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Combinatoire des transversales de la pyramide de Pascal

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N° 14/2021-C/MT

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

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T H E S I S

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By

Abdelghani MEHDAOUI

Combinatorics of finite rays crossing Pascal pyramid

Saturday, February 20, 2021

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بسم الله الرحمن الرحيم

In the name of Allah, the Most Gracious, the Most Merciful

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إلى روح قلبي وقرة عيني

إلى أمي الحبيبة...

ملخص:

توافقيات الأشعة المنتهية التي تعبر هرم باسكال

هذه الأطروحة تندرج ضمن مجال التوافقيات التعدادية.وهي تتعلق بدراسة السلاسل العددية التي يتم الحصول عليها من خلال جمع الأعداد التي تكون في اتجاه شعاع معين في مثلث المعاملات ذات الحدين المربعة، وكذلك في هرم باسكال. حيث نقوم بايجاد العلاقة التراجعية للسلاسل العددية التي يتم التعبير عنها بواسطة مجموع الأقطار الرئيسية لمثلث المعاملات ذات الحدين المربعة. كذلك نقوم بتقديم الدالة المولدة لهاته السلاسل، كما نعطي العديد من المعادلات المتطابقة التوافقية. بعد ذلك نقوم بدراسة مجموع الأعداد التي تكون في استقامية بالنسبة إلى شعاع معين في هرم باسكال، حيث نقوم بتعريف هاته الأشعة داخل هرم باسكال، كما نعطي العلاقة التراجعية للسلاسل المتحصل عليها من تلك المجاميع مع توفير الدالة المولدة الموافقة لها. ثم قمنا بدراسة ظاهرة تمسى مورقن-فويس والتي تسمح لنا باستنتاج العديد من العلاقات المتطابقة التوافقية، كما نثبت العديد من العلاقات التراجعية التى تم تخمينها في "الموسوعةالمباشرة لمتتاليات الأعداد الصحيحة". نقترح أيضا تفسيرا توافقيا للسلاسل المتحصل عليها. وفي الأخير ندرس السلاسل التي تكون نتيجة جمع الاعداد التي تقع في تقاطع مستو مع هرم باسكال (مقاطع) بعد تقديم تعريف لها بطريقة يسمح بالمحافظة على تناظر المعاملات ذات الثلاثة حدود، حيث نبرهن العلاقة التراجعية التي تعبر عن تلك السلاسل، ثم نختتم بايجاد العديد من العلاقات المتطابقة التوافقية وكذلك تفسيرا توافقيا مناسبا.

الكلمات الدالة: مثلث باسكال، هرم باسكال، سلسلة فيبوناشي، مقاطع، العلاقة التراجعية، المعاملات ذات الحدين، المعاملات ذات ثلاثة حدود، المعاملات ذات الحدين المربعة، الدالة المولدة.

ABSTRACT

Combinatorics of finite rays crossing Pascal pyramid.

This thesis is related to enumerative combinatorics. It concerns the study of sequences obtained by the summation of the elements of directions in square binomial triangle, as well as Pascal pyramid. We establish the linear recurrence relation associated with the sequences expressed as the sum of elements of the principal diagonal of the square binomial coefficients. We give the corresponding generating function, then we deduce several combinatorial identities. Then, we study the sums of elements of the rays in Pascal pyramid, where we define the directions in this pyramid, we give the linear recurrence relation for the sum of elements lying over the main diagonals, as well as its generating function. In addition, we study the Morgan-Voyce phenomemon, which allowed us to establish several combinatorial identities, prove and find several sequences in OEIS. We also propose a combinatorial interpretation of theses sequences. Finally, we define the plane sections in Pascal pyramid in a way that preserves the symmetry of the trinomial coefficients, we establish the associate linear recurrence relation, several combinatorial identities are given. We conclude with combinatorial interpretation.

Key words: Pascal triangle, Pascal pyramid, Fibonacci sequence, plane sections, recurrence relations, binomial coefficients, trinomial coefficients, square binomial coefficients, generating function.

RÉSUMÉ

Combinatoire des transversales de la pyramide de Pascal.

Cette thèse s'inscrit dans le domaine de la combinatoire énumérative. Elle porte sur l'étude des suites obtenues par la sommation des éléments des transversales dans le triangle des coefficients binomiaux au carré, ainsi que dans la pyramide de Pascal. Nous établissons la relation de récurrence linéaire associée aux suites qui s'expriment comme somme des éléments des diagonales principales du triangle des coefficients binomiaux au carré. Nous donnons la fonction génératrice correspondante, puis nous déduisons plusieurs identités combinatoires. Ensuite, nous étudions les sommes des éléments des transversales dans la pyramide de Pascal, où nous définissons les directions dans cette pyramide, nous donnons la relation de récurrence linéaire pour les sommes des éléments des diagonales principales et nous établissons la fonction génératrice adéquate. En outre, nous étudions le phénomène de Morgan-Voyce, cela nous a permis d"établir plusieurs identités combinatoires, ainsi nous prouvons et nous retrouvons plusieurs suites dans OEIS. Nous proposons aussi une interprétation combinatoire de ces suites. Enfin, nous définissons les sections planes dans la pyramide de Pascal d'une façon qui préserve la symétrie des coefficients trinomiaux, nous établissons la relation de récurrence linéaire associée, ainsi que plusieurs identités combinatoires. Nous concluons par des interprétations combinatoires.

Mots clés: Triangle de Pascal, pyramide de Pascal, suite de Fibonacci, section plane, récurrence linéair, coefficients binomial, coefficients trinomial, , coefficients binomial au carré, fonction génératrice.

LIST OF CONTRIBUTIONS

- Hacène Belbachir, **Abdelghani Mehdaoui**, and László Szalay. Diagonal sums in Pascal pyramid. *Journal of Combinatorial Theory. Series A*, 165:106–116, 2019.

- Hacène Belbachir, **Abdelghani Mehdaoui**, and László Szalay. Diagonal sums in the Pascal pyramid, II: Applications. *Journal of Integer Sequences*, 22(3):1–11, 2019.

- Hacène Belbachir and **Abdelghani Mehdaoui**. Diagonal sums in Pascal pyramid (1,2,r). *Les Annales RECITS*, 6:45–52, 2019.

- Hacène Belbachir and **Abdelghani Mehdaoui**. Plane section recurrences in the pascal pyramid. *Rocky Mountain Journal of Mathematics*, 50(5):1561–1566, 2020.

- Hacène Belbachir and **Abdelghani Mehdaoui**. Recurrence relation associated to the sums of square binomial coefficients. *Accepted in Quaestiones Mathematicae*.

- Hacène Belbachir and **Abdelghani Mehdaoui**. Summable identity between weighted square binomial and weighted trinomial coefficient. *Submitted*.

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Introduction

This thesis entitled *"Combinatorics of finite rays crossing Pascal pyramid"* is related to enumerative combinatorics, which is a branch of combinatorics, and deals with the ways of counting mathematical objects.

One of the most well-known arithmetic triangle is the Binomial coefficients triangle (BCT for short). The earliest mention of BCT, were known to ***Pingala*** (2nd century B.C), but only fragments of his work survived. In 505 ***Varãhamihira*** gave a description of the additive rule (See [39, 94]). Among also the earliest dealing with the binomial coefficients was in the works of ***"Al-Khalil ibn Aḣmad Al-Farahidi"*** (718-786) in his book *Kitab Al-'Ayn*, Al-Khalil was a linguist and tried to enumerate in an exhaustive way the words of the language by calculating the numbers of combination of letters of the alphabet (See [68], p.44). At the end of the tenth century, ***"Al-kharji"*** (953-1029) expressed and proved some of the relations of the triangle.

He used the recurrence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

and the explicit formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

It was later repeated by ***"Omar Al-khayam"*** (1048-1131) (See [53]), also by ***"Al-Samw'al"*** who came after ***"Al-kharji"*** by 200 hundred years (See Fig.1)[1]. After that, ***Al-Ṫũšĩ*** (1201-1273) established many fundamental relations of BCT. (See [67, 68, 69, 70, 71]).

Figure 1: BCT wrote by *Al-Samw'al*.

[1] The picture from AlSamwa'l's book *"Al-Bahir fi Al-Hissab"*, AyaSofya manuscript N° 2718.

Another discovery of the BCT was made by the Chinese in the 11^{th} century and preserved through the work of the Chinese mathematician ***Yang Hui*** (1238-1298), in Fig.2 we see the Chinese version of the BCT. (See [92], p.2169).

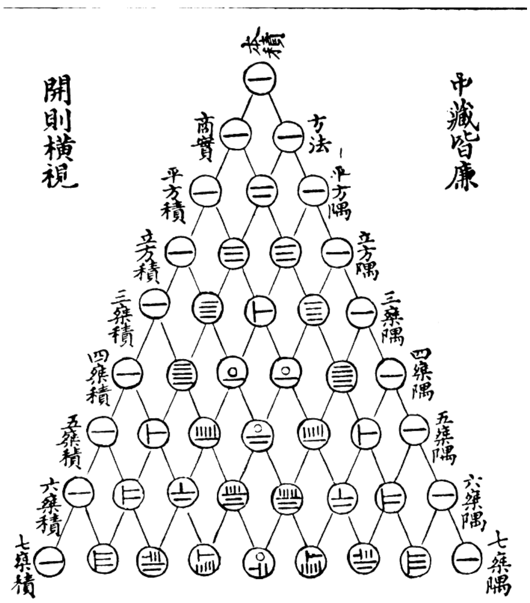


Figure 2: Yang Hui's version.

In the west, the BCT appeared in the early 13^{th} century in Arithmetic of ***Jordanus de Nemore***. ***Pertrus Apianus*** (1495-1552) published the full triangle (See [49]). In Italy, ***Tartaglia*** (1500-1577) wrote the triangle in his book *"General Trattato di Numeri et Misure"* published in 1556 (See [39]). Until the 17^{th} century ***Blaise Pascal*** published the BCT in his book *" Traité du triangle arithmétique "*, in Fig.3 we see the Pascal's version of the BCT, which is known by his name "Pascal triangle"[2].

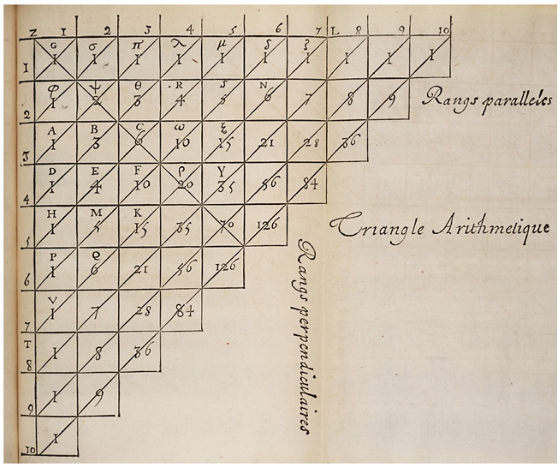


Figure 3: Pascal's version.

From then, many properties of Pascal triangle were studied by several authors and hundreds of papers were published dealing with this mysterious triangle. For instance, the unimodality and log-concavity properties were examined in the works of Tanny and Zucker [87], Benoumhani [26], Belbachir and Bencherif [9], and Belbachir et *al* [13, 14], we also find works dealt with binomial coefficients involving prime numbers (one can see [12, 47, 54, 81]). Other challenging question is the sum of elements lying along finite rays, for example, the sum of elements

[2]The pictures of Figures 1, 2 and 3 are in public domain ©.

of the principal diagonal gives one of the most well-known sequence in mathematics, the Fibonacci[3] sequence. It satisfies the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

with the initial values $F_0 = 0$ and $F_1 = 1$, and it can be expressed using binomial coefficients by

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}. \tag{1}$$

Raab [66] discussed the question of the sum of elements of other principal diagonals and gave the corresponding recurrence relation

$$U_{n+1} = \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \binom{n-qk}{k} x^{n-(q+1)k} y^k. \tag{2}$$

He showed that U_{n+1} satisfies

$$U_n = xU_{n-1} + yU_{n-q-1}.$$

Moreover, in 2014 Belbachir et *al.* [14] defined the concept of directions over Pascal triangle and generalized it for any given direction

$$T_{n+1} = \sum_{k=0}^{\lfloor (n-p)/(q+r) \rfloor} \binom{n-qk}{p+rk} x^{n-p-(q+r)k} y^{p+rk}, \tag{3}$$

and they showed that T_{n+1} satisfy a linear recurrence given by

$$T_n - x\binom{r}{1}T_{n-1} + x^2\binom{r}{2}T_{n-2} + \cdots + (-1)^r x^r \binom{r}{r} T_{n-r} = y^r\, T_{n-q-r}.$$

On the other hand, many generalization of the Pascal triangle can be found in literature. Probably, the most natural ones are the Pascal pyramid and hyperpyramids, which were constructed from trinomial and multinomial coefficients, respectively. Pascal pyramid introduced by Rosenthal in [73]. It was studied by several authors one can see [18, 32, 61, 82, 96]. The natural question arises what

[3] Leanardo Pisano also known as Fibonacci was born in Italy, studied and lived in Algeria for a period of time, exactly in Bejaïa (my hometown), (See [75]).

are (1), (2) and (3) equivalents in Pascal pyramid and hyperpyramids?.

This thesis is structured in five chapters as follows:

In chapter one

We give some notations and definitions regarding the binomial, multinomial and bi^{s}nomial coefficients. We present some of their properties such as recurrence relation and explicit formula. We give also a definition of directions in an arithmetic triangle, we cite some works applied the definition on the Pascal triangle. After that, we give some well-known sequences, more precisely, Fibonacci, *r*-Fibonacci, Pell and Jacobsthal sequences, which we will be derived later as particular cases of our results.

In chapter two

We establish a linear recurrence relation for the sequences obtained from the sums of the parallel diagonals of the square binomial triangle, we give the corresponding generating function, we present also the corresponding Morgan-Voyce phenomenon of the main recurrence, many combinatorial identities are given. As corollaries, we deduce the well-known central binomial coefficients, and the recurrence of the main diagonal.

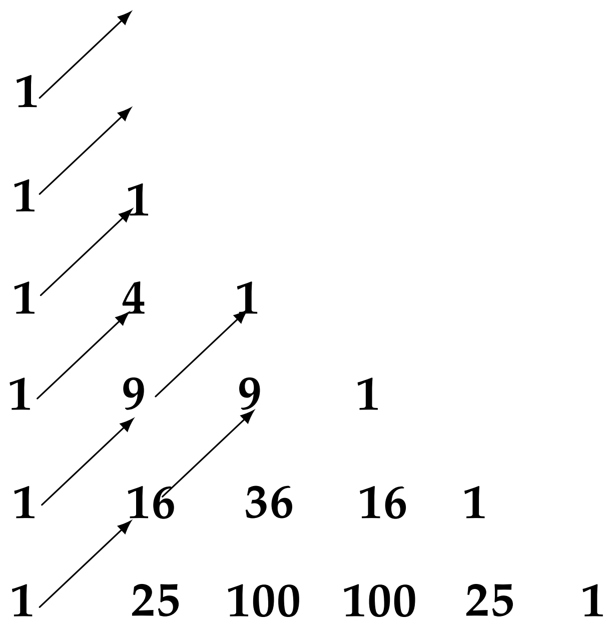


Figure 4: A direction in square binomial triangle.

In chapter three

We define the concept of directions in Pascal pyramid, we illustrate some well-known sequences, we establish also a linear recurrence relation for the sequences obtained from the sums of elements of the main diagonal, the corresponding generating function also is given.

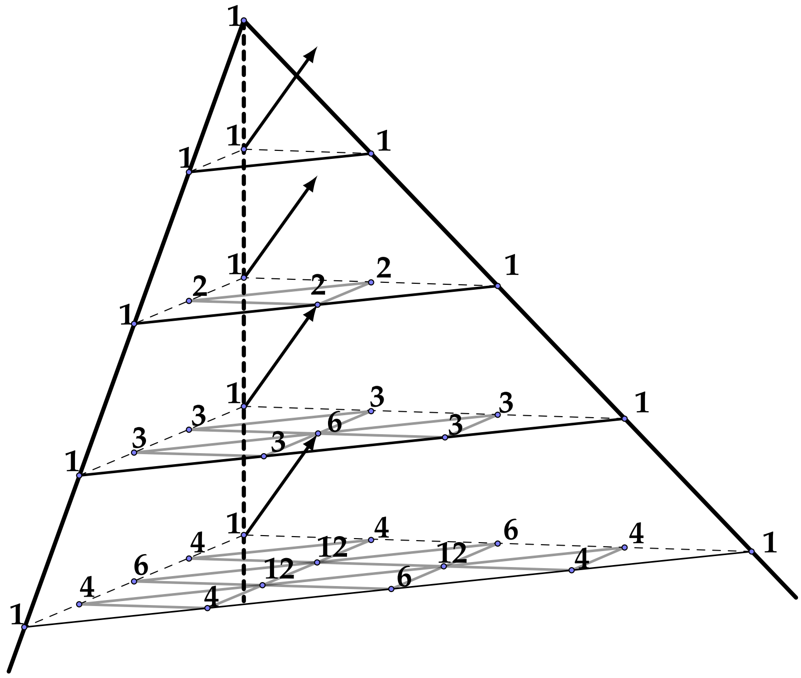


Figure 5: A direction in Pascal pyramid.

In chapter four

As continuation of chapter three, we dealt with the Morgan-Voyce phenomenon, which gives several combinatorial identities. We discovered and proved several recurrences and formulas in the Online Encyclopedia of Integer Sequences OEIS, finally we give the combinatorial interpretation for all the sequences satisfying the main recurrence.

In chapter five

We define the plane sections in Pascal pyramid in a way that preserves the symmetry of the trinomial coefficients, we establish the linear recurrence for the main plane sections, we provide some identities with r-Fibonacci recurrence, also an identity between plane sections and some direction in Pascal triangle is provided. We give also the combinatorial interpretation for all the sequences generated from the main recurrence.

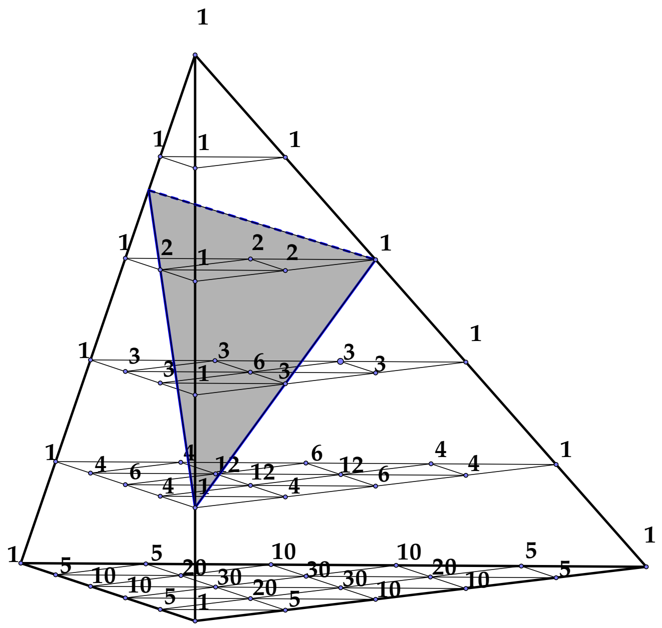


Figure 6: A plane section in Pascal pyramid.

CHAPTER

1

PRELIMINARIES

In this chapter, we give some notations and definitions of binomial, multinomial and bi^{s}nomial coefficients, with some of their properties such as recurrence relation and explicit formula. We give also a definition of directions in an arithmetic triangle, we cite some works applied the definition on the Pascal triangle. After that, we give some well-known sequences.

1.1 Binomial coefficients

The binomial coefficients defined as the number of distinct combinations of k elements out of n. They can also be derived from the expansion of the expression

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad 0 \leq k \leq n,$$

with the convention $\binom{n}{k} = 0$ for $k < 0$ or $k > n$.

1.1.1 Identities involving binomial coefficients

- $\binom{n}{k} = \binom{n}{n-k}$,
- $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$.
- $\sum_{k=0}^{n}\binom{n}{k} = 2^n$,
- $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.
- $\sum_{k=1}^{n}k\binom{n}{k} = n2^{n-1}$,
- $\sum_{m=k}^{n}\binom{m}{k} = \binom{n+1}{k+1}$.

The binomial coefficients can be arrange in a triangle as follows

1						
1	**1**					
1	**2**	**1**				
1	**3**	**3**	**1**			
1	**4**	**6**	**4**	**1**		
1	**5**	**10**	**10**	**5**	**1**	
1	**6**	**15**	**20**	**15**	**6**	**1**

Figure 1.1: The first six rows of binomial coefficients triangle.

It is well-known by the name of Pascal triangle, and can be constructed by using the recurrence relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

with $\binom{n}{0} = \binom{n}{n} = 1$.

1.2 Multinomial coefficients

The multinomial coefficients occur in the expansion of the polynomial $(x_1 + x_2 + \cdots + x_s)^n$, their notation is $\binom{n}{k_1,\, k_2,\, \ldots,\, k_s}$.

$$(x_1 + x_2 + \cdots + x_s)^n = \sum_{k_1+k_2+\cdots+k_s=n} \binom{n}{k_1,\ k_2,\ \ldots,\ k_s} x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s}, \tag{1.1}$$

where

$$\binom{n}{k_1,\ k_2,\ \ldots,\ k_s} = \frac{n!}{k_1!k_2!\cdots k_s!},$$

and $k_1 + k_2 + \cdots + k_s = n$, where $\binom{n}{k_1,\, k_2,\, \ldots,\, k_s} = 0$ for $k_1 + k_2 + \cdots + k_s \neq n$ or $k_i < 0, \quad i = 1, 2, \ldots, s$. One of the combinatorial interpretations of these coefficients is distribution of n distinguishable elements into s distinguishable cells, where the number of elements in the i^{th} cell is k_i, $i = 1, 2, \ldots, s$.
Also the multinomial coefficients can be expressed as binomial coefficients

$$\binom{n}{k_1,\ k_2,\ \ldots,\ k_s} = \binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-\cdots-k_{s-1}}{k_s},$$

and they satisfy the following recurrence relation

$$\binom{n}{k_1,\ k_2,\ \ldots,\ k_s} = \binom{n-1}{k_1-1,\ k_2,\ \ldots,\ k_s} + \binom{n-1}{k_1,\ k_2-1,\ \ldots,\ k_s} + \cdots + \binom{n-1}{k_1,\ k_2,\ \ldots,\ k_s-1}.$$

Setting $x_i = t^i$ $(0 \leq i \leq s)$ in (1.1), we obtain

$$(1 + t^2 + \cdots + t^s)^n = \sum_{k=0}^{sn} \binom{n}{k}_s t^k,$$

where $\binom{n}{k}_s$ known as bi^{s}nomial coefficients [10, 11] or ordinary multinomial coefficients [5, 28].

