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Par : **MASSOUN Youssouf**

Sujet

The homotopy analysis method for nonlinear differential equations

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M. Moulay Mohamed Said	Professeur à USTHB	Président
M. Benzine Rachid	Professeur à EPST Annaba	Directeur de thèse
M. Benmezai Abdelhamid	Professeur à USTHB	Co-directeur de thèse
M. Kessi Arezki	Professeur à USTHB	Examinateur
M. Talbi Mohamed elamine	MCA à université de Blida	Examinateur
M. Benayat Djilali	Professeur à ENS Kouba	Invité

بسم الله الرحمن الرحيم

رب صل على روح الوجود

الآية الكبرى مرآة الشهود

والآل والأصحاب بدور الكمال

والأئمة الكرام خواص البشر

A sidna chikh

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Contents

Chapter 1

Introduction

The homotopy analysis method is one of the important numerical methods used for solving all types of differential equations such as ordinary differential equations, partial differential equations, fractional differential equations, delay differential equations, stochastic equations, and along with others, this method namely HAM is created by Shijun Liao in his PhD in 1992, hwo was a mathematician at Shanghai university, he created the general rules of using this numerical technique in his books [56] both theorems and application of HAM for different type of differential equations. Many scientists used HAM for various differential equations and tried to improve it depending on the problems confronted.
The fractional differential equation is one of most equations modeled various phenomena in medicine, physics, chemistry, epidemics models, populations dynamics, SIR model, and many other areas. The scientists try to find the approximate solutions for this type of equations using different numerical methods such as the Adomian decomposition method, variational iteration method, collocation method and the homotopy analysis method. But it's difficult to find the approximate solution in majority of the time especially for fractional derivative.
In this work, gives the first chapter a preliminary about the fractional calculus including the Riemann Liouville fractional derivative and integral, Caputo fractional derivative, the generalized fractional derivative in the sense of Caputo and the Laplace operator for the fractional derivative. The second chapter is a brief description of the homotopy analysis method, giving the most important definitions and theorems briefly in the frame of theoretical side. In chapter three we present the application of the HAM and HATM

for the fractional pine wilt diseases model, the time-fractional Forenberg-Whitham equation, and fractional Korteweg-De Vries equation. The fourth chapter is an application of HAM for a generalized fractional Korteweg-De Vries equation. At last, the following work was determined with some perspective of research.

Chapter 2

Preliminaries

2.1 The fractional calculus

This section provides some definitions and properties of the fractional calculus

Definition 2.1.1 *[26] A real function $h(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p > l$, such that $h(t) = t^p h_1(t)$, where $h_1(t) \in C(0, \infty)$, and it is said to be in the space C_μ^n if and only if $h^n \in C_\mu, n \in \mathbb{N}$.*

2.1.1 The Riemann-Liouville fractional integrals

Definition 2.1.2 *[26] The Riemann-Liouville fractional integral operator J^α of order $\alpha \geq 0$ of a function $h \in C_\mu, \mu \geq -1$ is defined as*

$$\begin{aligned} J^\alpha h(t) &= \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) d\tau \quad (\alpha > 0) \\ J^0 h(t) &= h(t) \end{aligned}$$

$\Gamma(.)$ is the well-known Gamma function.

Some of the properties of the operator J^α, which are needed here are as follows:
For $h \in C_\mu, \mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma \geq -1$[26]

$$\begin{aligned} J^\alpha J^\beta h(t) &= J^{\alpha+\beta} h(t) \\ J^\alpha J^\beta h(t) &= J^\beta J^\alpha h(t) \\ J^\alpha t^\gamma &= \tfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma} \end{aligned}$$

2.1.2 The Riemann-Liouville fractional derivative

Definition 2.1.3 *[26] The Riemann-Liouville fractional derivative of a given function h is*

$$^{RL}D_{a^+}^{\alpha}h(t) = D^m J_{a^+}^{m-\alpha}h(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^t (t-s)^{m-\alpha-1}h(s)ds, \quad t > a,$$

2.1.3 The Caputo fractional derivative

Definition 2.1.4 *[26] The fractional derivative D^{α} of $h(t)$ in the Caputo's sense is defined as*

$$^{C}D^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}h^{(n)}(\tau)d\tau$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$, $h \in C_{-1}^n$.

The following are basic properties of the Caputo's fractional derivative:

1. Let $h \in C_{-1}^n$, $n \in \mathbb{N}$. Then $D^{\alpha}h, 0 < \alpha \leq 1$ is well defined and $D^{\alpha}h \in C_{-1}$

2. Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$ and $h \in C_{\mu}^n, \mu \geq -1$, then

$$(J^{\alpha}D^{\alpha})h(t) = h(t) + \sum_{k=0}^{n-1} h^{(k)}(0^+)\frac{t^k}{k!}$$

3. For $h(t) = t^{\beta}$ gets us

$$D^{\alpha}x^n = \begin{cases} 0 & for n \in \mathbb{N}, and n < [\alpha] \\ \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha} & for n \in \mathbb{N}, and n \geq [\alpha], or n \notin \mathbb{N} \end{cases}$$

.

2.2 The Laplace operator transform

Definition 2.2.1 *Let f a function defined on the interval $[0, +\infty[$, the Laplace transform of $f(t)$ designated by $L\{f\}(s)$ or $F(s)$ is*

$$L\{f\}(s) = \int_0^{\infty} e^{-st}f(t)dt$$

2.2.1 Properties of the Laplace Transform

Theorem 2.2.2 *Linearity*
The Laplace and the inverse Laplace operator are linear operator

$$L\{\alpha f + \beta g\}(s) = \alpha L\{f\}(s) + \beta L\{g\}(s)$$
$$L^{-1}[\alpha L\{f\}(s) + \beta L\{g\}(s)] = \alpha f + \beta g$$

where α *and* $\beta \in \mathbb{R}$

Example 2.2.3 *Consider the functions*

$$f(t) = c, \quad g(t) = t^n$$

we have

$$L\{f\}(s) = \int_0^\infty ce^{-st}dt = c\int_0^\infty e^{-st}dt = c\left[\frac{-1}{s}e^{-st}\right]_0^\infty = \frac{c}{s}$$

and

$$L\{g\}(s) = \int_0^\infty t^n e^{-st}dt = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x}\frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty x^n e^{-x}dx = \frac{\Gamma(n+1)}{s^{n+1}}$$

so

$$L\{f+g\}(s) = \frac{c}{s} + \frac{n!}{s^{n+1}}$$

Proposition 2.2.4 *Laplace transform of derivative*
The Laplace transform operator for the $n-th$ *order derivative* $(n = 1, 2, ...)$ *of* $f(t)$ *is*

$$L\{f^{(n)}(s)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$$

Example 2.2.5 *The Laplace transform operator for cases* $n = 2$*, and* $n = 3$ *is*

$$L\{f^{(2)}(s)\} = s^2F(s) - sf(0) - sf'(0)$$
$$L\{f^{(3)}(s)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

Proposition 2.2.6 *Laplace transform of Caputo fractional derivative*
The Laplace transform operator for the fractional derivative ${}^CD^\alpha f(t)$ *where* $n-1 < \alpha \leq n$ *is*

$$L\{{}^CD^\alpha f(t)(s)\} = s^nF(s) - \sum_{k=0}^{n-1} s^{n-k-1}f^{(k)}(0^+)$$

Chapter 3

The homotopy analysis method

This chapter presents a brief description of the homotopy analysis method (HAM), some definitions and theorems, as well as an illustration of the application of HAM into a simple problem.
Considering :

$$N[u(x,t)] = f(x,t) \tag{3.0.1}$$

where N is the nonlinear operator, x is the spatial variable, t is the temporal variable.

3.1 Properties of the homotopy derivative

Initially, mentioning that in the following part, the function ϕ is written as a convergent series

$$u(x,t) = u_0(x,t) + \sum_{i=1}^{\infty} u_i(x,t) \tag{3.1.1}$$

Definition 3.1.1 *[55] Let ϕ be a function of the homotopy parameter q, then the m-th order homotopy derivative of ϕ is defined as*

$$D_m(\phi) = \frac{1}{m!}\frac{\partial^m \phi}{\partial q^m}|_{q=0} \tag{3.1.2}$$

where $m \geq 0$ is an integer.

Theorem 3.1.2 *[55] Considering f and g two functions independent of the homotopy parameter q, for the homotopy series*

$$\phi = \sum_{k=0}^{\infty} u_k(x,t)q^k \qquad \psi = \sum_{k=0}^{\infty} v_k(x,t)q^k$$

we have

$$D_m\left[f\phi + g\psi\right] = fD_m[\phi] + gD_m[\psi] = fu_m + gv_m \tag{3.1.3}$$

Proof 1 *Consider f and g two functions independent of q, then*

$$\begin{aligned} D_m\left[f\phi + g\psi\right] &= \tfrac{1}{m!}\tfrac{\partial^m(f\phi+g\psi)}{\partial q^m}|_{q=0} \\ &= f\tfrac{1}{m!}\tfrac{\partial^m\phi}{\partial q^m}|_{q=0} + g\tfrac{1}{m!}\tfrac{\partial^m\psi}{\partial q^m}|_{q=0} \\ &= fD_m(\phi) + gD_m(\psi) = fu_m + gv_m \end{aligned}$$

Theorem 3.1.3 *[55] For two series*

$$\phi = \sum_{k=0}^{\infty} u_k(x,t)q^k \qquad \psi = \sum_{k=0}^{\infty} v_k(x,t)q^k$$

we have

$$D_m(\phi) = u_m(x,t) \tag{3.1.4}$$

$$D_m(q^k\phi) = D_{m-k}(\phi) \tag{3.1.5}$$

$$\begin{aligned} D_m(\phi\psi) &= \textstyle\sum_{k=0}^{m} D_k(\phi)D_{m-k}(\psi) &&= \textstyle\sum_{k=0}^{m} u_k v_{m-k} \\ &= \textstyle\sum_{k=0}^{m} D_k(\psi)D_{m-k}(\phi) &&= \textstyle\sum_{k=0}^{m} v_k u_{m-k} \end{aligned} \tag{3.1.6}$$

$$D_m(\phi^n\psi^l) = \sum_{k=0}^{m} D_k(\phi^n)D_{m-k}(\psi^l) = \sum_{k=0}^{m} D_k(\psi^l)D_{m-k}(\phi^n) \tag{3.1.7}$$

where $m \geq 0$, and $0 \leq k \leq m$

Proof 2 *For the purpose of simplification, consider $u(x,t) = u$, let*

$$\phi = \sum_{k=0}^{\infty} u_k q^k \qquad \psi = \sum_{k=0}^{\infty} v_k q^k$$

- *By Maclaurin series theorem u_k is unique in $\sum_{k=0}^{\infty} u_k q^k$, and is evaluated by*
$$\frac{1}{m!}\frac{\partial^m \phi}{\partial q^m}|_{q=0}$$
which is $D_m(\phi)$

- *Let*
$$q^k\phi = q^k\sum_{i=0}^{\infty} u_i q^i = \sum_{i=0}^{\infty} u_i q^{i+k} = \sum_{m=k}^{\infty} u_{m-k}q^m$$
by taking $i+k=m \Rightarrow i=m-k$, so if $i=0$ then $m=k$, and from the first properties in (3.1.4) $D_m(q^k\phi) = u_{m-k}$

- *According to Leibnitzs rule for derivatives of product,*
$$\frac{\partial^m(\phi\psi)}{\partial q^m} = \sum_{k=0}^{m}\frac{m!}{k!(m-k)!}\frac{\partial^k\phi}{\partial q^k}\frac{\partial^{m-k}\psi}{\partial q^{m-k}} = \sum_{k=0}^{m}\frac{m!}{k!(m-k)!}\frac{\partial^k\psi}{\partial q^k}\frac{\partial^{m-k}\phi}{\partial q^{m-k}}$$
using (3.1.1) and (3.1.4) ends in:
$$D_m(\phi\psi) = \tfrac{1}{m!}\tfrac{\partial^m(\phi\psi)}{\partial q^m}\ |_{q=0} = \textstyle\sum_{k=0}^{m}\left(\tfrac{1}{k!}\tfrac{\partial^k\phi}{\partial q^m}|_{q=0}\right)\left(\tfrac{1}{(m-k)!}\tfrac{\partial^{m-k}\psi}{\partial q^{m-k}}|_{q=0}\right)$$
$$= \textstyle\sum_{k=0}^{m} D_k(\phi)D_{m-k}(\psi) = \sum_{k=0}^{m} u_k v_{m-k}$$
Similar proof for
$$D_m(\phi\psi) = \sum_{k=0}^{m} D_k(\psi)D_{m-k}(\phi) = \sum_{k=0}^{m} v_k u_{m-k}$$

- *Write $\Phi = \phi^n$, and $\Psi = \psi^n$, by (3.1.6)*
$$D_m(\phi^n\psi^l) = D_m(\Phi\Psi) = \sum_{k=0}^{m} D_k(\Phi)D_{m-k}(\Psi) = \sum_{k=0}^{m} D_k(\phi^n)D_{m-k}(\psi^l)$$

Theorem 3.1.4 *[55] Consider L a linear operator independent of the homotopy parameter q, for the homotopy Maclaurin series*
$$\phi = \sum_{k=0}^{\infty} u_k(x,t)q^k \qquad \psi = \sum_{k=0}^{\infty} v_k(x,t)q^k$$

where ϕ, and ψ are analytic in $q \in [0,1]$, then

$$D_m[L(\phi)] = LD_m(\phi) = L(u_m) \tag{3.1.8}$$

and

$$D_m[\psi L(\phi)] = \sum_{k=0}^{m} D_k\psi D_{m-k}L(\phi) \tag{3.1.9}$$

where m is a integer.

Proof 3 *Since L is a linear operator independent of the parameter q, using theorem 3.1.3*

$$D_m[L(\phi)] = \frac{1}{m!}\frac{\partial^m L(\phi)}{\partial q^m}|_{q=0} = L(\frac{1}{m!}\frac{\partial^m \phi}{\partial q^m}|_{q=0}) = L(u_m)$$

and

$$D_m[\psi L(\phi)] = \sum_{k=0}^{m} D_k\psi D_{m-k}L(\phi) = \sum_{k=0}^{m} D_k\psi L(D_{m-k}\phi) = \sum_{k=0}^{m} v_k L(u_{m-k})$$

Theorem 3.1.5 *[55] For an arbitrary homotopy Maclaurin series*

$$\phi = \sum_{k=0}^{\infty} u_k(x,t)q^k$$

then

$$\begin{aligned}
&1.D_m(\phi^2) = \textstyle\sum_{k=0}^{m} u_k u_{m-k}\\
&2.D_m(\phi^3) = \textstyle\sum_{k=0}^{m} u_{m-k}\sum_{j=0}^{k} u_j u_{k-j}\\
&3.D_m(\phi^4) = \textstyle\sum_{k=0}^{m} u_{m-k}\sum_{j=0}^{k} u_{k-j}\sum_{i=0}^{j} u_i u_{j-i}\\
&4.D_m(\phi^\sigma) = \textstyle\sum_{r_1=0}^{m} u_{m-r_1}\sum_{r_2=0}^{r_1} u_{r_1-r_2}\sum_{r_3=0}^{r_2} u_{r_2-r_3}\ldots\sum_{r_{\sigma_1-1}=0}^{r_{\sigma-2}} u_{r_{\sigma-2}-r_{\sigma-1}}u_{r_{\sigma-1}}
\end{aligned} \tag{3.1.10}$$

Proof 4

- According to (3.1.6), having

$$D_m(\phi^2) = \sum_{k=0}^{m} D_k(\phi)D_{m-k}(\phi) = \sum_{k=0}^{m} u_k u_{m-k}$$

- According to (3.1.6), and using 1. in (3.1.10)

$$D_m(\phi^3) = D_m(\phi\phi^2) = \sum_{k=0}^{m} D_k(\phi^2)D_{m-k}(\phi) = \sum_{k=0}^{m} u_{m-k}\sum_{j=0}^{k} u_j u_{k-j}$$

- According to (3.1.6), and using 2. in (3.1.10)

$$D_m(\phi^4) = D_m(\phi\phi^3)\sum_{k=0}^{m} D_k(\phi^3)D_{m-k}(\phi) = \sum_{k=0}^{m} u_{m-k}\sum_{j=0}^{k} u_{k-j}\sum_{i=0}^{j} u_i u_{j-i}$$

- Suppose that this statement is true for $\sigma = k$ i.e

$$D_m(\phi^k) = \sum_{r_1=0}^{m} u_{m-r_1}\sum_{r_2=0}^{r_1} u_{r_1-r_2}\sum_{r_3=0}^{r_2} u_{r_2-r_3}\cdots\sum_{r_{k-1}=0}^{r_{k-2}} u_{r_{k-2}-r_{k-1}}u_{r_{k-1}}$$

where $m \geq 0$, and $k \geq 2$ are integer, replacing m by r'_1 and r_j by r'_{j+1}

$$D_{r'_1}(\phi^k) = \sum_{r'_2=0}^{r'_1} u_{r'_1-r'_2}\sum_{r'_3=0}^{r'_2} u_{r'_2-r'_3}\sum_{r'_4=0}^{r'_3} u_{r'_3-r'_4}\cdots\sum_{r'_k=0}^{r'_{k-1}} u_{r'_{k-1}-r'_k}u_{r'_k}$$

using the above expression and

$$D_m(\phi^{n+1}) = \sum_{k=0}^{m} D_k(\phi)D_{m-k}(\phi^n) = \sum_{k=0}^{m} D_k(\phi^n)D_{m-k}(\phi)$$

we get

$$\begin{aligned} D_m(\phi^{k+1}) &= \textstyle\sum_{r'_1=0}^{m} D_{r'_1}(\phi^k)D_{m-r'_1}(\phi) \\ &= \textstyle\sum_{r'_1=0}^{m} u_{m-r'_1}\sum_{r'_2=0}^{r'_1} u_{r'_1-r'_2}\sum_{r'_3=0}^{r'_2} u_{r'_2-r'_3}\cdots\sum_{r'_k=0}^{r'_{k-1}} u_{r'_{k-1}-r'_k}u_{r'_k} \\ &= \textstyle\sum_{r_1=0}^{m} u_{m-r_1}\sum_{r_2=0}^{r_1} u_{r_1-r_2}\cdots\sum_{r_{\sigma_1-1}=0}^{r_{\sigma-2}} u_{r_{\sigma-2}-r_{\sigma-1}}u_{r_{\sigma-1}} \end{aligned}$$

where $\sigma = k + 1$.

for more details about these theorems, please check [55][56][57]

3.2 The zero-order deformation equation

Consider the nonlinear (or linear) problem

$$N[u(x,t)] = 0 \tag{3.2.1}$$

where N is nonlinear operator, x is an independent spatial variable, t is an independent temporal variable, supposing that $u(x,t)$ is written as a convergent series

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t)$$

The concept of the homotopy analysis method is primarily proposed by Shijun Liao in 1992, he's construct a family of equation embedding a parameter $q \in [0,1]$ namely "the homotopy parameter",

$$(1-q)L[\phi(x,t,q) - u_0(x,t)] = qN[\phi(x,t,q)], \quad q \in [0,1] \tag{3.2.2}$$

where L is a linear auxiliary operator, $u_0(x,t)$ is the initial guess, as the parameter q increases from 0 to 1, i.e

$$(1-0)L[\phi(x,t,0) - u_0(x,t)] = 0.N[\phi(x,t,0)]$$

which implies that $\phi(x,t,0) = u_0(x,t)$ because of L is a linear operator, and

$$(1-1)L[\phi(x,t,1) - u_0(x,t)] = N[\phi(x,t,1)]$$

which implies that $N[\phi(x,t,1)] = 0$ it holds $\phi(x,t,1) = u(x,t)$, so as q increases from 0 to 1, $\phi(x,t,q)$ varies from initial guess $u_0(x,t)$ to the exact solution $u(x,t)$.
The generalized zeroth-order deformation equation is introduced by Liao in 1997 [57]

$$(1-q)L[\phi(x,t,q) - u_0(x,t)] = q\hbar N[\phi(x,t,q)], \quad q \in [0,1] \tag{3.2.3}$$

where $\hbar$ is an auxiliary parameter, note that the homotopy series is dependent to the spatial variable x, temporal variable t, and the auxiliary parameter $\hbar$. The often furthermore generalized deformation equation used is [57]

$$(1-q)L[\phi(x,t,q) - u_0(x,t)] = q\hbar H(x,t)N[\phi(x,t,q)], \quad q \in [0,1] \tag{3.2.4}$$

where $H(x,t)$ is a non-zero auxiliary function.

