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Study of some contact problems in elasticity and piezoelectricity

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Study of some contact problems in elasticity and piezoelectricity

Abstract

This thesis deals with the study of some boundary value problems describing the frictional contact between a deformable body and a foundation under the assumption of small deformations. In the first part, we consider a mathematical model that describes a bilateral contact between a nonlinear locking material and a rigid foundation. We derive the variational formulation and prove its unique weak solvability. We state an optimal control problem which consists of leading the stress tensor as close as possible to a given target by acting with a control on the boundary of the body. We then study a penalized control problem and prove a convergence result. The second part of the thesis is concerned with the rigorous establishment of the existence and the uniqueness of the solutions for two quasi-static adhesive contact problems with friction between a long-term memory piezoelectric body and an insulating foundation. In both cases we take the adhesion field as the dependent variable, which is governed by an ordinary differential equation. Thus, our main contribution in this part of the thesis is to demonstrate the solvability of the proposed problems.

Keywords: locking material, piezoelectric material, contact, adhesion, variational inequality, optimal control.

Étude de quelques problèmes de contact en élasticité et piézo-élasticité

Résumé

Cette thèse porte sur l'étude de certains problèmes aux limites décrivant le contact avec frottement entre un corps déformable et une fondation. Dans la première partie, nous considérons un modèle mathématique qui décrit un contact bilatéral entre un matériau élastique non-linéaire à verrouillage et une fondation rigide. Nous dérivons la formulation variationnelle et prouvons son unique solvabilité faible. Nous énonçons un problème de contrôle optimal qui consiste à conduire le tenseur des contraintes aussi près que possible d'une cible donnée en agissant avec un contrôle sur la frontière du corps. Nous étudions ensuite un problème de contrôle pénalisé et prouvons un résultat de convergence. La deuxième partie de la thèse concerne l'établissement rigoureux de l'existence et de l'unicité de la solution pour deux problèmes de contact adhésif quasi-statique avec frottement entre un corps piézoélectrique à mémoire à long terme et une fondation isolante. Dans les deux cas, nous prenons le champ d'adhésion comme variable dépendante, qui est régie par une équation différentielle ordinaire. Ainsi, notre principale contribution dans cette partie de la thèse est de démontrer la solvabilité des problèmes proposés.

Mots clés : matériau à verrouillage, matériau piézoélectrique, contact, adhésion, inégalité variationnelle, contrôle optimal.

Contents

List of Symbols

If X is a real Hilbert space and $d \in \mathbf{N}^*$, we use the following notations:

X^d the space defined by $X^d = \left\{x = (x_i)\,|x_i \in X, i = \overline{1,d}\right\}$.

$(.,.)_X$ the inner product in X.

$\|.\|_X$ the norm in X.

$X^{'}$ the dual space of X.

$\langle .,. \rangle_{X' \times X}$ stands for the duality pairing of $X^{'}$ and X.

$\mathcal{P}(X)$ the collection of all subsets of X

If Ω is a domain of $\mathbb{R}^d$ $(d = 2, 3)$, we denote by $\overline{\Omega}$ the adherence of Ω.

$\Gamma = \partial\Omega$ the boundary of Ω.

$\Gamma_i \left(i = \overline{1,3}, a, b\right)$ a part of the boundary Γ.

$meas\ \Gamma_i$ the Lebesgue measure of Γ_i.

ν is the outward unit normal to Γ.

v_ν, v_τ are the normal and tangential components of the vector field v defined on $\overline{\Omega}$.

$\sigma\nu$ is the Cauchy stress vector, σ_ν and σ_τ are respectively its normal and tangential components.

$C^1\left(\overline{\Omega}\right)$ is the space of continuously differentiable real functions on $\overline{\Omega}$.

$D\left(\Omega\right)$ is the space of real functions that are infinitely differentiable and have compact support contained in Ω.

$D^{'}\left(\Omega\right)$ the space of distributions on Ω.

$D = \left[D\left(\Omega\right)\right]^d$.

$D^{'} = \left[D^{'}\left(\Omega\right)\right]^d$.

$\mathcal{D} = \left[D\left(\Omega\right)\right]_s^{d\times d}$.

$\mathcal{D}^{'} = \left[D^{'}\left(\Omega\right)\right]_s^{d\times d}$.

$Q = \left[L^2\left(\Omega\right)\right]_s^{d\times d}$.

$H_1 = \left[H^1\left(\Omega\right)\right]^d$.

$Q_1 = \left\{\sigma \in Q\,|Div\ \sigma \in H\right\}$.

$H^{\frac{1}{2}}\left(\Gamma\right)$ is the Sobolev space of order $\frac{1}{2}$ on Γ.

$H_\Gamma = \left[H^{\frac{1}{2}}\left(\Gamma\right)\right]^d$.

$H_\tau = \left\{\xi \in H_\Gamma\,|\xi_\nu = 0 \text{ a.e. on } \Gamma\right\}$.

$H^{-\frac{1}{2}}(\Gamma)$ the dual space of $H^{\frac{1}{2}}(\Gamma)$.

H'_{Γ} the dual space of $H_{\Gamma} = \left[H^{-\frac{1}{2}}(\Gamma)\right]^d$.

γ : represents the trace operator for vector functions.

da is the surface measure element.

Df is the differential of f.

$\partial_i f$ the partial derivative of f with respect to the ith component x_i.

$\varepsilon(f)$ the symmetric part the gradient of f: $\varepsilon(f) = \frac{1}{2}\left(\nabla f + \nabla^T f\right)$.

$Div\ f$, $div\ f$ the divergence of f.

∂f the subdifferential of the function f

$\ddot{u}$ ($respectively,\ \dot{u}$) represents the second derivative (respectively, first derivative) of the displacement field with respect to time.

$\liminf$ represents the lower limit.

$\limsup$ represents the upper limit

$\operatorname{ess\,sup}$ the essential supremum.

$\mathbf{S}_d$ represents the space of second-order symmetric tensors on $\mathbb{R}^d$.

c is a generic strictly positive constant.

a.e. almost everywhere.

δ_{ij} the Kronecker delta

$r_+ = max\{r, 0\}$ the positive part of r

Introduction

Phenomena related to contact problems involving deformable bodies are common in everyday life and present in many industrial applications. Given the growing needs of industries faced with contact issues in mechanical systems, the contact between rigid or deformable solids has been a subject of significant scientific investment in recent years. The modeling of contact problems between two deformable bodies or between a deformable body and a rigid base essentially depends on the mechanical properties of the materials considered, as well as the contact boundary conditions. Among the various types of problems considered, unilateral or bilateral contact problems can be mentioned, with or without friction, for different types of bodies (elastic, viscoelastic, viscoplastic, ...) and particularly the case of locking materials. A locking material is material which is characterized by the fact that it is deformed under the effect of an external force but the deformation cannot continue when it reaches a certain value. The material is elastic if the deformation remains bounded and returns back to its initial shape if we stop to exercise any external force on it. These materials are part of a class of hyperelastic materials (Figure 1) in which the strain tensor is constrained to stay in a given convex set. The study of elastic materials with locking effects was first introduced in [92], [93], [95]. There, the constitutive law of such materials was derived and different mechanical interpretations have been presented. The theoretical study of variational problems of locking materials was introduced in [26] and [27].

Figure 1: Rigid rubber, polyurethane technical parts, titanium alloys.

Taking into account various conditions associated with increasingly complex behavior laws leads to the introduction of new models, especially in the realm of so-called piezoelectric contact problems, which consider the interaction between mechanical and electrical properties. In solid-state physics, piezoelectricity refers to materials that become electrically polarized when subjected to mechanical stresses and, conversely, change shape under an applied electric field. It was first discovered and explained experimentally by the Curie brothers in 1880. Subsequently, in 1910, a study of classes of asymmetric crystals and their piezoelectric properties was published by Woldemar Voigt [121].

In 1921, a significant improvement facilitated by piezoelectricity was achieved for the first time through the development of the electric quartz oscillator, and by the late 1930s, all high-frequency radio transmitters were under crystal control. The enduring success of this straightforward concept, even a century later, has solidified the quartz crystal-controlled oscillator as the secondary standard for synchronization and frequency control, [19]. In addition to crystals, piezoelectricity encompasses two other classes: one includes polymers made from materials such as rubber fibers, wool, hair, wood, and silk; the other includes ceramics that comprise various elements like barium titanates, lead titanates, and the family of PZT (lead zirconate titanate) (Figure2). Due to their broad range of applications, significant progress has been made in the processing and development of piezoelectric materials. A more detailed presentation of piezoelectricity can be found in reference [11], [12], [62], [82], [83], [111].

Figure 2: Quartz oscillator and crystal-controlled, (PZT), piezo-single-crystal.

However, in many situations, contact can also be accompanied by another phenomenon called adhesion. Adhesion is an interface phenomenon that occurs when movement is present, and an adhesive is introduced to prevent surfaces from undergoing relative motion. Several theories of adhesion exist, and various cohesive zone models have been established. More local formulations, derived from contact mechanics, have subsequently enabled the description of the evolution of the decohesion phenomenon. A modern theory of contact with adhesion, derived from continuum mechanics, is proposed by Frémond [40]–[43]. This theory builds upon the conditions of unilateral contact by Hertz-Signorini-Moreau and introduces an additional internal surface variable, ranging between zero and one, to describe the fractional density of active bonds.

The mathematical literature dedicated to the study of contact phenomena in general encompasses modeling, variational analysis, and numerical examination of models. One of the earliest mathematical publications regarding this topic is the paper by Signorini [102] in which the problem of unilateral contact (referred to as the Signorini problem) between a linearly elastic body and a rigid foundation is formulated. The associated variational formulation for this type of condition was mathematically studied by Fechera [39], where the Signorini problem has been resolved. This was followed by the work of Duvaut and Lions [33], who introduced friction to contact problems. The Signorini problem has been explored by other authors for various constitutive laws, with mathematical analysis results established by Hlaváček et Nečas [88], Kikuchi and Oden [65]. Unilateral contact problems with Coulomb friction have been investigated by Duvautt [32], Cocou [25], Shillor and Sofonea [103], Rocca [98] and Klarbring and Miklic [66]. In order to account for cases where the interfaces in contact are assumed to be penetrable, the normal compliance law was initially considered in [75], This case is also considered by Anderson [4], Motreanu-Sofonea [87] and Touzaline [118]. To the modeling and variational analysis is added the analysis and numerical implementation for a complete study of contact phenomena, see for example, [47], [56], and [54].

In addition to the numerical investigation of contact problems, there is also current interest in the optimal control study of such problems. It's worth noting that the theory of optimal control for variational inequalities is well-developed, as illustrated for example in [45] and [80]. In [81], the study of optimal control for linear or nonlinear elliptic problems and variational inequalities is addressed. However, optimal control problems for contact models are of significant importance, although they are not extensively developed, as mentioned in [1], [9], [10], [17], [20], [21], [28], [29], [67], [72], [89], [116], [117] and the references therein. Recently, in [77], [78], two optimal control problems for models of elastic frictional contact have been investigated. In particular, in [78], the authors studied the optimal control of a frictional contact problem with normal compliance.

Intense research efforts have also been devoted to the study of contact problems involving adhesive and piezoelectric models. Several authors have considered frictional contact problems involving the Signorini condition with or without adhesion, as studied in [30], [34], [35], [96], [113]. General models for linearly elastic materials with piezoelectric effects can be found in [61], [107], and [106], as well as models for viscoelastic adhesive materials and models with piezoelectric effects, as demonstrated in [7], [22], [58], [59], [71], [103]. More recent results on variational and numerical analysis can be found in [18], [38], [55], [70], [79], [101], [118], [119].

The subject of this thesis is the study of certain boundary problems involving contact with friction between a deformable body and a foundation under the assumption of small deformations. In the first part, we consider bilateral contact with Tresca's friction law between a locking material and a rigid foundation. The objective is to investigate an optimal control problem that aims to lead the stress tensor as close as possible to a given target by applying control on the body's boundary.

The goal of the second part is to rigorously establish the existence and uniqueness of the solution for two electro-elastic quasi-static contact problems with long-term memory. In both problems studied, we use the adhesion field as a dependent variable, with its evolution described by an ordinary differential equation. Thus, the main contribution of this part consists in proving the solvability of the two proposed models.

This thesis consists of three chapters. The first one is dedicated to the introduction and mathematical formulation of the mechanical problems under consideration. After introducing functional spaces, especially Sobolev-type spaces, and recalling trace theorems and essential properties, we also review standard abstract results concerning the existence and uniqueness of elliptic variational inequalities.

The second chapter deals with the optimal control of a contact problem. The contact is assumed to be static and described by Tresca's law of friction between a nonlinear elastic locking body and a rigid foundation. First, we derive a variational formulation of the problem and establish the existence and uniqueness result. Next, we define the optimal control problem related to this model, which consists in minimizing a cost functional, and prove the existence of a solution to this problem. We define a penalized and regularized problem, whose solution converges to the solution of the variational problem. Finally, we introduce a penalized and regularized optimal control problem, and we prove its convergence to the optimal control problem.

In the third chapter, we focus on the study of quasi-static adhesive contact processes with friction between a long-term memory piezoelectric body and an insulating foundation.

We consider a contact case modeled with a unilateral constraint and a non-local friction law, as well as another case modeled with a normal compliance condition and the associated Coulomb dry friction law, in which the adhesion of contact surfaces is taken into account. We establish a variational formulation in the form of a system involving displacement, stress field, electric displacement, electric potential, and adhesion field. We then prove that each problem admits a unique weak solution by demonstrating several technical lemmas and utilizing arguments involving variational inequalities, monotone operators, differential equations, and Banach's fixed-point theorem.

Relevant Papers

1- Guettaf Rachid and Touzaline Arezki. *Analysis of a contact problem with adhesion for electro-viscoelastic materials with long memory.* Rev. Roum. Math. Appl. 58(1), 67–84 (2013)

2- Guettaf Rachid and Touzaline Arezki. *Analysis of a frictional unilateral contact problem for piezoelectric materials with long-term memory and adhesion.* Mem. Differ. Equ. Math. Phys. Vol 83. (2021), 55–70.

3- Guettaf Rarchid and Touzaline Arezki. *Optimal control of a frictional contact problem for locking materials.* Rad Hrvat. Akad. Znan. Umjet. Mat. Znan

Introduction générale

Les phénomènes liés aux problèmes de contact impliquant des corps déformables sont courants dans la vie quotidienne et sont présents dans de nombreuses applications industrielles. Compte tenu des besoins croissants des industriels confrontés aux problèmes de contact dans les systèmes mécaniques, le contact entre solides rigides ou déformables a fait l'objet d'un investissement scientifique important ces dernières années. La modélisation des problèmes de contact entre deux corps déformables ou entre un corps déformable et une base rigide dépend essentiellement des propriétés mécaniques des matériaux considérés, ainsi que des conditions aux limites du contact. Parmi les différents types de problèmes considérés, on peut citer les problèmes de contact unilatéral ou bilatéral, avec ou sans frottement, pour différents types de corps (élastiques, viscoélastiques, viscoplastiques, ...) et en particulier le cas des matériaux à verrouillage. Un matériau à verrouillage est un matériau caractérisé par le fait qu'il se déforme sous l'effet d'une force extérieure, mais que la déformation ne peut se poursuivre lorsqu'elle atteint une certaine valeur. Le matériau est élastique si la déformation reste limitée et reprend sa forme initiale si l'on cesse d'exercer une force extérieure sur lui. Ces matériaux font partie d'une classe de matériaux hyperélastiques dans lesquels le tenseur de déformation est contraint de rester dans un ensemble convexe donné. L'étude des matériaux élastiques à verrouillage a été introduite pour la première fois dans [92], [93], [95] où la loi constitutive de ces matériaux a été dérivée et différentes interprétations mécaniques ont été présentées. L'étude théorique des problèmes variationnels des matériaux à effet de verrouillage a été introduite par Demengel [26], [27].

La prise en compte de diverses conditions associées à des lois de comportement de plus en plus complexes conduit à l'introduction de nouveaux modèles, en particulier dans le domaine des problèmes de contact dits piézoélectriques, qui considèrent l'interaction entre les propriétés mécaniques et électriques. En physique du solide, la piézoélectricité désigne les matériaux qui se polarisent électriquement lorsqu'ils sont soumis à des contraintes mécaniques et, inversement, changent de forme sous l'effet d'un champ électrique appliqué. Elle a été découverte et expliquée expérimentalement pour la première fois par les frères Curie en 1880. Par la suite, en 1910, Woldemar Voigt [121] a publié une étude sur les classes de cristaux asymétriques et leurs propriétés piézoélectriques.

Au début des années 1920, un progrès significatif a été réalisé grâce à la piézoélectricité avec le développement d'un oscillateur électrique en quartz, permettant un contrôle précis des fréquences des émetteurs radio. Depuis les années 1930, tous les émetteurs radio à haute fréquence utilisent le contrôle par cristal. Ce succès durable a solidifié l'oscillateur contrôlé par cristal de quartz comme le standard secondaire pour la synchronisation et le contrôle de fréquence [19]. En plus des cristaux, la piézoélectricité comprend deux autres classes de matériaux: les polymères fabriqués à partir de matières telles que les fibres de caoutchouc, la laine, les cheveux, le bois et la soie; et les céramiques qui comprennent plusieurs éléments comme les titanates de baryum, les titanates de plomb et la famille des PZT (plomb titanate de zirconate). Grâce à leur large éventail d'applications, des progrès importants ont été réalisés dans le traitement et le développement des matériaux piézoélectriques. Des références plus détaillées sur la piézoélectricité peuvent être trouvées dans [11], [12], [62], [82], [83], [111].

Cependant, dans de nombreuses situations, le contact peut également s'accompagner d'un autre phénomène appelé adhésion. L'adhésion est un phénomène d'interface qui se produit lorsqu'il y a un mouvement et qu'un adhésif est introduit pour empêcher les surfaces de subir un mouvement relatif. Plusieurs théories de l'adhésion existent et divers modèles de zone cohésive ont été établis. Des formulations plus locales, dérivées de la mécanique du contact, ont ensuite permis de décrire l'évolution du phénomène de décohésion. Une théorie moderne du contact avec adhésion, dérivée de la mécanique des milieux continus, est proposée par Frémond [40], [43]. Cette théorie repose sur les conditions de contact unilatéral de Hertz-Signorini-Moreau et introduit une variable interne de surface supplémentaire variant entre zéro et un, pour décrire la densité fractionnaire des liaisons actives.

La littérature mathématique dédiée à l'étude des phénomènes de contact en général comprend la modélisation, l'analyse variationnelle et l'examen numérique des modèles. L'une des premières publications mathématiques sur ce sujet est l'article de Signorini [102] dans lequel est formulé le problème du contact unilatéral (appelé problème de Signorini) entre un corps élastique linéaire et une fondation rigide. La formulation variationnelle associée à ce type de condition a été étudiée mathématiquement par Fechera [39] qui a résolu le problème de Signorini. Cette étude a été suivie par les travaux de Duvaut et Lions [33] qui ont introduit le frottement dans les problèmes de contact.

Le problème de Signorini a été étudié par d'autres auteurs pour diverses lois constitutives, avec des résultats d'analyse mathématique établis par Hlaváček et Nečas [88], Kikuchi et Oden [65]. Les problèmes de contact unilatéral avec frottement de Coulomb ont été étudiés par Duvaut [32], Cocou [25], Shillor and Sofonea [103], Rocca [98]. Pour tenir compte des cas où les interfaces en contact sont supposées pénétrables, la loi de compliance normale a été initialement considérée dans Martins [75], ce cas est également considéré par Anderson [4], Motreanu-Sofonea [87] and Touzaline [118]. En plus de la modélisation et de l'analyse variationnelle, l'analyse et la mise en œuvre numérique sont nécessaires pour une étude complète des phénomènes de contact. L'analyse numérique des modèles est destinée à l'étude des schémas discrétisés associés aux formulations faibles, le sujet est abordé dans le cas d'un corps élastique dans par exemple, [47], [56] et [54].

Outre l'étude numérique des problèmes de contact, il existe également un intérêt actuel pour l'étude du contrôle optimal de tels problèmes. Il convient de noter que la théorie du contrôle optimal pour les inéquations variationnelles est bien développée, comme illustré par exemple dans [45] et [80]. Dans [81], l'étude du contrôle optimal de problèmes elliptiques linéaires ou non linéaires et d'inéquations variationnelles est abordée. Cependant, les problèmes de contrôle optimal pour les modèles de contact sont d'une importance significative, bien qu'ils ne soient pas largement développés, comme mentionné dans [1], [9], [10], [17], [20], [21], [28], [29], [67], [72], [89], [116], [117] et les références qui y sont citées. Récemment, dans [77] et [78], deux problèmes de contrôle optimal pour des modèles de contact élastique avec frottement ont été étudiés. En particulier, dans [78], les auteurs ont étudié le contrôle optimal d'un problème de contact avec frottement et compliance normale.

Plusieurs recherches ont été réalisées sur les problèmes de contact impliquant des modèles adhésifs et piézoélectriques. Certains auteurs ont étudié des problèmes de contact avec frottement en prenant en compte la condition de Signorini avec ou sans adhésion, comme cela a été étudié dans les travaux de chercheurs tels que [30], [34], [35], [96], [113]. Des modèles généraux pour des matériaux élastiques linéaires avec effets piézoélectriques ont été développés par [61], [107] et [106], ainsi que des modèles pour des matériaux adhésifs viscoélastiques et des modèles avec effets piézoélectriques dans les travaux [7], [22], [58], [59], [71], [103]. Des résultats plus récents sur l'analyse variationnelle et numérique peuvent être trouvés dans les travaux de plusieurs chercheurs tels que [18], [38], [55], [70], [79], [101], [118], [119].

Cette thèse porte sur l'étude de problèmes de contact avec frottement entre un corps déformable et une fondation sous l'hypothèse des petites déformations. Dans la première partie, nous examinons un contact bilatéral avec la loi de frottement de Tresca entre un matériau à verrouillage et une fondation rigide.

L'objectif est de résoudre un problème de contrôle optimal qui consiste à rapprocher autant que possible le tenseur de contrainte d'une cible prédéfinie en appliquant un contrôle sur la frontière du corps. La deuxième partie de la thèse vise à justifier rigoureusement l'existence et l'unicité de la solution pour deux problèmes électro-élastiques quasi-statiques à mémoire à long terme. Dans ces deux problèmes, le champ d'adhésion est utilisé comme une variable dépendante, et son évolution est décrite par une équation différentielle ordinaire. Ainsi, notre contribution principale dans cette partie consiste à démontrer la solvabilité de deux modèles proposés.

Cette thèse se compose de trois chapitres. Le premier est consacré à l'introduction et à la formulation mathématique des problèmes mécaniques considérés. Après avoir introduit les espaces fonctionnels, en particulier les espaces de type Sobolev et rappelé les théorèmes de trace et les propriétés essentielles, nous passons également en revue quelques résultats abstraits standard concernant l'existence et l'unicité des inégalités variationnelles elliptiques.

Le deuxième chapitre est consacré au contrôle optimal d'un problème de contact. Le contact est supposé être statique et décrit par la loi de frottement de Tresca entre un corps à verrouillage élastique non linéaire et une fondation rigide. Tout d'abord, nous dérivons une formulation variationnelle du problème et nous établissons un résultat d'existence et d'unicité. Ensuite, nous définissons le problème de contrôle optimal lié à ce modèle, qui consiste à minimiser une fonctionnelle de coût et nous prouvons l'existence d'une solution à ce problème. Nous définissons également un problème pénalisé et régularisé, dont la solution converge vers la solution du problème variationnel. Enfin, nous introduisons un problème de contrôle optimal pénalisé et régularisé, et nous prouvons sa convergence vers le problème de contrôle optimal.

Dans le troisième chapitre, nous nous penchons sur l'étude des processus de contact adhésif quasi-statique avec frottement entre un corps piézoélectrique à mémoire à long terme et une fondation isolante. Nous considérons un cas de contact modélisé avec une contrainte unilatérale et une loi de frottement non locale, ainsi qu'un autre cas modélisé avec une condition de compliance normale et une loi de frottement sec de Coulomb associée dans laquelle l'adhésion des surfaces de contact est prise en compte. Nous établissons une formulation variationnelle sous la forme d'un système impliquant le déplacement, le champ de contrainte, le déplacement électrique, le potentiel électrique et le champ d'adhésion. Nous prouvons ensuite que chaque problème admet une solution faible unique en démontrant plusieurs lemmes techniques et en utilisant des arguments impliquant des inégalités variationnelles, des opérateurs monotones, des équations différentielles et le théorème du point fixe de Banach.

Chapitre 1

Formulation of Problems and Preliminaries

Contents

This chapter is dedicated to the mathematical formulation and introduction of the mechanical problems that are of interest to us. The first section involves an overview of the constitutive laws for locking materials and piezoelectric materials. We define the boundary conditions, frictional conditions for a deformable body in adhesive contact with a foundation, and the electrical conditions at the contact surface that we will use later. In the second section, we introduce functional spaces, particularly Sobolev-type spaces, while recalling trace theorems and fundamental properties of the main functional spaces that play a role in the theory of continuum mechanics. We also recall standard abstract results related to the existence and uniqueness of variational inequalities. Finally, in the last section, we present a version of the Cauchy-Lipschitz theorem and include the Gronwall lemmas. Additionally, to fulfill the requirements of Chapter 2, we will review a few notions of convex analysis.

1.1 Formulation of mechanical problems

A solid is a set of particles or material points in a given space. Each point of the solid occupies a point of space $\mathbb{R}^d\,(d = 2, 3)$ and the body is associated with a configuration represented by the domain $\Omega \subset \mathbb{R}^d$. When subjected to forces, a non-rigid body undergoes deformation in a specific manner. In the following, we will consider a deformable body occupying a bounded domain Ω of $\mathbb{R}^d\,(d = 2, 3)$, with the boundary $\partial\Omega = \Gamma$ assumed to be regular. Over a time interval $[0, T]$, we will study the evolution of the body due to the application of volume and surface forces.

1.1.1 Deformation tensor

Tensors, particularly second-order tensors, play a crucial role in continuum mechanics. This arises from the necessity to describe physical phenomena in a manner that is independent of the chosen coordinate system. When a body is subjected to external traction and volume forces, it undergoes deformation, i.e. every point inside the body will be in a different position after deformation.

Let us denote by $\mathbf{S}_d$ the space of symmetric second-order tensors on $\mathbb{R}^d$, we denote by "." and $\|.\|$ the inner product and Euclidean norm on $\mathbb{R}^d$ and $\mathbf{S}_d$ respectively. Thus,

$$
\begin{aligned}
u.v &= u_i v_i, \quad \|v\| = (v.v)^{\frac{1}{2}} \quad \forall u, v \in \mathbb{R}^d. \\
\sigma.\tau &= \sigma_{ij}\tau_{ij}, \quad \|\tau\| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \sigma, \tau \in \mathbf{S}_d.
\end{aligned}
$$

Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. We consider elastic solids under the assumption of small deformations. *The linearized strain tensor* is defined by:

$$
\varepsilon(u) = (\varepsilon_{ij}(u)), \, \varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \quad \text{in } \Omega \times (0, T) \tag{1.1.1}
$$

where ∂_j is the partial derivative operator with respect to variable x_j.

1.1.2 The equation of motion

In the study of a physical system, the initial step involves formulating the equation of motion for the bodies under consideration and mathematically expressing the boundary conditions.

The evolution of the body due to the application of volume and surface forces, as well as the influence of electric charges, within a time interval $[0, T]$, is described by:

$$\rho\ddot{u} = Div\ \sigma + f_0 \text{ in } \Omega \times (0, T)\,, \tag{1.1.2}$$

$$div\ D = q_0 \text{ in } \Omega \times (0, T)\,, \tag{1.1.3}$$

wheres (1.1.2) is the equation of motion describing the evolution of the displacement u and (1.1.3) is the equation describing the evolution of the electric displacement $D = (D_1, ..., D_d)$, where the mass density $\rho : \Omega \to \mathbb{R}_+$, the density of volume forces $f_0 : \Omega \times [0, T] \to \mathbb{R}^d$ and $q_0 : \Omega \times [0, T] \to \Omega$ are data for the problem, which consists of finding:
the displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, *the stress field* $\sigma : \Omega \times [0, T] \to \mathbf{S}_d$ *and the electric displacement field* $D : \Omega \times [0, T] \to \mathbb{R}^d$ verifying (1.1.2) and (1.1.3).
Evolutionary processes modeled by the equation (1.1.2) are called dynamic processes.
In the case where the velocity field $\dot{u}$ varies very slowly with respect to time, the $\rho\ddot{u}$ term becomes negligible. This is a quasistatic process, in which case equations (1.1.2) and (1.1.3) can be written:

$$Div\ \sigma + f_0 = 0 \text{ in } \Omega \times (0, T)\,. \tag{1.1.4}$$

$$div\ D = q_0 \text{ in } \Omega \times (0, T)\,. \tag{1.1.5}$$

The equation (1.1.4) is called *the equilibrium equation.*
In the static case the second member of the equation (1.1.2) is identically zero, in which case we look for *the displacement field* $u : \Omega \to \mathbb{R}^d$, *the stress field* $\sigma : \Omega \to \mathbf{S}_d$ *and the electric displacement field* $D : \Omega \to \mathbb{R}^d$ such that:

$$Div\ \sigma + f_0 = 0 \text{ in } \Omega. \tag{1.1.6}$$

$$div\ D = q_0 \text{ in } \Omega. \tag{1.1.7}$$

Remark 1.1. *A solution from (1.1.4) and (1.1.5) with the degree of regularity necessary for all derivation operations to be permitted is said to be regular.*

The equations above are not sufficient on their own to describe the motion of continuous media. Indeed, they need to be complemented by a material-specific description. This is the purpose of the constitutive law, which we will briefly introduce.

1.1.3 Constitutive law

In any elastic or piezoelectric material, the application of certain forces results in certain deformations. By understanding the correlation between the two, we can characterize and predict the mechanical behavior of the material. This correlation, often established through experimentation, is referred to as the constitutive law. Below, we present the constitutive laws discussed in this thesis, corresponding to a case of locking material and a specific category of materials known as electro-viscoelastic materials.