The bi^{s}nomial coefficients satisfy the recurrence relation, for $n \geq 1$

$$\binom{n}{k}_s = \binom{n-1}{k}_s + \binom{n-1}{k-1}_s + \cdots + \binom{n-1}{k-s}_s,$$

with $\binom{0}{0} = 1$ and $\binom{n}{k}_s = 0$ for $k < 0$ or $k > sn$. One can find several identities involving bi^{s}nomial coefficients (See [10, 11, 32]):

$$\binom{n}{k}_s = \sum_{j_1+j_2+\cdots+j_s=k} \binom{n}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-1}}{j_s}.$$

The symmetry relation

$$\binom{n}{k}_s = \binom{n}{sn-k}_s.$$

Diagonal recurrence relation

$$\binom{n}{k}_s = \sum_{j=0}^{n} \binom{n}{j}\binom{j}{k-j}_{s-1}.$$

1.3 Directions in arithmetic triangles

In this subsection, we define the directions in any arithmetic triangle, then we give directions in Pascal triangle as an example, which this thesis dealt with its generalization.

Definition 1. *Let $A(n,k)$ $(0 \le k \le n)$ be an arithmetic triangle, for $n \in \mathbb{N} \cup \{0\}$, $A(n-rk, \theta+\alpha k)$ define the elements lying over any given direction (θ, α, r), with $r \in \mathbb{Z}$, $\alpha \in \mathbb{N}$, $0 \le \theta < \alpha$ and $\alpha + r > 0$.*

Remark 2. The last condition in Definition 1 guarantees that the direction is finite, otherwise it will be infinite.

1.3.1 Directions (θ, r, α) in Pascal triangle

Using the Definition 1, the directions in the generalized Pascal triangle (See [13]) can be written as

$$\binom{n-rk}{\theta+\alpha k} x^{n-\theta-(r+\alpha)k} y^{\theta+\alpha k}. \tag{1.2}$$

Belbachir *et al.* [13, 14] defined and established the linear recurrence relation for all sequences for any given direction in Pascal triangle.

$$T_n^{(\theta,\,\alpha,\,r)} = \sum_{k=0}^{\lfloor (n-\theta)/(r+\alpha) \rfloor} \binom{n-rk}{\theta+\alpha k} x^{n-\theta-(r+\alpha)k} y^{\theta+\alpha k}, \tag{1.3}$$

they showed that (1.3) satisfy

$$T_n - x\binom{\alpha}{1}T_{n-1} + x^2\binom{\alpha}{2}T_{n-2} + \cdots + (-1)^{\alpha}x^{\alpha}\binom{\alpha}{\alpha}T_{n-\alpha} = y^{\alpha}\; T_{n-r-\alpha}.$$

The most well-known example is the principal direction $(\theta, \alpha, r) = (0, 1, 1)$ which gives the Fibonacci sequence (See Subsection 1.4.1).

Also (1.3) generalizes the work of Raab [66], who established the recurrence for all parallel diagonals in Pascal triangle (See Subsection 1.4.2)

$$T_n = \sum_{k=0}^{\lfloor n/(r+1) \rfloor} \binom{n-rk}{k} x^{(n-(r+1)k)} y^k. \tag{1.4}$$

He showed that (1.4) satisfy

$$T_n = xT_{n-1} + yT_{n-r-1}. \tag{1.5}$$

Figures 1.2 amd 1.3 are illustrations of different setting of the parameters (θ, α, r).

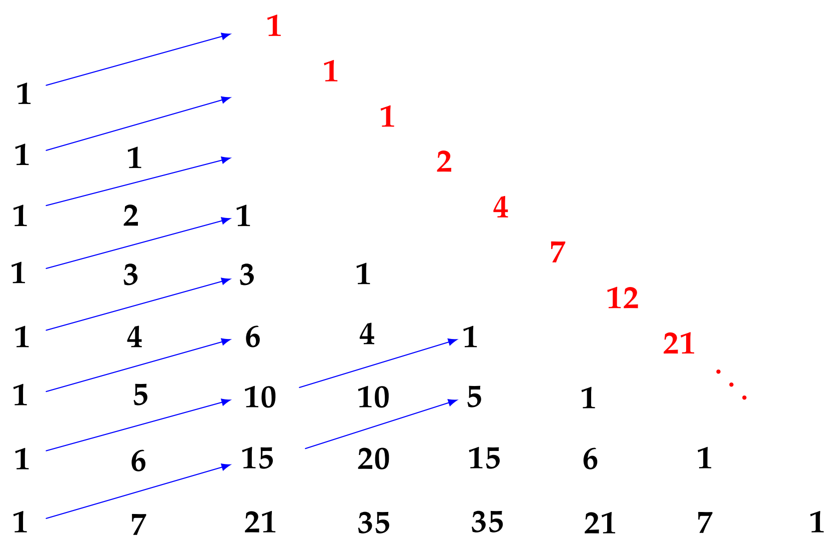


Figure 1.2: The sum of elements of the direction $(\theta,\ \alpha,\ r) = (0, 2, 1)$.

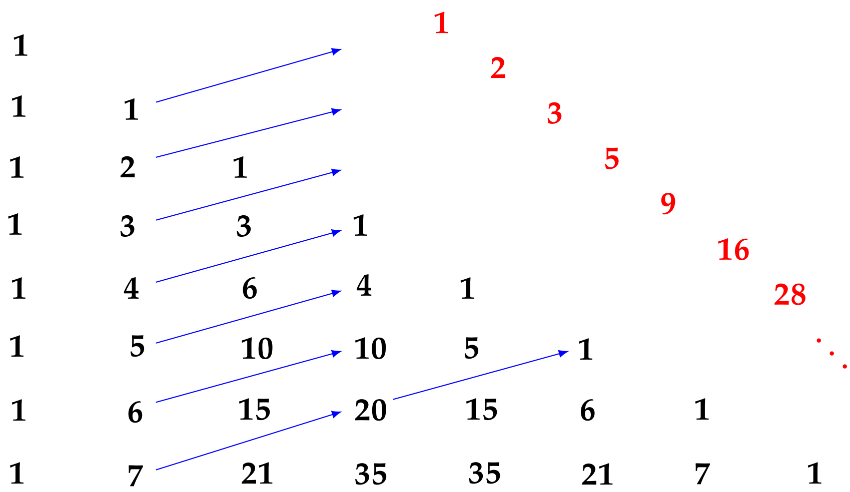


Figure 1.3: The sum of elements of the direction $(\theta,\ \alpha,\ r) = (1, 2, 1)$.

1.4 Classical sequences

In this section, we give some well-known sequences and some of their properties, these sequences can be expressed in different ways. We will see this in the next chapters.

1.4.1 Fibonacci Sequence

It is well-known that the Fibonacci sequence satisfy the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad n \geq 2, \tag{1.6}$$

with $F_0 = 0$ and $F_1 = 1$. The first values are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. It can also be expressed as a sum of binomial coefficients

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}.$$

An explicit formula for F_n is given by

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right], \tag{1.7}$$

where $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ are the roots of the characteristic polynomial of the relation (1.6). The illustration of F_n in Pascal triangle is

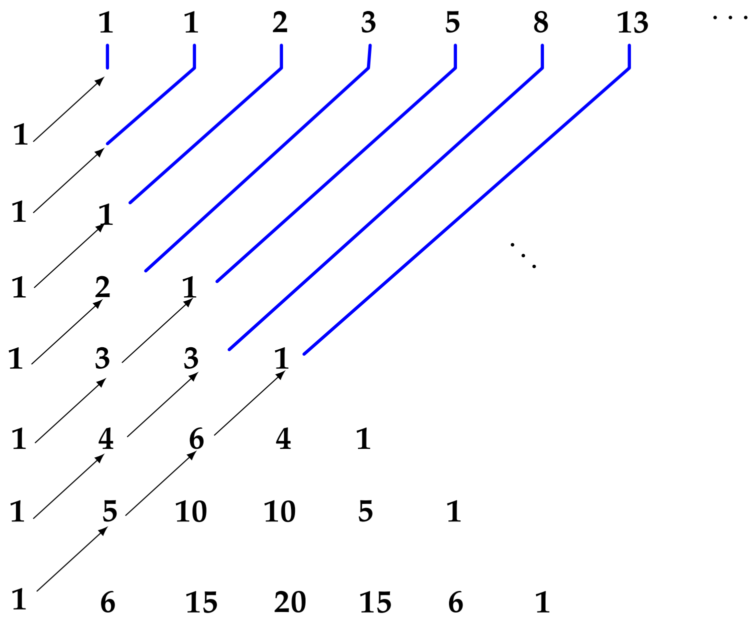


Figure 1.4: Fibonacci sequence in Pascal triangle.

1.4.2 Generalized Fibonacci and r-Fibonacci sequences

The generalized Fibonacci sequence defined by

$$F_n = xF_{n-1} + yF_{n-2} \quad n \geq 2,$$

with $F_0 = 0$, $F_1 = 1$, it is expressed as

$$F_{n+1}(x,y) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k} y^k.$$

For $(x,y) = (1,1)$ we get the Fibonacci sequence.
The r-Fibonacci is another generalization of F_n defined by

$$F_{n+1}^r = \sum_{k=0}^{\lfloor n/r+1 \rfloor} \binom{n-rk}{k} x^{n-(r+1)k} y^k.$$

It satisfies

$$F_n^r = xF_{n-1}^r + yF_{n-r-1}^r,$$

which describes the sequences lying over all parallel diagonals in Pascal triangle (See (1.4)).
Setting $(r, x, y) = (1, 1, 1)$, we obtain the Fibonacci sequence.

1.4.3 Pell and Jacobsthal sequences

The Pell sequence is defined by

$$P_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^{n-2k},$$

and it satisfies

$$P_n = 2P_{n-1} + P_{n-2} \quad n \geq 2,$$

with $P_0 = 0$ and $P_1 = 1$, the first values are $0, 1, 2, 5, 12, 29, 70, \ldots$.
The Jacobsthal sequence define by

$$J_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} 2^k,$$

and it satisfies

$$J_n = J_{n-1} + 2J_{n-2} \quad n \geq 2,$$

with $P_0 = 0$ and $P_1 = 1$, the first values $0, 1, 1, 3, 5, 11, 21, \ldots$, (For more details one can see [78])

CHAPTER

2

DIRECTIONS IN SQUARE BINOMIAL TRIANGLE

In this chapter, we present some results concerning square binomial coefficients (Shortly SBT). Several authors studied the question of the sums of the powers of binomial coefficients and their recurrence relations. Franel [43] gave the recurrence for the third and fourth powers, Perlstadt [62] established the recurrences for powers 5 and 6 using an algorithm in MACSYMA called *creative telescoping* known as *Zeilberger algorithm* [63, 95]. We also find several works on sum of negative powers of binomial coefficients, see for instance [16, 17, 86]. Our aim is to study the sums of the elements lying over SBT. We establish the linear recurrence relation for all sequences of the sums of elements of parallel diagonals, we give the corresponding generating function, we also present the corresponding Morgan-Voyce phenomenon of the main recurrence. As consequences, we derive many combinatorial identities (See [19]).

2.1 Square binomial triangle and sum of its elements

In this section, we recall the definition of SBT, and give some well-known results about the sums of its elements, then we present a formula that characterizes the sequences obtained from the sums of elements lying over parallel diagonals in SBT.

The square binomial triangle (see Fig. 2.1) consist of the second power of the Pascal triangle entries. That is

$$S(n,k) = \binom{n}{k}^2 x^{n-k}y^k, \quad n \in \mathbb{N}\cup\{0\} \quad \text{and} \quad 0 \leq k \leq n. \tag{2.1}$$

1						
1	**1**					
1	**4**	**1**				
1	**9**	**9**	**1**			
1	**16**	**36**	**16**	**1**		
1	**25**	**100**	**100**	**25**	**1**	
1	**36**	**225**	**400**	**225**	**36**	**1**

Figure 2.1: The first six rows of square binomial triangle.

It is well-known that the sum of rows of SBT gives the central binomial coefficients, more precisely

$$C_n := \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}, \tag{2.2}$$

and they satisfy the recurrence relation $nC_n = 2(2n-1)C_{n-1}$.
Verrill [89] showed that the terms

$$a_n \equiv a_n(x,y) = \sum_{k=0}^{n} \binom{n}{k}^2 x^{n-k}\, y^k \tag{2.3}$$

of $(a_n(x,y))_n$ satisfy, for x, y reals, the recurrence relation

$$na_n = (x+y)(2n-1)a_{n-1} - (x-y)^2(n-1)a_{n-2}. \tag{2.4}$$

Now, let $n \in \mathbb{N}\cup\{0\}$, moreover let $r \in \mathbb{N}$, $q \in \mathbb{Z}$, with $q + r > 0$ and $0 \leq p < r$. Then the direction (q, p, r) defines the elements lying over parallel diagonals of the SBT, which is given by

$$\binom{n-qk}{p+rk}^2 x^{n-p-(q+r)k}\, y^{p+rk}, \quad k = 0, \cdots, \lfloor (n-p)/(q+r) \rfloor, \tag{2.5}$$

where x and y are nonzero real parameters.
The sum of these elements gives the general term of the sequence

$$S_n = \sum_{k=0}^{\lfloor (n-p)/(q+r) \rfloor} \binom{n-qk}{p+rk}^2 x^{n-p-(q+r)k}\, y^{p+rk}. \tag{2.6}$$

Example 1. *Let* $(q,p,r,x,y) = (1,0,1,1,1)$*, then we get the sum*

$$b_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}^2 \tag{2.7}$$

of elements of the principal diagonal of SBT appears in OEIS [78] as A51286, the first few values are given in Fig. 2.2. It satisfies

$$\begin{cases} nb_n = (2n-1)\, b_{n-1} - (n-1)\, b_{n-2} + (2n-3)\, b_{n-3} - (n-2)\, b_{n-4}, \\ b_0 = b_1 = 1,\ b_2 = 2,\ b_3 = 5. \end{cases}$$

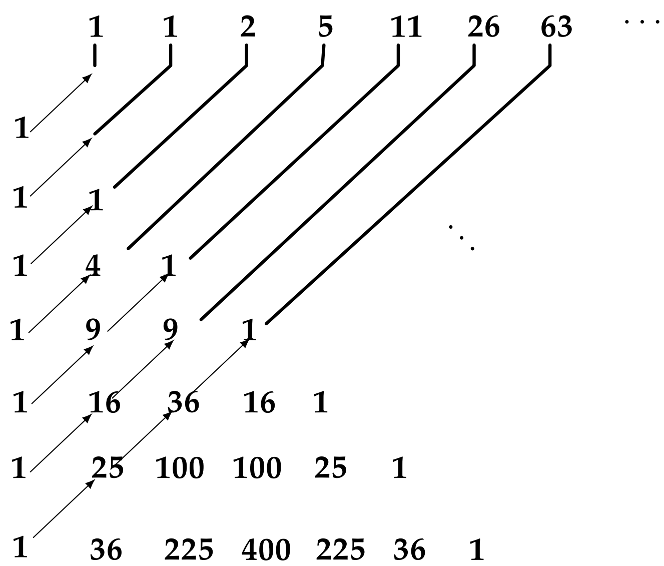


Figure 2.2: The sum of principal diagonal elements of SBT.

In the sequel of this chapter, we deal with the main parallel diagonals of SBT.

2.2 Characterization of linear recurrence relation associated to SBT

In this section, we establish the recurrence relation satisfied by the sums of elements of directions $(q, 0, 1)$, we also provide the corresponding generating function.

2.2.1 Linear recurrence relation

We give here the linear recurrence relation associated to the sum of elements lying over parallel diagonals of SBT.

Theorem 3. *The term of the sequence T_n, defined by*

$$S_n = \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \binom{n-qk}{k}^2 x^{n-(q+1)k}\, y^k \tag{2.8}$$

satisfy the linear recurrence

$$\begin{aligned} nS_n = x(2n-1)S_{n-1} - x^2(n-1)S_{n-2} + y(2n-(q+1))S_{n-(q+1)} \\ + xy(2n-(q+2))S_{n-(q+2)} - y^2(n-(q+1))S_{n-2(q+1)}, \end{aligned} \tag{2.9}$$

with $S_l = x^l \quad (0 \le l \le q),\ S_l = x^l + (l-q)^2 x^{l-q-1} y \quad (q+1 \le l \le 2q+1).$

Proof. By substituting (2.8) in the RHS of (2.9), then by simplification and change of variable the proof will be done.
According to the RHS of (2.9), we have

$$\begin{aligned} U_n :=& x(2n-1)S_{n-1} - x^2(n-1)S_{n-2} + y(2n-(q+1))S_{n-(q+1)} \\ & + xy(2n-(q+2))S_{n-(q+2)} - y^2(n-(q+1))S_{n-2(q+1)} \\ =& (2n-1) \sum_{k=0}^{\lfloor (n-1)/(q+1) \rfloor} \binom{n-1-qk}{k}^2 x^{n-(q+1)k}\, y^k \\ & - (n-1) \sum_{k=0}^{\lfloor (n-2)/(q+1) \rfloor} \binom{n-2-qk}{k}^2 x^{n-(q+1)k}\, y^k \\ & + (2n-(q+1)) \sum_{k=1}^{\lfloor (n+1-(q+1))/(q+1) \rfloor} \binom{n-1-qk}{k-1}^2 x^{n-(q+1)k}\, y^k \end{aligned}$$

$$+ (2n-(q+2)) \sum_{k=1}^{\lfloor (n+1-(q+2))/(q+1) \rfloor} \binom{n-2-qk}{k-1}^2 x^{n-(q+1)k}\, y^k$$
$$- (n-(q+1)) \sum_{k=2}^{\lfloor (n+2-2(q+1))/(q+1) \rfloor} \binom{n-2-qk}{k-2}^2 x^{n-(q+1)k}\, y^k.$$

Using the binomial convention, we have

$$\begin{aligned}
U_n = {} & (2n-1) \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \frac{(n-(q+1)k)^2}{k^2} \binom{n-1-qk}{k-1}^2 x^{n-(q+1)k}\, y^k \\
& - (n-1) \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \frac{(n-1-(q+1)k)^2}{k^2} \binom{n-2-qk}{k-1}^2 x^{n-(q+1)k}\, y^k \\
& + (2n-(q+1)) \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \binom{n-1-qk}{k-1}^2 x^{n-(q+1)k}\, y^k \\
& + (2n-(q+2)) \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \binom{n-2-qk}{k-1}^2 x^{n-(q+1)k}\, y^k \\
& - (n-(q+1)) \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \binom{n-2-qk}{k-2}^2 x^{n-(q+1)k}\, y^k \\
= {} & \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \frac{(n-1-qk)^2}{(k-1)^2} \binom{n-2-qk}{k-2}^2 x^{n-(q+1)k}\, y^k \\
& \left[(2n-1)\frac{(n-(q+1)k)^2}{k^2} + (2n-(q+1)) \right] \\
& + \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \frac{(n-(q+1)k)^2}{(k-1)^2} \binom{n-2-qk}{k-2}^2 x^{n-(q+1)k}\, y^k \\
& \left[(2n-(q+2)) - (n-1)\frac{(n-1-(q+1)k)^2}{k^2} \right] \\
& - (n-(q+1)) \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \binom{n-2-qk}{k-2}^2 x^{n-(q+1)k}\, y^k \\
= {} & \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \binom{n-2-qk}{k-2}^2 x^{n-(q+1)k}\, y^k \\
& \left[\frac{(n-1-qk)^2}{(k-1)^2} \left((2n-1)\frac{(n-(q+1)k)^2}{k^2} + (2n-(q+1)) \right) \right. \\
& + \frac{(n-(q+1)k)^2}{(k-1)^2} \left((2n-(q+2)) - (n-1)\frac{(n-1-(q+1)k)^2}{k^2} \right) \\
& \left. - (n-(q+1)) \right]
\end{aligned}$$

$$= \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \binom{n-2-qk}{k-2}^2 x^{n-(q+1)k}\, y^k$$
$$\left[\frac{(2n-1)(n-(q+1)k)^2(n-1-qk)^2}{k^2(k-1)^2} + \frac{k^2(n-1-qk)^2(2n-(q+1))}{k^2(k-1)^2} \right.$$
$$+ \frac{k^2(2n-(q+2))(n-(q+1)k)^2}{k^2(k-1)^2}$$
$$\left. - \frac{(n-(q+1)k)^2(n-1-(q+1)k)^2(n-1)}{k^2(k-1)^2} - \frac{k^2(k-1)^2(n-(q+1))}{k^2(k-1)^2} \right]$$
$$= \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \binom{n-2-qk}{k-2}^2 x^{n-(q+1)k}\, y^k \frac{n(n-qk)^2(n-1-qk)^2}{k^2(k-1)^2}$$
$$= n \sum_{k=0}^{\lfloor n/(q+1) \rfloor} \binom{n-qk}{k}^2 x^{n-(q+1)k}\, y^k$$
$$= nS_n.$$

By using (2.8), we can easily get the initial condition for (2.9).

