit after failure, respectively.

$$N\left(S\right)=N_0\left(S\right)+N_1\left(S\right),\ S^*=S+\sum_{i=1}^{N(S)}X_i+\sum_{i=1}^{N_1(S)}Y_i$$

Where X_i and Y_i represent the $i-th$ repair time and the $i-th$ reserved time, S is the service time. We know that $\left(N\left(S\right)/S\right)$, $\left(N_0\left(S\right)/S\right)$ and $\left(N_1\left(S\right)/S\right)$ are poisson processes with rates μ, μr, $\mu\left(1-r\right)$ and that $\left(N_1\left(S\right)/S,\ N\left(S\right)=n\right)$ has a binomial distribution with the parameters $n,\ (1-r)$. Then,

$$B^*(x)=P\left(S^*\leq x\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}P\left(S^*\leq x,\ N\left(S\right)=n,\ N_1\left(S\right)=k\right)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\int_0^{\infty}P\left(S^*\leq x,\ N\left(S\right)=n,\ N_1\left(S\right)=k/\ S=y\right)dB(y)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\int_{0}^{x} P\left(N\left(y\right)=n\right) P\left(N_1\left(y\right)=k/N\left(y\right)=n\right) P\left(\sum_{i=1}^{n}X_i+\sum_{i=1}^{k}Y_i\leq x-y\right) dB(y)$$

$$= \sum_{n\geq 0}\sum_{k=0}^{n}\int_{0}^{x}\binom{n}{k}e^{-\mu y}\frac{\left(\mu y\right)^n}{n!}\left(1-r\right)^k r^{n-k} C_{n,n-k}^{(2)}\left(x-y\right)dB(y)$$

Taking conditional expectation on S and $N(S)$, we have

$$E(e^{-sS^*}/S, N(S)) = e^{-sS}\left(L_C\left(s\right)\left\{r+(1-r)L_D\left(s\right)\right\}\right)^N$$

Since by hypothesis $\left(N\left(S\right)/S\right)$ is Poisson Process with rate μ, then

$$\begin{aligned} E(e^{-sS^*}/S) &= \sum_{n\geq 0} e^{-sS}\left(L_C\left(s\right)\left\{r+(1-r)L_D\left(s\right)\right\}\right)^n \frac{\left(\mu S\right)^n e^{-\mu S}}{n!} \\ &= e^{-S(s+\mu-\mu L_C(s)\{r+(1-r)L_D(s)\})} \end{aligned}$$

Finally

$$\begin{aligned} L_{B^*}\left(s\right) &= E(e^{-sS^*}) = \int_0^{\infty} E(e^{-sS^*}/S=t)dB\left(t\right) \\ &= L_B\left(s+\mu-\mu L_C\left(s\right)\left\{r+\left(1-r\right)L_D\left(s\right)\right\}\right) \end{aligned}$$

and

$$E(B^*) = -\frac{d}{ds}L_{B^*}(0) = \beta_1\left\{1+\mu\left[\left(1-r\right)\eta_1+\gamma_1\right]\right\}.$$

■

3.2 Stability Condition

As usual, the first question to be investigated in Queueing theory is the stability of the system. The performance measures of the system depend on its stability [76-77, 100]. In this section, we study the necessary and sufficient condition for the system to be stable. First, we establish the ergodicity of the embedded Markov chain at departure completion epocks. Then, we deduce the stability of the system using Burke's theorem (theorem 3.2.2).

3.2.1 Embedded Markov Chain

The method of the embedded Markov chain is used to construct a Markov model, not for all t, but for specific time points on the time axis. The idea is to simplify the description of state space, i.e. reduce its dimension. This method also allows to establish the stability condition of the whole system and determine the stationary distribution.

Let t_n be the time of the $n-th$ departure and Q_n the number of customers in the orbit just after the time t_n, then $J\left(t_n^+\right) = 0,\ Q\left(t_n^+\right) = Q_n, \forall n \in N$. We have the following fundamental recursive equation: $Q_{n+1} = Q_n - \beta^{(n+1)} + \nu^{(n+1)}$, where

- $\nu^{(n)}$ is the number of primary customers which arrive in the system during the generalized service time of the $(n)-th$ customer and do not leave the system after subsequent blocking . The flow of primary customers which arrived during the generalized service time and are not lost is the p-thinning of the original Poisson flow of primary customers. Thus it is Poisson with the rate λp. This implies that $\nu^{(n)}$ has distribution

$$k_j = \int_0^{+\infty} \frac{(\lambda p x)^j}{j!} e^{-\lambda p x} dB^*(x), \quad j \geq 0$$

- and

$$\beta^{(n)} = \begin{cases} 1 & if\ the\ retrial\ customer\ exits\ the\ orbit\ after\ the\ (n-1)-th\ departure \\ 0 & otherwise \end{cases}$$

  Notice that the retrial customer exits the orbit in two cases: it is the next customer to be served or it leaves the system if the next customer in service is a primary customer.

- If $Q_{n-1} = 0$ then $\beta^{(n)} = 0 \;\; a.s$

- If $Q_{n-1} = i > 0$ then

$$P(\beta^{(n)} = 0/Q_{n-1} = i) = q(1 - L_A(\lambda))$$
$$P(\beta^{(n)} = 1/Q_{n-1} = i) = L_A(\lambda) + (1 - L_A(\lambda))(1-q) = 1 - q(1 - L_A(\lambda))$$

The sequence of random variables $Q_n, n \geq 1$ forms an embedded Markov chain for our queue which is irreductible and aperiodic on the states-space $\mathbb{N}$.

The one step transition probabilities of the embedded Markov chain (Q_n) are given by the formula:

$$P_{ij} = P\left[Q_{n+1} = j/Q_n = i\right] = \begin{cases} k_j, & i = 0 \\ 0, & i \geq j+2 \\ \left[1 - q\left(1 - L_A(\lambda)\right)\right] k_0, & i = j+1 \\ \left[1 - q(1 - L_A(\lambda))\right] k_{j-i+1} + q(1 - L_A(\lambda))k_{j-i}, & 1 \leq i \leq j \end{cases}$$

3.2.2 Ergodicity of the Embedded Markov Chain

In the following theorem, we establish the ergodicity of the embedded Markov chain at the departure epochs.

Theorem 3.2.1 [127] *Let Q_n be the orbit length at the time of the $n-th$ departure, $n \geq 1$. Then $\{Q_n, n \geq 1\}$ is ergodic if and only if*

$$\lambda p \beta_1 \left\{1 + \mu\left[(1-r)\,\eta_1 + \gamma_1\right]\right\} < 1 - q + qL_A\left(\lambda\right)$$

Proof. To prove ergodicity, we shall use the Foster's criterion [97]: An irreductible and aperiodic Markov chain $\{Y_n, n \in \aleph\}$ with states space S is ergodic if there exists a nonnegative function $f\left(s\right),\ s \in S$, and $\epsilon > 0$ such that the mean drift $\chi_s = E\left(f\left(Y_{n+1}\right) - f\left(Y_n\right)/Y_n = s\right)$ is finite for all $s \in S$ and $\chi_s \leq -\epsilon$ for all $s \in S$ except perhaps a finite number.

In our case, we consider the function $f(j) = j$ for all $j \in \mathbb{N}$. Then the mean drift is given by

$$\chi_j = \begin{cases} \lambda p \beta_1 \left\{1 + \mu\left[(1-r)\,\eta_1 + \gamma_1\right]\right\} & if\ j = 0 \\ \lambda p \beta_1 \left\{1 + \mu\left[(1-r)\,\eta_1 + \gamma_1\right]\right\} - (1 - q + qL_A(\lambda)) & if\ j \geq 1 \end{cases}$$

Thus if $\lambda p \beta_1 \left\{1 + \mu\left[(1-r)\,\eta_1 + \gamma_1\right]\right\} < 1 - q + qL_A(\lambda)$, the Foster's criterion is fulfilled and so the chain $\{Q_n, n \geq 1\}$ is ergodic.

The same inequality is also necessary for ergodicity. As noted in [114], we can guarantee nonergodicity if the Markov chain $\{Q_n, n \geq 1\}$ satisfies Kaplan's condition: $\chi_j < +\infty$

for all $j \geq 0$, and there exists $j_0 \in \aleph$ such that $\chi_j \geq 0$ for $j \geq j_0$. In our case, $\lambda p \beta_1 \{1 + \mu [(1-r)\eta_1 + \gamma_1]\} \geq 1 - q + qL_A(\lambda)$ implies the nonergodicity of the Markov chain. ■

Since the arrival stream is a Poisson process, we can show from Burke's theorem [43] that the steady state probabilities (the limiting probabilities of the system state) of $\{X(t), t \geq 0\}$ exist and are positive if and only if

$$\lambda p \beta_1 \{1 + \mu [(1-r)\eta_1 + \gamma_1]\} < 1 - q + qL_A(\lambda).$$

Theorem 3.2.2 [43] *Let $N(t)$ be a stochastic process whose sample functions are (almost all) step functions with unit jumps. Let the points of increase after some time $t = 0$ be labeled T_n, and the points of decrease be labeled $T_n^{'}$, $n = 1, 2, ...$ Let $N(T_n^{-})$ be denoted A_n, and let $N(T_n^{'+})$ be denoted D_n. (Thus, if upward jumps correspond to arrivals and downward jumps correspond to departures then A_n is the state of the system just prior of the $n-th$ arrival epoch and D_n is the state of the system just after the $n-th$ departure epoch). Then if either $\lim_{n\to\infty} P(A_n \leq k)$ or $\lim_{n\to\infty} P(D_n \leq k)$ exists, the other does and these are equal. [Thus $N(t)$ has the same limiting distribution just prior to its points of increase as it does just after its points of decrease, if this limiting distribution exists].*

Interpretation: $1 - q + qL_A(\lambda)$ gives the expected number of orbiting customers who exit the retrial group (who enter service successfully or leave the system without service), while $\lambda p \beta_1 \{1 + \mu [(1-r)\eta_1 + \gamma_1]\}$ gives the expected number of new arrivals joining the orbit during the generalized service time. For stability, we require that new customers must arrive during a generalized service time more slowly than orbiting customers exiting orbit, that is $\lambda p \beta_1 \{1 + \mu [(1-r)\eta_1 + \gamma_1]\} < 1 - q + qL_A(\lambda)$.

Remark 3.2.1 *If $p = q = 1$ and $\mu = 0$, we find the stability condition given by Gómez-Corral [65].*

3.3 Steady-State Distribution

In this section, we study the limiting distribution of the system state. For the process $\{X(t), t \geq 0\}$, we introduce the following functions which describe the joint distribution of the server state and orbit size:

- $P_{(0,0)}(t) = P(J(t) = 0, Q(t) = 0)$
- $P_{(0,i)}(t, w)\, dw = P(J(t) = 0,\ Q(t) = i, w < \xi_0(t) < w + dw)$,
  for $i \geqslant 1$
- $P_{(1,i)}(t, x)\, dx = P(J(t) = 1,\ Q(t) = i, x < \xi_1(t) < x + dx)$, for $i \geqslant 0$
- $P_{(2,0,i)}(t, x, y)\, dxdy$
  $= P(J(t) = 2,\ J^*(t) = 0,\ Q(t) = i,\ x < \xi_1(t) < x + dx, y < \xi_2(t) < y + dy)$,
  for $i \geqslant 0$
- $P_{(2,1,i)}(t, x, y)\, dxdy$
  $= P(J(t) = 2,\ J^*(t) = 1,\ Q(t) = i, x < \xi_1(t) < x + dx, y < \xi_2(t) < y + dy)$,
  for $i \geqslant 0$
- $P_{(3,i)}(t, x, \tau)\, dxd\tau = P(J(t) = 3,\ Q(t) = i,\ x \prec \xi_1(t) < x + dx, \tau < \xi_3(t) < \tau + d\tau)$,
  for $i \geqslant 0$

By considering transitions of the process between time t and $t+\Delta t$ and letting $\Delta t \longrightarrow 0$, we derive the following system of equations that govern the dynamics of the system behaviour:

$$(\frac{\partial}{\partial t} + \lambda)P_{(0,0)}(t) = \int_0^\infty P_{(1,0)}(t, x)\, \beta(x)\, dx, \tag{3.3.4}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial w} + \lambda + \alpha(w)\right) P_{(0,i)}(t, w) = 0,\ \ i \geq 1, \tag{3.3.5}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + p\lambda + \mu + \beta(x)\right) P_{(1,i)}(t, x) = \int_0^\infty P_{(2,0,i)}(t, x, y)\, \gamma(y)\, dy$$

$$+ \int_0^\infty P_{(3,i)}(t, x, \tau)\, \eta(\tau)\, d\tau + p\lambda P_{(1,i-1)}(t, x),\ \ i \geq 0, \tag{3.3.6}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + p\lambda + \gamma(y)\right) P_{(2,0,i)}(t, x, y) = p\lambda P_{(2,0,i-1)}(t, x, y),\ \ i \geq 0, \tag{3.3.7}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+p\lambda+\gamma\left(y\right)\right)P_{(2,1,i)}\left(t,x,y\right)=p\lambda P_{(2,1,i-1)}\left(t,x,y\right),\ \ i\geq 0, \tag{3.3.8}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}+p\lambda+\eta\left(\tau\right)\right)P_{(3,i)}\left(t,x,\tau\right)=p\lambda P_{(3,i-1)}\left(t,x,\tau\right),\ \ i\geq 0, \tag{3.3.9}$$

The boundary conditions are given as follows:

$$P_{(0,i)}\left(t,0\right)=\int_{0}^{\infty}P_{(1,i)}\left(t,x\right)\beta\left(x\right)dx,\ \ i\geq 1, \tag{3.3.10}$$

$$\begin{aligned}P_{(1,i)}\left(t,0\right)=&\int_{0}^{\infty}P_{(0,i+1)}\left(t,w\right)\alpha\left(w\right)dw+\left(1-q\right)\lambda\int_{0}^{\infty}P_{(0,i+1)}\left(t,w\right)dw\\&+\left(1-\delta_{i,0}\right)q\lambda\int_{0}^{\infty}P_{(0,i)}\left(t,w\right)dw+\delta_{i,0}\lambda P_{(0,0)}(t),\ \ i\geq 0,\end{aligned} \tag{3.3.11}$$

$$P_{(2,0,i)}\left(t,x,0\right)=r\mu P_{(1,i)}\left(t,x\right),\ \ i\geq 0, \tag{3.3.12}$$

$$P_{(2,1,i)}\left(t,x,0\right)=\left(1-r\right)\mu P_{(1,i)}\left(t,x\right),\ \ i\geq 0, \tag{3.3.13}$$

$$P_{(3,i)}\left(t,x,0\right)=\int_{0}^{\infty}P_{(2,1,i)}\left(t,x,y\right)\gamma\left(y\right)dy,\ \ i\geq 0, \tag{3.3.14}$$

The normalizing condition is:

$$\begin{aligned}P_{(0,0)}(t)+\sum_{i=1}^{\infty}\int_{0}^{\infty}P_{(0,i)}\left(t,w\right)dw+\sum_{i=0}^{\infty}(\int_{0}^{\infty}P_{(1,i)}\left(t,x\right)dx+\int_{0}^{\infty}\int_{0}^{\infty}P_{(2,0,i)}\left(t,x,y\right)dxdy\\+\int_{0}^{\infty}\int_{0}^{\infty}P_{(2,1,i)}\left(t,x,y\right)dxdy+\int_{0}^{\infty}\int_{0}^{\infty}P_{(3,i)}\left(t,x,\tau\right)dxd\tau)=1,\end{aligned} \tag{3.3.15}$$

When the stability condition is met, we find that the limiting probabilities $\lim_{t\to\infty}P_{(0,0)}\left(t\right)=P_{(0,0)}$, $\lim_{t\to\infty}P_{(0,i)}\left(t,w\right)=P_{(0,i)}\left(w\right)$, $\lim_{t\to\infty}P_{(1,i)}\left(t,x\right)=P_{(1,i)}\left(x\right)$, $\lim_{t\to\infty}P_{(2,0,i)}\left(t,x,y\right)=P_{(2,0,i)}\left(x,y\right)$, $\lim_{t\to\infty}P_{(2,1,i)}\left(t,x,y\right)=P_{(2,1,i)}\left(x,y\right)$ and $\lim_{t\to\infty}P_{(3,i)}\left(t,x,\tau\right)=P_{(3,i)}\left(x,\tau\right)$ satisfy the equations of statistical equilibrium:

$$\lambda P_{(0,0)} = \int_0^\infty P_{(1,0)}(x)\,\beta(x)\,dx, \tag{3.3.16}$$

$$\left(\frac{\partial}{\partial w} + \lambda + \alpha(w)\right) P_{(0,i)}(w) = 0,\ \ i \geq 1, \tag{3.3.17}$$

$$\left(\frac{\partial}{\partial x} + p\lambda + \mu + \beta(x)\right) P_{(1,i)}(x) = \int_0^\infty P_{(2,0,i)}(x,y)\,\gamma(y)\,dy$$

$$+\int_0^\infty P_{(3,i)}(x,\tau)\,\eta(\tau)\,d\tau + p\lambda P_{1,i-1}(x)\,,\ \ i \geq 0, \tag{3.3.18}$$

$$\left(\frac{\partial}{\partial y} + p\lambda + \gamma(y)\right) P_{(2,0,i)}(x,y) = p\lambda P_{(2,0,i-1)}(x,y)\,,\ \ i \geq 0, \tag{3.3.19}$$

$$\left(\frac{\partial}{\partial y} + p\lambda + \gamma(y)\right) P_{(2,1,i)}(x,y) = p\lambda P_{(2,1,i-1)}(x,y)\,,\ \ i \geq 0, \tag{3.3.20}$$

$$\left(\frac{\partial}{\partial \tau} + p\lambda + \eta(\tau)\right) P_{(3,i)}(x,\tau) = p\lambda P_{(3,i-1)}(x,\tau)\,,\ \ i \geq 0, \tag{3.3.21}$$

under the boundary conditions

$$P_{(0,i)}(0) = \int_0^\infty P_{(1,i)}(x)\,\beta(x)\,dx,\ \ i \geq 0, \tag{3.3.22}$$

$$P_{(1,i)}(0) = \int_0^\infty P_{(0,i+1)}(w)\,\alpha(w)\,dw + (1-q)\,\lambda \int_0^\infty P_{(0,i+1)}(w)\,dw$$

$$+(1-\delta_{i,0})\,q\lambda \int_0^\infty P_{(0,i)}(w)\,dw + \delta_{i,0}\lambda P_{(0,0)},\ \ i \geq 0, \tag{3.3.23}$$

$$P_{(2,0,i)}(x,0) = r\mu P_{(1,i)}(x) >\ \ i \geq 0, \tag{3.3.24}$$

$$P_{(2,1,i)}(x,0) = (1-r)\,\mu P_{(1,i)}(x)\,,\ \ i \geq 0, \tag{3.3.25}$$

$$P_{(3,i)}(x,0) = \int_0^\infty P_{(2,1,i)}(x,y)\,\gamma(y)\,dy,\ \ i \geq 0, \tag{3.3.26}$$

together with the normalizing equation

$$P_{(0,0)} + \sum_{i=1}^{\infty}\int_0^{\infty} P_{(0,i)}(w)\,dw + \sum_{i=0}^{\infty}(\int_0^{\infty} P_{(1,i)}(x)\,dx + \int_0^{\infty}\int_0^{\infty} P_{(2,0,i)}(x,y)\,dxdy$$

$$+ \int_0^{\infty}\int_0^{\infty} P_{(2,1,i)}(x,y)\,dxdy + \int_0^{\infty}\int_0^{\infty} P_{(3,i)}(x,\tau)\,dxd\tau) = 1, \tag{3.3.27}$$

where $\delta_{i,0}$ is the Kronecker function, $P_{(1,-1)}(x) = 0,\ P_{(2,0,-1)}(x,y) = 0,$ $P_{(2,1,-1)}(x,y) = 0$ and $P_{(3,-1)}(x,\tau) = 0$ for $x \geq 0,\ y \geq 0,\ \tau \geq 0.$

To solve the system of equations (3.3.16) -(3.3.27) , we define the following generating functions:
$P_0(z,w) = \sum_{i=1}^{\infty} P_{(0,i)}(w)\,z^i,\ P_1(z,x) = \sum_{i=0}^{\infty} P_{(1,i)}(x)\,z^i,\ P_{20}(z,x,y) = \sum_{i=0}^{\infty} P_{(2,0,i)}(x,y)\,z^i,$
$P_{21}(z,x,y) = \sum_{i=0}^{\infty} P_{(2,1,i)}(x,y)\,z^i,\ P_3(z,x,\tau) = \sum_{i=0}^{\infty} P_{(3,i)}(x,\tau)\,z^i$
which are convergent for each $x \geq 0,\ y \geq 0,\ \tau \geq 0$ at least in the domain $|z| \leq 1$.

In the following theorem, we describe the steady state distribution of the system in terms of generating functions.