Constitutive laws for locking materials

$$\sigma(u) \in \mathcal{F}\varepsilon(u) + \partial I_B(\varepsilon(u)) \text{ in } \Omega, \tag{1.1.8}$$

where $\mathcal{F}$ is a given nonlinear function and I_B is the indicator function of the set defined by

$$B = \{\xi \in S_d; \ \|\xi\| \leq M_L\}$$

and we have

$$\begin{cases} I_B(\xi) = 0, & \text{if } \xi \in B, \\ I_B(\xi) = +\infty, & \text{if } \xi \notin B \end{cases} \quad \text{for } \xi \in S_d.$$

Electro-viscoelastic constitutive law with long memory

First, note that for electro-viscoelastic materials with short memory, the constitutive law takes the form:

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}\varepsilon(u) - \mathcal{E}^*E(\varphi) \qquad \text{in } \Omega \times (0,T),$$

$$D = \mathcal{E}\varepsilon(u) + \mathcal{C}E(\varphi) \qquad \text{in } \Omega \times (0,T),$$

where u, φ, $\varepsilon(u)$, $E(\varphi) = -\nabla\varphi$ are *the displacement field, the electric field, the linearized deformation tensor and the electric field*, respectively; $\mathcal{A}$ is the viscosity tensor and $\mathcal{B}$ is the elasticity tensor, which is not necessarily linear, $\mathcal{E} = (e_{ijk})$ is the third order piezoelectric tensor expressing the proportionality between charge and deformation at constant or zero field, $\mathcal{E}^* = (e^*_{ijk})$ is its transposed where $e^*_{ijk} = e_{kij}$; $\mathcal{C} = (c_{ij})$ is a positive definite symmetric tensor, called the electric permittivity. More details on the constitutive equations can be found in [6] and [11].

For the long memory electro-viscoelastic materials that are considered in this work, we consider the constitutive law of the form:

$$\sigma(t) = \mathcal{B}\varepsilon(u(t)) + \int_0^t \mathcal{F}(t-s)\varepsilon(u(s))ds - \mathcal{E}^* E(\varphi(t)) \tag{1.1.9}$$

$$D(t) = \mathcal{E}\varepsilon(u(t)) + \mathcal{C}E(\varphi(t)). \tag{1.1.10}$$

where $\mathcal{F}$ is the relaxation tensor.

1.1.4 Boundary conditions

Considering contact in the mechanics of deformable solids requires defining boundary conditions on displacements and forces at contact interfaces. This involves the normal and tangential components of the displacement vector field. For any vector field u on Γ, we define u_ν (resp.u_τ) the normal (resp. tangential) component of u as follows:

$$u_\nu = u.\nu = u_i\nu_i$$
$$u_\tau = (u_{\tau i}) \text{ such that: } u_{\tau i} = u_i - u_\nu \nu_i, \quad i = 1, ..., d.$$

Boundary conditions also involve the normal and tangential components of the Cauchy stress vector $\sigma\nu$. For simplicity, we denote:

$$\sigma_\nu = \sigma\nu.\nu = \sigma_{ij}\nu_i\nu_j$$
$$\sigma_\tau = (\sigma_{\tau i}) \text{ such that: } \sigma_{\tau i} = \sigma_{ij}\nu_j - \sigma_\nu\nu_i \quad i, j = 1, ..., d.$$

where σ_ν (resp.σ_τ) is the normal stress (resp. tangential stress) of σ.
We also have:

$$\sigma_{ij}\nu_j u_i = \sigma_\tau u_\tau + \sigma_\nu u_\nu \tag{1.1.11}$$

Traction-displacement, electric potential, and imposed electric charge conditions.

Consider a material body (Figure.1.1) occupying a bounded Lipschitz domain Ω in $\mathbb{R}^d$ $(d = 2, 3)$, with a boundary $\partial\Omega = \Gamma$ partitioned into three measurable parts Γ_1, Γ_2 and Γ_3 on one hand, and two measurable parts Γ_a and Γ_b such that $meas(\Gamma_1) > 0$, $meas(\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. On Γ_3 the body is in contact with a foundation, and let ν be the outward normal vector to Γ.

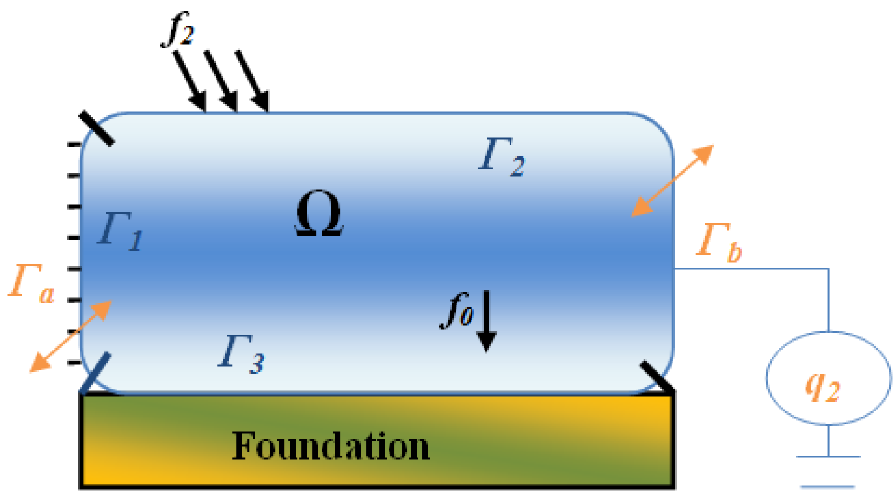

Figure 1.1: The contact problem.

We consider the following boundary conditions:

$$u = u_g \quad \text{on } \Gamma_1 \times (0,T)\,, \tag{1.1.12}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0,T)\,, \tag{1.1.13}$$

$$\varphi = 0 \quad \text{on } \ \Gamma_a \times (0,T)\,, \tag{1.1.14}$$

$$D\nu = q_b \quad \text{on } \Gamma_b \times (0,T)\,, \tag{1.1.15}$$

The condition (1.1.12) is called *the displacement boundary condition*, which implies that the displacement field is prescribed on the part Γ_1 of the boundary Γ. The function u_g is a given data of the problem, and if $u_g = 0$, the solid is clamped at Γ_1 within a fixed structure. The condition (1.1.13) is called *the traction boundary condition*, indicating that the Cauchy stress vector $\sigma\nu = (\sigma_{ij}\nu_j)$ is prescribed on the part Γ_2, where f_2 represents the density of applied surface forces. The condition (1.1.14) signifies that the electric potential vanishes on Γ_a and the condition (1.1.15) implies that a surface electric charge with density q_b is prescribed on Γ_b. To complete the mathematical models describing the evolution of the body in contact with a foundation, it is necessary to specify the boundary conditions on Γ_3, which are addressed by contact conditions and friction laws that we will describe in the following paragraph.

1.1.5 Contact laws with adhesion

Adhesive contact mechanics has become increasingly important in recent years. Applications can be found in nature, such as insect grasping forces, in nanofabrication, and in industrial systems involving rubber or polymer contacts. It is also widely used in various industrial assemblies where non-metallic components are bonded together. This area is of significant importance in the aerospace and automotive industries. Adhesion is an interfacial phenomenon that accompanies movement when glu is added to prevent the surfaces from moving relative to each other. As in [40] and [41] we introduce an internal surface variable, the "adhesion field," denoted in this thesis by β, which describes the point fraction of active adhesion density on the contact surface. The adhesion field satisfies the constraints $0 \leq \beta \leq 1$; when $\beta = 1$, adhesion is complete and all bonds are active; when $\beta = 0$, all bonds are inactive and there is no adhesion; when $0 < \beta < 1$ adhesion is partial, and only a fraction of the bonds are active. We refer the reader to the bibliography on the subject [44], [55], [96], [97], [103].

The evolution of the adhesion field is described by a differential equation of the form:

$$\begin{aligned} \dot{\beta}(t) &= -\left[\beta(t)\left((\gamma_\nu R_\nu u_\nu(t))^2 + \gamma_\tau \left\|R_\tau(u_\tau(t))\right\|^2\right) - \epsilon_a\right]_+ \quad \text{on } \Gamma_3, && (1.1.16) \\ \beta(0) &= \beta_0 \qquad \text{on } \Gamma_3. && (1.1.17) \end{aligned}$$

where γ_ν , γ_τ and ϵ_a are positive coefficients of adhesion, $r_+ = max\{r, 0\}$, R_ν and R_τ are the truncation operators defined by

$$R_\nu(s) = \begin{cases} L & \text{if } \ s < L \\ -s & \text{if } \ -L \leq s \leq 0 \\ 0 & \text{if } \ s > L \end{cases} \tag{1.1.18}$$

$$R_\tau(s) = \begin{cases} s & \text{if } \ \|s\| \leq L, \\ L\dfrac{s}{\|s\|} & \text{if } \ \|s\| > L. \end{cases} \tag{1.1.19}$$

where $L > 0$ is the characteristic length of the bond, beyond which it offers no additional traction (see, for example, [108]).

Equation (1.1.17) represents an initial adhesion condition. It is important to note that the adhesive process is irreversible, and once detachment occurs, adhesion cannot be restored as $\beta \leq 0$. Furthermore, it is easy to see that if $0 \leq \beta_0 \leq 1$ a.e. on Γ_3, then $0 \leq \beta \leq 1$ a.e. on Γ_3 during the process.

Unilateral contact with adhesion. This condition models contact with a rigid foundation and is given by the following equations:

$$\begin{aligned} u_\nu(t) \leq 0, \quad \sigma_\nu(t) - \gamma_\nu \beta^2 R_\nu(u_\nu(t)) \leq 0, \\ u_\nu(t)(\sigma_\nu(t) - \gamma_\nu \beta^2(t) R_\nu(u_\nu(t))) = 0 \end{aligned} \qquad \text{on } \Gamma_3, \tag{1.1.20}$$

By choosing L to be sufficiently large, we can assume that $R_\nu(u_\nu) = -u_\nu$, leading to the contact conditions:

$$\begin{aligned} u_\nu(t) \leq 0, \quad \sigma_\nu(t) - \gamma_\nu u_\nu(t) \beta^2(t) \leq 0, \\ u_\nu(t)(\sigma_\nu(t) - \gamma_\nu \beta^2(t) u_\nu(t)) = 0 \end{aligned} \qquad \text{on } \Gamma_3, \tag{1.1.21}$$

These conditions were used in [114] and [118] to model unilateral adhesive contact. We deduce from equation (1.1.20) that there is no interpenetration between the body and the obstacle, since $u \leq 0$ during the process. When the adhesion field is null, equation (1.1.21) becomes:

$$u_\nu(t) \leq 0, \quad \sigma_\nu(t) \leq 0, \quad \sigma_\nu(t) u_\nu(t) = 0, \qquad \text{on } \Gamma_3, \tag{1.1.22}$$

which are the classical Signorini contact conditions without adhesion.

Contact with normal compliance and adhesion.
The normal compliance condition models the interpenetration of the contact surface into the foundation. In this case, the contact area is not known a priori. The normal stress σ_ν satisfies the condition known as the normal compliance condition, which means

$$-\sigma_\nu = p_\nu \left(u_\nu - g\right), \tag{1.1.23}$$

where g represents the initial gap between the body and the foundation measured in the direction of the normal ν (Figure 1.2), p_ν is a given positive function. This condition illustrates that the foundation exerts a reaction on the body depending on the penetration $u_\nu - g$. For more details, see, for example, [56], [99], [87], [49]. The normal stress σ_ν satisfies the condition known as the normal compliance condition with adhesion, if

$$-\sigma_\nu \leq p(u_\nu - g) - \gamma_\nu \beta^2 R_\nu(u_\nu) \ \text{ on } \Gamma_3 \times (0, T), \tag{1.1.24}$$

The contribution of adhesion to the surface tension is represented by $\gamma_\nu \beta^2 R_\nu(u_\nu)$ in which γ_ν is an adhesion coefficient. However, by choosing L sufficiently large, that is, larger than the size of the system, we recover the case where tension is linear with extension. Thus, the contribution of adhesion to the normal tension $\gamma_\nu \beta^2 R_\nu(u_\nu)$ is proportional (with

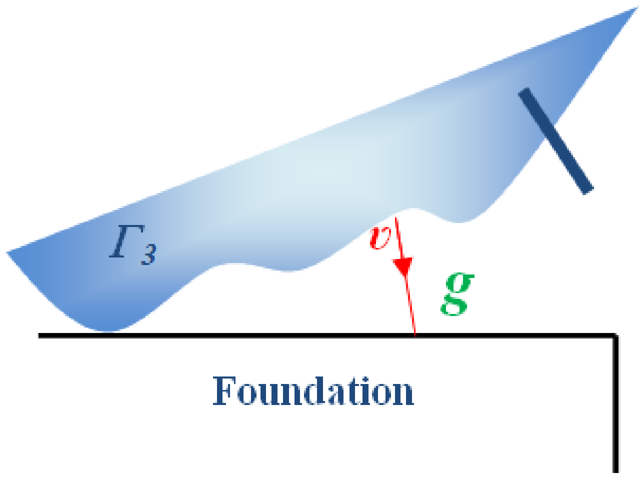

Figure 1.2: Contact with normal compliance.

the proportionality coefficient γ_ν) to the square of the adhesion intensity and the normal displacement, as long as it does not exceed the adhesion length L and the maximum stress is $\gamma_\nu L$.

As an example of normal compliance functions, we can consider

$$p(r) = cr_+ \tag{1.1.25}$$

and

$$p(r) = \begin{cases} cr_+ \\ c\alpha \end{cases} \tag{1.1.26}$$

where $c > 0$ and $\alpha > 0$ are surface parameters.

Laws of Friction, adhesion and friction coupling

Frictional boundary conditions define the interplay between the tangential forces (also referred to as frictional forces) at the contact zone Γ_3 and the relative tangential motion (sliding) of the body in contact with the foundation. The physical phenomena to be revealed in a friction law are the existence of a force threshold below which no sliding is possible, and any dependence of this threshold on the intensity of the normal forces. These laws include :

- *Coulomb's law:* This involves a direct proportionality between the normal force and the frictional force.

- *Tresca's law*: Used when significant normal forces are present, where the threshold is predetermined.

In this thesis, we use versions of Coulomb's law of dry friction and Tresca's law, which we will specify in the mathematical formulation of contact problems at the end of this section.

1.1.6 Mathematical formulation of the Problems

In the following, we will present the problem models studied in this thesis, specifying all the equations and conditions for each problem.

Problem $(P1)$. **(Bilateral contact with Tresca's friction law between a locking material and a rigid foundation):**

Find a displacement field $u:\Omega\to\mathbb{R}^d$ *such that*

$$Div\ \sigma(u)+f_0=0 \text{ in } \Omega,$$

$$\sigma(u)\in\mathcal{F}\varepsilon(u)+\partial I_B(\varepsilon(u)) \text{ in } \Omega,$$

$$u=0 \quad \text{on} \quad \Gamma_1,$$

$$\sigma(u)\nu=f_2 \quad \text{on } \Gamma_2,$$

$$\begin{cases} u_\nu=0 \\ \|\sigma_\tau(u)\|\le g \\ \|\sigma_\tau(u)\|<g\Longrightarrow u_\tau=0 \\ \|\sigma_\tau(u)\|=g\Longrightarrow \exists\lambda\ge 0 \text{ such that } \sigma_\tau(u)=-\lambda u_\tau \end{cases} \qquad \text{on } \Gamma_3.$$

Problem $(P2)$. **(Unilateral contact problem for long-memory piezoelectric materials with adhesion and non-local friction):**

Find a displacement field $u:\Omega\times[0,T]\to\mathbb{R}^d$, *a stress field* $\sigma:\Omega\times[0,T]\to\mathbf{S}_d$, *an electric potential* $\varphi:\Omega\times[0,T]\to\mathbb{R}$, *an electric displacement field* $D:\Omega\times[0,T]\to\mathbb{R}^d$ *and a bonding field* $\beta:\Gamma_3\times[0,T]\to\mathbb{R}$, *such that for all* $t\in[0,T]$,

$$\sigma(t)=\mathcal{B}\varepsilon(u(t))+\int_0^t\mathcal{F}(t-s)\varepsilon(u(s))ds-\mathcal{E}^*E(\varphi(t))$$

$$D(t)=\mathcal{E}\varepsilon(u(t))+\mathcal{C}E(\varphi(t))$$

$$Div\ \sigma(t)+f_0(t)=0 \quad \text{in } \Omega,$$

$$div\ D(t)+q_0(t)=0 \quad \text{in } \Omega,$$

$$u(t)=0 \quad \text{on } \Gamma_1,$$

$$\sigma\nu(t)=f_2(t) \quad \text{on } \Gamma_2,$$

$$\left.\begin{array}{c} u_\nu(t) \leq 0, \quad \sigma_\nu(t) - \gamma_\nu \beta^2 R_\nu(u_\nu(t)) \leq 0, \\ u_\nu(t)(\sigma_\nu(t) - \gamma_\nu \beta^2(t) R_\nu(u_\nu(t))) = 0 \end{array}\right. \quad \text{on } \Gamma_3,$$

$$\dot{\beta}(t) = -\left[\beta(t)\left((\gamma_\nu R_\nu u_\nu(t))^2 + \gamma_\tau \left\|R_\tau(u_\tau(t))\right\|^2\right) - \epsilon_a\right]_+ \quad \text{on } \Gamma_3,$$

$$\varphi(t) = 0 \quad \text{on } \Gamma_a,$$

$$D\nu(t) = q_2(t) \quad \text{on } \Gamma_b,$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3,$$

$$\left\{\begin{array}{l} \|\sigma_\tau(t) + \gamma_\tau \beta^2(t) R_\tau(u_\tau(t))\| \leqslant \mu \, |R\sigma_\nu(t)| \\ \\ \|\sigma_\tau(t) + \gamma_\tau \beta^2(t) R_\tau(u_\tau(t))\| < \mu \, |R\sigma_\nu(t)| \Longrightarrow u_\tau = 0 \\ \\ \|\sigma_\tau(t) + \gamma_\tau \beta^2(t) R_\tau(u_\tau(t))\| = \mu \, |R\sigma_\nu(t)| \Longrightarrow \exists \lambda \geqslant 0 \\ \text{such that: } \sigma_\tau\,(t) + \gamma_\tau^2 \beta^2(t) R_\tau(u_\tau(t)) = -\lambda u_\tau(t) \end{array}\right. \quad \text{on } \Gamma_3.$$

Problem $(P3)$. **(Contact problem for long-memory piezoelectric materials with normal compliance, adhesion and friction):**

Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, *a stress field* $\sigma : \Omega \times [0, T] \to \mathbf{S}_d$, *an electric potential* $\varphi : \Omega \times [0, T] \to \mathbb{R}$, *an electric displacement field* $D : \Omega \times [0, T] \to \mathbb{R}^d$ *and a bonding field* $\beta : \Gamma_3 \times [0, T] \to \mathbb{R}$, *such that for all* $t \in [0, T]$,

$$\sigma\,(t) = \mathcal{B}\varepsilon(u(t)) + \int_0^t \mathcal{F}(t-s)\varepsilon(u(s))ds - \mathcal{E}^* E(\varphi(t))$$

$$D(t) = \mathcal{E}\varepsilon(u(t)) + \mathcal{C}E(\varphi(t))$$

$$Div\ \sigma(t) + f_0(t) = 0 \quad \text{in } \Omega,$$

$$div\ D(t) + q_1(t) = 0 \quad \text{in } \Omega,$$

$$u(t) = 0 \quad \text{on } \Gamma_1,$$

$$\sigma\nu(t) = f_2(t) \quad \text{on } \Gamma_2,$$

$$-\sigma_\nu(t) = p_\nu(u_\nu(t) - q) - \gamma_\nu \beta^2 R_\nu(u_\nu(t)) \quad \text{on } \Gamma_3,$$

$$\dot{\beta}(t) = -\left[\beta(t)\left((\gamma_\nu R_\nu u_\nu(t))^2 + \gamma_\tau \left\|R_\tau(u_\tau(t))\right\|^2\right) - \epsilon_a\right]_+ \quad \text{on } \Gamma_3,$$

$$\varphi(t) = 0 \quad \text{on } \Gamma_a$$

$$D\nu(t) = q_2(t) \quad \text{on } \Gamma_b,$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3$$

$$\left\{\begin{array}{l} \|\sigma_\tau(t) + \gamma_\tau \beta^2 R_\tau(u_\tau(t))\| \leqslant \mu p\,(u_\nu(t) - q) \\ \\ \|\sigma_\tau(t) + \gamma_\tau \beta^2 R_\tau(u_\tau(t))\| < \mu p\,(u_\nu(t) - q) \Longrightarrow u_\tau\,(t) = 0 \\ \\ \|\sigma_\tau(t) + \gamma_\tau \beta^2 R_\tau(u_\tau(t))\| = \mu p\,(u_\nu(t) - q) \Longrightarrow \exists \lambda \geqslant 0 \\ \\ \quad \text{such that:} \;\; \sigma_\tau\,(t) + \gamma_\tau \beta^2\,(t)\, R_\tau(u_\tau(t)) = -\lambda u_\tau\,(t) \end{array}\right. \qquad \text{on } \Gamma_3.$$

1.2 Preliminaries on functional analysis

1.2.1 Functional spaces

In this section, we will introduce various Sobolev-type spaces and functional spaces used in continuum mechanics, particularly in contact problems. Here, we will limit our discussion to recalling and specifying certain properties that will be useful later on, without explicitly indicating the dependence of the various functions on the spatial variable. (For a detailed description, refer to, for instance [13], [33], [63], [74], [88]).

Let Ω be an open subset of $\mathbb{R}^d$ and $D(\Omega)$ the space of real functions defined on Ω infinitely differentiable and with a support compact in Ω. Let $D^{'}(\Omega)$ denote the space of continuous linear functionals on $D(\Omega)$. We introduce the following spaces:

$$D = \left\{\omega = (w_i)\,|w_i \in D(\Omega)\,,\ i = \overline{1,d}\right\} = D(\Omega)^d$$
$$\mathcal{D} = \left\{\phi = (\phi_{ij})\,|\phi_{ij} = \phi_{ji}\ ,\ i,j = \overline{1,d}\right\} = D(\Omega)_s^{d\times d}$$
$$D^{'} = \left\{u = (u_i)\,|u_i \in D^{'}(\Omega)\,,\ i = \overline{1,d}\right\}$$
$$\mathcal{D}^{'} = \left\{\sigma = (\sigma_{ij})\,|\sigma_{ij} = \sigma_{ji} \in D^{'}(\Omega)\,,\ i,j = \overline{1,d}\right\}$$

The dualities between D and $D^{'}$, $\mathcal{D}$ and $\mathcal{D}'$, will be denoted by $\langle .,. \rangle_{D'\times D}$ and $\langle .,. \rangle_{\mathcal{D}'\times\mathcal{D}}$. The symbol $\partial_i = \partial/\partial x_i$ will stand for differentiation with respect to x_i applied to both functions and distributions.

The first-order differential operators used are:

$$\varepsilon : D \to \mathcal{D}, \varepsilon\ (w) = (\varepsilon_{ij}(w))\,, \varepsilon_{ij}(w) = \frac{1}{2}(\partial_j w_i + \partial_i w_j) \quad \forall i,j = \overline{1,d},\ w \in D. \tag{1.2.1}$$

$$Div : \mathcal{D} \to D,\ Div\ \phi = (\partial_j \phi_{ij}) \quad \forall i = \overline{1,d},\ \phi \in \mathcal{D}. \tag{1.2.2}$$

The same notations are employed in the distribution spaces, namely:

$$\varepsilon : D' \to \mathcal{D}',\ \varepsilon(u) = (\varepsilon_{ij}(u))\,, \varepsilon_{ij}(u) = \tfrac{1}{2}(\partial_j u_i + \partial_i u_j) \quad \forall i,j = \overline{1,d},\ u \in D^{'}. \tag{1.2.3}$$

$$Div : \mathcal{D}' \to D',\ Div\ \sigma = (\partial_j \sigma_{ij}) \quad \forall i = \overline{1,d},\ \sigma \in \mathcal{D}^{'}. \tag{1.2.4}$$

The operator ε is called the *deformation operator*, and the operator Div is referred to as the divergence operator. The following equations apply:

$$\langle \partial_i \theta, w \rangle_{D'\times D} = -\langle \theta, \partial_i w \rangle_{D'\times D} \quad \forall \theta \in D^{'}(\Omega)\,,\ w \in D(\Omega) \tag{1.2.5}$$

$$\langle \varepsilon(u)\,, w \rangle_{\mathcal{D}'\times\mathcal{D}} = -\langle u, Div\ w \rangle_{D'\times D} \quad \forall u \in D^{'},\ w \in \mathcal{D} \tag{1.2.6}$$

$$\langle Div\ \sigma, w\rangle_{D'\times D} = -\langle \sigma, \varepsilon(w)\rangle_{\mathcal{D}'\times\mathcal{D}} \quad \forall\sigma \in \mathcal{D}',\ w \in D \tag{1.2.7}$$

Let $L^2(\Omega)$ denote the space of square-integrable functions on Ω equipped with the inner product:

$$(f, g)_{L^2(\Omega)} = \int_\Omega fgdx \quad \forall f, g \in L^2(\Omega).$$

Considering $H^1(\Omega)$ which denotes the space of functions in $L^2(\Omega)$ where all distributional derivatives belong to $L^2(\Omega)$.

In the context of analyzing mechanical problems, we also introduce the spaces:

$$H = \left\{u = (u_i)\,|u_i \in L^2(\Omega),\ i = \overline{1,d}\right\} = L^2(\Omega)^d \tag{1.2.8}$$

$$Q = \left\{\sigma = (\sigma_{ij})\,|\sigma_{ij} = \sigma_{ji} \in L^2(\Omega),\ i = \overline{1,d}\right\} = L^2(\Omega)_s^{d\times d}, \tag{1.2.9}$$

The spaces H and Q are real Hilbert spaces endowed with the respective inner products:

$$(u, v)_H = \int_\Omega u_i v_i dx \quad \forall u, v \in H,$$

$$(\sigma, \tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx \quad \forall \sigma, \tau \in Q.$$

For the deformation operator defined by (1.2.3), we introduce the space:

$$H_1 = \{u \in H \mid \varepsilon(u) \in Q\},$$

endowed with the inner product

$$\langle u, v\rangle_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_Q \quad \forall u, v \in H_1,$$

and the associated norm $\|.\|_{H_1}$.

H_1 is a real Hilbert space. The injection $H_1 \hookrightarrow H$ and the deformation operator $\varepsilon : H_1 \to Q$ are continuous.

Properties:

Assuming that the boundary Γ of Ω is of class C^1, we have the following results:

1) $C^1(\bar{\Omega})^d = \left[C^1(\bar{\Omega})\right]^d$ is dense in H_1, where $C^1(\bar{\Omega})$ denotes the space of functions that are continuously differentiable on $\bar{\Omega}$.

2) Relich's Theorem:

The canonical injection from H_1 into H is compact which means that any bounded set in H_1 is relatively compact in H.

3) For $v \in C^1\left(\bar{\Omega}\right)^d$, let $v\left|_\Gamma\right.$ be the value of the function on the boundary v on Γ. We have the following results:

First trace theorem:

Under the assumption Γ is of class C^1, then the application $v \to v\left|_\Gamma\right.$ defined on $C^1\left(\bar{\Omega}\right)^d$ in $L^2\left(\Gamma\right)^d$ extends to a continuous mapping γ from H_1 to $L^2\left(\Gamma\right)^d$, called the trace of v on Γ.

4) Generally, the trace operator is neither an injection nor a surjection from H_1 to $L^2\left(\Gamma\right)^d$. the range of the trace operator on the space H_1 is a subspace of $L^2\left(\Gamma\right)^d$ denoted $H_\Gamma = \left[H^{\frac{1}{2}}\left(\Gamma\right)\right]^d$ witch it can be shown to be a Hilbert space. We denote its inner product by $(.,.)_{1/2}$ and the kernel of the trace operator on H_1 by H_1^0.

Second trace theorem:[33]

Under the assumption Γ is of class C^1, then the application $\gamma : v \to \gamma v$ is linear, continuous, and surjective from H_1 to H_Γ.

5) The injection $H_\Gamma \hookrightarrow L^2\left(\Gamma\right)^d$ is continuous, and denoting by $\nu = (\nu_i)$ the outward unit normal to Γ, for every $w \in H_\Gamma$ we define the normal and tangential components as follows:

$$w_\nu = w.\nu \text{ and } w_\tau = w - w_\nu \nu \tag{1.2.10}$$

where $w_\nu \in H^{\frac{1}{2}}\left(\Gamma\right)$ the range of the trace operator on the space $L^2\left(\Gamma\right)$,
and $w_\tau \in H_\tau$, where H_τ is the closed subspace of H_Γ defined by:

$$H_\tau = \left\{w \in H_\Gamma \left| w_\nu = 0 \text{ a.e. on } \Gamma\right.\right\}.$$

We will respectively use the dual spaces $H^{-\frac{1}{2}}\left(\Gamma\right)$ of $H^{\frac{1}{2}}\left(\Gamma\right)$, $H_\Gamma^{'}$ of H_Γ and $H_\tau^{'}$ of H_τ. We denote their norms as well as their duality products by

$$\left\|.\right\|_{H^{-\frac{1}{2}}(\Gamma)}, \quad \langle .,.\rangle_\Gamma \quad ; \quad \left\|.\right\|_{H_\Gamma'}, \quad \langle .,.\rangle_{H_\Gamma' \times H_\Gamma}, \text{ and } \quad \left\|.\right\|_{H_\tau'}, \quad \langle .,.\rangle_{H_\tau' \times H_\tau}.$$

Using the trace application, we define for every $u \in H_1$ the elements $\gamma_\nu u \in H^{\frac{1}{2}}\left(\Gamma\right)$ and $\gamma_\tau u \in H_\tau$ by $\gamma_\nu u = \left(\gamma u\right)_\nu$ and $\gamma_\tau u = \left(\gamma u\right)_\tau$. While recalling that if $u \in C^1\left(\overline{\Omega}\right)^d$ then

$$\gamma u = u\left|_\Gamma\right., \quad \gamma_\nu u = u\left|_\Gamma\right..\nu, \quad \gamma_\tau u = u\left|_\Gamma\right. - \left(u\left|_\Gamma\right..\nu\right)\nu.$$

For the sake of simplicity, when no ambiguity may occur, we write u, u_ν and u_τ, instead of γu, $\gamma_\nu u$, and $\gamma_\tau u$.