□

2.2.2 Generating function

We will apply (2.11) and (2.12) to the main recurrence relation (2.9). Then we get a function which is the constant zero function, after that we integrate the two side of the equation to get the desired result.

Theorem 4. *The sequence $(S_n)_n$ admits the generating function*

$$g(z) = \frac{1}{\sqrt{\left(1 - xz - yz^{q+1}\right)^2 - 4xyz^{q+2}}}. \tag{2.10}$$

Proof. Let the function

$$g'(z) = S_1 + 2S_2 z + \cdots + nS_n z^{n-1} + \cdots = \sum_{j=1}^{\infty} jS_j z^{j-1} \tag{2.11}$$

be the derivative of the generating function

$$g(z) = S_0 + S_1 z + S_2 z^2 + \cdots + S_n z^n + \cdots = \sum_{j=0}^{\infty} S_j z^j. \tag{2.12}$$

Consider the function

$$\begin{aligned}G(z) =& g'(z) - 2xzg'(z) - xg(z) + x^2z^2g'(z) + x^2zg(z) - 2yz^{q+1}g'(z) \\ &- y(q+1)z^q g(z) - 2xyz^{q+2}g'(z) - xy(q+2)z^{q+1}g(z) \\ &+ y^2z^{2(q+1)}g'(z) + y^2(q+1)z^{2q+1}g(z). \quad (2.13)\end{aligned}$$

Applying (2.11) and (2.12), according to the recursive identity (2.9) the coefficient of z^j in $G(z)$ is vanishing for $j \geq 2q+1$, as we get the RHS minus LHS of (2.9). Then we have

$$\begin{aligned}G(z) =& (S_1 - xS_0) + (2S_2 - 3xS_1 + x^2S_0)z \\ &+ \sum_{u=3}^{q-1}\left[(u+1)S_{u+1} - x(2u+1)S_u + x^2uS_{u-1}\right]z^u \\ &+ \left((q+1)S_{q+1} - x(2q+1)S_q + x^2qS_{q-1} - (q+1)yS_0\right)z^q \\ &+ \sum_{u=q+1}^{2q}\Big[(u+1)S_{u+1} - x(2u+1)S_u + x^2uS_{u-1} \\ &- y(2u-q-1)S_{u-q} - xy(2u-q)S_{u-q-1}\Big]z^u.\end{aligned}$$

Now we show that $G(z)$ is the constant zero function. In order to prove it we need the initial values of the sequence (S_n) from (2.9). Hence

$$\begin{aligned}S_1 - xS_0 &= 0, \\ 2S_2 - 3xS_1 + x^2S_0 &= 0, \\ (u+1)S_{u+1} - x(2u+1)S_u + x^2uS_{u-1} &= 0 \\ &(3 \leq u \leq q-1), \\ (q+1)S_{q+1} - x(2q+1)S_q + x^2qS_{q-1} - (q+1)yS_0 &= 0, \\ (u+1)S_{u+1} - x(2u+1)S_u + x^2uS_{u-1} - y(2u-q-1)S_{u-q} - xy(2u-q)S_{u-q-1} &= 0 \\ &(q+1 \leq u \leq 2q),\end{aligned}$$

and indeed, $G(z)$ is identically zero. Thus by (2.13) we obtain

$$\begin{aligned}G(z) &= g'(z)\left(1 - 2xz + x^2z^2 - 2yz^{q+1} - 2xyz^{q+2} + y^2z^{2(q+1)}\right) \\ &+ g(z)\left(-x + x^2z - y(q+1)z^q - xy(q+2)z^{q+1} + y^2(q+1)z^{2q+1}\right),\end{aligned}$$

a first homogeneous linear differential equation in $g(z)$. Let

$$\phi(z) = 1 - 2xz + x^2z^2 - 2yz^{q+1} - 2xyz^{q+2} + y^2z^{2(q+1)} \quad (2.14)$$

and

$$\psi(z) \;=\; -x + x^2 z - y(q+1)z^q - xy(q+2)z^{q+1} + y^2(q+1)z^{2q+1}. \tag{2.15}$$

Thus,

$$G(z) \;= g'(z)\,\phi(z) + g(z)\,\psi(z).$$

From (2.14) and (2.15), it is clear that

$$\phi'(z) \;=\; 2\psi(z).$$

Consequently, the solution to the integrals

$$\int \frac{g'(z)}{g(z)} \;=\; \frac{-1}{2}\int \frac{\phi'(z)}{\phi(z)} \tag{2.16}$$

together with the initial value $g(0) = S_0 = 1$ implies the corresponding particular solution (2.10). Thus the proof is complete. □

2.3 Morgan-Voyce phenomenon

The Morgan-Voyce phenomenon was described in [1], and also studied in [2, 23]. The specializations of the main recurrence (2.9), when $q = 0$ gives the recurrence of the sum of rows of SBT, which yields the result of Verill [89]. For $q = 1$ we get the main diagonal.

Corollary 5. *For $q = 0$,*

$$S_n = \sum_{k=0}^{n} \binom{n}{k}^2 x^{n-k}\, y^k$$

satisfies

$$nS_n = (x+y)(2n-1)S_{n-1} - (x-y)^2(n-1)S_{n-2}. \tag{2.17}$$

Proof. It suffices to set $q = 0$ in (2.8) and (2.9) to get the result. □

Some sequences from OEIS [78] satisfy (2.17). See *A984*, *A1850*, *A6442*, *A12000*, *A59304*, *A69835*, *A84769*, *A84771*, *A84772*, *A84773*, *A84774*, *A98269*, *A98270*, *A98332*, *A98430*, *A98341*, *A98658*, *A98659*, *A115864*, *A116091*, *A119309*.

Corollary 6. *For $q = 1$, results (2.9)*

$$S_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}^2 x^{n-2k} y^k,$$

which satisfies

$$\begin{aligned} nS_n = x(2n-1)S_{n-1} &+ (x^2+y)(n-1)S_{n-2} \\ &+ xy(2n-3)S_{n-3} - y^2(n-2)S_{n-4}. \end{aligned} \tag{2.18}$$

Proof. It suffices to set $q = 1$ in (2.8) and (2.9) to get the desired result. □

The sequences *A*89165, *A*101500, *A*108489, *A*108490 satisfy (2.18).

2.4 Combinatorial identities

In this section, we give some well known sequences appear in OEIS [78], that satisfy the main recurrence, we also proved some recurrences conjectured, and find a few new ones.

2.4.1 Recurrences conjectured and proved

In Table 2.1, we list the sequences in OEIS [78] that we proved their recurrences using Theorem 2.9.

OEIS code	Formula	Recurrence
*A*84769	$\sum_k \binom{n}{k}^2 4^{n-k} 5^k$	$na_n = 9(2n-1)a_{n-1} - (n-1)a_{n-2}$
*A*84771	$\sum_k \binom{n}{k}^2 4^k$	$na_n = 5(2n-1)a_{n-1} - 9(n-1)a_{n-2}$
*A*98270	$\sum_k \binom{n}{k}^2 4^{n-k} 6^k$	$na_n = 10(2n-1)a_{n-1} - 4(n-1)a_{n-2}$
*A*98341	$\sum_k \binom{n}{k}^2 4^{n-k} (-1)^k$	$na_n = 3(2n-1)a_{n-1} - 25(n-1)a_{n-2}$
*A*101500	$\sum_k \binom{n-k}{k}^2 2^{n-2k}$	$na_n = 2(2n-1)a_{n-1} - 2(n-1)a_{n-2} +$ $2(2n-3)a_{n-3} - (n-2)a_{n-4}$

A108489	$\sum_k \binom{n-k}{k}^2 \ 3^k$	$na_n = (2n-1)a_{n-1} - 5(n-1)a_{n-2} + 3(2n-3)a_{n-3} - 9(n-2)a_{n-4}$
A108490	$\sum_k \binom{n-k}{k}^2 \ 2^{n-2k} \ 6^k$	$na_n = 2(2n-1)a_{n-1} - 8(n-1)a_{n-2} + 12(2n-3)a_{n-3} - 36(n-2)a_{n-4}$
A115864	$\sum_k \binom{n}{k}^2 \ 2^{n-k} \ 6^k$	$na_n = 8(2n-1)a_{n-1} - 16(n-1)a_{n-2}$

Table 2.1: Recurrences conjectured in [78], proved by Theorem 3.

2.4.2 Recurrences relation found for some OEIS's sequences

In Table 2.2, we list the sequences existed in OEIS and do not have a recurrence relation, which we found using (2.9).

OEIS code	Formula	Recurrence
A59304	$\sum_k \binom{n}{k}^2 \ 2^{n-k} \ 2^k$	$na_n = 4(2n-1)a_{n-1}$
A119309	$\sum_k \binom{n}{k}^2 \ 6^{n-k} \ 6^k$	$na_n = 12(2n-1)a_{n-1}$
A246840	$\sum_k \binom{n-2k}{k}^2$	$na_n = (2n-1)a_{n-1} - (n-1)a_{n-2}$ $+(2n-3)a_{n-3} + (2n-4)a_{n-4} - (n-3)a_{n-6}$
A246883	$\sum_k \binom{n-3k}{k}^2$	$na_n = (2n-1)a_{n-1} - (n-1)a_{n-2}$ $+(2n-4)a_{n-4} - (2n-5)a_{n-5} - (n-4)a_{n-8}$
A246884	$\sum_k \binom{n-4k}{k}^2$	$na_n = (2n-1)a_{n-1} - (n-1)a_{n-2}$ $+(2n-5)a_{n-5} - (2n-6)a_{n-6} - (n-5)a_{n-10}$
A248193	$\sum_k \binom{n-5k}{k}^2$	$na_n = (2n-1)a_{n-1} - (n-1)a_{n-2}$ $+(2n-6)a_{n-6} - (2n-7)a_{n-7} - (n-6)a_{n-12}$

Table 2.2: Recurrences found by the main recurrence.

2.4.3 Some well-known sequences that satisfy Theorem 3

In Table 2.3, we give the well-known sequences in OEIS [78] that satisfy (2.9).

OEIS code	Formula	Recurrence
A984	$\sum_k \binom{n}{k}^2$	$na_n = (2n-1)a_{n-1} - (n-1)a_{n-2}$
A1850	$\sum_k \binom{n}{k}^2 2^k$	$na_n = 3(2n-1)a_{n-1} - (n-1)a_{n-2}$
A6442	$\sum_k \binom{n}{k}^2 2^{n-k} 3^k$	$na_n = 5(2n-1)a_{n-1} - (n-1)a_{n-2}$
A12000	$\sum_k \binom{n}{k}^2 3^{n-k} (-1)^k$	$na_n = 2(2n-1)a_{n-1} - 16(n-1)a_{n-2}$
A69835	$\sum_k \binom{n}{k}^2 3^k$	$na_n = 4(2n-1)a_{n-1} - 4(n-1)a_{n-2}$
A84772	$\sum_k \binom{n}{k}^2 5^k$	$na_n = 6(2n-1)a_{n-1} - 16(n-1)a_{n-2}$
A84773	$\sum_k \binom{n}{k}^2 2^{n-k} 4^k$	$na_n = 6(2n-1)a_{n-1} - 4(n-1)a_{n-2}$
A84774	$\sum_k \binom{n}{k}^2 2^{n-k} 5^k$	$na_n = 7(2n-1)a_{n-1} - 9(n-1)a_{n-2}$
A89165	$\sum_k \binom{n-k}{k}^2 4^{n-2k}$	$na_n = (7n-3)a_{n-1} - (7n-4)a_{n-2}$
A98269	$\sum_k \binom{n}{k}^2 3^{n-k} 5^k$	$na_n = 8(2n-1)a_{n-1} - 4(n-1)a_{n-2}$
A98332	$\sum_k \binom{n}{k}^2 2^{n-k} (-1)^k$	$na_n = (2n-1)a_{n-1} - 9(n-1)a_{n-2}$
A98430	$\sum_k \binom{n}{k}^2 4^n$	$na_n = 8(2n-1)a_{n-1}$
A98658	$\sum_k \binom{n}{k}^2 3^n$	$na_n = 6(2n-1)a_{n-1}$
A98659	$\sum_k \binom{n}{k}^2 6^k$	$na_n = 7(2n-1)a_{n-1} - 25(n-1)a_{n-2}$
A108488	$\sum_k \binom{n-k}{k} 2^k$	$na_n = (2n-1)a_{n-1} + 3(n-1)a_{n-2}$ $+2(2n-3)a_{n-3} - 4(n-2)a_{n-4}$
A116091	$\sum_k \binom{n}{k}^2 (-3)^k$	$na_n = -2(2n-1)a_{n-1} - 16(n-1)a_{n-2}$
A116092	$\sum_k \binom{n}{k} 2^{n-k} (-6)^k$	$na_n = -4(2n-1)a_{n-1} - 64(n-1)a_{n-2}$
A307695	$\sum_{k=0} \binom{n}{k}^2 9^k$	$na_n = 10(2n-1)a_{n-1} - 64(n-1)a_{n-2}$

Table 2.3: Some well known sequences satisfy recurrence (2.9).

CHAPTER

3

DIAGONAL SUMS IN PASCAL PYRAMID

In this chapter, we define the concept of directions in Pascal pyramid trying to answer the question asked in the Introduction about the analogues of the Fibonacci sequence in the Pascal pyramid. This chapter is divided into two main sections. In the first section, we describe the concept of directions in Pascal pyramid and give some well-known examples with their illustrations. In the second section, we establish the main recurrence and its generating function (See [23]).

3.1 Trinomial coefficients

3.1.1 Definition

As we have seen in Chapter 1, the binomial coefficients $\binom{n}{k}$ arise also in the expansion of $(x+y)^n$, and can be arranged in the form of a triangle. We denote the trinomial coefficients by $\binom{n}{i,\, j,\, n-i-j}$, where i and j are non negative integers, and

$$\binom{n}{i,\ j,\ n-i-j} = \begin{cases} \frac{n!}{i!j!(n-i-j)!} & \textit{for } i \geq 0,\ j \geq 0 \textit{ and } i+j \leq n, \\ 0 & \textit{otherwise.} \end{cases}$$

These coefficients appear in the expansion of $(x+y+z)^n$ as

$$(x+y+z)^n = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i,\ j,\ n-i-j} x^i\, y^j\, z^{n-i-j}. \tag{3.1}$$

They satisfy the recurrence relation

$$\binom{n}{i,\ j,\ n-i-j} = \binom{n-1}{i-1,\ j,\ n-i-j} + \binom{n-1}{i,\ j-1,\ n-i-j} + \binom{n-1}{i,\ j,\ n-1-i-j}. \tag{3.2}$$

The trinomial coefficients can be expressed as a product of binomial coefficients and this is easy to get by applying the binomial theorem twice on the trinomial expansion

$$\begin{aligned}(x+(y+z))^n &= \sum_{i=0}^{n}\binom{n}{i} x^i\,(y+z)^{n-i} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j} x^i\, y^j\, z^{n-i-j}.\end{aligned} \tag{3.3}$$

Mary and Price in [7, 65] illustrated formula (3.3) by multiplying the n^{th} row of Pascal triangle by the n first rows, see Figure 3.1

1	$\times$	1					1				
3	$\times$	1	1				3	3			
3	$\times$	1	2	1		$=$	3	6	3		
1	$\times$	1	3	3	1		1	3	3	1	

Figure 3.1: The coefficients of $(x+y+z)^3$ using the first 3 rows of Pascal triangle.

Some special properties

- The sum of the coefficients

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i,\ j,\ n-i-j} = 3^n.$$

- The alternating sum

$$\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i,\ j,\ n-i-j}(-1)^{n-i-j} = 1.$$

- The symmetry

$$\binom{n}{i,\ j,\ n-i-j} = \binom{n}{\prod(i,\ n-i-j,\ j)},$$

where $\prod(i,\ j,\ n-i-j)$ is an arbitrary permutation of $i,\ j$, and $n-i-j$.

3.1.2 Combinatorial interpretation

Trinomial coefficients can be interpreted in many ways, now we give two well-known interpretations.

1) The trinomial coefficients count the number of paths from $(0,0,0)$ to (n,i,j) (where $i+j=n$) using three steps $(1,0,0),\ (0,1,0),\ (0,0,1)$, see Fig (3.2). As an example shown in the figure below, the number of paths from $(0,0,0)$ to $(1,1,1)$ is 6, all the possible paths are listed here

$$\begin{aligned}
(0,0,0) &\to (1,0,0) \to (1,1,0) \to (1,1,1),\\
(0,0,0) &\to (1,0,0) \to (1,0,1) \to (1,1,1),\\
(0,0,0) &\to (0,1,0) \to (1,1,0) \to (1,1,1),\\
(0,0,0) &\to (0,1,0) \to (0,1,1) \to (1,1,1),\\
(0,0,0) &\to (0,0,1) \to (1,0,1) \to (1,1,1),\\
(0,0,0) &\to (0,0,1) \to (0,1,1) \to (1,1,1).
\end{aligned}$$

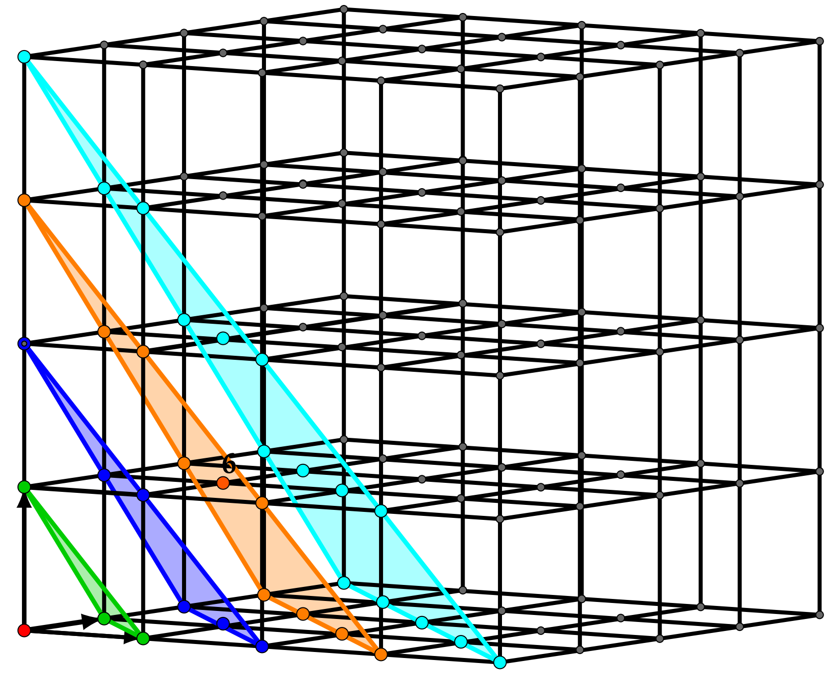


Figure 3.2: The view of trinomial coefficients inside the hyper-grid.

2) Also they can be interpreted as the number of ways to constitute 3 distinguishable groups from n candidates such that the first group contains i indistinguishable members, the second group contains j indistinguishable members and the third group contains $n-i-j$ indistinguishable members.

How many ways can we constitute 3 groups of sizes $1, 1, 2$ respectively, from a 4-elements set $\{a, b, c, d\}$?

The answer is $\binom{4}{1,\,1,\,2} = 12$, all the possible ways are

$$\{(a)(b)(c,d)\}, \{(a)(c)(b,d)\}, \{(a)(d)(b,c)\},$$
$$\{(b)(a)(c,d)\}, \{(b)(c)(a,d)\}, \{(b)(d)(a,c)\},$$
$$\{(c)(a)(b,d)\}, \{(c)(b)(a,d)\}, \{(c)(d)(a,b)\},$$
$$\{(d)(a)(b,c)\}, \{(d)(b)(a,c)\}, \{(d)(c)(a,b)\}.$$

3.2 Pascal pyramid

A three dimensional generalization of the Pascal triangle can be constructed using the recurrence (3.2) of trinomial coefficients, which gives the well-known Pascal pyramid, where each face of it is a Pascal triangle (See Figure 3.3). One of the first appearance of Pascal pyramid was in the work of E.B. Rosenthal [73], and then, many authors studied or rediscovered it, for instance in [82] J. Staib and L. Staib gave an algorithm to construct the cross section of Pascal pyramid, see also [28, 32, 52, 74, 91, 96].

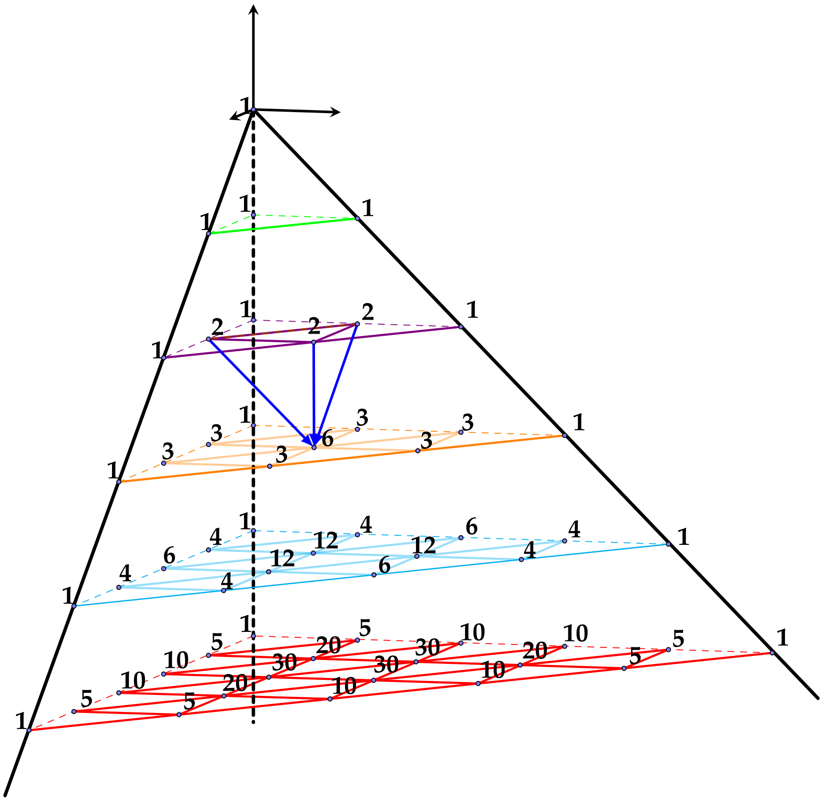


Figure 3.3: The first five layers of Pascal pyramid.