Theorem 3.3.1 [127] *If the stability condition is fulfilled then*

$$P_{(0,0)} = \frac{1 - q + qL_A(\lambda) - p\lambda\beta_1(1 + \mu((1-r)\eta_1 + \gamma_1))}{(1 - q + (q-p)L_A(\lambda))(1 + \lambda\beta_1(1 + \mu((1-r)\eta_1 + \gamma_1))) + pL_A(\lambda)}, \tag{3.3.28}$$

$$P_0(z,w) = \frac{\lambda z(1 - K(z))\exp(-\lambda w)(1 - A(w))\,P_{(0,0)}}{(1 - q + qL_A(\lambda))(1-z)K(z) - z(1 - K(z))}, \tag{3.3.29}$$

$$P_1(z,x) = \frac{\lambda(1 - q + qL_A(\lambda))(1-z)\exp(-G(p\lambda(1-z))x)(1 - B(x))P_{(0,0)}}{(1 - q + qL_A(\lambda))(1-z)K(z) - z(1 - K(z))}, \tag{3.3.30}$$

$$P_{2,0}(z,x,y) = \frac{r\lambda\mu(1 - q + qL_A(\lambda))(1-z)\,P_{(0,0)}}{(1 - q + qL_A(\lambda))(1-z)K(z) - z(1 - K(z))}$$
$$\times \exp(-G(p\lambda(1-z))x - p\lambda(1-z)y)(1 - B(x))(1 - C(y)), \tag{3.3.31}$$

$$P_{2,1}\left(z,x,y\right)=\frac{\left(1-r\right)\lambda\mu(1-q+qL_{A}\left(\lambda\right))\left(1-z\right)P_{(0,0)}}{\left(1-q+qL_{A}\left(\lambda\right)\right)\left(1-z\right)K\left(z\right)-z\left(1-K\left(z\right)\right)}$$
$$\times\exp\left(-G\left(p\lambda\left(1-z\right)\right)x-p\lambda\left(1-z\right)y\right)\left(1-B\left(x\right)\right)\left(1-C\left(y\right)\right), \tag{3.3.32}$$

$$P_{3}\left(z,x,\tau\right)=\frac{\left(1-r\right)\lambda\mu(1-q+qL_{A}\left(\lambda\right))\left(1-z\right)L_{C}\left(p\lambda\left(1-z\right)\right)P_{(0,0)}}{\left(1-q+qL_{A}\left(\lambda\right)\right)\left(1-z\right)K\left(z\right)-z\left(1-K\left(z\right)\right)}$$
$$\times\exp\left(-G\left(p\lambda\left(1-z\right)\right)x-p\lambda\left(1-z\right)\tau\right)\left(1-B\left(x\right)\right)\left(1-D\left(\tau\right)\right), \tag{3.3.33}$$

where

$$G\left(x\right)=x+\mu-\mu L_{C}\left(x\right)\left\{r+\left(1-r\right)L_{D}\left(x\right)\right\},$$

and

$$K\left(x\right)\equiv L_{B}\left(G\left(p\lambda\left(1-x\right)\right)\right),$$

Proof. When both sides of (3.3.17) -(3.3.26) are multiplied by z^{i} and summed over i, we obtain the following equations:

$$\left(\frac{\partial}{\partial w}+\lambda+\alpha\left(w\right)\right)P_{0}\left(z,w\right)=0, \tag{3.3.34}$$

$$\left(\frac{\partial}{\partial x}+p\lambda+\mu+\beta\left(x\right)\right)P_{1}\left(z,x\right)=\int_{0}^{\infty}P_{20}\left(z,x,y\right)\gamma\left(y\right)dy$$
$$+\int_{0}^{\infty}P_{3}\left(z,x,\tau\right)\eta\left(\tau\right)d\tau+p\lambda zP_{1}\left(z,x\right), \tag{3.3.35}$$

$$\left(\frac{\partial}{\partial y}+p\lambda+\gamma\left(y\right)\right)P_{20}\left(z,x,y\right)=p\lambda zP_{20}\left(z,x,y\right), \tag{3.3.36}$$

$$\left(\frac{\partial}{\partial y}+p\lambda+\gamma\left(y\right)\right)P_{21}\left(z,x,y\right)=p\lambda zP_{21}\left(z,x,y\right), \tag{3.3.37}$$

$$\left(\frac{\partial}{\partial \tau}+p\lambda+\eta\left(\tau\right)\right)P_{3}\left(z,x,\tau\right)=p\lambda zP_{3}\left(z,x,\tau\right), \tag{3.3.38}$$

$$P_{0}\left(z,0\right)=\int_{0}^{\infty}P_{1}\left(z,x\right)\beta\left(x\right)dx-\lambda P_{(0,0)}, \tag{3.3.39}$$

$$P_1(z,0) = \frac{1}{z}\int_0^\infty P_0(z,w)\,\alpha(w)\,dw + \frac{(1-q+qz)\,\lambda}{z}\int_0^\infty P_0(z,w)\,dw \tag{3.3.40}$$

$$P_{20}(z,x,0) = r\mu P_1(z,x), \tag{3.3.41}$$

$$P_{21}(z,x,0) = (1-r)\,\mu P_1(z,x), \tag{3.3.42}$$

$$P_3(z,x,0) = \int_0^\infty P_{21}(z,x,y)\,\gamma(y)\,dy, \tag{3.3.43}$$

Solving (3.3.36) and (3.3.37) , we obtain:

$$P_{20}(z,x,y) = r\mu P_1(z,x)\,e^{-\lambda p(1-z)y}\,(1-C(y)) \tag{3.3.44}$$

$$P_{21}(z,x,y) = (1-r)\mu P_1(z,x)\,e^{-\lambda p(1-z)y}\,(1-C(y)) \tag{3.3.45}$$

From (3.3.38) , (3.3.43) and (3.3.45) , we obtain:

$$P_3(z,x,\tau) = (1-r)\mu P_1(z,x)\,e^{-\lambda p(1-z)\tau}\,(1-D(\tau))\,L_C(\lambda p(1-z)) \tag{3.3.46}$$

Combining (3.3.35) , (3.3.44) , (3.3.46) , we have:

$$\begin{aligned}\left(\frac{\partial}{\partial x} + p\lambda + \mu + \beta(x)\right) P_1(z,x) \quad = \quad & P_1(z,x)\left[\mu L_C(\lambda p(1-z))\left\{(1-r)L_D(\lambda p(1-z)) + r\right\}\right. \\ & \left. + \lambda pz\right]\end{aligned}$$

which yields

$$P_1(z,x) = P_1(z,0)\,e^{-G(\lambda p(1-z))x}\,(1-B(x)) \tag{3.3.47}$$

From (3.3.39) and (3.3.47) , we have:

$$P_0\left(z,0\right)=P_1\left(z,0\right)K(z)-\lambda P_{(0,0)} \tag{3.3.48}$$

However,

$$P_0\left(z,w\right)=P_0\left(z,0\right)e^{-\lambda w}\left(1-A(w)\right) \tag{3.3.49}$$

This and (3.3.40) imply that

$$P_1\left(z,0\right)=\frac{\left[1-q(1-z)(1-L_A(\lambda))\right]}{z}P_0\left(z,0\right)+\lambda P_{(0,0)} \tag{3.3.50}$$

Then (3.3.48) and (3.3.50) yield

$$P_0\left(z,0\right)=\frac{\lambda z\left(1-K(z)\right)P_{(0,0)}}{\left(1-q+qL_A(\lambda)\right)K(z)-z\left(1-K(z)\right)} \tag{3.3.51}$$

and

$$P_1\left(z,0\right)=\frac{\lambda(1-z)\left(1-q+qL_A(\lambda)\right)P_{(0,0)}}{\left(1-q+qL_A(\lambda)\right)K(z)-z\left(1-K(z)\right)} \tag{3.3.52}$$

From $P_0\left(z,0\right)$, $P_1\left(z,0\right)$, we obtain $P_0\left(z,w\right)$, $P_1\left(z,x\right)$, $P_{20}\left(z,x,y\right)$, $P_{21}\left(z,x,y\right)$, $P_3\left(z,x,\tau\right)$ which depend on $P_{(0,0)}$. However, $P_{(0,0)}$ can be found using the normalization equation

$$\begin{aligned}P_{(0,0)}&+\int_0^{\infty}P_0\left(1,w\right)dw+\int_0^{\infty}P_1\left(1,x\right)dx+\int_0^{\infty}\int_0^{\infty}P_{20}\left(1,x,y\right)dxdy\\&+\int_0^{\infty}\int_0^{\infty}P_{21}\left(1,x,y\right)dxdy+\int_0^{\infty}\int_0^{\infty}P_3\left(1,x,\tau\right)dxd\tau)=1,\end{aligned} \tag{3.3.53}$$

Notice that Hospital's rule yields

$$P_0\left(1,w\right)=\frac{\lambda\exp\left(-\lambda w\right)\left(1-A\left(w\right)\right)P_{(0,0)}K^{'}(1)}{1-q+qL_A\left(\lambda\right)-K'(1)}$$

$$P_1\left(1,x\right)=\frac{\lambda\left(1-q+qL_A\left(\lambda\right)\right)\left(1-B(x)\right)P_{(0,0)}}{1-q+qL_A\left(\lambda\right)-K'(1)},$$

$$P_{20}\left(1,x,y\right)=\frac{r\lambda\mu\left(1-q+qL_A\left(\lambda\right)\right)\left(1-C(x)\right)\left(1-B(x)\right)P_{(0,0)}}{1-q+qL_A\left(\lambda\right)-K'(1)},$$

$$P_{21}\left(1,x,y\right)=\frac{(1-r)\lambda\mu\left(1-q+qL_A\left(\lambda\right)\right)\left(1-C(x)\right)\left(1-B(x)\right)P_{(0,0)}}{1-q+qL_A\left(\lambda\right)-K'(1)}$$

$$P_{3}\left(1,x,\tau\right)=\frac{(1-r)\lambda\mu\left(1-q+qL_A\left(\lambda\right)\right)\left(1-D(x)\right)\left(1-B(x)\right)P_{(0,0)}}{1-q+qL_A\left(\lambda\right)-K'(1)}$$

where $K^{'}(1)=\lambda p\beta_1\left(1+\mu\left\{\gamma_1+(1-r)\eta_1\right\}\right)$.

Thus:

$$\int_0^\infty P_0\left(1,w\right)dw=\frac{P_{(0,0)}\left(1-L_A\left(\lambda\right)\right)K^{'}(1)}{1-q+qL_A\left(\lambda\right)-K'(1)}$$

$$\int_0^\infty P_1\left(1,x\right)dx=\frac{\lambda\beta_1\left(1-q+qL_A\left(\lambda\right)\right)P_{(0,0)}}{1-q+qL_A\left(\lambda\right)-K'(1)}$$

$$\int_0^\infty\int_0^\infty P_{20}\left(1,x,y\right)dxdy=\frac{r\lambda\mu\beta_1\gamma_1\left(1-q+qL_A\left(\lambda\right)\right)P_{(0,0)}}{1-q+qL_A\left(\lambda\right)-K'(1)}$$

$$\int_0^\infty\int_0^\infty P_{21}\left(1,x,y\right)dxdy=\frac{(1-r)\lambda\mu\beta_1\gamma_1\left(1-q+qL_A\left(\lambda\right)\right)P_{(0,0)}}{1-q+qL_A\left(\lambda\right)-K'(1)}$$

$$\int_0^\infty\int_0^\infty P_{3}\left(1,x,\tau\right)dxd\tau)=\frac{(1-r)\lambda\mu\beta_1\eta_1\left(1-q+qL_A\left(\lambda\right)\right)P_{(0,0)}}{1-q+qL_A\left(\lambda\right)-K'(1)}$$

■

Corollary 3.3.1 [127] *In steady state*

(1) The marginal generating function of the orbit size when the server is free is given by

$$P_{(0,0)}+P_0(z)=\frac{\left(1-q+qL_A(\lambda)\right)(1-z)K(z)-z(1-K(z))L_A(\lambda)}{\left(1-q+qL_A(\lambda)\right)(1-z)K(z)-z(1-K(z))}P_{(0,0)}$$

(2) The marginal generating function of the orbit size when the server is busy is given by

$$P_1(z) = \frac{\lambda\,(1-q+qL_A(\lambda))\,(1-z)(1-K(z))}{G(\lambda p(1-z))\,[(1-q+qL_A(\lambda))\,(1-z)K(z)-z(1-K(z))]}P_{(0,0)}$$

(3) The marginal generating function of the orbit size when the server is under repair with a customer waiting in the server is given by

$$P_{20}(z) = \frac{r\mu\,(1-q+qL_A(\lambda))\,(1-K(z))\,(1-L_C(\lambda p(1-z))}{pG(\lambda p(1-z))\,[(1-q+qL_A(\lambda))\,(1-z)K(z)-z(1-K(z))]}P_{(0,0)}$$

(4) The marginal generating function of the orbit size when the server is under repair with a customer entering in the server orbit is given by

$$P_{21}(z) = \frac{(1-r)\mu\,(1-q+qL_A(\lambda))\,(1-K(z))\,(1-L_C(\lambda p(1-z))}{pG(\lambda p(1-z))\,[(1-q+qL_A(\lambda))\,(1-z)K(z)-z(1-K(z))]}P_{(0,0)}$$

(5) The marginal generating function of the orbit size when the server is reserved is given by

$$P_3(z) = \frac{(1-r)\mu\,(1-q+qL_A(\lambda))\,(1-K(z))\,(1-L_D(\lambda p(1-z))\,L_C(\lambda p(1-z)}{pG(\lambda p(1-z))\,[(1-q+qL_A(\lambda))\,(1-z)K(z)-z(1-K(z))]}P_{(0,0)}$$

Proof. We have

$$P_0(z) = \int_0^{+\infty} P_0(z,w)dw,\ \ P_1(z) = \int_0^{+\infty} P_1(z,x)dx,\ \ P_{20}(z) = \int_0^{+\infty}\int_0^{+\infty} P_{20}(z,x,y)dxdy$$

$$P_{21}(z) = \int_0^{+\infty}\int_0^{+\infty} P_{21}(z,x,y)dxdy\ ,\quad P_3(z) = \int_0^{+\infty}\int_0^{+\infty} P_3(z,x,\tau)dxd\tau \quad \blacksquare$$

In the next corollary we present some performance measures for the system in steady state, by using the above theorem.

Corollary 3.3.2 [127] *If the stability condition is fulfilled then*

(1) The system is empty with probability $P_{(0,0)}$

(2) The server is idle and the system is nonempty with probability

$$P_0 = \frac{\lambda p\beta_1\,(1-L_A(\lambda))\,(1+\mu((1-r)\,\eta_1+\gamma_1))}{(1-q+(q-p)\,L_A\,(\lambda))\,(1+\lambda\beta_1(1+\mu((1-r)\,\eta_1+\gamma_1)))+pL_A\,(\lambda)},$$

(3) The server is busy with probability

$$P_1 = \frac{(1-q+qL_A(\lambda))\lambda\beta_1}{(1-q+(q-p)L_A(\lambda))(1+\lambda\beta_1(1+\mu((1-r)\eta_1+\gamma_1)))+pL_A(\lambda)},$$

(4) The server is under repair and the customer in service is in the service position after service failure with probability

$$P_{2,0} = \frac{r(1-q+qL_A(\lambda))\lambda\beta_1\mu\gamma_1}{(1-q+(q-p)L_A(\lambda))(1+\lambda\beta_1(1+\mu((1-r)\eta_1+\gamma_1)))+pL_A(\lambda)},$$

(5) The server is under repair and the customer in service enters the service orbit with probability

$$P_{2,1} = \frac{(1-r)(1-q+qL_A(\lambda))\lambda\beta_1\mu\gamma_1}{(1-q+(q-p)L_A(\lambda))(1+\lambda\beta_1(1+\mu((1-r)\eta_1+\gamma_1)))+pL_A(\lambda)},$$

(6) The server is reserved with probability

$$P_3 = \frac{(1-r)(1-q+qL_A(\lambda))\lambda\beta_1\mu\eta_1}{(1-q+(q-p)L_A(\lambda))(1+\lambda\beta_1(1+\mu((1-r)\eta_1+\gamma_1)))+pL_A(\lambda)},$$

(7) The server is under repair with probability

$$P_{repair} = \frac{(1-q+qL_A(\lambda))\lambda\beta_1\mu\gamma_1}{(1-q+(q-p)L_A(\lambda))(1+\lambda\beta_1(1+\mu((1-r)\eta_1+\gamma_1)))+pL_A(\lambda)},$$

(8) The server is blocked with probability

$$P_{block} = \frac{(1-q+qL_A(\lambda))\lambda\beta_1[1+\mu\{\gamma_1+(1-r)\eta_1\}]}{(1-q+(q-p)L_A(\lambda))(1+\lambda\beta_1(1+\mu((1-r)\eta_1+\gamma_1)))+pL_A(\lambda)}$$

(9) The orbit is empty with probability

$$P_{emptyorbit} = \frac{1-(1-p)L_B(G(\lambda p))}{L_B(G(\lambda p))}\frac{P_{(0,0)}}{p}$$

Proof. We use the fact that

$P_0 = \int_0^\infty P_0(1,w)\,dw,\ P_1 = \int_0^\infty P_1(1,x)\,dx,$

$P_{2,0} = \int_0^\infty \int_0^\infty P_{2,0}(1,x,y)\,dxdy,\ P_{2,1} = \int_0^\infty \int_0^\infty P_{2,1}(1,x,y)\,dxdy,$

$P_3 = \int_0^\infty \int_0^\infty P_3(1,x,\tau)\,dxd\tau,\ \ P_{repair} = P_{2,0} + P_{2,1},$

$P_{block} = P_1 + P_{2,0} + P_{2,1} + P_3,\ P_{emptyorbit} = P_{(0,0)} + \lim_{z\longrightarrow 0} P_1(z) + \lim_{z\longrightarrow 0} P_{20}(z) + \lim_{z\longrightarrow 0} P_{21}(z) + \lim_{z\longrightarrow 0} P_3(z)$

where the expressions under integrals were given in the proof of theorem 3.3.1. ■

Corollary 3.3.3 [127] *Let N_q and N be number of customers in the retrial queue and in the system in steady state, respectively. Then the generating functions of N_q and N are given by*

$$P_q(z) = \frac{(1-q+qL_A(\lambda))(1-(1-p+pz)K(z)) - pL_A(\lambda)z(1-K(z))}{p((1-q+qL_A(\lambda))(1-z)K(z) - z(1-K(z)))} P_{(0,0)}$$

$$P(z) = \frac{(1-q+qL_A(\lambda))(z-(z-p+pz)K(z)) - pL_A(\lambda)z(1-K(z))}{p((1-q+qL_A(\lambda))(1-z)K(z) - z(1-K(z)))} P_{(0,0)}$$

Proof. We use here the following relations:

$P_q(z) = Ez^{N_q} = P_{(0,0)} + P_0(z) + P_1(z) + P_{20}(z) + P_{21}(z) + P_3(z).$

and

$P(z) = Ez^N = P_{(0,0)} + z\left(P_0(z) + P_1(z) + P_{20}(z) + P_{21}(z) + P_3(z)\right).$ ■

Corollary 3.3.4 [127] *If $\lambda p \beta_1 \{1 + \mu[(1-r)\eta_1 + \gamma_1]\} < 1 - q + qL_A(\lambda)$, then the expected values of N_q and N are given by*

$$E(N_q) = \frac{P_{(0,0)}}{2p\left[K'(1) - (1-q+qL_A(\lambda))\right]^2}\left[\Delta_1 + \Delta_2 + \Delta_3\right]$$

$$E(N) = \frac{P_{(0,0)}}{2p\left[K'(1) - (1-q+qL_A(\lambda))\right]^2}\left[\Lambda_1 + \Lambda_2 + \Lambda_3\right]$$

where

$$\Delta_1 = K''(1)(1-q+qL_A(\lambda))(1-(q-p)(1-L_A(\lambda)))$$

$$\Delta_2 = 2pK'(1)(1-q+qL_A(\lambda))(1-L_A(\lambda))$$

$$\Delta_3 = 2(1-L_A(\lambda))\left(K'(1)\right)^2\left[q(1-q+(q-p)L_A(\lambda)) - p(1-q)\right]$$

$$\Lambda_1 = K''(1)(1-q+qL_A(\lambda))(1-(q-p)(1-L_A(\lambda)))$$

$$\Lambda_2 = 2pK'(1)(1-q+qL_A(\lambda))(1+(1-q)(1-L_A(\lambda)))$$

$$\Lambda_3 = 2\left(K'(1)\right)^2\left[q(1-q+(q-p)L_A(\lambda))(1-L_A(\lambda)) - p(1+(1-L_A(\lambda))(1-2q))\right]$$

and

$$K''(1) = (\lambda p)^2\left\{\mu\beta_1[\gamma_2 + 2(1-r)\gamma_1\eta_1 + (1-r)\eta_2] + [1+\mu\{(1-r)\eta_1 + \gamma_1\}]^2\beta_2\right\}$$

Proof. We have $E(N_q) = P_q^{'}(1)$, $E(N) = P^{'}(1)$. Using Hospital's rule we obtain

$$E(N_q) = \frac{\left[-K^{"}(1)\left(1-q+(q-p)L_A(\lambda)\right) - 2p(1-q)K^{'}(1)\left(1-L_A(\lambda)\right)\right]}{2\left[K'(1)-(1-q+qL_A(\lambda))\right]^2}$$
$$\times \frac{P_{00}}{p}\left[K^{'}(1)-(1-q+qL_A(\lambda))\right] + \frac{P_{00}}{p}\left[2qK^{'}(1)(1-L_A(\lambda))+K^{"}(1)\right]$$
$$\times \frac{\left[p(1-q+qL_A(\lambda))+K^{'}(1)\left(1-q+(q-p)L_A(\lambda)\right)\right]}{2\left[K'(1)-(1-q+qL_A(\lambda))\right]^2}$$

$$E(N) = \frac{\left[p(1-q+qL_A(\lambda))+K^{'}(1)\left(1-q+(q-p)L_A(\lambda)\right)\right]}{2\left[K'(1)-(1-q+qL_A(\lambda))\right]^2}$$
$$\times \frac{P_{00}}{p}\left[2qK^{'}(1)(1-L_A(\lambda))+K^{"}(1)\right] + \frac{P_{00}}{p}\left[K^{'}(1)-(1-q+qL_A(\lambda))\right]$$
$$\times \frac{\left[-K^{"}(1)\left(1-q+(q-p)L_A(\lambda)\right) - 2pK^{'}(1)\left(2-2q+2qL_A(\lambda)-L_A(\lambda)\right)\right]}{2\left[K'(1)-(1-q+qL_A(\lambda))\right]^2}$$

After algebraic manipulations, formulas given in corollary 3.3.4 are obtained. ∎

Remark 3.3.1 *When $p = q = 1$ and $\mu = 0$, we have* $E(N_q) = \dfrac{2\lambda\beta_1\left(1-L_A(\lambda)\right)+\lambda^2\beta_2}{2\left(L_A(\lambda)-\lambda\beta_1\right)}$ *and* $E(N) = \dfrac{2\lambda\beta_1\left(1-\lambda\beta_1\right)+\lambda^2\beta_2}{2\left(L_A(\lambda)-\lambda\beta_1\right)}$ *which agree with the results given in [65].*

3.4 Number of Retrials Made by a Marked Customer

In this section we deal with R , the number of repeated attempts produced by a customer during his waiting time. This measure is the discrete counterpart of the waiting time W. In the case of classical retrial policy, Keilson, Cozzolino and Young [74] provide the expected value $E(R)$ in terms of the mean waiting time of a customer. Falin [53] studies the distribution of R for the $M/M/1$ retrial queue and derives expression for $E(R)$, $var(R)$ and the limiting behaviour of $P(R = i)$ when the retrial rate tends to 0. Falin and Fricker [55] assume general service times and obtain the generating function $E(z^R)$ in terms of an integral expression involving the Laplace transforms of the service time and the length of the busy period in a standard $M/G/1$ queue. Artalejo and Lopez-Herrero [22] focus on the numerical computation of the stationary distribution of the number of retrials made by a customer during its waiting time. They consider two basic models: the $M/G/1$ retrial queue and the $M/M/c$ retrial queue.