In Chapter 2, we will assume that the boundary Γ is partitioned into three measurable parts Γ_1, Γ_2 and Γ_3. In the following chapter, for the study of electromechanical problems, we will also assume that Γ is partitioned into two measurable parts Γ_a et Γ_b, such that $\Gamma_3 \subset \Gamma_b$. Then we have the following results:

Let V be the closed subspace of H_1 defined by:

$$V = \{v \in H_1 \,|\gamma v = 0 \text{ a.e. on } \Gamma_1\}. \tag{1.2.11}$$

Proposition 1.1. (Korn's inequality) *[88]. If meas* $\Gamma_1 > 0$. *Then there exists a positive constant* C_Ω *depending only on* Ω *and* Γ *such that:*

$$\|\varepsilon(v)\|_Q \geqslant C_\Omega \|v\|_V \quad \forall v \in V. \tag{1.2.12}$$

Consider in V the inner product $(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q$ and the associated norm $\|.\|_V$. Using Korn's inequality, it can be deduced that $\|.\|_V$ is a norm on V that is equivalent to the canonical norm $\|.\|_{H_1}$, and thus $(V, \|.\|_V)$ is a real Hilbert space.
In this case, considering the trace theorems, we have:

$$\|v\|_{L^2(\Gamma_3)^d} \leqslant d_\Omega \|v\|_V \quad \forall v \in V, \tag{1.2.13}$$

where d_Ω depends only on Ω and Γ.
We also introduce the space

$$Q_1 = H(div, \Omega) = \{\sigma \in Q \,|Div\ \sigma \in H\},$$

endowed with the inner product

$$(\sigma, \tau)_{Q_1} = (\sigma, \tau)_Q + (Div\ \sigma, Div\ \tau)_H \quad \forall \sigma, \tau \in Q_1.$$

The injection $Q_1 \hookrightarrow Q$ and the divergence operator $Div : Q_1 \to H$ are continuous mappings. H and Q_1 are subspaces of D' and $\mathcal{D}'$, and we have:

$$\langle Div\ \sigma, w\rangle_{D' \times D} + (\sigma, \varepsilon(w))_Q = 0 \quad \forall \sigma \in Q,\ w \in D \tag{1.2.14}$$

$$(Div\ \sigma, w)_H + (\sigma, \varepsilon(w))_Q = 0 \quad \forall \sigma \in Q_1,\ w \in D. \tag{1.2.15}$$

Consider the space $C^1\left(\overline{\Omega}\right)_s^{d\times d}$ defined by:

$$C^1\left(\overline{\Omega}\right)_s^{d\times d} = \left\{\sigma = (\sigma_{ij}) \,|\sigma_{ij} = \sigma_{ji} \in C^1\left(\overline{\Omega}\right), i, j = \overline{1, d}\right\}$$

$C^1\left(\overline{\Omega}\right)_s^{d\times d}$ is dense in Q, and we define the trace operator on the space Q_1 using the following result:

Proposition 1.2. *There exists a continuous and surjective linear mapping* $\overline{\gamma} : Q_1 \to H'_\Gamma$ *such that:*

$$\langle \overline{\gamma}\sigma, \xi \rangle_{H'_\Gamma \times H_\Gamma} = \int_\Gamma \sigma\nu.\xi da \quad \forall \xi \in H_\Gamma,\ \sigma \in C^1\left(\overline{\Omega}\right)_s^{d\times d}. \tag{1.2.16}$$

This leads to deducing the *Green's formula*:

For every $\sigma \in Q_1$, $\overline{\gamma}\sigma \in H'_\Gamma$ is the unique element satisfying the equality:

$$\langle \overline{\gamma}\sigma, v \rangle_{H'_\Gamma \times H_\Gamma} = (\sigma, \varepsilon(v))_Q + (Div\ \sigma, v)_H \quad \forall v \in H_1. \tag{1.2.17}$$

Given the trace operator, we define for every $\sigma \in Q_1$ the elements $\overline{\gamma}_\nu\sigma \in H^{-\frac{1}{2}}(\Gamma)$ and $\overline{\gamma}_\tau\sigma \in H'_\tau$ by:

$$\overline{\gamma}_\nu\sigma = (\overline{\gamma}\sigma)_\nu, \quad \overline{\gamma}_\tau\sigma = (\overline{\gamma}\sigma)_\tau.$$

If $\sigma \in C^1\left(\overline{\Omega}\right)_s^{d\times d}$, the elements $\overline{\gamma}\sigma$, $\overline{\gamma}_\nu\sigma$ and $\overline{\gamma}_\tau\sigma$ are defined by:

$$\overline{\gamma}\sigma = \sigma|_\Gamma .\nu, \quad \overline{\gamma}_\nu\sigma = (\sigma|_\Gamma .\nu)_\nu, \quad \overline{\gamma}_\tau\sigma = \sigma|_\Gamma .\nu - (\sigma|_\Gamma .\nu)\,\nu.$$

To simplify notation, we will use the following notation $\sigma\nu$, σ_ν and σ_τ instead of $\overline{\gamma}\sigma$, $\overline{\gamma}_\nu\sigma$ and $\overline{\gamma}_\tau\sigma$.

The vector $\sigma\nu$ is referred to as the *Cauchy stress vector*, σ_ν and σ_τ are respectively its *normal* and *tangential* components.

For every $\sigma \in Q_1$, we have the following double equality:

$$\langle \sigma\nu, \gamma v \rangle_{H'_\Gamma \times H_\Gamma} = \langle \sigma_\nu, v_\nu \rangle_\Gamma + \langle \sigma_\tau, v_\tau \rangle_{H'_\tau \times H_\tau} = (\sigma, \varepsilon(v))_Q + (Div\ \sigma, v)_H. \tag{1.2.18}$$

For the Sobolev spaces associated with the electric unknowns (electric potential and electric displacement field), we use the spaces:

$$\begin{aligned} \mathcal{W} &= \left\{ D = (D_i) \,\middle|\, D_i \in L^2(\Omega) \right\}, \\ W_a &= \left\{ D = (D_i) \,\middle|\, D_i \in L^2(\Omega),\ div\ D \in L^2(\Omega) \right\}, \end{aligned}$$

endowed with the inner products

$$\begin{aligned} (D, E)_{\mathcal{W}} &= \int_{\Gamma_b} D_i E_i dx, \\ (D, E)_{Wa} &= (D, E)_H + (div\ D, div\ E)_{L^2(\Omega)}, \end{aligned}$$

and their associated norms $\|.\|_{\mathcal{W}}$ and $\|.\|_{W_a}$.

We will require the admissible electric potential space W defined by:

$$W = \{\xi \in H_1 \,|\, \xi = 0 \quad \text{a.e on } \Gamma_a\}.$$

If $meas(\Gamma_a) > 0$, the Friedrichs-Poincaré inequality holds. Therefore, there exists a constant $C_F > 0$ depending only on Ω and Γ_a such that:

$$\|\nabla\xi\|_W \geq C_F \|\xi\|_{H^1(\Omega)} \quad \forall \xi \in W. \tag{1.2.19}$$

where ∇ is the gradient operator.

$$\nabla\psi = (\partial_i \psi) = (\psi_{,i}) \ \ \forall \psi \in W \tag{1.2.20}$$

Consider in W the inner product:

$$(\xi, \phi)_W = (\nabla\xi, \nabla\phi)_H$$

and the associated norm $\|.\|_W$.
Using the Friedrichs-Poincaré inequality, we conclude that$\|.\|_W$ is a norm on W that is equivalent to the canonical norm $\|.\|_{\mathcal{W}}$, and thus $(W, \|.\|_W)$ is a real Hilbert space. Considering the trace theorems, there exists a constant $\tilde{C}_0$ depending only on Ω and Γ such that:

$$\|\xi\|_{L^2(\Gamma_3)} \leqslant \tilde{C}_0 \|\xi\|_W \quad \forall \xi \in W \tag{1.2.21}$$

Furthermore, if $D \in W$ is sufficiently regular, the following Green's formula holds:

$$(D, \nabla\xi)_H + (div\ D, \xi)_{L^2(\Omega)} = \int_{\Gamma_b} D\nu.\xi da \quad \forall \xi \in W. \tag{1.2.22}$$

We will also require the Banach space of fourth-order tensors:

$$Q_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkh})\,;\ \mathcal{E}_{ijkh} = \mathcal{E}_{jikh} = \mathcal{E}_{khij} \in L^\infty(\Omega)\}$$

equipped with the norm $\|\mathcal{E}\|_{Q_\infty} = \max\limits_{0 \leq i,j,k,h \leq d} \|\mathcal{E}_{ijkh}\|_{L^\infty(\Omega)}$.
For additional details, we refer the reader, for example, to [13], [15], [33], [74], [103].

Let's consider X as a Hilbert space and T as a positive real number. We recall that $C([0,T]\,;X)$ is the space of continuous functions defined on $[0,T]$ with values in X, which is a Banach space equipped with the norm:

$$\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X\,,$$

and $C^1([0,T]\,;X)$ the space of continuously differentiable functions on $[0,T]$ into X is a Banach space equipped with the norm:

$$\|x\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X + \max_{t\in[0,T]} \|\dot{x}(t)\|_X\,.$$

We denote by $D_c(0,T;X)$ the set of continuous functions with compact support in $[0,T]$ with values in X. Let $f:[0,T]\longrightarrow X$, f is said to be:

- *Measurable* if there exists a subset $I\subset[0,T]$ of measure zero and a sequence of functions $(f_n)_{n\in\mathbb{N}}$ belonging to $D_c(0,T;X)$ such that $\|f_n(t)-f(t)\|_X\longrightarrow 0$ as $n\longrightarrow+\infty$, for all $t\in[0,T]\backslash I$.

- *Absolutely continuous* if for every $\varepsilon>0$, there exist $\delta>0$, such that: $\sum\limits_{i=1}^{n}\|f(b_i)-f(a_i)\|_X<\varepsilon$ for any n and any disjoint collection of intervals $\{(a_i,b_i)\}$ in $[0,T]$ satisfying $\sum\limits_{i=1}^{n}(b_i-a_i)<\delta$.

- *Strongly differentiable* at $t_0\in[0,T]$, if there exists $\frac{df}{dt}(t_0)\in X$ called the strong derivative of f at t_0, such that $\lim\limits_{h\longrightarrow 0}\left\|\frac{1}{h}f(t_0+h)-f(t)-\frac{df}{dt}(t_0)\right\|_X=0$.

- *Intégrable* if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of functions belonging to $D_c(0,T;X)$ such that $\lim\limits_{n\longrightarrow+\infty}\int_0^T\|f_n(t)-f(t)\|_X\,dt=0$.

Remark 1.2. *Clearly, an absolutely continuous function on $[0,T]$ is uniformly continuous. Moreover, a Lipschitz continuous function on $[0,T]$ is absolutely continuous.*

Let $1\leq p\leq\infty$, the Lebesgue space $L^p(0,T;X)$ is the set of equivalence classes of measurable functions $f:[0,T]\longrightarrow X$, such that the mapping $t\longrightarrow\|f(t)\|_X$ belongs to $L^p(X)$, which is a Banach space with the norm:

$$\begin{aligned}\|f\|_{L^p(0,T;X)} &= \left(\int_0^T\|f\|_X^p\,dt\right)^{\frac{1}{p}}\\ \|f\|_{L^\infty(0,T;X)} &= \operatorname*{ess\,sup}_{(0,T)}\|u\|_X=\inf\left\{c>0,\ \ \|f(t)\|_X\leq c\ \ \text{a.e. } t\in[0,T]\right\}\end{aligned}$$

Note that if $f:[0,T]\longrightarrow X$ is measurable and integrable, then:

$$\left\|\int_0^T f dt\right\|_X\leq\int_0^T\|f\|_X\,dt$$

Moreover, we have the following properties:

1- If $1\leq q\leq r\leq\infty$, $L^r(0,T;X)\subset L^p(0,T;X)$ with continuous injection.

2- $L^2(0,T;X)$ is a Hilbert space, equipped with the inner product

$$(u,v)=\int_0^T(u(t),v(t))_X dt$$

3- Let $L^p(0,T;X)'$ denote the dual of the space $L^p(0,T;X)$. Then,

$$
\begin{aligned}
L^p(0,T;X)' &= L^q(0,T;X) \ \text{ if } \ 1 < p,q < \infty \\
L^1(0,T;X)' &= L^\infty(0,T;X)
\end{aligned}
$$

4- Let $D_c'(0,T;X)$ denote the space of real functions that are infinitely differentiable, with compact support in $[0,T]$ and let $u \in L^1(0,T;X)$. A function $h \in L^1(0,T;X)$ is the generalized derivative of order n of u on $(0,T)$ if

$$
\int_0^T w^{(n)}(t)u(t)dt = (-1)^n \int_0^T w(t)h(t)dt, \quad \forall w \in D_c'(0,T;X)
$$

we denote $h = \dot{u}$ for $n = 1$ and $w = u^{(n)}$ for $n > 1$.

For $p \in [0,\infty]$, we define the Sobolev space $W^{1,p}(0,T;X)$ as follows:

$$
W^{1,p}(0,T;X) = \left\{u : [0,T] \longrightarrow X;\ u \in L^p(0,T;X) \text{ and } \dot{u} \in L^p(0,T;X)\right\},
$$

which is a Banach space equipped with the norm:

$$
\|u\|_{W^{1,p}(0,T;X)} = \|u\|_{L^p(0,T;X)} + \|\dot{u}\|_{L^p(0,T;X)}.
$$

Theorem 1.3. *Let $p \in [0,\infty]$, X be a reflexive Banach space, and $u \in L^p(0,T;X)$. The following properties are equivalent:*
1) $u \in W^{1,p}(0,T;X)$.
2) u has an almost everywhere absolutely continuous representative that is almost everywhere differentiable and has a strong derivative in $L^p(0,T;X)$.
3) There exist $u_0 \in X$ and $g \in L^p(0,T;X)$ such that

$$
u(t) = u_0 + \int_0^t g(s)ds
$$

Corollary 1.1. *Under the assumptions of the previous theorem, we have:*
- Every function $u \in W^{1,p}(0,T;X)$ is strongly differentiable a.e. on $(0,T)$ and $\dot{u} = \frac{du}{dt}$ a.e. on $(0,T)$.
- $W^{1,1}(0,T;X)$ coincides with the set of functions $u : [0,T] \longrightarrow X$ that are absolutely continuous, and $W^{1,\infty}(0,T;X)$ coincides with the set of functions $u : [0,T] \longrightarrow X$ that are Lipschitz.

We define by induction for $k \geq 2$ and $p \in [0, \infty]$ the space

$$W^{k,p}(0,T;X) = \left\{u \in W^{k-1,p}(0,T;X) \ , \ \dot{u} \in W^{k-1,p}(0,T;X) \ \right\},$$

and we verify that $u \in W^{k,p}(0,T;X)$ if there exist k functions $g_1, ..., g_k \in L^p(0,T;X)$ such that

$$\int_0^T \psi^{(i)}(t)u(t)dt = (-1)^i \int_0^T \psi(t)g_i(t)dt, \quad \forall i = \overline{1,k}.$$

We can thus consider the successive derivatives $\dot{u} = g_1$, $u^{(2)} = g_2$, ..., $u^{(k)} = g_k$.
Furthermore, $W^{k,p}(0,T;V)$ is a Banach space for the norm defined by:

$$\|u\|_{W^{k,p}(0,T;X)} = \|u\|_{L^p(0,T;X)} + \sum_{i=1}^{k} \left\|u^{(k)}\right\|_{L^p(0,T;X)}.$$

In particular, for $p = \infty$, $W^{k,\infty}(0,T;V)$, we have

$$\|u\|_{W^{k,\infty}(0,T;X)} = \sum_{j=0}^{k} \operatorname*{ess\,sup}_{(0,T)} \left\|u^{(k)}\right\|_X.$$

Now, for the normed space X we denote by X' its dual space, and $\langle .,. \rangle$ the duality pairing of X and X'.
A sequence $(u_n)_n \subset M$ is said to converge weakly to $u \in X$ if for every $f \in X'$,

$$\lim_{n \longrightarrow +\infty} f(u_n) \longrightarrow f(u) \tag{1.2.23}$$

u is called the weak limit of $(u_n)_n$ and we write $u_n \rightharpoonup u$ in X.
A subset $M \subset X$ is said to be weakly closed if it contains the limits of all weakly convergent sequences $(u_n)_n \subset M$.

We recall next the notions of convex and semicontinuous functionals (For more details, we refer the reader to [34]).

First, a set $M \subset X$ is said to be convex if and only if for every pair of elements (u,v) of M the segment $[u,v]$ is contained in M, where

$$[u,v] = \{ru + (1-r)v, \ r \in [0,1]\}$$

The whole of the space X is convex and, conventionally, so is the empty set and every intersection of convex sets is convex. Recall that if the subset M is convex, then it is closed if and only if it is weakly closed.

Next, let the functional $w : X \supset M \to]-\infty, +\infty]$, we have:

- The functional w is called convex if and only if M is convex and

$$w\left(ru+(1-r)v\right)\leq rw(u)+(1-r)w(v)$$

for all $u,v\in M$ and $r\in[0,1]$, w is called to be strictly convex if the inequality is strict.

- The effective domain of w is defined by:

$$dom\ w=\{u\in X:\ w(u)<+\infty\}$$

- w is said to be proper if $dom\ w\neq\emptyset$.
- The epigraph of w, denoted by $epi(w)$, is given by

$$epi(w)=\{(u,\lambda)\in X\times\mathbb{R}:\ w(u)<\lambda\}$$

- The functional w is said to be lower semicontinuous at u if

$$\lim_{n\longrightarrow\infty}\inf w(u_n)\geq w(u) \tag{1.2.24}$$

for each sequence $(u_n)_n\subset M$ converging to u in M. We say that w is lower semicontinuous if it is lower semicontinuous on M.
- The functional w is called upper semicontinuous if and only if $-w$ is lower semicontinuous.
- The functional w is continuous if and only if w is both lower and upper semicontinuous.

- The functional w is called weakly lower semicontinuous at $u\in M$ if and only if (1.2.24) holds for each weakly convergent sequence $(u_n)_n$ to u, *i.e.* $u_n\rightharpoonup u$.
- If M is closed and w is proper, then w is lower semicontinuous if and only if it is weakly lower semicontinuous.

- Assume $u\in M$ with $w(u)\neq\pm\infty$. Then w is lower semicontinuous at u if and only if for every $\varepsilon>0$, there exists $\delta>0$ such that, $\forall v\in M$

$$\|u-v\|<\varepsilon\Longrightarrow w(u)<w(v)+\varepsilon$$

- The functional w is convex, proper, and lower semicontinuous if and only if $epi(w)$ is, respectively, convex, nonempty, and closed in $X\times\mathbb{R}$.

- If w is convex, proper, and lower semicontinuous functional. Then w is locally Lipschitz on the interior of *dom* w.

- If X is also a reflexive space and $\lim\limits_{\|u\| \longrightarrow +\infty} w(u) = +\infty$, then the problem

$$u \in X : \; w(u) = \inf_{v \in X} w(v)$$

admits at least one solution.

Now, we proceed by recalling the notion of subgradient, which extends the classical concept of a derivative.
If w a proper, convex and lower semicontinuous function. The mapping $\partial w : X \longrightarrow \mathcal{P}(X')$ defined by

$$\partial w(x) = \left\{u^* \in X' : \langle u^*, v - u\rangle \leq w(v) - w(u) \;\; \text{for all} \;\; v \in X\right\},$$

is called the subdifferential of w. Any element $u^* \in \partial w(u)$ is called a subgradient of w in u. In particular, let $K \subset X$ be a nonempty subset, and I_K the indicator function of K. Then, the subdifferential of I_K is defined by

$$\partial w(u) = \left\{u^* \in X' : \langle u^*, v - u\rangle \geq 0 \;\; \text{for all} \;\; v \in X\right\}.$$

Among the properties of the subdifferential are the following:

- If w is convex and proper, then $\partial w(u)$ is convex and *weak**-closed.

- If w is continuous at $u \in$ *dom* (w), then $\partial w(u)$ is nonempty, convex, bounded, and *weak**-compact.

- Let $w_1, w_2 : X \to]-\infty, +\infty]$ two convex functionals. If there exists $\tilde{u} \in$ *dom* $(w_1) \cap$ *dom* (w_2) at which w_1 and w_2 are continuous, then the following holds:

$$\partial\left(w_1 + w_2\right)(u) = \partial w_1(u) + \partial w_2(u) \qquad \forall v \in X.$$

1.2.2 Abstract problems

Mathematical modeling for various classes of problems in continuum mechanics involves variational inequalities. This method allows both the derivation of theoretical results and the development of efficient numerical methods. In this section, we will present results concerning the existence and uniqueness of solutions for specific variational problems.

Let X be a real Hilbert functional space equipped with the inner product $(.,.)_X$ and the associated norm $\|.\|_X$. A bilinear form $a(.,.)$ is continuous and coercive, or elliptic on $X \times X$, if there are two constants $c_1 > 0$ and $c_2 > 0$ such that:

$$\begin{array}{ll} \text{(a)} & |a(u,v)| \leq c_1 \|u\|_X \|v\|_X, \forall u, v \in X. \\ \text{(b)} & a(v,v) \geq c_2 \|v\|_X^2, \forall v \in X. \end{array}$$

Consider the problem:

Problem $(\mathbf{S}_1)$. *Find u such: $u \in X$, and*

$$\boldsymbol{a}(u, v-u) + j(v) - j(u) \geqslant (f, v-u)_X \quad \forall v \in X. \tag{1.2.25}$$

where $j : X \to]-\infty, +\infty]$ is a functional, and $f \in X$.

A result concerning the existence and uniqueness of the solution to problem $(\mathbf{S}_1)$ is provided by: (see, for example, [64]).

Theorem 1.4. *If the bilinear form $\boldsymbol{a}(.,.)$ is symmetric, continuous, and coercive, and the functional j is proper, convex, and lower semicontinuous, then the elliptic variational inequality (1.2.25) has a unique solution.*

For the analysis of nonlinear situations, we present a concise review of strongly monotone and Lipschitz operators. For this purpose, let's consider X as a Hilbert space equipped with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|.\|_X$. Suppose $\mathcal{A} : X \longrightarrow X$ an operator. The operator $\mathcal{A}$ is called:

(a) monotone if

$$\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle_X \geqslant 0 \quad \forall u, v \in X, \tag{1.2.26}$$

strictly monotone if

$$\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle_X > 0 \quad \forall u, v \in X, \; u \neq v, \tag{1.2.27}$$

and *strongly monotone if there exists $m_{\mathcal{A}} > 0$ such that:*

$$\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle_X \geqslant m_{\mathcal{A}} \|u - v\|_X^2 \quad \forall u, v \in X \tag{1.2.28}$$

(b) nonexpansive if

$$\|\mathcal{A}u - \mathcal{A}v\|_X \leqslant \|u - v\|_X \quad \forall u, v \in X$$

(c) Lipschitz continuous if there exists $M_{\mathcal{A}} > 0$ such that:

$$\|\mathcal{A}u - \mathcal{A}v\|_X \leqslant M_{\mathcal{A}} \|u - v\|_X \quad \forall u, v \in X$$

Properties of the inverse $\mathcal{A}^{-1}$ are given by the following result:

Proposition 1.5. *Under the above assumptions, $\mathcal{A}^{-1}$ is also a strongly monotone Lipschitz continuous operator*

Now, let $j : X \to \mathbb{R}$, $f \in X$, and consider the following problem:

Problem $(\mathbf{S}_2)$ *Find $u \in X$ such that:*

$$(\mathcal{A}u, v - u) + j(v) - j(u) \geqslant (f, v - u)_X \quad \forall v \in X \tag{1.2.29}$$

Concerning this problem, we have the following existence and uniqueness result.

Theorem 1.6. *If the operator $\mathcal{A} : X \to X$ is both strongly monotone and Lipschitz continuous, and j is a proper, convex, and lower semicontinuous function, then the second-kind elliptic variational problem (1.2.29) has a unique solution.*

A proof of Theorem 1.4 can be found in ([58], p. 60). Choosing $j \equiv 0$ in Theorem 1.4 we obtain the following result:

Corollary 1.2. *Let $\mathcal{A} : X \to X$ a strongly monotone and Lipschitz continuous operator. Then, for any $f \in X$, there exists a unique element $u \in X$ such that $\mathcal{A}u = f$.*
Moreover, the mapping $u \longrightarrow f$ is Lipschitz continuous from X to X.

Many boundary value problems in continuum mechanics also result in mathematical problems with the following form:

Problem $(\mathbf{S}_3)$ *Find $u \in X$ such that:*

$$(\mathcal{A}u, v - u) + j(u, v) - j(u, u) \geqslant (f, v - u)_X \quad \forall v \in X \tag{1.2.30}$$

In contrast to problem $(\mathbf{S}_2)$, problem $(\mathbf{S}_3)$ has a functional that depends on the solution being searched for. To address problem $(\mathbf{S}_3)$, we present an existence and uniqueness result.

Suppose that $\mathcal{A} : X \longrightarrow X$ is a nonlinear operator that is both strongly monotone and Lipschitz, meaning that:

$$(h_1) \quad \begin{cases} (a) \text{ There exists } m > 0 \text{ such that:} \\ \langle \mathcal{A}u - \mathcal{A}v, u - v \rangle_X \geqslant m_{\mathcal{A}} \|u - v\|_X^2 \quad \forall u, v \in X \\ (b) \text{ There exists } M_{\mathcal{A}} > 0 \text{ such that:} \\ \|\mathcal{A}u - \mathcal{A}v\|_X \leqslant M_{\mathcal{A}} \|u - v\|_X \quad \forall u, v \in X. \end{cases} \tag{1.2.31}$$

And suppose that the functional j satisfies

$$(h_2)\begin{cases} (a)\ \textit{For all } \eta \in X,\ j(\eta,.)\ \textit{is } \text{convex and lower semi continuous on } X \\ (b)\ \ \textit{There exists } c>0 \textit{ such that:} \\ j(u_1,v_2)-j(u_1,v_1)+j(u_2,v_1)-j(u_2,v_2)\leq c\left\|u_1-u_2\right\|_X\left\|v_1-v_2\right\|_X \\ \forall u_1,u_2,v_1,v_2\in X \end{cases}$$

The main result obtained in this case is the following:

Theorem 1.7. *Under the assumptions (h_1) and (h_2). If $c < m_{\mathcal{A}}$ for all $f \in X$. Then, there exists a unique solution $u \in X$ to problem $(\mathbf{S}_3)$.*

The proof of this result is carried out in several steps, and can be found for example in [58].

1.3 Complements

The Banach fixed-point theorem will be repeatedly used in this work to prove the existence of solutions to variational problems.

Theorem 1.8. *Let K be a nonempty and closed set in a Banach space X. Assume that $\Lambda : K \longrightarrow K$ is a contraction mapping Then there exists a unique $u \in K$ such that $\Lambda u = u$, i.e. Λ has a unique fixed point in K.*

We will also need a variant of the Banach fixed-point theorem, which we recall next. To that end we note that the powers of an operator Λ are defined inductively by $\Lambda^n = \Lambda(\Lambda^{n-1})$ for $n \geq 2$.

Theorem 1.9. *Let K be a nonempty and closed set in a Banach space X, and $\Lambda : K \longrightarrow K$. If Λ^n is a contraction mapping for a positive integer n, then Λ has a unique fixed point in K*

We recall that the classic Fatou's lemma can be formulated in the following form (See Brezis [13])

Lemma 1.10. *Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfy*
(a) for all n, $f_n \geq 0$ a.e. on Ω.

(b) $\sup_n \int_\Omega f_n < +\infty$.

For all $x \in \Omega$ *we set* $f(x) = \liminf_{n \longrightarrow +\infty} f_n(x) \leq +\infty$. *Then* $f \in L^1(\Omega)$ *and*

$$\int_\Omega f \leq \liminf_{n \longrightarrow +\infty} \int_\Omega f_n. \tag{1.3.1}$$

Let's recall a version of the Cauchy-Lipschitz theorem (see [108])

Theorem 1.11. *Let* X *be a real Banach space, and let* $\mathcal{F}(t,.) : X \longrightarrow X$ *be an operator defined almost everywhere on* $(0,T)$ *satisfying the properties:*

- There exists $M_\mathcal{F} > 0$ *such that* $\|\mathcal{F}(t,x) - \mathcal{F}(t,y)\|_X \leqslant M_\mathcal{F} \|x-y\|_X \quad \forall x, y \in X$, *a.e.* $t \in (0,T)$

- There exists $p \in [1,+\infty)$ *such that* $\mathcal{F}(t,.) \in L^p(0,T;X)$ $\forall x \in X$.

Then, for any $x_0 \in X$, *there exists a unique function* $x \in W^{1,p}(0,T;X)$ *such that:*

$$\begin{cases} \dot{x}(t) = \mathcal{F}(t, x(t)), \ \text{a.e. } t \in (0,T) \\ x(0) = x_0. \end{cases} \tag{1.3.2}$$

The Gronwall's lemmas play a significant role in various problems related to bounding and estimation. For further details, refer to [63].

Lemma 1.12. (Gronwall) *Let* h, g, l *are continuous real functions such* $h(t) \geq 0$, $g(t) \geq 0$ *for all* $t \in [0,T]$, *and let* c *be a constant.*

1- If

$$l(t) \leq c + \int_0^t h(s)ds + \int_0^t g(s)l(s)ds, \quad \forall t \in [0,T],$$

then:

$$l(t) \leq \left(c + \int_0^t h(s)ds\right) + e^{\int_0^t g(s)ds}, \quad \forall t \in [0,T].$$

2- If

$$l(t) \leq h(s) + c\int_0^t l(s)ds, \quad \forall t \in [0,T],$$

then:

$$\int_0^t l(s)ds \leq (c+h) + e^{ct}\int_0^t h(s)ds, \quad \forall t \in [0,T].$$

In the particular case $h = 0$, we have:

Lemma 1.13. *Let* g, l *are continuous functions, and let* c *be a positive constant.*

If

$$l(t) \leq g(t) + c\int_0^t l(s)ds, \quad \forall t \in [0,T],$$

then:

$$l(t) \leq g(t) + c \int_0^t g(s) e^{c(t-s)} ds, \quad \forall t \in [0,T] \, .$$

The previous lemma is often used to demonstrate uniqueness. Assuming that two solutions exist, denoting by the norm of the difference between these solutions, we then attempt to bound l in the form $l(t) \leq \int_0^t g(s) l(s) ds$ using a certain positive function.

Lemma 1.14. *Let h, g, l are continuous real functions such that $h(t) \geq 0$, $g(t) \geq 0$ for all $t \in [0,T]$, and let c be a constant.*
IF

$$\frac{1}{2} l^2(t) \leq \frac{1}{2} a^2 + \int_0^t h(s) l(s) ds + \int_0^t g(s) l^2(s) ds, \quad \forall t \in [0,T] \, ,$$

then:

$$|l(t)| \leq \left(a + \int_0^t h(s) ds \right) + e^{\int_0^t g(s) ds}, \quad \forall t \in [0,T] \, .$$

In the particular case of $g = 0$, we have:

Lemma 1.15. *Let h, l c such that $h(t) \geq 0$ for all $t \in [0,T]$, and let c be a constant.*
If

$$\frac{1}{2} l^2(t) \leq \frac{1}{2} a^2 + \int_0^t h(s) l(s) ds \ , \quad \forall t \in [0,T] \, ,$$

then:

$$|l(t)| \leq a + \int_0^t h(s) ds \ , \quad \forall t \in [0,T] \, .$$

In the remaining part, V denotes a Hilbert space such that V is dense in X, and the injection from V in X is continuous. The space X is identified with its own dual and is a subspace of the dual V' of V, meaning that $V \subset X \subset V'$ algebraically and topologically.