Remark 7. We can see clearly the different layers of Pascal pyramid in the three dimensional hyper-grid in Figure 3.2.

E.B. Rosenthal in [73] showed that the sums of the numbers in vertical columns of the Pascal pyramid layers give the coefficients of the expansion of $(1 + x + x^2)^n$, (see figure 3.4), they are known as bi^{2}nomial coefficients, also as trinomial coefficients, which confuses sometimes people with the trinomial coefficients from the expansion of $(x + y + z)^n$. These coefficients and their generalization were studied by several authors see for instance [8, 10, 11, 28, 64, 80], we can also find some authors who are interested in the sums of elements lying over diagonals in these triangles, see for example [3, 41, 55].

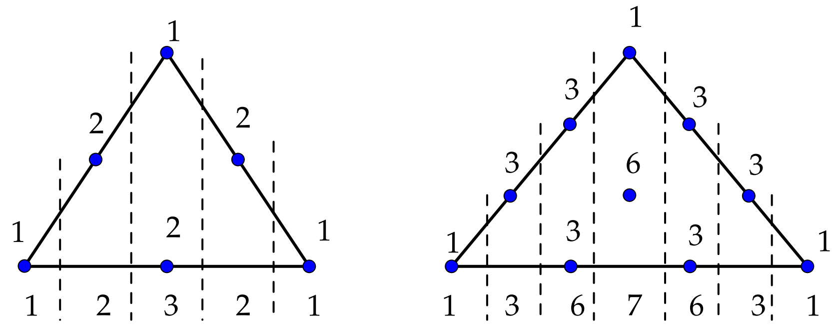


Figure 3.4: The column sum of layers 2 and 3 of Pascal pyramid.

The coefficients can be expressed by

$$(1 + x + x^2)^n = \sum_{k=0}^{2n} \binom{n}{k}_2 x^k,$$

which gives the following triangle

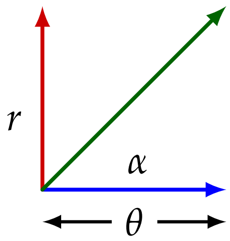
1										
1	1	1								
1	2	3	2	1						
1	3	6	7	6	3	1				
1	4	10	16	19	16	10	4	1		
1	5	15	30	45	61	45	30	15	5	1

Table 3.1: The first five rows of bi^{2}nomial coefficients triangle.

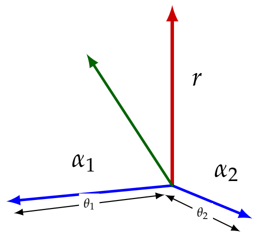
In the next subsection, we give the definition of directions in Pascal pyramid.

3.2.1 Definition of directions in Pascal pyramid

First of all, we give a simple illustration and comparison with directions in Pascal triangle, then we will explain how to construct the rays of any given direction in Pascal pyramid. We know that the directions in Pascal triangle defined by three parameters, where the first takes the number of jumps on the $x - axis$, the second does on the $y - axis$ and the third defines the starting point of the first parameter, as shown in Figure 3.5a. On the other hand, we see in Figure 3.5b the analogous directions in Pascal pyramid, where we have three axis and five parameters that define a direction $(\alpha_1, \alpha_2, r, \theta_1, \theta_2)$, where α_1, α_2 and r are the number of jumps according to $x - axis$, $y - axis$ and $z - axis$, respectively. θ_1 and θ_2 define the starting points for the first two parameters.



(a) The direction (α, r, θ)in Pascal triangle.



(b) The direction $(\alpha_1, \alpha_2, r, \theta_1, \theta_1)$ in Pascal pyramid.

Figure 3.5: The different schemes of directions.

Now we define how to construct any directions in Pascal pyramid.
Let $P_0(x_0, y_0, z_0) \in \mathbb{Z}^3$ and $P_1(x_1, y_1, z_1) \in \mathbb{Z}^3$ two arbitrary vertices of the pyramid, where $x_i \geq 0$, $y_i \geq 0$, $z_i \leq 0$ and $x_i + y_i + z_i \leq 0$, such that if $r = z_1 - z_0$ hold, then

$$\alpha_1 = x_1 - x_0 > 0, \qquad \alpha_2 = y_1 - y_0 > 0, \qquad \alpha_1 + \alpha_2 + r > 0$$

are fulfilled. We even prescribe $x_0 + y_0 + z_0 < 0$, excluding case when the vector $\vec{v} = \overrightarrow{P_0P_1}$ is on the plane $x + y + z = 0$.

Along the straight line determined by the points P_0 and P_1, we translate $-\vec{v}$ from P_0 as many time as possible: let $x_0 = s_1\alpha_1 + \ell_1$ with $0 \leq \ell_1 < \alpha_1$, further $y_0 = s_2\alpha_2 + \ell_2$ with $0 \leq \ell_2 < \alpha_2$. Put $s = \min\{s_1, s_2\}$, and let

$$\theta_1 = x_0 - s\alpha_1, \qquad \theta_2 = y_0 - s\alpha_2.$$

The point we obtained, say P, has the coordinates $(\theta_1, \theta_2, z_0 - sr)$. Now we push up the point P (parallel to axis z) till it reaches the plane $x + y + z = 0$. Then we obtained the uppermost ray defined by the quintuple $(\alpha_1, \alpha_2, r, \theta_1, \theta_2)$.

The direction $(\alpha_1, \alpha_2, r, \theta_1, \theta_2)$ defines diagonal rays in Pascal pyramid which contains the elements

$$A_k = \binom{n - rk}{\theta_1 + \alpha_1 k, \theta_2 + \alpha_2 k, n - \theta_1 - \theta_2 - (\alpha_1 + \alpha_2 + r)k} x^{\theta_1+\alpha_1 k} y^{\theta_2+\alpha_2 k} z^{n-\theta_1-\theta_2-(\alpha_1+\alpha_2+r)k}, \tag{3.4}$$

where $k = 0, ..., \lfloor \frac{n-\theta_1-\theta_2}{\alpha_1+\alpha_2+r} \rfloor$, with x, y and z are nonzero real parameters. Now we sum over all k to get the sum of the elements laying over the corresponding ray

$$T_n^{(\alpha_1,\alpha_2,r,\theta_1,\theta_2)} = \sum_{k=0}^{\lfloor \frac{n-\theta_1-\theta_2}{\alpha_1+\alpha_2+r} \rfloor} A_k. \tag{3.5}$$

Remark 8. We remark that some particular choices of α_1 and θ_1, α_2 and θ_2, lead to the faces of the pyramid. The directions $(\alpha_1, 0, r, \theta_1, 0)$ with $\theta_1 < \alpha_1$, and $(0, \alpha_2, r, 0, \theta_2)$ with $\theta_2 < \alpha_2$ return with the rays given in the Pascal triangle in [14]: More precisely, both

$$T_n^{(\alpha_1,0,r,\theta_1,0)} = \sum_{k=0}^{\lfloor \frac{n-\theta_1}{\alpha_1+r} \rfloor} \binom{n - rk}{\theta_1 + \alpha_1 k} x^{\theta_1+\alpha_1 k} z^{n-\theta_1-(\alpha_1+r)k}, \tag{3.6}$$

$$T_n^{(0,\alpha_2,r,0,\theta_2)} = \sum_{k=0}^{\lfloor \frac{n-\theta_2}{\alpha_2+r} \rfloor} \binom{n - rk}{\theta_2 + \alpha_2 k} y^{\theta_2+\alpha_2 k} z^{n-\theta_2-(\alpha_2+r)k} \tag{3.7}$$

have been described and studied in [13, 14].

3.2.2 Some well-known examples

In this subsection, we give some well-known sequences which can be expressed as the sum of trinomial coefficients, and can be obtained by setting the parameters of (3.5) to some particular values. We also show their illustration in Pascal pyramid.

- Let $r = -1$.
  - Putting $(x, y, z) = (1, 1, 1)$ we obtain the Central Delannoy numbers $(1, 3, 13, 63, 321, ...)$, see Figure 3.6, *A*001850 in OEIS [78]. This sequence

$$D_n = \sum_{k=0}^{n} \binom{n + k}{k, k, n - k} = \sum_{k=0}^{n} \binom{n + k}{k} \binom{n}{k} \tag{3.8}$$

satisfies

$$\begin{cases} nD_n = 3\,(2n-1)\,D_{n-1} - (n-1)\,D_{n-2}, \\ D_0 = 1, D_1 = 3, \end{cases}$$

and gives the number of paths in a grid from the point $(0,0)$ to (n,n) using only steps $(0,1)$, $(1,1)$ and $(1,0)$.

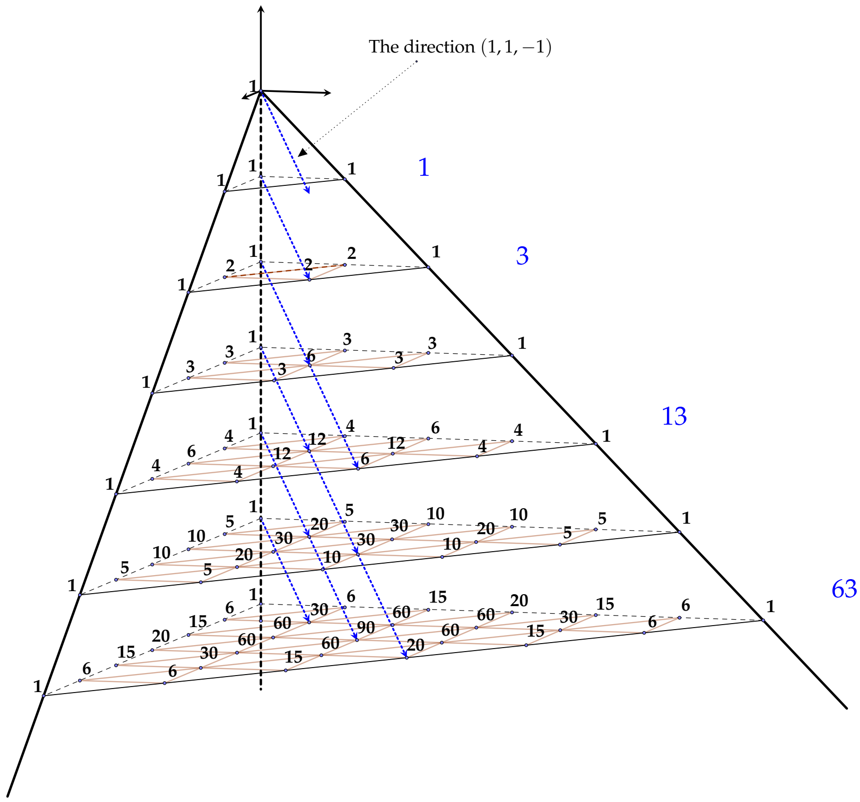


Figure 3.6: Illustration of the direction $(1,1,-1)$.

- For $(x,y,z)=(2,1,1)$ (or for $(1,2,1)$) we obtain the sequence *A*006442 in OEIS [78]. The first few values are $1, 5, 37, 305, 2641, \ldots$, and the sequence

$$V_n = \sum_{k=0}^{n} \binom{n+k}{k,k,n-k} 2^k = \sum_{k=0}^{n} \binom{n+k}{n-k}\binom{2k}{k} 2^k,$$

it satisfies

$$\begin{cases} nV_n = 5\,(2n-1)\,V_{n-1} - (n-1)\,V_{n-2}, \\ V_0 = 1, V_1 = 5. \end{cases}$$

- The triple $(x,y,z)=(4,1,1)$ (or $(2,2,1)$, or $(1,4,1)$) provides the sequence A084769 in OEIS [78]. The first few values are $1, 9, 121, 1809, \ldots$. The corresponding explicit formula is

$$W_n = \sum_{k=0}^{n} \binom{n+k}{k, k, n-k} 4^k = \sum_{k=0}^{n} \binom{n+k}{k} \binom{n}{k} 4^k,$$

it satisfies

$$\begin{cases} nW_n = 9\,(2n-1)\,W_{n-1} - (n-1)\,W_{n-2}, \\ W_0 = 1, W_1 = 9. \end{cases}$$

- For $r = 0$

  - For $(x, y, z) = (1, 1, 1)$ we get the Central trinomial coefficients $1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, \ldots$, see Figure 3.7, and the sequence

$$C_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k, k, n-2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}$$

appears as A002426 in OEIS [78] satisfies

$$\begin{cases} nC_n = (2n-1)\,C_{n-1} + 3(n-1)\,C_{n-2}, \\ C_0 = 1, C_1 = 5. \end{cases}$$

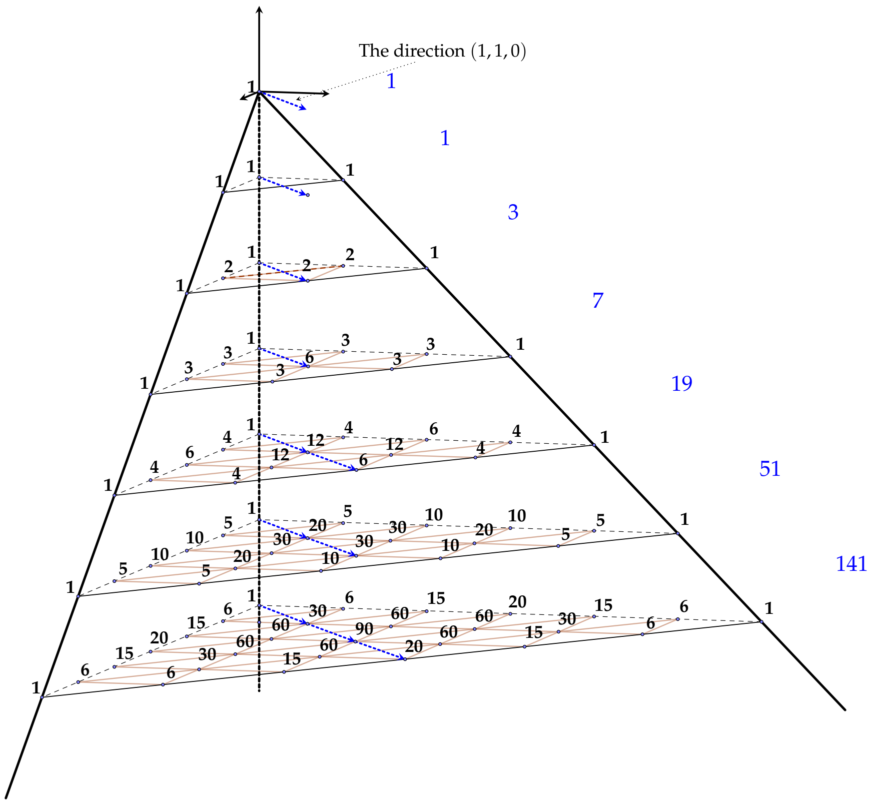


Figure 3.7: Illustration of the field of direction $(\alpha_1, \alpha_2, r) = (1, 1, 0)$.

3.3 Linear recurrence associated to diagonal sums in Pascal pyramid

In this section, we give the linear recurrence relation that describes the sequences obtained from the sums of directions of the main diagonal of Pascal pyramid, i.e. the parameters are $(\alpha_1, \alpha_2, r, \theta_1, \theta_2) = (1, 1, r, 0, 0)$, we also give its generating function.

By substituting the desired parameter of the main diagonals in (3.5), we have

$$T_n^{(1,1,r,0,0)} = \sum_{k=0}^{\lfloor \frac{n}{2+r} \rfloor} \binom{n - rk}{k, k, n-(2+r)k} x^k y^k z^{n-(2+r)k}. \tag{3.9}$$

For simplicity, we will write the expression $x^k y^k$ as t^k, where $t = xy$, and we will shortly use T_n instead of $T_n^{(1,1,r,0,0)}$.

Remark 9. We note in this particular case that $r \geq -1$.

3.3.1 The Linear recurrence of the main diagonal

The following theorem is the principal result of this chapter.

Theorem 10. *The terms of the sequence* $(T_n)_n$ *given by*

$$T_n = \sum_{k=0}^{\lfloor n/(r+2) \rfloor} \binom{n-rk}{k,k,n-(r+2)k} t^k z^{n-(r+2)k},$$

satisfy the linear recurrence relation

$$nT_n = (2n-1)zT_{n-1} - (n-1)z^2 T_{n-2} + 2(2n-(r+2))tT_{n-r-2}, \tag{3.10}$$

with $T_l = z^l \quad (0 \le l \le r)$.

Proof. Let $\varrho = r+2$. Thus $\varrho \ge 1$. Put $\nu_s = \lfloor (n-s)/\varrho \rfloor$, and for simplicity let $\nu = \nu_0$. Clearly, $\nu_\varrho = \nu - 1$. To prove the theorem we distinguish four cases according to

$$\nu = \nu_1 = \nu_2, \qquad (n = \nu\varrho + u,\ u \ge 2,\ \varrho \ge 3); \tag{3.11}$$

$$\nu = \nu_1 > \nu_2, \qquad (n = \nu\varrho + 1,\ \varrho \ge 2); \tag{3.12}$$

$$\nu > \nu_1 = \nu_2, \qquad (n = \nu\varrho,\ \varrho \ge 2); \tag{3.13}$$

$$\nu > \nu_1 > \nu_2, \qquad (\varrho = 1). \tag{3.14}$$

Putting

$$\begin{aligned}
\tau_n &= nT_n - (2n-1)zT_{n-1} + (n-1)z^2 T_{n-2} \\
&= n\sum_{k=0}^{\nu} \binom{n-(\varrho-2)k}{k,k,n-\varrho k} t^k z^{n-\varrho k} - (2n-1)z \sum_{k=0}^{\nu_1} \binom{n-1-(\varrho-2)k}{k,k,n-1-\varrho k} t^k z^{n-1-\varrho k} \\
&\qquad + (n-1)z^2 \sum_{k=0}^{\nu_2} \binom{n-2-(\varrho-2)k}{k,k,n-2-\varrho k} t^k z^{n-2-\varrho k}.
\end{aligned}$$

we need to show that $\tau_n = 2(2n-\varrho)tT_{n-\varrho}$.

Case 1.

$$\begin{aligned}
\tau_n &= nT_n - z(2n-1)T_{n-1} + z^2(n-1)T_{n-2} \\
&= n\sum_{k=0}^{\nu}\binom{n-(\varrho-2)k}{k,k,n-\varrho k}t^k z^{n-\varrho k} - (2n-1)\sum_{k=0}^{\nu}\binom{n-1-(\varrho-2)k}{k,k,n-1-\varrho k}t^k z^{n-\varrho k} \\
&\quad +(n-1)\sum_{k=0}^{\nu}\binom{n-2-(\varrho-2)k}{k,k,n-2-\varrho k}t^k z^{n-\varrho k} \\
&= \sum_{k=0}^{\nu}\frac{(n-2-(\varrho-2)k)!}{k!k!(n-2-\varrho k)!}t^k z^{n-\varrho k}\left[n\frac{(n-(\varrho-2)k)(n-1-(\varrho-2)k)}{(n-\varrho k)(n-1-\varrho k)}\right. \\
&\quad \left. -(2n-1)\frac{(n-1-(\varrho-2)k)}{n-1-\varrho k} + (n-1)\right] \\
&= \sum_{k=0}^{\nu}\frac{(n-2-(\varrho-2)k)!}{k!k!(n-2-\varrho k)!}t^k z^{n-\varrho k}\left[\frac{2k^2(2n-\varrho)}{(n-\varrho k)(n-1-\varrho k)}\right] \\
&= 2(2n-\varrho)\sum_{k=0}^{\nu}\frac{(n-2-(\varrho-2)k)!}{k!k!(n-\varrho k)!}k^2 t^k z^{n-\varrho k} \\
&= 2(2n-\varrho)\sum_{k=1}^{\nu}\frac{(n-\varrho-(\varrho-2)(k-1))!}{(k-1)!(k-1)!(n-\varrho k)!}t^k z^{n-\varrho k}.
\end{aligned}$$

Put $j = k-1$. Thus

$$\begin{aligned}
\tau_n &= 2t(2n-\varrho)\sum_{j=0}^{\nu-1}\frac{(n-\varrho-(\varrho-2)j)!}{j!j!(n-\varrho-\varrho j)!}t^j z^{n-\varrho-\varrho j} \\
&= 2t(2n-\varrho)\sum_{j=0}^{\nu_\varrho}\binom{n-\varrho-(\varrho-2)j}{j,j,n-\varrho-\varrho j}t^j z^{n-\varrho-\varrho j} \\
&= 2t(2n-\varrho)T_{n-\varrho}.
\end{aligned}$$

Case 2. Recall that $n = \nu\varrho + 1$, $\varrho \geq 2$. Preparing the computations, first observe that

$$\begin{aligned}
E_2 &= n\frac{(2\nu+1)!}{\nu!\nu!1!}t^\nu z - (2\nu\varrho+1)z\frac{(2\nu)!}{\nu!\nu!0!}t^\nu z^0 \\
&= ((\nu\varrho+1)(2\nu+1) - (2\nu\varrho+1))\frac{(2\nu)!}{\nu!\nu!0!}t^\nu z \\
&= 2(2\nu\varrho+2-\varrho)\frac{(2\nu-1)!}{(\nu-1)!(\nu-1)!1!}t^\nu z.
\end{aligned}$$

Then, together with $\nu - 1 = \nu_1 - 1 = \nu_2$, and the argument of Case 1 we see

$$\begin{aligned}\tau_n &= E_2 + 2(2n-\varrho)t\sum_{j=0}^{\nu-2}\frac{(n-\varrho-(\varrho-2)j)!}{j!j!(n-\varrho-\varrho j)!}t^j z^{n-\varrho-\varrho j} \\ & E_2 - 2(2n-\varrho)t\frac{(n-\varrho-(\varrho-2)(\nu-1))!}{(\nu-1)!(\nu-1)!(n-\varrho-\varrho(\nu-1))!}t^{\nu-1}z^{n-\varrho-\varrho(\nu-1)} \\ & +2(2n-\varrho)tT_{n-\varrho}.\end{aligned}$$

Finally, it is easy to see that the difference of first two terms is zero.

