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Intitulé de la thèse

# DÉFORMATIONS DE CERTAINES REPRÉSENTATIONS RÉDUCTIBLES DU GROUPE D'UN NŒUD DANS $\mathrm{SL}(d, \mathbb{C})$

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# Abstract

We study deformations of certain non-abelian, metabelian, reducible representations of the knot group into $\mathrm{SL}(d,\mathbb{C})$ which are associated to a simple root of Alexander polynomial.

Let $K$ be a knot in $\mathbb{S}^3$, $X = \overline{\mathbb{S}^3 \setminus V(K)}$ its complement, where $V(K)$ is a tubular neighborhood of $K$. Moreover, we let $\Gamma_K = \pi_1(X)$ denote the fundamental group of $X$. Let $\varphi\colon \pi_1(X) \to \mathbb{Z}$ denotes the canonical surjection which maps the meridian $\mu$ of $K$ to 1 i.e. $\varphi(\gamma) = \mathrm{lk}(\gamma, K)$. Let $\Delta_K(t) \in \mathbb{C}[t^{\pm 1}]$ denote the Alexander polynomial of $K$. We associate to $\lambda \in \mathbb{C}^*$, $\lambda \neq \pm 1$, a homomorphism

$$\lambda^{\varphi}\colon \Gamma_K \to \mathbb{C}^*, \quad \gamma \mapsto \lambda^{\varphi(\gamma)}\,.$$

Note that the meridian $\mu$ maps to $\lambda$. We obtain also a diagonal representation

$$\rho_\lambda\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C}), \quad \rho_\lambda(\gamma) = \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}.$$

Now, for a given map $z\colon \Gamma_K \to \mathbb{C}$ let $\rho_\lambda^z$ be a map defined by

$$\rho_\lambda^z\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C}), \quad \rho_\lambda^z(\gamma) = \begin{pmatrix} 1 & z(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}.$$

The map $\rho_\lambda^z$ is a representation if and only if $z\colon \Gamma_K \to \mathbb{C}_{\lambda^2}$ is a cocycle i.e. $z$ satisfies for all $\gamma_1, \gamma_2 \in \Gamma_K$ the following:

$$z(\gamma_1\gamma_2) = z(\gamma_1) + \lambda^{2\varphi(\gamma_1)} z(\gamma_2)\,.$$

Here, we define $\mathbb{C}_{\lambda^2}$ to be the $\Gamma$-module $\mathbb{C}$ with the action induced by $\lambda^2$, i.e. $\gamma \cdot x = \lambda^{2\varphi(\gamma)} x$ for all $\gamma \in \Gamma_K$ and all $x \in \mathbb{C}$. It is easy to see that in this case the representation $\rho_\lambda^z$ is conjugate to the diagonal (abelian) representation $\rho_\lambda\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C})$ if and only if

$z$ is a principal cocycle i.e. there exists an element $x_0 \in \mathbb{C}$ such that for all $\gamma \in \Gamma_K$ we have $z(\gamma) = \gamma \cdot x_0 - x_0$. Hence there exists a metabelian, non-abelian representation $\rho_\lambda^z$ if and only if there exists a non-principal cocycle $z\colon \Gamma_K \to \mathbb{C}_{\lambda^2}$. The representations $\rho_\lambda^z$ where first studied by G. Burde and G. de Rham [Bur67, deR67]. They proved that there exists such reducible, metabelian, non-abelian, representations of the knot group $\Gamma_K$ if and only if $\lambda^2$ is a root of the Alexander polynomial $\Delta_K(t)$.

In what follows, we let $r_d\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(d,\mathbb{C})$ denote the $d$-dimensional, irreducible, rational representation of $\mathrm{SL}(2,\mathbb{C})$ (see Chapter 2). We study the behavior of the representation $r_d \circ \rho_\lambda^z\colon \Gamma_K \to \mathrm{SL}(d,\mathbb{C})$ under the additional hypothesis that $\lambda^2$ is a simple root of the Alexander polynomial $\Delta_k(t)$. By making use of Theorem 1.1 in [HPS01], we prove in Proposition 4.1 that the representation $r_d \circ \rho_\lambda^z$ is the limit of irreducible representations.