Let $\mathcal{A} : V \longrightarrow V'$ be an operator, $\mathcal{A}$ is said to be:
(a) hemicontinuous if for all $u, v \in V$, the application $t \longmapsto A(u + tv)$ is continuous on $\mathbb{R}$.
(b) pseudomonotone if $\mathcal{A}$ is continuous and verifies the property: If $u_n \rightharpoonup u$ in V, and $\limsup\limits_{n \longrightarrow +\infty} (\mathcal{A} u_n, u_n - u) \leq 0$, then

$$\liminf_{n \longrightarrow +\infty} (\mathcal{A} u_n, u_n - v) \geq (\mathcal{A} u, u - v) \ \ \forall v \in V. \tag{1.3.3}$$

Remark 1.3. *1- If the operator $\mathcal{A} : V \to V'$ is a strongly continuous operator, then $\mathcal{A}$ is pseudomonotone.*
2- If $\mathcal{A} : V \to V'$ is monotone and hemicontinuous, then $\mathcal{A}$ is pseudomonotone. (see [122])

Theorem 1.16. *Let $\mathcal{A} : V \longrightarrow V'$ a pseudomonotone operator*
1 - If $u_n \rightharpoonup u$ in V, $\mathcal{A}(u_n) \rightharpoonup \eta$ and $(\mathcal{A}u_n, u_n) \leq (\eta, u_n)$, then $\mathcal{A}(u) = \eta$.
2- $\frac{(\mathcal{A}u,v)}{\|v\|} \longrightarrow +\infty$ when $v \longrightarrow +\infty$. Then $\forall f \in V'$, the equation $\mathcal{A}(u) = f$ has at least one solution.

To see the proof of the previous theorem as well as other properties and results, the reader can refer to [85] and [122].

Chapitre **2**

Optimal control of a frictional contact problem for locking materials

Contents

In this chapter, we study the optimal control of a bilateral contact between a nonlinear elastic locking material and a rigid foundation. The contact is assumed to be static and it is described by Tresca's friction law between a locking body and a rigid foundation. We state an optimal control problem that admits at least one solution. We also introduce the penalized and regularized optimal control problem for which we study the convergence when the penalization and regularization parameter tends to zero. The analysis of this problem has been the subject of the publication [52].

2.1 The contact problem and its weak solvability

We consider a locking body which initially occupies a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with a sufficiently smooth boundary $\partial\Omega = \Gamma$ partitioned into three disjoint measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $meas\,(\Gamma_1) > 0$. The body is clamped on Γ_1 and then the displacement vanishes there. It is acted upon by a volume force of density f_0 in Ω and a surface traction of density f_2 on Γ_2. On Γ_3 the body is in bilateral contact following Tresca's friction law with a rigid foundation. Thus, the classical formulation of the mechanical problem is written as follows.

Problem $(P1)$**.** *Find a displacement field* $u : \Omega \to \mathbb{R}^d$ *such that*

$$Div\,\sigma(u) + f_0 = 0 \quad \text{in } \Omega, \tag{2.1.1}$$

$$\sigma(u) \in \mathcal{F}\varepsilon(u) + \partial I_B(\varepsilon(u)) \quad \text{in } \Omega, \tag{2.1.2}$$

$$u = 0 \quad \text{on } \Gamma_1, \tag{2.1.3}$$

$$\sigma(u)\,\nu = f_2 \quad \text{on } \Gamma_2, \tag{2.1.4}$$

$$\begin{cases} u_\nu = 0 \\ \|\sigma_\tau(u)\| \le g \\ \|\sigma_\tau(u)\| < g \Longrightarrow u_\tau = 0 \\ \|\sigma_\tau(u)\| = g \Longrightarrow \exists\lambda \ge 0 \text{ s.t. } \sigma_\tau(u) = -\lambda u_\tau \end{cases} \quad \text{on } \Gamma_3. \tag{2.1.5}$$

Equation (2.1.1) represents the equilibrium equation. Equation (2.1.2) represents the elastic constitutive law where $\mathcal{F}$ is a given nonlinear function, and I_B is the indicator function of the set defined by

$$B = \{\xi \in S_d;\ \|\xi\| \le M_L\},$$

such that

$$I_B(\xi) = \begin{cases} 0 & \text{if } \xi \in B, \\ +\infty, & \text{if } \xi \notin B \end{cases} \quad \text{for } \xi \in S_d.$$

Equations (2.1.3) and (2.1.4) are the displacement and traction boundary conditions, respectively. Finally, (2.1.5) represents the bilateral contact with Tresca's friction law where g is a given friction bound.

Let V be the closed subspace of H_1 defined by:

$$V = \{v \in H_1;\ v = 0 \text{ on } \Gamma_1,\ v_\nu = 0 \text{ on } \Gamma_3\},$$

As in Chapter 1, we define by $\langle .,. \rangle$ the duality pairing between V' and V. Since $meas\,(\Gamma_1) > 0$, Korn's inequality (1.2.12) holds. We consider in V the inner product $(u,v)_V = (\varepsilon(u), \varepsilon(v))_Q$ and the associated norm $\|.\|_V$, which is equivalent to the canonical norm $\|.\|_{H_1}$. Thus, $(V, \|.\|_V)$ is a real Hilbert space, and (1.2.13) holds.
Let the closed convex subset of V

$$K = \{v \in V;\, \varepsilon\,(v\,(x)) \in B \ a.e. \ x \in \Omega\}\,.$$

We assume that the body forces and surface tractions have the regularity

$$f_0 \in H, \qquad f_2 \in \left(L^2\,(\Gamma_2)\right)^d \tag{2.1.6}$$

and we define the functional $j : V \to \mathbb{R}_+$ by

$$j\,(v) = \int_{\Gamma_3} g\,\|v_\tau\|\,da,$$

where g is assumed to satisfy

$$g \in L^\infty\,(\Gamma_3) \ \text{ and } g \geq 0 \ a.e. \text{ on } \Gamma_3. \tag{2.1.7}$$

Next, in the study of Problem $(P1)$ we assume that the nonlinear elasticity operator $\mathcal{F}$ satisfies

$$\begin{cases} (a) \ \ \mathcal{F} : \Omega \times S_d \to S_d; \\ (b) \text{ there exists } M > 0 \text{ such that} \\ \quad \|\mathcal{F}\,(x, \varepsilon_1) - \mathcal{F}\,(x, \varepsilon_2)\| \leq M\,\|\varepsilon_1 - \varepsilon_2\|\,, \forall \varepsilon_1, \varepsilon_2 \in S_d, \quad \text{a.e. } x \ \in \Omega; \\ (c) \text{ there exists } m > 0 \text{ such that} \\ \quad (\mathcal{F}\,(x, \varepsilon_1) - \mathcal{F}\,(x, \varepsilon_2))\,.\,(\varepsilon_1 - \varepsilon_2) \geq m\,\|\varepsilon_1 - \varepsilon_2\|^2\,, \ \ \forall \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } x \ \in \Omega; \\ (d) \text{ the mapping } x \to \mathcal{F}\,(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \quad \text{for any } \varepsilon \in S_d; \\ (e) \ \ \mathcal{F}\,(x, 0) = 0 \text{ for a.e. } x \in \Omega. \end{cases} \tag{2.1.8}$$

Examples of nonlinear law that satisfy (2.1.8) can be found in [91] and [110], (see Appendix).

Now, we derive the variational formulation of Problem $(P1)$. To this end, let $(u, \sigma\,(u))$ be a pair of smooth functions which satisfies (2.1.1)-(2.1.5), and let $v \in V$.
Multiplying the equilibrium equation (2.1.1) by $v - u$ and using the Green formula (1.2.12), we deduce that

$$(\sigma\,(u)\,, \varepsilon\,(v) - \varepsilon\,(u))_Q = (f_0, v - u)_H + \int_\Gamma \sigma\,(u)\,\nu.\,(v - u)\,da.$$

Using the boundary conditions (2.1.3) and (2.1.4), we have

$$\begin{aligned}(\sigma(u), \varepsilon(v) - \varepsilon(u))_Q = (f_0, v-u)_H + (f_2, v-u)_{(L^2(\Gamma_2))^d} \\ + \textstyle\int_{\Gamma_3} \sigma_\tau(u) \,.\, (v_\tau - u_\tau)\, da.\end{aligned} \tag{2.1.9}$$

On the other hand, condition (2.1.5) implies

$$\int_{\Gamma_3} \sigma_\tau(u) \,.\, (v_\tau - u_\tau)\, da \geq \int_{\Gamma_3} g\left(\|u_\tau\| - \|v_\tau\|\right) da. \tag{2.1.10}$$

From the constitutive law (2.1.2), we have

$$\sigma(u) = \mathcal{F}\varepsilon(u) + \zeta(u) \text{ and } \zeta(u) \in \partial I_B(\varepsilon(u)) \text{ in } \Omega.$$

The latter, for $v, u \in K$, implies

$$\zeta(u) \,.\, (\varepsilon(v) - \varepsilon(u)) \leq I_B(\varepsilon(v)) - I_B(\varepsilon(u)) = 0 \text{ in } \Omega.$$

Hence, we obtain

$$(\sigma(u), \varepsilon(v) - \varepsilon(u))_Q \leq (\mathcal{F}\varepsilon(u), \varepsilon(v) - \varepsilon(u))_Q. \tag{2.1.11}$$

We define the operator $A : V \to V$ by

$$(Au, v)_V = (\mathcal{F}\varepsilon(u), \varepsilon(v))_Q, \ \forall u, v \in V. \tag{2.1.12}$$

Inserting (2.1.10) and (2.1.11) in (2.1.9) and taking into account (2.1.12), we obtain the following variational formulation of

Problem $(P1_V)$**.** Find $u \in K$ such that

$$(Au, v-u)_V + j(v) - j(u) \geq (f_0, v-u)_H + (f_2, v-u)_{(L^2(\Gamma_2))^d} \quad \forall v \in K. \tag{2.1.13}$$

Theorem 2.1. *Assume (2.1.6), (2.1.7) and (2.1.8) hold. Then, there exists a unique solution of Problem* $(P1_V)$.

Proof. By (2.1.8), the operator A is Lipschitz continuous and strongly monotone; using (2.1.7), the functional j is proper, convex and lower semicontinuous, then moreover by using (2.1.6), since K is a non-empty closed convex, it follows (see Chap1) that the inequality (2.1.13) has a unique solution. ■

2.2 The optimal control problem

We now suppose that $f_0 \in H$ is fixed and consider the following state variational problem.

Problem $(\mathbf{Q_1})$. For $f_2 \in (L^2(\Gamma_2))^d$ (called control), find $u \in K$ such that

$$\begin{aligned}(Au, v-u)_V &+ j(v) - j(u) \\ &\geq (f_0, v-u)_H + (f_2, v-u)_{(L^2(\Gamma_2))^d} \quad \forall v \in K.\end{aligned} \tag{2.2.1}$$

Following the existence and uniqueness of Problem $(P1_V)$, we deduce that for every control $f_2 \in (L^2(\Gamma_2))^d$, the state variational Problem $(\mathbf{Q_1})$ has a unique solution $u \in K$.

For $\alpha, \beta > 0$ and $u_d \in K$, we define the cost functional

$$\mathcal{L} : V \times \left(L^2(\Gamma_2)\right)^d \to \mathbb{R}_+,$$

by

$$\mathcal{L}(u, f_2) = \alpha \|u - u_d\|_V^2 + \beta \|f_2\|^2_{(L^2(\Gamma_2))^d}. \tag{2.2.2}$$

We have that $\sigma_d = \sigma(u_d) = \mathcal{F}\varepsilon(u_d)$, then for $u \in K$, we have $\sigma(u) = \mathcal{F}\varepsilon(u)$, and

$$\|\sigma(u) - \sigma(u_d)\|_Q \leq M \|u - u_d\|_V,$$

so $\sigma(u)$ is close to $\sigma(u_d)$.

Next, we define the set of admissible pairs U_{ad} as

$$U_{ad} = \left\{(u, f_2) \in K \times \left(L^2(\Gamma_2)\right)^d, \text{ such that (2.2.1) is satisfied}\right\},$$

and we consider the optimal control problem below.

Problem $(\mathbf{C_1})$. Find $(u^*, f_2^*) \in U_{ad}$ such that

$$\mathcal{L}(u^*, f_2^*) = \min_{(u, f_2) \in U_{ad}} \mathcal{L}(u, f_2).$$

Theorem 2.2. *Assume (2.1.6), (2.1.7) and (2.1.8)(c). Then Problem $(\mathbf{C_1})$ has at least one solution.*

Proof. Take $v = 0_V$ in (2.2.1), using (1.2.13) and (2.1.8)(c), we deduce that the solution u of Problem $(\mathbf{Q_1})$ is bounded in V as

$$\|u\|_V \leq \frac{c_0}{m}\left(\|f_0\|_H + d_\Omega \|f_2\|_{(L^2(\Gamma_2))^d}\right),$$

where $c_0 > 0$. This inequality implies that

$$0 \le \inf_{(u,f_2)\in U_{ad}} \mathcal{L}\left(u, f_2\right) < +\infty.$$

Then, there exists a sequence $(u^n, \{_\in{}^n) \subset U_{ad}$ such that

$$\mathcal{L}\left(u^n, f_2^n\right) \to \inf_{(u,f_2)\in U_{ad}} \mathcal{L}\left(u, f_2\right) \text{ as } n \to +\infty.$$

The sequence (u^n, f_2^n) is bounded in $V \times \left(L^2\left(\Gamma_2\right)\right)^d$, so there exists an element

$$\left(u^*, f_2^*\right) \in V \times \left(L^2\left(\Gamma_2\right)\right)^d$$

such that passing to a subsequence still denoted by (u^n, f_2^n), we have

$$\begin{cases} (a) \;\; u^n \to u^* \text{ weakly in } V, \quad \text{as } n \to +\infty, \\ (b) \;\; f_2^n \to f_2^* \text{ weakly in } \left(L^2\left(\Gamma_2\right)\right)^d \text{ as } n \to +\infty, . \end{cases} \tag{2.2.3}$$

But for the remainder of the proof, we had to show that

$$u^n \to u^* \text{ strongly } in\ V\ as\ n \to +\infty. \tag{2.2.4}$$

In fact, as $(u^n, f_2^n) \in U_{ad}$, u^n satisfies the inequality:

$$\begin{aligned} &\left(Au^n, v - u^n\right)_V + j\left(v\right) - j\left(u^n\right) \\ &\qquad \ge \left(f_0, v - u^n\right)_H + \left(f_2^n, v - u^n\right)_{(L^2(\Gamma_2))^d} \quad \forall v \in K. \end{aligned} \tag{2.2.5}$$

Using (2.1.8)(c) and (2.2.5), we deduce that

$$\begin{cases} m\left\|u^n - u^*\right\|_V^2 \le \left(Au^n - Au^*, u^n - u^*\right)_V \\ \qquad \le \left(Au^n, u^n - u^*\right)_V - \left(Au^*, u^n - u^*\right)_V \\ \qquad \le \left(Au^*, u^* - u^n\right)_V + j\left(u^*\right) - j\left(u^n\right) \\ \qquad + \left(f_0, u^n - u^*\right)_H + \left(f_2^n, u^n - u^*\right)_{(L^2(\Gamma_2))^d}. \end{cases} \tag{2.2.6}$$

Now from (2.2.3)(a), we have that $\left(Au^*, u^n - u^*\right)_V \to 0$ as $n \to +\infty$. Next, using that $u^n \to u^*$ weakly in V implies that $u^n \to u^*$ strongly in $\left(L^2\left(\Gamma_2\right)\right)^d$ and as $\left(f_2^n\right)$ is bounded in $\left(L^2\left(\Gamma_2\right)\right)^d$, then

$$j\left(u^*\right) - j\left(u^n\right) + \left(f_0, u^n - u^*\right)_H + \left(f_2^n, u^n - u^*\right)_{(L^2(\Gamma_2))^d} \to 0 \text{ as } n \to +\infty.$$

This suggests that the last member on the right side of the last inequality tends to be zero. Hence, from (2.2.6), we get (2.2.4). On the other hand, K is closed convex of V, then $u^* \in K$. Moreover, using (2.2.5)(b), (2.2.4) and passing to the limit as $n \to +\infty$ in (2.2.5), we get that $(u^*, f_2^*) \in U_{ad}$ and then this is a solution to Problem $(\mathbf{C_1})$. ■

2.3 The penalized and regularized problem

In this part we consider the regularized problem that can be exploited numerically. The interest is to approximate the nondifferentiable term by a sequence of differentiable ones. This regularization is obtained by substituting the functional j with a regularized function $j_\delta : V \to \mathbb{R}$ defined by

$$j_\delta(v) = \int_{\Gamma_3} g\sqrt{v_\tau^2 + \delta^2}da,$$

where $\delta > 0$ is a small parameter.

Recall (see [1]) that the functional j_δ is proper, convex, lower semicontinuous and satisfies $j_\delta \in C^2(V)$. Indeed, denoting by $Dj_\delta(u)$ the differential of j_δ at the point u, we have

$$(Dj_\delta(u), v)_V = \int_{\Gamma_3} g\frac{u_\tau v_\tau}{\sqrt{u_\tau^2 + \delta^2}}da \quad \forall v \in V \tag{2.3.1}$$

and

$$\left(D^2 j_\delta(u, v), w\right)_V = \int_{\Gamma_3} g\frac{\delta^2 v_\tau w_\tau}{(u_\tau^2 + \delta^2)\sqrt{u_\tau^2 + \delta^2}}da \quad \forall v, w \in V. \tag{2.3.2}$$

Next, in the penalty problem, we will replace $I_B(\xi)$ by $\frac{2}{\delta}\left[(\|\tau\| - M_L)_+\right]^2$, where $(\|\tau\| - M_L)_+ = \max\{0, \|\tau\| - M_L\}$. For this purpose, we denote

$$P_\delta(\tau) = \frac{2}{\delta}\left[(\|\tau\| - M_L)_+\right]\frac{\tau}{\|\tau\|}, \ \forall \tau \in S_d, \tag{2.3.3}$$

We define the operator A_δ as

$$(A_\delta u, v)_V = (P_\delta(\varepsilon(u)), \varepsilon(v))_Q, \ \forall u, v \in V.$$

Then, we have the lemma below.

Lemma 2.3. *Operator A_δ verifies the following properties:*

(1) *A_δ is monotone:*

$$(A_\delta u - A_\delta v, u - v)_V \geq 0, \ \forall u, v \in V. \tag{2.3.4}$$

(2) *A_δ is Lipschitz continuous:*

$$|(A_\delta u - A_\delta v, w)_V| \leq \frac{4}{\delta}\|u - v\|_V \|w\|_V, \ \forall u, v, w \in V. \tag{2.3.5}$$

Proof. The mapping $\Psi_\delta : S_d \to \mathbb{R}_+;\ \xi \longmapsto \frac{1}{\delta}\left((\|\xi\| - M_L)_+\right)^2$ is a convex function and continuously differentiable and

$$\begin{aligned}(\Psi_\delta'(\xi);\tau) &= \lim_{\lambda\to 0}\frac{\Psi_\delta(\xi+\lambda\tau)-\Psi_\delta(\xi)}{\lambda}\\ &= \frac{2}{\delta}(\|\xi\|-M_L)_+\frac{\xi}{\|\xi\|}.\tau = P_\delta(\xi).\tau,\ \forall\xi,\tau\in S_d.\end{aligned}$$

Then, the mapping $G : V \to \mathbb{R}$, $u \longmapsto \Psi_\delta(\varepsilon(u))$ is also a convex function and continuously differentiable and

$$\begin{aligned}&\left\langle G'(u), v\right\rangle = (\Psi_\delta'(\varepsilon(u)), \varepsilon(v)) = \tfrac{2}{\delta}(\|\varepsilon(u)\| - M_L)_+\tfrac{\varepsilon(u)}{\|\varepsilon(u)\|}.\varepsilon(v)\\ &= P_\delta(\varepsilon(u)).\varepsilon(v)),\ \forall u,v\in V.\end{aligned}$$

The property of convexity of G implies that $G'$is monotone, then

$$\left\langle G'(u) - G'(v), u-v\right\rangle \geq 0,\ \forall u,v\in V.$$

By integrating the two sides of the previous inequality on Ω, one gets inequality (2.3.4).
Now, to prove (2), we have

$$(A_\delta u - A_\delta v, w)_V = \tfrac{2}{\delta}\int_\Omega\left(\frac{(\|\varepsilon(u)\| - M_L)_+\varepsilon(u)\,\|\varepsilon(v)\| - (\|\varepsilon(u)\| - M_L)_+\varepsilon(v)\,\|\varepsilon(u)\|}{\|\varepsilon(u)\|\,\|\varepsilon(v)\|}\right)\varepsilon(w)\,dx.$$

We see that there are three cases.
The first case: if $\|\varepsilon(u)\| \leq M_L$, $\|\varepsilon(v)\| \leq M_L$, then $(T_\delta u - T_\delta v, w)_V = 0$,
and $(A_\delta u - A_\delta v, w)_V = 0$.
The second case: if $\|\varepsilon(u)\| > M_L$, $\|\varepsilon(v)\| \leq M_L$, then

$$\begin{aligned}&|(A_\delta u - A_\delta v, w)_V| = \tfrac{2}{\delta}\left|\int_\Omega(\|\varepsilon(u)\| - M_L)\tfrac{\varepsilon(u)}{\|\varepsilon(u)\|}\varepsilon(w)\,dx\right|\\ &\leq \tfrac{2}{\delta}\int_\Omega \|\varepsilon(u)\| - \|\varepsilon(v)\|\,\|\varepsilon(w)\|\,dx\\ &\leq \tfrac{2}{\delta}\int_\Omega\|\varepsilon(u)-\varepsilon(v)\|\,\|\varepsilon(w)\|\,dx\\ &\leq \tfrac{2}{\delta}\|u-v\|_V\,\|w\|_V\,.\end{aligned}$$

The third case: if $\|\varepsilon(u)\| > M_L$, $\|\varepsilon(v)\| > M_L$, we have

$$|(A_\delta u - A_\delta v, w)_V| = \frac{2}{\delta}\left|\int_\Omega ((\varepsilon(u)-\varepsilon(v)) - M_L(\tfrac{\varepsilon(u)}{\|\varepsilon(u)\|} - \tfrac{\varepsilon(v)}{\|\varepsilon(v)\|}))\varepsilon(w)\,dx\right|$$

$$\leq \frac{2}{\delta}\int_\Omega \|\varepsilon(u)-\varepsilon(v)\|\,\|\varepsilon(w)\|\,dx + \frac{2}{\delta}\left|\int_\Omega M_L\left(\tfrac{\varepsilon(v)}{\|\varepsilon(v)\|} - \tfrac{\varepsilon(u)}{\|\varepsilon(u)\|}\right)\varepsilon(w)\,dx\right|$$

$$\leq \frac{2}{\delta}\int_\Omega \|\varepsilon(u)-\varepsilon(v)\|\,\|\varepsilon(w)\|\,dx + \frac{2}{\delta}\int_\Omega \left|M_L\left(\tfrac{\varepsilon(v)}{\|\varepsilon(v)\|} - \tfrac{\varepsilon(u)}{\|\varepsilon(u)\|}\right)\right|\,\|\varepsilon(w)\|\,dx$$

$$\leq \frac{4}{\delta}\int_\Omega \|\varepsilon(u)-\varepsilon(v)\|\,\|\varepsilon(w)\|\,dx.$$

$$\leq \frac{4}{\delta}\|u-v\|_V\,\|w\|_V\,.$$

Then, it follows that in all the cases, (2.3.22) is satisfied. Hence, we end the proof of Lemma 2.3. ■

We now consider the penalized and regularized problem below.

Problem (P_δ)**.** Find $u^\delta \in V$ such that

$$\begin{aligned}&\left(Au^\delta, v\right)_V + \left(Dj_\delta\left(u^\delta\right), v\right)_V + \left(P_\delta\left(\varepsilon\left(u^\delta\right)\right), \varepsilon(v)\right)_Q \\ &= (f_0, v)_H + (f_2, v)_{(L^2(\Gamma_2))^d} \quad \forall v \in V.\end{aligned} \tag{2.3.6}$$

Theorem 2.4. *Let (2.1.6), (2.1.7), (2.1.8), (2.3.4) and (2.3.5) hold. Then, Problem* (P_δ) *has a unique solution.*

Proof. We define the operator $B_\delta : V \to V$ by

$$(B_\delta u, v)_V = (Au, v)_V + (Dj_\delta(u), v)_V + (P_\delta(\varepsilon(u)), \varepsilon(v))_Q\,,\ \forall u, v \in V.$$

Using (2.1.6), (2.1.8)(b) and (2.3.5), we have that for all $u, v, w \in V$:

$$|(B_\delta u - B_\delta v, w)_V| \leq (M + \frac{4}{\delta})\|u-v\|_V\,\|w\|_V + |(Dj_\delta(u) - Dj_\delta(v), w)_V|\,. \tag{2.3.7}$$

On the other hand, by the mean value theorem and (2.3.2), there exist $\theta \in (0,1)$ such that

$$(Dj_\delta(u) - Dj_\delta(v), w)_V = \int_{\Gamma_3} \frac{g\delta^2(u_\tau - v_\tau)w_\tau}{\left((v_\tau + \theta(u_\tau - v_\tau))^2 + \delta^2\right)\sqrt{\left((v_\tau + \theta(u_\tau - v_\tau))^2 + \delta^2\right)}}da, \tag{2.3.8}$$

which means

$$\left|\left(Dj_\delta\left(u\right)-Dj_\delta\left(v\right),w\right)_V\right|\le\frac{1}{\delta}\left\|g\right\|_{L^\infty(\Gamma_3)}\left\|u_\tau-v_\tau\right\|_{(L^2(\Gamma_3))^d}\left\|w_\tau\right\|_{(L^2(\Gamma_3))^d}.$$

Then, using (1.2.13, we deduce

$$\left|\left(Dj_\delta\left(u\right)-Dj_\delta\left(v\right),w\right)_V\right|\le\frac{1}{\delta}d_\Omega^2\left\|g\right\|_{L^\infty(\Gamma_3)}\left\|u-v\right\|_V\left\|w\right\|_V. \tag{2.3.9}$$

Thus, by combining (2.3.7) and (2.3.9), we obtain

$$\left\|B_\delta u-B_\delta v\right\|_V\le\left(M+\frac{4+d_\Omega^2\left\|g\right\|_{L^\infty(\Gamma_3)}}{\delta}\right)\left\|u-v\right\|_V.$$

Then, the operator B_δ is Lipschitz continuous.

Now, to prove that B_δ is strongly monotone, we use (2.1.8)(c) and (2.3.8), we get for all $u,v\in V$:

$$\begin{aligned}\left(B_\delta u-B_\delta v,u-v\right)_V &\ge m\left\|u-v\right\|_V^2+\int_{\Gamma_3}\frac{g\delta^2\left(u_\tau-v_\tau\right)^2}{\left(\left(v_\tau+\theta\left(u_\tau-v_\tau\right)\right)^2+\delta^2\right)\sqrt{\left(\left(v_\tau+\theta\left(u_\tau-v_\tau\right)\right)^2+\delta^2\right)}}da\\ &\ge m\left\|u-v\right\|_V^2,\quad \text{where } \theta\in(0,1).\end{aligned}$$

This implies that Problem (P_δ) has a unique solution. ■

The next convergence is demonstrated below.

Theorem 2.5. *The following convergence holds:*

$$u^\delta\to u \text{ strongly in } V \text{ as } \delta\to 0, \tag{2.3.10}$$

where u is a solution of Problem $(P1_V)$.