3.4.1 Distribution of the Number of Retrials

First of all, we mark a primary customer arriving at the system. Our objective is to determine the probability distribution of the number of retrials made by this tagged customer.

Theorem 3.4.1 *If* $\lambda p\beta_1\{1+\mu[(1-r)\eta_1+\gamma_1]\}<1-q+qL_A(\lambda)$ *then*

$$P(R=0)=\frac{1-q+qL_A(\lambda)+\lambda\beta_1[(1-p)(1-q+qL_A(\lambda))-pL_A(\lambda)]}{(1-q+(q-p)L_A(\lambda))(1+\lambda\beta_1(1+\mu((1-r)\eta_1+\gamma_1)))+pL_A(\lambda)}$$

$$P(R=n)=pP_{block}((1-L_A(\lambda))q)^{n-1}(1-q+qL_A(\lambda)),\quad n\geq 1$$

Proof. By observing the state of the server when the tagged customer arrives, we have:

- $R=0$ if (i) the server is free or (ii) the server is blocked and the tagged customer decides to balk,
- $R>0$ if the server is blocked and the tagged customer decides to enter the retrial group.

In the first case, we derive $P(R=0)=P_{(0,0)}+P_0+(1-p)P_{block}$.

In the latter case the tagged customer will arrive during a generalized service time and he is allowed to access the server when he is at the head of the orbit and the retrial begins when the server is idle. If this customer produces n retrials, then: in the last one he finds (i) a free server or (ii) a blocked server and he decides to leave the system, and the first $(n-1)$ unsuccessful retrials are made during generalized service times where the tagged customer returns to the retrial orbit after each unsuccessful retrial.

Notice that unsuccessful retrial means that a primary customer is served before the tagged customer with probability $(1-L_A(\lambda))$. Thus

$$P(R=n)=pP_{block}(1-L_A(\lambda))^{n-1}q^{n-1}(L_A(\lambda)+(1-q)(1-L_A(\lambda)))$$

■

Following the same reasoning, we find that the number of retrials $R^{(q)}$ made by a secondary or retrial customer is geometrically distributed with parameter $(1-q+qL_A(\lambda))$.

Corollary 3.4.1 *The generating funcion of* R*, the expected values of* R *and* $R^{(q)}$ *are given by*

$$g_R(z) = E(z^R) = 1 - pP_{block}\frac{1-z}{1-q(1-L_A(\lambda))z}, \quad z \in [0,1]$$
$$E(R) = pP_{block}\frac{1}{1-q+qL_A(\lambda)}$$
$$E(R^{(q)}) = \frac{1}{1-q+qL_A(\lambda)}$$

3.4.2 Loss Probability

Let P_{loss} and $P_{loss}^{(q)}$ be the loss probability of a customer and the loss probability of a retrial customer.

Theorem 3.4.2 *The loss probabilities* P_{loss} *and* $P_{loss}^{(q)}$ *are given by*

$$P_{loss} = P_{block}\frac{1-p+(p-q)(1-L_A(\lambda))}{1-q+qL_A(\lambda)} = P_{block}\frac{1-q+(q-p)L_A(\lambda)}{1-q+qL_A(\lambda)}$$
$$P_{loss}^{(q)} = \frac{(1-L_A(\lambda))(1-q)}{1-q+qL_A(\lambda)}$$

Proof. A customer arriving at the system is lost if and only if (i) he finds the server blocked and decides to balk or (ii) finds the server blocked, enters into retrial group, makes in average $E(R^{(q)})$ trials and leaves the system without service after the last unsuccessful repeated attempt. This yields

$$P_{loss} = P_{block}(1-p) + pP_{block}E(R^{(q)})(1-L_A(\lambda))(1-q)$$

In the same manner, we get

$$P_{loss}^{(q)} = E(R^{(q)})(1-L_A(\lambda))(1-q)$$

■

3.4.3 Number of Primary Customers that get Service before the Customer at the Head of the Orbit

After service completion, there is a competition between a primary customer and the retrial customer at the head of the orbit which will determine the next customer to be served. Consider a secondary customer at the head of the orbit and let ϑ be the number of primary customers that get service before this secondary customer. We give in the following theorem the distribution of ϑ.

Theorem 3.4.3 *The distribution of the number of primary customers that get service before the customer at the head of the orbit is given by*

$$P\left(\vartheta=0\right)=L_A\left(\lambda\right)$$

$$P\left(\vartheta=n\right)=\left(1-L_A\left(\lambda\right)\right)^n q^{n-1}(1-q+qL_A\left(\lambda\right)), \quad n\geq 1$$

Proof. The event $(\vartheta=0)$ can occur if and only if the customer at the head of the orbit is the next customer who accesses the server. Thus $P\left(\vartheta=0\right)=L_A\left(\lambda\right).$
The event $(\vartheta=n)$ can occur if and only if (i) the secondary customer makes n unsuccessful trials and in the last unsuccessful attempt he leaves the system or (ii) the secondary customer makes $(n+1)$ trials and the last one is successful. This yields $P\left(\vartheta=n\right)=\left(1-L_A\left(\lambda\right)\right)^{n-1}q^{n-1}\left(1-L_A\left(\lambda\right)\right)\left(1-q\right)+\left(1-L_A\left(\lambda\right)\right)^n q^n L_A\left(\lambda\right).$ ■

Corollary 3.4.2 *The expected value and generating function of ϑ are given by*

$$E\left(\vartheta\right)=\frac{1-L_A\left(\lambda\right)}{1-q+qL_A\left(\lambda\right)}$$

$$g_\vartheta(z)=E(z^\vartheta)=\frac{L_A\left(\lambda\right)+\left(1-q\right)\left(1-L_A\left(\lambda\right)\right)z}{1-q\left(1-L_A\left(\lambda\right)\right)z}, \quad z\in[0,1]$$

3.5 Waiting Time

The waiting time is the time that the customer spends waiting at the queue (standard model) or at the orbit (retrial model) exluding the service time. The waiting time analysis constitutes a fundamental part of the queueing literature and its study in retrial models is much more difficult than the analysis of the system state. It is much more difficult for retrial queues with the random order policy; a customer joining the orbit at time t_2 may access to the server before another customer who joined the orbit earlier, say at t_1 with $t_1 < t_2$. Several papers have addressed the problem of getting the waiting time distribution [18, 55-56, 108].

In this section, we are interested in the study of the joint distribution of the waiting time that a customer spends in the retrial queue and the number of customers served during this period.

Let W and N be the waiting time that a customer spends in the retrial queue and the number of customers served during this waiting time, respectively.

Theorem 3.5.1 [127] *In steady state the generating function Laplace transform for W and N is given by*

$$\Phi(s,z) = E\left(e^{-sW}z^N\right) = 1 - p + p\left(P_{0,0} + P_0\right)$$
$$+\frac{p\lambda\left(1-q+qL_A(\lambda)\right)z\left(1-Q^*(s,z)\right)Q(s,z)}{\left(1-q+qL_A(\lambda)\right)\left(1-Q^*(s,z)\right)K\left(Q^*(s,z)\right)-Q^*(s,z)\left(1-K\left(Q^*(s,z)\right)\right)}P_{0,0}$$
$$\times\frac{L_B(G(s))-K\left(Q^*(s,z)\right)}{p\lambda\left(1-Q^*(s,z)\right)-s} \tag{3.5.54}$$

where

$$Q(s,z) = \frac{(1-q)\lambda + (s+q\lambda)L_A(s+\lambda)}{s+\lambda-q\lambda\left(1-L_A(s+\lambda)\right)zL_B(G(s))} \tag{3.5.55}$$

and

$$Q^*(s,z) = zL_B(G(s))Q(s,z) \tag{3.5.56}$$

Proof. When a primary customer arrives to the system he will find one of the following situations:

(i) the system is empty,

(ii) the system is nonempty and the server is idle,

(iii) the server is busy,

(iv) the server is under repair and the customer in service remains in the service position,

(v) the server is under repair and the customer in service is in the service orbit,

or (vi) the server is reserved.

We can write

$$\Phi(s,z) = P_{(0,0)} E\left(e^{-sW} z^N / \ J=0, Q=0\right)$$
$$+\sum_{i=1}^{\infty} \int_0^{\infty} P_{(0,i)}(w)\, E\left(e^{-sW} z^N / \ J=0, Q=i, \xi_0=w\right) dw$$
$$+\sum_{i=0}^{\infty}\Big(\int_0^{\infty} P_{(1,i)}(x)\, E\left(e^{-sW} z^N / J=1, Q=i, \xi_1=x\right) dx$$
$$+\int_0^{\infty}\int_0^{\infty} P_{(2,0,i)}(x,y)\, E\left(e^{-sW} z^N / J=2, J^*=0, Q=i, \xi_1=x, \xi_2=y\right) dxdy$$
$$+\int_0^{\infty}\int_0^{\infty} P_{(2,1,i)}(x,y)\, E\left(e^{-sW} z^N / J=2, J^*=1, Q=i, \xi_1=x, \xi_2=y\right) dxdy$$
$$+\int_0^{\infty}\int_0^{\infty} P_{(3,i)}(x,\tau)\, E\left(e^{-sW} z^N / J=3, Q=i, \xi_1=x, \xi_3=\tau\right) dxd\tau\Big) \tag{3.5.57}$$

• If the server is idle then $W=0$ and $N=0$. Thus, the first two terms of (3.5.57) are $P_{(0,0)}$ and P_0.

• If the server is blocked, the arrival customer either enters the retrial queue or leaves the system. In this last case $W=0$ and $N=0$. If the blocked customer enters the retrial queue and finds i customers into it, then we write $W = W^{(\star)} + W^{(i+1)}$ where $W^{(\star)}$ represents the remaining generalized service time of the customer in service at the instant this primary customer arrives and $W^{(i+1)}$ represents the waiting time of the newly arrived customer in the retrial queue from the instant that the current customer (in service) completes service.

We have :

$$E\left(e^{-sW} z^N / J=1, Q=i, \xi_1=x\right) = 1-p+pE\left(e^{-sW^{(i+1)}} z^{N^{(i+1)}}\right)$$
$$\times E\left(e^{-sW^{(*)}} z / J=1, Q=i, \xi_1=x, E=1\right), \tag{3.5.58}$$

$$E\left(e^{-sW} z^N / J=2, J^*=0, Q=i, \xi_1=x, \xi_2=y\right) = 1-p+pE\left(e^{-sW^{(i+1)}} z^{N^{(i+1)}}\right)$$
$$\times E\left(e^{-sW^{(*)}} z / J=2, J^*=0, Q=i, \xi_1=x, \xi_2=y, E=1\right), \tag{3.5.59}$$

$$E\left(e^{-sW} z^N / J=2, J^*=1, Q=i, \xi_1=x, \xi_2=y\right) = 1-p+pE\left(e^{-sW^{(i+1)}} z^{N^{(i+1)}}\right)$$

$$\times E\left(e^{-sW^{(*)}}z/J=2,J^{*}=1,Q=i,\xi_1=x,\xi_2=y,E=1\right), \tag{3.5.60}$$

$$E\left(e^{-sW}z^{N}/J=3,Q=i,\xi_1=x,\xi_3=\tau\right)=1-p+pE\left(e^{-sW^{(i+1)}}z^{N^{(i+1)}}\right)$$

$$\times E\left(e^{-sW^{(*)}}z/J=3,Q=i,\xi_3=\tau,E=1\right), \tag{3.5.61}$$

In order to get

$$E\left(e^{-sW^{(*)}}/J=1,Q=i,\xi_1=x,E=1\right),$$

$$E\left(e^{-sW^{(*)}}/J=2,J^{(*)}=0,Q=i,\xi_1=x,\xi_2=y,E=1\right),$$

$$E\left(e^{-sW^{(*)}}/J=2,J^{*}=1,Q=i,\xi_1=x,\xi_2=y,E=1\right),$$

and

$$E\left(e^{-sW^{(*)}}/J=3,\ Q=i,\ \xi_1=x,\ \xi_3=\tau,E=1\right)$$

we use the following formulae

$$P\left[y<\xi_1^{(r)}<y+dy \mid S>x\right]=\frac{b(x+y)}{1-B(x)}dy$$

$$P\left[y<\xi_2^{(r)}<y+dy \mid X>x\right]=\frac{c(x+y)}{1-C(x)}dy$$

$$P\left[y<\xi_3^{(r)}<y+dy \mid Y>x\right]=\frac{d(x+y)}{1-D(x)}dy$$

where $S,\ X$ and Y represent the service time, repair time and reserved time respectively, $\xi_1^{(r)},\ \xi_2^{(r)}$ and $\xi_3^{(r)}$ represent the remaining service time of the customer in service, the remaining repair time of the customer in service and the remaining reserved time of the customer in service at the instant the primary customer arrives.

- If $J=1,Q=i,\xi_1=x,E=1$ then $W^{(\star)}=\xi_1^{(r)}+\sum_{i=1}^{N\left(\xi_1^{(r)}\right)}X_i+\sum_{i=1}^{N_1\left(\xi_1^{(r)}\right)}Y_i$

$$E\left(e^{-sW^{(*)}}/J=1,Q=i,\xi_1=x,E=1\right)$$

$$=\int_0^{+\infty} E\left(e^{-sW^{(*)}}/J=1,Q=i,\xi_1=x,\xi_1^{(r)}=t,E=1\right)\frac{b(x+t)}{1-B(x)}dt$$

$$=\frac{1}{1-B(x)}\int_x^{\infty} b(t)\,e^{-G(s)(t-x)}dt, \tag{3.5.62}$$

- If $J=2,J^*=0,Q=i,\xi_1=x,\xi_2=y,E=1$ then

$$W^{(\star)}=\xi_1^{(r)}+\xi_2^{(r)}+\sum_{i=1}^{N\left(\xi_1^{(r)}\right)}X_i+\sum_{i=1}^{N_1\left(\xi_1^{(r)}\right)}Y_i$$

$$E\left(e^{-sW^{(*)}}/J=2,J^{(*)}=0,Q=i,\xi_1=x,\xi_2=y,E=1\right)$$

$$=\int_0^{+\infty}\int_0^{+\infty} E\left(e^{-sW^{(*)}}/J=2,J^{(*)}=0,Q=i,\xi_1=x,\xi_2=y,\xi_1^{(r)}=t,\xi_2^{(r)}=v,E=1\right)$$
$$\times\frac{b(x+t)}{1-B(x)}\frac{c(v+y)}{1-C(y)}dtdv$$

$$=\frac{1}{(1-B(x))(1-C(y))}\int_x^{\infty}\int_y^{\infty} b(t)\,c(v)\,e^{-G(s)(t-x)-s(v-y)}dtdv, \tag{3.5.63}$$

- If $J=2,J^*=1,Q=i,\xi_1=x,\xi_2=y,E=1$ then

$$W^{(\star)}=\xi_1^{(r)}+\xi_2^{(r)}+Y+\sum_{i=1}^{N\left(\xi_1^{(r)}\right)}X_i+\sum_{i=1}^{N_1\left(\xi_1^{(r)}\right)}Y_i$$

$$E\left(e^{-sW^{(*)}}/J=2,J^*=1,Q=i,\xi_1=x,\xi_2=y,E=1\right)$$

$$=\int_0^{+\infty}\int_0^{+\infty} E\left(e^{-sW^{(*)}}/J=2,J^{(*)}=1,Q=i,\xi_1=x,\xi_2=y,\xi_1^{(r)}=t,\xi_2^{(r)}=v,E=1\right)$$
$$\times\frac{b(x+t)}{1-B(x)}\frac{c(v+y)}{1-C(y)}dtdv$$

$$= \frac{L_D(s)}{(1-B(x))(1-C(y))} \int_x^{\infty} \int_y^{\infty} b(t)\, c(v)\, e^{-G(s)(t-x)-s(v-y)} dt dv, \qquad (3.5.64)$$

• If $J=3,\ Q=i,\ \xi_1=x,\ \xi_3=\tau, E=1$ then $W^{(\star)} = \xi_1^{(r)} + \xi_3^{(r)} + \sum_{i=1}^{N\left(\xi_1^{(r)}\right)} X_i + \sum_{i=1}^{N_1\left(\xi_1^{(r)}\right)} Y_i$
$E\left(e^{-sW^{(*)}}/J=3,\ Q=i,\ \xi_1=x,\ \xi_3=\tau, E=1\right)$

$$= \int_0^{+\infty}\int_0^{+\infty} E\left(e^{-sW^{(*)}}/J=3, Q=i, \xi_1=x, \xi_3=\tau, \xi_1^{(r)}=t, \xi_3^{(r)}=v, E=1\right) \times \frac{b(x+t)}{1-B(x)} \frac{d(v+y)}{1-D(\tau)} dt dv$$

$$= \frac{1}{(1-B(x))(1-D(\tau))} \int_x^{+\infty}\int_\tau^{+\infty} b(t)\, d(v) e^{-G(s)(t-x)-s(v-\tau)} dt dv, \qquad (3.5.65)$$

Notice that $E=1$ means that this primary customer enters the retrial queue finding the server blocked.

It remains for us to find $E\left(e^{-sW^{(i+1)}} z^{N^{(i+1)}}\right)$. We can write $W^{(i+1)} = W_*^{(i)} + W^{(1)}$ where $W_*^{(i)}$ is the waiting time (after service completion of the current customer) of newly customer until the i customers leave the system and the server becomes idle. Let $N_*^{(i)}$ be the number of customers served during $\left(0,\ W_*^{(i)}\right)$. Then

$$E\left(e^{-sW^{(i+1)}} z^{N^{(i+1)}}\right) = E\left(e^{-sW^{(1)}} z^{N^{(1)}}\right) E\left(e^{-sW_*^{(i)}} z^{N_*^{(i)}}\right) \qquad (3.5.66)$$

Define $Q^*(s,z) = E\left(e^{-sW_*^{(1)}} z^{N_*^{(1)}}\right)$. Then (we consider the competition between exponential law of rate λ, and the general retrial time distribution which will determine the next customer to be served)

$$Q^*(s,z) = \int_0^{+\infty}\int_0^{+\infty} E\left(e^{-sW_*^{(1)}} z^{N_*^{(1)}}/Z=y, \xi^{(r)}=x\right) \lambda e^{-\lambda x} dx dA(y)$$

$$= \int_0^{+\infty}\int_0^{y} E\left(e^{-sW_*^{(1)}} z^{N_*^{(1)}}/Z=y, \xi^{(r)}=x\right) \lambda e^{-\lambda x} dx dA(y)$$
$$+ \int_0^{+\infty}\int_y^{+\infty} E\left(e^{-sW_*^{(1)}} z^{N_*^{(1)}}/Z=y, \xi^{(r)}=x\right) \lambda e^{-\lambda x} dx dA(y)$$

$$= \int_{0}^{+\infty} \int_{0}^{y} \lambda e^{-\lambda x - sx} \left[1 - q + qQ^*(s,z)\right] zL_B(G(s)) dx dA(y) + \int_{0}^{+\infty} \int_{y}^{+\infty} \lambda e^{-\lambda x - sy} zL_B(G(s)) dx dA(y)$$

$$= \left[1 - q + qQ^*(s,z) zL_B(G(s))\right] \frac{\lambda}{\lambda + s} \left(1 - L_A(s+\lambda)\right) + zL_B(G(s)) L_A(s+\lambda) \quad (3.5.67)$$

which yields equation (3.5.56) . Consequently,

$$E\left(e^{-sW_*^{(i)}} z^{N_*^{(i)}}\right) = \left(Q^*(s,z)\right)^i . \quad (3.5.68)$$

In the same way, define $Q(s,z) = E\left(e^{-sW^{(1)}} z^{N^{(1)}}\right)$.

$$Q(s,z) = \int_{0}^{+\infty} \int_{0}^{y} \lambda e^{-\lambda x - sx} \left[1 - q + qQ(s,z) zL_B(G(s))\right] dx dA(y) + \int_{0}^{+\infty} \int_{y}^{+\infty} \lambda e^{-\lambda x - sy} dx dA(y)$$

$$= \left[1 - q + qQ(s,z) zL_B(G(s))\right] \frac{\lambda}{\lambda + s} \left(1 - L_A(s+\lambda)\right) + L_A(s+\lambda)$$

which yields equation (3.5.55) (where Z is the retrial time and $\xi^{(r)}$ is the remaining interarrival time).