3.3 The $m-$th order deformation equation

Lemma 3.3.1 *[57] Let*

$$\phi(x,t,q) = \sum_{m=0}^{\infty} u_m(x,t) q^m$$

a homotopy-Maclaurin series, where $q \in [0,1]$ is embedding namely a homotopy parameter, consider L an auxiliary linear operator independent to q, then

$$D_m[(1-q)L(\phi - u_0)] = L(u_m - \chi_m u_{m-1}) \tag{3.3.1}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{3.3.2}$$

Proof 5 *Since L is a linear operator independent of q,*

$$(1-q)L(\phi - u_0) = L(\phi - q\phi - u_0 + qu_0)$$

According to the theorems 3.1.11,3.1.12 and 3.1.15

$$\begin{aligned} D_m[(1-q)L(\phi - u_0)] &= D_m[L(\phi - q\phi - u_0 + qu_0)] \\ &= L[D_m(\phi - q\phi - u_0 + qu_0)] \\ &= L[D_m(\phi) - D_{m-1}(\phi) + u_0 D_m(q)] \\ &= L[u_m - u_{m-1} + u_0 D_m(q)] \end{aligned}$$

which equal $L(u_m)$ when $m = 1$, and $L(u_m - u_{m-1})$ when $m > 1$, so according to (3.3.2) *we get*

$$D_m[(1-q)L(\phi - u_0)] = L(u_m - \chi_m u_{m-1})$$

Theorem 3.3.2 *[55] Suppose*

$$\phi(x,t) = \sum_{k=0}^{\infty} u_k(x,t) q^k$$

where $q \in [0,1]$ is the homotopy parameter, let L an auxiliary linear operator, N a nonlinear operator, $u_0(x,t)$ the initial guess, $\hbar$ the convergence control *parameter independent of q, and $H(x,t)$ an auxiliary function independent of q, for the zeroth-order deformation equation*

$$(1-q)L[\phi(x,t,q)-u_0(x,t)] = q\hbar H(x,t)N[\phi(x,t,q)], \quad q\in[0,1] \tag{3.3.3}$$

the corresesopndance $m-th$ order deformation equation

$$L(u_m-\chi_m u_{m-1}) = \hbar H(x,t)D_{m-1}(N[\phi(x,t,q)]) \tag{3.3.4}$$

where D_{m-1} is defined in (3.1.2)*, and χ_n is defined by* (3.3.2)*.*

Proof 6 *Using* (3.1.6)

$$D_m((1-q)L(\phi-u_0)) = D_m(q\hbar H(x,t)N[\phi(x,t,q)])$$

According to lemma 3.1.6

$$D_m((1-q)L(\phi-u_0)) = L(u_m-\chi_m u_{m-1})$$

According to theorem 3.1.2,and 3.1.5

$$D_m(q\hbar H(x,t)N[\phi(x,t,q)]) = \hbar H(x,t)D_{m-1}(N[\phi(x,t,q)])$$

thus

$$L(u_m-\chi_m u_{m-1}) = \hbar H.D_{m-1}(N(\phi))$$

3.4 Convergence theorems

The convergence of the series solution is important for numerical solution, in this section providing theorems about the convergence of the homotopy series.

Theorem 3.4.1 *If the series*

$$\sum_{n=0}^{\infty} u_n(x,t)$$

is convergent, where $u_n(x,t)$ is governed by the high-order deformation equation (3.3.1)*, then it must be a solution for* (3.2.1)

Proof 7 *Let*

$$S_n(t) = \sum_{n=0} u_n(x,t)$$

and

$$S(t) = \sum_{n=0}^{\infty} u_n(x,t)$$

with

$$\lim_{n \longrightarrow \infty} u_n = 0 \tag{3.4.1}$$

using (3.3.2) *and* (3.3.1) *results in*

$$\begin{aligned} \textstyle\sum_{k=0}^{n} L(u_k - \chi_k u_{k-1}) &= u_1 + (u_2 - u_1) + ...(u_{n-1} - u_{n-2}) + (u_n - u_{n-1}) \\ &= u_n \end{aligned}$$

According to (3.4.1),

$$\sum_{k=1}^{\infty}(u_k - \chi_k u_{k-1}) = \lim_{n \longrightarrow \infty} u_n = 0$$

Using the property of linearity of L results in

$$\sum_{k=1}^{\infty} L(u_k - \chi_k u_{k-1}) = 0$$

According to (3.3.4), *and the above expression*

$$\sum_{k=0}^{\infty} L(u_k - \chi_k u_{k-1}) = \sum_{k=1}^{\infty} \hbar H(x,t) D_{k-1}(N[\phi(x,t,q)]) = 0$$

which gives, since $\hbar \neq 0$, and $H \neq 0$ that

$$D_{m-1}(N[\sum_{k=1}^{\infty} u_k]) = 0$$

which must be the solution of (3.2.1) *i.e*

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t)$$

3.4.1 Solution expression

In the frame of HAM, we have a great freedom to choose the initial guess u_0, the auxiliary linear parameter L, the auxiliary function H, moreover the choice of a base of function that the series of solutions can be expressed, mentioning that Shijun Liao put a rule namely the rule expression or solution expression [54], expressing the HAM solution via polynomial functions, fractional functions, and exponential functions. Consider

$$\{e_k(x,t), k = 0, 1, 2, ...\} \tag{3.4.2}$$

a base of functions that

$$u(x,t) = \sum_{n=0}^{\infty} c_n e_n(x,t) \tag{3.4.3}$$

where c_n is a coefficient, so the initial guess $u_0(x,t)$ can be expressed by

$$u_0(x,t) = \sum_{n=0}^{m_0} c'_n e_n(x,t) \tag{3.4.4}$$

where m_0 is an integer and c'_n is a coefficient.

Definition 3.4.2 *Let $e_n(x,t)$ denote a base of functions, $u(x,t)$ is a solution of the problem $N[u(x,t)] = 0$, then*

$$u(x,t) \sim \sum_{n=1}^{\infty} c_n e_n(x,t)$$

is called the solution expression of $u(x,t)$ if

$$\lim_{n \longrightarrow \infty} \| u(x,t) - \sum_{n=1}^{\infty} c_n e_n(x,t) \| = 0$$

Some rules are given to choice as the solution expression for $N[u(x,t)] = 0$

1. Equations defined in a finite domain:

   - Polynomials and Fourier series can be always used as the solution-expression i.e

$$\{t^m, m = 0, 1, 2, ...\}$$
$$\{\cos(mt), \sin(mt), m = 0, 1, 2, ...\}$$

- Chebyshev series gives the best approximation for arbitrary continuous solutions of $N[u] = 0$, i.e

$$\begin{aligned}
&T_n(t) = \cos(nt) \\
&T_0(t) = 1, \quad T_1(t) = t \\
&T_n(t) = 2tT_{n-1}(t) - T_{n-2}(t), n = 2, 3, ...
\end{aligned}$$

2. Equations defined in an infinite domain:

   - Periodic base functions should be used, if the solution is periodic
   - Non-periodic base functions should be used, if the solution is not periodic
   - The solution-expression should satisfy as many asymptotic properties of solution as possible, if the solution is not periodic

The concept of the so-called solution-expression mentioned above is important in the frame of the HAM, because it is the start-point for us to choose the initial approximation u_0 and the auxiliary linear operator L.

3.4.2 Choice of the initial guess u_0

The philosophy of the homotopy analysis method is based on the continuous deformation between the initial guess and the exact solution. In the frame of HAM, we have great freedom to choose the initial guess, but under the rule of expression we could choose such a kind of initial approximation

$$u_0(x,t) = \sum_{k=1}^{n_0} a_k e_k(x,t) \tag{3.4.5}$$

where $e_k(x,t)$ is the base function, a_k is unknown constant, and $n_0 \geq$ the number of linear boundary/initial conditions, denoted by κ

3.4.3 Choice of the auxiliary linear operator L

To solve the problem $N[u(x,t)] = 0$ using HAM, this method is based on the initial guess u_0, the auxiliary linear operator L, and the convergence control parameter $\hbar$, and there is a great freedom choice because there is no theorem about it, just some rules it will be mentioned in the following [57]

- If the solution expression of u is a polynomial, then

$$L(u) = \frac{\partial^n u}{\partial t^n}$$

where n is the highest order of derivative of $N(u) = 0$.

- If the u is a periodic function with a known frequency ω, then

$$L(u) = u'' + \omega^2 u$$

- If the u is a periodic function with an unknown frequency ω, then

$$L(u) = u'' + u$$

3.4.4 The convergence-control parameter $\hbar$

The homotopy analysis method contains an auxiliary parameter $\hbar$ as shown in (3.2.4), which provides us a simple way to adjust and control the convergent region and the rate of convergence of the series solution. there are no rigorous theories to direct us to choose the initial approximations, auxiliary linear operators, auxiliary functions, and auxiliary parameter $\hbar$. Primarily considering the so-called the square residual error defined as [57]

$$E_m(\hbar_1, \hbar_2, ..., \hbar_k) = \int_\Omega \left[N\left(\sum_{n=0}^{m} u_n \right) \right]^2 d\Omega \tag{3.4.6}$$

The optimal value of the convergence control parameter $\hbar$ is determined by the solution of

$$\frac{\partial E_m}{\partial \hbar_n} = 0 \qquad 0 \leq n \leq k \tag{3.4.7}$$

3.5 Examples

Presenting the general framework for solving two parameters fractional differential equation using the homotopy analysis method, taking into consideration the following form:

$$\mathfrak{R}u(x,t) + \mathfrak{N}u(x,t) = f(x,t), \tag{3.5.1}$$

where $\mathfrak{R}$ and $\mathfrak{N}$ are the linear and nonlinear operator respectively, $f(x,t)$ is the source term. By applying the homotopy analysis method shown in [55] and [57], defining the non-linear operator

$$N[\phi(x,t,q)] = \mathfrak{R}\phi(x,t,q) + \mathfrak{N}\phi(x,t,q) - f(x,t) \tag{3.5.2}$$

where $\phi(t,q)$ is a real-valued function of t and $q \in [0,1]$. The zeroth-order deformation constructed by Liao [55],[57] is

$$(1-q)\mathcal{L}[\phi(x,t,q) - u_0(x,t)] = \hbar q H(x,t) N[\phi(x,t,q)] \tag{3.5.3}$$

where $\hbar \neq 0$ is nonzero convergent control parameter, $u_0(x,t)$ is the initial guess, N is the nonlinear operator and $\mathcal{L}$ is an injective linear operator, and $H(x,t)$ is the auxiliary function. When $q = 0$ (3.5.3) becomes

$$\mathcal{L}[\phi(x,t,0) - u_0(x,t)] = 0 \tag{3.5.4}$$

which give by the property of linear operator

$$\phi(x,t,0) = u_0(x,t)$$

For $q = 1$, since $q \neq 0$ and $H(x,t) \neq 0$, we get

$$N[\phi(x,t,1)] = 0 \tag{3.5.5}$$

which is the same as (3.5.1), that means

$$\phi(x,t,1) = u(x,t)$$

we expand $\phi(x,t,q)$ in Taylor series with respect q,

$$\phi(x,t,q) = \sum_{i=0}^{\infty} u_i(x,t) q^i$$

where

$$u_i(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t,q)}{\partial q^m}|_{q=0}, \tag{3.5.6}$$

by differentiating (3.5.3) m times with respect q, and taking $q = 0$, we obtain the m-th order deformation equation

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(\vec{u}_{m-1}(x,t)) \tag{3.5.7}$$

where the vector $\vec{u}_{m-1} = \{u_0, u_1, u_2, \cdots, u_{m-1}\}$. Applying $\mathcal{L}^{-1}$ in (3.5.7), results in:

$$u_m(x,t) \;=\; \chi_m u_{m-1}(x,t) + L^{-1}[R_m(h_{m-1}(x,t))]$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Example 3.5.1 (HAM for ODE's) *Considering a simple linear differential equation*

$$\begin{cases} u^{'}(t) = u(t), & 0 \leq t \leq 1, \\ u(0) = 1. \end{cases} \tag{3.5.8}$$

the exact solution of (3.5.8) *is*

$$u(t) = e^t$$

Define

$$N[\phi(t,q)] = \phi^{'}(t,q) - \phi(t,q)$$

where the prime denotes the differentiation with respect to t*, and*

$$\phi(t,q) = \sum_{k=0}^{\infty} u_k(t) q^k$$

the zeroth-order deformation equation is

$$(1-q)L[\phi(t,q) - u_0(t)] = q\hbar H(t) N[\phi(t,q)], \quad q \in [0,1]$$

where $u_0(t)$ *is the initial guess satisfy the initial conditions, the corresponding* $m-th$ *order deformation equation is*

$$L(u_m(t) - \chi_m u_{m-1}(t)) = \hbar H(t) D_{m-1}(N[\phi(t,q)])$$

subject to the initial condition

$$u_m(0) = 1$$

According to the theorem 3.1.2 and 3.1.5

$$D_{m-1}(N[\phi(t,q)]) = u'_{m-1}(t) - u_{m-1}$$

so the $m-th$ order deformation equation becomes

$$L(u_m - \chi_m u_{m-1}) = u'_{m-1} - u_{m-1} \tag{3.5.9}$$

Choosing $u_0(t) = 1$, $H = 1$ and $L = \frac{\partial}{\partial t}$, by applying L^{-1} on (3.5.8) *results in*

$$u_m(t) = \chi_m u_{m-1} + \hbar L^{-1}(u'_{m-1}(t) - u_{m-1})$$

Using Mathematica, leads to obtaining successively the first terms of the homotopy series solutions

$$\begin{array}{lcl}
u_0(t) & = & 1 \\
u_1(t) & = & -\hbar t \\
u_2(t) & = & \frac{1}{2}\hbar t(\hbar(t-2)-2) \\
u_3(t) & = & -\frac{1}{6}\hbar t\left(\hbar^2\left(t^2-6t+6\right)-6\hbar(t-2)+6\right) \\
u_4(t) & = & \frac{1}{24}\hbar t\left(\hbar^3\left(t^3-12t^2+36t-24\right)-12\hbar^2\left(t^2-6t+6\right)+36\hbar(t-2)-24\right)
\end{array}$$

and for $\hbar = -1$, the $10-th$ order HAM series solution becomes

$$\sum_{k=0}^{\infty} u_k(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} + \frac{t^7}{5040} + ... = \sum_{k=0}^{\infty} \frac{t^n}{n!} = e^t$$

which is the exact solution.

Example 3.5.2 (HAM for ODE's) *To illustrate the application of the homotopy analysis method (HAM) for ODE's, consider the nonlinear problem*

$$\begin{cases} u'(t) = u^2(t), & 0 \leq t \leq 1, \\ u(0) = 1. \end{cases} \tag{3.5.10}$$

the exact solution of (3.5.10) *is*

$$u(t) = \frac{1}{1-t}$$

Define

$$N[\phi(t,q)] = \phi'(t,q) - \phi^2(t,q)$$

where the prime denotes the differentiation with respect to t, and

$$\phi(t,q) = \sum_{k=0}^{\infty} u_k(t) q^k$$

the zeroth-order deformation equation is

$$(1-q)L[\phi(t,q)-u_0(t)] = q\hbar H(t)N[\phi(t,q)], \quad q\in[0,1]$$

where $u_0(t)$ *is the initial guess satisfy the initial conditions, the corresponding* $m-th$ *order deformation equation is*

$$L(u_m(t)-\chi_m u_{m-1}(t)) = \hbar H(t)D_{m-1}(N[\phi(t,q)])$$

According to the theorem 3.1.2 and 3.1.5

$$D_{m-1}(N[\phi(t,q)]) = u'_{m-1}(t) - \sum_{k=0}^{m-1} u_k(t)u_{m-1-k}(t)$$

so the $m-th$ *order deformation equation becomes*

$$L(u_m-\chi_m u_{m-1}) = u'_{m-1} - \sum_{k=0}^{m-1} u_k u_{m-1-k} \tag{3.5.11}$$

Choosing $u_0(t)=1$, $H=1$ *and* $L=\frac{\partial^2}{\partial t^2}$, *by applying* L^{-1} *on* (3.5.11) *leads to*

$$u_m(t) = \chi_m u_{m-1} + \hbar L^{-1}(u'_{m-1} - \sum_{k=0}^{m-1} u_k u_{m-1-k})$$

obtaining successively the first terms of the homotopy series solutions

$$\begin{aligned}
u_0(t) &= 1\\
u_1(t) &= -\hbar t\\
u_2(t) &= -\hbar t + \hbar\left(\hbar t^2 - \hbar t\right)\\
u_3(t) &= -\hbar t + \hbar\left(\hbar t^2 - \hbar t\right) + h\left(-\hbar^2 t^3 + 2\hbar^2 t^2 - \hbar^2 t + \hbar t^2 - \hbar t\right)\\
u_4(t) &= \hbar t(\hbar(t-1)-1)^3
\end{aligned}$$

and for $\hbar=-1$ *the HAM approximation solution becomes*

$$u(t) = 1+t+t^2+t^3+t^4+t^5+t^6+\ldots = \sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$$

which is the exact solution.

Example 3.5.3 (HAM for PDE's) *Considering the nonlinear homogeneous gas dynamics equation (Evans 2002) [22]*

$$u_t + \frac{1}{2}u^2 u_{xx} = u(1-u) \tag{3.5.12}$$

with initial condition $u(x,0) = ae^{-x}$. *The exact solution is* $u(x,t) = ae^{x-t}$, *Choosing* $u_0 = ae^{-x}$, *and* $L = \frac{\partial}{\partial t}$. *Defines*

$$N[\phi(t,q)] = \phi_t + \frac{1}{2}\phi^2\phi_{xx} - \phi(1-\phi)$$

where $_*$ *denotes the differentiation with respect to* $*$, *and*

$$\phi(x,t,q) = \sum_{k=0}^{\infty} u_k(x,t)q^k$$

the zeroth-order deformation equation is

$$(1-q)L[\phi(x,t,q) - u_0(x,t)] = q\hbar H(x,t)N[\phi(x,t,q)], \quad q \in [0,1]$$

where $u_0(t)$ *is the initial guess satisfy the initial conditions, the corresponding* $m-th$ *order deformation equation is*

$$L(u_m(x,t) - \chi_m u_{m-1}(x,t)) = \hbar H(x,t)D_{m-1}(N[\phi(x,t,q)])$$

subject to the initial condition

$$u_m(x,0) = ae^{-x}$$

According to the theorem 3.1.2 and 3.1.5

$$D_{m-1}(N[\phi(t,q)]) = R_m(\overrightarrow{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial t} + \sum_{i=0}^{m-1} u_i \frac{\partial u_{m-1-i}}{\partial x} + \sum_{i=0}^{m-1} u_i u_{m-1-i} - u_{m-1}$$

so the $m-th$ *order deformation equation becomes*

$$L(u_m - \chi_m u_{m-1}) = \frac{\partial u_{m-1}}{\partial t} + \sum_{i=0}^{m-1} u_i \frac{\partial u_{m-1-i}}{\partial x} + \sum_{i=0}^{m-1} u_i u_{m-1-i} - u_{m-1} \tag{3.5.13}$$

the solution of the $m-th$ *deformation equation* (3.5.13)

$$u_m(t) = \chi_m u_{m-1} + \hbar L^{-1}(R_m(\overrightarrow{u}_{m-1}))$$

Using $u_m(x,0) = 0, \forall m \geq 1$, *results in*

$$\begin{aligned} u_0(t) &= ae^{-x} \\ u_1(t) &= -ae^{-x}\hbar t \\ u_2(t) &= \tfrac{1}{2}ae^{-x}t^2\hbar - ae^{-x}t\hbar^2 - ae^{-x}t\hbar \\ &\vdots \end{aligned}$$

So

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ... \\ &= ae^{-x}(1 - 4\hbar t - 6t\hbar^2 + 3t^2\hbar^2 - 4t\hbar^3 - \tfrac{2t^3\hbar^3}{3} - t\hbar^4 + \tfrac{3t^2\hbar^4}{3} + ...) \end{aligned}$$

for $\hbar = -1$, *obtaining*

$$u(x,t) = ae^{-x}(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + ...) = ae^{-x}\sum_{k=0}^{\infty}\frac{t^k}{k!} = ae^{t-x}$$

which is the exact solution.