Proof. We take $v=u^\delta$ in (2.3.6), then as $\left(Dj_\delta\left(u^\delta\right),u^\delta\right)_V\ge 0$, $\left(P_\delta\left(\varepsilon\left(u^\delta\right)\right),\varepsilon\left(u^\delta\right)\right)_Q\ge 0$, we deduce that

$$\left(Au^\delta,u^\delta\right)_V\le\left(f_0,u^\delta\right)_H+\left(f_2,u^\delta\right)_{(L^2(\Gamma_2))^d}.$$

Thus, by (1.2.13) and (2.1.8)(c), we get

$$\left\|u^\delta\right\|_V\le\frac{c_0}{m}\left(\left\|f_0\right\|_H+d_\Omega\left\|f_2\right\|_{\left((L^2(\Gamma_2))^d\right)}\right). \tag{2.3.11}$$

This estimate implies that there exists an element $\bar{u}$ such that

$$u^\delta\to\bar{u} \text{ weakly in } V \text{ as } \delta\to 0. \tag{2.3.12}$$

Taking $v \in K$ in (2.3.6) and $v = u^\delta$ in the same equality, by adding the resulting inequalities, we deduce

$$\begin{aligned} & \left(Au^\delta, v - u^\delta\right)_V + \left(Dj_\delta\left(u^\delta\right), v - u^\delta\right)_V + \left(P_\delta\left(\varepsilon\left(u^\delta\right)\right), \varepsilon(v) - \varepsilon\left(u^\delta\right)\right)_Q \\ & = \left(f_0, v - u^\delta\right)_H + \left(f_2, v - u^\delta\right)_{(L^2(\Gamma_2))^d} \quad \forall v \in K. \end{aligned} \tag{2.3.13}$$

From (2.3.4), we have

$$\left(P_\delta\left(\varepsilon\left(u^\delta\right)\right) - P_\delta\left(\varepsilon(v)\right), \varepsilon(v) - \varepsilon\left(u^\delta\right)\right)_Q \leq 0,$$

thus

$$\left(P_\delta\left(\varepsilon\left(u^\delta\right)\right), \varepsilon(v) - \varepsilon\left(u^\delta\right)\right)_Q \leq \left(P_\delta\left(\varepsilon(v)\right), \varepsilon(v) - \varepsilon\left(u^\delta\right)\right)_Q \leq 0 \tag{2.3.14}$$

On the other hand, since $j_\delta(u^\delta,)$ is convex, we obtain

$$\left(Dj_\delta\left(u^\delta\right), v - u^\delta\right)_V \leq j_\delta(v) - j_\delta\left(u^\delta\right), \tag{2.3.15}$$

Then, it follows from (2.3.13), (2.3.14), and (2.3.15), that

$$\left(Au^\delta, v - u^\delta\right)_V + j_\delta(v) - j_\delta\left(u^\delta\right) \geq \left(f_0, v - u^\delta\right)_H + \left(f_2, v - u^\delta\right)_{(L^2(\Gamma_2))^d}. \tag{2.3.16}$$

Taking now $v = \bar{u}$ in (2.3.16), we see that

$$\left(Au^\delta, u^\delta - \bar{u}\right)_V \leq j_\delta(\bar{u}) - j_\delta\left(u^\delta\right) + \left(f_0, u^\delta - \bar{u}\right)_H + \left(f_2, u^\delta - \bar{u}\right)_{(L^2(\Gamma_2))^d}.$$

As the right hand side of the above inequality tends to zero when $\delta \to 0$, then we get

$$\limsup_{\delta \to 0} \left(Au^\delta, u^\delta - \bar{u}\right)_V \leq 0.$$

Thus, using the pseudomonotonicity of A, we deduce that

$$(A\bar{u}, \bar{u} - v)_V \leq \liminf_{\delta \to 0} \left(Au^\delta, u^\delta - v\right)_V, \forall v \in K.$$

Keeping in mind (2.3.12) and passing to the limit as $\delta \to 0$ in (2.3.16), we get

$$j_\delta(v) = \int_{\Gamma_3} g\sqrt{v_\tau^2 + \delta^2}\,da \to j(v), \tag{2.3.17}$$

We have

$$\begin{aligned} \left|\int_{\Gamma_3} g\left(\sqrt{(u_\tau^\delta)^2 + \delta^2} - \|\bar{u}_\tau\|\right)da\right| & \leq \left|\int_{\Gamma_3} g\left(\left\|u_\tau^\delta - \bar{u}_\tau\right\| + \delta\right)da\right| \\ & \leq \|g\|_{L^\infty(\Gamma_3)} \left\|u_\tau^\delta - \bar{u}_\tau\right\|_{(L^2(\Gamma_2))^d} + \|g\|_{L^\infty(\Gamma_3)}\, \delta mes(\Gamma_3) \end{aligned}$$

however, $\|g\|_{L^\infty(\Gamma_3)} \delta mes\,(\Gamma_3) \to 0$, as $\delta \to 0$, thus

$$j_\delta\left(u^\delta\right) = \int_{\Gamma_3} g(\sqrt{(u_\tau^\delta)^2 + \delta^2} - \|\bar{u}_\tau\|)da + \int_{\Gamma_3} g\,\|\bar{u}_\tau\|\,da \to \int_{\Gamma_3} g\,\|\bar{u}_\tau\|\,da = j\,(\bar{u})\,, \quad (2.3.18)$$

Also, according to the fact that the Sobolev trace is compact, $u^\delta \to u$ strongly in $\left(L^2\left(\Gamma_2\right)\right)^d$. Using (2.3.11), we have

$$\left(f_0, v - u^\delta\right)_H + \left(f_2, v - u^\delta\right)_{(L^2(\Gamma_2))^d} \to \left(f_0, v - \bar{u}\right)_H + \left(f_2, v - \bar{u}\right)_{(L^2(\Gamma_2))^d}\,, \quad (2.3.19)$$

thus, by (2.3.18), (2.3.19), and (2.3.20), we obtain

$$(A\bar{u}, v - \bar{u})_V + j\,(v) - j\,(\bar{u}) \geq (f_0, v - \bar{u})_H + (f_2, v - \bar{u})_{(L^2(\Gamma_2))^d}\,. \quad (2.3.20)$$

Now we claim to prove that $\overline{u} \in K$. Indeed, take $v = u^\delta$ in (2.3.6), we get

$$\left(P_\delta\left(\varepsilon\left(u^\delta\right)\right), \varepsilon\left(u^\delta\right)\right)_Q \leq \left(f_0, u^\delta\right)_H + \left(f_2, u^\delta\right)_{(L^2(\Gamma_2))^d}\,.$$

This inequality with (2.3.11) implies that

$$\int_\Omega \left(\left\|\varepsilon\left(u^\delta\right)\right\| - M_L\right)_+ \left\|\varepsilon\left(u^\delta\right)\right\| dx \leq \frac{\delta c_0}{m}\left(\|f_0\|_H + d_\Omega\,\|f_2\|_{\left((L^2(\Gamma_2))^d\right)}\right)^2. \quad (2.3.21)$$

However

$$\begin{aligned} \int_\Omega \left(\left\|\varepsilon\left(u^\delta\right)\right\| - M_L\right)_+ \left\|\varepsilon\left(u^\delta\right)\right\| dx &= \int_\Omega \left(\left\|\varepsilon\left(u^\delta\right)\right\| - M_L\right)_+ \left(\left\|\varepsilon\left(u^\delta\right)\right\| - M_L + M_L\right) dx \\ &= \int_\Omega \left(\left\|\varepsilon\left(u^\delta\right)\right\| - M_L\right)_+^2 dx + M_L \int_\Omega \left(\left\|\varepsilon\left(u^\delta\right)\right\| - M_L\right)_+ dx \end{aligned}$$

which implies that

$$\int_\Omega \left(\left\|\varepsilon\left(u^\delta\right)\right\| - M_L\right)_+^2 dx \leq \int_\Omega \left(\left\|\varepsilon\left(u^\delta\right)\right\| - M_L\right)_+ \left\|\varepsilon\left(u^\delta\right)\right\| dx.$$

So from the inequality (2.3.21), we deduce that

$$\limsup_{\delta \to 0} \int_\Omega \left(\left\|\varepsilon\left(u^\delta\right)\right\| - M_L\right)_+^2 dx = 0.$$

The mapping $h : V \to \mathbb{R}$, $u \longmapsto \left(\|\varepsilon\,(u)\| - M_L\right)_+^2$ is a convex and continuous function, thus h is weakly lower semicontinuous. Then, by (2.3.11), we have

$$\left(\|\varepsilon\,(\bar{u})\| - M_L\right)_+^2 \leq \liminf_{\varepsilon \to 0} \left(\left\|\varepsilon\left(u^\delta\right)\right\| - M_L\right)_+^2.$$

Thus, by Fatou's lemma, we get

$$\begin{aligned}\int_\Omega (\|\varepsilon(\bar{u})\| - M_L)_+^2\, dx &\leq \liminf_{\delta\to 0} \int_\Omega \left(\left\|\varepsilon\left(u^\delta\right)\right\| - M_L\right)_+^2 dx \\ &\leq \limsup_{\delta\to 0} \int_\Omega \left(\left\|\varepsilon\left(u^\delta\right)\right\| - M_L\right)_+^2 dx = 0.\end{aligned}$$

We therefore obtain

$$\int_\Omega (\|\varepsilon(\bar{u})\| - M_L)_+^2\, dx = 0.$$

Then, we deduce that $(\|\varepsilon(\bar{u})\| - M_L)_+ = 0$ a.e. in Ω, i.e. $\|\varepsilon(\bar{u})\| \leq M_L$ a.e. in Ω, i.e. $\bar{u} \in K$. So, we deduce that $\bar{u}$ is a solution of Problem $(P1_V)$. Following the uniqueness part of Theorem 2.1, we obtain that $\bar{u} = u$.

Now, we claim to prove the strong convergence. Indeed, we have by using (2.1.8)(c) and (2.3.16),

$$\begin{aligned}&m\|u_\delta - u\|_V^2 \leq (Au_\delta - Au, u_\delta - u)_V \\ &= (Au_\delta, u_\delta - u)_V - (Au, u_\delta - u)_V \\ &\leq (Au, u - u_\delta)_V + j_\delta(u) - j_\delta(u_\delta) + (f_0, u_\delta - u)_H + (f_0, u_\delta - u)_{(L^2(\Gamma_2))^d}.\end{aligned} \tag{2.3.22}$$

Then, using (2.3.11), and that $u^\delta \to u$ strongly in $(L^2(\Gamma))^d$, we have, as $\delta \to 0$,

$$(Au, u - u_\delta)_V + j_\delta(u) - j_\delta(u_\delta) + (f_0, u_\delta - u)_H + (f_0, u_\delta - u)_{(L^2(\Gamma_2))^d} \to 0,$$

then, from (2.3.22), we obtain (2.3.10). ■

2.4 The penalized and regularized optimal control problem

For $\delta > 0$ and a fixed $f_0 \in H$, we introduce the following penalized and regularized state problem.

Problem $(\mathbf{Q_2})$. For $f_2 \in (L^2(\Gamma_2))^d$ (called control), find $u^\delta \in V$ such that

$$\begin{aligned}&\left(Au^\delta, v\right)_V + \left(Dj_\delta\left(u^\delta\right), v\right)_V + \left(P_\delta\left(\varepsilon\left(u^\delta\right)\right), \varepsilon(v)\right)_Q \\ &= (f_0, v)_H + (f_2, v)_{(L^2(\Gamma_2))^d} \quad \forall v \in V.\end{aligned} \tag{2.4.1}$$

According to Theorem 2.4, the state Problem $(\mathbf{Q_2})$ has a unique solution. Next, we define the set U_{ad}^δ as

$$U_{ad}^{\delta} = \left\{ (u, f_2) \in V \times \left(L^2 (\Gamma_2) \right)^d, \text{ such that } \ (5.11) \ \text{ is satisfied} \right\}.$$

Hence, we introduce the optimal control problem below.

Problem $(\mathbf{C_2})$. Find $\left(\tilde{u}^{\delta}, \tilde{f}_2\right) \in U_{ad}^{\delta}$ such that $\tilde{f}_2$

$$\mathcal{L}\left(\tilde{u}^{\delta}, \tilde{f}_2^{\delta}\right) = \min_{(u, f_2) \in U_{ad}^{\delta}} \mathcal{L}\left(u, f_2\right).$$

We have the following result.

Theorem 2.6. *Let (2.1.6), (2.1.7) and (2.1.8)(c) hold. Then, Problem $(\mathbf{C_2})$ has at least one solution.*

Proof. We refer the reader to the arguments used in the proof of Theorem 2.2. ■

Next, we aim to demonstrate the convergence results concerning the solutions of Problems $(\mathbf{C_1})$ and $(\mathbf{C_2})$. To this end, we have the theorem below.

Theorem 2.7. *We have*

$$\lim_{\delta \to 0} \mathcal{L}\left(\tilde{u}^{\delta}, \tilde{f}_2^{\delta}\right) = \mathcal{L}\left(u^*, f_2^*\right).$$

The proof of this theorem is carried out in several steps. First, we prove the following lemma.

Lemma 2.8. *Let $\left(\tilde{u}^{\delta}, \tilde{f}_2^{\delta}\right)$ be a solution of Problem $(\mathbf{C_2})$. Then, there exists a solution $\left(\tilde{u}, \tilde{f}_2\right)$ of Problem $(\mathbf{C_1})$, such that passing to a subsequence still denoted $\left(\tilde{u}^{\delta}, \tilde{f}_2^{\delta}\right)$, we have as $\delta \to 0$, the following convergences:*

$$\begin{cases} (a) \ \ \tilde{u}^{\delta} \to \tilde{u} \ \textit{strongly in } V, \\ (b) \ \ \tilde{f}_2^{\delta} \to \tilde{f}_2 \ \textit{weakly in } \ \left(L^2 (\Gamma_2)\right)^d. \end{cases} \tag{2.4.2}$$

Proof. Let u_0^{δ} the solution of Problem $(\mathbf{Q_2})$ obtained for $f_2 = 0_{(L^2(\Gamma_2))^d}$. We have

$$\mathcal{L}\left(\tilde{u}^{\delta}, \tilde{f}_2^{\delta}\right) \leq \mathcal{L}\left(u_0^{\delta}, 0_{(L^2(\Gamma_2))^d}\right) \leq c\left(\|f_0\|_H^2 + \|u_d\|_V^2\right),$$

where $c > 0$. Thus, it follows that the sequence $\left(\tilde{u}^{\delta}, \tilde{f}_2^{\delta}\right)$ is bounded in $V \times \left(L^2 (\Gamma_2)\right)^d$. Then, there exists an element

$$\left(\tilde{u}, \tilde{f}_2\right) \in V \times \left(L^2 (\Gamma_2)\right)^d$$

such that passing to a subsequence still denoted $\left(\tilde{u}^\delta, \tilde{f}_2^\delta\right)$, we have

$$\tilde{u}^\delta \to \tilde{u} \text{ weakly in } V,$$
$$\tilde{f}_2^\delta \to \tilde{f}_2 \text{ weakly in } \left(L^2\left(\Gamma_2\right)\right)^d \text{ (then (2.4.2)}\,(b) \text{ is proved).}$$

Now, to prove that $\tilde{u} \in K$, it suffices to sketch the proof of Theorem 2.5. Moreover, using (2.1.8)(c), we have

$$\begin{cases} m\left\|\tilde{u}^\delta - \tilde{u}\right\|_V^2 \leq \left(A\tilde{u} - A\tilde{u}^\delta, \tilde{u} - \tilde{u}^\delta\right)_V \\ \leq \left(A\tilde{u}, \tilde{u} - \tilde{u}^\delta\right)_V + j_\delta\left(\tilde{u}\right) - j_\delta\left(\tilde{u}^\delta\right) + \left(P_\delta\left(\varepsilon\left(\tilde{u}^\delta\right)\right) - P_\delta\left(\varepsilon\left(\tilde{u}\right)\right), \varepsilon(\tilde{u} - \tilde{u}^\delta)\right)_Q \\ + \left(P_\delta\left(\varepsilon\left(\tilde{u}\right)\right), \varepsilon(\tilde{u} - \tilde{u}^\delta)\right)_Q + \left(f_0, \tilde{u} - \tilde{u}^\delta\right)_H + \left(\tilde{f}_2^\delta, \tilde{u} - \tilde{u}^\delta\right)_{(L^2(\Gamma_2))^d}. \end{cases} \tag{2.4.3}$$

Keeping in mind (2.3.4), we have

$$\left(P_\delta\left(\varepsilon\left(\tilde{u}^\delta\right)\right) - P_\delta\left(\varepsilon\left(\tilde{u}\right)\right), \varepsilon(\tilde{u} - \tilde{u}^\delta)\right)_Q \leq 0,$$

and since $\tilde{u} \in K$,

$$\left(P_\delta\left(\varepsilon\left(\tilde{u}\right)\right), \varepsilon(\tilde{u} - \tilde{u}^\delta)\right)_Q = 0.$$

Then, (2.4.3) implies

$$\begin{aligned} & m\left\|\tilde{u}^\delta - \tilde{u}\right\|_V^2 \\ & \leq \left(A\tilde{u}, \tilde{u} - \tilde{u}^\delta\right)_V + j_\delta\left(\tilde{u}\right) - j_\delta\left(\tilde{u}^\delta\right) + \left(f_0, \tilde{u} - \tilde{u}^\delta\right)_H + \left(\tilde{f}_2^\delta, \tilde{u} - \tilde{u}^\delta\right)_{(L^2(\Gamma_2))^d}. \end{aligned} \tag{2.4.4}$$

Hence, as $\tilde{u}^\delta \to \tilde{u}$ weakly in V, when $\delta \to 0$, implies that $\tilde{u}^\delta \to \tilde{u}$ strongly in $\left(L^2\left(\Gamma_2\right)\right)^d$, then $j_\delta\left(\tilde{u}\right) - j_\delta\left(u^\delta\right) \to 0$. Thus, we deduce that the right hand side of the previous inequality (2.4.4) tends to zero. Then, we obtain (2.4.2)(a).

Now, we still have to prove that $\left(\tilde{u}, \tilde{f}_2\right) \in U_{ad}$. Indeed, from the inequality (2.3.6), we deduce by (2.3.4)

$$\begin{aligned} & \left(A\tilde{u}^\delta, v - \tilde{u}^\delta\right)_V + j_\delta\left(v\right) - j_\delta\left(\tilde{u}^\delta\right) \\ & \qquad \geq \left(f_0, v - \tilde{u}^\delta\right)_H + \left(\tilde{f}_2^\delta, v - \tilde{u}^\delta\right)_{(L^2(\Gamma_2))^d}, \ \forall v \in K. \end{aligned} \tag{2.4.5}$$

Then, using (2.4.2)(a), it follows that the following convergences hold:

$$\begin{aligned} & \lim_{\delta\to 0}\left(A\tilde{u}^\delta, v - \tilde{u}^\delta\right)_V = \lim_{\delta\to 0}\left(\left(A\tilde{u}^\delta - Au, v - \tilde{u}^\delta\right)_V + \left(A\tilde{u}, v - \tilde{u}^\delta\right)_V\right) \\ & = \lim_{\delta\to 0}\left(A\tilde{u}^\delta - Au, v - \tilde{u}^\delta\right)_V + \lim_{\delta\to 0}\left(A\tilde{u}, v - \tilde{u}^\delta\right)_V = \left(A\tilde{u}, v - \tilde{u}\right)_V, \\ & \lim_{\delta\to 0}(j_\delta\left(v\right) - j_\delta\left(\tilde{u}^\delta\right)) = j\left(v\right) - j\left(\tilde{u}\right), \\ & \lim_{\delta\to 0}(\left(f_0, v - \tilde{u}^\delta\right)_H + \left(\tilde{f}_2^\delta, v - \tilde{u}^\delta\right)_{(L^2(\Gamma_2))^d}) = \left(f_0, v - \tilde{u}\right)_H + \left(\tilde{f}_2, v - \tilde{u}\right)_{(L^2(\Gamma_2))^d}. \end{aligned}$$

Hence, using these convergences and passing to the limit as $\delta \to 0$ in (2.4.5), we deduce that $\left(\tilde{u}, \tilde{f}_2\right)$ satisfies (2.2.1). ■

Now, we shall complete the proof of Theorem 2.7 by proving the following lemma.

Lemma 2.9. *We have that*

$$\mathcal{L}\left(\tilde{u}, \tilde{f}_2\right) = \mathcal{L}\left(u^*, f_2^*\right).$$

Proof. Consider the sequence $\left(u^\delta\right)$ such that for each $\delta > 0$, u^δ is the unique solution of Problem $(\mathbf{Q_2})$ written for $f_2^* \in \left(L^2\left(\Gamma_2\right)\right)^d$. Hence, for each $\delta > 0$, $\left(u^\delta, f_2^*\right) \in U_{ad}^\delta$ and by Lemma 2.8, it follows that

$$\left(u^\delta, f_2^*\right) \to \left(u^*, f_2^*\right) \text{ strongly in } V \times \left(L^2\left(\Gamma_2\right)\right)^d \text{ as } \delta \to 0. \tag{2.4.6}$$

As the functional $\mathcal{L}$ is convex and continuous, then

$$\mathcal{L}\left(\tilde{u}, \tilde{f}_2\right) \leq \liminf_{\delta \to 0} \mathcal{L}\left(\tilde{u}^\delta, \tilde{f}_2^\delta\right). \tag{2.4.7}$$

We also have, as $\left(\tilde{u}^\delta, \tilde{f}_2^\delta\right)$ is a solution of Problem $(\mathbf{C_2})$

$$\limsup_{\delta \to 0} \mathcal{L}\left(\tilde{u}^\delta, \tilde{f}_2^\delta\right) \leq \limsup_{\delta \to 0} \mathcal{L}\left(u^\delta, f_2^*\right). \tag{2.4.8}$$

Using (2.4.6), we have

$$\limsup_{\delta \to 0} \mathcal{L}(u^\delta, f_2^*) = \mathcal{L}\left(u^*, f_2^*\right) \tag{2.4.9}$$

and as (u^*, f_2^*) is a solution of Problem C1, it follows

$$\mathcal{L}\left(u^*, f_2^*\right) \leq \mathcal{L}\left(\tilde{u}, \tilde{f}_2\right). \tag{2.4.10}$$

Also, from (2.4.8), we deduce

$$\limsup_{\delta \to 0} \mathcal{L}\left(\tilde{u}^\delta, \tilde{f}_2^\delta\right) \leq \mathcal{L}\left(u^*, f_2^*\right). \tag{2.4.11}$$

Hence, from (2.4.7)-(2.4.11), we obtain

$$\mathcal{L}\left(\tilde{u}, \tilde{f}_2\right) = \mathcal{L}\left(u^*, f_2^*\right).$$

Finally, we conclude that the solution of the penalized and regularized optimal control Problem $(\mathbf{C_2})$ converges to the solution of the optimal control Problem $(\mathbf{C_1})$, which completes the proof of Theorem 2.7. ■

2.5 *Conclusion*

In this chapter we have investigated the optimal control of contact problems for nonlinear elastic materials with Tresca's friction law. We have introduced a variational formulation of the problem and, under suitable assumptions, we have established the existence and uniqueness results. We have obtained results for the optimal control problem formulated in terms of the variational inequality related to the displacement. Furthermore, we have proved the convergence of the penalized and regularized problems to the original optimal control problem, basing our proofs on arguments of monotonicity and lower semicontinuity. This initial work lays the foundation for studying necessary optimality conditions, and it would be of significant interest to extend our results to dynamic or viscoelastic contact problems.

Chapitre **3**

Frictional contact with adhesion for long-memory piezoelectric materials

Contents

In this chapter, we will study two mathematical models that describe the frictional contact between a piezoelectric body and a foundation, accounting for adhesive interactions at the contact surfaces. In these models, the material behavior is described by a long-memory electro-viscoelastic constitutive law. For each model, we establish a variational formulation in the form of a system involving displacement, stress, electric displacement, electric potential, and adhesion field. Subsequently, we prove that each problem admits a unique weak solution. The proofs of these results require demonstrating several technical lemmas using arguments related to variational inequalities, monotone operators, differential equations, and the Banach fixed-point theorem. The analysis of these two problems has been the subject of two separate publications [50] et [51].

3.1 Unilateral contact problem with non-local friction and adhesion

In this section, we delve into the mathematical modeling of an adhesive contact problem with friction between a long-memory electro-viscoelastic body and an isolated foundation. We consider a unilateral constraint to model the contact and introduce a non-local friction law. The evolution of the adhesive field is characterized by a first-order differential equation. Similar to the approach taken in [40]- [41], we employ it as an internal surface variable that ranges between zero and one, describing the fractional density of active bonds.

3.1.1 The contact problem statement

We consider the following physical setting. An electro-elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz boundary $\partial\Omega = \Gamma$. The boundary Γ is partitioned into three disjoint measurable parts Γ_1, Γ_2, Γ_3, and into two disjoint measurable parts Γ_a and Γ_b, such that $meas(\Gamma_1) > 0$, $meas(\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. Let $T > 0$, and the time interval of interest is denoted as $[0, T]$. We assume thatthe body is clamped on Γ_1, resulting in a vanishing displacement field there. Volume forces with density f_0 act in Ω , while surface tractions with density f_2 act on Γ_2. The body is also subjected to electrical constraints, where we assume the electric potential is zero on Γ_a. Additionally, the body is subject to the influence of an electric charge density q_0 within Ω and a surface electric charge of density q_2 act on Γ_b. Moreover, on Γ_3, the body is in unilateral contact with adhesion following the non-local friction law with an insulator obstacle, the so-called foundation.

Thus, the formulation of the mechanical problem is written as follows:

Problem $(P2)$. *Find a displacement field* $u : \Omega \times [0, T] \to \mathbb{R}^d$, *a stress field* $\sigma : \Omega \times [0, T] \to \mathbf{S}_d$, *an electric potential* $\varphi : \Omega \times [0, T] \to \mathbb{R}$, *an electric displacement field* $D : \Omega \times [0, T] \to \mathbb{R}^d$ *and a bonding field* $\beta : \Gamma_3 \times [0, T] \to \mathbb{R}$, *such that for all* $t \in [0, T]$,

$$\sigma(t) = \mathcal{B}\varepsilon(u(t)) + \int_0^t \mathcal{F}(t-s)\varepsilon(u(s))ds - \mathcal{E}^*E(\varphi(t)) \tag{3.1.1}$$

$$D(t) = \mathcal{E}\varepsilon(u(t)) + \mathcal{C}E(\varphi(t)) \tag{3.1.2}$$

$$Div\ \sigma(t) + f_0(t) = 0 \quad \text{in } \Omega, \tag{3.1.3}$$

$$div\ D(t) + q_0(t) = 0 \quad \text{in } \Omega, \tag{3.1.4}$$

$$u(t) = 0 \quad \text{on } \Gamma_1, \tag{3.1.5}$$

$$\sigma\nu(t) = f_2(t) \quad \text{on } \Gamma_2, \tag{3.1.6}$$

$$\left.\begin{array}{c} u_\nu(t) \le 0, \quad \sigma_\nu(t) - \gamma_\nu\beta^2 R_\nu(u_\nu(t)) \le 0 \\ u_\nu(t)(\sigma_\nu(t) - \gamma_\nu\beta^2(t)R_\nu(u_\nu(t))) = 0 \end{array}\right. \quad \text{on } \Gamma_3, \tag{3.1.7}$$

$$\dot{\beta}(t) = -\left[\beta(t)\left((\gamma_\nu R_\nu u_\nu(t))^2 + \gamma_\tau \left\|R_\tau(u_\tau(t))\right\|^2\right) - \epsilon_a\right]_+ \quad \text{on} \Gamma_3, \tag{3.1.8}$$

$$\varphi(t) = 0 \quad \text{on } \Gamma_a, \tag{3.1.9}$$

$$D\nu(t) = q_2(t) \quad \text{on } \Gamma_b, \tag{3.1.10}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3, \tag{3.1.11}$$

$$\begin{cases} \|\sigma_\tau(t) + \gamma_\tau\beta^2(t)R_\tau(u_\tau(t))\| \leqslant \mu\,|R\sigma_\nu(t)| \\ \\ \|\sigma_\tau(t) + \gamma_\tau\beta^2(t)R_\tau(u_\tau(t))\| < \mu\,|R\sigma_\nu(t)| \Longrightarrow u_\tau = 0 \\ \\ \|\sigma_\tau(t) + \gamma_\tau\beta^2(t)R_\tau(u_\tau(t))\| = \mu\,|R\sigma_\nu(t)| \Longrightarrow \exists\lambda \geqslant 0 \ \text{ such that:} \\ \qquad \sigma_\tau\,(t) + \gamma_\tau^2\beta^2(t)R_\tau(u_\tau(t)) = -\lambda u_\tau(t) \end{cases} \quad \text{on } \Gamma_3. \tag{3.1.12}$$

Let's start by recalling that equations (3.1.1) and (3.1.2) represent the elastic constitutive law with long memory, where $\mathcal{B}$ is the elasticity operator, $\mathcal{F}$ is the relaxation tensor, $\mathcal{E} = (e_{ijk})$ represents the piezoelectric operator, and $\mathcal{C} = (\mathcal{C}_{ij})$ is the permittivity tensor. Equation (3.1.8) describes the evolution of the adhesion field β. Here γ_ν, γ_τ and ϵ_a are positive adhesion coefficients where $[r]_+ = \max\{0, r\}$, R_ν and R_τ are defined by (1.1.18) and (1.1.19). Lastly, equation (3.1.12) represents the Coulomb law of dry friction with adhesion, where μ denotes the friction coefficient.

Now, to derive a variational formulation for problem $(P2)$, we will use the space of admissible displacement fields given by:

$$U_{ad} = \{v \in V : v_\nu \le 0 \ \text{ a.e. on } \Gamma_3\},$$

where V is the closed subspace of H_1 defined by (1.2.11).

We adopt the following assumptions on the data:

The operator $\mathcal{B}$ and the tensors $\mathcal{F}$, $\mathcal{C}$, $\mathcal{E}$, and $\mathcal{E}^*$ satisfy the following assumptions:

$$\begin{cases} \text{(a)} \quad \mathcal{B} : \Omega \times S_d \longrightarrow S_d \\ \text{(b)} \;\; \mathcal{B} \in Q_\infty \text{ and there exists a constant } M_{\mathcal{B}} > 0 \text{ such that:} \\ \qquad \|\mathcal{B}(x,\xi_1) - \mathcal{B}(x,\xi_2)\| \leq M_{\mathcal{B}} \|\xi_1 - \xi_2\| \quad \forall \xi_1, \xi_2 \in S_d, \quad \text{a.e. on } \Omega. \\ \text{(c)} \;\; \text{There exists a constant } m_{\mathcal{B}} > 0 \text{ such that:} \\ \qquad \mathcal{B}\xi.\xi \geqslant m_{\mathcal{B}} \|\xi\|^2 \quad \forall \xi \in S_d \;\; \text{a.e. on } \Omega. \\ \text{(d)} \;\; \text{The function } x \longrightarrow \mathcal{B}(x,\xi) \text{ is measurable on } \Omega \;\; \text{a.e. } \xi \in S_d. \end{cases} \tag{3.1.13}$$

$$\mathcal{F} \in C([0,T]; Q_\infty). \tag{3.1.14}$$

$$\begin{cases} \text{(a)} \;\; \mathcal{C} : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \\ \text{(b)} \;\; \mathcal{C}(x,E) = (c_{ij}(x)E_j) \quad \forall E = (E_i) \in \mathbb{R}^d, \quad \text{a.e. on } \Omega \\ \qquad c_{ij} = c_{ji} \in L^\infty(\Omega). \\ \text{(c)} \;\; \text{There exists a constant } m_{\mathcal{C}} > 0 \\ \quad \text{such that: } c_{ij}(x)E_iE_j \geqslant m_{\mathcal{C}} \|E\|^2 \quad \forall E = (E_i) \in \mathbb{R}^d, \quad \text{a.e. on } \Omega. \end{cases} \tag{3.1.15}$$

$$\begin{cases} \text{(a)} \;\; \mathcal{E} : \Omega \times S_d \longrightarrow \mathbb{R}^d \\ \text{(b)} \;\; \mathcal{E}(x,\xi) = (e_{ijk}(x)\xi_{ij}) \quad \forall \xi = (\xi_{ij}) \in \mathbf{S}_d, \quad \text{a.e. in } \Omega \\ \text{(c)} \qquad e_{ijk} = e_{ikj} \in L^\infty(\Omega) \end{cases} \tag{3.1.16}$$

$$\mathcal{E}\sigma.\upsilon = \sigma.\mathcal{E}^*\upsilon \quad \forall \sigma \in \mathbf{S}_d, \; \forall \upsilon \in \mathbb{R}^d. \tag{3.1.17}$$

Moreover, we assume that the adhesion coefficients satisfy the following condition:

$$\gamma_\tau, \gamma_\nu \in L^\infty(\Gamma_3), \; \epsilon_a \in L^2(\Gamma_3), \; \gamma_\tau, \gamma_\nu, \epsilon_a \geqslant 0 \quad \text{a.e. } x \in \Gamma_3, \tag{3.1.18}$$

and the following regularity on f_0 and q_0

$$f_0 \in C([0,T]; H), \;\; f_2 \in C\left([0,T]; L^2(\Gamma_2)^d\right), \tag{3.1.19}$$

$$q_0 \in C([0,T]; H), \;\; q_2 \in C\left([0,T]; L^2(\Gamma_b)^d\right). \tag{3.1.20}$$

To reflect that the foundation is isolated, we assume

$$q_0(t) = 0 \quad \text{on } \Gamma_3 \;\; \forall t \in [0,T]. \tag{3.1.21}$$

So, the electrical conditions on Γ_3 do not change according to the contact state; they remain the same in the separation zone and are characterized by:

$$D.\nu = 0 \quad \text{on } \Gamma_3 \;\; \forall t \in [0,T]. \tag{3.1.22}$$

The initial data β_0 satisfy

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \text{ a.e. on } \Gamma_3. \tag{3.1.23}$$

The friction coefficient μ is such that

$$\mu \in L^\infty(\Gamma_3), \ \ \mu(x) \geq 0 \text{ a.e. on } \Gamma_3. \tag{3.1.24}$$

Finally, R is linear and continuous mapping, where

$$R : H^{-\frac{1}{2}}(\Gamma) \longrightarrow L^2(\Gamma_3). \tag{3.1.25}$$

3.1.2 Variational formulation

By the representation theorem of *Riesz–Fréchet*, for all $t \in [0, T]$, we define $f(t) \in V$ and $q(t) \in W$ as follows:

$$(f(t), v)_V = \int_\Omega f_0(t).vdx + \int_{\Gamma_2} f_2(t).vda \ \ \forall v \in V, \tag{3.1.26}$$

$$(q(t), \psi)_V = \int_\Omega q_0(t).\psi dx + \int_{\Gamma_2} q_2(t).\psi da \ \ \forall \psi \in W. \tag{3.1.27}$$