Combining (3.5.55) , (3.5.56) , (3.5.58) -(3.5.61) , (3.5.62) -(3.5.65) and (3.5.68) , we obtain

$$\Phi(s,z) = P_{(0,0)} + \int_0^\infty P_0(1,w) dw + (1-p)\times$$

$$\left[\int_0^\infty P_1(1,x) dx + \int_0^\infty \int_0^\infty P_{2,0}(1,x,y) dx dy + \int_0^\infty \int_0^\infty P_{2,1}(1,x,y) dx dy + \int_0^\infty \int_0^\infty P_3(1,x,\tau) dx d\tau\right]$$

$$+ pzQ(s,z) \times \Big(\int_{0}^{+\infty} \frac{P_1(Q^*(s,z),x)}{1 - B(x)} \int_x^\infty b(t) e^{-G(s)(t-x)} dt dx$$

$$+ \int_{0}^{+\infty} \int_{0}^{+\infty} \frac{P_{20}(Q^*(s,z),x,y)}{(1-B(x))(1-C(y))} \int_x^\infty \int_y^\infty b(t) c(v) e^{-G(s)(t-x) - s(v-y)} dt dv dx dy$$

$$+ L_D(s) \int_{0}^{+\infty} \int_{0}^{+\infty} \frac{P_{21}(Q^*(s,z),x,y)}{(1-B(x))(1-C(y))} \int_x^\infty \int_y^\infty b(t) c(v) e^{-G(s)(t-x) - s(v-y)} dt dv dx dy$$

$$+ \int_{0}^{+\infty} \int_{0}^{+\infty} \frac{P_3(Q^*(s,z),x,\tau)}{(1-B(x))(1-D(\tau))} \frac{1}{(1-B(x))(1-D(\tau))} \int_{x}^{+\infty} \int_{\tau}^{+\infty} b(t) d(v) e^{-G(s)(t-x) - s(v-\tau)} dt dv \Big)$$

$$= 1 - p + p(P_{(0,0)} + P_0) + pzQ(s,z) \times (\int_0^{+\infty} \frac{P_1(Q^*(s,z),x)}{1-B(x)} \int_x^{\infty} b(t)\, e^{-G(s)(t-x)} dtdx +$$

$$\int_0^{+\infty}\int_0^{+\infty} \frac{P_{20}(Q^*(s,z),x,y)}{(1-B(x))(1-C(y))} \int_x^{\infty}\int_y^{\infty} b(t)\, c(v)\, e^{-G(s)(t-x)-s(v-y)} dtdvdxdy$$

$$+L_D(s) \int_0^{+\infty}\int_0^{+\infty} \frac{P_{21}(Q^*(s,z),x,y)}{(1-B(x))(1-C(y))} \int_x^{\infty}\int_y^{\infty} b(t)\, c(v)\, e^{-G(s)(t-x)-s(v-y)} dtdvdxdy$$

$$+ \int_0^{+\infty}\int_0^{+\infty} \frac{P_3(Q^*(s,z),x,\tau)}{(1-B(x))(1-D(\tau))} \frac{1}{(1-B(x))(1-D(\tau))} \int_x^{+\infty}\int_\tau^{+\infty} b(t)\, d(v) e^{-G(s)(t-x)-s(v-\tau)} dtdv)$$

Using theorem 3.3.1 and after algebraic manipulation we get (3.5.54) ■

The mean waiting time of W in the steady state can be obtained from $L_W(s) = E(e^{-sW}) = \lim_{z\to 1} \Phi(s,z)$.

3.6 Busy Period

A system busy period is defined as the period that starts at an epoch when an arriving customer finds an empty system and ends at the next departure epoch at which the system is empty. The main characteristics of a system busy period are: L the length of a system busy period, K the number of customers served during the busy period. The system busy period of a retrial queue consists of alternating service periods and periods in which the server is free and there are customers in retrial orbit; in our model the server is free or blocked. The system busy period can also be expressed in terms of a sequence of orbit busy periods and orbit idle periods [65]. In the following theorem we give the generating function Laplace transform of L and K,the mean length of a system busy period $E(L)$ and the mean number of customers served during the busy period $E(K)$.

In the following theorem, we give the generating function Laplace transform of T and K, expressed as a functional equation which can be differentiated to find the first moments of T and K.

Theorem 3.6.1 [127] *If* $\lambda p \beta_1 \{1 + \mu [(1-r)\eta_1 + \gamma_1]\} < 1 - q + qL_A(\lambda)$ *then the gen-*

erating function Laplace transform of T and K is given by

$$\psi(s,z) \equiv E\left(e^{-sL}z^{K}\right) = zL_B\left(G\left(s+\lambda-\lambda\left(1-p+p\psi^{(q)}(s,z)\right)\right)\right) \tag{3.6.69}$$

where

$$\psi^{(q)}(s,z) = \frac{((1-q)\lambda+(s+q\lambda)L_A(s+\lambda))\psi(s,z)}{s+\lambda-q\lambda(1-L_A(s+\lambda))\psi(s,z)} \tag{3.6.70}$$

$$E(L) = \frac{\beta_1[1-(1-L_A(\lambda))(q-p)][1+\mu((1-r)\eta_1+\gamma_1)]}{1-q(1-L_A(\lambda))-\lambda p\beta_1[1+\mu((1-r)\eta_1+\gamma_1)]} \tag{3.6.71}$$

$$E(K) = \frac{1-q+qL_A(\lambda)}{1-q(1-L_A(\lambda))-\lambda p\beta_1[1+\mu(\gamma_1+(1-r)\eta_1)]} \tag{3.6.72}$$

Proof. The derivation of $\psi(s,z)$ and $\psi^{(q)}(s,z)$ is similar to that of [138]. If we condition on the first generalized service time S^* and the number ν of primary customers entering the retrial orbit during this time, we obtain

$$\psi(s,z) = \int_0^{+\infty}\sum_{n\geq 0} E\left(e^{-sL}z^{K}\ /S^*=x,\nu=n\right)\frac{(\lambda px)^n}{n!}e^{-\lambda px}dB^*(x) \tag{3.6.73}$$

$$E\left(e^{-sL}z^{K}\ /S^*=x,\nu=n\right) = e^{-sx}zE\left(e^{-s(L_1^{(q)}+L_2^{(q)}+...+T_n^{L(q)})}z^{(K_1^{(q)}+K_2^{(q)}+...+K_n^{(q)})}\right) \tag{3.6.74}$$

where $L_i^{(q)}$ and $K_i^{(q)}$ denote respectively, the length of the quasi-busy period generated by the $i-th$ retrial customer and the number of customers served during the corresponding time, $i=1,2,...$

The random vectors $\left(L_1^{(q)},K_1^{(q)}\right),\left(L_2^{(q)},K_2^{(q)}\right),...,\left(L_n^{(q)},K_n^{(q)}\right)$ are independent and identically distributed with the generating function Laplace transform

$$\psi^{(q)}(s,z) = E\left(e^{-sL_i^{(q)}}z^{K_i^{(q)}}\right)$$

In the same way as in the waiting time, we consider the competition between exponential law of rate $\lambda>0$, and the general retrial time distribution which will determine the next customer to be served. Then

$$\begin{aligned}\psi^{(q)}(s,z) &= \int_0^{+\infty}\int_y^{+\infty} \lambda e^{-\lambda x - sy}\psi(s,z)\,dA(y)dx \\ &+ \int_0^{+\infty}\int_0^{y} \lambda e^{-\lambda x - sx}\left[(1-q)\psi(s,z) + q\psi(s,z)\,\psi^{(q)}(s,z)\right] dA(y)dx \\ &= \psi(s,z)\,L_A(s+\lambda) + \left[(1-q)\psi(s,z) + q\psi(s,z)\,\psi^{(q)}(s,z)\right]\frac{\lambda}{\lambda+s}\left(1 - L_A(s+\lambda)\right)\end{aligned}$$

which yields equation (3.6.70) . Consequently

$$E\left(e^{-s(L_1^{(q)}+L_2^{(q)}+\ldots+L_n^{(q)})}z^{(K_1^{(q)}+K_2^{(q)}+\ldots+K_n^{(q)})}\right) = \left(\psi^{(q)}(s,z)\right)^n \tag{3.6.75}$$

Combining (3.6.70) , (3.6.73) -(3.6.75) , we obtain (3.6.69) .

In order to compute $E(L)$ we use the following properties of Laplace transforms

$\psi_L(s) = E(e^{-sL}) = \lim_{z\to 1}\psi(s,z)$, $\psi_L^{'(q)}(0) = \dfrac{L_A(\lambda) + \lambda\psi_L^{'}(0) - 1}{\lambda - q\lambda\left(1 - L_A(\lambda)\right)}$,

$\psi_L^{'}(0) = \left[1 - \lambda p\psi_L^{'(q)}(0)\right] G^{'}(0)L_B^{'}(0)$, $E(L) = -\,\psi_L^{'}(0)$

where $\psi_L^{(q)}(s) = \lim_{z\to 1}\psi^{(q)}(s,z) = \dfrac{\left((1-q)\,\lambda + (s+q\lambda)\,L_A(s+\lambda)\right)\psi_L(s)}{s + \lambda - q\lambda\left(1 - L_A(s+\lambda)\right)\psi_L(s)}$

Similarly for $E(K)$

$\psi_K(z) = E(z^K) = \lim_{s\to 0}\psi(s,z)$, $\psi_K^{'}(1) = 1 - \lambda p\psi_K^{'(q)}(1)G^{'}(0)L_B^{'}(0)$

$\psi_K^{'(q)}(1) = \dfrac{\psi_K^{'}(1)}{1 - q(1 - L_A(\lambda))}$, $E(K) = \psi_K^{'}(1)$.

where $\psi_K^{(q)}(z) = \lim_{s\to 0}\psi^{(q)}(s,z) = \dfrac{\left(1 - q + qL_A(\lambda)\right)\psi_K(z)}{1 - q\left(1 - L_A(\lambda)\right)\psi_K(z)}$ ■

The mean length of a system busy period and the mean number of customers served during a busy period can be obtained with the help of Wald's equality as follows. We can write

$$L = S^{*} + \sum_{j=1}^{\sum_{k=1}^{\nu}\vartheta_k + \delta_\nu} L_j + \sum_{j=1}^{\sum_{k=1}^{\nu}\vartheta_k + \delta_\nu} I_j \tag{3.6.76}$$

$$K = 1 + \sum_{j=1}^{\sum_{k=1}^{\nu} \vartheta_k + \delta_\nu} K_j \tag{3.6.77}$$

where

•$S^*$is the first generalized service time

• ν is the number of customers entering the retrial group during the first generalized service time with distribution $k_n,\ n \geq 0$,

• ϑ_k is the number of customers that get service before the customer at the head of the orbit, for $k = 1, ..., \nu$. The random variables $(\vartheta_k)$are independent and identically distributed with distribution $P\left(\vartheta = n\right),\ n \geq 0$.

• δ_ν is the number of customers from orbit among the ν customers which are served (aren't lost) where $(\delta_\nu/\nu = n)$ has a binomial distribution with parameter $\left(1 - P_{loss}^{(q)}\right)$

• The length I_j of the idle period of the server is determined by the competition between an exponential law of rate λ and the general retrial time distribution which determines the next customer who accesses the server. The random variables (I_j) are independent and identically distributed with distribution function

$$P\left(I \leq t\right) = 1 - e^{-\lambda t}(1 - A(t)), \quad t > 0$$

• L_j is the length of the busy period generated by the customer who accesses the server. The random variables (L_j) are independent and identically distributed with the same distribution as L.

• K_j is the number of customers served during the busy period of length L_j. The random variables (K_j) are independent and identically distributed with the same distribution as K.

We have

$$E(\nu) = \lambda p \beta_1 \left\{1 + \mu\left[\left(1 - r\right)\eta_1 + \gamma_1\right]\right\}$$
$$E(\ \delta_\nu) = \lambda p \beta_1 \left\{1 + \mu\left[\left(1 - r\right)\eta_1 + \gamma_1\right]\right\} \frac{L_A(\lambda)}{1 - q\left(1 - L_A(\lambda)\right)}$$
$$E(I_j) = \frac{1 - L_A(\lambda)}{\lambda}$$

Wald's identity yields:

$$E(L) = \beta_1 \{1 + \mu [(1-r)\eta_1 + \gamma_1]\} + \lambda p \beta_1 \{1 + \mu [(1-r)\eta_1 + \gamma_1]\} \frac{1}{1 - q(1 - L_A(\lambda))} E(L)$$
$$+ \lambda p \beta_1 \{1 + \mu [(1-r)\eta_1 + \gamma_1]\} \frac{1}{1 - q(1 - L_A(\lambda))} \frac{1 - L_A(\lambda)}{\lambda}$$
$$E(K) = 1 + \lambda p \beta_1 \{1 + \mu [(1-r)\eta_1 + \gamma_1]\} \frac{1}{1 - q(1 - L_A(\lambda))} E(K)$$

Remark 3.6.1 *If T_{00} is the amount of time in a regenerative cycle during which the system is in the state $(0,0)$, then $P_{(0,0)} = \dfrac{E(T_{00})}{E(T_{00}) + E(L)}$ which yields $E(L) = E(T_{00}) \left(\dfrac{1}{P_{(0,0)}} - 1 \right)$. However, T_{00} has an exponential law of rate λ. Using (3.3.28), we get (3.6.71)*

We give, as follows, the first three probabilities of the number of customers K served during the busy period by direct probability statements. The expressions are given in terms of (k_n) and $L_A()$.

Proposition 3.6.1 [127] *Let K be the number of customers served during a busy period. Then*

$$P(K = 1) = k_0, \tag{3.6.78}$$

$$P(K = 2) = k_0 k_1 (1 - q + q L_A(\lambda)), \tag{3.6.79}$$

$$\begin{aligned} P(K &= 3) = (1 - q + q L_A(\lambda))^2 k_2 k_0^2 + k_0 k_1^2 (1 - q + q L_A(\lambda))^2 \\ &+ k_1 k_0^2 (1 - L_A(\lambda)) q (1 - q + q L_A(\lambda)) \end{aligned} \tag{3.6.80}$$

Proof.

• The event $(K = 1)$ can occur if and only if no primary customers enter into the orbit during the first generalized service.

$$P(K = 1) = k_0$$

• The event $(K = 2)$ can occur if and only if (i) one primary customer enters into the orbit during the first generalized service and the next customer in service comes from the retrial queue and no customer enters into the orbit during the second generalized service or (ii) one primary customer enters into the orbit during the first generalized service, the next customer in service is a primary customer, the customer in orbit leaves the system and no customer enters into the orbit during the second generalized service.

$$P\left(K=2\right)=k_{1}L_{A}\left(\lambda\right)k_{0}+k_{1}\left(1-L_{A}(\lambda)\right)\left(1-q\right)k_{0}$$

• There are ten different possibilities for serving three customers during the busy period as indicated below.

1. One primary customer enters into the orbit during the first generalized service, the customer in orbit goes into the server, one primary customer enters into the orbit during the second generalized service, the next customer in service comes from the orbit and no customer enters into the orbit during the third generalized service.

2. One primary customer enters into the orbit during the first generalized service, the next customer in service comes from the orbit, one customer enters into the orbit during the second generalized service, the next customer served is a primary customer, the customer in orbit leaves the system and no customer enters into the orbit during the third generalized service.

3. One primary customer enters into the orbit during the first generalized service, the next customer in service is a primary customer, the customer in orbit leaves the system, one primary customer enters into the orbit during the second generalized service, the next customer in service comes from the retrial and no customer enters into the orbit during the third generalized service.

4. One primary customer enters into the orbit during the first generalized service, the next customer in service is a primary customer , the customer in orbit leaves the system, one primary customer enters into the orbit during the second generalized service, the third customer served is a primary customer, the customer in orbit leaves the system and no customer enters into the orbit during the third generalized service.

5. One primary customer enters into the orbit during the first generalized service, the next customer in service is a primary customer , the customer in orbit remains in the

orbit, no customer enters into the orbit during the second generalized service, the third customer served is a primary customer, the customer in orbit leaves the system and no customer enters into the orbit during the third generalized service.

6. One primary customer enters into the orbit during the first generalized service, the next customer in service is a primary customer , the customer in orbit remains in the orbit, no customer enters into the orbit during the second generalized service, the next customer in service comes from the retrial queue and no customer enters into the orbit during the third generalized service.

7. Two primary customers enter into the orbit during the first generalized service time, the next customer in service comes from the retrial orbit, no customer enters into the orbit during the second generalized service, the next customer in service comes from the orbit and no customer enters into the orbit during the third generalized service.

8. Two primary customers enter into the orbit during the first generalized service time, the next customer in service comes from the retrial orbit, no customer enters into the orbit during the second generalized service, the next customer in service is a primary customer, the customer in orbit leaves the system and no customer enters into the orbit during the third generalized service.

9. Two primary customers enter into the orbit during the first generalized service time, the next customer in service is a primary customer, the customer at the head of the orbit leaves the system, no customer enters into the orbit during the second generalized service, the next customer in service comes from the retrial orbit and no customer enters into the orbit during the third generalized service.

10. Two primary customers enter into the orbit during the first generalized service time, the next customer in service is a primary customer, the customer at the head of the orbit leaves the system, no customer enters into the orbit during the second generalized service, the next customer in service is a primary customer, the customer in orbit leaves the system and no customer enters into the orbit during the third generalized service.

Further,

$$P(K=3) = k_1 L_A(\lambda) k_1 L_A(\lambda) k_0 + k_1 L_A(\lambda) k_1 (1-L_A(\lambda))(1-q) k_0$$

$$+k_1(1-L_A(\lambda))(1-q) k_1 L_A(\lambda) k_0 + k_1(1-L_A(\lambda))(1-q)^2 k_1(1-L_A(\lambda)) k_0$$

$$+k_1(1-L_A(\lambda))(1-q) q k_0 (1-L_A(\lambda)) k_0 + k_1(1-L_A(\lambda)) q k_0 L_A(\lambda) k_0 + k_2 L_A(\lambda) k_0 L_A(\lambda) k_0$$

$$+k_2 L_A(\lambda) k_0 (1-L_A(\lambda))(1-q) k_0 + k_2(1-L_A(\lambda))(1-q) k_0 L_A(\lambda) k_0$$

$$+k_2(1-L_A(\lambda))(1-q) k_0 (1-L_A(\lambda))(1-q) k_0$$

After simplification , we obtain equations (3.6.79) -(3.6.80) . ■

Remark 3.6.2 *Lopez-Herrero [99] presents a recursive method of computation for the probability of the number of customers served during a busy period for an $M/G/1$ retrial queues under classical retrial policy. Atencia, Bouza and Moreno [23] follow the same reasonings. In the case of persistent customers ($q = 1$), we can use the same algorithm with slight modification.*

3.7 Reliability Analysis

The server breakdown affects the quality of service in almost all the fields including computer and communication systems, manufacturing and production process. The measurement of reliability indices of unreliable systems can provide a basic idea about server's operating efficiency. In this section we focus on the time to the first failure of the server and its reliability

We assume that the system is empty at time $t = 0$. Denote ς the time to the first failure of the server. The reliability of the server is defined as

$$R(t) = P(\varsigma > t).$$

In order to find the reliability of the server, we consider a new queueing system where the failure states of the server are assumed to be absorbent states. In this new system, we use the same notations as in the section 3, namely $P_{(0,0)}(t),\ P_{(0,i)}(t,w),\ P_{(1,i)}(t,x)$. Then we can get the following set of equations:

$$(\frac{\partial}{\partial t}+\lambda)P_{(0,0)}(t)=\int_0^\infty P_{(1,0)}\left(t,x\right)\beta\left(x\right)dx, \tag{3.7.81}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial w}+\lambda+\alpha\left(w\right)\right)P_{(0,i)}\left(t,w\right)=0,\ i\geq 1, \tag{3.7.82}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+p\lambda+\mu+\beta\left(x\right)\right)P_{(1,i)}\left(t,x\right)=p\lambda P_{1,i-1}\left(t,x\right),\ i\geq 0, \tag{3.7.83}$$

$$P_{(0,i)}\left(t,0\right)=\int_0^\infty P_{(1,i)}\left(t,x\right)\beta\left(x\right)dx,\ i\geq 1, \tag{3.7.84}$$

$$\begin{aligned}P_{(1,i)}\left(t,0\right)=&\int_0^\infty P_{(0,i+1)}\left(t,w\right)\alpha\left(w\right)dw+\left(1-q\right)\lambda\int_0^\infty P_{(0,i+1)}\left(t,w\right)dw\\&+\left(1-\delta_{i,0}\right)q\lambda\int_0^\infty P_{(0,i)}\left(t,w\right)dw+\delta_{i,0}\lambda P_{(0,0)}(t),\ i\geq 0,\end{aligned} \tag{3.7.85}$$

with the initial condition

$$P_{(0,0)}(0)=1 \tag{3.7.86}$$

Using the Laplace transforms: $P^*_{(0,0)}(s)=\int\limits_0^{+\infty}e^{-st}P_{(0,0)}(t)dt$,

$P^*_{(0,i)}(s,w)=\int\limits_0^{+\infty}e^{-st}P_{(0,i)}(t,w)dt,\ P^*_{(1,i)}(s,x)=\int\limits_0^{+\infty}e^{-st}P_{(0,i)}(t,x)dt$, we obtain

$$(s+\lambda)P^*_{(0,0)}(s)-1=\int_0^\infty P^*_{(1,0)}\left(s,x\right)\beta\left(x\right)dx \tag{3.7.87}$$

$$\left(s+\frac{\partial}{\partial w}+\lambda+\alpha\left(w\right)\right)P^*_{(0,i)}\left(s,w\right)=0,\ i\geq 1, \tag{3.7.88}$$

$$\left(s+\frac{\partial}{\partial x}+p\lambda+\mu+\beta\left(x\right)\right)P^*_{(1,i)}\left(s,x\right)=p\lambda P^*_{(1,i-1)}\left(s,x\right),\ i\geq 0, \tag{3.7.89}$$

$$P^*_{(0,i)}\left(s,0\right)=\int_0^\infty P^*_{(1,i)}\left(s,x\right)\beta\left(x\right)dx,\ i\geq 1, \tag{3.7.90}$$

$$P^*_{(1,i)}\left(s,0\right)=\int_0^\infty P^*_{(0,i+1)}\left(s,w\right)\alpha\left(w\right)dw+\left(1-q\right)\lambda\int_0^\infty P^*_{(0,i+1)}\left(s,w\right)dw$$

$$+\left(1-\delta_{i,0}\right)q\lambda\int_0^\infty P^*_{(0,i)}\left(s,w\right)dw+\delta_{i,0}\lambda P^*_{(0,0)}(s),\ \ i\geq 0, \qquad (3.7.91)$$