Example 3.5.4 (HAM for fractional initial value problems) *Consider the linear fractional initial value problem [26]*

$$\begin{cases} D^\alpha u(t) = -u(t), & 0 \leq t \leq 1, 1 \leq \alpha \leq 2 \\ u(0) = 1 \qquad u'(0) = 0. \end{cases} \tag{3.5.14}$$

the exact solution of (3.5.14) *for* $\alpha = 2$ *is*

$$u(t) = \cos t$$

Define

$$N[\phi(t,q)] = D^\alpha\phi(t,q) + \phi(t,q)$$

where $1 < \alpha \leq 2$, *the zeroth-order deformation equation is*

$$(1-q)L[\phi(t,q) - u_0(t)] = q\hbar H(t)N[\phi(t,q)], \quad q \in [0,1]$$

the corresponding $m-th$ *order deformation equation is*

$$L(u_m(t) - \chi_m u_{m-1}(t)) = \hbar H(t)D_{m-1}(N[\phi(t,q)])$$

According to the two theorems 3.1.2 and 3.1.5

$$D_{m-1}(N[\phi(t,q)]) = D^\alpha u_{m-1}(t) + u_{m-1}(t)$$

so the m−th order deformation equation becomes

$$L(u_m - \chi_m u_{m-1}) = D^\alpha u_{m-1}(t) + u_{m-1}(t) \quad (3.5.15)$$

Choosing $u_0(t) = 1$, $H = 1$ *and* $L = D^\alpha$, *applying* $L^{-1} = I^\alpha$ *on* (3.5.15) *ends as*

$$u_m(t) = \chi_m u_{m-1}(t) + \hbar L^{-1}(D^\alpha u_{m-1}(t) + u_{m-1}(t))$$

obtaining successively the first terms of the homotopy series solutions

$$\begin{aligned}
u_0(t) &= 1 \\
u_1(t) &= \tfrac{\hbar t^\alpha}{\Gamma(\alpha+1)} \\
u_2(t) &= \hbar t^\alpha \left(\tfrac{\hbar+1}{\Gamma(\alpha+1)} + \tfrac{\hbar t^\alpha}{\Gamma(2\alpha+1)} \right) \\
u_3(t) &= \hbar t^\alpha \left(\tfrac{(\hbar+1)^2}{\Gamma(\alpha+1)} + \hbar t^\alpha \left(\tfrac{2(h+1)}{\Gamma(2\alpha+1)} + \tfrac{\hbar t^\alpha}{\Gamma(3\alpha+1)} \right) \right) \\
u_4(t) &= \hbar t^\alpha \left(\tfrac{(\hbar+1)^3}{\Gamma(\alpha+1)} + \hbar t^\alpha \left(\tfrac{3(\hbar+1)^2}{\Gamma(2\alpha+1)} + \hbar t^\alpha \left(\tfrac{3(\hbar+1)}{\Gamma(3\alpha+1)} + \tfrac{\hbar t^\alpha}{\Gamma(4\alpha+1)} \right) \right) \right)
\end{aligned}$$

and for $\hbar = -1$ *and* $\alpha = 2$ *the HAM solution becomes*

$$u(t) = 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720} + \frac{t^8}{40320} ... = \sum_{k=0}^{\infty} \frac{(-1)^k (t)^{2k}}{(2k)!} = \cos t$$

Example 3.5.5 (HAM for the Fokker-Planck fractional differential equation) *Consider the Fokker-Planck fractional equation [29]*

$${}^C D_t^\alpha u(x,t) = D_x u(x,t) + D_{xx} u(x,t) \quad (3.5.16)$$

with

$$u(x,0) = x \quad (3.5.17)$$

We obtain by applying the Laplace transform to the TFFP (3.5.16) *results in*

$$s^\alpha L[u(x,t)] - s^{\alpha-1} u(x,0) - L[u_x(x,t) + u_{xx}(x,t)] = 0$$

that means

$$L[u(x,t)] - \frac{1}{s} u(x,0) - \frac{1}{s^\alpha} L[u_x(x,t) + u_{xx}(x,t)] = 0$$

defining the linear operator as

$$N[\phi(x,t,q)] = L[\phi(x,t,q)] - \frac{1}{s} u(x,0) - \frac{1}{s^\tau} L\left[\phi_x(x,t,q) + \phi_{xx}(x,t,q)\right]$$

the m-th order deformation equation is

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(u_{m-1}(x,t))]$$

where

$$R_m[u_{m-1}] = L[u_{m-1}] - \frac{1}{s}u(x,0)(1-\chi_m) - \frac{1}{s^\alpha}L\left[(u_{m-1})_x + (u_{m-1})_{xx}\right]$$

Choosing $u_0(x,t) = x$, the first terms of HATM solution of TFFP are:

$$\begin{array}{l} u_1(x,t) = \hbar L^{-1}\left[R_1(u_0(x,t))\right] \\ u_2(x,t) = \chi_2 u_1(x,t) + \hbar L^{-1}\left[R_2(u_1(x,t), u_0(x,t))\right] \\ u_3(x,t) = \chi_3 u_2(x,t) + \hbar L^{-1}\left[R_3(u_2(x,t), u_1(x,t), u_0(x,t))\right] \end{array}$$

where,

$$\begin{array}{lcl} R_1(u_0) & = & L[u_0] - \frac{1}{s}u(x,0) - \frac{1}{s^\alpha}L\left[(u_0)_x + (u_0)_{xx}\right] \\ R_2(u_1,u_0) & = & L[u_1] - \frac{1}{s}u(x,0)(1-\chi_2) - \frac{1}{s^\alpha}L\left[(u_1)_x + (u_1)_{xx}\right] \\ R_3(u_2,u_1,u_0) & = & L[u_2] - \frac{1}{s}u(x,0)(1-\chi_3) - \frac{1}{s^\alpha}L\left[(u_2)_x + (u_2)_{xx}\right] \end{array}$$

So, the first terms of equation (3.5.16) *are given after simplification as*

$$\begin{array}{l} u_1(x,t) = -\frac{\hbar t^\alpha}{\Gamma(\alpha+1)} \\ u_2(x,t) = -\frac{\hbar(\hbar+1)t^\alpha}{\Gamma(\alpha+1)} \\ u_3(x,t) = -\frac{\hbar(\hbar+1)^2 t^\alpha}{\Gamma(\alpha+1)} \end{array}$$

the HATM solution for this equation is given as

$$u(x,t) = \sum_{i=1}^{n} u_i(x,t) = \frac{x\Gamma(\alpha+1) - \hbar\left(10 + 45\hbar + 120\hbar^2 + 210\hbar^3 + 252\hbar^4 + \ldots\right)t^\alpha}{\Gamma(\alpha+1)}$$

and for $\alpha = 1$ and $\hbar = -1$, the HATM solution is

$$u(x,t) = \sum_{i=1}^{\infty} u_i(x,t) = x + t$$

Chapter 4

Application

This chapter gives a proper application of HAM on various type of differential equations, the results are published in different journals, the first is entitled "Analytic study of pine wilt disease model with Caputo-Fabrizio fractional derivative" in the journal of Mathematical Methods in the Applied Sciences, Wiley. The second "Comparative numerical study of Forenberg-Whitham equation" is published in the International Journal of Applied and Computational Mathematics. The third is "Comparative numerical study of time-fractional coupled Korteweg-De Vries equation" published in the Italian journal of pure and applied mathematics. The fourth entitled "Numerical solution of time-fractional coupled Korteweg-de Vries equation with a Caputo fractional derivative in two parameters"

4.1 Analytic study of pine wilt disease model with Caputo-Fabrizio fractional derivative

4.1.1 Introduction

The pine wilt disease epidemic was discovered in Japan in 1905 in Nagasaki, which is the spread of the string worms in pinewood called Aphelenchoides xylophilus kills trees in a few days, this epidemic has spread to the east of Asia (Japan, China, and Korea and others), it caused heavy losses in Japan by the weathering of black and red pine forests.
The pine wilt diseases is one of the causes why forests are destroyed and damaged, so the researchers and scientists try to find solutions for this phenomena; because of the importance of the forests for the human and animal.
The pine wild disease is governed by a system of a differential equation [43],[44]

$$\begin{cases} S'_H(t) &= \Pi_H - \frac{K_1 S_H I_V}{1+\theta_1 I_V} - \frac{K_2 \psi S_H I_V}{1+\theta_1 I_V} - \gamma_H S_H, \\ E'_H(t) &= \frac{K_1 S_H I_V}{1+\theta_1 I_V} - \gamma_H E_H - \delta_H E_H, \\ I'_H(t) &= \frac{K_2 \psi S_H I_V}{1+\theta_1 I_V} + \delta_H E_H - \gamma_H I_H \\ S'_V(t) &= \Pi_V - \frac{\beta_1 S_V I_H}{1+\theta_2 I_H} - \gamma_V S_V, \\ E'_V(t) &= \frac{\beta_1 S_V I_H}{1+\theta_2 I_H} - \gamma_V E_V - \delta_V E_V, \\ I'_V(t) &= \delta_V E_V - \gamma_V I_V \end{cases} \tag{4.1.1}$$

To protect the forests, scientists and researchers are always based on their studies, they focus on understanding this natural phenomena and their methods of spreading and analyzing it from all sides. This epidemic kills the pine trees in a few days, and can't be recovered.
Currently, many epidemics have been modeled into a system of fractional differential equations such as Muhammad Altaf Khan[43], Yongjin Li[67], and Sunil Kumar[65],[64]
Mathematical modeling of many diseases and epidemics have contributed in recent years to understanding the behavior and how it spreads, as well as helping in knowledge or prediction, for example, concerning diseases such as covid-19, in knowing the peak of spread and the days of the epidemic fading about this model, many researchers carried out various studies, such as Muhammad Altaf Khan[43][44], who conducted an analytical study using the Caputo-Fabrizio fractional derivative. Yongjin Li[67] also presented a study of the spread of this epidemic. Kamal Shah[32] used the Laplace Adomian decomposition method in his study.

In this paper, the homotopy analysis transform method is used based on Laplace operator, and the homotopy analysis method proposed in 1992 by Shijun Liao[56], and [57], this method has many applications in various differential equations, the homotopy analysis transform method is mostly used for solving the fractional differential equation[64][62][3][4][53], this method simplifies fractional differential equations and makes the calculations simpler and faster compared to the homotopy analysis method as shown in this work.

This paper models mathematically this epidemic into a system of fractional differential equations as shown follows [1][43][45] :

$$\begin{cases} {}^{CF}D^{\tau}S_H(t) &=& \Pi_H - \frac{K_1 S_H I_V}{1+\theta_1 I_V} - \frac{K_2 \psi S_H I_V}{1+\theta_1 I_V} - \gamma_H S_H, \\ {}^{CF}D^{\tau}E_H(t) &=& \frac{K_1 S_H I_V}{1+\theta_1 I_V} - \gamma_H E_H - \delta_H E_H, \\ {}^{CF}D^{\tau}I_H(t) &=& \frac{K_2 \psi S_H I_V}{1+\theta_1 I_V} + \delta_H E_H - \gamma_H I_H \\ {}^{CF}D^{\tau}S_V(t) &=& \Pi_V - \frac{\beta_1 S_V I_H}{1+\theta_2 I_H} - \gamma_V S_V, \\ {}^{CF}D^{\tau}E_V(t) &=& \frac{\beta_1 S_V I_H}{1+\theta_2 I_H} - \gamma_V E_V - \delta_V E_V, \\ {}^{CF}D^{\tau}I_V(t) &=& \delta_V E_V - \gamma_V I_V \end{cases} \tag{4.1.2}$$

with initial condition

$$\begin{array}{lll} S_H(0) = \eta_1 & E_H(0) = \eta_2 & I_H(0) = \eta_3 \\ S_V(0) = \eta_4 & E_V(0) = \eta_5 & I_V(0) = \eta_6 \end{array}$$

where $S_H(t)$ is the total number of trees, $E_H(t)$, $I_H(t)$, and The total number of beetles is divided into three classes susceptible $S_V(t)$, Exposed $E_V(t)$, and infected $I_V(t)$, Π_H is The recruitment rates of pine trees, Π_V vector population. K_1 is the rate of contact between suspectable trees and infected, K_2 is the contact rate between suspectable trees and infected vectors when the nematode is transmitted by the infected vector at oviposition. The quantity γ_H the natural death rates of pine trees. The quantity γ_V the death rates population, ψ the natural death rate of pine trees which are uninfected.
θ_1 and θ_2 are the constants of saturation, δ_H rates the exposed pine trees join the infected class, δ_V is the rate of transfer of an exposed vector to become an infected vector, β_1 is the contact rate of a suspectable vector with infected pine trees.

4.1.2 Basic definitions of fractional calculus

This section gives some basic definitions of fractional calculus theory used in this paper:

Definition 4.1.1 *A real function $h(t)$ is said C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$ where $h_1(t) \in C(0,\infty)$ and it is said to be in space C_n if and only if $h^{(n)} \in C_\mu, n \in \mathbb{N}$*

Definition 4.1.2 *The Riemann-Liouville fractional operator of order $\tau \geq 0$, of a function $h \in C_\tau, \tau \geq -1$ is defined as [68]*

$$\begin{aligned} I^\tau h(t) &= \tfrac{1}{\Gamma(\tau)} \int_0^t (t-\alpha)^{(\tau-1)} h(\alpha) d\alpha, \mu > 0, t > 0 \\ I^0 h(t) &= h(t) \end{aligned}$$

where $\Gamma(.)$ is well-known Gamma function

Definition 4.1.3 *The Caputo fractional derivative of h, $h \in C_{-1}^m$ is defined as*

$$D_C^\tau h(t) = \frac{1}{\Gamma(m-\tau)} \int_0^t (t-\varsigma)^{(m-\tau-1)} h^{(m)}(\varsigma) d\varsigma,$$

where $m-1 < \tau < m, m \in \mathbb{N}$

Definition 4.1.4 *Consider $h \in H^1(a,b)$, and $\tau \in (0,1)$, then the Caputo-Fabrizio fractional derivative of h is defined as follows*

$$D_{CF}^\tau h(t) = \frac{(2-\tau)A(\tau)}{2(1-\tau)} \int_0^t h'(s) e^{(\frac{-\tau(t-s)}{1-\tau})} ds$$

Definition 4.1.5 *The Laplace transform L for the function $h(t)$ of the Caputo fractional derivative is[68]:*

$$L[D_t^{n\tau} h(t)] = s^{n\tau} L[h(t)] - \sum_{k=0}^{n-1} s^{(n\tau-k-1)} h^{(k)}(0), \quad n-1 < n\tau \leq n \quad (4.1.3)$$

Definition 4.1.6 *The Laplace transform for the function $h(t)$ with $\tau \in (0,1)$ of the Caputo-Fabrizo fractional derivative is [48][21][62]:*

$$L[D_{CF}^\tau h(t)] = \frac{1}{2} \frac{A(\tau)(2-\tau)}{(1-\tau)} \frac{sL[h(t)] - h(0)}{s + \frac{\tau}{1-\tau}} \quad (4.1.4)$$

4.1.3 Basic idea of the homotopy analysis transform method

Considering the following fractional differential equation to illustrate the application of the HATM for the fractional differential equation:

$$^{CF}D^{\tau}h(t) + \mathfrak{R}h(t) + \mathfrak{N}h(t) = f(t), \quad 0 < \tau < 1 \tag{4.1.5}$$

where $^{CF}D^{\tau}h(t)$ is the Caputo-Fabrizo derivative of $h(t)$, $\mathfrak{R}$ and $\mathfrak{N}$ are the linear and nonlinear operator respectively, $f(t)$ is the source term.
initially, by applying the Laplace transform operator on the (4.1.5) results in

$$\frac{1}{2}\frac{A(\tau)(2-\tau)}{(1-\tau)}\frac{sL[h(t)] - h(0)}{s + \frac{\tau}{1-\tau}} + L[\mathfrak{R}h(t)] + L[\mathfrak{N}h(t)] = L[f(t)], \quad 0 < \tau < 1 \tag{4.1.6}$$

after simplification, using an explicit formula $A(\tau) = \frac{2}{2-\tau}$ proved by Losada [28] we get:

$$L[h(t)] = \frac{1}{s}h(0) + \frac{s + \tau(1-s)}{s}(L[f(t)] - L[\mathfrak{R}h(t)] - L[\mathfrak{N}h(t)]), \quad 0 < \tau < 1 \tag{4.1.7}$$

By applying the homotopy analysis method [62],[55], and [57], defining the nonlinear operator

$$N[\phi(t,q)] = L[\phi(t,q)] - \frac{1}{s}h(0) - \frac{s + \tau(1-s)}{s}(L[f(t)] + L[\mathfrak{R}h(t)] + L[\mathfrak{N}h(t)]) \tag{4.1.8}$$

where $\phi(t,q)$ is the real function of t and $q \in [0,1]$, the zeroth-order deformation constructed by Liao [55],[57] is

$$(1-q)\mathcal{L}[\phi(t,q) - h_0(t)] = \hbar q H(t) N[\phi(t,q)] \tag{4.1.9}$$

where $\mathcal{L}$ is the auxiliary linear operator, for the chosen system $\mathcal{L}$ as a Laplace operator L, $\hbar$ is a nonzero auxiliary parameter, the auxiliary function $H(t) \neq 0$, $h_0(t)$ is the initial guess of $h(t)$ and $\phi(t,q)$ is an unknown function. Obviously $\phi(t,0) = u_0(t)$ and $\phi(t,1) = h(t)$ expanding $\phi(t,q)$ in Taylor series with respect q,

$$\phi(t,q) = \sum_{i=0}^{n} h_i(t)q^i$$

where

$$h_i(t) = \frac{1}{m!}\frac{\partial^m \phi(t,q)}{\partial q^m}|_{q=0}$$

by differentiating (4.4.12) m times with respect q, and taking $q=0$, obtaining the mth order deformation equation

$$L[h_m(t) - \ell_m h_{m-1}(t)] = \hbar H(t) R_m(h_{m-1}(t)) \qquad (4.1.10)$$

Applying L^{-1} in (4.1.10) results in:

$$h_m(t) = \ell_m h_{m-1}(t) + \hbar L^{-1}[R_m(h_{m-1}(t))] \qquad (4.1.11)$$

where

$$\ell_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Now, for the epidemic model (4.3.1) determining a system of the nonlinear operator as:

$$\begin{aligned}
N_1[\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6] &= L[\phi_1] - \tfrac{\eta_1}{s} - \tfrac{s+\tau(1-s)}{s}(L[\Pi_H] + L[(K_1\phi_1\phi_6)(1-\theta_1\phi_6) - \\
&\quad (K_2\psi\phi_1\phi_6)(1-\theta_1\phi_6) - \gamma_H\phi_1]) \\
N_2[\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6] &= L[\phi_1] - \tfrac{\eta_2}{s} - \tfrac{s+\tau(1-s)}{s}(L[(K_1\phi_1\phi_6)(1-\theta_2\phi_6) - \gamma_H\phi_2 - \delta_H\phi_2]) \\
N_3[\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6] &= L[\phi_1] - \tfrac{\eta_3}{s} - \tfrac{s+\tau(1-s)}{s}L[(K_2\psi\phi_1\phi_6)1 - \theta_1\phi_6 + \delta_H\phi_2 - \gamma_H\phi_3] \\
N_4[\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6] &= L[\phi_1] - \tfrac{\eta_4}{s} - \tfrac{s+\tau(1-s)}{s}L[\Pi_V - (\beta_1\phi_4\phi_3)(1-\theta_2\phi_3) - \gamma_V\phi_4] \\
N_5[\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6] &= L[\phi_1] - \tfrac{\eta_5}{s} - \tfrac{s+\tau(1-s)}{s}L[(\beta_1\phi_4\phi_3)(1-\theta_2\phi_3) - \gamma_V\phi_5 - \delta_V\phi_5] \\
N_6[\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6] &= L[\phi_1] - \tfrac{\eta_6}{s} - \tfrac{s+\tau(1-s)}{s}L[\delta_V\phi_5 - \gamma_V\phi_6]
\end{aligned}$$