Conditions (3.1.19) and (3.1.20) imply

$$f \in C\left([0, T]; H\right) \quad \text{and} \quad q \in C\left([0, T]; W\right). \tag{3.1.28}$$

Next, we consider V_0, the subset of regularity defined by

$$V_0 = \left\{(u, \varphi) \in H_1 \times W : \ Div\ \sigma(u, \varphi) \in H\right\}. \tag{3.1.29}$$

We introduce the space of bonding field noted B by

$$B = \left\{\beta : [0, T] \longrightarrow L^2(\Gamma_3); \ 0 \leq \beta \leq 1 \quad \forall t \in [0, T], \quad a.e. \text{ on } \Gamma_3\right\}$$

Let us denote by $j_{ad} : L^\infty(\Gamma_3) \times V_0 \times V \longrightarrow \mathbb{R}$ and $j_{fr} : V_0 \times V \longrightarrow \mathbb{R}$, respectively, the functionals given by

$$j_{ad}(\beta, u, v) \ = \ \int_{\Gamma_3} (-\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu + \gamma_\tau \beta^2 R_\tau(u_\tau).v_\tau) da, \tag{3.1.30}$$

$$j_{fr}((u, \varphi), v) \ = \ \int_{\Gamma_3} \mu \left|R\sigma_\nu(u, \varphi)\right| \|v_\tau\| \, da \tag{3.1.31}$$

Using the properties of the truncation operators R_ν and R_τ, it can be shown that the functional j_{ad} satisfies the following properties

- j_{ad} is linear with respect to the third term and we have:

$$j_{ad}(\beta, u, -v) = -j_{ad}(\beta, u, v). \tag{3.1.32}$$

$$j_{ad}(\beta, u_1, u_2 - u_1) + j_{ad}(\beta, u_2, u_1 - u_2) \leq 0. \tag{3.1.33}$$

$$j_{ad}(\beta, v, v) \geq 0. \tag{3.1.34}$$

- There exists a constant $c > 0$, such that:

$$j_{ad}(\beta_1, u_1, u_2 - u_1) + j_{ad}(\beta_2, u_2, u_1 - u_2) \leq c \int_{\Gamma_3} \|\beta_1 - \beta_2\| \, \|u_1 - u_2\|_V \, ds. \tag{3.1.35}$$

- There exists a constant $C > 0$, such that:

$$|j_{ad}(\beta, u_1, v) - j_{ad}(\beta, u_2, v)| \leq C \|u_1 - u_2\|_V \|v\|_V . \tag{3.1.36}$$

Note that if (u, φ) is a solution of the problem stated below, then $\sigma(t) = \sigma(u(t), \varphi(t)) \in Q$ a.e. $t \in [0, T]$, and therefore

$$j_{fr}((u(t), \varphi(t)), v) = \int_{\Gamma_3} \mu \, |R\sigma_\nu(u(t), \varphi(t))| \, \|v_\tau\| \, da \quad \forall (u, \varphi) \in V_0, \quad \forall v \in V.$$

Using the *Green's formulas* (1.2.16) and (1.2.22), we obtain the following proposition:

Proposition 3.1. *If u, σ, φ and D are regular and satisfy the equations and conditions (3.1.1) – (3.1.12), then*

$$\begin{aligned} (\sigma(t), \varepsilon(u(t)))_Q + j_{ad}(\beta(t), u(t), v) + j_{fr}((u(t), \varphi(t)), v) - j_{fr}((u(t), \varphi(t), u(t)) \\ \geq (f(t), v - u(t))_V \quad \forall v \in V, t \in [0, T]. \end{aligned} \tag{3.1.37}$$

$$(D(t), \nabla\psi)_H + (q(t), \psi)_W = 0 \quad \forall \psi \in W. \tag{3.1.38}$$

Proof. Using the formula (1.2.18), we get

$$\begin{aligned} (\sigma, \varepsilon(v) - \varepsilon(u(t)))_Q &= \langle \sigma\nu, v - u(t) \rangle_{H'_\Gamma \times H_\Gamma} - (Div \ \sigma, v - u(t))_H \\ &= (f_0(t), v - u(t))_H + (\sigma\nu, \gamma(v - u(t)))_{L^2(\Gamma_2)^d} + (\sigma\nu, \gamma(v - u(t)))_{L^2(\Gamma_3)^d} \\ &= (f_0(t), v - u(t))_H + (f_2(t), \gamma(v - u(t)))_{L^2(\Gamma_2)^d} + (\sigma\nu, \gamma(v - u(t)))_{L^2(\Gamma_3)^d}, \end{aligned}$$

and according to (1.1.11), (3.1.7), and (3.1.12), we have

$$\begin{aligned} (\sigma\nu, v)_{L^2(\Gamma_3)^d} &= (\sigma_\tau, v_\tau)_{L^2(\Gamma_3)^d} + (\sigma_\nu, v_\nu)_{L^2(\Gamma_3)} \\ &= \textstyle\int_{\Gamma_3} \sigma_\tau . v_\tau da + \int_{\Gamma_3} \sigma_\nu . v_\tau da \\ &= \textstyle\int_{\Gamma_3} \mu \, |R\sigma_\nu(u(t), \varphi(t))| \, \|v_\tau\| \, da - \int_{\Gamma_3} (p_\nu(u(t)) - \gamma_\nu \beta^2 R_\nu(u_\nu(t)) v_\nu) da \\ &= -j_{fr}(u(t), \varphi(t), v) - j_{ad}(\beta, u(t), v), \end{aligned}$$

and thus using (3.1.30) and (3.1.31), it follows that

$$\begin{aligned}(\sigma, \varepsilon(v) - \varepsilon(u(t)))_Q \;\; &\geq \;\; (\boldsymbol{f}(t), v - u(t))_V - j_{ad}(\beta, u(t), v) - j_{ad}(\beta, u(t), u(t)) \\ &\quad - j_{fr}(u(t), \varphi(t), v) + j_{fr}(u(t), \varphi(t), u(t)),\end{aligned}$$

from which we have (3.1.37).

To obtain (3.1.38), we use Green's formula (1.2.22) and we have

$$\begin{aligned}(D(t), \nabla\xi)_H &= -\left(div\ D(t), \xi\right)_{L^2(\Omega)} + \textstyle\int_{\Gamma_b} D\nu.\xi da \quad \forall \xi \in W. \\ &= (q_0(t), \xi)_W + \textstyle\int_{\Gamma_b} q_2(t).\xi da \\ &= (\boldsymbol{q}(t), \xi)_W\end{aligned}$$

■

Let's substitute $\sigma(t)$ in (3.1.37) with its expression given by (3.1.1), and replace $D(t)$ with its expression given by (3.1.2). This leads us to derive the following variational formulation of problem $(P2)$:

Problem $(P2_V)$. *Find a displacement field $u \in C([0,T]; V)$, an electric potential $\varphi \in C([0,T]; W)$ and a bonding field $\beta \in W^{1,\infty}(0,T; L^2(\Gamma_3)) \cap B$, such that $u(t) \in U_{ad}$ for all $t \in [0,T]$, and*

$$\begin{aligned}&(\mathcal{B}\varepsilon(u(t)), \varepsilon(v - u(t)))_Q + \left(\textstyle\int_0^t \mathcal{F}(t-s)\varepsilon(u(s))ds, \varepsilon(v - u(t))\right)_Q \\ &+ (\mathcal{E}^*\nabla\varphi(t), \varepsilon(v - u(t)))_Q + j_{ad}(\beta(t), u(t), v - u(t)) \\ &+ j_{fr}((u(t), \varphi(t)), v) - j_{fr}((u(t), \varphi(t)), u(t)) \geq (f(t), v - u(t))_V \\ &\qquad\qquad \forall v \in U_{ad},\ t \in [0,T],\end{aligned} \tag{3.1.39}$$

$$(\mathcal{C}\nabla\varphi(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u(t), \nabla\psi)_H = (q(t), \psi)_W \quad \forall \psi \in W, \quad t \in [0,T], \tag{3.1.40}$$

$$\dot{\beta}(t) = -\left[\beta(t)\left((\gamma_\nu R_\nu u_\nu(t))^2 + \gamma_\tau \left\|R_\tau(u_\tau(t))\right\|^2\right) - \epsilon_a\right]_+, \quad t \in [0,T], \tag{3.1.41}$$

$$\beta(0) = \beta_0. \tag{3.1.42}$$

3.1.3 Existence and uniqueness of the weak solution of the Problem $(P2)$

Our main existence and uniqueness result that we state and prove is the following

Theorem 3.2. *Assume that assumptions (3.1.13)–(3.1.25) hold. Then there exists a constant $\mu_0 > 0$ such that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, the Problem $(P2_V)$ has a unique solution (u, φ, β).*

We carry out the proof of Theorem 3.2 in several steps. We define intermediate problems and prove their unique solvability, and then we construct a contraction mapping whose unique fixed point is the solution of $(P2_V)$.

First, we consider the closed subset,

$$Z = \left\{\theta \in C([0,T]\,; L^2(\Gamma_3)) \cap B, \quad \theta(0) = \beta_0\right\},$$

where the Banach space $C([0,T]\,; L^2(\Gamma_3))$ is endowed with the norm

$$\|\theta\|_k = \max_{t\in[0,T]} \left[e^{-kt}\, \|\theta\|_{L^2(\Gamma_3)}\right], \; k > 0.$$

For a given $\beta \in Z$, we consider the following auxiliary problem.

Problem $(P2_V^\beta)$. *Find a displacement field $u_\beta \in C\left([0,T]\,;V\right)$ and an electric potential $\varphi_\beta \in C\left([0,T]\,;W\right)$, such that $u_\beta(t) \in U_{ad}$ for all $t \in [0,T]$, and*

$$\begin{aligned}
&(\mathcal{B}\varepsilon(u_\beta(t)), \varepsilon(v - u_\beta(t)))_Q + \left(\int_0^t \mathcal{F}(t-s)\varepsilon(u_\beta(s))ds, \varepsilon(v - u_\beta(t))\right)_Q \\
&\quad + (\mathcal{E}^*\nabla\varphi_\beta(t), \varepsilon(v - u_\beta(t)))_Q + j_{ad}(\beta(t), u_\beta(t), v - u_\beta(t)) \\
&\qquad + j_{fr}((u_\beta(t), \varphi_\beta(t)), v) - j_{fr}((u_\beta(t), \varphi_\beta(t)), u_\beta(t)) \geq (f(t), v - u_\beta(t))_V \\
&\qquad\qquad \forall v \in U_{ad}, \; t \in [0,T],
\end{aligned} \tag{3.1.43}$$

$$\begin{aligned}
(\mathcal{C}\nabla\varphi_\beta(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u_\beta(t), \nabla\psi)_H = (q(t), \psi)_W \\
\forall\psi \in W, \; t \in [0,T].
\end{aligned} \tag{3.1.44}$$

We have the following result.

Theorem 3.3. *Problem $(P2_V^\beta)$ has a unique solution $(u_\beta, \varphi_\beta) \in C([0,T]\,; V \times W)$.*

In order to prove Theorem 3.3, we consider the product Hilbert space $X = V \times W$ equipped with the inner product defined by

$$\langle x, y\rangle = \langle (u,\varphi), (v,\psi)\rangle = \langle u, v\rangle + \langle \varphi, \psi\rangle\,, \quad x, y \in X \tag{3.1.45}$$

and the associated norm $\|.\|_X$. In the sequel, let $X_1 = U_{ad} \times W$, and consider the following problem defined for any $\eta \in C([0,T]\,;Q)$ by

Problem $(P2_\eta^1)$. *Find $x_{\beta\eta} \in C([0,T]\,;X)$ such that $x_{\beta\eta}(t) \in X_1$ for all $t \in [0,T]$, and*

$$\begin{aligned}
&(\mathcal{B}\varepsilon(u_{\beta\eta}(t)), \varepsilon(v - u_{\beta\eta}(t)))_Q + (\mathcal{E}^*\nabla\varphi_{\beta\eta}(t), \varepsilon(v - u_{\beta\eta}(t)))_Q + (\mathcal{C}\nabla\varphi_{\beta\eta}(t), \nabla\psi)_H \\
&\quad - (\mathcal{E}\varepsilon(u_{\beta\eta}(t), \nabla\psi)_H + (\eta(t), \varepsilon(v - u_{\beta\eta}(t)))_Q + j_{ad}(\beta(t), u_{\beta\eta}(t), v - u_{\beta\eta}(t)) \\
&\quad + j_{fr}(x_{\beta\eta}(t), v) - j_{fr}(x_{\beta\eta}(t), u_{\beta\eta}(t)) \geq (f(t), v - u_{\beta\eta}(t))_V + (q(t), \psi)_W\,, \\
&\qquad\qquad \forall v \in U_{ad}, \forall\psi \in W, \; t \in [0,T].
\end{aligned} \tag{3.1.46}$$

The Riesz's representation theorem, implies that there exists an element $f_\eta(t) \in X$ defined for all $x = (u, \varphi)$ by

$$\langle f_\eta(t), x\rangle = (f(t), u)_V + (q(t), \varphi)_W - (\eta(t), \varepsilon(u))_Q\,. \tag{3.1.47}$$

And let introduce the operator $\Lambda_\beta : [0, T] \times X \longrightarrow X$ defined as

$$\begin{aligned}\langle \Lambda_\beta(t)x, y\rangle &= (\mathcal{B}\varepsilon(u), \varepsilon(v))_Q + (\mathcal{E}^*\nabla\varphi, \varepsilon(v))_Q + (\mathcal{C}\nabla\varphi, \nabla\psi)_H \\ &\quad - (\mathcal{E}\varepsilon(u), \nabla\psi)_H + j_{ad}(\beta(t), u, v) \\ \textit{for all } x &= (u, \varphi) \textit{ and } y = (v, \psi) \in X,\end{aligned} \tag{3.1.48}$$

and $j_{fr} : X \times X \longrightarrow \mathbb{R}$ defined by

$$j_{fr}(x, y) = \; j_{fr}(x, v) \text{ for all } x = (u, \varphi),\; y = (v, \psi) \in X.$$

Then, we introduce the following problem

Problem $(P2^2_\eta)$. *Find $x_{\beta\eta} : [0, T] \longrightarrow X_1$ such that:*

$$\begin{aligned}&\langle \Lambda_\beta(t)x_{\beta\eta}(t), y - x_{\beta\eta}(t)\rangle + j_{fr}(y, x_{\beta\eta}(t)) - j_{fr}(x_{\beta\eta}(t), x_{\beta\eta}(t)) \\ &\quad \geq \langle f_\eta(t), y - x_{\beta\eta}(t)\rangle\,, \quad \forall y \in X,\; t \in [0, T].\end{aligned} \tag{3.1.49}$$

Proposition 3.4. *The two problems $(P2^1_\eta)$ and $(P2^2_\eta)$ are equivalent in the following way: if $x_{\beta\eta} = (u_{\beta\eta}, \varphi_{\beta\eta}) \in C([0, T]\,; X)$ is a solution to one of the problems, it is also a solution to the other problem.*

Proof. Suppose $x_{\beta\eta} = (u_{\beta\eta}, \varphi_{\beta\eta}) \in C([0, T]\,; X)$ is a solution to Problem $(P2^1_\eta)$. For $t \in [0, T]$, taking $\psi = \psi - \varphi_{\beta\eta}(t)$ in (3.1.46), we get

$$\begin{aligned}&(\mathcal{B}\varepsilon(u_{\beta\eta}(t)), \varepsilon(v - u_{\beta\eta}(t)))_Q + (\mathcal{E}^*\nabla\varphi_{\beta\eta}(t), \varepsilon(v - u_{\beta\eta}(t)))_Q + (\mathcal{C}\nabla\varphi_{\beta\eta}(t), (\psi - \varphi_{\beta\eta}(t))_H \\ &- (\mathcal{E}\varepsilon(u_{\beta\eta}(t), \nabla(\psi - \varphi_{\beta\eta}(t))_H + (\eta(t), \varepsilon(v - u_{\beta\eta}(t)))_Q + j_{ad}(\beta(t), u_{\beta\eta}(t), v - u_{\beta\eta}(t)) \\ &+ j_{fr}(x_{\beta\eta}(t), v) - j_{fr}(x_{\beta\eta}(t), u_{\beta\eta}(t)) \geq (f(t), v - u_{\beta\eta}(t))_V + (q(t), (\psi - \varphi_{\beta\eta}(t))_W\,,\end{aligned}$$

Using (3.1.45), (3.1.47), and (3.1.48) and simplifying, we check that $x_{\beta\eta}$ verifies the inequality (3.1.49).

Conversely, let $x_{\beta\eta} = (u_{\beta\eta}, \varphi_{\beta\eta}) \in C([0, T]\,; X)$ is a solution of the Problem $(P2^2_\eta)$.

For $t \in [0, T]$, taking $y = (v, \varphi_{\beta\eta}(t))$ in (3.1.46), we get

$$
\begin{aligned}
&((\mathcal{B}\varepsilon(u_{\beta\eta}(t)),\varepsilon(v-u_{\beta\eta}(t)))_Q+(\mathcal{E}^*\nabla\varphi_{\beta\eta}(t),\varepsilon(v-u_{\beta\eta}(t)))_Q\\
&+j_{ad}(\beta(t),u_{\beta\eta}(t),v-u_{\beta\eta}(t))+j_{fr}((u_{\beta\eta}(t),\varphi_{\beta\eta}(t)),v)-j_{fr}((u_{\beta\eta}(t),\varphi_{\beta\eta}(t)),u_{\beta\eta}(t))\\
&\geq (f(t),v-u_{\beta\eta}(t))_V-(\eta(t),\varepsilon(v-u_{\beta\eta}(t)))_Q
\end{aligned}
$$

Then, by successively inserting $y=(u_{\beta\eta}(t),\varphi_{\beta\eta}(t)+\psi)$ and then $y=(u_{\beta\eta}(t),\varphi_{\beta\eta}(t)-\psi)$ in (3.1.46), we obtain

$$
(\mathcal{C}\nabla\varphi_{\beta\eta}(t),\nabla\psi-\nabla\varphi_{\beta\eta}(t))_H-(\mathcal{E}\varepsilon(u_{\beta\eta}(t)),\nabla\psi-\nabla\varphi_{\beta\eta}(t))_H\geq (q(t),\psi-\varphi_{\beta\eta}(t))_W
$$

Adding the last two results, we see that $x_{\beta\eta}$ is a solution to the problem.$(P2^1_\eta)$. ■

We now have the following

Lemma 3.5. *There exists a constant $\mu_0>0$ such that if $\|\mu\|_{L^\infty(\Gamma_3)}<\mu_0$, Problem $(P2^2_\eta)$ has a unique solution $x_{\beta\eta}\in C([0,T];X)$.*

We will prove Lemma 3.4 by steps.

For $t\in[0,T]$, and for all $x_1,x_2\in X$, $x_1=(u_1,\varphi_1)$ and $x_2=(u_2,\varphi_2)$, using (3.1.49), we have

$$
\begin{aligned}
\langle\Lambda_\beta(t)x_1-\Lambda_\beta(t)x_2,x_1-x_2\rangle \;=\;& (\mathcal{B}\varepsilon(u_1)-\mathcal{B}\varepsilon(u_2),\varepsilon(u_1)-\varepsilon(u_2))_Q\\
&+(\mathcal{E}^*\nabla\varphi_1-\mathcal{E}^*\nabla\varphi_2,\varepsilon(u_1)-\varepsilon(u_2))_Q\\
&+(\mathcal{C}\nabla\varphi_1-\mathcal{C}\nabla\varphi_2,\nabla\varphi_1-\nabla\varphi_2)_H\\
&-(\mathcal{E}\varepsilon(u_1)-\mathcal{E}\varepsilon(u_2),\nabla\varphi_1-\nabla\varphi_2)_H\\
&+j_{ad}(\beta,u_1,u_2)-j_{ad}(\beta,u_2,u_1),
\end{aligned}
\tag{3.1.50}
$$

and, by (3.1.17) we get

$$
(\mathcal{E}^*\nabla\varphi_1-\mathcal{E}^*\nabla\varphi_2,\varepsilon(u_1)-\varepsilon(u_2))_Q=(\mathcal{E}\varepsilon(u_1)-\mathcal{E}\varepsilon(u_2),\nabla\varphi_1-\nabla\varphi_2)_H
$$

Then, by (3.1.34), (3.1.13) (c) and (3.1.15) (c) we deduce

$$
\begin{aligned}
\langle\Lambda_\beta(t)x_1-\Lambda_\beta(t)x_2,x_1-x_2\rangle \;\geq\;& (\mathcal{B}\varepsilon(u_1)-\mathcal{B}\varepsilon(u_2),\varepsilon(u_1)-\varepsilon(u_2))_Q\\
&+(\mathcal{C}\nabla\varphi_1-\mathcal{C}\nabla\varphi_2,\nabla\varphi_1-\nabla\varphi_2)_H\\
\geq\;& m_{\mathcal{B}}\|u_1-u_2\|_V^2+m_{\mathcal{C}}\|\varphi_1-\varphi_2\|_W^2.
\end{aligned}
$$

Then the operator $\Lambda_\beta(t)$ is strongly monotone, and for $C_m=\min(m_{\mathcal{B}},m_{\mathcal{C}})$ it satisfies

$$
\langle\Lambda_\beta(t)x_1-\Lambda_\beta(t)x_2,x_1-x_2\rangle\geq C_m\|x_1-x_2\|_X^2\qquad\forall x,y\in X.
\tag{3.1.51}
$$

For $y=(v,\psi)$, using (1.2.13), (3.1.13) (b), (3.1.15) and (3.1.36), we have

$$\langle \Lambda_\beta(t)x_1-\Lambda_\beta(t)x_2,y\rangle \leq c\left(\|u_1-u_2\|_V\left(\|v\|_V+\|\psi\|_W\right)+\|\varphi_1-\varphi_2\|_W\left(\left(\|v\|_V+\|\psi\|_W\right)\right)\right),$$

thus, $\Lambda_\beta(t)$ is a Lipschitz continuous operator and there exists a constant $L_0>0$ such that:

$$\|\Lambda_\beta(t)x_1-\Lambda_\beta(t)x_2\| \leq L_0\,\|x_1-x_2\|_X \qquad \forall x,y\in X. \tag{3.1.52}$$

Next, let the non-empty subset $L^2_+(\Gamma_3)$ be defined by:

$$L^2_+(\Gamma_3)=\left\{g\in L^2(\Gamma_3);\ g\geqslant 0 \quad \text{a.e. on } \Gamma_3\right\}.$$

For each $g\in L^2_+(\Gamma_3)$, we define the functional $h(g,.) : X\longrightarrow \mathbb{R}$ by

$$h(g,y)=\int_{\Gamma_3}\mu g\,\|w_\tau\|\,da \quad \forall y=(w,\varphi)\in X,$$

and introduce an intermediate problem as follows

Problem $(P2^g_1)$. *Find $x_{\beta\eta}:[0,T]\longrightarrow X_1$ such that*

$$\begin{aligned}&\langle \Lambda_\beta(t)x_{\beta\eta g}(t),y-x_{\beta\eta g}(t)\rangle+h(g,y)-h(g,x_{\beta\eta g}(t))\geqslant (f,y-x_{\beta\eta g}(t))_X\\ &\quad \forall y\in X.\end{aligned} \tag{3.1.53}$$

Lemma 3.6. *Problem $(P2^g_1)$ has a unique solution.*

Proof. The functional $h(g,.)$ is convex and lower semi-continuous, Λ_β is Lipschitz continuous and strongly monotone, we deduce that Problem $(P2^g_1)$ has a unique solution (see Chap 1). ■

Now, to prove Lemma 3.4, for each $t\in[0,T]$, we define on $L^2_+(\Gamma_3)$ the map:

$$\Psi_t : g\longmapsto \Psi_t(g)=\left|R\sigma_\nu(x_{_{\beta\eta g}}(t))\right|.$$

Next, we prove the following lemma

Lemma 3.7. *There exists a constant $\mu_1>0$ such that if $\|\mu\|_{L^\infty(\Gamma_3)}<\mu_1$, the mapping Ψ has a unique fixed point g^* and $x_{\beta\eta g^*}$ is a unique solution to Problem $(P2^2_\eta)$.*

Proof. For $i=1,2$, define the following

Problem. $(P^2_{\eta gi})$. *Find $x_{\beta\eta gi}=(u_{_{\beta\eta g_i}},\varphi_{_{\beta\eta g_i}})\in X_1$ such that*

$$\langle \Lambda_\beta(t)x_{\beta\eta gi}(t),y\rangle+h(g_i,y)-h(g_i,x_{\beta\eta gi}(t))\geqslant (f,y-x_{\beta\eta gi}(t))_V \quad \forall y\in V.$$

Take $y = x_{\beta\eta g_2}$ in (3.1.53) written for $g = g_1$, then take $y = x_{\beta\eta g_1}$ in (3.1.53) written for $g = g_2$, by adding the resulting inequalities, we get

$$\begin{aligned}\langle \Lambda_\beta(t)(x_{\beta\eta g_1}(t) - x_{\beta\eta g_2}(t)), x_{\beta\eta g_1}(t) - x_{\beta\eta g_2}(t)\rangle \ &\leq \ h\left(g_1, x_{\beta\eta g_1}(t)\right) - h\left(g_1, x_{\beta\eta g_2}(t)\right) \\ &\quad + h\left(g_2, x_{\beta\eta g_2}(t)\right) - h\left(g_2, x_{\beta\eta g_1}(t)\right).\end{aligned}$$

Then using, (1.2.13) and (3.1.51), we have

$$C_m \left\| x_{\beta\eta g_1}(t) - x_{\beta\eta g_2}(t) \right\|_X^2 \leqslant C_\Omega \left\| g_1 - g_2 \right\|_{L^2(\Gamma_3)} \int_{\Gamma_3} \mu\left(\|u_{\beta\eta g 1\tau}(t)\| - \|u_{\beta\eta g 2\tau}(t)\|\right) da. \tag{3.1.54}$$

Using (3.1.25), it follows that there exists a constant c_0 such that

$$\left\| \Psi\left(g_1\right) - \Psi\left(g_2\right) \right\|_{L^2(\Gamma_3)} \leqslant c_0 \left\| \sigma_\nu(x_{\beta\eta g_1}(t)) - \sigma_\nu(x_{\beta\eta g_2}(t)) \right\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{3.1.55}$$

Indeed, by using (3.1.13), R is a bounded linear operator, so there exists a constant $c > 0$ such that

$$\left\| R\sigma_\nu \right\|_{L^2(\Gamma_3)} \leq c \left\| \sigma_\nu \right\|_{H^{-\frac{1}{2}}(\Gamma)},$$

and from the continuity of the trace operator $\overline{\gamma}_\nu \sigma = \sigma_\nu$, there exists a constant $c > 0$ such that

$$\left\| \sigma_\nu \right\|_{H^{-\frac{1}{2}}(\Gamma)} \leq c \left\| \sigma \right\|_{Q_1},$$

Then, there exists a constant $c > 0$ such that

$$\left\| R\sigma_\nu \right\|_{L^2(\Gamma_3)} \leq c \left\| \sigma \right\|_{Q_1}$$

from which we obtain the inequality (3.1.55).

Now, using (3.1.13), (3.1.16), we prove the existence of a positive constant $c_1 > 0$ such that

$$\left\| \sigma_\nu(x_{\beta\eta g_1}(t)) - \sigma_\nu(x_{\beta\eta g_2}(t)) \right\|_{H^{-\frac{1}{2}}(\Gamma)} \leq c_1 \left\| x_{\beta\eta g_1}(t) - x_{\beta\eta g_2}(t) \right\|_X. \tag{3.1.56}$$

Thus, taking into account (1.2.13), combining (3.1.54), (3.1.55) and (3.1.56), and performing some algebraic calculations, we obtain

$$\left\| \Psi\left(g_1\right) - \Psi\left(g_2\right) \right\|_{L^2(\Gamma_3)} \leqslant \frac{c_0 c_1 C_\Omega}{C_m} \left\| \mu \right\|_{L^\infty(\Gamma_3)} \left\| g_1 - g_2 \right\|_{L^2(\Gamma_3)}$$

Let $\mu_1 = \dfrac{C_m}{c_0 c_1 C_\Omega}$, then we deduce that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$, Ψ is a contraction and, so, it admits a unique fixed point denoted by g^*.

Keeping in mind that there is a unique element $x_{\beta\eta g^*}$ satisfying the inequality:

$$\langle \Lambda_\beta(t) x_{\beta\eta g^*}, y - x_{\beta\eta g^*} \rangle + h\left(\Psi(g^*), y\right) - h\left(\Psi(g^*), x_{\beta\eta g^*}\right) \geqslant \left(f, y - x_{\beta\eta g^*}\right)_V \quad \forall y \in X,$$

and $h \circ \Psi = j_{fr}$, we prove that $x_{\beta\eta}(t) = x_{\beta\eta g^*}$ is a unique solution of Problem $(P2^2_\eta)$.
We will now see that $x_{\beta\eta} \in C([0,T];X)$. Indeed, let t_1, $t_2 \in [0,T]$, take $y = x_{\beta\eta}(t_2)$ in (3.1.46) written for $t = t_1$ and take $y = x_{\beta\eta}(t_1)$ in the same inequality written for $t = t_2$. Using (3.1.13), (3.1.25) and the properties of R_ν and R_τ we prove that there exists a constant $c > 0$ such that

$$\|x_{\beta\eta}(t_1) - x_{\beta\eta}(t_2)\|_X \leq c(\|\beta(t_1) - \beta(t_2)\|_{L^2(\Gamma_3)} + \|f(t_1) - f(t_2)\|_H + \|\eta(t_1) - \eta(t_2)\|_Q).$$

Then, as $f \in C([0,T];H)$, $\eta \in C([0,T];Q)$ and $\beta \in C([0,T];L^2(\Gamma_3))$, we immediately conclude that $x_{\beta\eta} \in C([0,T];X)$.
We also have that $u_{\beta\eta}(t) \in U_{ad} \cap V$, $\forall t \in [0,T]$. Indeed, for each $t \in [0,T]$ denote

$$\sigma(u_{\beta\eta}(t), \varphi_{\beta\eta}(t)) = \mathcal{B}\varepsilon(u_{\beta\eta}(t), \varphi_{\beta\eta}(t)) - \mathcal{E}^* E(\varphi_{\beta\eta}(t)) + \eta(t)$$

and using *Green's formula* with the regularity $f_0(t) \in H$ we get $\operatorname{div}\big(\sigma(u_{\beta\eta}(t), \varphi_{\beta\eta}(t))\big) \in H$ and then $(u_{\beta\eta}(t), \varphi_{\beta\eta}(t)) \in V_0$. Therefore, $u_{\beta\eta}(t) \in V$. ■

Now, for the next step in the proof, we define the operator $\digamma_\beta : C([0,T];Q) \longrightarrow C([0,T];Q)$ by

$$\digamma_\beta \eta(t) = \int_0^t \mathcal{F}(t-s)\varepsilon(u_{\beta\eta}(s))ds \qquad \forall \eta \in C(0,T;Q),\ t \in [0,T]. \tag{3.1.57}$$

We have the following

Lemma 3.8. *The operator $\digamma_\beta$ has a unique fixed point η_β.*

Proof. Let η_1, $\eta_1 \in C([0,T];Q)$. By a standard computation based on (3.1.14) and (3.1.46), we prove that there exists a constant $c_2 > 0$ such that:

$$\left\|\digamma_\beta \eta_1(t) - \digamma_\beta \eta_2(t)\right\|_Q \leq c_2 \int_0^t \|\eta_1(t) - \eta_2(t)\|_Q \, ds, \quad \forall t \in [0,T].$$

By iteration, for any positive integer n we deduce the estimate

$$\left\|\digamma_\beta^n \eta_1 - \digamma_\beta^n \eta_2\right\|_{C([0,T];Q)} \leq \frac{c_2^n T^n}{n!} \|\eta_1 - \eta_2\|_{C([0,T];Q)}.$$

As $\lim\limits_{n \longrightarrow +\infty} \frac{c_2^n T^n}{n!} = 0$, it follows that for a positive integer n sufficiently large, $\digamma_\beta^n$ is a contraction on the space $C([0,T];Q)$. Then, by using the Banach fixed point theorem, $\digamma_\beta^n$ has a unique fixed point $\eta_\beta \in C([0,T];Q)$ which is also a unique fixed point of $\digamma_\beta$, i.e.,

$$\digamma_\beta \eta_\beta(t) = \eta_\beta(t), \ \forall t \in [0,T].$$

■

Next, we denote $u_\beta = u_{\beta\eta}$ and $\varphi_\beta = \varphi_{\beta\eta}$, and deduce that the couple (u_β, φ_β) is a solution of Problem $(P2_V^\beta)$. The uniqueness follows from the fixed point of the operator $\digamma$, which completes the proof of Theorem 3.3.■

In the following step, we use u_β the solution obtained by Theorem 3.3, to state the following Cauchy problem.