Define the following generating functions $P_0^*(s,z,w)=\sum_{n\geq 1}z^nP^*_{(0,n)}(s,w)$, $P_1^*(s,z,x)=\sum_{n\geq 0}z^nP^*_{(1,n)}(s,x)$. Then the equations (3.7.88) -(3.7.91) are transformed into

$$\left(s+\frac{\partial}{\partial w}+\lambda+\alpha\left(w\right)\right)P_0^*(s,z,w)=0, \qquad (3.7.92)$$

$$\left(s+\frac{\partial}{\partial x}+p\lambda+\mu+\beta\left(x\right)\right)P_1^*(s,z,x)=p\lambda zP_1^*(s,z,x), \qquad (3.7.93)$$

$$P_0^*(s,z,0)=\int_0^\infty P_1^*(s,z,x)\beta\left(x\right)dx-(s+\lambda)P^*_{0,0}(s)+1, \qquad (3.7.94)$$

$$P_1^*(s,z,0)=\frac{1}{z}\int_0^\infty P_0^*(s,z,w)\alpha\left(w\right)dw+\frac{\lambda\left(1-q+qz\right)}{z}\int_0^\infty P_0^*(s,z,w)dw$$
$$+\lambda P^*_{(0,0)}(s). \qquad (3.7.95)$$

From (3.7.92) and (3.7.93) we obtain

$$P_0^*(s,z,w)=P_0^*(s,z,0)e^{-(s+\lambda)w}(1-A(w)) \qquad (3.7.96)$$

$$P_1^*(s,z,x)=P_1^*(s,z,0)e^{-(\lambda p(1-z)+\mu+s)x}(1-B(x)) \qquad (3.7.97)$$

Substituting (3.7.97) into (3.7.94) leads to

$$P_0^*(s,z,0)=P_1^*(s,z,0)L_B(\lambda p(1-z)+\mu+s)-(s+\lambda)P^*_{0,0}(s)+1 \qquad (3.7.98)$$

However (3.7.95) gives

$$P_1^*(s,z,0)=\frac{1}{z(s+\lambda)}P_0^*(s,z,0)\left[(s+\lambda)L_A(s+\lambda)+\lambda\left(1-q+qz\right)\left(1-L_A(s+\lambda)\right]\right.$$
$$+\lambda P^*_{(0,0)}(s) \qquad (3.7.99)$$

Substituting (3.7.99) into (3.7.98) yields

$$F(s,z)P_0^*(s,z,0)=z(s+\lambda)\left[\{(s+\lambda)-\lambda L_B(\lambda p(1-z)+\mu+s)\}P^*_{(0,0)}(s)-1\right] \quad (3.7.100)$$

where

$$F(s,z)=\left[(s+\lambda)L_A(s+\lambda)+\lambda\left(1-q+qz\right)\left(1-L_A(s+\lambda)\right]L_B(\lambda p(1-z)+\mu+s)-z(s+\lambda)\right.$$

It is easy to see that:

- $F(s,0) = [(s+\lambda)L_A(s+\lambda) + \lambda(1-q)(1-L_A(s+\lambda)]\, L_B(\lambda p+\mu+s) > 0$
- $F(s,1) = [sL_A(s+\lambda)+\lambda]\, L_B(\mu+s) - (s+\lambda) \leq s(L_A(s+\lambda)-1) \leq 0$
- $\dfrac{d^2}{dz^2}F(s,z) = -2\lambda^2 pq(1-L_A(s+\lambda)L_B'(\lambda p(1-z)+\mu+s)$

$+\lambda^2 p^2\,[(s+\lambda)L_A(s+\lambda)+\lambda\,(1-q+qz)\,(1-L_A(s+\lambda)]\, L_B^{''}(\lambda p(1-z)+\mu+s) > 0$

and therefore $F(s,z)$ is convex function for each fixed s.

Hence, for each fixed s, the function $F(s,z)$ has exactly a root $z = f(s)$ in the interval $]0,1]$. Choosing $z = f(s)$ in (3.7.100) yields

$$P^*_{(0,0)}(s) = \frac{1}{s+\lambda-\lambda L_B(\lambda p(1-f(s))+\mu+s)} \qquad (3.7.101)$$

We summarize our results in the following theorem.

Theorem 3.7.1 *The Laplace transform of $R(t)$ is given by*

$$R^*(s) = \frac{1}{(\mu+s)(s+\mu-\{\lambda+sL_A(s+\lambda)\}\, L_B(\mu+s))}\times$$

$$[\lambda+s+\mu-\mu L_A(s+\lambda)-(\lambda+sL_A(s+\lambda))\, L_B(\mu+s)$$

$$+\{(\lambda-\mu)\,[s+\lambda-\lambda L_B(\mu+s)]-\lambda s L_B(\mu+s)L_A(\lambda+s)\}P^*_{(0,0)}(s)] \qquad (3.7.102)$$

Proof. We have the relation

$$R^*(s) = P^*_{(0,0)}(s) + \int_0^{+\infty} P_0^*(s,1,w)dw + \int_0^{+\infty} P_1^*(s,1,x)dx \qquad (3.7.103)$$

where from (3.7.96) -(3.7.97)

$$\int_0^{+\infty} P_0^*(s,1,w)dw = P_0^*(s,1,0)\frac{1-L_A(s+\lambda)}{s+\lambda} \qquad (3.7.104)$$

$$\int_0^{+\infty} P_1^*(s,1,x)dx = P_1^*(s,1,0)\frac{1-L_B(s+\mu)}{s+\mu} \qquad (3.7.105)$$

$$P_0^*(s,1,0) = \frac{(s+\lambda)\left[\{(s+\lambda)-\lambda L_B(\mu+s)\}P^*_{(0,0)}(s)-1\right]}{[sL_A(s+\lambda)+\lambda]\, L_B(\mu+s)-(s+\lambda)} \qquad (3.7.106)$$

$$P_1^*(s,1,0) = \frac{\left[\{(s+\lambda) - \lambda L_B(\mu+s)\} P_{(0,0)}^*(s) - 1\right](sL_A(s+\lambda)+\lambda)}{[sL_A(s+\lambda)+\lambda]\, L_B(\mu+s) - (s+\lambda)} + \lambda P_{(0,0)}^*(s) \tag{3.7.107}$$

Combining (3.7.103) -(3.7.105) , (3.7.106) , (3.7.107) and after algebraic manipulation, we get (3.7.102) . ■

Corollary 3.7.1 *The mean time to the first failure $MTTFF$ of the server is given by*

$$\begin{aligned} MTTFF &= \frac{1}{\mu(\mu - \lambda L_B(\mu))} \times \\ &\left\{ \lambda\,(1 - L_B(\mu)) + \mu\,(1 - L_A(\lambda)) + \frac{(\lambda-\mu)\,(1-L_B(\mu))}{1 - L_B(\lambda p(1-f(0)) + \mu)} \right\} \end{aligned} \tag{3.7.108}$$

Proof. We need only to use the relation

$$MTTFF = \int_0^{+\infty} R(t)dt = R^*(s)\mid_{s=0}$$

■

Chapter 4

Unreliable M/G/1 Retrial Queue : Monotonicity and Comparability

In the literature many results have been obtained on the stochastic comparison of standard queues but few papers are devoted to retrial queues. Monotonicity is considered as a descriptive approach to the study of the system. Instead of studying a performance measure in a quantitative fashion, this approach attempts to reveal the relation between the performance measures and the parameters of the system. This approach is first used for the retrial queues by Liang and Kulkarni [95]. The proof of results is based on sample path arguments. Liang [94], Shin [119], Shin and Kim [120] follow the same resonings. Khalil and Falin [78]study stochastic inequalities for the number of customers in an $M/G/1$ retrial queue using explicit formulas. Boualen, Djellab and Aissani [32] derive several stochastic comparison properties for an $M/G/1$ retrial queue with vacations and constant retrial policy. Oukid and Aissani [110] obtain lower and upper bounds for the mean busy period of $GI/GI/1$ retrial queue with breakdowns and classical retrial policy.

In this chapter we investigate the monotonicity properties of the unreliable $M/G/1$ retrial queue with geometric loss and random reserved time described in chapter 2 using the general theory of stochastic ordering. The comparability of generalized service times and number of customers entering the retrial queue during generalized service times are discussed in section 1. Then in section 2, we focus on monotonicity of the transition operator of the embedded Markov chain and give the comparability conditions of two transition operators. Stochastic inequalities for the stationary number of customers in the system are discussed in section 3. In section 4, we study inequalities for the mean characteristics of the busy period and waiting times. In section 5 we derive inequalities for the steady state distribution of the server state. Finally, illustrative examples are presented in section 6.

4.1 Comparability of Generalized Service Times

To establish the relation between the performance measures and the parameters of the system, we need first to give the conditions under which the generalized service times are comparable in $\leq_{st}$, $\leq_{icx}$ and $\leq_L$ orderings.

Consider two stable unreliable $M/G/1$ retrial queues with parameters $\lambda^{(i)}$, $p^{(i)}$, $q^{(i)}$, $r^{(i)}$, $\mu^{(i)}, B^{(i)}$, $C^{(i)}$, $D^{(i)}$, $A^{(i)}$, $i = 1, 2$. Let $S^{(i)}$ and $S^{(i)*}$ be the service time and the generalized service time in the $i-th$ system. Also, let $N^{(i)}(S^{(i)})$, $N_1^{(i)}(S^{(i)})$, $\left(k_j^{(i)}\right)$ be the number of failures during a generalized service time, the number of failures during the same period where the customer in service enters the service orbit, and the distribution of the number of customers entering the orbit during this period in the $i-th$ system.

Proposition 4.1.1 [126] *If* $\mu^{(1)} \leq \mu^{(2)}$, $r^{(1)} \geq r^{(2)}$ *and* $B^{(1)} \leq_{st} B^{(2)}$ *then*

$$N^{(1)}(S^{(1)}) \leq_{st} N^{(2)}(S^{(2)}).$$

$$N_1^{(1)}(S^{(1)}) \leq_{st} N_1^{(2)}(S^{(2)}).$$

Proof. We have

$$\begin{aligned}
\sum_{j\geq n} P\left[N^{(i)}(S^{(i)}) = j\right] &= \sum_{j\geq n} \int_0^{+\infty} \frac{\left(\mu^{(i)}x\right)^j}{j!} e^{-\mu^{(i)}x} dB^{(i)}(x) \\
&= \int_0^{+\infty} \left[\sum_{j\geq n} \frac{\left(\mu^{(i)}x\right)^j}{j!} e^{-\mu^{(i)}x}\right] dB^{(i)}(x)
\end{aligned}$$

The function $f(\mu, x) = \sum_{j\geq n} \frac{(\mu x)^j}{j!} e^{-\mu x}$ is increasing in x and μ. In fact $\frac{d}{dx} f(\mu, x) =$

$$\mu\frac{(\mu x)^{n-1}}{(n-1)!}e^{-\mu x} \geq 0 \text{ and } \frac{d}{d\mu}f(\mu,x) = x\frac{(\mu x)^{n-1}}{(n-1)!}e^{-\mu x} \geq 0.$$

Since $B^{(1)} \leq_{st} B^{(2)}$ then

$$\int_0^{+\infty} f(\mu^{(1)},x)dB^{(1)}(x) \leq \int_0^{+\infty} f(\mu^{(2)},x)dB^{(1)}(x) \leq \int_0^{+\infty} f(\mu^{(2)},x)dB^{(2)}(x)$$

On the other hand

$$\begin{aligned}\sum_{j\geq n} P\left[N_1^{(i)}(S^{(i)}) = j\right] &= \sum_{j\geq n}\int_0^{+\infty} \frac{\left(\mu^{(i)}\left(1-r^{(i)}\right)x\right)^j}{j!}e^{-\mu^{(i)}\left(1-r^{(i)}\right)x}dB^{(i)}(x)\\ &= \int_0^{+\infty}\left[\sum_{j\geq n}\frac{\left(\mu^{(i)}\left(1-r^{(i)}\right)x\right)^j}{j!}e^{-\mu^{(i)}\left(1-r^{(i)}\right)x}\right]dB^{(i)}(x)\end{aligned}$$

The function $g(\mu,r,x) = \sum_{j\geq n}\frac{(\mu\,(1-r)\,x)^j}{j!}e^{-\mu(1-r)x}$ is increasing in x and μ and decreasing in r. Therefore

$$\int_0^{+\infty} g(\mu^{(1)},r^{(1)},x)dB^{(1)}(x) \leq \int_0^{+\infty} g(\mu^{(2)},r^{(2)},x)dB^{(1)}(x) \leq \int_0^{+\infty} g(\mu^{(2)},r^{(2)},x)dB^{(2)}(x)$$

■

Proposition 4.1.2 [126] *If* $\mu^{(1)} \leq \mu^{(2)}$, $r^{(1)} \geq r^{(2)}$, $B^{(1)} \leq_{st} B^{(2)}$, $C^{(1)} \leq_{st} C^{(2)}$ *and* $D^{(1)} \leq_{st} D^{(2)}$ *then*

$$S^{(1)*} \leq_{st} S^{(2)*}.$$

Proof. Let f be a bounded differentiable increasing function. We can write

$$\begin{aligned}E\left[f(S^{(i)*})\right] &= \int_0^{+\infty} g^{(i)}(x)dB^{(i)}(x)E(f(S^{(i)} + \sum_{k=1}^{N^{(i)}(S^{(i)})} X_k^{(i)} + \sum_{k=1}^{N_1^{(i)}(S^{(i)})} Y_k^{(i)}))\\ &= \int_0^{+\infty} E(f(x + \sum_{k=1}^{N^{(i)}(S^{(i)})} X_k^{(i)} + \sum_{k=1}^{N_1^{(i)}(S^{(i)}} Y_k^{(i)}))dB^{(i)}(x)\\ &= \int_0^{+\infty} g^{(i)}(x)dB^{(i)}(x)\end{aligned}$$

Consider the funcction

$$g(x) = E(f(x + \sum_{k=1}^{N(x)} X_k + \sum_{k=1}^{N_1(x)} Y_k))$$

which can be written as follows

$$g(x) = \sum_{n\geq 0} \sum_{j=0}^{n} f_{n,j}(x) C_n^j \left(1-r\right)^j r^{n-j} \frac{(\mu x)^n}{n!} e^{-\mu x}$$

where

$$f_{n,j}(x) = E(f(x + \sum_{k=1}^{n} X_k + \sum_{k=1}^{j} Y_k))$$

Differentiating in x, we have

$$g^{'}(x) = \sum_{n\geq 0} \sum_{j=0}^{n} C_n^j \left(1-r\right)^j r^{n-j} \left[f_{n,j}^{'}(x) + \mu \left(f_{n+1,j}(x) - f_{n,j}(x) \right) \right] \frac{(\mu x)^n}{n!} e^{-\mu x}$$

Since $f \implies$ and (X_k) are positive random variables, then $f_{n,j}(x)$ is increasing in n and x, i.e., $f_{n+1,j}(x) - f_{n,j}(x) \geq 0$ and $f_{n,j}^{'}(x) \geq 0$. Therefore $g(x)$ is increasing.

By proposition 4.1.1 and theorem 1.1.6,

$$\sum_{k=1}^{N^{(1)}(x)} X_k^{(1)} \leq_{st} \sum_{k=1}^{N^{(2)}(x)} X_k^{(2)}$$

and

$$\sum_{k=1}^{N_1^{(1)}(x)} Y_k^{(1)} \leq_{st} \sum_{k=1}^{N_1^{(2)}(x)} Y_k^{(2)}$$

Since the function $f(x+y+z)$ is increasing in x, y and z, we obtain $g^{(1)}(x) \leq g^{(2)}(x)$ and

$$\int_0^{+\infty} g^{(1)}(x) dB^{(1)}(x) \leq \int_0^{+\infty} g^{(2)}(x) dB^{(1)}(x) \leq \int_0^{+\infty} g^{(2)}(x) dB^{(2)}(x).$$

■

Lemma 4.1.1 [126] *If* $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)} \leq p^{(2)}, \mu^{(1)} \leq \mu^{(2)}$, $r^{(1)} \geq r^{(2)}$, $B^{(1)} \leq_{st} B^{(2)}$, $C^{(1)} \leq_{st} C^{(2)}$ *and* $D^{(1)} \leq_{st} D^{(2)}$ *then* $\left(k_j^{(1)}\right) \leq_{st} \left(k_j^{(2)}\right)$.

Proof. By definition

$$\begin{aligned}
\overline{k_n^{(i)}} &= \sum_{j\geq n} k_j^{(i)} \\
&= \int_0^{+\infty} \left[\sum_{j\geq n} \frac{\left(\lambda^{(i)} p^{(i)} x\right)^j e^{-\lambda^{(i)} p^{(i)} x}}{j!} \right] dB^{(i)*}(x).
\end{aligned}$$

The function $f(\lambda,p,x) = \sum_{j\geq n} \frac{(\lambda p x)^j e^{-\lambda p x}}{j!}$ is increasing in λ, p and x and $B^{(1)*} \leq_{st} B^{(2)*}$ by proposition 4.1.2. Then

$$\int_0^{+\infty} f(\lambda^{(1)}, p^{(1)}, x) dB^{(1)*}(x). \leq \int_0^{+\infty} f(\lambda^{(2)}, p^{(2)}, x) dB^{(1)*}(x) \leq \int_0^{+\infty} f(\lambda^{(2)}, p^{(2)}, x) dB^{(2)*}(x).$$

■

Proposition 4.1.3 [126] *If* $\mu^{(1)} \leq \mu^{(2)}$, $r^{(1)} \geq r^{(2)}$ *and* $B^{(1)} \leq_{icx} B^{(2)}$ *then*

$$N^{(1)}(S^{(1)}) \leq_{icx} N^{(2)}(S^{(2)}).$$

$$N_1^{(1)}(S^{(1)}) \leq_{icx} N_1^{(2)}(S^{(2)}).$$

Proof. $\sum_{j\geq n} \sum_{l\geq j} P\left[N^{(i)}(S^{(i)}) = l\right] = \int_0^{+\infty} \left[\sum_{j\geq n} \sum_{l\geq j} \frac{\left(\mu^{(i)} x\right)^l}{l!} e^{-\mu^{(i)} x} \right] dB^{(i)}(x)$

Consider the function

$$f(\mu, x) = \sum_{j\geq n} \sum_{l\geq j} \frac{(\mu x)^l}{l!} e^{-\mu x}$$

Differentiating twice in x, we obtain

$$\frac{d}{dx} f(\mu,x) = \mu \sum_{j\geq n-1} \frac{(\mu x)^j}{j!} e^{-\mu x} \geq 0 \text{ and } \frac{d^2}{dx^2} f(\mu,x) = \mu^2 \frac{(\mu x)^{n-2}}{(n-2)!} e^{-\mu x} \geq 0$$

Therefore $f(\mu, x)$is increasing and convex in x and μ. Since $B^{(1)} \leq_{icx} B^{(2)}$ then

$$\int_0^{+\infty} f(\mu^{(1)}, x) dB^{(1)}(x) \leq \int_0^{+\infty} f(\mu^{(2)}, x) dB^{(1)}(x) \leq \int_0^{+\infty} f(\mu^{(2)}, x) dB^{(2)}(x)$$

In the same way

$$\sum_{j\geq n}\sum_{l\geq j} P\left[N_1^{(i)}(S^{(i)})=l\right]=\int_0^{+\infty}\left[\sum_{j\geq n}\sum_{l\geq j}\frac{\left(\mu^{(i)}\left(1-r^{(i)}\right)x\right)^l}{l!}e^{-\mu^{(i)}\left(1-r^{(i)}\right)x}\right]dB^{(i)}(x)$$

where the function $g(\mu,r,x)=\sum_{j\geq n}\sum_{l\geq j}\frac{(\mu(1-r)x)^l}{l!}e^{-\mu(1-r)x}$ is increasing and convex in x and μ and decreasing in r.Hence

$$\int_0^{+\infty} g(\mu^{(1)},r^{(1)},x)dB^{(1)}(x)\leq\int_0^{+\infty} g(\mu^{(2)},r^{(2)},x)dB^{(2)}(x)$$

■

Proposition 4.1.4 [126] *If* $\mu^{(1)}\leq\mu^{(2)},\ \ r^{(1)}\geq r^{(2)},\ B^{(1)}\leq_{icx}B^{(2)},\ C^{(1)}\leq_{icx}C^{(2)}$ *and* $D^{(1)}\leq_{icx}D^{(2)}$, *then*

$$S^{(1)*}\leq_{icx}S^{(2)*}.$$

Proof. Let f be a twice differentiable increasing convex function. As in the proof of proposition 4.1.2, we consider the function

$$g(x)=\sum_{n\geq 0}\sum_{j=0}^{n} f_{n,j}(x)C_n^j(1-r)^j r^{n-j}\frac{(\mu x)^n}{n!}e^{-\mu x}$$

Differentiating twice in x, we obtain

$$\begin{aligned} g''(x) &= \sum_{n\geq 0}\sum_{j=0}^{n} C_n^j(1-r)^j r^{n-j}[f''_{n,j}(x)+2\mu\left(f'_{n+1,j}(x)-f'_{n,j}(x)\right)\\ &\qquad +\mu^2\left(f_{n+2,j}(x)-2f_{n+1,j}(x)+f_{n,j}(x)\right)]\frac{(\mu x)^n}{n!}e^{-\mu x}\end{aligned}$$

Since f is increasing and convex, then $f_{n,j}(x)$ is increasing and convex in x and n, i.e. $f''_{n,j}(x)\geq 0$ and $f_{n+2,j}(x)-2f_{n+1,j}(x)+f_{n,j}(x)\geq 0$. Because f' is increasing, $f'_{n,j}(x)$ is increasing in n, i.e. $f'_{n+1,j}(x)-f'_{n,j}(x)\geq 0$. Therefore $g(x)$ is increasing and convex.
By proposition 4.1.3 and theorem 1.1.11,

$$\sum_{k=1}^{N^{(1)}(x)}X_k^{(1)}\leq_{icx}\sum_{k=1}^{N^{(2)}(x)}X_k^{(2)}\quad\text{and}\quad\sum_{k=1}^{N_1^{(1)}(x)}Y_k^{(1)}\leq_{icx}\sum_{k=1}^{N_1^{(2)}(x)}Y_k^{(2)}$$