The m-th order deformation equation is

$$\begin{aligned}
L[S_{H,m} - l_m S_{H,m-1}] &= \hbar H_1(t) R_{1,m}[\overrightarrow{S}_{H,m-1}, \overrightarrow{E}_{H,m-1}, \overrightarrow{I}_{H,m-1}, \overrightarrow{S}_{V,m-1}, \overrightarrow{E}_{V,m-1}, \overrightarrow{I}_{V,m-1}] \\
L[E_{H,m} - l_m E_{H,m-1}] &= \hbar H_2(t) R_{2,m}[\overrightarrow{S}_{H,m-1}, \overrightarrow{E}_{H,m-1}, \overrightarrow{I}_{H,m-1}, \overrightarrow{S}_{V,m-1}, \overrightarrow{E}_{V,m-1}, \overrightarrow{I}_{V,m-1}] \\
L[I_{H,m} - l_m I_{H,m-1}] &= \hbar H_3(t) R_{3,m}[\overrightarrow{S}_{H,m-1}, \overrightarrow{E}_{H,m-1}, \overrightarrow{I}_{H,m-1}, \overrightarrow{S}_{V,m-1}, \overrightarrow{E}_{V,m-1}, \overrightarrow{I}_{V,m-1}] \\
L[S_{V,m} - l_m S_{V,m-1}] &= \hbar H_4(t) R_{4,m}[\overrightarrow{S}_{H,m-1}, \overrightarrow{E}_{H,m-1}, \overrightarrow{I}_{H,m-1}, \overrightarrow{S}_{V,m-1}, \overrightarrow{E}_{V,m-1}, \overrightarrow{I}_{V,m-1}] \\
L[E_{V,m} - l_m E_{V,m-1}] &= \hbar H_5(t) R_{5,m}[\overrightarrow{S}_{H,m-1}, \overrightarrow{E}_{H,m-1}, \overrightarrow{I}_{H,m-1}, \overrightarrow{S}_{V,m-1}, \overrightarrow{E}_{V,m-1}, \overrightarrow{I}_{V,m-1}] \\
L[I_{V,m} - l_m I_{V,m-1}] &= \hbar H_6(t) R_{6,m}[\overrightarrow{S}_{H,m-1}, \overrightarrow{E}_{H,m-1}, \overrightarrow{I}_{H,m-1}, \overrightarrow{S}_{V,m-1}, \overrightarrow{E}_{V,m-1}, \overrightarrow{I}_{V,m-1}]
\end{aligned} \qquad (4.1.12)$$

where

$$
\begin{aligned}
R_{1,m} &= L[S_{H,n-1}(t)] - (1-l_m)L[S_{H,0}(t)] + \hbar\tfrac{s+\tau(1-s)}{s}L[\Pi_H(1-l_m) - K_1(1-\psi) \\
&\quad (\textstyle\sum_{i=0}^{n-1} S_{H,i}(t)I_{V,n-1-i}(t) - \theta_1 \sum_{i=0}^{n-1} S_{H,i}(t)\sum_{j=0}^{i} I_{V,i-j}(t)I_{V,j}(t)) \\
&\quad -\gamma_H S_{H,n-1}(t)] \\
R_{2,m} &= L[E_{H,n-1}(t)] - (1-l_m)L[E_{H,0}(t)] \\
&\quad +\hbar\tfrac{s+\tau(1-s)}{s}L[K_1(\textstyle\sum_{i=0}^{n-1} S_{H,i}(t)I_{V,n-1-i}(t) - \theta_1 \\
&\quad \textstyle\sum_{i=0}^{n-1} S_{H,n-1-i}(t)\sum_{j=0}^{i} I_{V,i-j}(t)I_{V,j}(t)) - (\gamma_H+\delta_H)E_{H,n-1}(t) \\
R_{3,m} &= L[I_{H,n-1}(t)] - (1-l_m)L[I_{H,0}(t)] + \hbar\tfrac{s+\tau(1-s)}{s} \\
&\quad L[K_2\psi(\textstyle\sum_{i=0}^{n-1} S_{H,i}(t)I_{V,n-1-i}(t) - \theta_1 \\
&\quad \textstyle\sum_{i=0}^{n-1} S_{H,n-1-i}(t)\sum_{j=0}^{i} I_{V,i-j}(t)I_{V,j}(t)) - \gamma_H I_{H,n-1}(t) + \delta_H)E_{H,n-1}(t) \\
R_{4,m} &= L[S_{V,n-1}(t)] - (1-l_m)L[S_{V,0}(t)] + \hbar\tfrac{s+\tau(1-s)}{s}L[\Pi_V(1-l_m) + \beta_1 \\
&\quad (\textstyle\sum_{i=0}^{n-1} S_{V,i}(t)I_{H,n-1-i}(t) \\
&\quad -\theta_2 \textstyle\sum_{i=0}^{n-1} S_{V,i}(t)\sum_{j=0}^{i} I_{H,i-j}(t)I_{H,j}(t)) - \gamma_V S_{V,n-1}(t)] \\
R_{5,m} &= L[E_{V,n-1}(t)] - (1-l_m)L[E_{V,0}(t)] + \hbar\tfrac{s+\tau(1-s)}{s}L[\beta_1(\textstyle\sum_{i=0}^{n-1} S_{V,i}(t)I_{H,n-1-i}(t) \\
&\quad -\theta_2 \textstyle\sum_{i=0}^{n-1} S_{V,n-1-i}(t)\sum_{j=0}^{i} I_{H,i-j}(t)I_{H,j}(t)) - \gamma_V S_{V,n-1}(t) - \delta_V E_{V,n-1}(t)] \\
R_{6,m} &= L[I_{V,n-1}(t)] - (1-l_m)L[I_{V,0}(t)] + \hbar\tfrac{s+\tau(1-s)}{s}L[\delta_V E_{V,n-1}(t) - \gamma_V I_{V,n-1}(t)]
\end{aligned}
$$

Using the method shown above, leads to the m-th order deformation equation

$$
\begin{aligned}
S_{H,m}(t) &= \ell_m S_{H,m-1}(t) + \hbar L^{-1}R_{1,m}[\overrightarrow{S}_{H,m}, \overrightarrow{E}_{H,m}, \overrightarrow{I}_{H,m}, \overrightarrow{S}_{V,m}, \overrightarrow{E}_{V,m}, \overrightarrow{I}_{V,m}] \\
E_{H,m}(t) &= \ell_m E_{H,m}(t) + \hbar L^{-1}R_{2,m}[\overrightarrow{S}_{H,m}, \overrightarrow{E}_{H,m}, \overrightarrow{I}_{H,m}, \overrightarrow{S}_{V,m}, \overrightarrow{E}_{V,m}, \overrightarrow{I}_{V,m}] \\
I_{H,m}(t) &= \ell_m I_{H,m}(t) + \hbar L^{-1}R_{3,m}[\overrightarrow{S}_{H,m}, \overrightarrow{E}_{H,m}, \overrightarrow{I}_{H,m}, \overrightarrow{S}_{V,m}, \overrightarrow{E}_{V,m}, \overrightarrow{I}_{V,m}] \\
S_{V,m}(t) &= \ell_m S_{V,m}(t) + \hbar L^{-1}R_{4,m}[\overrightarrow{S}_{H,m}, \overrightarrow{E}_{H,m}, \overrightarrow{I}_{H,m}, \overrightarrow{S}_{V,m}, \overrightarrow{E}_{V,m}, \overrightarrow{I}_{V,m}] \\
E_{V,m}(t) &= \ell_m E_{V,m}(t) + \hbar L^{-1}R_{5,m}[\overrightarrow{S}_{H,m}, \overrightarrow{E}_{H,m}, \overrightarrow{I}_{H,m}, \overrightarrow{S}_{V,m}, \overrightarrow{E}_{V,m}, \overrightarrow{I}_{V,m}] \\
I_{V,m}(t) &= \ell_m I_{V,m}(t) + \hbar L^{-1}R_{6,m}[\overrightarrow{S}_{H,m}, \overrightarrow{E}_{H,m}, \overrightarrow{I}_{H,m}, \overrightarrow{S}_{V,m}, \overrightarrow{E}_{V,m}, \overrightarrow{I}_{V,m}]
\end{aligned}
\tag{4.1.13}
$$

where L^{-1} in inverse Laplace transform operator, and in this case, mentioning that each of $S_H(t), E_H(t), I_H(t), S_V(t), E_V(t)$, and $I_V(t)$ is written as a series

$$
\begin{array}{lll}
S_H(t) = \sum_{i=0}^{\infty} S_{H,i}(t) & E_H(t) = \sum_{i=0}^{\infty} E_{H,i}(t) & I_H(t) = \sum_{i=0}^{\infty} I_{H,i}(t) \\
S_V(t) = \sum_{i=0}^{\infty} S_{V,i}(t) & E_V(t) = \sum_{i=0}^{\infty} E_{V,i}(t) & I_V(t) = \sum_{i=0}^{\infty} I_{V,i}(t)
\end{array}
\tag{4.1.14}
$$

with the initial conditions,

$$
\begin{array}{lll}
S_{H,0}(0) = \eta_1 = 600 & E_{H,0}(0) = \eta_2 = 200 & I_{H,0}(0) = \eta_3 = 100 \\
S_{V,0}(0) = \eta_4 = 600 & E_{V,0}(0) = \eta_5 = 300 & I_{V,0}(0) = \eta_6 = 20
\end{array}
\tag{4.1.15}
$$

deciphering the above system (4.1.13), leads to:

$$\begin{array}{rcl} S_{H,1}(t) & = & -\hbar(1-\tau+t\tau)(-600\gamma_H - K_1(12000-2400000\theta_1) \\ & & -\psi K_1(12000-2400000\theta_1)+\Pi_H) \\ E_{H,1}(t) & = & -\hbar(1-\tau+t\tau)(-200\gamma_H - 200\delta_H + K_1(12000-2400000\theta_1)) \\ I_{H,1}(t) & = & -\hbar(1-\tau+t\tau)(-100\gamma_H + 200\delta_H + \psi K_2(12000-2400000\theta_1)) \\ S_{V,1}(t) & = & -\hbar(1-\tau+t\tau)(-600\gamma_V - \beta_1(60000-6000000\theta_2)+\Pi_V) \\ E_{V,1}(t) & = & -\hbar(1-\tau+t\tau)(-600\gamma_V - 300\delta_V + \beta_1(60000-6000000\theta_2)) \\ I_{V,1}(t) & = & 20\hbar\gamma_V - 20h\tau\gamma_V + 20\hbar t\tau\gamma_V - 300\hbar\delta_V + 300\hbar\tau\delta_V - 300\hbar t\tau\delta_V \end{array}$$

4.1.4 Convergence analysis

Theorem 4.1.7 *if the series solutions* $\sum_{i=0}^{\infty} S_{H,i}(t), \sum_{i=0}^{\infty} E_{H,i}(t), \sum_{i=0}^{\infty} I_{H,i}(t)$, $\sum_{i=0}^{\infty} S_{V,i}(t), \sum_{i=0}^{\infty} E_{V,i}(t)$, *and* $\sum_{i=0}^{\infty} I_{V,i}(t)$ *are convergent, where* $S_{H,i}, E_{H,i}, I_{H,i}$, $S_{V,i}, E_{V,i}$, *and* $I_{V,i}$ *is governed by equation* (4.1.13), *then must be solutions of* (4.3.1).

Proof 8 *Suppose that* $\sum_{i=0}^{\infty} S_{H,i}(t), \sum_{i=0}^{\infty} E_{H,i}(t), \sum_{i=0}^{\infty} I_{H,i}(t)$, $\sum_{i=0}^{\infty} S_{V,i}(t), \sum_{i=0}^{\infty} E_{V,i}(t)$, *and* $\sum_{i=0}^{\infty} I_{V,i}(t)$ *are convergent, i.e*

$$\lim_{n\longrightarrow\infty} S_{H,n}(t) = \lim_{n\longrightarrow\infty} E_{H,n}(t) = \lim_{n\longrightarrow\infty} I_{H,n}(t) = 0$$
$$\lim_{n\longrightarrow\infty} S_{V,n}(t) = \lim_{n\longrightarrow\infty} E_{V,n}(t) = \lim_{n\longrightarrow\infty} I_{V,n}(t) = 0$$

From the equation (4.1.12), *we have:*

$$\begin{array}{rcl} \hbar H_1 \sum_{m=1}^{\infty} R_{1,m} & = & \lim_{k\longrightarrow\infty} \sum_{m=0}^{k} L[S_{H,m} - \ell_m S_{H,m-1}] \\ & = & L[\lim_{k\longrightarrow\infty} \sum_{m=0}^{k} [S_{H,m} - \ell_m S_{H,m-1}]] \quad = \quad L[\lim_{k\longrightarrow\infty} S_{H,k}] \end{array}$$

Where L is a Laplace operator. Since $\lim_{k\longrightarrow\infty} S_{H,k} = 0$, $H_1 \neq 0$ *and* $\hbar \neq 0$, *this implies* $\sum_{m=1}^{\infty} R_{1,m} = 0$.*Now, expand* $N_1[\Phi_i(x,t,q)], i = 1, ..., 6$ *about* $q = 0$, *then set* $q = 1$,

$$N[\Phi_i(x,t,1)] = 0,$$

Observing that $S_H(x,t) = \Phi_1(x,t,1) = \sum_{i=0}^{\infty} S_{H,i}(x,t)$ *solve equation* (4.3.1).

Theorem 4.1.8 *Consider the solution component* $S_{H,0}(t), S_{H,1}(t), S_{H,2}(t), ...$ *be defined as* (4.1.13). *The series solution* $\sum_{m=0}^{\infty} S_{H,m}(t)$ *defined in* (4.1.14) *converge if there exist* $0 < \gamma < 1$ *such that* $\|S_{H,m+1}(t)\| \leq \gamma \|S_{H,m}(t)\|, \forall m > m_0$, *where* $m_0 \in \mathbb{N}$.

Proof 9 *consider* $(T_n)_{n\in\mathbb{N}}$ *to be a sequence, defined as* $T_n = S_{H,0} + S_{H,1} + ... + S_{H,n}$. *Proving that* T_n *is a Cauchy sequence in Hilbert space* $\mathbb{R}$.
For every $p, q \in \mathbb{N}, p > q \geq m_0$, *having*

$$\begin{aligned} \| T_p - T_q \| &= \| (S_{H,p} - S_{H,p-1}) + (S_{H,p-1} - S_{H,p-2}) + ... + (S_{H,q+1} - S_{H,q}) \| \\ &\leq \| (S_{H,p} - S_{H,p-1}) \| + \| (S_{H,p-1} - S_{H,p-2}) \| + ... + \| (S_{H,q+1} - S_{H,q}) \| \\ &\leq \gamma^{p-m_0} \| S_{H,m_0} \| + \gamma^{p-m_0-1} \| S_{H,m_0} \| + ... + \gamma^{q-m_0+1} \| S_{H,m_0} \| \\ &\leq (\gamma^{p-m_0} + \gamma^{p-m_0-1} + ... + \gamma^{q-m_0+1}) \| S_{H,m_0} \| \\ &\leq \gamma^{q-m_0+1} \tfrac{\gamma^{p-q}}{1-\gamma} \| S_{H,m_0} \| \end{aligned} \tag{4.1.16}$$

Because of $0 < \gamma < 1$, *ends in*

$$\lim_{p,q\to\infty} \| T_p - T_q \| = 0$$

So, (T_n) *is a Cauchy sequence in* $\mathbb{R}$, *which implies the convergence of* (T_n), *which means that the series solution* $\sum S_{H,i}(t)$ *converge.*

Theorem 4.1.9 *Suppose that the series solution* $\sum_{m=0}^{\infty} S_{H,m}(t)$ *is convergent to the solution* $S_H(t)$. *If the truncated series* $\sum_{m=0}^{k} S_{H,m}(t)$ *is used as approximation to the solution* $S_H(t)$, *then the truncated error satisfies*

$$\| S_H(t) - \sum_{m=0}^{k} S_{H,m}(t) \| \leq \frac{1}{1-\gamma} \gamma^{k+1} \| S_{H,0}(t) \| .$$

Proof 10 *Let* $n, m \in \mathbb{N}$, *using the following inequality :*

$$\| T_n - T_m \| \leq \gamma^{m+1} \frac{\gamma^{n-m}}{1-\gamma} \| S_{H,0} \|$$

4.1.5 Numerical result and discussion

This section applied successfully the homotopy analysis transform method for the problem (4.3.1), giving the first and second-order of the approximation for [43] $\Pi_H = 0.3, K_1 = 0.00022, K_2 = 0.00043, \theta_1 = 0.01, \gamma_H = 0.0048, \delta_H = 0.00317, \psi = 0.0000083, \theta_2 = 0.001, \gamma_V = 0.0028, \delta_V = 0.002, \Pi_V = 0.8, \beta_1 =$

0.0031183 also for different value of τ $\{0.2, 0.4, 0.6, 0.8, 1\}$ resulted in:

$$\begin{aligned}
S_{H,1}(t) &= 1.90802\hbar(1-\tau+t\tau) \\
E_{H,1}(t) &= -1.19\hbar(1-\tau+t\tau) \\
I_{H,1}(t) &= -0.586034\hbar(1-\tau+t\tau) \\
S_{V,1}(t) &= 172.762\hbar(1-\tau+t\tau) \\
E_{V,1}(t) &= -169.602\hbar(1-\tau+t\tau) \\
I_{V,1}(t) &= -5.044\hbar+0.544\hbar\tau-0.544\hbar t\tau
\end{aligned}$$

and

$$\begin{aligned}
S_{H,2}(t) &= 1.90802\hbar(1-\tau+t\tau)+\hbar(-0.0321468\hbar+0.0642936\hbar\tau-0.0321468\hbar\tau^2 \\
&\quad -0.0160734\hbar t^2\tau^2+1.90802\hbar(1-\tau+t\tau)-t(0.0642936\hbar\tau-0.0642936\hbar\tau^2)) \\
E_{H,2}(t) &= -1.19\hbar(1-\tau+t\tau)+\hbar(0.0287188\hbar-0.0574377\hbar\tau+0.0287188\hbar\tau^2 \\
&\quad +0.0143594\hbar t^2\tau^2-1.19\hbar(1-\tau+t\tau)-t(-0.0574377\hbar\tau+0.0574377\hbar\tau^2)) \\
I_{H,2}(t) &= -0.586034\hbar(1-\tau+t\tau)+\hbar(0.00349159\hbar-0.00698319\hbar\tau+0.00349159\hbar\tau^2 \\
&\quad +0.0017458\hbar t^2\tau^2-0.586034\hbar(1-\tau+t\tau)-t(-0.00698319\hbar\tau+0.00698319\hbar\tau^2)) \\
S_{V,2}(t) &= 172.762\hbar(1-\tau+t\tau)+\hbar(49.0795\hbar-98.159\hbar\tau+49.0795\hbar\tau^2+24.5397\hbar t^2\tau^2 \\
&\quad +172.762\hbar(1-\tau+t\tau)-t(-98.159\hbar\tau+98.159\hbar\tau^2)) \\
E_{V,2}(t) &= -169.602\hbar(1-\tau+t\tau)+\hbar(-169.602\hbar(1-\tau+t\tau)-48.4512\hbar(1-2\tau \\
&\quad +\tau^2+\tfrac{t^2\tau^2}{2}-2t(-\tau+\tau^2)) \\
I_{V,2}(t) &= -5.044\hbar-0.206319\hbar^2+0.544\hbar\tau-0.131362\hbar^2\tau-0.544\hbar t\tau+ \\
&\quad 0.131362\hbar^2 t\tau+0.337681\hbar^2\tau^2-0.675362\hbar^2 t\tau^2+0.16884\hbar^2 t^2\tau^2
\end{aligned}$$

HATM solution for S_H, E_H, I_H, S_V, E_V, and I_V is presented in Fig4.12, 4.11 and 4.13 for $\tau = 1$, For different value of τ ($\tau = 0.8, 0.6, 0.4, 0.2$) seeing that the result of model depends on the value of fractional order of derivative τ.

4.1.6 Concluding remarks

This work, the pine wild disease is modeling into a system of fractional differential equations, in sense of Caputo-Fabrizio fractional derivative. Construct a successful of an analytical study of the epidemic model using the homotopy analysis transform method, giving an approximation of solutions, and plotting the curve-solutions for different values of τ. Seeing that the presented method is very efficient for this type of differential equations.