Case 3. Recall that $n = \nu\varrho$, $\varrho \geq 2$. Now put

$$E_3 = n\binom{\nu\varrho-(\varrho-2)\nu}{\nu,\nu,n-\nu\varrho}t^\nu z^{n-\nu\varrho} = \nu\varrho\frac{(2\nu)!}{\nu!\nu!0!}t^\nu z^0. \tag{3.15}$$

Thus,

$$\begin{aligned}\tau_n &= E_3 + 2(2n-\varrho)t\sum_{j=0}^{\nu-2}\frac{(n-\varrho-(\varrho-2)j)!}{j!j!(n-\varrho-\varrho j)!}t^j z^{n-\varrho-\varrho j} \\ &= E_3 - 2(2n-\varrho)t\frac{(n-\varrho-(\varrho-2)(\nu-1))!}{(\nu-1)!(\nu-1)!(n-\varrho-\varrho(\nu-1))!}t^{\nu-1}z^{n-\varrho-\varrho(\nu-1)} \\ & +2(2n-\varrho)tT_{n-\varrho},\end{aligned}$$

where the first two terms simplify to $\mathcal{T}_n = 2(2n-\varrho)tT_{n-\varrho}$.

Case 4. Now $\varrho = 1$, and $\nu - 2 = \nu_1 - 1 = \nu_2$. Since $n - \varrho = n - 1$, we need to show

$$nT_n = (2n-1)(2t+z)T_{n-1} - (n-1)z^2T_{n-2}, \tag{3.16}$$

where

$$T_n = \sum_{k=0}^{n}\binom{n+k}{k,k,n-k}t^k z^{n-k}.$$

We will compare the coefficients of the term $t^k z^{n-k}$ on the right hand side and left hand side of (3.16), respectively.

If $1 \leq k \leq n-2$ we have

$$2(2n-1)\binom{n+k-2}{k-1,k-1,n-k} + (2n-1)\binom{n+k-1}{k,k,n-k-1} - (n-1)\binom{n+k-2}{k,k,n-k-2},$$

which results, after a short computation,

$$n\frac{(n+k)!}{k!k!(n-k)!} = n\binom{n+k}{k,k,n-k}.$$

The specific cases $k=0$, $k=n-1$, and $k=n$ are easy to handle.

□

3.3.2 Generating function

Now we give the generating function for the recurrence (3.10).

Theorem 11. *The generating function of* (3.10) *is given by*

$$g(u) = \frac{1}{\sqrt{(1-zu)^2 - 4tu^{r+2}}} \tag{3.17}$$

Proof. Define $m = n-(r+2)k$, and then change the directions of the summation as follows.

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n u^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2+r} \rfloor} \binom{n-rk}{k,k,n-(2+r)k} t^k z^{n-(2+r)k} u^n \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{m+2k}{k,k,m} t^k z^m u^{m+(r+2)k} \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{2k}{k}\binom{m+2k}{2k} t^k z^m u^{m+(r+2)k} \\
&= \sum_{k=0}^{\infty} \binom{2k}{k} t^k u^{(r+2)k} \sum_{m=0}^{\infty} \binom{m+2k}{2k} z^m u^m.
\end{aligned}$$

Now apply the identity (see [44]),

$$\sum_{m=0}^{\infty} \binom{m+2k}{2k} z^m u^m = \frac{1}{(1-zu)^{2k+1}}.$$

Then we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_n u^n &= \frac{1}{1-zu} \sum_{k=0}^{\infty} \binom{2k}{k} \left(\frac{tu^{r+2}}{(1-zu)^2} \right)^k \\ &= \frac{1}{1-zu} \frac{1}{\sqrt{1-\frac{4tu^{r+2}}{(1-zu)^2}}} \\ &= \frac{1}{\sqrt{(1-zu)^2 - 4tu^{r+2}}}.\end{aligned}$$

□

CHAPTER 4

DIAGONAL SUMS OF PASCAL PYRAMID II: APPLICATIONS

In the previous chapter, we established the recurrence relation satisfied by the sums of elements lying along the main diagonal in Pascal pyramid. In the first section of this chapter, we discuss a phenomenon that appears in recurrence relations that have a dynamic order (subscript), this phenomenon allows us to deduce many combinatorial identities which are described in the second section. In the third section, we establish the combinatorial interpretation for all the sequences that satisfy the main recurrence relation. In the last section, we present a link between the second and the third chapters, we show that in some particular cases, we can express trinomial coefficients as square binomial coefficients and vice-versa (See [24]).

4.1 Morgan-Voyce phenomenon

In this section, we study the Morgan-Voyce phenomemon which appears in the recurrences that have a dynamic order (or subscript), this phenomenon is due to the subscript of the last term of the RHS of (3.10), which is $n-r-2$. If $r = -1$, or $r = 0$, the last term of RHS can be merged to T_{n-1} or T_{n-2}, respectively, which means that the terms $(2n-1)z$ or $-n(n-1)z^2$ in (3.10) is modified by the coefficient $2t(2n-(r+2))$ of the last term. This phenomenon defined in the case of Pascal triangle in [1], also studied in [2] on the generalized Delannoy triangle. The two cases described above can be expressed as follows.

Corollary 12. *If $r = -1$, then the sequence T_n can be expressed by the sum*

$$T_n = \sum_{k=0}^{n} \binom{n+k}{k, k, n-3k} t^k z^{n-3k}. \tag{4.1}$$

Formula (4.1) satisfies the recurrence relation

$$\begin{aligned} &nT_n = (2n-1)(2t+z)T_{n-1} - (n-1)z^2 T_{n-2}, \\ &T_0 = 1 \text{ and } T_1 = 2t+z. \end{aligned} \tag{4.2}$$

Proof. It suffices to substitute $r = -1$ in (3.10) to get the desired result. □

By using a program implemented with SageMath[1], we were able to search in the Encyclopedia of Integer Sequences (OEIS) [78], and get the sequences existed for different parameters.

A000984, *A002426*, *A004454*, *A006139*, *A012000*, *A026375*, *A059304*, *A080609*, *A081671*, *A084601*, *A084603*, *A084605*, *A084609*, *A084770*, *A084771*, *A084772*, *A084773*, *A084774*, *A098264*, *A098265*, *A098269*, *A098270*, *A098329*, *A098331*, *A098332*, *A098333*, *A098334*, *A098335*, *A098336*, *A098337*, *A098338*, *A098339*, *A098340*, *A098341*, *A098409*, *A098410*, *A098411*, *A098430*, *A098439*, *A098440*, *A098441*, *A098442*, *A098443*, *A098444*, *A098453*, *A098455*, *A098456*, *A098658*, *A098659*, *A106258*, *A106259*, *A106260*, *A106261*, *A115864*, *A115865*, *A116091*, *A116092*, *A116093*, *A122868*, *A248168*, *A258723*.

Corollary 13. *Suppose $r = 0$. Then the sequence T_n can be expressed as*

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k, k, n-2k} t^k z^{n-2k}, \tag{4.3}$$

and it satisfies

[1]SageMath is a free open-source mathematics software system licensed under the GPL. It builds on top of many existing open-source packages: NumPy, SciPy, matplotlib, Sympy, Maxima, GAP, FLINT, R and many more. Access their combined power through a common, Python-based language or directly via interfaces or wrappers (see *https : //www.sagemath.org*).

$$nT_n = (2n-1)zT_{n-1} + (n-1)(4t - z^2)z^2T_{n-2}, \tag{4.4}$$
$$T_0 = 1 \; and \; T_1 = z.$$

Proof. It suffices to substitute $r = 0$ in (3.10) to have the result. □

In OEIS [78] we identify some sequences satisfied by (4.4):
*A*001850, *A*006442, *A*069835, *A*084768, *A*084769, *A*084772, *A*084773.

4.2 Same recurrence for two different directions

As we saw in the two cases of the Morgan-Voyce phenomenon. The recurrences we investigate have order 2. We will show that by choosing the parameters t and z carefully, we can unify the two cases, which yields to an identity concerning two directions in Pascal pyramid. Clearly, we have to find the solutions for the following system

$$z_0 = 2t_1 + z_1, \tag{4.5}$$
$$4t_0 - z_0^2 = -z_1^2. \tag{4.6}$$

The value $t_1 = 0$ does not give a distinct pairs of parameters. Thus we assume that $t_1 \neq 0$. By combining the two equations (4.5) and (4.6), we have

$$z_0 + z_1 = \frac{2t_0}{t_1}, \tag{4.7}$$

and then

$$z_0 = \frac{t_0}{t_1} + t_1, \qquad z_1 = \frac{t_0}{t_1} - t_1. \tag{4.8}$$

Thus, for arbitrary $t_1 \neq 0$ and t_0, we can determine distinct z_0 and z_1 by (4.8). Now we need to be sure that the initial conditions are the same in (4.2) and (4.4). It is clear that both $T_0 = 1$. If (4.5) fulfills, then it guarantees that the initial value T_1 is the same, which leads us to the following identity between (4.1) and (4.3).

Corollary 14. *For an arbitrary* t_0 *and* $t_1 \neq 0$, *let* $\tau = \frac{t_0}{t_1}$. *Then we have*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k, k, n-2k} t_0^k (\tau + t_1)^{n-2k} = \sum_{k=0}^{n} \binom{n+k}{k, k, n-k} t_1^k (\tau - t_1)^{n-k}. \tag{4.9}$$

4.2.1 Combinatorial identities

In this subsection, we will list several identities resulted by (4.9) for different parameters, as well as the corresponding sequences existed in OEIS [78], which are proved, found or discovered by (3.10). In Table 4.1 we list the sequences that their recurrences conjectured in OEIS [78], and proved using Theorem 10.

OEIS code	Sum	Conjectured recurrence relations
A084768	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 5^k\, 6^{n-2k}$	$na_n = 7(2n-1)a_{n-1} - (n-1)a_{n-2}$
A084769	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 4^k\, 5^{n-2k}$	$na_n = 9(2n-1)a_{n-1} - (n-1)a_{n-2}$
A084771	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 4^k\, 5^{n-2k}$	$na_n = 5(2n-1)a_{n-1} - 9(n-1)a_{n-2}$
A098270	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 24^k\, 10^{n-2k}$	$na_n = 10(2n-1)a_{n-1} - 4(n-1)a_{n-2}$
A098333	$\sum_k \binom{n}{k,\, k,\, n-2k} (-3)^k$	$na_n = (2n-1)a_{n-1} - 13(n-1)a_{n-2}$
A098334	$\sum_k \binom{n}{k,\, k,\, n-2k} (-4)^k\, 2^{n-2k}$	$na_n = (2n-1)a_{n-1} - 17(n-1)a_{n-2}$
A098336	$\sum_k \binom{n}{k,\, k,\, n-2k} (-2)^k\, 2^{n-2k}$	$na_n = 2(2n-1)a_{n-1} - 12(n-1)a_{n-2}$
A098337	$\sum_k \binom{n}{k,\, k,\, n-2k} (-4)^k\, 2^{n-2k}$	$na_n = 2(2n-1)a_{n-1} - 20(n-1)a_{n-2}$
A098338	$\sum_k \binom{n}{k\ k,\, n-2k} (-1)^k\, 3^{n-2k}$	$na_n = 3(2n-1)a_{n-1} - 13(n-1)a_{n-2}$
A098340	$\sum_k \binom{n}{k,\, k,\, n-2k} (-3)^k\, 3^{n-2k}$	$na_n = 3(2n-1)a_{n-1} - 21(n-1)a_{n-2}$
A098341	$\sum_k \binom{n+k}{k,\, k,\, n-k} (-1)^k\, 5^{n-k}$	$na_n = 3(2n-1)a_{n-1} - 25(n-1)a_{n-2}$
A098411	$\sum_k \binom{n}{k,\, k,\, n-2k} 4^k\, 8^{n-2k}$	$na_n = 8(2n-1)a_{n-1} - 48(n-1)a_{n-2}$
A098439	$\sum_k \binom{n}{k,\, k,\, n-2k} 12^k$	$na_n = (2n-1)a_{n-1} + 47(n-1)a_{n-2}$

A098456	$\sum_k \binom{n}{k,\,k,\,n-2k} 17^k\, 2^{n-2k}$	$na_n = 2(2n-1)a_{n-1} + 64(n-1)a_{n-2}$
A098479	$\sum_k \binom{n-k}{k,\,k,\,n-3k}$	$na_n = (2n-1)a_{n-1} - (n-1)a_{n-2} + 2(2n-3)a_{n-3}$
A098480	$\sum_k \binom{n-k}{k,\,k,\,n-3k} 2^k$	$na_n = (2n-1)a_{n-1} - (n-1)a_{n-2} + 4(2n-3)a_{n-3}$
A106186	$\sum_k \binom{n-k}{k,\,k,\,n-3k} 4^k\, 2^{n-3k}$	$na_n = 2(2n-1)a_{n-1} - 4(n-1)a_{n-2} + 8(2n-3)a_{n-3}$
A115864	$\sum_k \binom{n+k}{k,\,k,\,n-k} 2^k\, 4^{n-k}$	$na_n = 8(2n-1)a_{n-1} - 16(n-1)a_{n-2}$
A116092	$\sum_k \binom{n+k}{k,\,k,\,n-2k} (-6)^k\, 8^{n-k}$	$na_n = -4(2n-1)a_{n-1} - 64(n-1)a_{n-2}$
A116093	$\sum_k \binom{n}{k,\,k,\,n-k} (-2)^k\, (-2)^{n-2k}$	$na_n = -2(2n-1)a_{n-1} - 12(n-1)a_{n-2}$
A122868	$\sum_k \binom{n}{k,\,k,\,n-2k} 3^k\, 3^{n-2k}$	$na_n = 3(2n-1)a_{n-1} + 3(n-1)a_{n-2}$

Table 4.1: Conjectured recurrence relations proved.

In Table 4.2, we collect at least two new trinomial sums for some sequences, the columns 2-4 of the table provide new equivalent sum identities for the sequences of the first column, respectively.

OEIS code	Sum 1	Sum 2	Sum 3
A012000	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} (-3)^k\, 2^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} (-1)^k\, 4^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 3^k\, (-4)^{n-k}$
A084771	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 4^k\, 5^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 3^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 4^k\, (-3)^{n-k}$
A084772	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 5^k\, 6^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 4^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 5^k\, (-4)^{n-k}$
A084773	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 8^k\, 6^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 2^k 2^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 4^k\, (-2)^{n-k}$
A084774	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 10^k\, 7^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 2^k 3^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 5^k\, (-3)^{n-k}$
A098269	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 15^k\, 8^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 3^k 2^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 5^k\, (-2)^{n-k}$
A098270	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 24^k\, 10^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 4^k 2^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 6^k\, (-2)^{n-k}$
A098341	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} (-4)^k\, 3^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} (-1)^k 5^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 4^k\, (-5)^{n-k}$
A098659	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 6^k\, 7^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 5^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 6^k\, (-5)^{n-k}$
A115864	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 12^k\, 8^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 2^k 4^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 6^k\, (-4)^{n-k}$
A115865	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 27^k\, 12^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 3^k 6^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 9^k\, (-6)^{n-k}$
A0116091	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} (-3)^k\, (-2)^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} (-3)^k 4^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} (-4)^{n-k}$
A116092	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} (-12)^k\, (-4)^{n-2k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} (-6)^k 8^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 2^k\, (-8)^{n-k}$
A069835	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 3^k\, 4^{n-2k}$	known	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 3^k\, (-2)^{n-k}$
A084768	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 12^k\, 7^{n-2k}$	known	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 4^k\, (-1)^{n-k}$
A084769	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 20^k\, 9^{n-2k}$	known	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 5^k\, (-1)^{n-k}$
A098332	known	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} (-1)^k 3^{n-k}$	$\sum_k^n \binom{n+k}{k,\,k,\,n-k} 2^k (-3)^{n-k}$

Table 4.2: Identities discovered.

In the following table, we list three sequences for which the corresponding recurrence relations were not given in OEIS [78].

OEIS code	Recurrence relation
A113179	$na_n = 2(2n-1)a_{n-1} - 4(n-1)a_{n-2} + 4(2n-3)a_{n-3}$
A248168	$na_n = 7(2n-1)a_{n-1} - 33(n-1)a_{n-2}$
A258723	$na_n = 6(2n-1)a_{n-1} - 48(n-1)a_{n-2}$

Table 4.3: Recurrence relation discovered.

Table 4.4 contains new identities for certain sequences in OEIS [78], we found them using formula (3.5).

OEIS code	Sum OEIS code	Sum	
A001850	$\sum_k^n \binom{n+k}{k,\, k,\, n-k} 2^k (-1)^{n-k}$	A006134	$\sum_k^{\lfloor n/3 \rfloor} \binom{n-k}{k,\, k,\, n-3k} 3^{n-3k}$
A006442	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 6^k\, 5^{n-2k}$	A026375	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 3^{n-2k}$
A059304	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 4^k\, 4^{n-2k}$	A080609	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 2^k\, 4^{n-2k}$
A081671	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 4^{n-2k}$	A084605	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 4^k$
A084770	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 5^k\, 2^{n-2k}$	A098339	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} (-2)^k 3^{n-2k}$
A098409	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 5^{n-2k}$	A098410	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 6^{n-2k}$
A098411	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 4^k 8^{n-2k}$	A098430	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 16^k 8^{n-2k}$
A098443	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 5^k 4^{n-2k}$	A098444	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 5^k 3^{n-2k}$
A098453	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 4^k 2^{n-2k}$	A098455	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 10^k 2^{n-2k}$
A098456	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 17^k 2^{n-2k}$	A098658	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 9^k 6^{n-2k}$
A106186	$\sum_k^{\lfloor n/3 \rfloor} \binom{n-k}{k,\, k,\, n-3k} 4^k 2^{n-3k}$	A106258	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 6^k 4^{n-2k}$

A106259	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 12^k 6^{n-2k}$	A106260	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 20^k 8^{n-2k}$
A106261	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 30^k 10^{n-2k}$	A122868	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 3^k 3^{n-2k}$
A248168	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} 4^k 7^{n-2k}$	A258723	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\,k,\,n-2k} (-3)^k 6^{n-2k}$

Table 4.4: Sums discovered.

4.3 Combinatorial interpretation

A combinatorial interpretation of the sequence (3.9) corresponding to the direction $(r, 1, 1)$ in the Pascal pyramid follows easily from the sum (3.9).

Theorem 15. *The sequence T_n in (3.9) counts the sum of products of weights of lattice paths from the point $(0, 0)$ to the point (n, n) using steps $\{(2 + r, 0), (0, 2 + r), (1, 1)\}$, where the first and the second steps are weighted with $\sqrt{t}$ and the last one with z.*

Proof. Let k be the number of steps of the type $(r + 2, 0)$ weighted $\sqrt{t}$ we use, it is clear that $0 \leq k \leq \lfloor n/(r + 2)\rfloor$. We must use the same number of steps of type $(0, r + 2)$ weighted $\sqrt{t}$ to go back to the main diagonal, where the destination point (n, n) is. In order to reach the final point we need to go $(n - (r + 2)k)$ steps of type $(1, 1)$ weighted z, (see Figure (4.1)). The numbers of steps used in total is $k + k + (n - (r + 2)k) = n - rk$, the number of all possible configuration of the path is given by $(n - rk)!$. Subtracting the configurations of the same type, we obtain $\frac{(n-rk)!}{k!k!(n-(r+2)k)!} t^k z^{n-(r+2)k}$. The total number is the sum over all possible k, given by

$$T_n = \sum_{k=0}^{\lfloor \frac{n}{2+r} \rfloor} \binom{n - rk}{k, k, n - (2 + r)k} t^k z^{n-(2+r)k}.$$

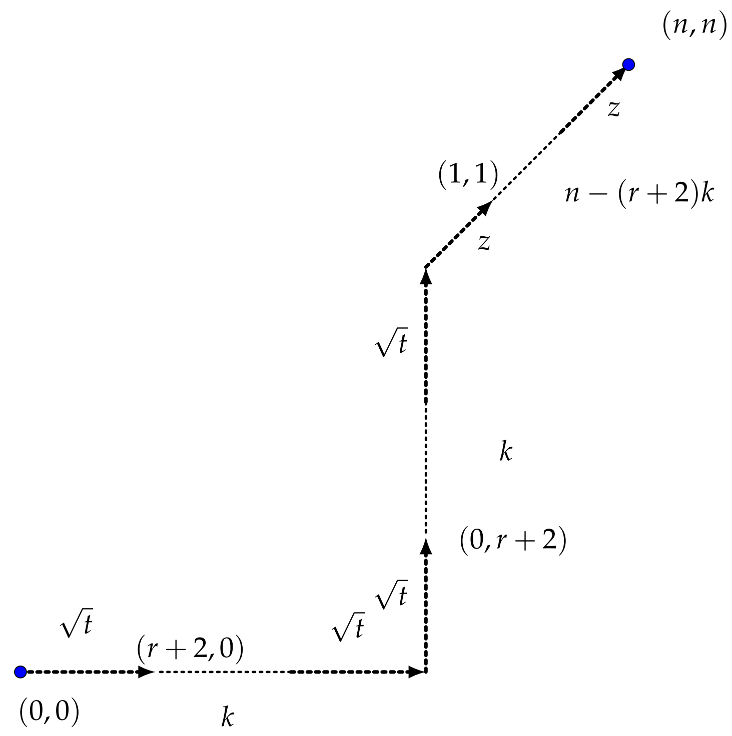


Figure 4.1: Lattice paths using three steps $\{(2+r,0),(0,2+r),(1,1)\}$.