Problem $(P2_{ad})$ *Find a bonding field $\theta_\beta : [0,T] \longrightarrow L^\infty(\Gamma_3)$ such that:*

$$\dot{\theta}_\beta(t) = -\left[\theta_\beta(t)\left((\gamma_\nu R_\nu u_{\beta\nu}(t))^2 + \gamma_\tau \left\|R_\tau(u_{\beta\tau}(t))\right\|^2\right) - \epsilon_a\right]_+ \quad \text{a.e. } t \in [0,T], \tag{3.1.58}$$

$$\theta_\beta(0) = \beta_0. \tag{3.1.59}$$

Lemma 3.9. *Problem $(P2_{ad})$ has a unique solution θ_β which satisfies*

$$\theta_\beta \in W^{1,\infty}(0,T; L^\infty(\Gamma_3)) \cap Z.$$

Proof. Consider the mapping $\mathcal{F} : [0,T] \times L^2(\Gamma_3) \longrightarrow L^2(\Gamma_3)$ defined by

$$\mathcal{F}_\beta(t,\theta) = -\left[\theta\left((\gamma_\nu R_\nu u_{\beta\nu}(t))^2 + \gamma_\tau \left\|R_\tau(u_{\beta\tau}(t))\right\|^2\right) - \epsilon_a\right]$$

For all $t \in [0,T]$ and $\theta \in L^2(\Gamma_3)$, it follows from the properties of the truncation operators R_ν and R_τ that $\mathcal{F}_\beta$ is Lipschitz continuous uniformly in time with respect to β.

Moreover, for any $\theta \in L^2(\Gamma_3)$, the mapping $t \to \mathcal{F}_\beta(t,\theta)$ belongs to $L^\infty(0,T; L^2(\Gamma_3))$.

Using now a version of the Cauchy–Lipschitz theorem (see Chap 1), we obtain a unique function $\theta_\beta \in W^{1,\infty}(0,T; L^2(\Gamma_3))$ satisfying (3.1.58) and (3.1.59).

We note that the restriction $0 \le \theta_\beta \le 1$ is implicitly included in the variational Problem $(P2_V)$ and, therefore, from the definition of the sets B and Z, we find that $\theta_\beta \in Z$, which concludes the proof of Lemma 3.9. ■

Consider the mapping $\Phi : Z \longrightarrow Z$ defined by

$$\Phi\beta = \theta_\beta \tag{3.1.60}$$

The third step consists in the following result.

Lemma 3.10. *There exists a unique element $\beta^* \in Z$ such that $\Phi\beta^* = \beta^*$.*

Proof. Let β_i, $i = 1, 2$ be two elements of Z, Denote by $x_i = (u_i, \varphi_i)$, θ_i the functions obtained in Theorem 3.3 and denote $\theta_{\beta_i} = \beta_i$. It follows from (3.1.58) that

$$\theta_i(t) = \beta_0 - \int_0^t \left[\beta_i(s)\left((\gamma_\nu R_\nu u_{\beta_i\nu}(s))^2 + \gamma_\tau \left\|R_\tau(u_{\beta_i\tau}(s))\right\|^2\right) - \epsilon_a\right]_+ ds, \tag{3.1.61}$$

and there exists a constant $c > 0$ such that

$$\begin{aligned} \|\theta_1(t) - \theta_2(t)\|_{L^2(\Gamma_3)} &\le c \textstyle\int_0^t \|\beta_1(s)R_\nu(u_{\beta_{1\nu}}(s))^2 - \beta_2(s)R_\nu(u_{\beta_{2\nu}}(s))^2\|_{L^2(\Gamma_3)}\, ds \\ &+ \textstyle\int_0^t \left\|\beta_1(s)\left\|R_\beta(u_{\beta_{1\tau}}(s))\right\|^2 - \beta_2(s)\left\|R_\tau(u_{\beta_{2\tau}}(s))\right\|^2\right\|_{L^2(\Gamma_3)} ds. \end{aligned} \tag{3.1.62}$$

Using the properties of the operators R_ν and R_τ, we get

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Gamma_3)} \le c_3 \left(\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, ds + \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_{L^2(\Gamma_3)^d}\, ds\right), \tag{3.1.63}$$

for a certain constant $c_3 > 0$.

Now, to continue the proof, we need to prove the following

Lemma 3.11. *There exists a constant $\mu_2 > 0$ such that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_2$, then*

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_{L^2(\Gamma_3)^d} \le c\,\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \quad \forall t \in [0, T]. \tag{3.1.64}$$

Proof. Let $t \in [0, T]$. We take $\psi = \psi - \varphi_\beta(t)$ in (3.1.44), and by adding with (3.1.43) we get

$$\begin{aligned} &(\mathcal{B}\varepsilon(u_\beta(t)), \varepsilon(v - u_\beta(t)))_Q + \left(\textstyle\int_0^t \mathcal{F}(t-s)\varepsilon(u(s))ds, \varepsilon(v - u(t))\right)_Q \\ &\quad + (\mathcal{E}^*\nabla\varphi_\beta(t), \varepsilon(v - u_\beta(t)))_Q + j_{ad}(\beta(t), u_\beta(t), v - u_\beta(t)) \\ &+ (\mathcal{C}\nabla\varphi_\beta(t), \nabla\psi - \nabla\varphi_\beta(t))_H - (\mathcal{E}\varepsilon(u_\beta(t), \nabla\psi - \nabla\varphi_\beta(t))_H \\ &+ j_{fr}(u_\beta(t), v) - j_{fr}(u_\beta(t), u_\beta(t)) \ge (f(t), v - u_\beta(t))_V + (q(t), \psi - \varphi_\beta(t))_W \\ &\qquad\qquad \forall v \in U_{ad},\ \forall \psi \in W,\ t \in [0, T]. \end{aligned} \tag{3.1.65}$$

Taking $v = u_{\beta_2}(t)$ and $\psi = \varphi_{\beta_2}$ in (3.1.65) satisfied by $(u_{\beta_1}(t), \varphi_{\beta_1}(t))$, and then taking $v = u_{\beta_1}(t)$ and $\psi = \varphi_{\beta_1}$ in the same inequality satisfied by $(u_{\beta_2}(t), \varphi_{\beta_2}(t))$, by adding the resulting inequalities and using (3.1.17), we have

$$\begin{aligned} &(\mathcal{B}\varepsilon(u_{\beta_1}(t)) - \mathcal{B}\varepsilon(u_{\beta_2}(t)), \varepsilon(u_{\beta_1}(t)) - \varepsilon(u_{\beta_2}(t)))_Q \\ &+ (\mathcal{C}\nabla\varphi_{\beta_1}(t) - \mathcal{C}\nabla\varphi_{\beta_2}(t), \nabla\varphi_{\beta_1}(t) - \nabla\varphi_{\beta 2}(t))_H \\ &\le \left(\textstyle\int_0^t \mathcal{F}(t-s)\left(\varepsilon(u_{\beta_1}(t)) - \varepsilon(u_{\beta_2}(t))\right) ds, \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t))\right)_Q \\ &+ j_{ad}(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)) + j_{fr}(u_{\beta_1}(t), u_{\beta_2}(t)) \\ &+ j_{ad}(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)) + j_{fr}(u_{\beta_2}(t), u_{\beta_1}(t)) \\ &- j_{fr}(u_{\beta_1}(t), u_{\beta_1}(t)) - j_{fr}(u_{\beta_2}(t), u_{\beta_2}(t)). \end{aligned} \tag{3.1.66}$$

Using (3.1.13) (c) and (3.1.15) (c), we deduce

$$\begin{aligned}
&m_{\mathcal{B}}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2+m_{\mathcal{C}}\left\|\varphi_{\beta_1}(t)-\varphi_2(t)\right\|_W \leq \\
&+\left(\int_0^t \mathcal{F}(t-s)\left(\varepsilon(u_{\beta_1}(s))-\varepsilon(u_{\beta_2}(s))\right) ds, \varepsilon(u_{\beta_2}(t))-\varepsilon(u_{\beta_1}(t))\right)_Q \\
&+j_{ad}(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t)-u_{\beta_1}(t))+j_{fr}(u_{\beta_1}(t), u_{\beta_2}(t)) \\
&+j_{ad}(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t)-u_{\beta_2}(t))+j_{fr}(u_{\beta_2}(t), u_{\beta_1}(t)) \\
&-j_{fr}(u_{\beta_1}(t), u_{\beta_1}(t))-j_{fr}(u_{\beta_2}(t), u_{\beta_2}(t)),
\end{aligned} \tag{3.1.67}$$

thus

$$\begin{aligned}
&m_{\mathcal{B}}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2 \leq \left(\int_0^t \mathcal{F}(t-s)\left(\varepsilon(u_{\beta_1}(s))-\varepsilon(u_{\beta_2}(s))\right) ds, \varepsilon(u_{\beta_2}(t))-\varepsilon(u_{\beta_1}(t))\right)_Q \\
&+j_{ad}(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t)-u_{\beta_1}(t))+j_{ad}(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t)-u_{\beta_2}(t)) \\
&+j_{fr}(u_{\beta_1}(t), u_{\beta_2}(t))-j_{fr}(u_{\beta_1}(t), u_{\beta_1}(t))+j_{fr}(u_{\beta_2}(t), u_{\beta_1}(t))-j_{fr}(u_{\beta_2}(t), u_{\beta_2}(t)),
\end{aligned} \tag{3.1.68}$$

Hence, we have

$$\begin{aligned}
&\left(\int_0^t \mathcal{F}(t-s)\left(\varepsilon(u_{\beta_1}(s))-\varepsilon(u_{\beta_2}(s))\right) ds, \varepsilon(u_{\beta_2}(t))-\varepsilon(u_{\beta_1}(t))\right)_Q \\
\leq\ &\left(\int_0^t\|\mathcal{F}(t-s)\|_{Q_\infty}\left\|u_{\beta_1}(s)-u_{\beta_2}(s)\right\|_V ds\right)\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V \\
\leq\ &c_4\left(\int_0^t\left\|u_{\beta_1}(s)-u_{\beta_2}(s)\right\|_V ds\right)\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V
\end{aligned} \tag{3.1.69}$$

for some positive constant $c4$. Using Young's inequality ($ab \leq \delta a^2+\frac{1}{4\delta}b^2$), we find that

$$\begin{aligned}
&\left(\int_0^t \mathcal{F}(t-s)\left(\varepsilon(u_{\beta_1}(s))-\varepsilon(u_{\beta_2}(s))\right) ds, \varepsilon(u_{\beta_2}(t))-\varepsilon(u_{\beta_1}(t))\right)_Q \\
\leq\ &\frac{c_4^2}{m_{\mathcal{B}}}\left(\int_0^t\left\|u_{\beta_1}(s)-u_{\beta_2}(s)\right\|_V^2 ds\right)+\frac{m_{\mathcal{B}}}{4}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2
\end{aligned} \tag{3.1.70}$$

Using (3.1.35), and Young's inequality, we deduce that there exists a positive constant c_5 such that

$$\begin{aligned}
j_{ad}(\beta_1, u_1, u_2-u_1)+j_{ad}(\beta_2, u_2, u_1-u_2) \leq c_5\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)}^2 \\
+\frac{m_{\mathcal{B}}}{4}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2.
\end{aligned} \tag{3.1.71}$$

Moreover, we have

$$\begin{aligned}
&j_{fr}(u_{\beta_1}(t), u_{\beta_2}(t))-j_{fr}(u_{\beta_1}(t), u_{\beta_1}(t)) \\
&+j_{fr}(u_{\beta_2}(t), u_{\beta_1}(t))-j_{fr}(u_{\beta_2}(t), u_{\beta_2}(t)) \\
\leq\ &\int_{\Gamma_3}\mu R\left\|\sigma_\nu(u_{\beta_{1\nu}}(t), \varphi_{\beta_1}(t))-\sigma_\nu(u_{\beta_{2\nu}}(t), \varphi_{\beta_2}(t))\right\|\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\| da.
\end{aligned} \tag{3.1.72}$$

Keeping in mind (3.1.55)-(3.1.56) and using (1.2.13), we get

$$\begin{aligned} & j_{fr}(u_{\beta_1}(t), u_{\beta_2}(t)) - j_{fr}(u_{\beta_1}(t), u_{\beta_1}(t)) \\ & + j_{fr}(u_{\beta_2}(t), u_{\beta_1}(t)) - j_{fr}(u_{\beta_2}(t), u_{\beta_2}(t)) \\ \leq \;& c_1 C_\Omega \|\mu\|_{L^\infty(\Gamma_3)} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 . \end{aligned} \tag{3.1.73}$$

We now combine inequalities (3.1.68), (3.1.70), (3.1.71) and (3.1.73) to deduce

$$\begin{aligned} m_{\mathcal{B}} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \;\leq\; & c_5 \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)}^2 + \frac{m_{\mathcal{B}}}{4} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \\ & + c_1 C_\Omega^2 \|\mu\|_{L^\infty(\Gamma_3)} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \\ & + \frac{c_4^2}{m_{\mathcal{B}}} \left(\int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V \, ds \right)^2 \\ & + \frac{m_{\mathcal{B}}}{4} \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 . \end{aligned} \tag{3.1.74}$$

Hence, we have

$$\begin{aligned} \left(\frac{m_{\mathcal{B}}}{2} - c_1 C_\Omega^2 \|\mu\|_{L^\infty(\Gamma_3)} \right) \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \leq & \; c_5 \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2 \\ & + \frac{c_4^2}{m_{\mathcal{B}}} \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V^2 \, ds \end{aligned} \tag{3.1.75}$$

Further, if

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_2 = \frac{m_{\mathcal{B}}}{2 c_1 C_\Omega^2}, \tag{3.1.76}$$

we deduce that there exists a constant $c_8 > 0$ such that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \leq c_8 \left(\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2 + \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V^2 \, ds \right). \tag{3.1.77}$$

Hence, using Cornwall's argument (see Chap 1), it follows that there exists a constant $c_9 > 0$ such that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \leq c_9 \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}^2 , \; \forall t \in [0, T]. \tag{3.1.78}$$

■

Now, to end the proof of Lemma 3.10, we use (3.1.63) and (3.1.78) to obtain

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma_3)} \leq c_9 \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} \, ds, \; \forall t \in [0, T], \tag{3.1.79}$$

where $c_7 > 0$. We have

$$e^{-kt} \|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma_3)} \leq c_9 e^{-kt} \int_0^t e^{ks} e^{-ks} \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} \, ds, \tag{3.1.80}$$

then

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_k \le c_9 e^{-kt} \|\beta_1(t) - \beta_2(t)\|_k \int_0^t e^{ks} ds,\ \forall t \in [0,T]. \tag{3.1.81}$$

So, we deduce that

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_k \le \frac{c_{10}}{k} \|\beta_1(t) - \beta_2(t)\|_k,\ \forall t \in [0,T] \tag{3.1.82}$$

where $c_{10} > 0$.

Inequality (3.1.82) shows that for $k > c_{10}$, Φ is a contraction Z. Then Φ has a unique fixed point which satisfies (3.1.58) and (3.1.59). ■

Thus, we have all the ingredients to prove Theorem 3.2.

Existence. Consider β^* the fixed point of the operator Φ and $x^* = (u^*, \varphi^*)$ the solution of Problem $(P2_V^{\beta^*})$, i.e., $u^* = u_{\beta^*}$ and $\varphi^* = \varphi_{\beta^*}$.

By (3.1.43), (3.1.44), (3.1.58) and (3.1.59) we conclude that the triple $(u^*, \varphi^*, \beta^*)$ is a solution to Problem $(P2_V)$.

Uniqueness. The uniqueness arises from the uniqueness of the fixed point of the operator Φ, which completes the proof of Theorem 3.2.

Indeed, let (u, φ, β) be a solution of Problem $(P2_V)$, it follows from (3.1.43) and (3.1.44) that u is a solution of Problem $(P2_V^{\beta})$ and, by Theorem 3.3, this problem has a unique solution (u_β, φ_β), where $u_\beta = u$ and $\varphi_\beta = \varphi$.

Taking $u = u_\beta$ and $\varphi = \varphi_\beta$ in Problem $(P2_V)$, we deduce that β is a solution of Problem $(P2_{ad})$. From the result of Lemma 3.10, the problem $(P2_{ad})$ has a unique solution β^*, so we find $\beta^* = \beta$, and then, we conclude that $(u^*, \varphi^*, \beta^*)$ is a unique solution to Problem (P_V).

Let now σ^* et D^* be the functions defined by (3.1.1) and (3.1.2), respectively, which correspond to (u^*, φ^*). Then it results from (3.1.13) –(3.1.17) that $\sigma^* \in C([0,T]; Q)$ and $D^* \in C([0,T]; H)$.

Using also a standard argument, it follows from (3.1.39) and (3.1.40) that

$$Div\ \sigma^*(t) + f_0(t) = 0 \qquad in\ \Omega, \tag{3.1.83}$$

$$div\ D^*(t) + q_0(t) = 0 \qquad in\ \Omega, \tag{3.1.84}$$

Therefore, using (3.1.19), we deduce that $Div\ \sigma^*(u^*(t), \varphi^*(t)) \in H$ for each $t \in [0,T]$, and then $(u^*, \varphi^*) \in V_0$, and using (3.1.20), we have $div\ D^* \in C([0,T]; L^2(\Omega))$, which implies that $\sigma^* \in C([0,T]; Q_1)$ and $D^* \in C([0,T]; W_a)$.

The triple $(u^*, \varphi^*, \beta^*)$ which satisfies (3.1.39)–(3.1.42) is called a weak solution of Problem $(P2)$. We conclude that under stated assumptions, Problem $(P2)$ has a unique weak solution $(u^*, \varphi^*, \beta^*, \sigma^*, D^*)$ with the regularity

$$
\begin{aligned}
u^* &\in C([0,T]\,;V),\\
\varphi^* &\in C([0,T]\,;W)\\
\beta^* &\in W^{1,\infty}\big(0,T;L^2(\Gamma_3)\big)\cap B\\
\sigma^* &\in C([0,T]\,;Q_1)\\
D^* &\in C([0,T]\,;W_a)
\end{aligned}
$$

3.2 Contact problem with normal compliance, adhesion and friction

In this section, we explore an alternative mathematical model that characterizes a quasi-static interaction between a piezoelectric material and a deformable foundation. A non-linear electro-viscoelastic constitutive law with long memory effects is applied. The contact is represented using a normal compliance condition combined with the Coulomb law of dry friction. This formulation also incorporates the adhesion between the contact surfaces.

3.2.1 The contact problem statement

We are examining the same previous physical framework; however, along the part of bandary Γ_3, the body is in adhesive contact with a deformable foundation. The contact is represented through a normal compliance condition coupled with Coulomb friction. This translates to investigating the $(P3)$, stated as follows:

Problem $(P3)$. *Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to \mathbf{S}_d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field $D : \Omega \times [0,T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$, such that for all $t \in [0,T]$,*

$$\sigma\,(t) = \mathcal{B}\varepsilon(u(t)) + \int_0^t \mathcal{F}(t-s)\varepsilon(u(s))ds - \mathcal{E}^* E(\varphi(t)) \tag{3.2.1}$$

$$D(t) = \mathcal{E}\varepsilon(u(t)) + \mathcal{C}E(\varphi(t)) \tag{3.2.2}$$

$$Div\ \sigma(t) + f_0(t) = 0 \quad \text{in } \Omega, \tag{3.2.3}$$

$$div\ D(t) + q_1(t) = 0 \quad \text{in } \Omega, \tag{3.2.4}$$

$$u(t) = 0 \quad \text{on } \Gamma_1, \tag{3.2.5}$$

$$\sigma\nu(t) = f_2(t) \quad \text{on } \Gamma_2, \tag{3.2.6}$$

$$-\sigma_\nu(t) = p_\nu(u_\nu(t) - g) - \gamma_\nu \beta^2 R_\nu(u_\nu(t)) \quad \text{on } \Gamma_3, \tag{3.2.7}$$

$$\dot{\beta}(t) = -\left[\beta(t)\left((\gamma_\nu R_\nu u_\nu(t))^2 + \gamma_\tau \left\|R_\tau(u_\tau(t))\right\|^2\right) - \epsilon_a\right]_+ \quad \text{on } \Gamma_3, \tag{3.2.8}$$

$$\varphi(t) = 0 \quad \text{on } \Gamma_a \tag{3.2.9}$$

$$D\nu(t) = q_2(t) \quad \text{on } \Gamma_b, \tag{3.2.10}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3 \tag{3.2.11}$$

$$\begin{cases} \|\sigma_\tau(t) + \gamma_\tau \beta^2 R_\tau(u_\tau(t))\| \leqslant \mu p\,(u_\nu(t) - g) \\ \\ \|\sigma_\tau(t) + \gamma_\tau \beta^2 R_\tau(u_\tau(t))\| < \mu p\,(u_\nu(t) - g) \Longrightarrow u_\tau\,(t) = 0 \\ \\ \|\sigma_\tau(t) + \gamma_\tau \beta^2 R_\tau(u_\tau(t))\| = \mu p\,(u_\nu(t) - g) \Longrightarrow \exists \lambda \geqslant 0 \\ \\ \textit{such that:}\;\; \sigma_\tau\,(t) + \gamma_\tau \beta\,(t)^2\, R_\tau(u_\tau(t)) = -\lambda u_\tau\,(t) \end{cases} \text{on } \Gamma_3. \tag{3.2.12}$$

Recall that the equation (3.2.7) reflects the fact that the body is in adhesive contact with a deformable foundation. The normal stress complies with the condition referred to as the normal compliance, which models the interpenetration of the contact surface into the foundation. The contact surface is not known beforehand, where g is the initial gap between the body and the foundation, measured in the direction of the normal ν, and p is a given non-negative function. This condition demonstrates that the foundation exerts a reaction on the body which depends on the penetration $(u_\nu - g)$. Finally, (3.2.12) represents a version of the Coulomb dry friction law in its static form, where μ is the coefficient of friction and $\mu p\,(u_\nu(t) - g)$ represents the so-called friction bound.

In the study of the problems $(P3)$, we impose the following assumptions on the problem's data:
The operator $\mathcal{B}$, and the tensors $\mathcal{F}$, $\mathcal{C}$, $\mathcal{E}$, and $\mathcal{E}^*$ satisfy the assumptions (3.1.13)–(3.1.17)
The normal compliance function $p : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies:

$$\begin{cases} \text{(a)} & \text{There exists a constant } M_{\mathfrak{p}} > 0, \text{ such that:} \\ & |p(x, r_1) - p(x, r_2)| \leq M_{\mathfrak{p}}\, |r_1 - r_2| \;\; \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. on } \Gamma_3, \\ \text{(b)} & (p(x, r_1) - p(x, r_2)\,(r_1 - r_2\;) \geq 0 \;\; \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. on } \Gamma_3, \\ \text{(c)} & \text{The function } x \to p(x, r) \text{ is measurable on } \Gamma_3, \text{ for all } r \in \mathbb{R}, \\ \text{(d)} & x \to p(x, r) = 0, \text{ for all } r \leq 0. \end{cases} \tag{3.2.13}$$

As an example satisfying condition (3.2.13), we can consider the function $p(r) = [r]_+$.
The adhesion coefficients satisfy (3.1.18) and the initial gap g satisfies:

$$g \in L^2\,(\Gamma_3)\,, \;\; g \geqslant 0 \;\; \text{a.e. } x \in \Gamma_3. \tag{3.2.14}$$

The volume and surface forces, as well as the electric charges q_0 and q_2 possess the following respective regularities:

$$f_0 \in C\left([0,T];H\right), \quad f_2 \in C\left([0,T];L^2\left(\Gamma_2\right)^d\right). \tag{3.2.15}$$

$$q_0 \in C\left([0,T];H\right), \quad q_2 \in C\left([0,T];L^2\left(\Gamma_b\right)^d\right). \tag{3.2.16}$$

To reflect the fact that the foundation is insulating, we assume that:

$$q_0(t) = 0 \quad \text{ on } \Gamma_3 \quad \forall t \in [0,T]. \tag{3.2.17}$$

And finally, the initial data β_0 satisfies:

$$\beta_0 \in L^2(\Gamma_3), \qquad 0 \leq \beta_0 \leq 1 \quad \text{a.e.} \quad \text{on } \Gamma_3. \tag{3.2.18}$$

The friction coefficient μ is such that

$$\mu \in L^\infty(\Gamma_3), \;\; \mu(x) \geq 0 \;\; \text{a.e. on } \Gamma_3. \tag{3.2.19}$$

3.2.2 Variational formulation

It follows from the de *Riesz-Fréchet* representation theorem that there exist a function $f : [0,T] \longrightarrow V$ and a function $q : [0,T] \longrightarrow W$ such that:

$$(f(t), v)_V = \int_\Omega f_0(t).v dx + \int_{\Gamma_2} f_2.v da \;\; \forall v \in V. \tag{3.2.20}$$

$$(q(t), \psi)_V = \int_\Omega q_0(t).\psi dx + \int_{\Gamma_2} q_2.\psi da \;\; \forall \psi \in W \tag{3.2.21}$$

(3.2.15) and (3.2.16) imply that:

$$f \in C\left([0,T];H\right) \text{ and } q \in C\left([0,T];W\right). \tag{3.2.22}$$

We define the adhesion functional $j_{ad} : L^\infty\left(\Gamma_3\right) \times V \times V \longrightarrow \mathbb{R}$, the normal compliance functional $j_{cn} : V \times V \longrightarrow \mathbb{R}$, and the frictional functional $j_{fr} : V \times V \longrightarrow \mathbb{R}$ as follows:

$$j_{ad}(\beta, u, v) = \int_{\Gamma_3} -(\gamma_\nu \beta^2 R_\nu(u_\nu)v_\nu + \gamma_\tau \beta^2 R_\tau(u_\tau).v_\tau)da, \tag{3.2.23}$$

$$j_{cn}(u, v) = \int_{\Gamma_3} p(u_\nu - g)v_\nu da, \tag{3.2.24}$$

$$j_{fr}(u, v) = \int_{\Gamma_3} \mu p\left(u_\nu(t) - g\right).\left\|v_\tau\right\| da. \tag{3.2.25}$$

Using the standard procedure based on the *Green's formula* as employed in the previous problems, we demonstrate that if u, σ, φ and D are sufficiently regular and satisfy the equations and conditions (3.2.1)-(3.2.12), then

$$(\sigma(t), \varepsilon(u(t)))_Q + j_{ad}(\beta(t), u(t), v) + j_{cn}(u(t), v - u(t))$$

$$+j_{fr}(u(t), v) - j_{fr}(u(t), u(t)) \geq (f(t), v - u(t))_V \tag{3.2.26}$$

$$\forall v \in V, t \in [0, T]$$

$$(D(t), \nabla\psi)_H + (q(t), \psi)_W = 0 \quad \forall \psi \in W. \tag{3.2.27}$$

We then associate the following variational formulation to the problem $(P3)$:

Problem $(P3_V)$. *Find a displacement field* $u : [0, T] \longrightarrow V$, *an electric potential* φ : $[0, T] \longrightarrow W$, *and an adhesion field* $\beta : [0, T] \longrightarrow L^\infty(\Gamma_3)$, such that $u(t) \in V$, $\varphi(t) \in W$ *for all* $t \in [0, T]$, *and*

$$\begin{aligned}
&(\mathcal{B}\varepsilon(u(t)), \varepsilon(v - u(t)))_Q + \left(\int_0^t \mathcal{F}(t-s)\varepsilon(u(s))ds, \varepsilon(v - u(t))\right)_Q \\
&+ (\mathcal{E}^*\nabla\varphi(t), \varepsilon(v - u(t)))_Q + j_{ad}(\beta(t), u(t), v - u(t)) \ + j_{cn}(u(t), v - u(t)) \\
&+j_{fr}(u(t), v) - j_{fr}(u(t), u(t)) \geq (f(t), v - u(t))_V \\
&\qquad\qquad\qquad\qquad \forall v \in V, \ t \in [0, T],
\end{aligned} \tag{3.2.28}$$

$$(\mathcal{C}\nabla\varphi(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u(t), \nabla\psi)_H = (q(t), \psi)_W \quad \forall \psi \in W, \ t \in [0, T], \tag{3.2.29}$$

$$\begin{aligned}
\dot\beta(t) = -\left[\beta(t)\left((\gamma_\nu R_\nu u_\nu(t))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2\right) - \epsilon_a\right]_+ \ t \in [0, T], , \\
\beta(0) = \beta_0.
\end{aligned} \tag{3.2.30}$$

3.2.3 Existence and uniqueness of the weak solution to Problem (P3)

In this section, we state and prove the existence and uniqueness of the solution to the variational problem $(P3_V)$.