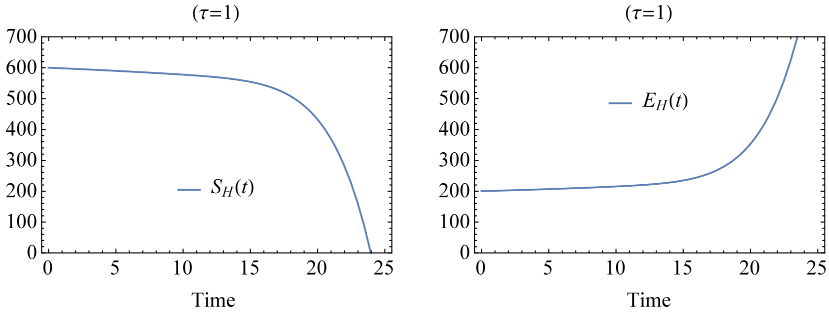


Figure 4.1: The HATM solution of S_H, and E_H

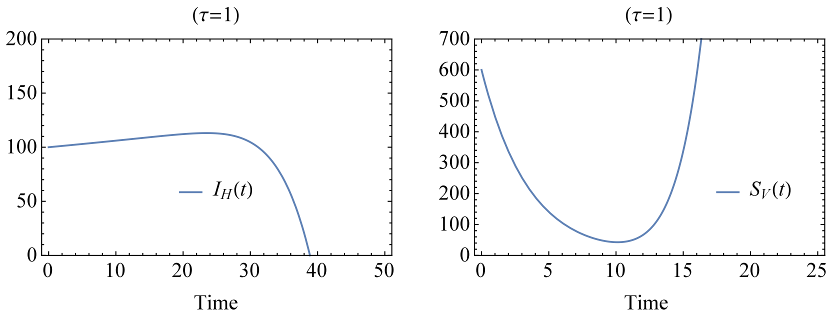


Figure 4.2: The HATM solution of I_H, and S_V

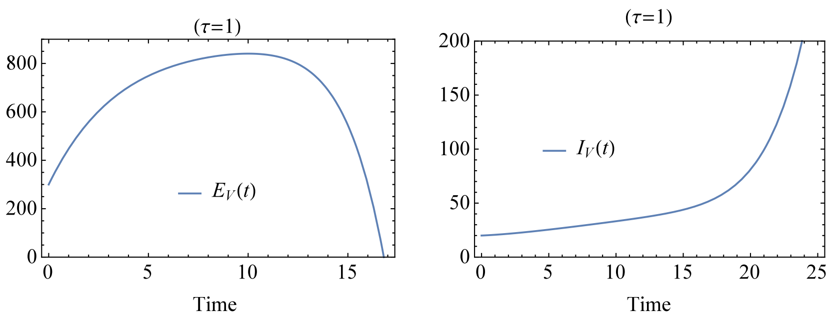


Figure 4.3: The HATM solution of E_V, and I_V

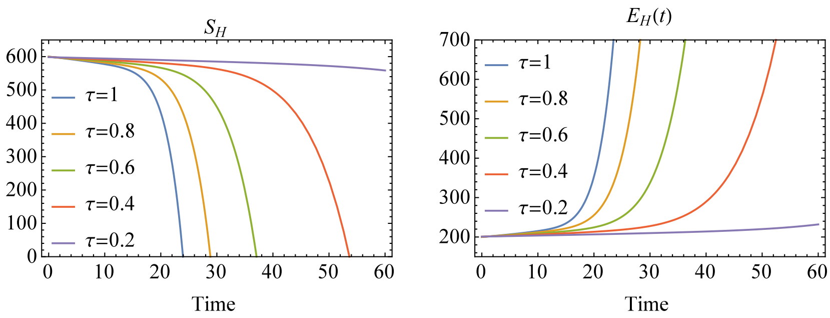


Figure 4.4: The HATM solution of S_H, and E_H for different value of τ

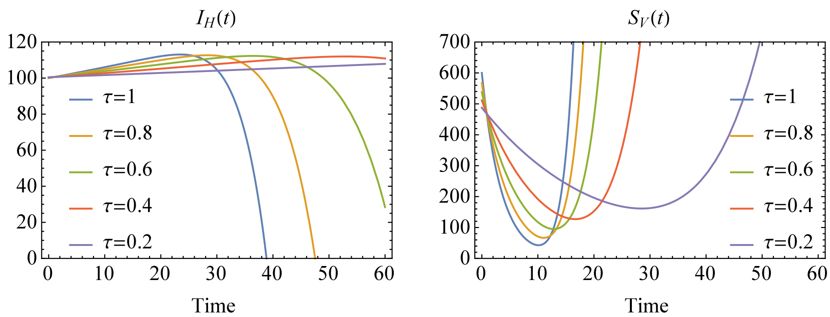


Figure 4.5: The HATM solution of I_H, and S_V for different value of τ

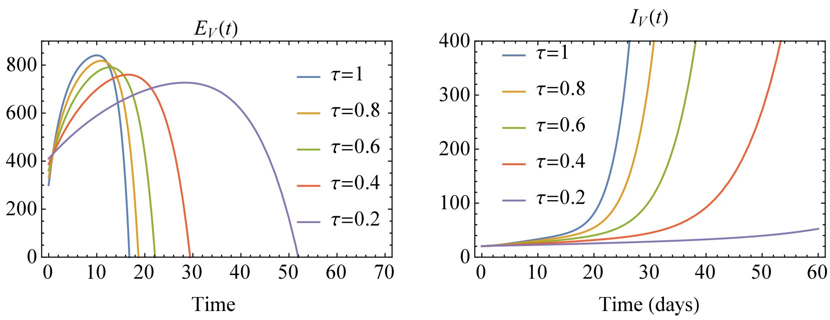


Figure 4.6: The HATM solution of E_V, and I_V for different value of τ

4.2 Comparative numerical study of Forenberg-Whitham equation

4.2.1 Introduction

Differential equations are one of the most effective tools for formulating the real-life problem in several disciplines [9]. Furthermore, fractional differential equations are one of the important branches in the field of differential equations, because of various phenomena in physics and chemistry, electronic and electrical, medicine and epidemiology spread, engineering, can be modeled into a fractional differential equation [7, 8, 2, 27]. Fornberg-Whitham in 1978 proposed their equation while they studied the solitary wave phenomena

$$u_t - u_{xxt} + u_x = u_{xxx}u - u_x u + 3u_x u_{xx} \tag{4.2.1}$$

Many mathematician researchers deal with the solution of Fornberg-Whitham using different numerical methods, to obtain good approximations and their properties, such as the variational iteration method used by Lu (2011)[30], Sakar et al. (2012), Sakar and Ergren (2015)[37][38], Ibis and Bayram (2014)[16], and Merdan et al. (2012)[39] used the differential transformation method. Lanre Akinyemi applied the Shehu transform method Akinyemi and Iyiola (2020), homotopy perturbation method investigated by Gupta and Singh (2011)[49], and homotopy analysis method (HAM) in Abidi and Omrani (2010)[23] was used by Sakar and Erdogan (2013)[40]

The homotopy analysis method is one of the most important numerical methods for solving linear and nonlinear differential equations, including fractional differential equations. HAM proposed in Ph.D. dissertation in 1992 of Liao (1992)[56],Liao (2009)[57], and Liao (2012)[57]. The method has many applications in various classes of famous differential equations. For fractional differential equations, the scientists used the coupled Laplace transform with the homotopy analysis method (HATM) to write a simple algorithm corresponding to those equations, easily solved by Mathematica and Maple. The method introduced by Kumar (2014a)[62] Kumar (2014b)[63], Kumar et al. (2014)[19], Khan et al. (2012)[35].

In this paper, an investigation of the HATM for the following fractional differential equation Alderremy et al. (2020)[10]

$$D_t^{\alpha}u - D_{xxt}u + D_x u = uD_{xxx}u - uD_x u + 3D_x u D_{xx}u, \tag{4.2.2}$$

where D^{α} is the fractional derivative operator of order α in the sense of Caputo. Convergence analysis is provided and a comparison of results with the result obtained by Alderremy [20], who used the natural decomposition method (NDM).

4.2.2 Basic definitions of fractional calculus

This section gives some basic definitions of fractional calculus theory used in this paper:

Definition 4.2.1 *A real function $h(t)$ is said C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$ where $h_1(t) \in C(0, \infty)$ and it is said to be in space C_n if and only if $h^{(n)} \in C_\mu, n \in \mathbb{N}$*

Definition 4.2.2 *The Riemann-Liouville fractional operator of order $\alpha \geq 0$, of a function $h \in C_\alpha, \alpha \geq -1$ is defined as Lu (2001) [27]*

$$\begin{array}{rcl} I^{\alpha}h(t) & = & = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} h(s)ds, \mu > 0, t > 0 \\ I^0 h(t) & = & h(t) \end{array}$$

where $\Gamma(.)$ is well-known Gamma function.

Definition 4.2.3 *The Caputo fractional derivative of h, $h \in C_{-1}^m$ is defined as*

$$^{C}D^{\alpha}h(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\varsigma)^{(m-\alpha-1)} h^{(m)}(\varsigma)d\varsigma,$$

where $m - 1 < \alpha < m, m \in \mathbb{N}$

Definition 4.2.4 *The Laplace transform for the function $h(t)$ of the Caputo fractional derivative given by Kumar (2014b)[63]:*

$$L[D_t^{\alpha}h(t)] = s^{\alpha}L[h(t)] - \sum_{k=0}^{n-1} s^{(\alpha-k-1)}h^{(k)}(0), \quad n-1 < \alpha \leq n \tag{4.2.3}$$

4.2.3 The homotopy analysis transform method

initially giving some basic ideas of the homotopy analysis transform method (HATM), considering the following fractional differential equation to illustrate the application of the HATM for the fractional differential equation:

$$D_t^{\alpha}h(x,t) + \mathfrak{R}h(x,t) + \mathfrak{N}h(x,t) = f(x,t), \quad 0 < \alpha \leq 1 \tag{4.2.4}$$

where $D_t^{\alpha}h(x,t)$ is the Caputo derivative of $h(x,t)$, $\mathfrak{R}$ and $\mathfrak{N}$ are the linear and nonlinear operator respectively, $f(x,t)$ is the source term. firstly, by applying the Laplace transform L on the (4.2.2), resulting in

$$s^{\alpha}L[h(x,t)] - s^{\alpha-1}h(x,0) + L[\mathfrak{R}h(x,t)] + L[\mathfrak{N}h(x,t)] = L[f(x,t)], \quad 0 < \alpha \leq 1 \tag{4.2.5}$$

after simplification,

$$L[h(x,t)] = \frac{1}{s}h(x,0) + \frac{1}{s^{\alpha}}(L[f(x,t)] - L[\mathfrak{R}h(x,t)] - L[\mathfrak{N}h(x,t)]), \quad 0 < \alpha \leq 1 \tag{4.2.6}$$

By applying the homotopy analysis method shown in Gupta and Singh (2011)[49], Abidi and Omrani (2010)[23] and Sakar and Erdogan (2013)[40], defining the non-linear operator

$$N[\phi(x,t,q)] = L[\phi(x,t,q)] - \frac{1}{s}h(x,0) - \frac{1}{s^{\alpha}}(L[f(x,t)] + L[\mathfrak{R}h(x,t)] + L[\mathfrak{N}h(x,t)]) \tag{4.2.7}$$

where $\phi(t,q)$ is a real-valued function of t and $q \in [0,1]$. The zeroth-order deformation constructed by Liao Gupta and Singh (2011)[**?**],Abidi and Omrani (2010)[**?**], is

$$(1-q)\mathcal{L}[\phi(x,t,q) - h_0(x,t)] = \hbar q N[\phi(x,t,q)] \tag{4.2.8}$$

where $\hbar \neq 0$ is nonzero convergent control parameter, $h_0(x,t)$ is the initial guess, N is the nonlinear operator and $\mathcal{L}$ is an injective linear operator. Here, choosing the linear operator as a Laplace operator $\mathcal{L} = L$. Obviously $\phi(x,t,0) = u_0(x,t)$ and $\phi(x,t,1) = h(x,t)$ expanding $\phi(x,t,q)$ in Taylor series with respect q and t,

$$\phi(x,t,q) = \sum_{i=0}^{n} h_i(x,t)q^i$$

where

$$h_i(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t,q)}{\partial q^m}|_{q=0}, \tag{4.2.9}$$

by differentiating (4.2.8) m times with respect q, and taking $q = 0$, resulting in the mth order deformation equation

$$L[h_m(x,t) - \ell_m h_{m-1}(x,t)] = \hbar H(x,t) R_m(h_{m-1}(x,t)) \tag{4.2.10}$$

Applying L^{-1} in (4.2.10) ends in:

$$h_m(x,t) = \ell_m h_{m-1}(x,t) + \hbar L^{-1}[R_m(h_{m-1}(x,t))] \tag{4.2.11}$$

where

$$\ell_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Now, for equation (4.2.2) defining a nonlinear operator as:

$$N[\phi(x,t,q)] = L[\phi] - \frac{1}{s}\phi(x,0,q) + \frac{1}{s^\alpha}\left(-\frac{\partial^3\phi}{\partial x^2 \partial t}\phi + \frac{\partial\phi}{\partial x} - \phi\frac{\partial^3\phi}{\partial x^3} + \phi\frac{\partial\phi}{\partial x} - 3\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2}\right) \tag{4.2.12}$$

the mth order deformation equation is

$$L[h_m(x,t) - l_m h_{m-1}(x,t)] = \hbar H(x,t) K_m[\overrightarrow{h}_{m-1}(x,t)] \tag{4.2.13}$$

where

$$\begin{aligned} K_n \;=\; & h_{n-1}(x,t) - \tfrac{(1-l_m)}{s} h_{n-1}(x,0) + \tfrac{1}{s^\alpha}(L[\tfrac{\partial^3 h_{n-1}}{\partial x^2 \partial t} + \tfrac{\partial h_{n-1}}{\partial x} - \textstyle\sum_{i=0}^{n-1} h_i \tfrac{\partial^3 h_{n-1-i}}{\partial x^3}] \\ & + L[\textstyle\sum_{i=0}^{n-1} h_i \tfrac{\partial h_{n-1-i}}{\partial x} - 3\sum_{i=0}^{n-1} \tfrac{\partial h_i}{\partial x}\tfrac{\partial^2 h_{n-1-i}}{\partial x^2}]) \end{aligned}$$

By applying the inverse Laplace operator for (4.2.13), results in:

$$h_m(x,t) = \ell_m h_{m-1}(x,t) + \hbar L^{-1} K_m[\overrightarrow{h}_m(x,t)], \tag{4.2.14}$$

where L^{-1} in inverse Laplace transform operator, and in this case, a mention that $u(x,t)$ is written as a series

$$u(x,t) = h_0(x,t) + \sum_{i=1}^{\infty} h_i(x,t) \tag{4.2.15}$$

with the initial conditions,

$$u(x,0) = h_0(x,t)$$

Example 4.2.5 *A solution to the above equation* (4.2.14) *with the initial condition:*

$$u(x,0) = e^{\frac{x}{2}}, \tag{4.2.16}$$

obtaining the first-order approximation :

$$\begin{aligned} h_1(x,t) &= \frac{\hbar e^{\frac{x}{2}} t^{\alpha}}{2\Gamma[\alpha+1]} \\ h_2(x,t) &= \frac{\hbar e^{\frac{x}{2}} t^{\alpha}}{2\Gamma[\alpha+1]} + \frac{\hbar^2 e^{\frac{x}{2}} t^{\alpha}}{2\Gamma[\alpha+1]} - \frac{\hbar^2 e^{\frac{x}{2}} t^{2\alpha-1}}{8\Gamma[2\alpha]} + \frac{\hbar^2 e^{\frac{x}{2}} t^{2\alpha}}{4\Gamma[1+2\alpha]}. \end{aligned}$$

So the approximate solution becomes

$$u(x,t) = \sum_{i=0}^{M} h_i(x,t).$$

Example 4.2.6 *For the second test example, consider* (4.3.2)*, with the new initial condition*

$$u(x,0) = cosh^2(\frac{x}{4}) \tag{4.2.17}$$

Applying the HATM, ends in

$$u_1(x,t) = -\hbar e^{\frac{x}{2}} + \hbar \cosh^2(\frac{x}{4}) + \frac{\hbar t^{\alpha} \cosh(\frac{x}{4}) \sinh(\frac{x}{4})}{2\Gamma[1+\alpha]} + \frac{(3\hbar t^{\alpha} \cosh(\frac{x}{4}) \sinh(\frac{x}{4}))(\cosh^2(\frac{x}{4}) - \sinh^2(\frac{x}{4}))}{16\Gamma[1+\alpha]}$$

$$u_2(x,t) = -\hbar e^{\frac{x}{2}}(1+\hbar) + \hbar \cosh^2(\frac{x}{4})(1+\hbar) - \frac{11\hbar^2 \cosh(\frac{x}{2})}{16\Gamma[1+\alpha]} + \frac{121\hbar^2 t^{2\alpha} \cosh(\frac{x}{2})}{512\Gamma[1+2\alpha]} + \ldots.$$

So the approximate solution becomes

$$u(x,t) = \sum_{i=0}^{M} h_i(x,t).$$

4.2.4 Convergence analysis

Theorem 4.2.7 *If the series solutions* $\sum_{i=0}^{n} h_i(x,t)$ *is convergent, where* h_m *is governed by equation* (4.2.9)*, then must be solutions of* (4.2.2).

Proof 11 *Suppose that* $\sum_{i=0}^{n} h_i(x,t)$ *is convergent, i.e* $\lim_{n\longrightarrow\infty} h_n(x,t) = 0$. *From the equation* (4.2.13)*, we get:*

$$\begin{aligned} \hbar H \sum_{m=1}^{\infty} K_m &= \lim_{k\longrightarrow\infty} \sum_{m=0}^{k} L[h_m - \ell_m h_{m-1}] \\ &= L[\lim_{k\longrightarrow\infty} \sum_{m=0}^{k} [h_m - \ell_m h_{m-1}]] \;=\; L[\lim_{k\longrightarrow\infty} h_k] \end{aligned}$$

Where L is a Laplace operator. Since $\lim_{k\longrightarrow\infty} h_k = 0$, $H \neq 0$ and $\hbar \neq 0$, this implies $\sum_{m=1}^{\infty} K_m = 0$.Now, expand $N[\Phi(x,t,q)]$ about $q = 0$, then set $q = 1$,

$$N[\Phi(x,t,1)] = 0,$$

witnessing that $h(x,t) = \Phi(x,t,1) = \sum_{i=0}^{\infty} h_i(x,t)$ solve equation (4.2.2).

Theorem 4.2.8 *Alomari et al. (2020)[12] Let the solution component*

$$h_0(x,t), h_1(x,t), h_2(x,t), ...$$

be defined as (4.2.11). The series solution $\sum_{m=0}^{\infty} h_m(x,t)$ defined in (4.2.15) converge if there exist $0 < \gamma < 1$ such that $\|h_{m+1}(x,t)\| \leq \gamma \|h_m(x,t)\|, \forall m > m_0$, for some $m_0 \in \mathbb{N}$.

Theorem 4.2.9 *Alomari et al. (2020)[12] Suppose that the series solution $\sum_{m=0}^{\infty} h_m(x,t)$ is convergent to the solution $h(x,t)$. If the truncated series $\sum_{m=0}^{k} h_m(x,t)$ is used as an approximation to the solution $h(x,t)$, then the truncated error satisfies*

$$\| \, h(x,t) - \sum_{m=0}^{k} h_m(x,t) \, \| \leq \frac{1}{1-\gamma} \gamma^{k+1} \, \| \, h_0(x,t) \, \| \, .$$

Theorem 4.2.10 *Suppose that J is Lipschitz with Lipschitz constant γ, and $\hbar$ is optimally chosen. The solution of the IVP (4.2.2) subject to $u(x,0) = f(x)$ by mean of HATM, $0 \leq t \leq T$, is unique whenever $K = (1+\hbar) - \hbar\gamma T^{\alpha}/\Gamma(\alpha+1) < 1$.*

Proof 12 *Suppose that u and u^* both solve the equations (4.2.2) subject to $u(x,0) = f(x)$ for $0 \leq t \leq T$. By adding h_0 for both side of Eq. (4.2.13)and take the sum, the convolution property follows that*

$$\begin{aligned} u - u^* &= (1+\hbar)(u-u^*) - \hbar L^{-1}\left[\frac{1}{s^{\alpha}} L\left[J(u) - J(u^*)\right]\right] && (4.2.18) \\ &= (1+\hbar)(u-u^*) - \hbar \int_0^t \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)} \left(J(u) - J(u^*)\right) d\xi && (4.2.19) \end{aligned}$$

Then from the Lipschitz condition seeing that

$$\max|u-u^*| \leq (1+\hbar)\max|u-u^*| - \hbar\gamma \max|u-u^*| \int_0^t \frac{(t-\xi)^{\tau-1}}{\Gamma(\alpha)} d\xi \leq K \max|u-u^*|.$$

Since $K < 1$, it must be that $u = u^*$ showing the uniqueness of the solution.

4.2.5 Numerical result and discussion

Initially discussing the example 1. The HATM solution has convergent control parameter $\hbar$, to choose proper value of $\hbar$, firstly plot the so-called $\hbar$-curve for example 1 in Fig.1. It can be seen from this figure that the best value of $\hbar$ is in the convergence region $-1.6 \leq \hbar \leq -0.6$, so there is a freedom of choice $\hbar$ in this region to get a convergence series. To find a proper value in this region there must be a definition of the residual error

$$\text{Residualerror}(x,t,\hbar) = D_t^{\alpha}u - D_{xxt}u + D_x u = uD_{xxx}u - uD_x u + 3D_x u D_{xx}u,$$

where u is the HATM solution. Using the least-square method, defining the average residual error

$$ARE(\hbar) = \frac{1}{(M+1)(N+1)} \sum_{i=0}^{M} \sum_{j=0}^{N} (\text{Residualerror}(\frac{ai}{M}, \frac{j}{N}, \hbar))^2$$

where a is the Maximum value of x. By minimizing the ARE the optimal value of $\hbar$ can be obtained. At this line, finding that the optimal value of $\hbar = -1.29427$. In Fig. 2, ploting the 10 order approximation solutions and the exact solution of example 1, it is obvious that the two graphs coincide.

In table 4.2.11, comparing HATM solution with other results obtained using the NDM solutions Alderremy et al. (2020)[10], and the LDM solutions Kumar et al. (2018)[20]. In Table 4.2.12 the approximation solutions for different value of α are given.

Table 4.2.11 *Comparison of HATM solution with exact solution and the* *absolute error for with NDM solution for $\alpha = 1$ for example 1.*

t	x	*Exact*	*HATM($\alpha = 1$)*	*Error (HATM)*	*Error(NDM)*
	0.5	1.123744785	1.123744771	1.43576×10^{-8}	6.620×10^{-5}
0.2	1	1.442916866	1.442916848	1.84355×10^{-8}	8.500×10^{-5}
	1.5	1.852741930	1.852741907	2.36717×10^{-8}	1.090×10^{-4}
	2	2.378967729	2.378967699	3.03951×10^{-8}	1.400×10^{-4}
	0.5	0.983471453	0.983471409	4.45751×10^{-8}	1.070×10^{-4}
0.4	1	1.262802343	1.262802286	5.72355×10^{-8}	1.380×10^{-4}
	1.5	1.621470305	1.621470231	7.34919×10^{-8}	1.770×10^{-4}
	2	2.082009084	2.082008989	9.43654×10^{-8}	2.270×10^{-4}
	0.5	0.659240630	0.659240710	7.99801×10^{-8}	1.84×10^{-4}
1	1	0.846481724	0.846481827	1.02696×10^{-7}	2.370×10^{-4}
	1.5	1.086904049	1.086904181	1.31865×10^{-7}	3.004×10^{-4}
	2	1.395612425	1.395612594	1.69388×10^{-7}	3.900×10^{-4}

Table 4.2.12 *The exact solution ($\alpha = 1$) and HATM solution for different values of α for example 1.*

t	x	*Exact*	*HATM* $\alpha = 0.5$	*HATM* $\alpha = 0.6$	*HATM* $\alpha = 0.8$
	0.5	0.920044414	0.826307039	0.828603386	0.854518952
0.5	1	1.181360412	1.060999240	1.063947808	1.097342211
	1.5	1.516896796	1.362349992	1.366136028	1.409015289
	2	1.947734041	1.749292016	1.754153383	1.809211444
	0.5	0.659240630	0.732037711	0.710800444	0.672664365
1	1	0.846481724	0.939955077	0.912685837	0.863718142
	1.5	1.086904049	1.206926145	1.171911812	1.109036047
	2	1.395612425	1.549723846	1.504764553	1.424030472
	0.5	0.472366552	0.675945925	0.633909353	0.553160181
1.5	1	0.606530659	0.867931749	0.813955721	0.710271731
	1.5	0.778800783	1.114446425	1.045139834	0.912006956
	2	1	1.430977536	1.341986111	1.171040112

The above tables show that the approximation solution obtained by HATM is agreed on and appropriate with the exact solution more than the NDM method.