□

4.3.1 Illustration of some examples

In this subsection, we illustrate some examples given in section 3.2.2.

Example 2. *The first example is the Central Delannoy numbers D_n (see formula (3.8)), they count the sums of product of weights related to the lattice paths from $(0,0)$ to (n,n) using the steps from $\{(1,0)\rightarrow,(0,1)\uparrow,(1,1)\nearrow\}$, with weight 1 for all the steps. For $n=2$, we illustrate all corresponding paths below.*

$$\begin{aligned} D_2 &= (1\times1\times1\times1)+(1\times1\times1\times1)+(1\times1\times1\times1)+(1\times1\times1\times1) \\ &\quad +(1\times1\times1\times1)+(1\times1\times1\times1)+(1\times1\times1\times1)+(1\times1\times1\times1) \\ &\quad +(1\times1\times1\times1)+(1\times1\times1\times1)+(1\times1\times1\times1)+(1\times1\times1\times1) \\ &\quad +(1\times1\times1\times1)=13 \end{aligned}$$

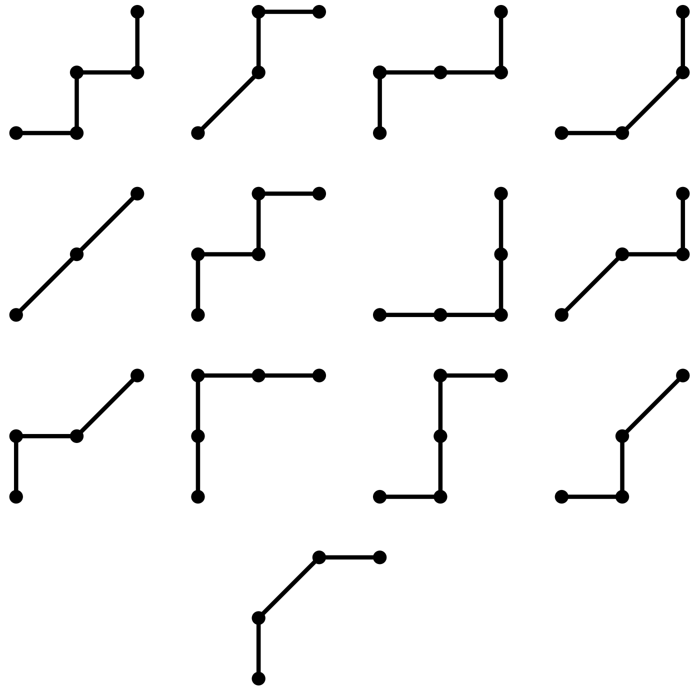


Figure 4.2: All lattice paths correspond to D_2.

Example 3. *The second example given by setting* $(r, t, z) = (0, 1, 2)$*, which gives the central binomial coefficients,* $(1, 2, 6, 20, 70, 252, ...)$*, A000984 (see [78]). They can be expressed as*

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k,\ k,\ n-2k} 2^{n-2k},$$

They count the sum of products of weights of lattice paths from $(0, 0)$ *to* (n, n) *using the steps* $\{(2, 0), (0, 2), (1, 1)\}$*, with weight* 1 *for the two first steps and with weight* 2 *for the last step. Where* T_3 *is the sum of product of weights of each path, then*

$$\begin{aligned} T_3 =& (1 \times 1 \times 2) + (1 \times 1 \times 2) + (1 \times 2 \times 1) + (1 \times 2 \times 1) + (2 \times 1 \times 1) \\ &+ (2 \times 1 \times 1) + (2 \times 2 \times 2) = 20 \end{aligned}$$

We illustrate that as follows

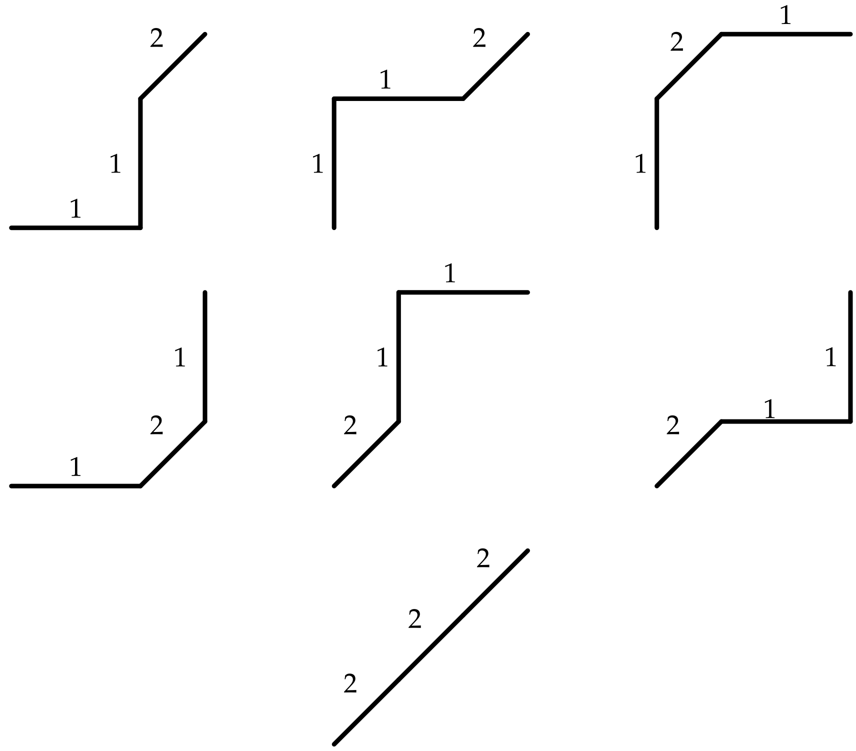


Figure 4.3: Illustration of T_3 of $A000984$

4.4 An identity involving square binomial and trinomial coefficients

In this section, we establish an identity between a particular case of square binomial coefficients that we have seen in Chapter 2 and two particular cases of trinomial coefficients (directions in Pascal pyramid).

MacMahon in [58] established the following identity

$$\sum_{k=0}^{n} \binom{n}{k}^3 x^k y^{n-k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} \binom{n-k}{k} \binom{n+k}{k} (xy)^k (x+y)^{n-2k}. \tag{4.10}$$

It is known that the central binomial coefficient can be expressed as

$$T_n := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k,\ k,\ n-2k} 2^{n-2k} = \binom{2n}{n}. \tag{4.11}$$

also by

$$T_n := \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}, \tag{4.12}$$

It is clear from (4.11) and (4.12) that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k,\ k,\ n-2k} 2^{n-2k}. \tag{4.13}$$

Using the same argument that we have used to prove the identity (4.9) on the formulas (2.17) and (4.2)

Theorem 16. *For x, y real numbers, $x \neq -y$, we have*

$$\sum_{k=0}^{n} \binom{n}{k}^2 x^{n-k}\, y^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k,\ k,\ n-2k} (x+y)^{n-2k}\, (xy)^k. \tag{4.14}$$

Proof. By the using the identification between the coefficients of the two recurrences (2.17) and (4.2), and by verifying that the initial conditions are equal, we get the result. □

Theorem 17. *For x, y real numbers, $x \neq y$, we have*

$$\sum_{k=0}^{n} \binom{n}{k}^2 x^{n-k}\, y^k = \sum_{k=0}^{n} \binom{n+k}{k,\ k,\ n-k} (x-y)^{n-k}\, y^k. \tag{4.15}$$

Proof. It suffices to use the identity (4.9) to have the result. □

Corollary 18. *For $x = y = 1$, the central binomial coefficients A000984 see [78]*

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k,\ k,\ n-2k} 2^{n-2k} = \binom{2n}{n}.$$

Corollary 19. *For $(x, y) = (1, 2)$, we get the central Delannoy coefficients A001850*

$$\sum_{k=0}^{n} \binom{n}{k}^2 2^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k,\ k,\ n-2k} 3^{n-2k}\, 2^k = \sum_{k=0}^{n} \binom{n+k}{k,\ k,\ n-k} (-1)^{n-k}\, 2^k.$$

Corollary 20. *For $(x,y) = (1,-1)$, we have*

$$\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k = \sum_{k=0}^{n} \binom{n+k}{k,\ k,\ n-k} 2^{n-k} (-1)^k = \begin{cases} 0 & \text{for } n \text{ odd}, \\ \binom{n}{n/2} & \text{for } n \text{ even}. \end{cases}$$

4.4.1 Combinatorial identities

In Tables 4.5 and 4.6, we give at least two equivalent sums for several sequences that can found in OEIS [78], the sums were derived from Theorems (17) and (16).

OEIS code	Sum 1	Sum 2
*A*984	$\sum_k \binom{n}{k}^2$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\ k,\ n-2k} 2^{n-2k}$;
*A*59304	$\sum_k \binom{n}{k}^2 2^{n-k} 2^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\ k,\ n-2k} 4^{n-2k} 4^k$;
*A*98430	$\sum_k \binom{n}{k}^2 4^{n-k} 4^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\ k,\ n-2k} 8^{n-2k} 16^k$;
*A*98658	$\sum_k \binom{n}{k}^2 3^{n-k} 3^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\ k,\ n-2k} 6^{n-2k} 9^k$;
*A*119309	$\sum_k \binom{n}{k}^2 6^{n-k} 6^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\ k,\ n-2k} 12^{n-2k} 36^k$.

Table 4.5: Identities discovered.

OEIS code	Sum 1	Sum 2	Sum 3
A1850	$\sum_k \binom{n}{k}^2 2^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 3^{n-2k}\, 2^k$	$\sum_k^n \binom{n+k}{k,\, k,\, n-k} (-1)^{n-k}\, 2^k$;
A6442	$\sum_k \binom{n}{k}^2 2^{n-k}\, 3^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 5^{n-2k}\, 6^k$	$\sum_k^n \binom{n+k}{k,\, k,\, n-k} (-1)^{n-k}\, 3^k$;
A12000	$\sum_k \binom{n}{k}^2 3^{n-k}\, (-1)^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-k} 2^{n-2k}\, (-3)^k$	$\sum_k^n \binom{n}{k,\, k,\, n-k} 4^{n-k}\, (-1)^k$;
A69835	$\sum_k \binom{n}{k}^2 3^{n-k}$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 4^{n-2k}\, 3^k$	$\sum_k^n \binom{n+k}{k,\, k,\, n-k} 2^{n-k}$;
A84768	$\sum_k \binom{n}{k}^2 3^{n-k}\, 4^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 7^{n-2k}\, 12^k$	$\sum_k^n \binom{n+k}{k,\, k,\, n-k} (-1)^{n-k}\, 4^k$;
A84769	$\sum_k \binom{n}{k}^2 4^{n-k}\, 5^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 9^{n-2k}\, 20^k$	$\sum_k^n \binom{n+k}{k,\, k,\, n-k} (-1)^{n-k}\, 5^k$;
A84771	$\sum_k \binom{n}{k}^2 4^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 5^{n-2k}\, 4^k$	$\sum_k^n \binom{n+k}{k,\, k,\, n-k} (-3)^{n-k}\, 4^k$;
A84772	$\sum_k \binom{n}{k}^2 5^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-k} 6^{n-2k}\, 5^k$	$\sum_k^{n/2} \binom{n+k}{k,\, k,\, n-k} (-4)^{n-k}\, 5^k$;
A84773	$\sum_k \binom{n}{k}^2 2^{n-k}\, 4^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 6^{n-2k}\, 8^k$	$\sum_k^n \binom{n+k}{k,\, k,\, n-k} (-2)^{n-k}\, 4^k$;
A84774	$\sum_k \binom{n}{k}^2 2^{n-k}\, 5^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 7^{n-2k}\, 10^k$	$\sum_k^n \binom{n+k}{k,\, k,\, n-k} (-3)^{n-k}\, 5^k$;
A98269	$\sum_k \binom{n}{k}^2 3^{n-k}\, 5^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 8^{n-2k}\, 15^k$	$\sum_k^n \binom{n+k}{k,\, k,\, n-k} (-2)^{n-k}\, 5^k$;
A98270	$\sum_k \binom{n}{k}^2 4^{n-k}\, 6^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} 10^{n-2k}\, 24^k$	$\sum_k^n \binom{n+k}{k,\, k,\, n-k} (-2)^{n-k}\, 6^k$;
A98332	$\sum_k \binom{n}{k}^2 2^{n-k}\, (-1)^k$	$\sum_k^{\lfloor n/2 \rfloor} \binom{n}{k,\, k,\, n-2k} (-2)^k$	$\sum_k^n \binom{n+k}{k,\, k,\, n-k} 3^{n-k}\, (-1)^k$;

Table 4.6: Identities discovered.

CHAPTER

5

PLANE SECTIONS IN PASCAL PYRAMID

As we saw in the previous chapters, we established a formula and a recurrence for the sum of elements lying over diagonals in Pascal pyramid using a one dimensional vector as a direction. In this chapter, we study another type of objects in Pascal pyramid, which are known as plane sections. In the first section, we quote the preceding works dealt with the plane sections in Pascal pyramid. In the second section, we give a new definition for plane sections and give some well-known examples with their illustrations. In the third section, we establish the recurrence relation. After that, we list many corollaries and combinatorial identities and at the end of the chapter we give a combinatorial interpretation for all the sequences obtained using the main recurrence (See [22]).

5.1 Plane sections

Muller [61] studied a particular case of plane sections in Pascal pyramid, and gave the recurrence of their sums, where he considered the following characterization of the plane sections (see Fig. 5.1):

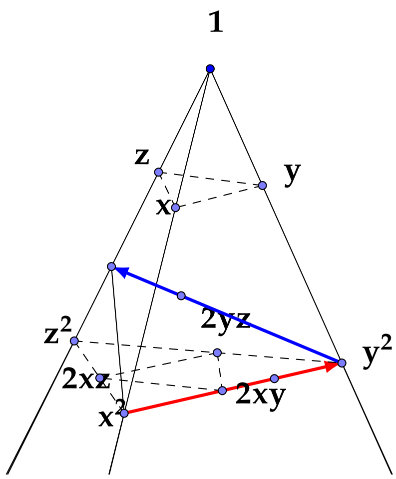


Figure 5.1: Muller characterization of plane sections in Pascal pyramid.

$$M_n = \sum_{i=0}^{\left[\frac{n}{q_1+r_1}\right]} \sum_{j=0}^{\left[\frac{n-q_1 i}{q_2+r_2}\right]} \binom{n-q_1 i-q_2 j}{r_1 i-q_2 j}\binom{r_1 i-q_2 j}{r_2 j} x^{n-i(q_1+r_1)} y^{r_1 i-j(q_2+r_2)} z^{r_2 j}. \quad (5.1)$$

He showed that M_n satisfies

$$M_n = xM_{n-1} + yM_{n-(q_1+1)} + zM_{n-(q_1+1)(q_2+1)}. \quad (5.2)$$

Shannon [76] expressed the Tribonacci sequence as a sum of plane sections in Pascal pyramid by

$$T_n = \sum_{i=0}^{[n/2]} \sum_{j=0}^{[n/3]} \binom{n-i-2j}{i+j}\binom{i+j}{j}. \quad (5.3)$$

Remark 21. The expression (5.3) given by Shannon in [76] can not be deduced from the general expression (5.1) of Muller in [61].

On the other hand, Kuzmin and Seregina [57] studied the plane sections in a generalized Pascal pyramid (see [56]) by using the equation of the plane sections to describe the sum of their elements.

5.2 Definition of symmetric plane sections

Here, we introduce another approach in the Pascal pyramid by describing the plane sections with two main diagonals (see Fig.5.2) and provide an explicit formula of the sums of their elements. We also establish the associated recurrence relation for some particular cases.

Any plane section on the pyramid is determined by two diagonals (see Fig. 5.2). Let (q_1, r_1) be the first diagonal from the $x - axis$ towards the $y - axis$ with $q_1 \in \mathbb{Z}, r_1 \in \mathbb{N}$ and $q_1 + r_1 > 0$, and let (q_2, r_2) be the second diagonal from the $x - axis$ to the $z - axis$ with $q_2 \in \mathbb{Z}, r_2 \in \mathbb{N}$ and $q_2 + r_2 > 0$. The elements lying on the plane section defined for a fixed (q_1, q_2, r_1, r_2) are given by

$$A_{n,i,j} = \binom{n - q_1 i - q_2 j}{r_1 i,\ r_2 j,\ n - (q_1 + r_1)i - (q_2 + r_2)j} x^{n-(q_1+r_1)i-(q_2+r_2)j} y^{r_1 i} z^{r_2 j},$$

where x, y and z are real parameters, and we denote its sum over i, j by

$$T_{n,q_1,q_2,r_1,r_2} = \sum_{i \geq 0} \sum_{j \geq 0} A_{n,i,j}. \tag{5.4}$$

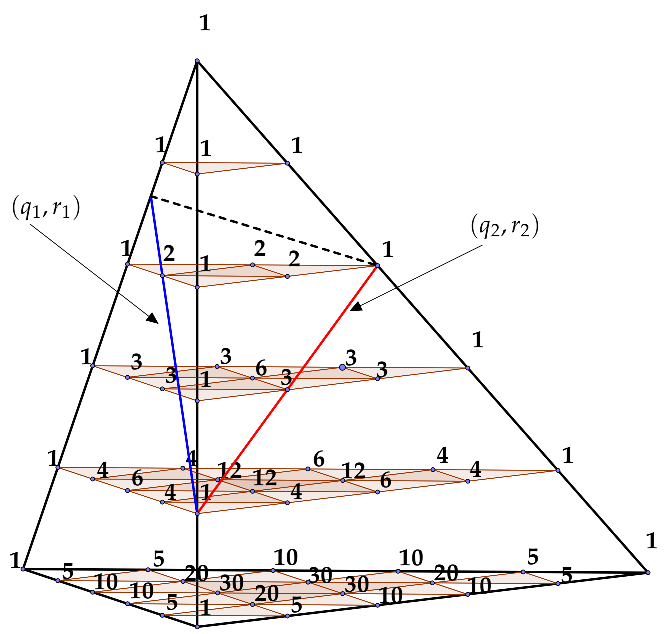


Figure 5.2: The diagonals which defines a plane section.

In the sequel of this chapter, we consider only the specific case $(q_1, q_2, r_1, r_2) =$

$(q_1, q_2, 1, 1)$, for brevity we write T_{n,q_1,q_2}, instead of T_{n,q_1,q_2,r_1,r_2}.

5.2.1 Classical examples

In this subsection, we give some examples which yield some well-know sequences, and we illustrate them in the Pascal pyramid (see Figures 5.3, 5.4 and 5.5).
Put $(x, y, z) = (1, 1, 1)$.

- Let $(q_1, q_2) = (1, 2)$, we obtain the Tribonacci sequence *A*000073 in OEIS [78], the first values are $0, 0, 1, 1, 2, 4, 7, 13, \ldots$, the corresponding explicit formula (see [76]) is

$$T_{n+2,1,1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor n/3 \rfloor} \binom{n-i-2j}{i,\ j,\ n-2i-3j}, \tag{5.5}$$

and it satisfies

$$\begin{cases} T_n = T_{n-1} + T_{n-2} + T_{n-3}, \\ T_0 = T_1 = 0, T_2 = 1. \end{cases}$$

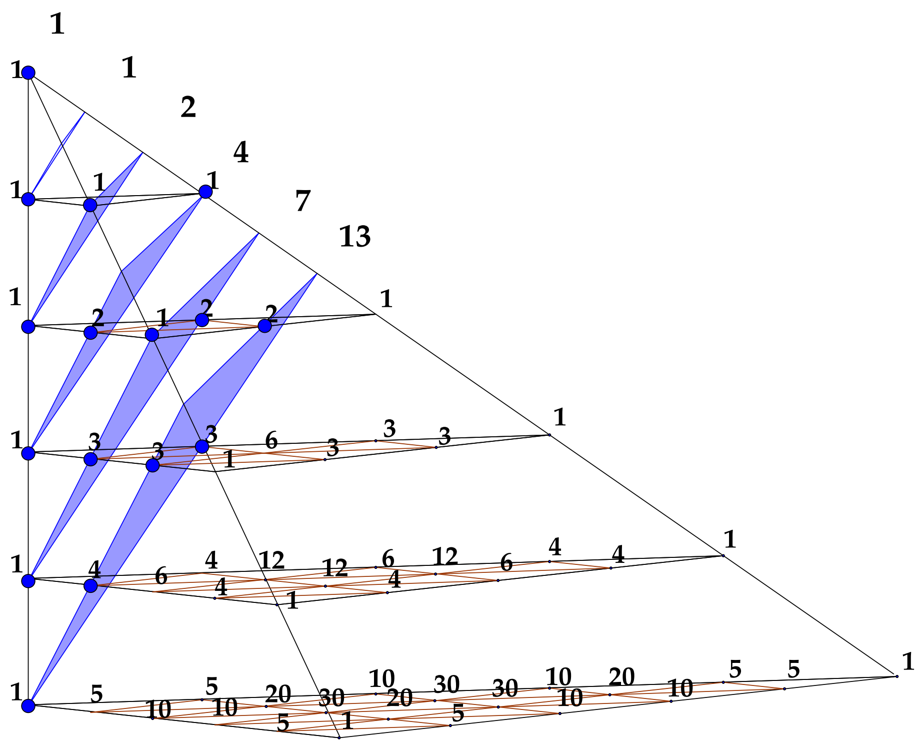


Figure 5.3: Plane section which gives the Tribonacci sequence.