Since the function $f(x+y+z)$ is increasing and convex in $x,\ y$ and z, then $g^{(1)}(x) \leq g^{(2)}(x)$ and

$$\int_0^{+\infty} g^{(1)}(x)dB^{(1)}(x) \leq \int_0^{+\infty} g^{(2)}(x)dB^{(1)}(x) \leq \int_0^{+\infty} g^{(2)}(x)dB^{(2)}(x).$$

■

Lemma 4.1.2 [126] *If* $\lambda^{(1)} \leq \lambda^{(2)},\ p^{(1)} \leq p^{(2)}, \mu^{(1)} \leq \mu^{(2)},\ r^{(1)} \geq r^{(2)},\ B^{(1)} \leq_{icx} B^{(2)}$, $C^{(1)} \leq_{icx} C^{(2)}$ *and* $D^{(1)} \leq_{icx} D^{(2)}$ *then* $\left(k_j^{(1)}\right) \leq_{icx} \left(k_j^{(2)}\right)$

Proof. We have

$$\overline{\overline{k_n^{(i)}}} = \int_0^{+\infty} \left[\sum_{j\geq n} \sum_{l \geq j} \frac{\left(\lambda^{(i)} p^{(i)} x\right)^l}{l!} e^{-\lambda^{(i)} p^{(i)} x} \right] dB^{(i)*}(x)$$

The function $f(\lambda, p, x) = \sum_{j\geq n} \sum_{l\geq j} \frac{(\lambda p x)^l}{l!} e^{-\lambda p x}$ is increasing in λ, p and x and is convex in x. Then by proposition 4.1.4

$$\int_0^{+\infty} f(\lambda^{(1)}, p^{(1)}, x)dB^{(1)*}(x) \leq \int_0^{+\infty} f(\lambda^{(2)}, p^{(2)}, x)dB^{(2)*}(x)$$

■

Proposition 4.1.5 [126] *If* $\mu^{(1)} \leq \mu^{(2)},\ r^{(1)} \geq r^{(2)}$ *and* $B^{(1)} \leq_L B^{(2)}$ *then*

$$N^{(1)}(S^{(1)}) \leq_L N^{(2)}(S^{(2)}).$$

$$N_1^{(1)}(S^{(1)}) \leq_L N_1^{(2)}(S^{(2)}).$$

Proof. It is well known that

$$\begin{aligned} \sum_{n\geq 0} z^n P\left[N^{(i)}(S^{(i)}) = n\right] &= L_{B^{(i)}}\left(\mu^{(i)}(1-z)\right) \\ \sum_{n\geq 0} z^n P\left[N_1^{(i)}(S^{(i)}) = n\right] &= L_{B^{(i)}}\left(\mu^{(i)}\left(1-r^{(i)}\right)(1-z)\right) \end{aligned}$$

Since $\mu^{(1)} \leq \mu^{(2)},\ r^{(1)} \geq r^{(2)},\ B^{(1)} \leq_L B^{(2)}$ and $L_{B^{(i)}}(s)$ is decreasing in s, then

$$\begin{aligned} L_{B^{(1)}}\left(\mu^{(1)}(1-z)\right) &\geq L_{B^{(2)}}\left(\mu^{(2)}(1-z)\right) \\ L_{B^{(1)}}\left(\mu^{(1)}\left(1-r^{(1)}\right)(1-z)\right) &\geq L_{B^{(2)}}\left(\mu^{(2)}\left(1-r^{(2)}\right)(1-z)\right) \end{aligned}$$

■

Proposition 4.1.6 [126] *If* $\mu^{(1)} \leq \mu^{(2)},\ \ r^{(1)} \geq r^{(2)},\ B^{(1)} \leq_L B^{(2)},\ C^{(1)} \leq_L C^{(2)}$ *and* $D^{(1)} \leq_L D^{(2)}$ *then*

$$S^{(1)*} \leq_L S^{(2)*}.$$

Proof. The function $s + \mu - \mu\left\{r + (1-r)\, L_D(s)\right\} L_C(s)$ is increasing in μ and decreasing in r. Since $C^{(1)} \leq_L C^{(2)}$ and $D^{(1)} \leq_L D^{(2)}$, then

$$\begin{aligned} & s + \mu^{(1)} - \mu^{(1)}\left\{r^{(1)} + \left(1-r^{(1)}\right) L_{D^{(1)}}(s)\right\} L_{C^{(1)}}(s) \\ & \leq s + \mu^{(2)} - \mu^{(2)}\left\{r^{(2)} + \left(1-r^{(2)}\right) L_{D^{(1)}}(s)\right\} L_{C^{(1)}}(s) \\ & \leq s + \mu^{(2)} - \mu^{(2)}\left\{r^{(2)} + \left(1-r^{(2)}\right) L_{D^{(2)}}(s)\right\} L_{C^{(2)}}(s) \end{aligned}$$

Finally $B^{(1)} \leq_L B^{(2)}$ yields

$$L_{B^{(1)*}}(s) \geq L_{B^{(2)*}}(s)$$

■

Lemma 4.1.3 [126] *If* $\lambda^{(1)} \leq \lambda^{(2)},\ p^{(1)} \leq p^{(2)}, \mu^{(1)} \leq \mu^{(2)},\ r^{(1)} \geq r^{(2)},\ B^{(1)} \leq_L B^{(2)},\ C^{(1)} \leq_L C^{(2)}$ *and* $D^{(1)} \leq_L D^{(2)}$ *then* $\left(k_j^{(1)}\right) \leq_L \left(k_j^{(2)}\right)$

Proof. We have

$$\sum_{n\geq 0} z^n k_n^{(i)} = L_{B^{(i)*}}\left(\lambda^{(i)} p^{(i)}(1-z)\right)$$

By proposition 4.1.6 and $\lambda^{(1)} \leq \lambda^{(2)},\ p^{(1)} \leq p^{(2)}$ we obtain

$$L_{B^{(1)*}}\left(\lambda^{(1)} p^{(1)}(1-z)\right) \geq L_{B^{(2)*}}\left(\lambda^{(2)} p^{(2)}(1-z)\right)$$

■

4.2 Monotonicity Properties of the Embedded Markov Chain

The one step transition probabilities of the embedded Markov chain Q_n are given by the formula

$$\begin{aligned} P_{ij} &= [1-q(1-L_A(\lambda))]\,k_{j-i+1} + q(1-L_A(\lambda))\,k_{j-i}, \quad \text{for } i \neq 0 \text{ and } j \geq 0 \\ P_{0j} &= k_j, \quad \text{for } j \geq 0 \end{aligned}$$

To every distribution $\alpha = (\alpha_n)$, the transition operator T of the embedded Markov chain associates a distribution $T\alpha = \beta = (\beta_m)$ where

$$\beta_m = \sum_{n\geq 0} \alpha_n P_{nm}.$$

In the following theorem, we focus on the monotonicity of the transition operator T relatively to $\leq_{st}$, and $\leq_{icx}$ orders.

Theorem 4.2.1 [126] *The operator T is monotone with respect to the orders $\leq_{st}$, $\leq_{icx}$.*

Proof. We have

$$\begin{aligned} \overline{P}_{nm} &= [1-q(1-L_A(\lambda))]\,\overline{k}_{m-n+1} + q(1-L_A(\lambda))\,\overline{k}_{m-n} \\ &= \overline{k}_{m-n} - [1-q(1-L_A(\lambda))]\,k_{m-n} \\ &= \overline{k}_{m-n+1} + q(1-L_A(\lambda))\,k_{m-n} \end{aligned}$$

Thus

$$\begin{aligned} \overline{P}_{nm} - \overline{P}_{n-1m} &= \overline{k}_{m-n+1} + q(1-L_A(\lambda))\,k_{m-n} - \overline{k}_{m-n+1} + [1-q(1-L_A(\lambda))]\,k_{m-n+1} \\ &= q(1-L_A(\lambda))\,k_{m-n} + [1-q(1-L_A(\lambda))]\,k_{m-n+1} > 0 \end{aligned}$$

On the other hand

$$\begin{aligned} \overline{\overline{P}}_{n-1m} + \overline{\overline{P}}_{n+1m} - 2\overline{\overline{P}}_{nm} &= \overline{\overline{k}}_{m-n} + q(1-L_A(\lambda))\,\overline{k}_{m-n+1} + \overline{\overline{k}}_{m-n+1} \\ &\quad - [1-q(1-L_A(\lambda))]\,\overline{k}_{m-n} - \overline{\overline{k}}_{m-n+1} + q(1-L_A(\lambda))\,\overline{k}_{m-n+1} \\ &\quad - \overline{\overline{k}}_{m-n} + [1-q(1-L_A(\lambda))]\,\overline{k}_{m-n} \\ &= 2q(1-L_A(\lambda))\,\overline{k}_{m-n+1} > 0. \end{aligned}$$

■

Corollary 4.2.1 [126] *If at time $t = 0$ the system were empty then the number of customers in the orbit would form a monotonically increasing sequence with respect to the orders $\leq_{st}$ and $\leq_{icx}$.*

Proof. If $\alpha^{(0)} = (1, 0, 0, ...)$ is the initial probability vector, then
$\alpha^{(1)} = T\alpha^{(0)} = (k_0,\ k_1,,\ k_2,\ ...)$, $\overline{\alpha_k^{(0)}} = \begin{cases} 1 & if\ k = 0 \\ 0 & if\ k \neq 0 \end{cases}$,
$\overline{\alpha_k^{(1)}} = \begin{cases} 1 & if\ k = 0 \\ \sum\limits_{i \geq k} k_i & if\ k \neq 0 \end{cases}$, $\overline{\overline{\alpha_n^{(0)}}} = \begin{cases} 1 & if\ n = 0 \\ 0 & if\ n \neq 0 \end{cases}$ and $\alpha^{(0)} \leq_s \alpha^{(1)}$ where s is one of the symbols $st,\ icx$. By induction and using the monotonicity of the operator T we show that $\alpha^{(n)} \leq_s \alpha^{(n+1)}$. Thus $Q_n \leq_s Q_{n+1}$. ■

Remark 4.2.1 *The operator T is not monotone with respect to the order $\leq_L$. In fact for $p = q = 1$ and for $\alpha^{(1)} = (1, 0, 0, ...)$, $\alpha^{(2)} = (0, 1, 0, ...)$ we have $\alpha^{(1)} \leq_L \alpha^{(2)}$ but $T\alpha^{(1)} \nleq_L T\alpha^{(2)}$.*

In the following three theorems, we give comparability conditions of two transition operators relatively to $\leq_{st}$, $\leq_{icx}$ and $\leq_L$ orders. Consider two stable unreliable $M/G/1$ retrial queues with parameters $\lambda^{(1)},\ p^{(1)},\ q^{(1)},\ r^{(1)},\ \mu^{(1)},\ B^{(1)},\ C^{(1)},\ D^{(1)},\ A^{(1)}$ and $\lambda^{(2)},\ p^{(2)},\ q^{(2)},\ r^{(2)},\ \mu^{(2)},\ B^{(2)},\ C^{(2)},\ D^{(2)},\ A^{(2)}$ respectively. Let T^1, T^2 be the transition operators of the corresponding embedded Markov chains.

Theorem 4.2.2 [126] *If $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)} \leq p^{(2)}$, $q^{(1)} \leq q^{(2)}$, $\mu^{(1)} \leq \mu^{(2)}$, $r^{(1)} \geq r^{(2)}$, $B^{(1)} \leq_{st} B^{(2)}$, $C^{(1)} \leq_{st} C^{(2)}$, $D^{(1)} \leq_{st} D^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then*

$$T^1 \leq_{st} T^2$$

Proof. We have

$$\overline{P}_{nm}^{(1)} = \overline{k}_{m-n+1}^{(1)} + q^{(1)}(1 - L_{A^{(1)}}(\lambda^{(1)}))k_{m-n}^{(1)}$$

Since $\lambda^{(1)} \leq \lambda^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$ and $q^{(1)} \leq q^{(2)}$ then

$L_{A^{(1)}}(\lambda^{(1)}) \geq L_{A^{(2)}}(\lambda^{(2)})$ and $\overline{P}^{(1)}_{nm} \leq \overline{k}^{(1)}_{m-n+1} + q^{(2)}\left(1 - L_{A^{(2)}}(\lambda^{(2)})\right) k^{(1)}_{m-n}$

But

$$\begin{aligned}\overline{k}^{(1)}_{m-n+1} + q^{(2)}\left(1 - L_{A^{(2)}}(\lambda^{(2)})\right) k^{(1)}_{m-n} &= \left[1 - q^{(2)}\left(1 - L_{A^{(2)}}(\lambda^{(2)})\right)\right] \overline{k}^{(1)}_{m-n+1} \\ &\quad + q^{(2)}\left(1 - L_{A^{(2)}}(\lambda^{(2)})\right) \overline{k}^{(1)}_{m-n}\end{aligned}$$

By lemma 4.1.1, $\overline{k}^{(1)}_j \leq \overline{k}^{(2)}_j$ for all $j \geq 0$.

Finally $\overline{P}^{(1)}_{nm} \leq \left[1 - q^{(2)}\left(1 - L_{A^{(2)}}(\lambda^{(2)})\right)\right] \overline{k}^{(2)}_{m-n+1} + q^{(2)}(1 - L_{A^{(2)}}(\lambda^{(2)}))\overline{k}^{(2)}_{m-n} = \overline{P}^{(2)}_{nm}$. ■

Theorem 4.2.3 [126] *If* $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)} \leq p^{(2)}$, $q^{(1)} \leq q^{(2)}$, $\mu^{(1)} \leq \mu^{(2)}$, $r^{(1)} \geq r^{(2)}$, $B^{(1)} \leq_{icx} B^{(2)}$, $C^{(1)} \leq_{icx} C^{(2)}$, $D^{(1)} \leq_{icx} D^{(2)}$ *and* $A^{(1)} \leq_L A^{(2)}$, *then*

$$T^1 \leq_{icx} T^2$$

Proof. The proof is similar to that of theorem 4.2.2. It is sufficient to substitute $\overline{k}^{(1)}_j \leq \overline{k}^{(2)}_j$ by $\overline{\overline{k}}^{(1)}_j \leq \overline{\overline{k}}^{(2)}_j$ using lemma 4.1.2. ■

Theorem 4.2.4 [126] *If* $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)} \leq p^{(2)}$, $q^{(1)} \leq q^{(2)}$, $\mu^{(1)} \leq \mu^{(2)}$, $r^{(1)} \geq r^{(2)}$, $B^{(1)} \leq_L B^{(2)}$, $C^{(1)} \leq_L C^{(2)}$, $D^{(1)} \leq_L D^{(2)}$ *and* $A^{(1)} \leq_L A^{(2)}$, *then*

$$T^1 \leq_L T^2$$

Proof. Let $\alpha = (\alpha_n)$ be a distribution and $T\alpha = \beta = (\beta_n)$ where

$$\beta_n = \alpha_0 k_n + \sum_{i\geq 1} \alpha_i P_{in}, \text{ for all } n \geq 0.$$

Let $Q(z) = \sum_{n\geq 0} z^n k_n$ and $g(z) = \sum_{n\geq 0} z^n \alpha_n$ be the generating functions of (k_n) and (α_n) respectively.The generating function $G(z)$ of β is given by

$$\begin{aligned}G(z) &= \sum_{n\geq 0} z^n \beta_n = \sum_{n\geq 0} z^n \left[\alpha_0 k_n + \sum_{i\geq 1} \alpha_i P_{in}\right] \\ &= \alpha_0 \sum_{n\geq 0} z^n k_n + \sum_{n\geq 0} z^n \sum_{i\geq 1} \alpha_i P_{in} \\ &= \alpha_0 Q(z) + \sum_{i\geq 1} \alpha_i \sum_{n\geq 0} z^n \left[\left[1 - q\left(1 - L_A(\lambda)\right)\right] k_{n-i+1} + q\left(1 - L_A(\lambda)\right) k_{n-i}\,\right] \\ &= \alpha_0 Q(z) + \left[1 - q\left(1 - L_A(\lambda)\right)\right] \sum_{i\geq 1} \alpha_i \sum_{n\geq 0} z^n k_{n-i+1} + q\left(1 - L_A(\lambda)\right) \sum_{i\geq 1} \alpha_i \sum_{n\geq 0} z^n k_{n-i}\end{aligned}$$

After algebraic manipulation, we obtain

$$G(z) = \alpha_0 Q(z) + \frac{1}{z} Q(z)(g(z) - \alpha_0) - q\left(1 - L_A(\lambda)\right) Q(z)(g(z) - \alpha_0) \frac{1-z}{z}$$

If the conditions of theorem 4.2.4 are fulfilled then $Q^{(1)}(z) \geq Q^{(2)}(z)$ by lemma 4.1.3. and $-q^{(1)}\left(1 - L_{A^{(1)}}(\lambda^{(1)})\right) \geq -q^{(2)}\left(1 - L_{A^{(2)}}(\lambda^{(2)})\right)$. Hence $G^{(1)}(z) \geq G^{(2)}(z)$ ■

4.3 Stochastic Inequalities for the Stationary Number of Customers in the System

Consider two stable unreliable $M/G/1$ retrial queues with patient customers,$p = q = 1$, and parameters $\lambda^{(i)}$, $r^{(i)}$, $\mu^{(i)}, B^{(i)}$, $C^{(i)}$, $D^{(i)}$, $A^{(i)}$, $i = 1, 2$, and let $(\pi_n^{(1)})$, $(\pi_n^{(2)})$ be the corresponding stationary distributions of the number of customers in the system.

Theorem 4.3.1 [126] *If $\lambda^{(1)} \leq \lambda^{(2)}$, $\mu^{(1)} \leq \mu^{(2)}$, $r^{(1)} \geq r^{(2)}$, $B^{(1)} \leq_s B^{(2)}$, $C^{(1)} \leq_s C^{(2)}$, $D^{(1)} \leq_s D^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then $(\pi_n^{(1)}) \leq_s (\pi_n^{(2)})$ where s is one of the symbols $\leq_{st}$, $\leq_{icx}$.*

Proof. We know that the distribution of the number of customers in the system at steady state coincides with the distribution of the number of customers in the system at departure epochs. Since the corresponding embedded Markov chain is ergodic, then the stationary distribution coincides with the limit distribution. Using theorems 4.2.1, 4.2.2 and 4.2.3 saying that T^i are monotone with respect to the order $\leq_s$ and $T^1 \leq_s T^2$, we have by induction $T^{1,n}\alpha \leq_s T^{2,n}\alpha$ for any distribution α, where $T^{i,n}\alpha = T^i\left(T^{i,n-1}\alpha\right)$. Taking the limit, we obtain the stated result. ■

Theorem 4.3.2 [126] *If in the unreliable $M/G/1$ retrial queue the service time distribution, the repair time distribution and the reserved time distribution are $HNBUE$ and the retrial time distribution is $\overline{\mathcal{L}}$ then $(\pi_n) \leq_{icx} (\pi_n^*)$ where (π_n^*) is the stationary distribution* *of the number of customers in the unreliable $M/M/1$ retrial queue with exponential retrial, repair and reserved times with the same parameters.*

Proof. Consider an auxiliary unreliable $M/M/1$ retrial queue with the same arrival rate λ, parameters $p,\ q$,r, mean retrial time α_1, mean service time β_1, mean repair time γ_1 and mean reserved time η_1, but with exponentially distributed retrial time $A^*(x) = 1 - e^{-x/\alpha_1}$, service time $B^*(x) = 1 - e^{-x/\beta_1}$, repair time $C^*(x) = 1 - e^{-x/\gamma_1}$ and reserved time $D^*(x) = 1 - e^{-x/\eta_1}$ for $x > 0$. If B, C and D are $HNBUE$ and A is $\overline{\mathcal{L}}$ then $B \leq_{icx} B^*$, $C \leq_{icx} C^*$, $D \leq_{icx} D^*$ and $A \leq_L A^*$.Using theorem 4.3.1, we obtain $(\pi_n) \leq_{icx} (\pi_n^*)$. ■

Remark 4.3.1 *For $p = q = 1$ and $\mu = 0$, the previous theorem implies that the mean number of customers in the system in steady state $E(N)$ is less than or equals to* $\dfrac{\lambda\left(\lambda\beta_2 + 2(1-\lambda\beta_1)\beta_1\right)(1+\lambda\alpha_1)}{2\left(1-\lambda\beta_1\left(1+\lambda\alpha_1\right)\right)}$.

4.4 Inequalities for the Mean Characteristics of the Busy Period and Waiting Time

Assume that we have two unreliable $M/G/1$ retrial queues with parameters $\lambda^{(i)}$, $p^{(i)}$, $q^{(i)}$, $r^{(i)}$, $\mu^{(i)}$, $B^{(i)}$, $C^{(i)}$, $D^{(i)}$, $A^{(i)}$, $i = 1, 2$. Let $L^{(i)}$, $K^{(i)}$, be the length busy period and the number of customers served during a busy period respectively in the $i - th$ system, $i = 1, 2$.

Theorem 4.4.1 [126] *If $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)} \leq p^{(2)}$, $q^{(1)} \leq q^{(2)}$, $\mu^{(1)} \leq \mu^{(2)}$, $r^{(1)} \geq r^{(2)}$, $B^{(1)} \leq_s B^{(2)}$, $C^{(1)} \leq_s C^{(2)}$, $D^{(1)} \leq_s D^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$, then*

$$E(L^{(1)}) \leq E(L^{(2)}),$$

$$E(K^{(1)}) \leq E(K^{(2)}),$$

where $\leq_s$ is one of the symbols $\leq_{st}$, $\leq_{icx}$, $\leq_L$.

Proof. The expected values of L and K are increasing with respect to $\lambda,\ p,\ q,\ \mu,\ \beta_1,\ \gamma_1$ and η_1, decreasing with respect to r and $L_A(.)$. Under conditions of theorem 4.1.1, we obtain the desired inequalities. ■

Theorem 4.4.2 [126] *For any unreliable $M/G/1$ retrial queue under study*

$$E(L) \leq \frac{\beta_1 \left[1-(1-e^{-\lambda\alpha_1})(q-p)\right]\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right]}{1-q(1-e^{-\lambda\alpha_1})-\lambda p\beta_1\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right]},$$

$$E(K) \leq \frac{1-q+qe^{-\lambda\alpha_1}}{1-q+qe^{-\lambda\alpha_1}-\lambda p\beta_1\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right]},$$

If A is $\mathcal{L}$ then

$$E(L) \geq \frac{\beta_1\left[1+\alpha_1\lambda\left(1-q+p\right)\right]\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right]}{1+\alpha_1\lambda\left(1-q\right)-\lambda p\beta_1(1+\alpha_1\lambda)\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right]},$$

$$E(K) \geq \frac{1+\alpha_1\lambda(1-q)}{1+\alpha_1\lambda(1-q)-\lambda p\beta_1(1+\alpha_1\lambda)\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right]},$$

where α_1 is the mean retrial time.