For the second example, follow the same manner for obtaining the optimal value of $\hbar$. The $\hbar-$curve is plotted in Fig. 4.14. The least-square method gives the optimal value of $\hbar = -1.33034$. Comparing with the exact solution and the approximate solutions for different values of α in Table 4.2.13. In Fig. 4, plotting the 10th order of HATM with the exact solution $u(x,t) = \cosh^2(\frac{x}{2} - \frac{11t}{24})$.

Table 4.2.13 *The exact solution ($\alpha = 1$) and HATM solution for different values of α for example 2.*

t	x	*Exact*	*HATM* $\alpha = 1$	*Error (HATM)*	*HATM* $\alpha = 0.9$	*HATM* $\alpha = 0.7$
	0.5	1.001111522	1.001111125	2.63096×10^{-7}	1.002172640	1.028552241
0.2	1	1.025279638	1.025279428	2.09726×10^{-7}	1.019156915	1.025170889
	1.5	1.082449077	1.082448907	1.69532×10^{-7}	1.068757845	1.054784028
	2	1.176211578	1.176211438	1.39989×10^{-7}	1.154091670	1.119252140
	0.5	1.003406639	1.003404733	1.90616×10^{-6}	1.015654733	1.075188533
0.4	1	1.004451032	1.004449548	1.48449×10^{-6}	1.007948014	1.045867157
	1.5	1.037188167	1.037187011	1.15608×10^{-6}	1.032153739	1.050840540
	2	1.103674793	1.103673893	9.00307×10^{-7}	1.089792660	1.090421141
	0.5	1.115287790	1.115215382	7.24072×10^{-5}	1.170012652	1.304089624
1	1	1.044034356	1.043977948	5.64083×10^{-5}	1.088255393	1.198428590
	1.5	1.006960534	1.006916581	4.39533×10^{-5}	1.043455985	1.136647169
	2	1.001737116	1.001702856	3.42598×10^{-5}	1.032799851	1.114863871

Concluding remarks

This work successfully applied the HATM for a Fornberg-Whitham equation, and shows that this method is efficient and applicable for this type of fractional differential equation. Compared results with others obtained by different numerical methods. Several results for convergent and unique solutions are provided.

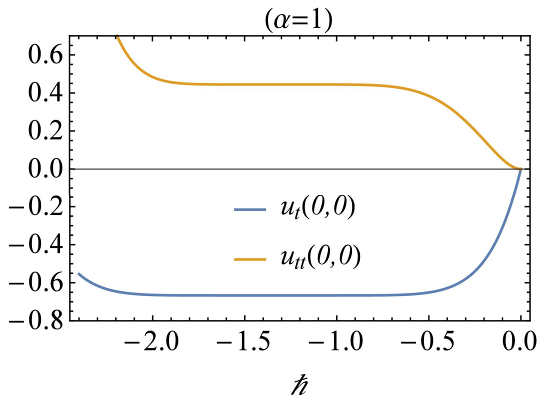


Figure 4.7: The $\hbar-$curve for the example 1

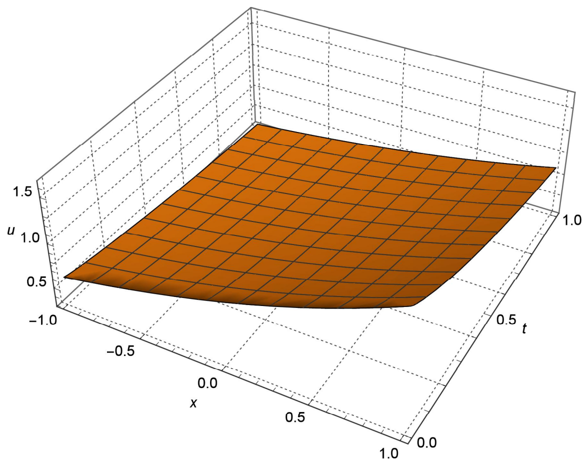


Figure 4.8: The approximate solution for example 1

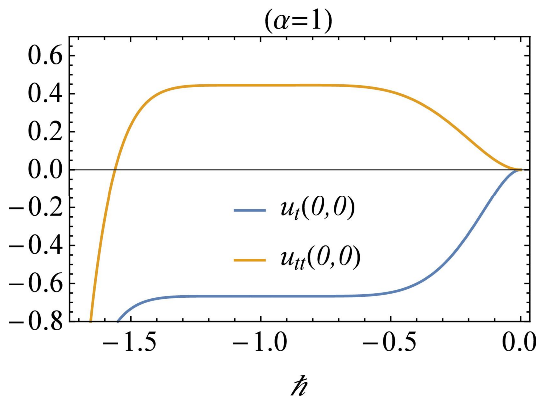


Figure 4.9: The $\hbar-$curve for the example 2

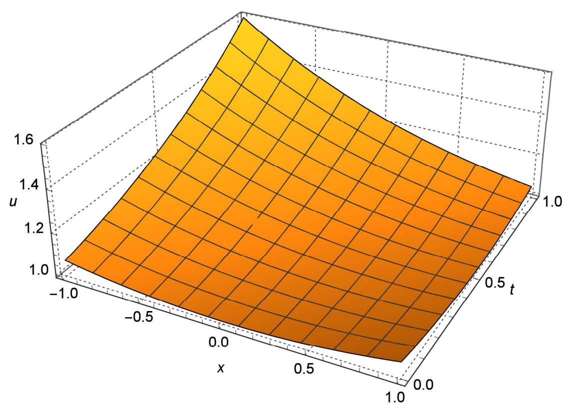


Figure 4.10: The approximate solution for example 2

4.3 Comparative numerical study of time fractional coupled Korteweg-De Vries equation

4.3.1 Introduction

Fractional differential equations are one of the important branches in the field of differential equations, because of various phenomena in physics and chemistry, electronic and electrical, medicine and epidemiology spread, engineering, can be formulated into a fractional differential equation such as the coupled Korteweg-de Vries equation[41]:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \eta \frac{\partial^3 u(x,t)}{\partial x^3} + \gamma u(x,t) \frac{\partial u(x,t)}{\partial x} + \mu v(x,t) \frac{\partial v(x,t)}{\partial x}, \\ \frac{\partial v(x,t)}{\partial t} &= \lambda \frac{\partial^3 v(x,t)}{\partial x^3} - \nu v(x,t) \frac{\partial v(x,t)}{\partial x}. \end{aligned} \tag{4.3.1}$$

The solution of coupled Korteweg-de Vries is introduced using different numerical methods, to obtain the accurate approximations and their properties, such as the Spectral collocation method used by Khader [42], and Hussain [36]. Bashan [5] used the Finite difference method and differential quadrature method. Bakodah applied the decomposition method [**?**], homotopy perturbation method investigated by Bothayna S.Kashkari [17], homotopy analysis method (HAM) in [51] used by Abbsbandy and homotopy analysis transform method [31] by Saad.
The homotopy analysis method is one of the most important approximate methods for solving the linear and nonlinear differential equations, as partial differential equations[13], delay differential equtions[47], fuzzy differential equations[6], including the fractional differential equations[52]. HAM proposed in Ph.D. dissertation in 1992 of Shijun Liao [56],[57]. The method has many applications in various classes of famous differential equations. For fractional differential equations, the scientists used the coupled Laplace transform with the homotopy analysis method (HATM), to write a simple algorithm corresponding to this type of equations, easily solved by Mathematica and Maple. The method introduced by Sunil Kumar [62] [63], Devendra Kumar [19], Majid Khan [35].
In this paper, an investigation of the HATM for the following fractional dif-

ferential equation [41]

$$\begin{array}{rcl} {}^{C}D_t^{\alpha}u(x,t) & = & \eta\frac{\partial^3 u(x,t)}{\partial x^3} + \gamma u(x,t)\frac{\partial u(x,t)}{\partial x} + \mu v(x,t)\frac{\partial v(x,t)}{\partial x}, \\ {}^{C}D_t^{\beta}v(x,t) & = & \lambda\frac{\partial^3 v(x,t)}{\partial x^3} - \nu u(x,t)\frac{\partial v(x,t)}{\partial x}, \end{array} \tag{4.3.2}$$

where ${}^{C}D^{\alpha}$, ${}^{C}D^{\beta}$ are the fractional derivative operator of order α, and β in the sense of Caputo. Comparing the numerical solutions with the result obtained by Mohamed Elbadri [41], who used the Naturel decomposition method (NDM), and spectral collection method [15] .

4.3.2 Basic definitions of fractional calculus

This section gives some basic definitions of fractional calculus theory used in this paper:

Definition 4.3.1 *A real function $h(t)$ is said C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $h(t) = t^p h_1(t)$ where $h_1(t) \in C(0, \infty)$ and it is said to be in space C_n if and only if $h^{(n)} \in C_\mu, n \in \mathbb{N}$.*

Definition 4.3.2 *The Riemann-Liouville fractional operator of order $\alpha \geq 0$, of a function $h \in C_\alpha, \alpha \geq -1$ is defined as [41]*

$$\begin{array}{rcl} I^{\alpha}h(t) & = & = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\alpha)^{(\alpha-1)}h(\tau)d\tau, \mu > 0, t > 0, \\ I^0 h(t) & = & h(t), \end{array}$$

where $\Gamma(.)$ is well-known Gamma function.

Definition 4.3.3 *The Caputo fractional derivative of h, $h \in C_{-1}^m$ is defined as*

$$D_C^{\alpha}h(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\varsigma)^{(m-\alpha-1)}h^{(m)}(\varsigma)d\varsigma,$$

where $m - 1 < \alpha < m, m \in \mathbb{N}$

Definition 4.3.4 *The Laplace transform for the function $h(t)$ of the Caputo fractional derivative given by [63]:*

$$L[D_t^{\alpha}h(t)] = s^{\alpha}L[h(t)] - \sum_{k=0}^{n-1} s^{(\alpha-k-1)}h^{(k)}(0), \quad n-1 < \alpha \leq n. \tag{4.3.3}$$

4.3.3 The homotopy analysis transform method

Giving some basic ideas of the homotopy analysis transform method, by considering the following fractional differential equation to illustrate the application of the HATM for the fractional differential equation:

$$ {}^{C}D^{\alpha}h(x,t) + \mathfrak{R}h(x,t) + \mathfrak{N}h(x,t) = f(x,t), \quad 0 < \alpha \leq 1, \tag{4.3.4}$$

where ${}^{C}D_{t}^{\alpha}h(x,t)$ is the Caputo derivative of $h(x,t)$, $\mathfrak{R}$ and $\mathfrak{N}$ are the linear and nonlinear operator respectively, $f(x,t)$ is the source term.
Initially, by applying the Laplace transform L on the (4.3.4), results in

$$ s^{\alpha}L[h(x,t)] - s^{\alpha-1}h(x,0) + L[\mathfrak{R}h(x,t)] + L[\mathfrak{N}h(x,t)] = L[f(x,t)], \quad 0 < \alpha \leq 1 \tag{4.3.5}$$

after simplification,

$$ L[h(x,t)] = \frac{1}{s}h(x,0) + \frac{1}{s^{\alpha}}(L[f(x,t)] - L[\mathfrak{R}h(x,t)] - L[\mathfrak{N}h(x,t)]), \quad 0 < \alpha \leq 1. \tag{4.3.6}$$

By applying the homotopy analysis method shown in [18],[51] and [31], defines the non-linear operator

$$ N[\phi(x,t,q)] = L[\phi(x,t,q)] - \frac{1}{s}h(x,0) - \frac{1}{s^{\alpha}}(L[f(x,t)] + L[\mathfrak{R}\phi(x,t,q)] + L[\mathfrak{N}\phi(x,t,q)]). \tag{4.3.7}$$

where ϕ is a real-valued function of x, t, and $q \in [0,1]$. The zeroth-order deformation constructed by Liao [18],[51] is

$$ (1-q)\mathcal{L}[\phi(x,t,q) - h_0(x,t)] = \hbar q N[\phi(x,t,q)], \tag{4.3.8}$$

where $\hbar \neq 0$ is nonzero convergent control parameter, $h_0(x,t)$ is the initial guess, N is the nonlinear operator and $\mathcal{L}$ is an injective linear operator. Here, choosing the linear operator as a Laplace operator $\mathcal{L} = L$. Obviously $\phi(x,t,0) = u_0(x,t)$ and $\phi(x,t,1) = h(x,t)$. Expanding $\phi(x,t,q)$ in Taylor series with respect q,

$$ \phi(x,t,q) = \sum_{i=0}^{n} h_i(x,t)q^i, $$

where

$$ h_i(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t,q)}{\partial q^m}|_{q=0}, \tag{4.3.9}$$

by differentiating (4.3.8) m times with respect q, and taking $q = 0$, obtaining the mth order deformation equation

$$L[h_m(x,t) - \ell_m h_{m-1}(x,t)] = \hbar H(x,t) R_m(h_{m-1}(x,t)). \qquad (4.3.10)$$

Applying L^{-1} in (4.3.10) we get:

$$h_m(x,t) = \ell_m h_{m-1}(x,t) + \hbar L^{-1}[R_m(h_{m-1}(x,t))], \qquad (4.3.11)$$

where

$$\ell_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

Now, for the system (4.3.2), defines a system of nonlinear operator as:

$$\begin{cases} N_1[\phi_(x,t,q), \psi_(x,t,q)] &= L[\phi] - \frac{1}{s}u(x,0) - \frac{1}{s^\alpha}(\eta \frac{\partial^3 \phi}{\partial x^3} + \gamma \phi \frac{\partial \phi}{\partial x} + \mu \psi \frac{\partial \psi}{\partial x}), \\ N_2[\phi_(x,t,q), \psi_(x,t,q)] &= L[\psi] - \frac{1}{s}v(x,0) - \frac{1}{s^\beta}(\lambda \frac{\partial^3 \psi}{\partial x^3} - \nu \phi \frac{\partial \psi}{\partial x}) \end{cases} \qquad (4.3.12)$$

the mth order deformation equation is

$$\begin{aligned} L[u_m(x,t) - l_m u_{m-1}(x,t)] &= L[u_m(x,t) - l_m u_{m-1}(x,t)] \\ &= \hbar H(x,t) K_m[\overrightarrow{u}_{m-1}(x,t), \overrightarrow{v}_{m-1}(x,t)], \\ L[v_m(x,t) - l_m v_{m-1}(x,t)] &= L[u_m(x,t) - l_m u_{m-1}(x,t)] \\ &= \hbar H(x,t) R_m[\overrightarrow{u}_{m-1}(x,t), \overrightarrow{v}_{m-1}(x,t)], \end{aligned} \qquad (4.3.13)$$

where

$$\begin{aligned} K_m &= u_{n-1}(x,t) - \frac{(1-l_m)}{s} u_0(x,t) + \frac{1}{s^\alpha}(L[\eta \frac{\partial^3 u_{n-1}}{\partial x^3} \\ &\quad + \gamma \textstyle\sum_{i=0}^{n-1} u_i(x,t) \frac{\partial u_{n-1-i}}{\partial x} + \mu \sum_{i=0}^{n-1-i} v_i \frac{\partial v_{n-1-i}}{\partial x}]), \\ R_m &= v_{n-1}(x,t) - \frac{(1-l_m)}{s} v_0(x,t) + \frac{1}{s^\beta}(L[\lambda \frac{\partial^3 v_{n-1}}{\partial x^3} - \nu \textstyle\sum_{i=0}^{n-1} u_i(x,t) \frac{\partial v_{n-1-i}}{\partial x}]). \end{aligned}$$

By applying the inverse Laplace operator for (4.3.13), results in:

$$\begin{aligned} u_m(x,t) &= \ell_m u_{m-1}(x,t) + \hbar L^{-1} K_m[\overrightarrow{u}_m(x,t), \overrightarrow{v}_m(x,t)], \\ v_m(x,t) &= \ell_m v_{m-1}(x,t) + \hbar L^{-1} R_m[\overrightarrow{u}_m(x,t), \overrightarrow{v}_m(x,t)], \end{aligned} \qquad (4.3.14)$$

where L^{-1} is the inverse Laplace transform operator, and in this case, mentioning that $u(x,t)$, and $v(x,t)$ is written as a series

$$\begin{aligned} u(x,t) &= u_0(x,t) + \textstyle\sum_{i=1}^{\infty} u_i(x,t), \\ v(x,t) &= v_0(x,t) + \textstyle\sum_{i=1}^{\infty} v_i(x,t), \end{aligned} \qquad (4.3.15)$$

with the initial conditions,

$$u(x,0) = u_0(x,t), \qquad v(x,0) = v_0(x,t).$$

4.3.4 Application

This section evaluates the coupled Korteweg-de Vries (4.3.2) using HATM with two different initial conditions.