Theorem 3.12. *Suppose that (3.1.13)–(3.1.18), (3.2.13) and (3.2.14)–(3.2.19) are satisfied. Then, there exists a constant* $L_0 > 0$ *such that if* $M_{\mathfrak{p}}\left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right) < L_0$, *the problem*

$(P3_V)$ *has a unique solution* (u, φ, β) *with the regularity:*

$$u \in C([0,T]\,;V), \tag{3.2.31}$$

$$\varphi \in C([0,T]\,;W), \tag{3.2.32}$$

$$\beta \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right) \cap B. \tag{3.2.33}$$

The proof of this theorem will proceed similarly to the previous case, that is, in several steps. In the following, we assume that (3.1.13)–(3.1.18), (3.2.13) and (3.2.14)–(3.2.21) hold true. Also, throughout this section, c will represent a strictly positive constant independent of time, and its value may vary from one location to another. First, for a given $\beta \in B$, we formulate the following auxiliary problem:

Problem $(P3_V^\beta)$. *Find a displacement field* $u_\beta : [0,T] \longrightarrow V$ *and an electric potential* $\varphi_\beta : [0,T] \longrightarrow W$ *such that:*

$$\begin{aligned}
&(\mathcal{B}\varepsilon(u_\beta(t)), \varepsilon(v - u_\beta(t)))_Q + \left(\textstyle\int_0^t \mathcal{F}(t-s)\varepsilon(u(s))ds, \varepsilon(v-u(t))\right)_Q \\
&+ (\mathcal{E}^*\nabla\varphi_\beta(t), \varepsilon(v - u_\beta(t)))_Q + j_{ad}(\beta(t), u_\beta(t), v - u_\beta(t)) \\
&+ j_{cn}(u_\beta(t), v - u_\beta(t)) + j_{fr}(u_\beta(t), v) - j_{fr}(u_\beta(t), u_\beta(t)) \\
&\geq (f(t), v - u_\beta(t))_V \qquad \forall v \in V, t \in [0,T]
\end{aligned} \tag{3.2.34}$$

$$\begin{aligned}
&(\mathcal{C}\nabla\varphi_\beta(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u_\beta(t), \nabla\psi)_H = (q(t), \psi)_W \\
&\forall \psi \in W, t \in [0,T].
\end{aligned} \tag{3.2.35}$$

We have the following lemma:

Lemma 3.13. *The problem* $(P3_V^\beta)$ *has a unique solution.* $(u_\beta, \varphi_\beta) \in C([0,T]\,;V \times W)$.

Proof. To prove this lemma, we consider the Hilbert product space $X = V \times W$ with the inner product defined as

$$\langle x, y\rangle = \langle (u,\varphi), (v,\psi)\rangle = (u,v) + (\varphi,\psi), \qquad x, y \in X \tag{3.2.36}$$

and the associated norm$\|.\|_X$.

For all $\eta \in C([0,T]\,;Q)$ et $t \in [0,T]$, we introduce the operator $\Lambda_\beta : X \longrightarrow X$ and the element $f_\eta(t) \in X$ defined for all $x = (u, v)$ and $y = (\varphi, \psi)$ by:

$$\begin{aligned} \langle \Lambda_\beta(t)x, y\rangle &= (\mathcal{B}\varepsilon(u), \varepsilon(v))_Q + (\mathcal{E}^*\nabla\varphi, \varepsilon(v))_Q + (\mathcal{C}\nabla\varphi, \nabla\psi)_H \\ &\quad - (\mathcal{E}\varepsilon(u), \nabla\psi)_H + j_{ad}(\beta(t), u, v), \end{aligned} \tag{3.2.37}$$

$$\langle f_\eta(t), y\rangle = (f(t), v)_V + (q(t), \psi)_W - (\eta(t), \varepsilon(v))_Q\,, \tag{3.2.38}$$

and let

$$j(x, y) = j_{cn}(u, v) + j_{fr}(u, v). \tag{3.2.39}$$

■

We introduce the following two problems:

Problem $(P3^1_\eta)$. *Find* $x_{\beta\eta} : [0, T] \longrightarrow X$ *such that*

$$\begin{aligned} &(\mathcal{B}\varepsilon(u_{\beta\eta}(t)), \varepsilon(v - u_{\beta\eta}(t)))_Q + (\mathcal{E}^*\nabla\varphi_{\beta\eta}(t), \varepsilon(v - u_{\beta\eta}(t)))_Q \\ &+ (\mathcal{C}\nabla\varphi_{\beta\eta}(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u_{\beta\eta}(t), \nabla\psi)_H + (\eta(t), \varepsilon(v - u_{\beta\eta}(t)))_Q \\ &+ j_{ad}(\beta(t), u_{\beta\eta}(t), v) + j_{cn}(u_{\beta\eta}(t), v - (u_{\beta\eta}(t)) \\ &+ j_{fr}(u_{\beta\eta}(t), v) - j_{fr}(u_{\beta\eta}(t), u_{\beta\eta}(t)) \geq (f(t), v - u_{\beta\eta}(t))_V \\ &\qquad \forall v \in V,\ t \in [0, T], \end{aligned} \tag{3.2.40}$$

$$\begin{aligned} &(\mathcal{C}\nabla\varphi_{\beta\eta}(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u_{\beta\eta}(t), \nabla\psi)_H = (q(t), \psi)_W \\ &\qquad \forall\psi \in W, t \in [0, T]. \end{aligned} \tag{3.2.41}$$

Problem $(P3^2_\eta)$. *Find* $x_{\beta\eta} : [0, T] \longrightarrow X$ *such that*

$$\begin{aligned} &\langle \Lambda_\beta(t)x_{\beta\eta}(t), y - x_{\beta\eta}(t)\rangle + j(y, x_{\beta\eta}(t)) - j(x_{\beta\eta}(t), x_{\beta\eta}(t)) \\ &\qquad \geq \langle f_\eta(t), y - x_{\beta\eta}(t)\rangle\ \ \forall y \in X,\ t \in [0, T]. \end{aligned} \tag{3.2.42}$$

Remark 3.1. *The two problems mentioned above are equivalent in the sense that if* $x_{\beta\eta} = (u_{\beta\eta}, \varphi_{\beta\eta}) \in C([0, T]\,; X)$ *is a solution to either of the two problems, it is also a solution to the other problem.*

We have the following lemma:

Lemma 3.14. *There exists a constant $L_0 > 0$ such that if $M_{\mathfrak{p}}(1 + \|\mu\|_{L^\infty(\Gamma_3)}) < L_0$, the problem $(P3^2_\eta)$ has a unique solution $x_{\beta\eta} \in C([0,T]\,; X)$.*

Proof. To prove Lemma 3.13, we proceed in steps:

The functional j_{ad} is linear in the third term and satisfies (3.1.32)-(3.1.36)
As in the previous problem, the following assumptions (3.1.13), (3.1.15)-(3.1.17), and the properties of R_ν, R_τ given by (1.1.18)-(1.1.19), show that Λ_β is a Lipschitz continuous and strongly monotone operator. In other words, there exists a positive constant $c > 0$ such that:

$$\langle \Lambda_\beta(t) x_{\beta\eta}(t), x_{\beta\eta}(t)\rangle \geq c\,\|x_{\beta\eta}(t)\|^2_X \qquad \forall x_{\beta\eta}(t) \in X. \tag{3.2.43}$$

We note that:

$$j(y, x_{\beta\eta}(t)) - j(x_{\beta\eta}(t), x_{\beta\eta}(t)) = j(y, u_{\beta\eta}(t)) - j(u_{\beta\eta}(t), u_{\beta\eta}(t))$$

In the following, let's consider the set $L^2_+(\Gamma_3)$ defined by:

$$L^2_+(\Gamma_3) = \left\{\varphi \in L^2(\Gamma_3)\,;\; \varphi \geqslant 0 \text{ a.e. on } \Gamma_3\right\}.$$

For each $g = (g_1, g_2) \in L^2_+(\Gamma_3)^2$, we define the function $h(g,.) : X \longrightarrow \mathbb{R}$ by

$$h(g,y) = \int_{\Gamma_3} g_1 w_\nu da + \int_{\Gamma_3} g_2 \|w_\tau\|\, da \quad \forall y = (w,\psi) \in X, \tag{3.2.44}$$

and we introduce another intermediate problem as follows:

Problem $(P3^g_1)$. *Find $x_{\beta\eta} : [0,T] \longrightarrow X$ such that*

$$\langle \Lambda_\beta(t) x_{\beta\eta}(t), y - x_{\beta\eta}(t)\rangle + h(g,y) - h(g, x_{\beta\eta}(t)) \geqslant (f, y - x_{\beta\eta}(t))_V$$
$$\forall y \in X. \tag{3.2.45}$$

Lemma 3.15. *The problem $(P3^g_1)$ has a unique solution.*

Indeed, as the functional $h(g,.)$ is convex and lower semi-continuous, and since Λ_β is Lipschitz continuous and strongly monotone, then, problem $(P3^g_1)$ has a unique solution. (see Chap1).

Now, to prove Lemma (3.14.), we define the following mapping:

$$\Psi : L^2_+ (\Gamma_3)^2 \longrightarrow L^2_+ (\Gamma_3)^2$$

$$g \longmapsto \Psi (g) = (p (u_{\beta\eta g\nu} - g) , \mu p (u_{\beta\eta g\nu} - g)),$$

and we establish the following lemma.

Lemma 3.16. *Ψ has a unique fixed point g^*, and $x_{\beta\eta g^*}$ is a unique solution to problem $(P3^2_\eta)$.*

Proof. For $i = 1, 2$, we define the following problem.

Problem $(P3^2_{\eta gi})$. *Find $u_{gi} \in V$ such that:*

$$\langle \Lambda_\beta(t) x_{\beta\eta gi}, y\rangle + h (g_i, y) - h (g_i, x_{\beta\eta gi}) \geqslant (f, y - x_{\beta\eta gi})_V \quad \forall y \in V.$$

Let's denote $x_{\beta\eta gi}$ by x_{g_i} and take $y = x_{g_2}$ in the preceding inequality written for $g = g_1$. Then, substitute $y = x_{g_1}$ in the same inequality written for $g = g_2$. By adding the resulting two inequalities, we obtain:

$$\langle \Lambda_\beta(t)(x_{g_1} - x_{g_2}), x_{g_1} - x_{g_2}\rangle \leq h (g_1, x_{g_1}) - h (g_1, y_{g_2}) + h (g_2, x_{g_2}) - h (g_2, y_{g_1}) .$$

Next, using (1.2.13), (3.2.43) and (3.2.44), it follows that there exists a constant $c > 0$ such that

$$\|u_{g_1} - u_{g_2}\|_V \leqslant c \|g_1 - g_2\|_{L^2(\Gamma_3)^2} . \tag{3.2.46}$$

Moreover, based on the assumption (3.2.13) regarding the function p and (1.2.13), we confirm that there exists a constant $c > 0$ such that:

$$\|\Psi (g_1) - \Psi (g_2)\|_{L^2(\Gamma_3)^2} \leqslant cM_{\mathfrak{p}} \left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right) \|x_{g_1} - x_{g_2}\|_X .$$

Consequently, employing (3.2.46) we have:

$$\|\Psi (g_1) - \Psi (g_2)\|_{L^2(\Gamma_3)^2} \leqslant c'cM_{\mathfrak{p}} \left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right) \|g_1 - g_2\|_{L^2(\Gamma_3)^2} .$$

By setting $L_0 = 1/c'c$, we deduce that if $M_{\mathfrak{p}} \left(1 + \|\mu\|_{L^\infty(\Gamma_3)}\right) < L_0$, Ψ is a contraction, implying it has a unique fixed point g^*.
Considering the unique element x_{g^*} that satisfies the equality:

$$\langle \Lambda_\beta(t) x_{g^*}, y - x_{g^*}\rangle + h (\Psi(g^*), y) - h (\Psi(g^*), x_{g^*}) \geqslant (f, y - x_{g^*})_V \;\; \forall y \in X x_{g^*},$$

and since $h \circ \Psi = j$, we conclude that $x_{\beta\eta}(t) = x_{g^*}$ is a unique solution to the problem $(P3^2_\eta)$. ■

We now define the operator $\digamma_\beta : C([0,T]\,;Q) \longrightarrow C([0,T]\,;Q)$ by

$$\digamma_\beta \eta(t) = \int_0^t \mathcal{F}(t-s)\varepsilon(u_{\beta\eta}(s))ds \qquad \forall \eta \in C(0,T;Q) \ \ \forall t \in [0,T]. \tag{3.2.47}$$

Lemma 3.17. *The operator $\digamma_\beta$ possesses a unique fixed point u_β.*

Proof. The proof of this lemma is analogous to that of lemma 3.7.

Now, denote $u_\beta = u_{\beta\eta}$ and $\varphi_\beta = \varphi_{\beta\eta}$. The pair (u_β, φ_β) is the unique solution to the problem $(P3^\beta_V)$ whose existence and uniqueness follow Lemma 3.14 and Lemma 3.17. To prove Theorem 3.12, we require further intermediate results.

Let u_β be the solution obtained above and consider the following Cauchy problem:

Problem $(P3_{ad})$ *Find an adhesion field $\beta^* : [0,T] \longrightarrow L^\infty(\Gamma_3)$ such that:*

$$\begin{aligned} \dot{\beta}^*(t) &= -\left[\beta^*(t)\left((\gamma_\nu R_\nu u_{\beta^*\nu}(t))^2 + \gamma_\tau \left\|R_\tau(u_{\beta^*\tau}(t))\right\|^2\right) - \epsilon_a\right]_+ \\ \text{a.e. } t &\in [0,T], \end{aligned} \tag{3.2.48}$$

$$\beta^*(0) = \beta_0, \tag{3.2.49}$$

and we have:

Lemma 3.18. *The problem $(P3_{ad})$ has a unique solution $\beta^* \in W^{1,\infty}(0,T;L^\infty(\Gamma_3)) \cap B$.*

Proof. Similar to the proof of Lemma 3.8. ■

The solution β^* belongs to the subspace Z_0 of the space $C([0,T]\,;L^2(\Gamma_3)$ defined by:

$$Z_0 = \left\{\beta \in C([0,T]\,;L^2(\Gamma_2)) \cap B\ ;\ \beta(0) = \beta_0\right\}. \tag{3.2.50}$$

Let's consider the mapping: $\Phi : Z_0 \longrightarrow Z_0$ defined by

$$\begin{gathered}\Phi(t) = \beta_0 - \textstyle\int_0^t \left[\beta(s)\left((\gamma_\nu R_\nu u_{\beta\nu}(s))^2 + \gamma_\tau \left\|R_\tau(u_{\beta\tau}(s))\right\|^2\right) - \epsilon_a\right]_+ ds \\ \forall t \in [0,T], \end{gathered} \tag{3.2.51}$$

The next step is to establish the following result.

Lemma 3.19. *There is a unique element $\beta^* \in Z_0$ such that $\Phi\beta^* = \beta^*$.*

Proof. Let β_i, $i = 1, 2$ be two elements of Z_0. Denoting $u_{\beta_i} = u_i$ and $x_i = (u_i, \varphi_i)$, let $t \in [0, T]$, from (3.1.35) , there exists a constant $c > 0$ such that

$$j_{ad}(\beta_1(t), u_1(t), u_2(t) - u_1(t)) + j_{ad}(\beta_2(t), u_2(t), u_1(t) - u_2(t)) \tag{3.2.52}$$

$$\leq \; c \, \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma 3)} \|u_1(t) - u_2(t)\|_V \tag{3.2.53}$$

Using the assumptions (3.1.13)–(3.1.18) and (3.2.13), we also demonstrate that there exists a constant $c > 0$ such that

$$\|x_1(t) - x_2(t)\|_V \leq c \, \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma 3)} \|u_1(t) - u_2(t)\|_V \, ,$$

Consequently,

$$\|u_1(t) - u_2(t)\|_V + \|\varphi_1(t) - \varphi_2(t)\|_W \leq c \, \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma 3)} \, , \tag{3.2.54}$$

Hence,

$$\|u_1(t) - u_2(t)\|_V \leq c \, \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma 3)} \, . \tag{3.2.55}$$

We have

$$\begin{array}{r}\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma 3)} \leq c \int_0^t \|\beta_1(s) R_\nu(u_{1\nu})^2 - \beta_2(s) R_\nu(u_{2\nu})^2\|_{L^2(\Gamma 3)} \, ds \\ + \int_0^t \left\|\beta_1(s) \, \|R_\tau(u_{1\tau})\|^2 - \beta_2(s) \, \|R_\tau(u_{2\tau})\|^2\right\|_{L^2(\Gamma 3)} ds.\end{array} \tag{3.2.56}$$

By using the properties (1.1.18) and (1.1.19) of the operators R_ν and R_τ, we have:

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma 3)} \leq c \left(\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma 3)} \, ds + \int_0^t \|u_1(s) - u_2(s)\|_{L^2(\Gamma 3)^d} \, ds \right). \tag{3.2.57}$$

Finally, by applying the Gronwall's argument and (1.2.13) we obtain:

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma 3)} \leq c \int_0^t \|u_1(s) - u_2(s)\|_V \, ds, \tag{3.2.58}$$

Hence,

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma 3)} \leq c \int_0^t \|u_1(s) - u_2(s)\|_V \, ds$$

Taking into account (3.2.54), we get:

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma 3)} \leq c \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma 3)} \, ds. \tag{3.2.59}$$

Through iteration, for all integer n, we have the estimate:

$$\|\Phi^n\beta_1 - \Phi^n\beta_2\|_{C([0,T],L^2(\Gamma3))} \leq \frac{c^nT^n}{n!}\|\beta_1 - \beta_2\|_{C([0,T],L^2(\Gamma3))}. \tag{3.2.60}$$

Therefore, for sufficiently large n, the operator Φ^n is contractive on $C([0,T]\,;L^2(\Gamma3))$, and consequently, Φ^n has a unique fixed point $\beta^* \in Z_0$ which is also a unique fixed point of Φ. ■

We now have all we need for proving Theorem 3.

Let β^* be the fixed point of the operator Φ and $x^* = (u^*, \varphi^*)$ be the solution to problem $(P_V^{\beta^*})$, Using similar arguments as those shown in (3.2.55) and on p, we establish that there exists a constant $c_1 > 0$ such that:

$$\begin{aligned}\|u^*(t_1) - u^*(t_2)\|_V \leq c_1 \Big(&\|\beta^*(t_1) - \beta^*(t_2)\|_{L^2(\Gamma3)} + \|q(t_1) - q(t_2)\|_W \\ &+ \|f(t_1) - f(t_2)\|_V\big), \quad \forall t_1, t_2 \in [0,T]\end{aligned} \tag{3.2.61}$$

and that there exists a constant $c_2 > 0$ such that:

$$\begin{aligned}\|\varphi^*(t_1) - \varphi^*(t_2)\|_V \leq c_2 \left(\|q(t_1) - q(t_2)\|_W + \|u^*(t_1) - u^*(t_2)\|_V\right), \\ \forall t_1, t_2 \in [0,T].\end{aligned} \tag{3.2.62}$$

It follows from Lemmas (3.14.), (3.16.), (3.17.) and (3.18.) that the triplet $(u^*, \varphi^*, \beta^*)$ is a solution to problem $(P3_V)$, and the uniqueness follows from the uniqueness of the fixed point of the operator Φ. The regularity is justified by the last two inequalities

3.3 *Conclusion*

Under the assumption of the smallness of the friction coefficient and with appropriate regularity of the data, we have established the existence of a unique weak solution for each problem ($P2$ and $P3$) using variational methods, specifying the regularity of the solution. The proofs are based on arguments involving variational inequalities, differential equations and Banach's fixed point theorem. However, the exploration of numerical simulations represents an intriguing line for future research.

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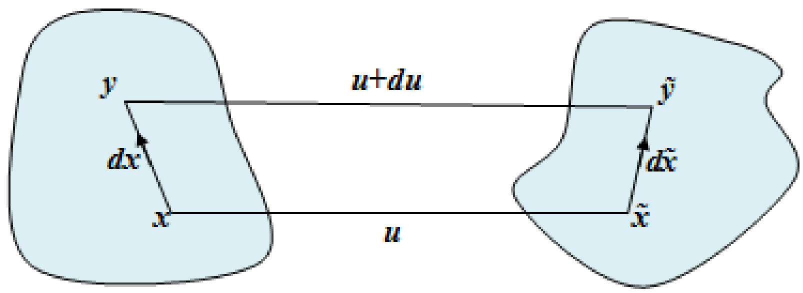

Figure 3.1: The displacement field.

3.4 Deformations

In order to define deformations, we study the displacement of a point and its neighbors at each point, so we introduce a field. Let $\mathfrak{D}$ denote the transformation function between the initial configuration (the reference configuration) and the deformed configuration. In the original state, a particle P in this field occupies the position $x(x_1, x_2, x_3)$. In the final state, it occupies the position:

$$\tilde{x}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \mathfrak{D}(x_1, x_2, x_3).$$

The displacement field in the undeformed reference system is given by:

$$u(x) = \tilde{x} - x = \mathfrak{D}(x) - x.$$

Let $y = x + dx$, then $d\tilde{x} = dx + du$ and the variation of the distance between the pair of points under consideration:

$$d\tilde{l}^2 = \sum_{i=1}^{1=3} (dx_i + du_i)^2$$

We can express du as follows:

$$du_i = \sum_{j=1}^{j=3} u_{i,j} dx_j; \quad du = \nabla(u) dx$$

The displacement gradient tensor is then defined as

$$\nabla(u) = \begin{pmatrix} u_{1,1} & u_{2,1} & u_{3,1} \\ u_{1,2} & u_{2,2} & u_{3,2} \\ u_{1,3} & u_{2,3} & u_{3,3} \end{pmatrix}$$

The Green-Lagrange deformation tensor is obtained directly from the displacements by the expression

$$E = \frac{1}{2}\left(\nabla(u) + \nabla^t(u) + \nabla^t(u).\nabla(u)\right), \quad E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$$

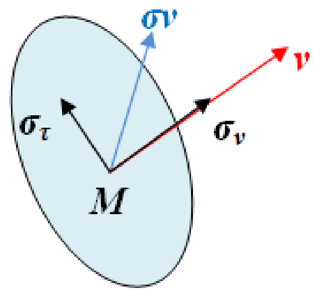

Figure 3.2: Cauchy stress vector.

In the case of small deformations, the second order terms of the Green-Lagrange tensor are negligible, and we often refer to this as the linearized deformation tensor or the small displacement deformation tensor.

$$\varepsilon = \frac{1}{2}(u_{i,j} + u_{j,i})$$

or

$$\varepsilon = (\varepsilon_{ij}) = \begin{pmatrix} u_{1,1} & \frac{1}{2}(u_{1,2} + u_{2,1}) & \frac{1}{2}(u_{1,3} + u_{3,1}) \\ \frac{1}{2}(u_{1,2} + u_{2,1}) & u_{2,2} & \frac{1}{2}(u_{2,3} + u_{3,3}) \\ \frac{1}{2}(u_{1,3} + u_{3,1}) & \frac{1}{2}(u_{2,3} + u_{3,3}) & u_{3,3} \end{pmatrix}$$

Spherical tensor and deviator

Note that I is the identity tensor of the second order and $tr\varepsilon = \varepsilon_{ii}$ represents the trace of ε. In the main frame of reference, the spherical tensor and the deviator are given by the following expression.

$$\varepsilon = \varepsilon^S + \varepsilon^D. \tag{3.4.1}$$

where:

- $\varepsilon^S = \frac{1}{d}tr\varepsilon I$, is called the spherical part and represents the transformation responsible for the volume expansion.

- $\varepsilon^D = \varepsilon - \frac{1}{d}tr\varepsilon I$, is called the deviatoric part, which by construction has a zero trace. It represents constant-volume deformation, i.e., pure shear deformation.

Stress Tensor - Stress Vector

The use of Cauchy's postulate and theorem to represent the forces acting on an elastic body has led to the definition, at each point x of a body Ω and, at each instant t, of a symmetric tensor $\sigma(x,t)$ called the stress tensor . This tensor associates with each facet M of the normal ν enclosing the point x the surface density vector of the forces $\sigma(x,t).\nu$, called the Cauchy stress vector.

3.5 Example of a nonlinear elasticity operator

In this section, we present an example of a nonlinear elasticity operator that satisfies equations (2.1.8). It concerns a class of deformable materials known as Hencky materials (refer to [11] and [28]), where the constitutive law can be written as

$$\sigma(u) = K_0 tr\varepsilon(u)I + \psi(\left\|\varepsilon(u)^D\right\|^2)\varepsilon(u)^D. \tag{3.5.1}$$

where $K_0 > 0$ is a material coefficient and $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}$.

In this example, the elasticity operator is given by:

$$\mathcal{F}(\varepsilon) = K_0 tr\varepsilon I + \psi(\left\|\varepsilon^D\right\|^2)\varepsilon^D, \tag{3.5.2}$$

One approach that has already been explored in some research involving a family of nonlinear Hencky materials is when the function is defined as continuously piecewise differentiable. This function is defined by:

$$\psi(\xi) = \begin{cases} 1 & \text{if } 0 \le \xi \le \xi_0 \\ \beta + (1-\beta)\xi_0^{1-\gamma}\xi^{\gamma-1} & \text{if } \xi > \xi_0 \end{cases} \tag{3.5.3}$$

with parameters $\beta \in (0,1)$ and $0 \neq \gamma \in (\frac{1-2\beta}{2-2\beta}, 1)$.
More generally (see [11]), if the function ψ is assumed to be piecewise continuously differentiable and there exist positive constants c_1, c_2, d_1, and d_2 such that

$$\psi(\xi) \le d_1, \quad -c_1 \le \psi^{'}(\xi) \le 0, \;\; c_2 \le \psi(\xi) + 2\psi^{'}(\xi)\xi \le d_2 \text{ for all } \xi > 0, \tag{3.5.4}$$

then the elasticity operator $\mathcal{F}$, as defined by (3.5.2), satisfies the conditions (2.1.8).

Here, we present a demonstration of this result:

Proof. Firstly, note that conditions (2.1.8) (d), (e) clearly satisfied.
We will show that condition (2.1.8) (b), is fulfilled.
For all $\varepsilon_1, \varepsilon_2 \in S_d$, we have

$$\mathcal{F}(\varepsilon_1) - \mathcal{F}(\varepsilon_2) = K_0 tr\left(\varepsilon_1 - \varepsilon_2\right) I + \psi(\left\|\varepsilon_1^D\right\|^2)\varepsilon_1^D - 2\psi(\left\|\varepsilon_2^D\right\|^2)\varepsilon_2^D$$

Thereafter,

$$\begin{aligned} \psi(\left\|\varepsilon_1^D\right\|^2)\varepsilon_1^D - \psi(\left\|\varepsilon_2^D\right\|^2)\varepsilon_2^D &= \int_0^1 \frac{d}{dt}\left[\psi(\left\|\varepsilon_2^D + t\left(\varepsilon_1^D - \varepsilon_2^D\right)\right\|^2\left(\varepsilon_2^D + t\left(\varepsilon_1^D - \varepsilon_2^D\right)\right)\right] dt \\ &= \int_0^1 [2\psi'(\left\|\varepsilon_2^D + t\left(\varepsilon_1^D - \varepsilon_2^D\right)\right\|^2)\left(\varepsilon_2^D + t\left(\varepsilon_1^D - \varepsilon_2^D\right)\right) \\ &\quad \times\left(\varepsilon_1 - \varepsilon_2\right)\left(\varepsilon_2^D + t\left(\varepsilon_1^D - \varepsilon_2^D\right)\right) \\ &\quad + \psi(\left\|\varepsilon_2^D + t\left(\varepsilon_1^D - \varepsilon_2^D\right)\right\|^2) \times \left(\varepsilon_1 - \varepsilon_2\right)]dt \end{aligned}$$

So, by setting $\varepsilon_2^D + t\left(\varepsilon_1^D - \varepsilon_2^D\right) = \xi$, we obtain

$$\mathcal{F}(\varepsilon_1) - \mathcal{F}(\varepsilon_2) = K_0 tr\left(\varepsilon_1 - \varepsilon_2\right) I + \left(\varepsilon_1 - \varepsilon_2\right)\int_0^1 [2\psi'(\xi^2)\xi^2 + \psi(\xi^2)]dt$$

Then, using equation (3.5.4), it is shown that

$$\left\|\mathcal{F}(x, \varepsilon_1) - \mathcal{F}(x, \varepsilon_2)\right\| \le M\left\|\varepsilon_1 - \varepsilon_2\right\|, \forall \varepsilon_1, \varepsilon_2 \in S_d,$$

where M is a positive constant that depends on K_0, d_1, d_2 and c_1. This proves (2.1.8) (b).
Next, for all $\varepsilon_1, \varepsilon_2 \in S_d$ we have,

$$\begin{aligned}
\left(\mathcal{F}\left(\varepsilon_1\right)-\mathcal{F}\left(\varepsilon_2\right)\right)\left(\varepsilon_1-\varepsilon_2\right) \quad = \quad & K_0\left\|tr\left(\varepsilon_1-\varepsilon_2\right)\right\|^2 \\
& +\int_0^1 [2\psi'(\left\|\varepsilon_2^D+t\left(\varepsilon_1^D-\varepsilon_2^D\right)\right\|^2)\left\|\left(\varepsilon_1-\varepsilon_2\right)\left(\varepsilon_2^D+t\left(\varepsilon_1^D-\varepsilon_2^D\right)\right)\right\|^2 \\
& +\psi(\left\|\varepsilon_2^D+t\left(\varepsilon_1^D-\varepsilon_2^D\right)\right\|^2)\left\|\varepsilon_1-\varepsilon_2\right\|^2]dt
\end{aligned}$$

So, by writing $\varepsilon_2^D+t\left(\varepsilon_1^D-\varepsilon_2^D\right)=\xi$, we obtain

$$\left(\mathcal{F}\left(\varepsilon_1\right)-\mathcal{F}\left(\varepsilon_2\right)\right)\left(\varepsilon_1-\varepsilon_2\right) \geq K_0\left\|tr\left(\varepsilon_1-\varepsilon_2\right)\right\|^2+\int_0^1 [2\psi'(\xi^2)\xi^2+\psi(\xi^2)]\left\|\varepsilon_1-\varepsilon_2\right\|^2 dt,$$

and using (3.5.4), we get

$$\left(\mathcal{F}\left(\varepsilon_1\right)-\mathcal{F}\left(\varepsilon_2\right)\right)\left(\varepsilon_1-\varepsilon_2\right) \geq K_0^2\left\|tr\left(\varepsilon_1-\varepsilon_2\right)\right\|^2+c_2^2\left\|\varepsilon_1-\varepsilon_2\right\|^2 dt$$

Thus,

$$\left(\mathcal{F}\left(\varepsilon_1\right)-\mathcal{F}\left(\varepsilon_2\right)\right)\left(\varepsilon_1-\varepsilon_2\right) \geq m\left\|\varepsilon_1-\varepsilon_2\right\|^2$$

with m depending on K_0 and c_2, which fulfills the condition (2.1.8) (c). ■