- Putting $(q_1, q_2) = (1, 0)$ we obtain the Pell sequence, *A*000129 in OEIS [78]. The first values are $0, 1, 2, 5, 12, 29, 70, 169, \ldots$ and its explicit formula is given by

$$P_{n+1,1,0} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{n} \binom{n-i}{i,\ j,\ n-2i-j}. \tag{5.6}$$

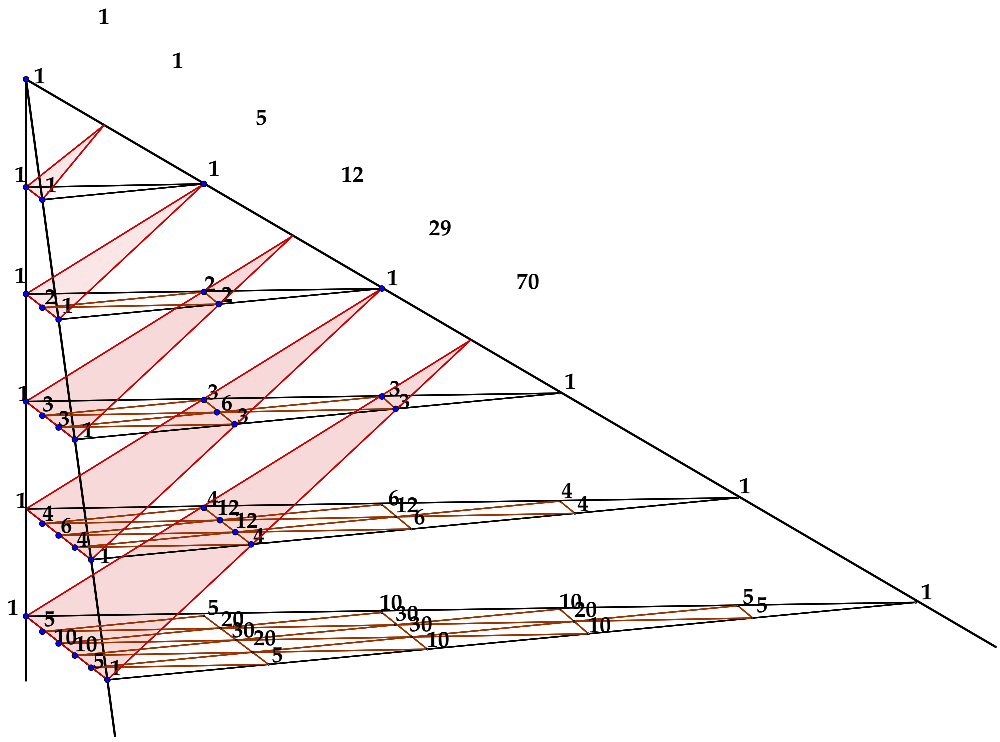


Figure 5.4: Plan section which gives Pell sequence.

- Let $(q_1, q_2) = (1, 1)$. It yields the Jacobsthal sequence, and the sequence appears as *A*001045 in OEIS [78], the first few values are $0, 1, 1, 3, 5, 11, 21, \ldots$. The corresponding explicit formula is

$$J_{n+1,1,1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-i-j}{i,\ j,\ n-2i-2j}. \tag{5.7}$$

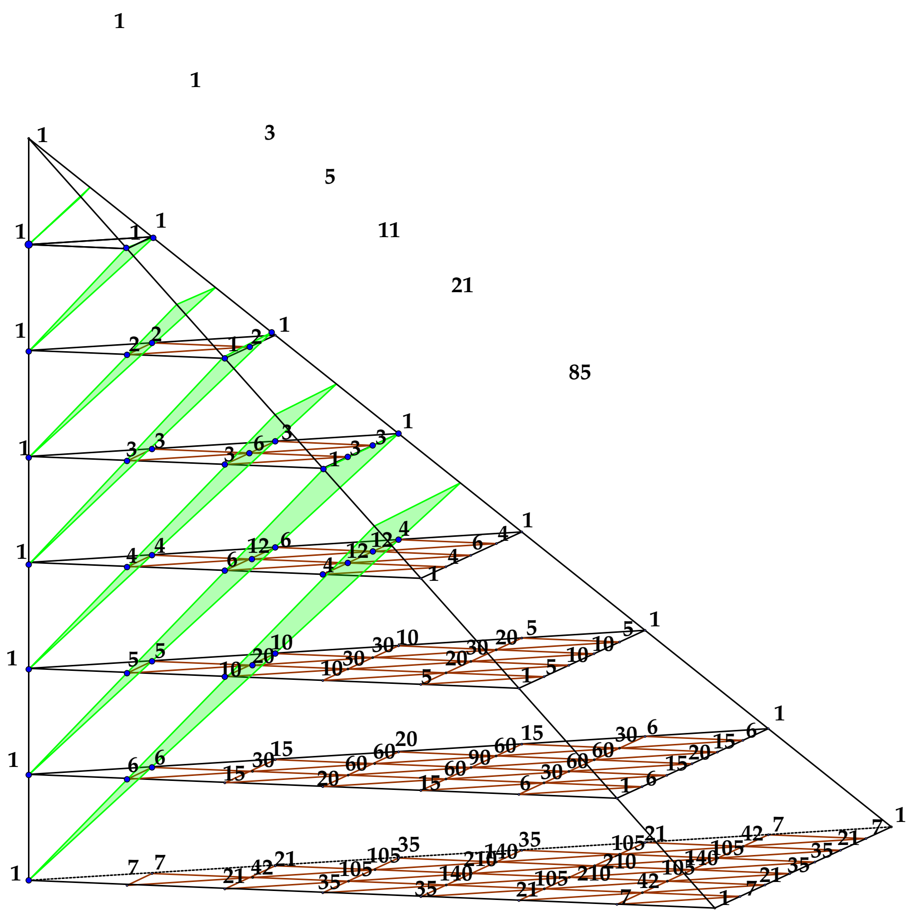


Figure 5.5: Plan section which gives Jacobsthal sequence.

5.3 Linear recurrence associated to plane sections in Pascal pyramid

In what follows, we use T_n instead of T_{n,q_1,q_2}.

Theorem 22. *Let $n \in \mathbb{N} \cup \{0\}$. Then the terms of the sequence $(T_n)_n$ defined by*

$$T_n = \sum_{i \geq 0} \sum_{j \geq 0} \binom{n - q_1 i - q_2 j}{i,\ j,\ n - (q_1+1)i - (q_2+1)j} x^{n-(q_1+1)i-(q_2+1)j} y^i z^j \tag{5.8}$$

satisfy the linear recurrence

$$T_n = xT_{n-1} + yT_{n-(q_1+1)} + zT_{n-(q_2+1)}. \tag{5.9}$$

Proof. We simplify the RHS of (5.9) as follows. We obtain

$$
\begin{aligned}
&xT_{n-1} + yT_{n-(q_1+1)} + zT_{n-(q_2+1)} = \\
&\sum_{i\geq 0}\sum_{j\geq 0}\binom{n-1-q_1 i-q_2 j}{i,\ j,\ n-1-(q_1+1)i-(q_2+1)j} x^{n-(q_1+1)i-(q_2+1)j}\, y^i\, z^j \\
&\quad + \sum_{i\geq 0}\sum_{j\geq 0}\binom{n-1-q_1(i+1)-q_2 j}{i,\ j,\ n-(q_1+1)(i+1)-(q_2+1)j} x^{n-(q_1+1)(i+1)-(q_2+1)j}\, y^{i+1}\, z^j \\
&\qquad + \sum_{i\geq 0}\sum_{j\geq 0}\binom{n-1-q_1 i-q_2(j+1)}{i,\ j,\ n-(q_1+1)i-(q_2+1)(j+1)} x^{n-(q_1+1)i-(q_2+1)(j+1)}\, y^i\, z^{j+1}.
\end{aligned}
$$

Let $i := i+1$ and $j := j+1$, and the last formula is equivalent to

$$
\begin{aligned}
&\sum_{i\geq 0}\sum_{j\geq 0}\binom{n-1-q_1 i-q_2 j}{i,\ j,\ n-1-(q_1+1)i-(q_2+1)j} x^{n-(q_1+1)i-(q_2+1)j}\, y^i\, z^j \\
&\quad + \sum_{i\geq 1}\sum_{j\geq 0}\binom{n-1-q_1 i-q_2 j}{i-1,\ j,\ n-(q_1+1)i-(q_2+1)j} x^{n-(q_1+1)i-(q_2+1)j}\, y^i\, z^j \\
&\qquad + \sum_{i\geq 0}\sum_{j\geq 1}\binom{n-1-q_1 i-q_2 j}{i,\ j-1,\ n-(q_1+1)i-(q_2+1)j} x^{n-(q_1+1)i-(q_2+1)j}\, y^i\, z^j.
\end{aligned}
$$

According to the definition of the trinomial coefficients, we can rewrite the lower bounds as follow.

$$
\begin{aligned}
&\sum_{i\geq 0}\sum_{j\geq 0}\binom{n-1-q_1 i-q_2 j}{i,\ j,\ n-1-(q_1+1)i-(q_2+1)j} x^{n-(q_1+1)i-(q_2+1)j}\, y^i\, z^j \\
&\quad + \sum_{i\geq 0}\sum_{j\geq 0}\binom{n-1-q_1 i-q_2 j}{i-1,\ j,\ n-(q_1+1)i-(q_2+1)j} x^{n-(q_1+1)i-(q_2+1)j}\, y^i\, z^j \\
&\qquad + \sum_{i\geq 0}\sum_{j\geq 0}\binom{n-1-q_1 i-q_2 j}{i,\ j-1,\ n-(q_1+1)i-(q_2+1)j} x^{n-(q_1+1)i-(q_2+1)j}\, y^i\, z^j
\end{aligned}
$$

$$
\begin{aligned}
= \sum_{i\geq 0}\sum_{j\geq 0}\Bigg(&\binom{n-1-q_1 i-q_2 j}{i,\ j,\ n-1-(q_1+1)i-(q_2+1)j} \\
&+ \binom{n-1-q_1 i-q_2 j}{i-1,\ j,\ n-(q_1+1)i-(q_2+1)j} \\
&+ \binom{n-1-q_1 i-q_2 j}{i,\ j-1,\ n-(q_1+1)i-(q_2+1)j}\Bigg) x^{n-(q_1+1)i-(q_2+1)j}\, y^i\, z^j.
\end{aligned}
$$

Using (3.2) we get

$$
\sum_{i\geq 0}\sum_{j\geq 0}\binom{n-q_1 i-q_2 j}{i,\ j,\ n-(q_1+1)i-(q_2+1)j} x^{n-(q_1+1)i-(q_2+1)j}\, y^i\, z^j = T_n.
$$

□

5.4 Combinatorial identities

The main result is expressed according to the choice of the couple (q_1, q_2). Some special cases are of interest.

Corollary 23. *For the case* $(q_1, q_2) = (q, q)$*, we have*

$$T_n = \sum_{i\geq 0}\sum_{j\geq 0} \binom{n - q(i+j)}{i,\ j,\ n-(q+1)(i+j)} x^{n-(q+1)(i+j)}\, y^i\, z^j, \tag{5.10}$$

and it satisfies

$$T_n = xT_{n-1} + (y+z)T_{n-q-1}. \tag{5.11}$$

Thus (5.11) *is equivalent to* (1.5) *for* $a = x$ *and* $b = y + z$*.*

Proof. From (1.5) we have $U_p = a^p$ for $0 \leq p \leq q$, and from (5.10) we have $T_p = x^p$ for $0 \leq p \leq q$. By identification between the coefficients of the two recurrences we get $a = x$, $b = y + z$, and the proof is complete. □

Corollary 24. *For* $(q_1, q_2) = (q, 0)$ *we have*

$$T_n = \sum_{i\geq 0}\sum_{j\geq 0} \binom{n - qi}{i,\ j,\ n-qi-j} x^{n-qi-j}\, y^i\, z^j, \tag{5.12}$$

and it satisfies

$$T_n = (x+z)T_{n-1} + yT_{n-q-1}. \tag{5.13}$$

Proof. It suffices to set the parameters $(q1, q2) = (q, 0)$ in Theorem 22. □

Corollary 25. *For* $(q_1, q_2) = (0, q)$ *we have*

$$T_n = \sum_{i\geq 0}\sum_{j\geq 0} \binom{n - i - qj)}{i,\ j,\ n-i-qj} x^{n-i-qj}\, y^i\, z^j, \tag{5.14}$$

and it satisfies

$$T_n = (x+y)T_{n-1} + zT_{n-q-1}. \tag{5.15}$$

Proof. Choose the parameters $(q1, q2) = (0, q)$ in Theorem 22 to get the desired result. □

Remark 26. Corollaries 23, 24 and 25 describe the q-Fibonacci in terms of plane sections in Pascal pyramid.

Corollary 27. *From Corollary 23 it follows that*

$$\sum_{i\geq 0}\sum_{j\geq 0}\binom{n-q(i+j)}{i,\ j,\ n-(q+1)(i+j)} x^{n-(q+1)(i+j)}\, y^i\, z^j = \sum_{k\geq 0}\binom{n-qk}{k} x^{n-(q+1)k}\,(y+z)^k. \tag{5.16}$$

Proof. From Corollary 23, the RHS of (5.12) satisfies the linear recurrence (5.11). Using Corollary 1 in [9] and (5.11) we get the result. □

In particular, $(q, x, y, z) = (1, 1, y, 1-y)$ leads to the Fibonacci sequence.

5.4.1 Same sequence with different plane sections

Applying the same argument that we used to prove Corollary 23, one can establish a relation between the three Corollaries 23, 24 and 25, which allows us to express each sequence that satisfy one of them in three ways. Using different notation to distinguish the parameters of different cases. From Corollary 23 we have (5.11), from Corollary 24 (5.13), and finally, from Corollary 25 we obtain (5.15).

Corollary 28. *For $x = x' + y' = x'' + y''$ and $y + z = y' = z''$ the formulas*

$$\begin{aligned}
&\sum_{i\geq 0}\sum_{j\geq 0}\binom{n-q(i+j)}{i,\ j,\ n-(q+1)(i+j)} x^{n-(q+1)(i+j)}\, y^i\, z^j \\
&\qquad = \sum_{i\geq 0}\sum_{j\geq 0}\binom{n-qi}{i,\ j,\ n-qi-j} x'^{n-qi-j}\, y'^i\, z'^j \\
&\qquad = \sum_{i\geq 0}\sum_{j\geq 0}\binom{n-qj}{i,\ j,\ n-i-qj} x''^{n-i-qj}\, y''^i\, z''^j
\end{aligned} \tag{5.17}$$

are equivalent.

5.4.2 Some well-known sequences

In Table 5.1, we list several sequences that both their explicit formula and recurrence relation were found by using the theorem 22.

OEIS	Formula	Recurrence
A297583	$\sum_i \sum_j \binom{n-2i-3j}{i,j,n-3i-4j} 4^i 2^j$	$T_n = T_{n-1} + 4T_{n-3} + 2T_{n-4}$
A335718	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 2^{n-2i-3j} 3^i 5^j$	$T_n = 2T_{n-1} + 3T_{n-2} + 5T_{n-3}$
A304723	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 20^{n-2i-2j} (-24)^i (-51)^j$	$T_n = 20T_{n-1} - 75T_{n-2}$
A291779	$\sum_i \sum_j \binom{n-2i-3j}{i,j,n-3i-4j} 2^{n-3i-4j} 4^i (-8)^j$	$T_n = 2T_{n-1} + 4T_{n-3} - 8T_{n-4}$
A282641	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 3^i 4^j$	$T_n = T_{n-1} + 3T_{n-2} + 4T_{n-3}$
A285393	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 20^{n-2i-2j} (-14)^i (-34)^j$	$T_n = 20T_{n-1} - 48T_{n-2}$
A272931	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j}(-2)^i (-2)^j$	$T_n = T_{n-1} - 4T_{n-2}$
A258109	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 5^{n-2i-3j} (-7)^i 2^j$	$T_n = 5T_{n-1} - 7T_{n-2} + 2T_{n-3}$
A218836	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 2^{n-2i-3j} 3^i$	$T_n = 2T_{n-1} + 3T_{n-2} + T_{n-3}$
A218992	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 7^{n-2i-3j} (-5)^i (-1)^j$	$T_n = 7T_{n-1} - 5T_{n-2} - T_{n-3}$
A206727	$\sum_i \sum_j \binom{n-2i-4j}{i,j,n-3i-5j} 3^{n-3i-5j} (-2)^i$	$T_n = 3T_{n-1} - 2T_{n-3} + T_{n-5}$
A206452	$\sum_i \sum_j \binom{n-5j}{i,j,n-i-6j} 11^{n-i-6j} (-5)^i (-1)^j$	$T_n = 6T_{n-1} - T_{n-6}$
A206451	$\sum_i \sum_j \binom{n-4j}{i,j,n-i-5j} 11^{n-i-5j} (-6)^i (-1)^j$	$T_n = 5T_{n-1} - T_{n-5}$
A206450	$\sum_i \sum_j \binom{n-3j}{i,j,n-i-4j} 10^{n-i-4j} (-6)^i (-1)^j$	$T_n = 4T_{n-1} - T_{n-4}$
A200676	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 5^{n-2i-3j} (-3)^i$	$T_n = 5T_{n-1} - 3T_{n-2} + T_{n-3}$
A186244	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 5^{n-2i-3j} (-4)^i (-6)^j$	$T_n = 5T_{n-1} - 4T_{n-2} - 6T_{n-3}$
A183435	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 4^j$	$T_n = T_{n-1} + T_{n-2} + 4T_{n-3}$
A180844	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 29^{n-2i-2j} (-19)^i (-35)^j$	$T_n = 29T_{n-1} - 54T_{n-2}$
A143281	$\sum_i \sum_j \binom{n-4i-3j}{i,j,n-5i-4j} 2^{n-5i-4j} (-1)^i (-1)^j$	$T_n = 2T_{n-1} - T_{n-4} - T_{n-5}$
A123006	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 2^{n-2i-2j} 63^i 58^j$	$T_n = 2T_{n-1} + 12T_{n-2}$
A27439	$\sum_i \sum_j \binom{n-2i-j}{i,j,n-3i-2j} 4^{n-3i-2j} 3^i (-5)^j$	$T_n = 4T_{n-1} - 5T_{n-2} + 3T_{n-3}$

A16223	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 12^{n-2i-3j} (-39)^i 28^j$	$T_n = 12T_{n-1} - 39T_{n-2} + 28T_{n-3}$
A16214	$\sum_i \sum_j \binom{n-2i-j}{i,j,n-3i-2j} 12^{n-3i-2j} 24^i (-35)^j$	$T_n = 12T_{n-1} - 35T_{n-2} + 24T_{n-3}$
A16203	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 11^{n-2i-3j} (-26)^i 16^j$	$T_n = T_{n-1} - 26T_{n-2} + 16T_{n-3}$
A16174	$\sum_i \sum_j \binom{n-i}{i,j,n-2i-j} (-66)^i 16^j$	$T_n = 17T_{n-1} - 66T_{n-2}$
A16165	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 7^{n-i-2j} 9^i (-55)^j$	$T_n = 16T_{n-1} - 55T_{n-2}$
A16164	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 17^{n-i-2j} (-2)^i (-50)^j$	$T_n = 15T_{n-1} - 50T_{n-2}$
A175005	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 4^{n-2i-3j} (-3)^i 2^j$	$T_n = 4T_{n-1} - 3T_{n-2} + 2T_{n-3}$
A90399	$\sum_i \sum_j \binom{n-3j}{i,j,n-i-4j} (-2)^j$	$T_n = 2T_{n-1} - 2T_{n-4}$
A77988	$\sum_i \sum_j \binom{n-2j}{i,j,n-i-3j} 12^{n-i-3j} (-14)^i 2^j$	$T_n = -2T_{n-1} + 2T_{n-3}$
A52942	$\sum_i \sum_j \binom{n-3j}{i,j,n-i-4j} 10^{n-i-4j} (-9)^i 2^j$	$T_n = T_{n-1} + 2T_{n-4}$
A5708	$\sum_i \sum_j \binom{n-5j}{i,j,n-i-6j} 2^{n-i-6j} (-1)^i$	$T_n = T_{n-1} + T_{n-6}$
A98589	$\sum_i \sum_j \binom{n-2j}{i,j,n-i-3j} 10^{n-i-3j} (-7)^i 2^j$	$T_n = 3T_{n-1} + 2T_{n-3}$
A264570	$\sum_i \sum_j \binom{n-3j}{i,j,n-i-4j} (-10)^{n-i-4j} 12^i 8^j$	$T_n = 2T_{n-1} + 8T_{n-4}$
A317509	$\sum_i \sum_j \binom{n-4j}{i,j,n-i-5j} (-2)^i 2^j$	$T_n = -T_{n-1} + 2T_{n-5}$
A99211	$\sum_i \sum_j \binom{n-2j}{i,j,n-i-3j} 10^{n-i-3j} (-12)^i 4^j$	$T_n = -2T_{n-1} + 4T_{n-3}$
A77940	$\sum_i \sum_j \binom{n-2j}{i,j,n-i-3j} 10^{n-i-3j} (-8)^i (-2)^j$	$T_n = 2T_{n-1} - 2T_{n-3}$
A45883	$\sum_i \sum_j \binom{n-2j}{i,j,n-i-3j} (-10)^{n-i-3j} 13^i (-4)^j$	$T_n = 3T_{n-1} - 4T_{n-3}$

Table 5.1: Sequences that we provide both recurrence relation and explicit formula.

In Table 5.2, we give sequences that their recurrence relation were known, and we provide an explicit formula for them via the main theorem 22.