Example 4.3.5 *Considering* (4.3.2) *with* $\eta = -a, \gamma = -6a, \mu = 2b, \lambda = -r$ *and* $\nu = 3r$*:*

$$u(x,0) = \tfrac{\zeta}{a}\left(sech(\tfrac{1}{2}\sqrt{\tfrac{\zeta}{a}}x)\right)^2, \qquad v(x,0) = \tfrac{\zeta}{\sqrt{2a}}\left(sech(\tfrac{1}{2}\sqrt{\tfrac{\zeta}{a}}x)\right)^2. \tag{4.3.16}$$

for $\eta = -1, \gamma = -6, \mu = 6, \lambda = -1$ *and* $\nu = 3$*, results in obtaining the first-order approximation :*

$$\begin{aligned}
u_0(x,t) &= Sech(\tfrac{x}{2})^2,\\
u_1(x,t) &= -\tfrac{\hbar t^{\alpha}(Sech[\frac{x}{2}]^4\tanh[\frac{x}{2}]^2+Sech[\frac{x}{2}]^2\tanh[\frac{x}{2}]^3)}{Gamma[\alpha+1]},\\
u_2(x,t) &= -\tfrac{\hbar t^{\alpha}(Sech[\frac{x}{2}]^4\tanh[\frac{x}{2}]^2+Sech[\frac{x}{2}]^2\tanh[\frac{x}{2}]^3)}{Gamma[\alpha+1]}+\\
&= \hbar(\tfrac{\hbar t^{\alpha}sech^6\left(\frac{x}{2}\right)\left(t^{\alpha}(22\cosh(x)+\cosh(2x)-39)\Gamma(\alpha+\beta+1)-12\Gamma(2\alpha+1)t^{\beta}(2\cosh(x)-3)\right)}{8\Gamma(2\alpha+1)\Gamma(\alpha+\beta+1)}\\
&\quad -\tfrac{\hbar t^{\alpha}\left(\tanh\left(\frac{x}{2}\right)sech^4\left(\frac{x}{2}\right)+\tanh^3\left(\frac{x}{2}\right)sech^2\left(\frac{x}{2}\right)\right)}{\Gamma(\alpha+1)}),
\end{aligned}$$

for $v(x,t)$*, there is:*

$$\begin{aligned}
v_0(x,t) &= \tfrac{sech^2\left(\frac{x}{2}\right)}{\sqrt{2}},\\
v_1(x,t) &= \tfrac{\sqrt{2}ht^{\beta}\tanh\left(\frac{x}{2}\right)sech^4\left(\frac{x}{2}\right)}{\Gamma(\beta+1)} - \tfrac{3ht^{\beta}\tanh\left(\frac{x}{2}\right)sech^4\left(\frac{x}{2}\right)}{\sqrt{2}\Gamma(\beta+1)} - \tfrac{ht^{\beta}\tanh^3\left(\frac{x}{2}\right)sech^2\left(\frac{x}{2}\right)}{\sqrt{2}\Gamma(\beta+1)},\\
v_2(x,t) &= -\tfrac{3h^2sech^6\left(\frac{x}{2}\right)t^{\alpha+\beta}}{2\sqrt{2}\Gamma(\alpha+\beta+1)} + \tfrac{3h^2\cosh(x)sech^6\left(\frac{x}{2}\right)t^{\alpha+\beta}}{2\sqrt{2}\Gamma(\alpha+\beta+1)} + \tfrac{9h^2t^{2\beta}sech^6\left(\frac{x}{2}\right)}{8\sqrt{2}\Gamma(2\beta+1)} - \tfrac{7h^2t^{2\beta}\cosh(x)sech^6\left(\frac{x}{2}\right)}{4\sqrt{2}\Gamma(2\beta+1)}\\
&\quad + \tfrac{h^2t^{2\beta}\cosh(2x)sech^6\left(\frac{x}{2}\right)}{8\sqrt{2}\Gamma(2\beta+1)} - \tfrac{3ht^{\beta}\tanh\left(\frac{x}{2}\right)sech^4\left(\frac{x}{2}\right)}{\sqrt{2}\Gamma(\beta+1)} + \tfrac{\sqrt{2}ht^{\beta}\tanh\left(\frac{x}{2}\right)sech^4\left(\frac{x}{2}\right)}{\Gamma(\beta+1)} + \cdots
\end{aligned}$$

Example 4.3.6 *For the second test example, consider* (4.3.2)*, with the new initial conditions*

$$u(x,0) = \tfrac{4c^2e^{cx}}{(1+e^{cx})^2} \qquad v(x,0) = \tfrac{4c^2e^{cx}}{(1+e^{cx})^2}. \tag{4.3.17}$$

Applying the HATM, with $\eta = -1, \gamma = -6, \mu = 3, \lambda = -1,$ *and* $\nu = 3,$

obtaining

$$
\begin{aligned}
u_0(x,t) &= \tfrac{4c^2e^{cx}}{(e^{cx}+1)^2},\\
u_1(x,t) &= -\tfrac{1}{\Gamma(\alpha+1)}\left(-\frac{12c^2e^{cx}\left(\frac{4c^3e^{cx}}{(e^{cx}+1)^2}-\frac{8c^3e^{2cx}}{(e^{cx}+1)^3}\right)}{(e^{cx}+1)^2}-4c^2\left(\frac{c^3e^{cx}}{(e^{cx}+1)^2}-\frac{6c^3e^{2cx}}{(e^{cx}+1)^3}+3ce^{cx}\ldots\right.\right.\\
&= \left.\left.+\hbar t^{\alpha}e^{cx}\left(-\frac{2c^3e^{cx}}{(e^{cx}+1)^3}+\frac{18c^3e^{2cx}}{(e^{cx}+1)^4}-\frac{24c^3e^{3cx}}{(e^{cx}+1)^5}\right)\right)\right).\\
u_2(x,t) &= -\tfrac{ht^{\alpha}}{\Gamma(\alpha+1)}\left(-\frac{12c^2e^{cx}\left(\frac{4c^3e^{cx}}{(e^{cx}+1)^2}-\frac{8c^3e^{2cx}}{(e^{cx}+1)^3}\right)}{(e^{cx}+1)^2}-4c^2\left(\frac{c^3e^{cx}}{(e^{cx}+1)^2}-\frac{6c^3e^{2cx}}{(e^{cx}+1)^3}+\right.\right.\\
&\quad \left.\left. e^{cx}\left(-\frac{2c^3e^{cx}}{(e^{cx}+1)^3}+\frac{18c^3e^{2cx}}{(e^{cx}+1)^4}-\frac{24c^3e^{3cx}}{(e^{cx}+1)^5}\right)+3ce^{cx}\left(\frac{6c^2e^{2cx}}{(e^{cx}+1)^4}-\frac{2c^2e^{cx}}{(e^{cx}+1)^3}\right)\right)\right)+\ldots
\end{aligned}
$$

and for $v(x,t)$, getting

$$
\begin{aligned}
v_0(x,t) &= \tfrac{4c^2e^{cx}}{(e^{cx}+1)^2},\\
v_1(x,t) &= \tfrac{4c^5\hbar e^{cx}t^{\beta}}{\Gamma(\beta+1)(e^{cx}+1)^2}-\tfrac{56c^5\hbar e^{2cx}t^{\beta}}{\Gamma(\beta+1)(e^{cx}+1)^3}+\tfrac{48c^5\hbar e^{2cx}t^{\beta}}{\Gamma(\beta+1)(e^{cx}+1)^4}+\tfrac{144c^5\hbar e^{3cx}t^{\beta}}{\Gamma(\beta+1)(e^{cx}+1)^4}\\
&\quad -\tfrac{96c^5\hbar e^{3cx}t^{\beta}}{\Gamma(\beta+1)(e^{cx}+1)^5}-\tfrac{96c^5\hbar e^{4cx}t^{\beta}}{\Gamma(\beta+1)(e^{cx}+1)^5},\\
v_2(x,t) &= \tfrac{48c^5\hbar e^{2cx}t^{\beta}}{\Gamma(\beta+1)(e^{cx}+1)^4}+\tfrac{144c^5\hbar e^{3cx}t^{\beta}}{\Gamma(\beta+1)(e^{cx}+1)^4}-\tfrac{96c^5\hbar e^{3cx}t^{\beta}}{\Gamma(\beta+1)(e^{cx}+1)^5}-\tfrac{96c^5\hbar e^{4cx}t^{\beta}}{\Gamma(\beta+1)(e^{cx}+1)^5}\\
&\quad +\tfrac{48c^8\hbar^2e^{4cx}t^{\alpha+\beta}}{(e^{cx}+1)^6\Gamma(\alpha+\beta+1)}+\tfrac{4c^8\hbar^2e^{cx}t^{2\beta}}{\Gamma(2\beta+1)(e^{cx}+1)^6}-\tfrac{56c^8\hbar^2e^{2cx}t^{2\beta}}{\Gamma(2\beta+1)(e^{cx}+1)^6}+\tfrac{72c^8\hbar^2e^{3cx}t^{2\beta}}{\Gamma(2\beta+1)(e^{cx}+1)^6}+\cdots.
\end{aligned}
$$

4.3.5 Convergence analysis

Theorem 4.3.7 *If the series solutions $\sum_{i=0}^{n}u_i(x,t)$, and $\sum_{i=0}^{n}v_i(x,t)$ are convergent, where u_m and v_m are governed by equation (4.3.9), then must be solutions of (4.3.11).*

Proof 13 *Suppose that $\sum_{i=0}^{n}u_i(x,t)$ and $\sum_{i=0}^{n}v_i(x,t)$ are convergent, i.e $\lim\limits_{n\longrightarrow\infty}u_n(x,t)=\lim\limits_{n\longrightarrow\infty}v_n(x,t)=0$. From the equation (4.3.13), results in:*

$$
\begin{aligned}
\hbar H\sum_{m=1}^{\infty}K_m &= \lim_{k\longrightarrow\infty}\sum_{m=0}^{k}L[u_m-\ell_m u_{m-1}]\\
&= L[\lim_{k\longrightarrow\infty}\sum_{m=0}^{k}[u_m-\ell_m u_{m-1}]] \;=\; L[\lim_{k\longrightarrow\infty}u_k],
\end{aligned}
$$

where L is a Laplace operator. Since $\lim\limits_{k\longrightarrow\infty}u_k=0$, $H\neq 0$ and $\hbar\neq 0$, this implies $\sum_{m=1}^{\infty}K_m=0$.Now, expand $N_1[\Phi(x,t,q),\psi(x,t,q)]$ about $q=0$, then set $q=1$,

$$N_1[\Phi(x,t,1),\psi(x,t,q)]=0,$$

it is seen that $u(x,t)=\Phi(x,t,1)=\sum_{i=0}^{\infty}u_i(x,t)$, and $v(x,t)=\psi(x,t,1)=\sum_{i=0}^{\infty}v_i(x,t)$ solve equation (4.3.2).

Theorem 4.3.8 *[14] Let the solution component* $h_0(x,t), h_1(x,t), h_2(x,t), \ldots$ *be defined as* (4.3.11)*. The series solution* $\sum_{m=0}^{\infty} h_m(x,t)$ *defined in* (4.3.15) *converge if there exist* $0 < \gamma < 1$ *such that* $\|h_{m+1}(x,t)\| \leq \gamma \|h_m(x,t)\|, \forall m > m_0$*, for some* $m_0 \in \mathbb{N}$.

Theorem 4.3.9 *[14] Suppose that the series solution* $\sum_{m=0}^{\infty} h_m(x,t)$ *is convergent to the solution* $h(x,t)$*. If the truncated series* $\sum_{m=0}^{k} h_m(x,t)$ *is used as approximation to the solution* $h(x,t)$*, then the truncated error satisfies*

$$\| h(x,t) - \sum_{m=0}^{k} h_m(x,t) \| \leq \frac{1}{1-\gamma} \gamma^{k+1} \| h_0(x,t) \| .$$

4.3.6 Numerical result and discussion

In Fig. 2 and 3, a plot for the 5-th order approximation solutions and the exact solution of (4.3.2), it is observed that the graphs coincide with the graphs of the exact solution.
Now, to control the convergence of the series in the frame of HATM, by the auxiliary convergence parameter $\hbar$, a plot to the so-called $\hbar$-curve for $u_{tt}(0,0)(\hbar)$ and $v_{tt}(0,0)(\hbar)$ In Fig1,4,6 and 8. it is seen from those figures that the best value of $\hbar$ for the first example is in the convergence region $-1.2 \leq \hbar \leq -0.6$, and for the second example is $-1.3 \leq \hbar \leq -0.7$ so there is a freedom of choice $\hbar$ in this region to get a convergence series. In table 6.1, and 6.3 compare our results with other results obtained using the NDM solutions [41], and the spectral collection method [15]. In Table 6.2, the approximation solutions for different value of α, and β are given.

Table 4.3.10 *Comparison of personal results of* $u(x,t)$*, and* $v(x,t)$ *with other result obtained by other numerical methods for different values of* $\alpha =$

$\beta = 1.$

x	t	*Exact($u(x,t)$)*	*HATM*
-10	0.1	0.000164304	0.000164304
-10	0.2	0.000148670	0.000148670
-5	0.1	0.024092321	0.024092321
-5	0.2	0.021824797	0.021824797
5	0.1	0.029347625	0.029347625
5	0.2	0.032383774	0.032383773
10	0.1	0.000200678	0.000200678
10	0.2	0.000221781	0.000221781

x	t	*Exact($v(x,t)$)*	*HATM*
-10	0.1	0.000116180	0.000116180
-10	0.2	0.000105125	0.000105125
-5	0.1	0.017035843	0.017035843
-5	0.2	0.015432462	0.015432462
5	0.1	0.020751905	0.020751905
5	0.2	0.022898786	0.022848786
10	0.1	0.000141901	0.000141901
10	0.2	0.000156823	0.000156823

x	t	*Error (HATM)*	*Error(NDM)[41]*	*Error[15]*
-10	0.1	2.47239×10^{-13}	2.95039×10^{-8}	2.99039×10^{-8}
-10	0.2	1.56036×10^{-11}	2.30335×10^{-7}	2.33335×10^{-7}
-5	0.1	8.91166×10^{-12}	3.93592×10^{-6}	3.96592×10^{-6}
-5	0.2	5.83638×10^{-10}	3.08049×10^{-5}	3.38049×10^{-5}
5	0.1	8.43378×10^{-12}	2.02966×10^{-4}	3.97592×10^{-6}
5	0.2	5.22271×10^{-10}	5.15558×10^{-4}	3.78049×10^{-5}
10	0.1	2.54363×10^{-13}	2.11254×10^{-8}	2.96039×10^{-8}
10	0.2	1.65158×10^{-11}	2.28173×10^{-7}	2.37335×10^{-7}

x	t	*Error (HATM)*	*Error(NDM)[41]*	*Error[15]*
-10	0.1	1.74825×10^{-13}	2.08624×10^{-8}	2.18624×10^{-8}
-10	0.2	1.10334×10^{-11}	1.62872×10^{-7}	1.64872×10^{-7}
-5	0.1	6.30149×10^{-12}	2.78312×10^{-6}	2.88312×10^{-6}
-5	0.2	4.12694×10^{-10}	2.21782×10^{-5}	2.87824×10^{-5}
5	0.1	5.96358×10^{-12}	2.9094×10^{-6}	2.98312×10^{-6}
5	0.2	3.69301×10^{-10}	2.238036×10^{-5}	2.47824×10^{-5}
10	0.1	1.79862×10^{-13}	2.19313×10^{-8}	2.09624×10^{-8}
10	0.2	1.16785×10^{-11}	1.79991×10^{-7}	1.72872×10^{-7}

In the above tables, it is witnessed that the approximation solution obtained *by HATM, is agreed on and appropriate with the exact solution more than the NDM and spectral collection methods.*

Table 4.3.11 *Numerical solutions of $u(x,t)$, and $v(x,t)$ for different values of α, β. Numerical solutions of $u(x,t)$, and $v(x,t)$ for different values of α, β*

x	t	*Exact* $u(x,t)$	$\alpha=\beta=0.9$	$\alpha=\beta=0.8$	$\alpha=\beta=0.6$
-10	0.1	0.000164304	0.000159449	0.000153648	0.000139462
-10	0.2	0.000148670	0.000142868	0.000136314	0.000123400
-5	0.1	0.024092321	0.023388290	0.022546135	0.020472920
-5	0.2	0.021824797	0.020954214	0.020024033	0.018074633
5	0.1	0.029347625	0.030919078	0.031537740	0.035657141
5	0.2	0.032383774	0.033919078	0.035900711	0.041940955
10	0.1	0.000200678	0.000207156	0.000215926	0.000244977
10	0.2	0.000221781	0.000232514	0.000246460	0.000290077
x	t	*Exact* $v(x,t)$	$\alpha=\beta=0.9$	$\alpha=\beta=0.8$	$\alpha=\beta=0.6$
-10	0.1	0.000116180	0.000112747	0.000108646	0.000098614
-10	0.2	0.000105125	0.000100894	0.000096389	0.000087257
-5	0.1	0.017035843	0.016538018	0.015942525	0.014476541
-5	0.2	0.015432462	0.014816867	0.014159129	0.012780695
5	0.1	0.020751905	0.021410870	0.022300550	0.025213406
5	0.2	0.022898786	0.023984410	0.025385636	0.029656733
10	0.1	0.000141901	0.000146481	0.000152682	0.000173225
10	0.2	0.000156823	0.000164412	0.000174273	0.000205115

For the second example, a comparison with the exact solution, the approximate solutions for [50] in Table 5.3. In Fig. 4 the 10-th order of solution with the exact solution $u(x,t) = \frac{\zeta}{a}\left(\operatorname{sech}\left(\frac{1}{2\sqrt{\frac{\zeta}{a}}(x-\zeta t)}\right)\right)^2$,$u(x,t) = \frac{\zeta}{\sqrt{2a}}\left(\operatorname{sech}\left(\frac{1}{2\sqrt{\frac{\zeta}{a}}(x-\zeta t)}\right)\right)^2$ for example1 is presented, and $u(x,t) = v(x,t) = \frac{4c^2 e^{c(x-c^2t)}}{(1+e^{c(x-c^2t)})^2}$ for the second example. In Fig 6, the graphs traced for different values of α, β.

Table 4.3.12 *Comparison of numerical solutions of $u(x,t)$, and $v(x,t)$ with other obtained by other numerical methods for different value of $\alpha = \beta = 1$.*

x	t	*Exact*	*HATM*($\alpha = \beta = 1$)
-10	0.1	0.000164304	0.000164304
-10	0.2	0.000148670	0.000148670
-5	0.1	0.024092321	0.024092321
-5	0.2	0.021824797	0.021824797
5	0.1	0.029347625	0.029347625
5	0.2	0.032383774	0.032383773
10	0.1	0.000200678	0.000200678
10	0.2	0.000221781	0.000221781

x	t	*Exact*	*HATM*($\alpha = \beta = 1$)
-10	0.1	0.000164304	0.000164304
-10	0.2	0.000148670	0.000148670
-5	0.1	0.024092321	0.024092321
-5	0.2	0.021824797	0.021824797
5	0.1	0.029347625	0.029347625
5	0.2	0.032383774	0.032383773
10	0.1	0.000200678	0.000200678
10	0.2	0.000221781	0.000221781

x	t	*Error (HATM)*	*Error(NDM)[41]*	*Error[50]*
-10	0.1	2.47240×10^{-13}	2.95039×10^{-8}	2.95039×10^{-8}
-10	0.2	1.56036×10^{-11}	2.30335×10^{-7}	2.30335×10^{-7}
-5	0.1	8.91164×10^{-12}	3.93592×10^{-6}	3.93592×10^{-6}
-5	0.2	5.83638×10^{-10}	3.08049×10^{-5}	3.08049×10^{-5}
5	0.1	8.43377×10^{-12}	4.11452×10^{-6}	4.11452×10^{-6}
5	0.2	5.22271×10^{-10}	3.36633×10^{-5}	3.36633×10^{-5}
10	0.1	2.54363×10^{-13}	3.10155×10^{-8}	3.10155×10^{-8}
10	0.2	1.65158×10^{-11}	2.54546×10^{-7}	2.54546×10^{-7}

x	t	*Error (HATM)*	*Error(NDM)[41]*	*Error[50]*
-10	0.1	2.47240×10^{-13}	2.95039×10^{-8}	2.95039×10^{-8}
-10	0.2	1.56036×10^{-11}	2.30335×10^{-7}	2.30335×10^{-7}
-5	0.1	8.91164×10^{-12}	3.93592×10^{-6}	3.93592×10^{-6}
-5	0.2	5.83638×10^{-10}	3.08049×10^{-5}	3.08049×10^{-5}
5	0.1	8.43375×10^{-12}	4.11452×10^{-6}	4.11452×10^{-6}
5	0.2	5.22277×10^{-10}	3.36633×10^{-5}	3.36633×10^{-5}
10	0.1	2.54363×10^{-13}	3.10155×10^{-8}	3.10155×10^{-8}
10	0.2	1.65158×10^{-11}	2.54546×10^{-7}	2.54546×10^{-7}

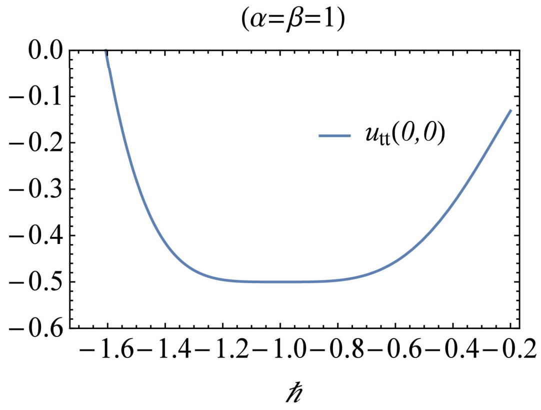


Figure 4.11: The $\hbar-$curve for example 1

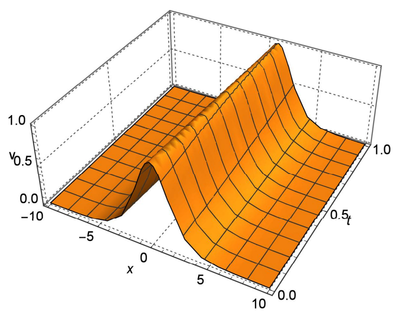


Figure 4.12: The approximate solution $u(x,t)$ for example 1

Concluding remarks

This work, we successfully applied the HATM for a coupled Korteweg-de Vries equation, and shows that this method is efficient and applicable for this type of fractional differential equations, comparing personal results with others obtained by different numerical method. The algorithm is powerfully for solving these kinds of equations.