Proof. Consider auxiliary unreliable $M/D/1$ and $M/M/1$ retrial queues with the same arrival rates λ, mean service times β_1, mean repair times γ_1, mean reserved times η_1 and mean retrial times α_1. The distributions C, D and A are Dirac distributions at γ_1, η_1 and α_1 respectively for the $M/D/1$ system and are exponential distributions for the $M/M/1$ system. Recall that for the class of distribution functions with mean m, θ_m is its $\leq_L$ −maximum and for the class $\mathcal{L}$, $Exp(m^{-1})$ is its $\leq_L$ −minimum. Using theorem above we obtain the stated results. ■

Consider a special case with patient customers and without failure, i.e. $p = q = 1$ and $\mu = 0$. Theorems 4.4.3 and 4.4.4 are concerned with this case. Let $N_b^{(i)}$ and $W^{(i)}$ be the number of orbit busy periods which take place in $\left]0, L^{(i)}\right]$ and the waiting time in the $i-th$ system, $i = 1, 2$.

Theorem 4.4.3 [126] *If $\lambda^{(1)} \leq \lambda^{(2)}$, $B^{(1)} \leq_L B^{(2)}$ and $A^{(1)} \leq_L A^{(2)}$ then*

$$E(N_b^{(1)}) \leq E(N_b^{(2)})$$

$$E(W^{(1)}) \leq E(W^{(2)})$$

Proof. Gomez-Corral (99) shows that

$$E(N_b) = \frac{1 - L_B(\lambda)}{L_B(\lambda)}$$

and

$$E(W) = \frac{\lambda\beta_2 + 2\beta_1 (1 - L_A(\lambda))}{2 (L_A(\lambda) - \lambda\beta_1)}$$

These quantities are increasing with respect to λ, β_1 and β_2, decreasing with respect to $L_B(.)$ and $L_A(.)$. Under the conditions of theorem 4.4.3 we obtain the desired inequalities.

■

Theorem 4.4.4 [126] *For any $M/G/1$ retrial queue*

$$E(N_b) \leq \exp(\lambda\beta_1) - 1,$$

$$E(W) \leq \frac{\lambda\beta_2 + 2\beta_1 \left(1 - e^{-\lambda\alpha_1}\right)}{2 (e^{-\lambda\alpha_1} - \lambda\beta_1)}.$$

If B and A are $\mathcal{L}$ then

$$E(N_b) \geq \lambda\beta_1,$$

$$\frac{\lambda\beta_2 (1 + \alpha_1\lambda) + 2\lambda\beta_1\alpha_1}{2 (1 - \lambda\beta_1 (1 + \alpha_1\lambda))} \leq E(W) \leq . \frac{2\lambda\beta_1^2 + 2\beta_1 \left(1 - e^{-\lambda\alpha_1}\right)}{2 (e^{-\lambda\alpha_1} - \lambda\beta_1)}$$

Proof. The proof is similar to that of the theorem 4.4.2. In addition, if F is L then $\beta_2 \leq 2\beta_1^2$. ■

4.5 Inequalities for the Steady State Distribution of the Server State

Let $P_{(0,0)}^{(i)}$, $P_0^{(i)}$ be the probability that the system is empty and the probability that the server is idle and the system is nonempty respectively in the $i-th$ system, $i = 1, 2$.

Theorem 4.5.1 [126] *If* $\lambda^{(1)} \leq \lambda^{(2)}$, $p^{(1)} \leq p^{(2)}$, $q^{(1)} \leq q^{(2)}$, $\mu^{(1)} \leq \mu^{(2)}$, $r^{(1)} \geq r^{(2)}$, $B^{(1)} \leq_s B^{(2)}$, $C^{(1)} \leq_s C^{(2)}$, $D^{(1)} \leq_s D^{(2)}$ *and* $A^{(1)} \leq_L A^{(2)}$, *then*

$$P^{(1)}_{(0,0)} \geq P^{(2)}_{(0,0)},$$

$$P^{(1)}_{0} \leq P^{(2)}_{0},$$

where $\leq_s$ *is one of the symbols* $\leq_{st}$, $\leq_{icx}$, $\leq_L$

Proof. The measure $P_{(0,0)}$ is increasing with respect to $L_A(.)$ and r and is decreasing with respect to λ, p, q, μ, β_1, γ_1 and η_1 while the measure P_0 is increasing with respect to λ, p, q, μ, β_1, γ_1 and η_1 and decreasing with respect to $L_A(.)$ and r. Under conditions of theorem 4.5.1, the stated inequalities are obtained. ■

Theorem 4.5.2 [126] *For any unreliable* $M/G/1$ *retrial queue*

$$P_{(0,0)} \geq \frac{1-q+qe^{-\lambda\alpha_1}-p\lambda\beta_1\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right]}{(1-q+(q-p)\,e^{-\lambda\alpha_1})(1+\lambda\beta_1\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right])+pe^{-\lambda\alpha_1}}$$

and

$$P_0 \leq \frac{\lambda p\beta_1\left(1-e^{-\lambda\alpha_1}\right)\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right]}{\left(1-q+(q-p)\,e^{-\lambda\alpha_1}\right)\left(1+\lambda\beta_1\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right]\right)+pe^{-\lambda\alpha_1}}$$

If A *is* $\mathcal{L}$ *then*

$$P_{(0,0)} \leq \frac{1+(1-q)\,\lambda\alpha_1-p\lambda\beta_1\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right](1+\lambda\alpha_1)}{\left[1-p+(1-q)\lambda\alpha_1\right]\left(1+\lambda\beta_1\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right]\right)+p}$$

and

$$P_0 \geq \frac{\lambda^2 p\beta_1\alpha_1\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right]}{\left[1-p+(1-q)\lambda\alpha_1\right]\left(1+\lambda\beta_1\left[1+\mu\left(\gamma_1+(1-r)\eta_1\right)\right]\right)+p}$$

Proof. The proof follows the steps given in theorem 4.4.2, and consequently it is omitted. ■

There are theoritical and practical implications of the results. For example, usually the state that the server is idle and the retrial orbit is not empty is not desirable. The system controller, unable to control the arrivals or the retrial attempts, might be able to control the service time, repair time or the reserved time to improve the system performance. If any of these times is shortened stochastically the probability of this less desirable state will be reduced as it is proved in the theorem 4.5.1. If the systtem is completely empty, this is a good opportunity to close the system for maintenance. It is interesting to know how frequently such opportunities occur. If the server is idle and the retrial orbit is not empty, it can be assigned to other tasks while waiting for a customer arrival. We can make the system more efficient if we know how often it is idle.

Remark 4.5.1 *For the measures P_1, $P_{2,0}$, $P_{2,1}$, P_{repair}, P_3 and P_{block}, we have:*

- P_1 *is increasing with respect to* λ, p, q, r, β_1,*and* $L_A(.)$, *decreasing with respect to* μ, η_1 *and* γ_1*;*
- $P_{2,0}$ *and* p_{repair} *are increasing with respect to* λ, p, q, r, β_1, μ, γ_1 *and* $L_A(.)$, *decreasing with respect to* η_1*;*
- $P_{2,1}$ *is increasing with respect to* λ, p, q, β_1, μ, γ_1 *and* $L_A(.)$, *decreasing with respect to* r *and* η_1*;*
- P_3 *is increasing with respect to* λ, p, q, β_1, μ, η_1 *and* $L_A(.)$, *decreasing with respect to* r *and* γ_1*;*
- P_{block} *is increasing with respect to* λ, p, q, β_1, μ, γ_1, η_1 *and* $L_A(.)$, *decreasing with respect to* r*.*

4.6 Numerical Examples

In this section, we give numerical illustration concerning theorems 4.4.2 and 4.5.2. To this end, we consider the model with $\lambda = 0.2$, $\beta_1 = 1$, $\gamma_1 = 2$, $\eta_1 = 1$, and patient customers, i.e. $p = q = 1$. Four distributions of the retrial time are considered $A_1(x)$, $A_2(x)$, $A_3(x)$, $A_4(x)$ which are all in the class $\mathcal{L}$. The parameters r and μ are chosen in such a way that the ergodicity condition holds. Through these examples, we will show how close the bounds are to the exact values. It should be noted that $E(K)$ and $E(L)$ behave in a similar fashion in all aspects, and the distance between the exponential bound of P_0 and its exact value is equal to the distance between the exponential bound of P_{00} and its exact value. In the following Figures, $X^{(Exp)}$, $X^{(D)}$, $X^{(Exact)}$ represent respectively the exponential bound, deterministic bound and exact value of the measure X where $X = P_{00},\quad P_0,\quad E(L)$.

Example 1:

$$A_1(x) = \begin{cases} 0 & \text{if } x < 0.5 \\ 0.5 & \text{if } 0.5 \leq x < 1.5 \\ 1 & \text{if } x \geq 1.5 \end{cases}$$

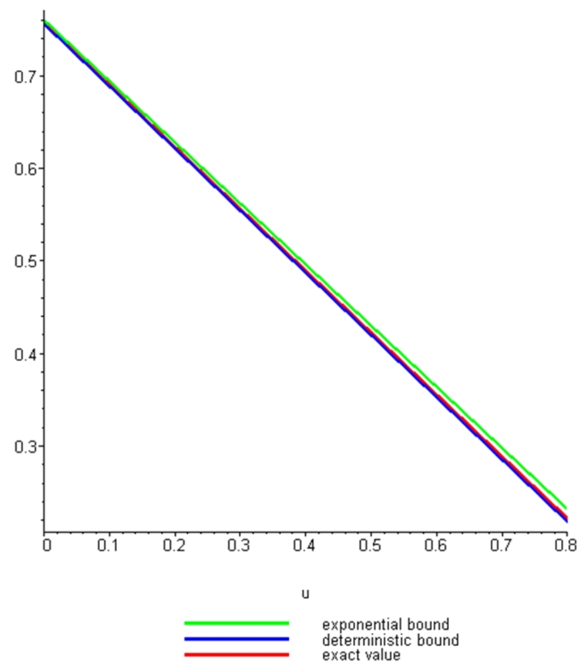

Figure 1: Exact value, upper and lower bounds of P_{00} for $r = 0.25$

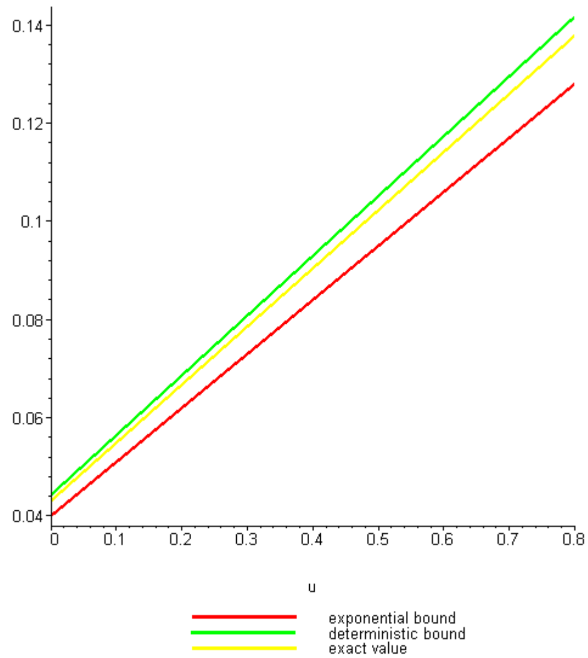

Figure 2: Exact value, upper and lower bounds of P_0 for $r = 0.25$

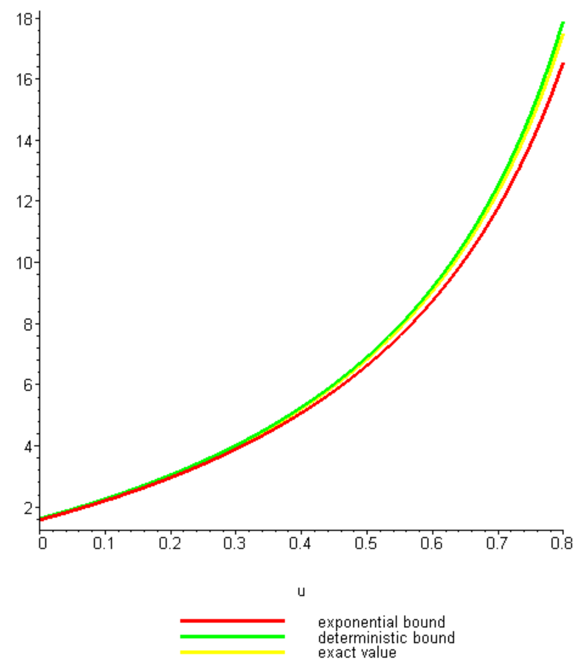

Figure 3: Exact value, upper and lower bounds of $E(L)$ for $r = 0.2.5$

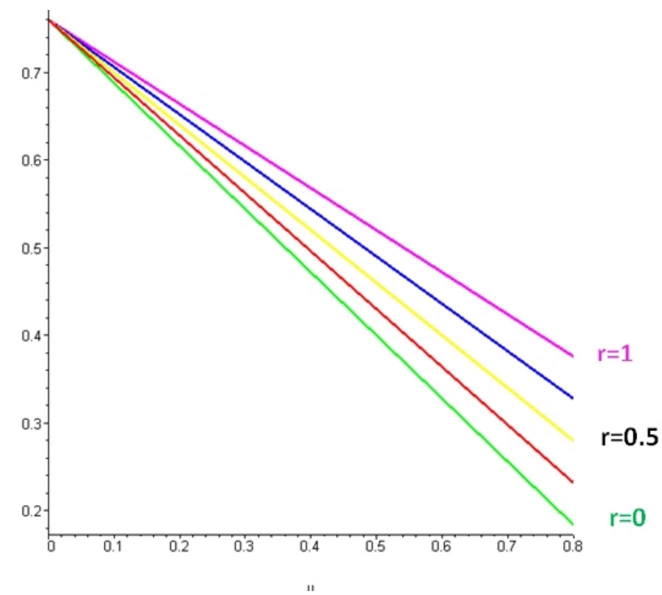

Figure 4: Exponential bound of P_{00} versus μ

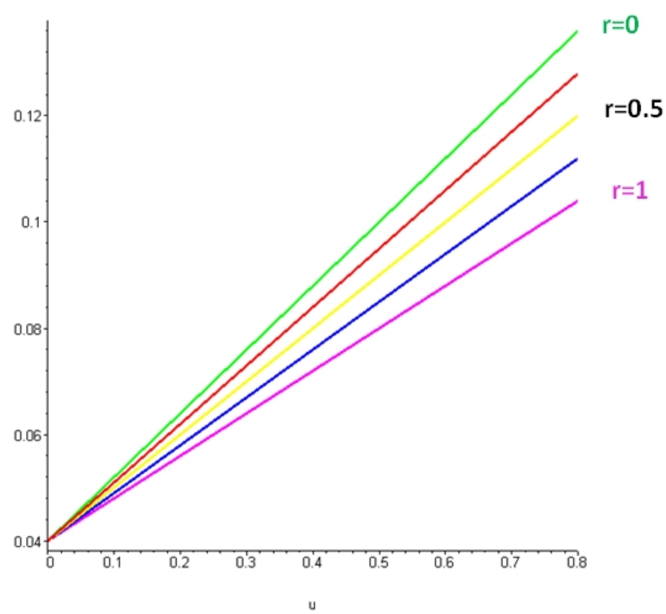

Figure 5: Exponential bound of P_0 versus μ

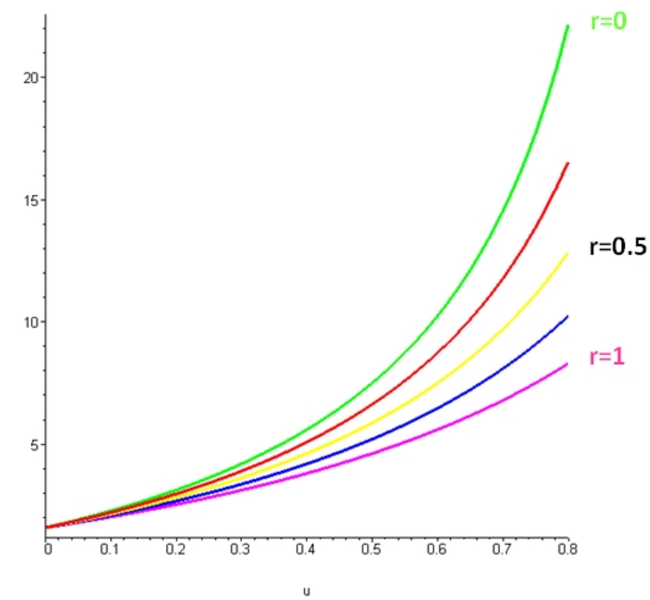

Figure 6: Exponential bound of $E(L)$ versus μ

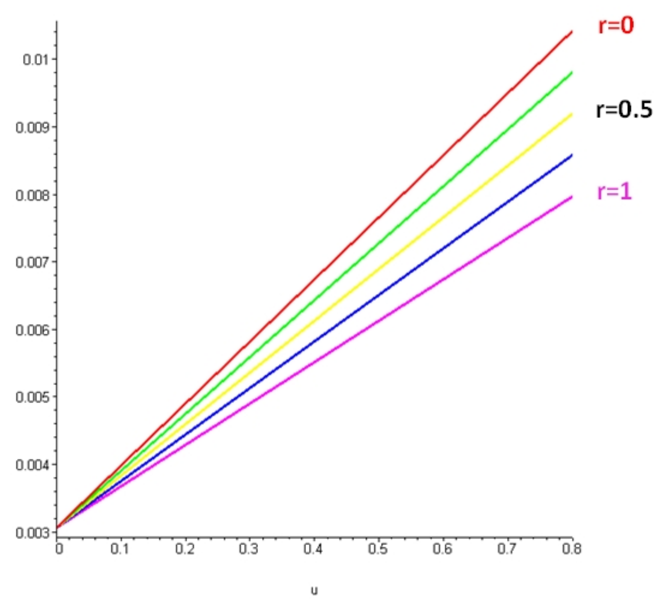

Figure 7: $P_{00}^{(Exp)} - P_{00}^{(Exact)}$ versus μ

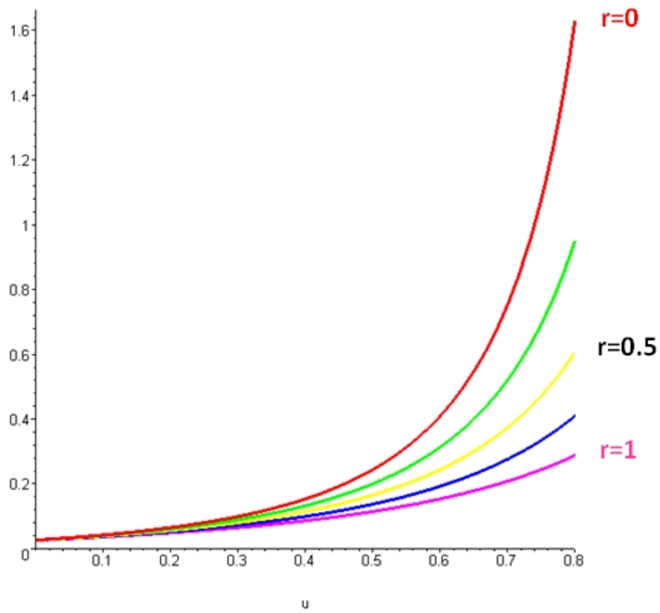

Figure 8: $E(L)^{(Exact)} - E(L)^{(Exp)}$ versus μ

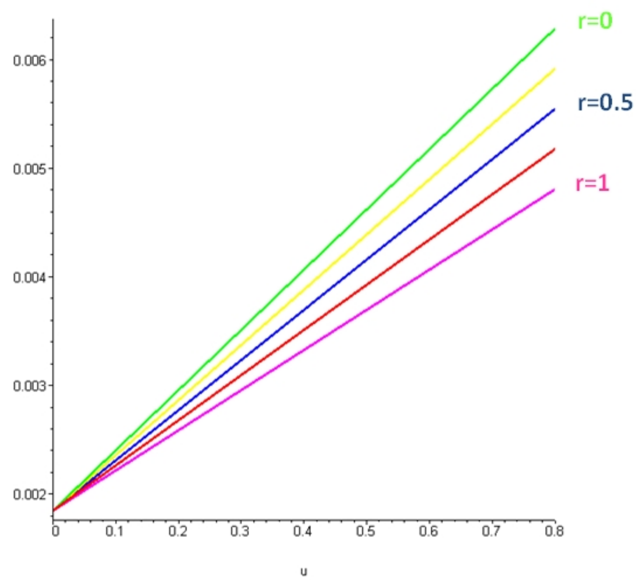

Figure 9: $P_{00}^{(Exp)} - 2P_{00}^{(Exact)} + P_{00}^{(D)}$ versus μ

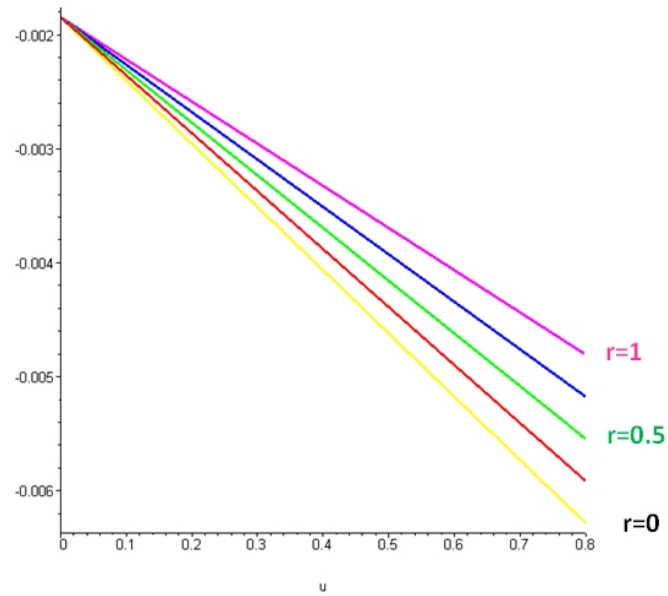

Figure 10: $P_0^{(D)} - 2P_0^{(Exact)} + P_0^{(Exp)}$ versus μ

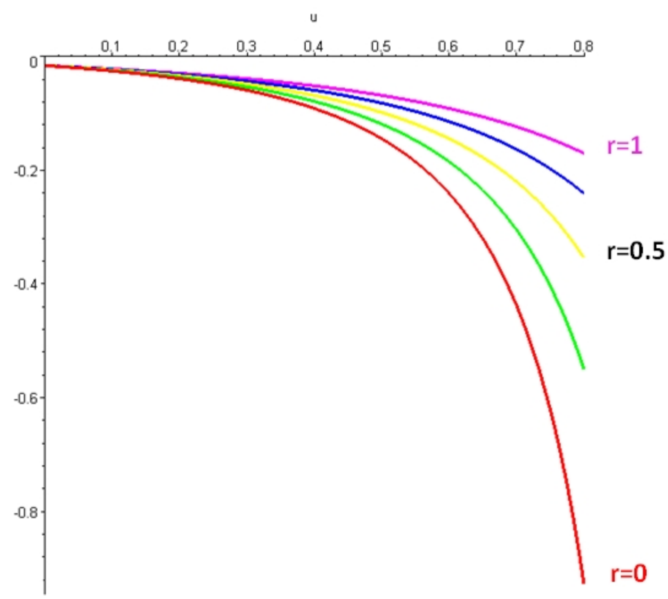

Figure 11: $E(L)^{(D)} - 2E(L)^{(Exact)} + E(L)^{(Exp)}$ versus μ

Example 2: $A_2(x) = \int_0^x \frac{y^{1/2}\left(\frac{3}{2}\right)^{3/2}}{\Gamma\left(\frac{3}{2}\right)} \exp(-\tfrac{3}{2}y)dy, \quad x > 0$