OEIS	Formula	Recurrence
A77878	$\sum_i \sum_j \binom{n-3i-j}{i,j,n-4i-2j} 2^{n-4i-2j}\, 2^i\, (-3)^j$	$T_n = 2T_{n-1} - 3T_{n-2} + 2T_{n-4}$
A77872	$\sum_i \sum_j \binom{n-i-3j}{i,j,n-2i-4j} 2^{n-2i-4j}\, (-2)^i$	$T_n = 2T_{n-1} - 2T_{n-2} + T_{n-4}$
A77867	$\sum_i \sum_j \binom{n-2i-3j}{i,j,n-3i-4j} 2^{n-3i-4j}\, (-3)^i\, 2^j$	$T_n = 2T_{n-1} - 3T_{n-3} + 2T_{n-4}$
A77849	$\sum_i \sum_j \binom{n-i-3j}{i,j,n-2i-4j} 3^{n-2i-4j}\, (-1)^i\, (-1)^j$	$T_n = 3T_{n-1} - T_{n-2} - T_{n-4}$
A77831	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 3^{n-2i-3j}\, 2^i\, 2^j$	$T_n = 3T_{n-1} + 2T_{n-2} + 2T_{n-3}$
A52992	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 4^i\, (-4)^j$	$T_n = T_{n-1} + 4T_{n-2} - 4T_{n-3}$
A35344	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 5^{n-2i-3j}\, (-6)^i\, 2^j$	$T_n = 5T_{n-1} - 6T_{n-2} + 2T_{n-3}$
A249993	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 10^i\, 8^j$	$T_n = T_{n-1} + 10T_{n-2} + 8T_{n-3}$
A248088	$\sum_i \sum_j \binom{n-3i-3j}{i,j,n-4i-4j} 4^{n-4i-4j}\, (-11)^i\, (-16)^j$	$T_n = 4T_{n-1} - 27T_{n-4}$
A218987	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 5^{n-2i-3j}\, (-1)^i\, (-3)^j$	$T_n = 5T_{n-1} - T_{n-2} - 3T_{n-3}$
A200781	$\sum_i \sum_j \binom{n-2i-3j}{i,j,n-3i-4j} 5^{n-3i-4j}\, (-10)^i\, 5^j$	$T_n = 5T_{n-1} - 10T_{n-3} + 5T_{n-4}$
A164393	$\sum_i \sum_j \binom{n-4i-3j}{i,j,n-5i-4j} 2^{n-5i-4j}\, (-2)^j$	$T_n = 2T_{n-1} - 2T_{n-4} + T_{n-5}$
A137500	$\sum_i \sum_j \binom{n-i-2j}{i,j,n-2i-3j} 5^{n-2i-3j}\, (-8)^i\, 6^j$	$T_n = 5T_{n-1} - 8T_{n-2} + 6T_{n-3}$
A135248	$\sum_i \sum_j \binom{n-i-3j}{i,j,n-2i-4j} 4^{n-2i-4j}\, (-4)^i\, 2^j$	$T_n = 4T_{n-1} - 4T_{n-2} + 2T_{n-4}$
A123888	$\sum_i \sum_j \binom{n-2i-4j}{i,j,n-3i-5j} 3^{n-3i-5j}\, (-3)^i$	$T_n = 3T_{n-1} - 3T_{n-3} + T_{n-5}$
A122439	$\sum_i \sum_j \binom{n-i-3j}{i,j,n-2i-4j} 2^{n-2i-4j}\, (-4)^j$	$T_n = 2T_{n-1} + T_{n-2} - 4T_{n-4}$
A106511	$\sum_i \sum_j \binom{n-i-3j}{i,j,n-2i-4j} (-2)^{n-2i-4j}\, (-1)^i$	$T_n = -2T_{n-1} - T_{n-2} + T_{n-4}$
A99526	$\sum_i \sum_j \binom{n-3j}{i,j,n-i-4j} 10^{n-i-4j}\, (-8)^i\, 3^j$	$T_n = 2T_{n-1} + 3T_{n-4}$
A97831	$\sum_i \sum_j \binom{n-2i-j}{i,j,n-3i-2j} 18^{n-3i-2j}\, (-18)^j$	$T_n = 18T_{n-1} - 18T_{n-2} + T_{n-3}$
A96979	$\sum_i \sum_j \binom{n-2i-3j}{i,j,n-3i-4j} 6^{n-3i-4j}\, (-6)^i$	$T_n = 6T_{n-1} - 6T_{n-3} + T_{n-4}$

A77989	$\sum_i \sum_j \binom{n-2i-j}{i,j,n-3i-2j} (-2)^{n-3i-2j}\, 2^i\, (-1)^j$	$T_n = -2T_{n-1} - T_{n-2} + 2T_{n-3}$
A77895	$\sum_i \sum_j \binom{n-i-3j}{i,j,n-2i-4j} (-2)^i\, 2^j$	$T_n = T_{n-1} - 2T_{n-2} + 2T_{n-4}$
A59633	$\sum_i \sum_j \binom{n-2i-3j}{i,j,n-3i-4j} 2^{n-3i-4j}\, (-1)^i$	$T_n = 2T_{n-1} - T_{n-3} + T_{n-4}$
A33139	$\sum_i \sum_j \binom{n-2i-3j}{i,j,n-3i-4j} 3^{n-3i-4j}\, (-3)^j$	$T_n = 3T_{n-1} + T_{n-3} - 3T_{n-4}$
A23435	$\sum_i \sum_j \binom{n-4i-j}{i,j,n-5i-2j} (-1)^i$	$T_n = T_{n-1} + T_{n-2} - T_{n-5}$
A15523	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 2^i\, 5^j$	$T_n = 3T_{n-1} + 5T_{n-2}$
A10904	$\sum_i \sum_j \binom{n-2i-j}{i,j,n-3i-2j} 4^{n-3i-2j}\, (-2)^j$	$T_n = 4T_{n-1} - 2T_{n-2} + T_{n-3}$

Table 5.2: Sequences that we provide an explicit formula.

In Table 5.3, we show the application of the Corollary 27, which allow us to provide a different explicit formula for the sequences.

OEIS	Formula 1	Formula 2
A65874	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} (-11)^{n-i-2j}\, 12^i\, 42^j$	$\sum_k \binom{n}{k} 42^k$
A53540	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 18^{n-2i-2j}\, (-34)^i\, (-47)^j$	$\sum_k \binom{n-k}{k} 18^{n-2k}\, (-81)^k$
A27473	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 13^i\, (-49)^j$	$\sum_k \binom{n}{k} 14^{n-k}\, (-49)^k$
A168579	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 10^{n-i-2j}\, (-9)^i\, 16^j$	$\sum_k \binom{n}{k} 16^k$
A162670	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 34^i\, 66^j$	$\sum_k \binom{n-k}{k} 100^k$
A161007	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} (-10)^{n-i-2j}\, 12^i\, 16^j$	$\sum_k \binom{n}{k} 2^{n-k}\, 16^k$
A155458	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 10^{n-i-2j}\, (-4)^i\, 25^j$	$\sum_k \binom{n}{k} 6^{n-k}\, 25^k$
A153600	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 18^{n-2i-2j}\, (-10)^i\, (-68)^j$	$\sum_k \binom{n-k}{k} 18^{n-2k}\, (-78)^k$
A145978	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} (-10)^{n-i-2j}\, 11^i\, (-8)^j$	$\sum_k \binom{n}{k} (-8)^k$
A138395	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 10^{n-i-2j}\, (-4)^i\, (-3)^j$	$\sum_k \binom{n}{k} 6^{n-k}\, (-3)^k$

A125905	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 12^{n-i-2j} (-16)^i (-1)^j$	$\sum_k \binom{n}{k} (-4)^{n-k} (-1)^k$
A99139	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 12^{n-2i-2j} 62^i 46^j$	$\sum_k \binom{n-k}{k} 12^{n-2k} 108^k$
A87584	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 9^{n-2i-2j} 10^i 31^j$	$\sum_k \binom{n-k}{k} 9^{n-2k} 41^k$
A75921	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 21^{n-2i-2j} (-69)^i (-29)^j$	$\sum_k \binom{n-k}{k} 21^{n-2k} (-98)^k$
A41545	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 33^i$	$\sum_k \binom{n}{k} 34^{n-k}$
A16190	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 20^{n-2i-2j} (-34)^i (-65)^j$	$\sum_k \binom{n-k}{k} 20^{n-2k} (-99)^k$
A16173	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} (-10)^{n-i-2j} 26^i (-60)^j$	$\sum_k \binom{n}{k} 16^{n-k} (-60)^k$
A15583	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 10^{n-i-2j} (-1)^i 7^j$	$\sum_k \binom{n}{k} 9^{n-k} 7^k$
A154250	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 18^{n-2i-2j} (-36)^i (-38)^j$	$\sum_k \binom{n-k}{k} 18^{n-2k} (-74)^k$
A154240	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 16^{n-2i-2j} (-17)^i (-41)^j$	$\sum_k \binom{n-k}{k} 16^{n-2k} (-58)^k$
A153598	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 14^{n-2i-2j} (-11)^i (-35)^j$	$\sum_k \binom{n-k}{k} 14^{n-2k} (-46)^k$
A154244	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 10^{n-i-2j} (-4)^i (-2)^j$	$\sum_k \binom{n}{k} 6^{n-k} (-2)^k$
A214733	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} (-2)^i (-3)^j$	$\sum_k \binom{n}{k} (-1)^{n-k} (-3)^k$
A268413	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 10^{n-i-2j} (-23)^i 14^j$	$\sum_k \binom{n}{k} (-13)^{n-k} 14^k$
A246645	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 22^{n-2i-2j} (-43)^i (-38)^j$	$\sum_k \binom{n-k}{k} 22^{n-2k} (-81)^k$
A191014	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 11^{n-i-2j} (-1)^i 2^j$	$\sum_k \binom{n}{k} 10^{n-k} 2^k$
A91903	$\sum_i \sum_j \binom{n-j}{i,j,n-i-2j} 14^{n-i-2j} (-9)^i 50^j$	$\sum_k \binom{n}{k} 5^{n-k} 50^k$
A16183	$\sum_i \sum_j \binom{n-i-j}{i,j,n-2i-2j} 18^{n-2i-2j} (-37)^i (-40)^j$	$\sum_k \binom{n-k}{k} 18^{n-2k} (-77)^k$
A118005	$\sum_i \sum_j \binom{n-i}{i,j,n-2i-j} 23^{n-2i-j} 45^i (-19)^j$	$\sum_k \binom{n-k}{k} 4^{n-2k} 45^k$
A93144	$\sum_i \sum_j \binom{n-i}{i,j,n-2i-j} 45^{n-2i-j} (-50)^i (-25)^j$	$\sum_k \binom{n-k}{k} 20^{n-2k} (-50)^k$

Table 5.3: Sequences that we provide two explicit formulas.

5.5 Combinatorial Interpretation

In this section, we present the combinatorial interpretation for all the sequences obtained by the sum of elements of plane sections in Pascal pyramid. We also give some examples and their illustrations.

Theorem 29. *T_n counts the number of tilings of $1 \times n$ rectangle, using 1×1, $1 \times (q_1 + 1)$ and $1 \times (q_2 + 1)$ -ominoes with intensity[1] x, y, z, respectively.*

Proof. Let i be the number of $q1 + 1$-ominoes with intensity y. It is clear that $0 \leq i \leq \lfloor n/(q_1 + 1) \rfloor$. Let j be the number of $(q_2 + 1)$-ominoes with intensity z, where $0 \leq j \leq \lfloor (n - i)/(q_2 + 1) \rfloor$, and then we complete the tiling with monominoes with intensity x. Thus we have $n - (q_1 + 1)i - (q_2 + 1)j$ (see Figure 5.6) monominoes. The number of monominoes, $(q_1 + 1)$-ominoes and $(q_2 + 1)$-ominoes used is $i + j + n - (q_1 + 1)i - (q_2 + 1)j = n - q_1 i - q_2 j$. Considering all the permutations possible between the monominoes, $(q_1 + 1)$-ominoes and $(q_2 + 1)$-ominoes to create different tilings, which is given by $(n - q_1 i - q_2 j)!$, now subtracting the permutations of the same kind of $(1, 1)$ and $(1, q_1 + 1)$, $(1, q_2 + 1)$, which gives $(n - (q_1 + 1)i - (q_2 + 1)j)!$, $i!$ and $j!$, respectively. Now we sum over all possible combinations of i and j, we have

$$\sum_{i=0}^{\lfloor n/(q_1+1) \rfloor} \sum_{j=0}^{\lfloor (n-i)/(q_2+1) \rfloor} \binom{n - q_1 i - q_2 j}{i,\ j,\ n - (q_1 + 1)i - (q_2 + 1)j} x^{n-(q_1+1)i-(q_2+1)j}\, y^i\, z^j$$

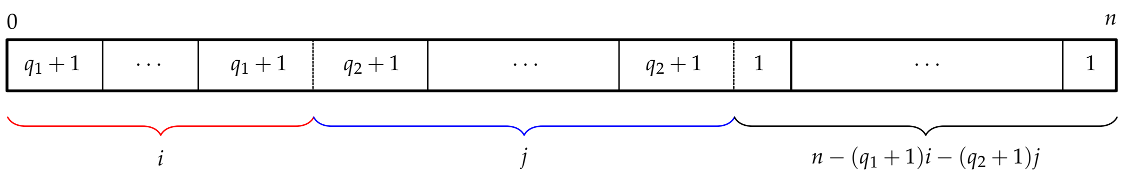


Figure 5.6: Tiling using monominoes,$q_1 + 1$-ominoes and $q_2 + 1$-ominoes.

□

[1]Color intensity refers to how bright or dark a color looks.

5.5.1 Examples

In these examples, we illustrate the combinatorial interpretation of three sequences A000073, A335718 and A206727.

Example 4. *The Tribonacci sequence A000073, (see [40, 76]) counts the number of tilings of* $1 \times n$ *rectangle, using* 1×1, 1×2, *and* 1×3 *rectangles with intensity* $1, 1, 1$, *respectively (see Fig. 5.7).*

$$T_{n+2} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor n/3 \rfloor} \binom{n-i-2j}{i,\ j,\ n-2i-3j},$$

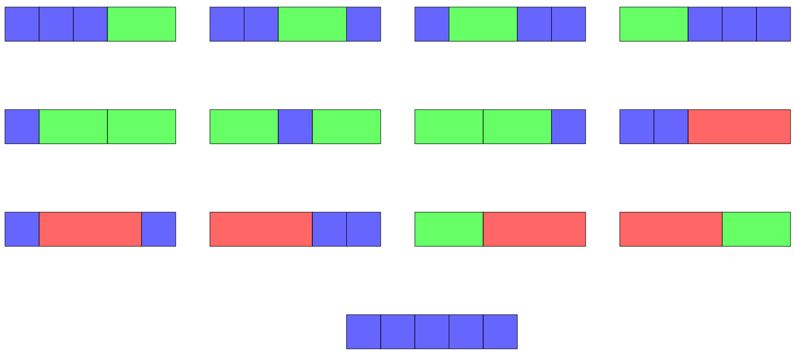


Figure 5.7: Example of tiling $T_5 = 13$ of tribonacci sequence.

Example 5. *The sequence A206727 (see [78]) counts the number of tilings of* $1 \times n$ *rectangle, using* 1×1, 1×3, *and* 1×5 *rectangles with intensity* $3, -2, 1$, *respectively (see Fig. 5.8).*

$$F_n = \sum_{i \geq 0} \sum_{j \geq 0} \binom{n-2i-4j}{i, j, n-3i-5j} 3^{n-3i-5j} (-2)^i,$$

and it satisfies

$$F_n = 3F_{n-1} - 2F_{n-3} + F_{n-5}.$$

The sixth term of the sequence is calculated by

$$\begin{aligned} & 3 \times 3 \times 3 \times (-2) \ + 3 \times 3 \times (-2) \times 3 \ + 3 \times (-2) \times 3 \times 3 \\ & + (-2) \times 3 \times 3 \times 3 \ + (-2) \times (-2) \ + 3 \times 1 \ + 1 \times 3 \\ & + 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 1440 \end{aligned}$$

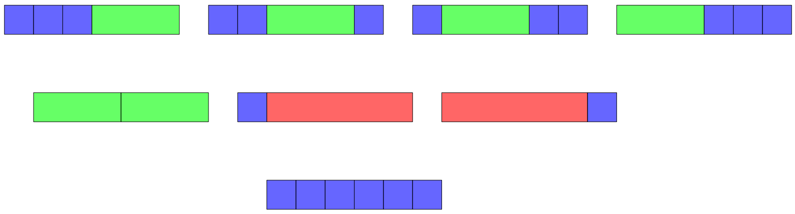


Figure 5.8: Example of tiling $F_6 = 1440$.

Conclusion and perspectives

Along this thesis, we explored the linear recurrence relation associated with the sum of elements lying over the parallel diagonals of the square binomial triangle, we gave also the corresponding generating function. By using a phenomemon called Morgen-Voyce we derived many combinatorial identities. After that, we defined the concept of directions in the Pascal pyramid, and established a linear recurrence relation which characterizes the sequences obtained from the sums of elements of the main diagonal in the pyramid, as well as its generating function. We studied also the phenomemon of Morgan-Voyce in case of Pascal pyramid, which allowed us to prove and discover several sequences known in the Online Encyclopedia of Integer Sequences OEIS, we give also a combinatorial interpretation for all the sequences satisfying the main recurrence. In the last chapter, we dealt with plane sections in Pascal pyramid, we gave a definition in a way that preserves the symmetry of the trinomial coefficients, we established a linear recurrence relation for the sequences obtained from the sums of the elements of the main plane sections, we gave also many combinatorial identities, one of them relate some particular cases of plane sections in Pascal pyramid with the directions in Pascal triangle, which answers the question that we asked in the introduction as follows, the natural generalization of directions in Pascal triangle is the plane sections in Pascal pyramid. As an example, we showed that we can get the Tribonacci sequence using plane sections which is the natural generalization of the Fibonacci sequence found in the Pascal triangle. A combinatorial interpretation for the sequences was also provided.

Perspectives

Some questions and open problems are of interest:
We have established the recurrence relation for the sums of elements lying over the main parallel diagonals in square binomial triangle, but still to find the gen-

eral recurrence relation for any given directions, which its explicit formula is given by

$$Q_n = \sum_{k}^{\lfloor n/(q+r) \rfloor} \binom{n-qk}{p+rk}^2 x^{n-(q+r)k}\, y^{p+rk}.$$

More general, is it possible to established a recurrence that characterizes the sequences of any given power of binomial coefficients, for $d \in \mathbb{N}$

$$O_n = \sum_{k}^{\lfloor n/(q+r) \rfloor} \binom{n-qk}{p+rk}^d x^{n-(q+r)k}\, y^{p+rk}.$$

On the other hand, we have established a recurrence relation for the sequences of the sum of elements lying along finite rays in Pascal pyramid. In [21], we studied another case of directions in Pascal pyramid, where we showed that the directions $(1, 2, r)$ which is given by

$$C_n = \sum_{k=0}^{\lfloor n/(r+3) \rfloor} \binom{n-rk}{k, 2k, n-(r+3)k} x^k y^{2k} z^{n-(r+2)k},$$

satisfy the linear recurrence relation

$$\begin{aligned}
&2n(2n-(r+3))(3n-(r+6))C_n \\
&-z\Big(36n^3-(30r+144)n^2+2(3r^2+35r+87)n-4(r^2+8r+15)\Big)C_{n-1} \\
&+z^2\Big(36n^3-(30r+162)n^2+2(3r^2+43r+117)n-4(2r^2+15r+27)\Big)C_{n-2} \\
&-z^3\Big(2(n-2)(2n-(r+4))(3n-(r+3))\Big)C_{n-3} \\
&=3xy^2(3n-(r+6))(3n-(r+3))(3n-2(r+3))C_{n-r-3} \qquad (5.18)
\end{aligned}$$

As we can see in (5.18) the small change of the parameters gives a very complicated recurrence relation.

The general recurrence relation still a challenging question as well as the generating function, the characterization of theses sequences is given by

$$P_n = \sum_{k=0}^{\lfloor \frac{n-\theta_1-\theta_2}{\alpha_1+\alpha_2+r} \rfloor} \binom{n-rk}{\theta_1+\alpha_1 k, \theta_2+\alpha_2 k, n-\theta_1-\theta_2-(\alpha_1+\alpha_2+r)k} x^{\theta_1+\alpha_1 k} y^{\theta_2+\alpha_2 k} z^{n-\theta_1-\theta_2-(\alpha_1+\alpha_2+r)k},$$

with x, y and z are nonzero real parameters. More general, what is the recurrence relation for the sequences obtained by the sums of elements lying over finite rays in Pascal hyperpyramids for any given dimension,

$$D_n = \sum_{k=0}^{\lfloor \frac{n-\theta_1-\cdots-\theta_s}{\alpha_1+\cdots+\alpha_s+r} \rfloor} \binom{n-rk}{\theta_1+\alpha_1 k, \ldots, \theta_s+\alpha_s k, n-\theta_1-\cdots-\theta_2-(\alpha_1+\cdots+\alpha_s+r)k} x_1^{\theta_1+\alpha_1 k} \cdots x_s^{n-\theta_1-\cdots-\theta_s-(\alpha_1+\cdots+\alpha_s+r)k}.$$

Another challenging question is to established the recurrence relation for the generalized plane sections in Pascal pyramid that are given by

$$A_n = \sum_{i\geq 0}\sum_{j\geq 0} \binom{n-q_1 i-q_2 j}{r_1 i,\ r_2 j,\ n-(q_1+r_1)i-(q_2+r_2)j} x^{n-(q_1+r_1)i-(q_2+r_2)j}\, y^{r_1 i}\, z^{r_2 j},$$

which can also generalized to other dimension of Pascal pyramid as follows

$$B_n = \sum_{i_1\geq 0} \cdots \sum_{i_s\geq 0} \binom{n-q_1 i_1-\cdots-q_s i_s}{r_1 i_1, \ldots,\ n-(q_1+r_1)i_1-\cdots-(q_s+r_s)i_s} x_1^{n-(q_1+r_1)i_1-\cdots-(q_s+r_s)i_s} \cdots x_s^{r_s i_s}.$$

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