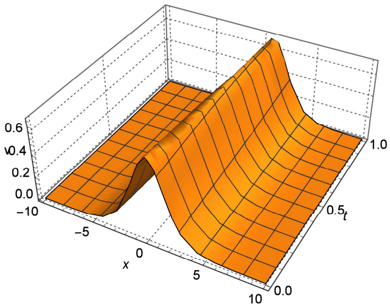


Figure 4.13: The approximate solution $v(x,t)$ for example 1

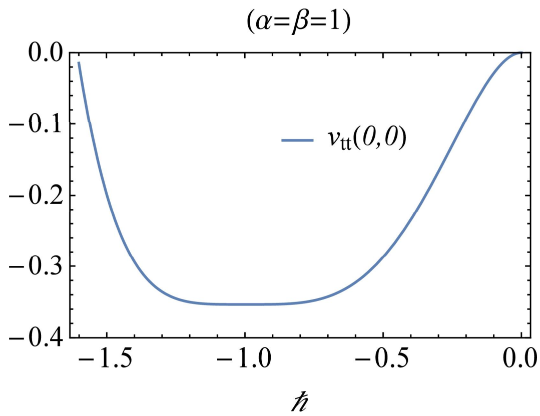


Figure 4.14: The $\hbar-$curve the example 1

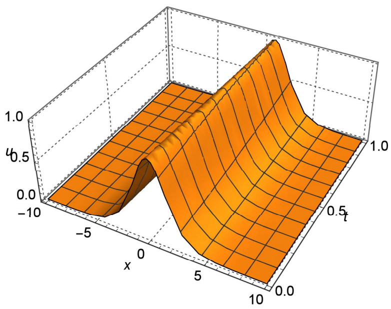


Figure 4.15: The approximate solution $u(x,t)$ for example 2

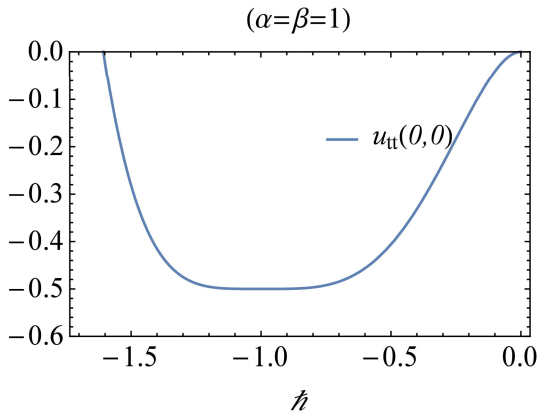


Figure 4.16: The $\hbar$−curve for example 1

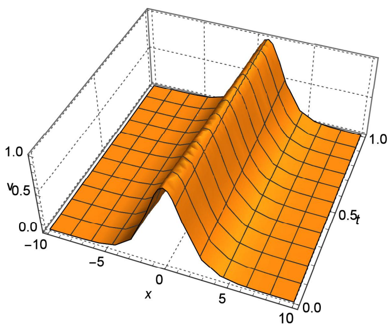


Figure 4.17: The approximate solution $v(x,t)$ for example 2

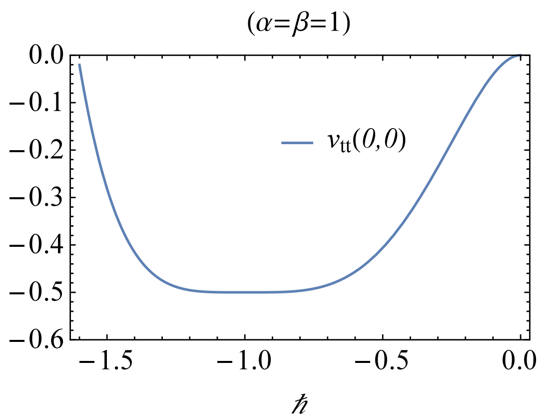


Figure 4.18: The $\hbar$−curve for example 1

4.4 Numerical study of the seventh-order boundary value problems

This section consider the general seventh-order boundary value problems of the type:

$$\begin{cases} u^{(7)}(x) = f(x, u, u', u'', ..., u^{(6)}), & a \leq x \leq b, \\ u^{(i)}(a) = \alpha_i \quad u^{(j)}(b) = \beta_j & i = 0, 1, 2, 3 \quad j = 0, 1, 2. \end{cases} \tag{4.4.1}$$

where f is known function, and α_i and β_j ($i = 0, 1, 2, 3$ and $j = 0, 1, 2$) are real constants.
the boundary value problems have many applications in sciences and many fields, such as the induction motors with two rotor circuits, the synchronous generator model, mathematical physics, medicine .
Several numerical methods have been applied to solve the linear and nonlinear seventh-order boundary value problems, such as a homotopy perturbation method [24], differential transform method [59], variational iteration method [58] and [60].
The homotopy analysis method was proposed and developed by Shijun Liao in 1992 to solve linear and nonlinear problems[57]. In recent years, this method has been successfully employed to solve many types of nonlinear problems in science and engineering. All of these successful application verified the validity, effectiveness and flexibility of the homotopy analysis method.

Liao in [34], [56], presented the homotopy analysis method for solving different problems in several dimensions. The homotopy analysis method contains the auxiliary parameter $\hbar$ which provides a simple way to adjust and control the convergence region of solution series.
In this paper using the transformation

$$u_i(x) = u^{(i-1)}(x), \qquad i = 1, ..., 7$$

of problem (4.4.1) to a system of first-order differential equation such as

$$\begin{cases} u_i'(x) &= u_{i+1}(x) \qquad i = 1, ..., 6 \\ u_7'(x) &= f(x, u_1(x), ..., u_7(x)) \end{cases}$$

with the initial conditions :

$$u_i(0) = \alpha_i \qquad i = 1, ..., 7$$

and applying the homotopy analysis method [1] with take $\hbar = -1$ to solve a system of first-order initial value problems to get a series solution of problem (4.4.1), gets an approximation of α_5, α_6 and α_7 at $x = b$. After that, applying the homotopy analysis method for a system of differential equation with a new initial condition $u_5(0) = u^{(4)}(0) = \alpha_5, u_6(0) = u^{(5)}(0) = \alpha_6$ and $u_7(0) = u^{(6)}(0) = \alpha_7$.
Illustrating with examples the advantage and the effectiveness of the proposed method.

4.4.1 The homotopy analysis method

Consider:

$$u^{(i)}(x) = u_{i+1}(x), \qquad i = 0, ..., 6 \tag{4.4.2}$$

rewriting (4.4.1) as the system of ordinary differential equations:

$$\begin{cases} u_i'(x) &= u_{i+1}(x) \\ u_7'(x) &= f(x, u_1(x), ..., u_6(x)) \end{cases} \tag{4.4.3}$$

with the initial conditions

$$u_i(a) = u^{(i-1)}(a) \qquad i = 1, ..., 7 \tag{4.4.4}$$

considering the differential equation

$$N_i[u_i(x)] = g_i(x), \qquad i = 1, ..., 7 \tag{4.4.5}$$

where N_i is the nonlinear operators, $u_i(x)$ is the unknown functions and $g_i(x)$ is a known function.
The zeroth-order deformation equation [1]:

$$(1-q)L[\phi_i(x,q)-u_{i,0}(x)] = \hbar_i q H_i(x)\{N_i[\phi_i(x,q)-g_i(x)]\}, \qquad (4.4.6)$$

for i=1,...,7,where L_i, N_i are both linear and nonlinear operator, $q \in [0,1]$ is the homotopy parameter, $\hbar_i \neq 0$ is the convergence-control parameter, and $H_i(x)$ is an auxiliary function.
Having:

$$\phi_i(x,0) = u_{i,0}(x) \quad and \quad \phi_i(x,1) = u_i(x), \qquad i = 1,...,7, \qquad (4.4.7)$$

thus as q increases from 0 to 1, the solution $\phi_i(x,q)$ varies from the initial guesses $u_{i,0}(x)$ to the exact solutions $u_i(x)$.
Expanding $\phi_i(x,q)$ in Taylor series with respect to q one has:

$$\phi_i(x,q) = u_{i,0}(x) + \sum_{m=1}^{n} u_{i,m}(x)q^m, \qquad (4.4.8)$$

where

$$u_{i,m}(x) = \frac{1}{m!}\frac{\partial^m \phi_i(x,q)}{\partial q^m}|_{q=0}, \qquad (4.4.9)$$

if L, $u_0(x)$, $\hbar$, and $H(x)$ are so properly chosen, the series solutions (4.4.8) converge at $q=1$, and has

$$\phi_i(x,1) = u_{i,0}(x) + \sum_{i=1}^{+\infty} u_{i,m}(x),$$

Define the vectors

$$\overrightarrow{u}_{i,n} = \{u_{i,0}(x), u_{i,1}(x), ..., u_{i,n}(x),\}$$

Differentiating (4.4.4) m times with respect to the embedding parameter q and then setting $q=0$ and finally dividing them by $m!$, gives the m-th-order deformation equation

$$L[u_{i,m}(x) - \chi_m u_{i,m-1}(x)] = \hbar_i H_i(x) R_{i,m}(\overrightarrow{u}_{i,m-1}), \qquad (4.4.10)$$

where

$$R_{i,m}(\overrightarrow{u}_{i,m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N_i[\phi_i(x,q)]}{\partial q^{m-1}} \qquad (4.4.11)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

4.4.2 Application

Example 4.4.1 *Consider the following seventh-order boundary value problem[60][61]*

$$u^{(7)}(x) = -u(x) - e^x(35 + 12x + 2x^2), \qquad 0 \leq x \leq 1 \tag{4.4.12}$$

with:

$$\begin{array}{llll} u(0) = 0 & u'(0) = 1 & u''(0) = 0 & u'''(0) = -3 \\ u(1) = 0 & u'(1) = -e & u''(1) = -4e & \end{array} \tag{4.4.13}$$

The exact solution is

$$u(x) = x(1-x)e^x$$

Using (4.4.2), *results in:*

$$\begin{cases} u_i'(x) &= u_{i+1}(x) \quad i = 1, ..., 6 \\ u_7'(x) &= -u_1(x) - e^x(35 + 12x + 2x^2) \end{cases} \tag{4.4.14}$$

with the initial conditions

$$\begin{array}{llll} u_1(0) = 0 & u_2(0) = 1 & u_3(0) = 0 & u_4(0) = -3 \\ u_5(0) = \alpha_1 & u_6(0) = \alpha_2 & u_7(0) = \alpha_3 & \end{array} \tag{4.4.15}$$

Defining a system of linear operators:

$$\begin{cases} N_i[\phi_i(x,q)] &= \frac{\partial \phi_i(x,q)}{\partial x} - \phi_{i+1}(x), \qquad i = 1, ..., 6 \\ N_7[\phi_i(x,q)] &= \frac{\partial \phi_7(x,q)}{\partial x} - \phi_1(x,q) - e^x(35 + 12x + 2x^2). \end{cases} \tag{4.4.16}$$

The zeroth-order deformation equation is

$$(1-q)L[\phi_i(x,q) - u_{i,0}(x)] = \hbar_i q H(x)\{N_i[\phi_i(x,q) - g_i(x)]\}, \qquad i = 1, ..., 7, \tag{4.4.17}$$

where

$$L[\phi_i(x,q)] = \frac{\partial \phi_i(x,q)}{\partial x}, \qquad i = 1, ..., 7 \tag{4.4.18}$$

and the mth-order deformation equation is

$$L[u_{i,m}(x) - \chi_m u_{i,m-1}(x)] = \hbar_i R_{i,m}(\overrightarrow{u}_{i,m-1}) \tag{4.4.19}$$

with the initial conditions

$$\begin{array}{llll} u_{1,0}(0) = 0 & u_{2,0}(0) = 1 & u_{3,0}(0) = 0 & u_{4,0}(0) = -3 \\ u_{5,0}(0) = \alpha_1 & u_{6,0}(0) = \alpha_2 & u_{7,0}(0) = \alpha_3 & \end{array} \tag{4.4.20}$$

where

$$\begin{cases} R_{i,m}(\overrightarrow{u}_{i,m-1}) & = & u'_{i,m-1}(x) - u_{i+1,m-1}(x), \qquad i = 1, ..., 6 \\ R_{7,m}(\overrightarrow{u}_{i,m-1}) & = & u'_{7,m-1}(x) + u_{1,m-1}(x) + e^{x}(35 + 12x + 2x^2) \end{cases} \tag{4.4.21}$$

Now, the solution of the mth-order deformation Eq. (4.4.19) *for* $m \geq 1$ *is*

$$u_{i,m}(x) = \chi_m u_{i,m-1}(x) + \hbar_i L^{-1}\left[R_{i,m}(\overrightarrow{u}_{i,m-1})\right] dx, \qquad i = 1, ..., 7 \tag{4.4.22}$$

setting $\hbar = -1$*, and using the boundary conditions at* $x = 1$ (4.4.13)*, obtaining:*

$$\begin{aligned} \alpha_5 &\simeq -8.00000000001599548955547560856... \\ \alpha_6 &\simeq -14.9999999998269020884963 19961... \\ \alpha_7 &\simeq -24.0000000005657917014479 78441... \end{aligned}$$

Example 4.4.2 *Now consider the following nonlinear problem :*

$$u^{(7)}(x) = e^{-x}u^2(x), \qquad 0 \leq x \leq 1, \tag{4.4.23}$$

with:

$$u(0) = u'(0) = u''(0) = u'''(0) = 1 \quad u(1) = u'(1) = u''(1) = e, \tag{4.4.24}$$

the exact solution is:

$$u(x) = e^x$$

Using (4.4.2)*:*

$$\begin{cases} u'_i(x) & = & u_{i+1}(x) \qquad i = 1, ..., 6 \\ u'_7(x) & = & e^{-x}y_1^2(x), \end{cases} \tag{4.4.25}$$

subject to the initial conditions

$$\begin{array}{cccc} u_1(0) = 1 & u_2(0) = 1 & u_3(0) = 1 & u_4(0) = 1 \\ u_5(0) = \alpha_1 & u_6(0) = \alpha_2 & u_7(0) = \alpha_3 & \end{array} \tag{4.4.26}$$

applying HAM with $\hbar = -1$ *and using the conditions at* $x = 1$*, ends in:*

$$\begin{aligned} \alpha_1 &\simeq 1.00000078149334070641102499567289 2976... \\ \alpha_2 &\simeq 0.99999127398056317492095753082029 1208... \\ \alpha_3 &\simeq 1.00002962834895129668407112692075 8841... \end{aligned}$$

Example 4.4.3 *Consider the nonlinear nonhomogeneous test example*

$$u^{(7)}(x) = -u(x)u'(x) + e^x(-35 + (e^x - 13)x - x^2(1 + 2e^x)x^2 + e^x x^4), \qquad 0 \le x \le 1, \tag{4.4.27}$$

subject to:

$$\begin{array}{llll} u(0) = 0 & u'(0) = 1 & u''(0) = 0 & u'''(0) = 3 \\ u(1) = 0 & u'(1) = -e & u''(1) = -4e & \end{array} \tag{4.4.28}$$

Using (4.4.3), *and setting* $\hbar = -1$ *using the boundary conditions at* $x = 1$, *gives:*

$$\begin{array}{l} \alpha_1 \simeq -8.0000681937767504353435183121152 53.... \\ \alpha_2 \simeq -14.9992456856484749687126762343278 0.... \\ \alpha_3 \simeq -24.0025319575497091212378016044187 3.... \end{array}$$

4.4.3 Numerical discussion

Using the residual error [31]

$$\Delta(\hbar) = \int N(u_n(x))^2 dx \tag{4.4.29}$$

ends with the optimal value of $\hbar$, $\hbar \simeq -0.991519....$, we can see from the table 4.4.4 that the maximum error obtained by HAM is better than the variation of parameter method [61]. Fig4.19 show that both of HAM and exact solution are coincide, and a comparison of 10-th order HAM with exact solution. and a comparison of the HAM solution with exact, VIM and VPM solution for different value of x in $[0, 1]$ is given in table 4.4.5.
For example 2, fig 4.20 shows the $\hbar-$ curve for (4.4.23) of $u'_h(0)$, and $u''_h(0)$, the valid region of $\hbar$ is $-1.2 \le \hbar \le -0.8$ the optimal value of $\hbar$ can be obtained using 4.4.29. Fig 4.21 shows the HAM solution and Absolute error for (4.4.23)

Table 4.4.4 *The maximum of Absolute error for HAM, VIM, and VPM method.*

HAM	*VIM[60]*	*VPM[61]*
9.6019×10^{-16}	2.46782×10^{-10}	2.1729×10^{-09}

Table 4.4.5 *Comparison of HAM solution $u(x)$ with other result obtained by other numerical methods for different values of x*

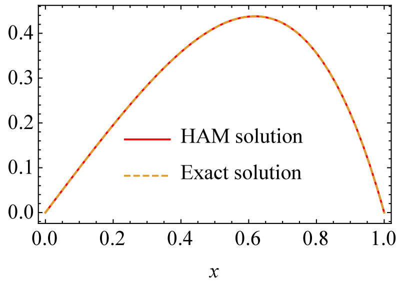
x	*Exacte solution*	*HAM solution*	*Error*
0	0	0.	0.
0.1	0.099465382	0.099465382	4.69467×10^{-12}
0.2	0.195424441	0.195424441	2.30461×10^{-11}
0.3	0.283470349	0.283470349	2.10734×10^{-10}
0.4	0.358037927	0.358037928	8.59685×10^{-10}
0.5	0.412180317	0.412180320	2.50462×10^{-9}
0.6	0.437308512	0.437308517	5.54043×10^{-9}
0.7	0.422888068	0.422888077	8.4732×10^{-9}
0.8	0.356086548	0.356086551	2.59727×10^{-9}
0.9	0.221364280	0.221364237	4.21248×10^{-8}
1	0	0.000000294	2.09418×10^{-7}

For the second example, as shown in the table 4.4.6 the HAM solution approach to the exact solution $y(x) = e^x$ more than VIM and VPM method.

Table 4.4.6

x	*Exacte solution*	*Series solution*	*Absolute error HAM*	*VPM[61]*
0	1	1.	6.99218×10^{-14}	0.0000
0.1	1.105170918	1.105170918	4.24171×10^{-13}	2.86257×10^{-7}
0.2	1.221402758	1.221402758	1.26573×10^{-11}	4.38942×10^{-7}
0.3	1.349858808	1.349858807	4.96001×10^{-11}	6.1274×10^{-7}
0.4	1.491824698	1.491824698	1.08651×10^{-9}	7.71759×10^{-7}
0.5	1.648721271	1.648721277	6.90129×10^{-9}	7.71759×10^{-7}
0.6	1.822118800	1.822118828	2.78252×10^{-8}	7.37682×10^{-7}
0.7	2.013752707	2.013752793	8.58459×10^{-8}	6.25932×10^{-7}
0.8	2.225540928	2.225541149	2.21205×10^{-7}	4.68244×10^{-7}
0.9	2.459603111	2.459603611	5.00586×10^{-7}	2.95852×10^{-7}
1	2.718281828	2.718282858	1.02714×10^{-6}	1.25922×10^{-7}

Table 4.4.7 shows also the HAM solutions and the error corresponding to the example 3.3, and numerical results obtained by other method.



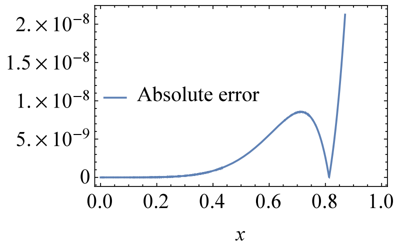


Figure 4.19: The HAM solution (left) and absolute error (right) of (4.4.12)

Table 4.4.7 *Numerical result of example 3*

x	*Exacte solution*	*HAM solution*	*Error*
0	0	-1.47526×10^{-11}	1.47526×10^{-11}
0.1	0.099465382	0.099465382	6.58523×10^{-12}
0.2	0.195424441	0.195424441	7.97423×10^{-11}
0.3	0.283470349	0.283470349	1.53912×10^{-10}
0.4	0.358037927	0.358037928	6.50187×10^{-10}
0.5	0.412180317	0.412180320	2.53009×10^{-9}
0.6	0.437308512	0.437308518	6.3726×10^{-9}
0.7	0.422888068	0.422888082	1.37726×10^{-8}
0.8	0.356086548	0.356086575	2.71929×10^{-8}
0.9	0.221364280	0.221364329	4.96526×10^{-8}
1	0	8.67891×10^{-8}	8.67891×10^{-8}

Concluding remarks

This work successfully applied the HAM for a seventh-order boundary value problems, and shows that this method is efficient and applicable for this type of differential equation, The graphical representation shows that the method depends notably on the convergence parameter $\hbar$, compared personal results with others obtained by B-spline method. Concluding that this analytical technique is more effective and it can be used for boundary value problems for any order, and fractional order.

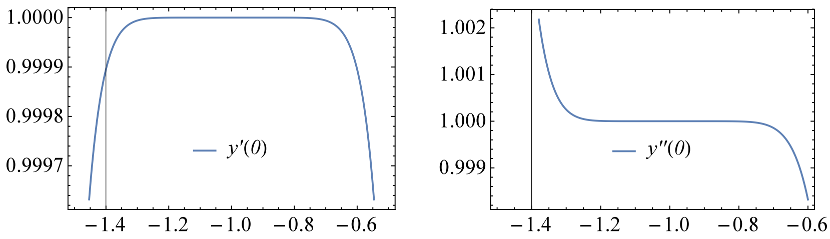


Figure 4.20: The $\hbar-$curve for $y_x(\hbar)$ (left) and $y_{xx}(\hbar)$ (right) of (4.4.23)

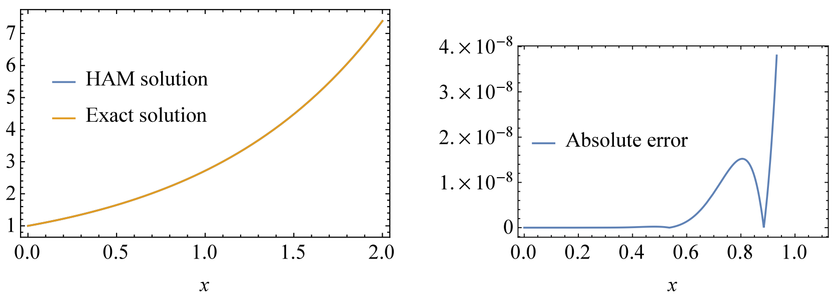


Figure 4.21: The HAM solution (left) and absolute error (right) of (4.4.23)

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