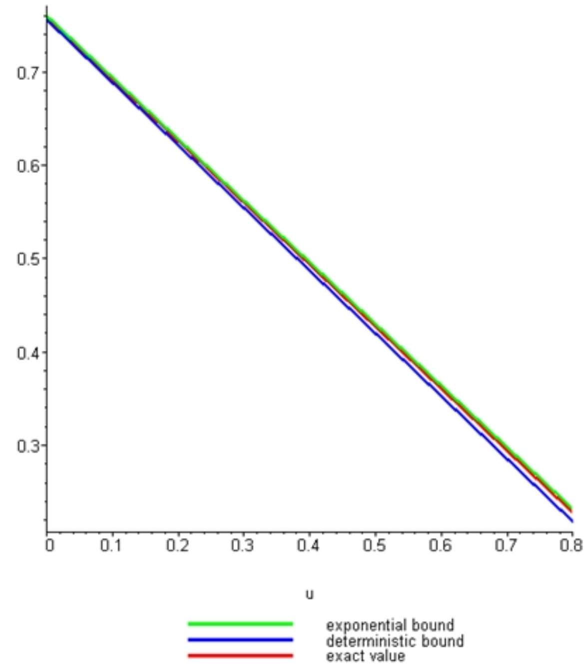

Figure 12: Exact value, upper and lower bounds of P_{00} for $r = 0.25$

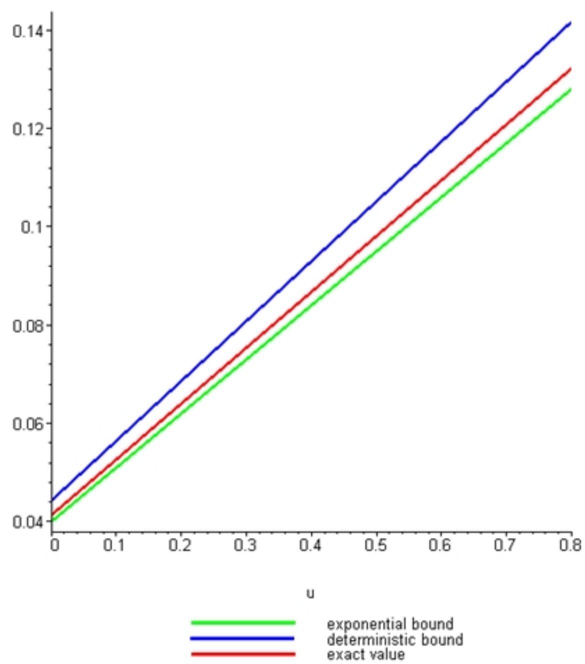

Figure 13: Exact value, upper and lower bounds of P_0 for $r = 0.25$

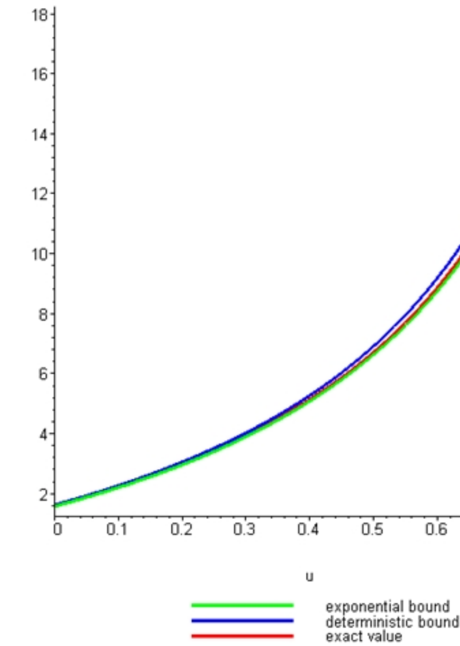

Figure 14: Exact value, upper and lower bounds of $E(L)$ for $r = 0.25$

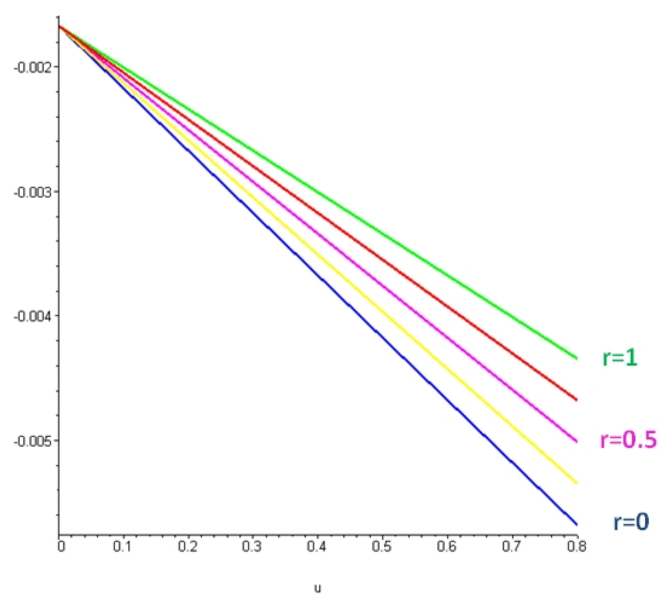

Figure 15: $P_{00}^{(Exp)} - 2P_{00}^{(Exact)} + 2P_{00}^{(D)}$ versus μ

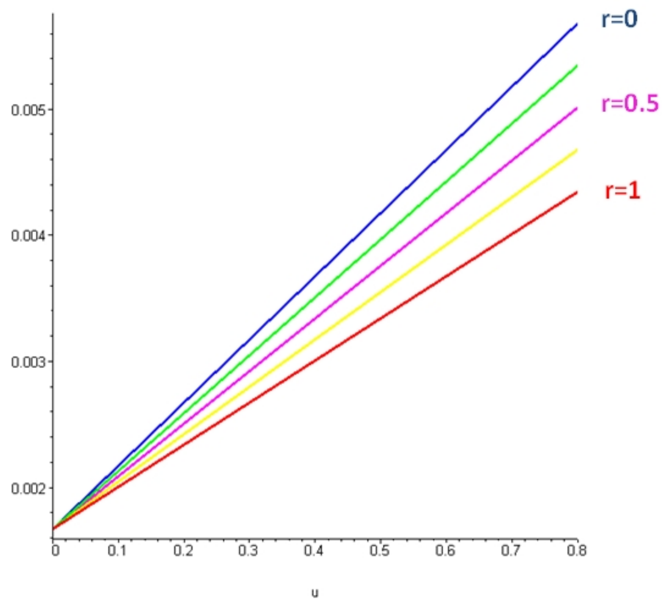

Figure 16: $P_0^{(D)} - 2P_0^{(Exact)} + P_0^{(Exp)}$ versus μ

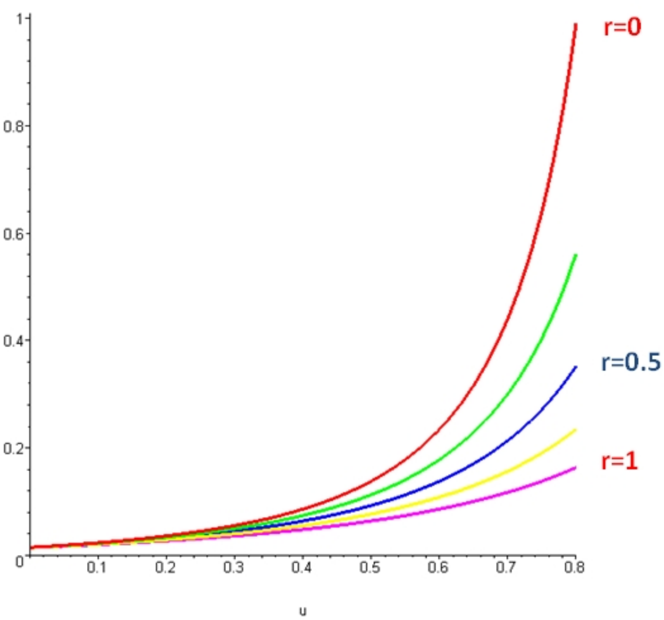

Figure 17: $E(L)^{(D)} - 2E(L)^{(Exact)} + E(L)^{(Exp)}$ versus μ

Example 3: $A_3(x) = \int_0^x \frac{y^2 3^3}{\Gamma(3)} \exp(-3y)dy, \quad x > 0$

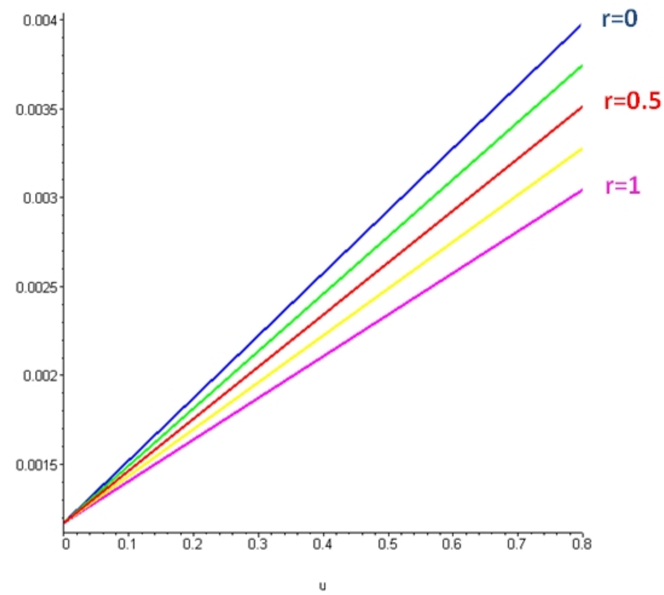

Figure 18: $P_{00}^{(Exp)} - 2P_{00}^{(Exact)} + 2P_{00}^{(D)}$ versus μ

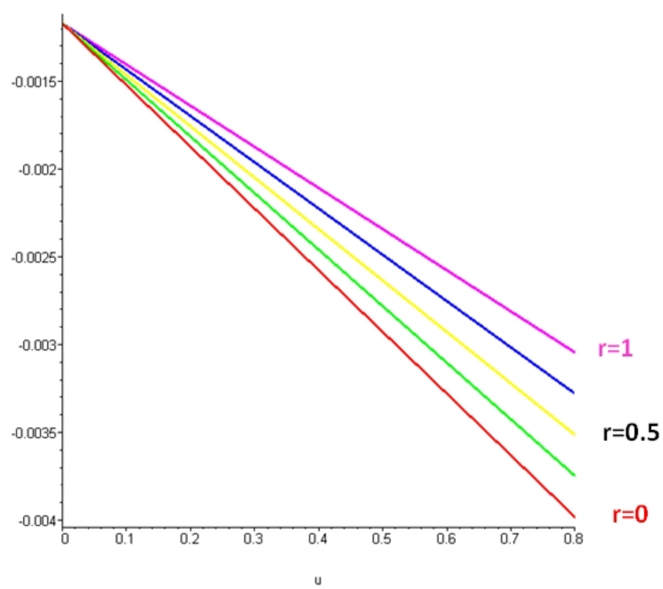

Figure 19: $P_0^{(D)} - 2P_0^{(Exact)} + P_0^{(Exp)}$ versus μ

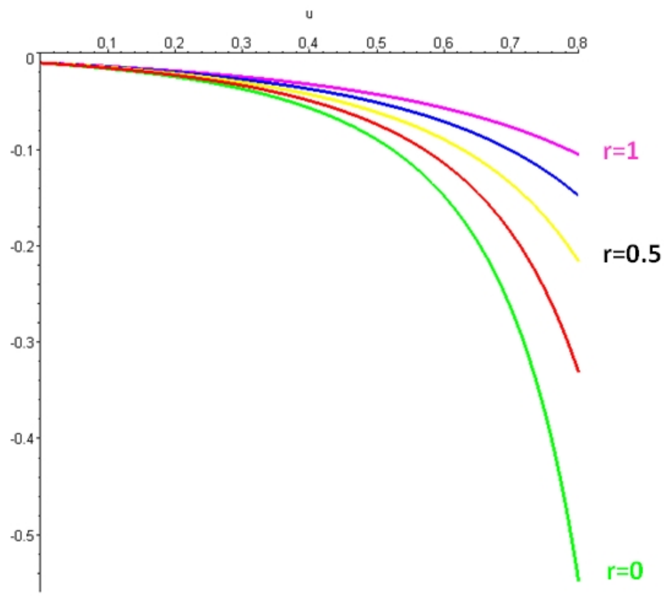

Figure 20: $E(L)^{(D)} - 2E(L)^{(Exact)} + E(L)^{(Exp)}$ versus μ

Example 4: Klar [80]

$$\overline{A}_4(x) = \begin{cases} 1, & x < \dfrac{81}{100} \\ c, & \dfrac{81}{100} \leq x < 3 \\ \dfrac{d}{t^3}, & x \geq 3 \end{cases}$$

where $c = \dfrac{7361}{179361}$ and $d = \dfrac{124}{91}$.

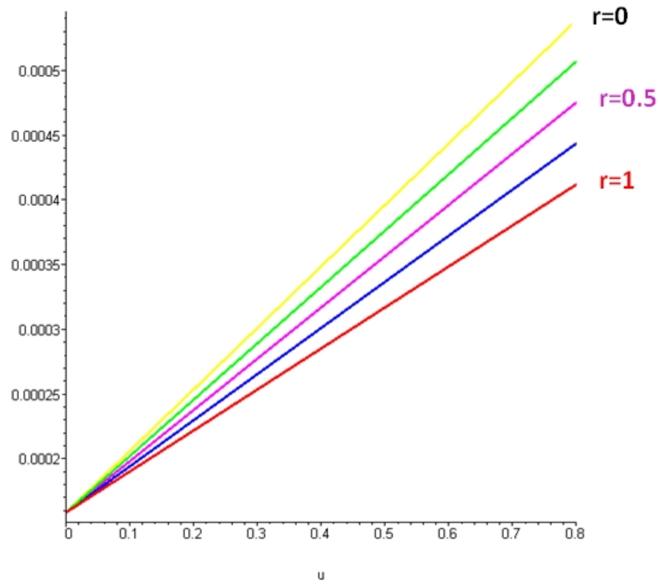

Figure 21: $P_{00}^{(Exp)} - 2P_{00}^{(Exact)} + 2P_{00}^{(D)}$ versus μ

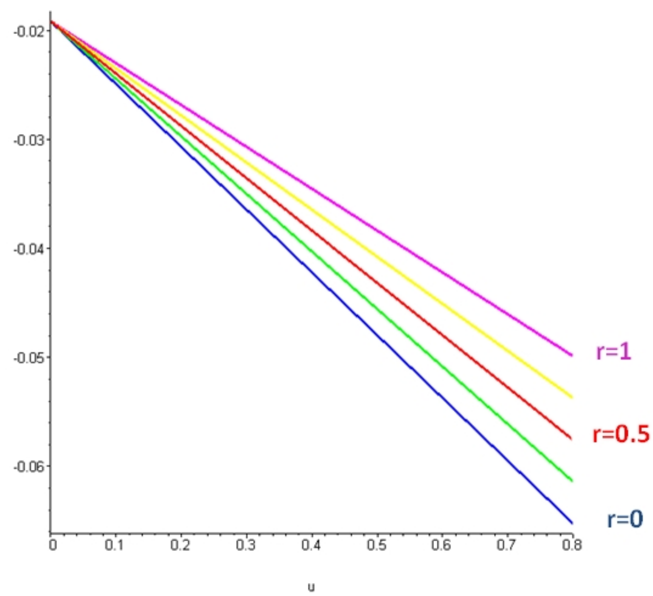

Figure 22: $P_0^{(D)} - 2P_0^{(Exact)} + P_0^{(Exp)}$ versus μ

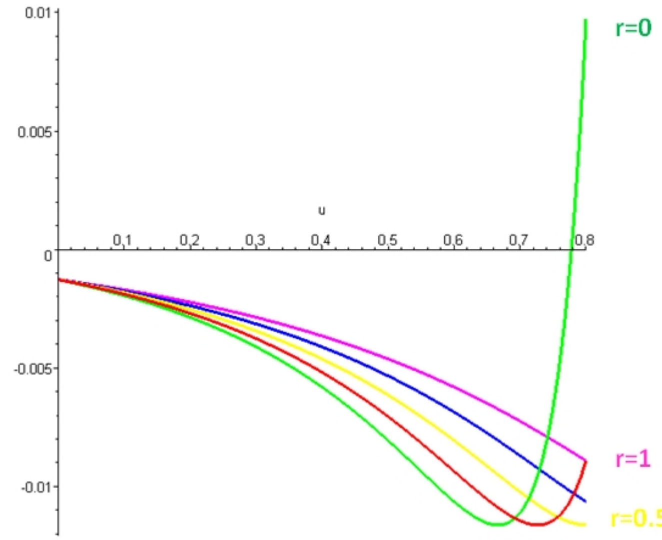

Figure 23: $E(L)^{(D)} - 2E(L)^{(Exact)} + E(L)^{(Exp)}$ versus μ

Comments: We observe from Figures 4-6 that P_{00} is increasing with respect to r, and decreasing with respect to μ, while P_0 and $E(L)$ are increasing as functions of μ and decreasing as functions of r. If r increases and μ decreases, the exponential bounds of $P_{0,0}$, P_0 and $E(L)$ become closer to the exact values (Figures 7-9). It emerges from Figures 10-12 and 18-23 that the deterministic bound is closer to the exact value than the exponential bound for all the measures. If the retrial time has a Gamma distribution with parameters $\left(\frac{3}{2}, \frac{3}{2}\right)$, then the exponential bound is closer to the exact value than the deterministic bound for all the measures as showed in figures 15-17.

Conclusion: If the distribution of the retrial time is close to the exponential distribution in Laplace transform (at point λ), then the exponential bound is closer to the exact value than the deterministic bound. Otherwise, the deterministic bound is better. Besides, the lower and upper bounds seem to be good approximations.

Conclusion

In this thesis we studied a new version of an unreliable retrial queue with geometric loss, FCFS orbit, random reserved time and general retrial time. In a first part, we gave extensive analysis from queueing and reliability viewpoints. We obtained the necessary and sufficient condition for the system to be stable using an embedded Markov chain, the expressions for the partial generating functions of the server and the number of customers in the retrial group and derived some performance measures. We were interested in the study of the joint distribution of the waiting time that a customer spends in the retrial queue and the number of customers served during this period. The generating function Laplace transform of the length of a busy period and the number of customers served during this period was also obtained as well as the corresponding means. We gave the first three probabilities of the number of customers served during the busy period by direct probability statements. From the reliability viewpoint, we analysed the time to the first failure of the server.

On the other hand, we derived monotonicities of the major performance measures in terms of strong stochastic ordering and increasing convex ordering. The model was compared with a simpler counterpart of unreliable $M/M/1$ retrial queue where all distributions are exponential and hence bounds of performance measures are derived. We also discussed the conditions under which the comparison is made. The monotonicity properties are derived via the monotonicity of the embedded Markov chain.

The work suggests several directions of research such as:

- The retrial times were measured from the time of a service completion. This may not be reasonable, since it assumes that the retrial customer has information about the system. In future work, we will assume that the repeated customer can attempt to access the server at any time;

- We have assumed that, after repair, the server is required to search for customer in service orbit. What happens if we allow the customer in service orbit to retry;

- The server is assumed to be subject to active breakdowns. We can introduce the passive breakdowns or vacations;

- The stochastic comparisons have been made in steady state. The transient case can be studied using the sample path approach. The processes can change states only when one of the following events take place in either systems: (i) arrival, (ii) service completion, (iii) failure, (iv) repair completion, (v) reserved completion, (vi) retrial;

- We have assumed that the primary source generates customers with exponential interarrival times. It would be interesting to generalize and exploit the relations (3.6.76) and (3.6.77) to determine bounds and inequalities for the busy period and the number of customers served during this period using ageing distributions.

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