Finally, we show in Theorem 4.1 that if $\lambda^2$ is a simple root of the Alexander polynomial $\Delta_K(t)$ and if $\Delta_K(\lambda^{2k}) \neq 0$ for $2 \leq k \leq d-1$ then the representation $r_d \circ \rho_\lambda^z$ is a smooth point of the representation variety $R_d\big(\Gamma_K\big) := R\big(\Gamma_K, \mathrm{SL}(d,\mathbb{C})\big)$ of knot group $\Gamma_K$ into $\mathrm{SL}(d,\mathbb{C})$; it is contained in a unique $(d-1)(d+2)$-dimensional component $R_{\lambda,d} \subset R_d\big(\Gamma_K\big)$.

# Résumé


Nous nous intéressons à l'étude de certaines représentations réductibles non-abéliennes et métabéliennes du groupe d'un nœud dans $\mathrm{SL}(d,\mathbb{C})$ qui sont associées à une racine simple du polynôme d'Alexander.

Soit $K$ un nœud dans $\mathbb{S}^3$, $X = \overline{\mathbb{S}^3 \setminus V(K)}$ son complémentaire, où $V(K)$ est un voisinage tubulaire de $K$. Soit $\Gamma_K = \pi_1(X)$ le groupe fondamental de $X$. Soit $\varphi\colon \pi_1(X) \to \mathbb{Z}$ la surjection canonique qui envoie le méridien $\mu$ de $K$ sur 1, i.e. $\varphi(\gamma) = \mathrm{lk}(\gamma, K)$. Soit $\Delta_K(t) \in \mathbb{C}[t^{\pm 1}]$ le polynôme d'Alexander de $K$. On associe à $\lambda \in \mathbb{C}^*$, $\lambda \neq 1$, un homomorphisme

$$\lambda^\varphi\colon \Gamma_K \to \mathbb{C}^*, \quad \gamma \mapsto \lambda^{\varphi(\gamma)}.$$

Notons que $\lambda^\varphi$ envoie le méridien $\mu$ de $K$ sur $\lambda$. On obtient ainsi une représentation diagonale

$$\rho_\lambda\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C}), \quad \rho_\lambda(\gamma) = \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}.$$

Pour une application donnée $z\colon \Gamma_K \to \mathbb{C}$ on définit l'application $\rho_\lambda^z$ par

$$\rho_\lambda^z\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C}), \quad \rho_\lambda^z(\gamma) = \begin{pmatrix} 1 & z(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}.$$

L'application $\rho_\lambda^z$ est une repésentation si et seulement si $z\colon \Gamma_K \to \mathbb{C}_{\lambda^2}$ est un cocycle i.e. $z$ satisfait, pour tous $\gamma_1, \gamma_2 \in \Gamma_K$, la relation

$$z(\gamma_1\gamma_2) = z(\gamma_1) + \lambda^{2\varphi(\gamma_1)} z(\gamma_2).$$

Ici, Nous définissons $\mathbb{C}_{\lambda^2}$ comme étant le $\Gamma$-module $\mathbb{C}$ muni de l'action induite par $\lambda^2$, i.e. $\gamma \cdot x = \lambda^{2\varphi(\gamma)} x$ pour tout $\gamma \in \Gamma_K$ et tout $x \in \mathbb{C}$. Il est facile de voir que, dans ce cas, la représentation $\rho_\lambda^z$ est conjugée à la représentation diagonale ( abélienne) $\rho_\lambda\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C})$ si et seulement si $z$ est un cobord i.e. il existe un élément $x_0 \in \mathbb{C}$ tel que pout tout $\gamma \in \Gamma_K$ on a $z(\gamma) = \gamma \cdot x_0 - x_0$. Il existe donc une représentation non abélienne métabélienne $\rho_\lambda^z$ si et seulement si il existe un cocyle principal $z\colon \Gamma_K \to \mathbb{C}_{\lambda^2}$. La représentation $\rho_\lambda^z$ était étudiée pour la première fois par G. Burde et G. de Rham [Bur67, deR67]. Ils ont montré qu'il existe de telles représentations non abéliennes métabéliennes et réductibles du groupe de nœuds $\Gamma_K$ si et seulement si $\lambda^2$ est une racine du polynôme d'Alexander $\Delta_K(t)$.

Dans ce qui suit, $r_d\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(d,\mathbb{C})$ dénote la représentation rationnelle irréductible de dimension $d$ de $\mathrm{SL}(2,\mathbb{C})$ (see Chapter 2). Nous étudions le comportement de la représentation $r_d \circ \rho_\lambda^z\colon \Gamma_K \to \mathrm{SL}(d,\mathbb{C})$ sous l'hypothèse supplémentaire que $\lambda^2$ est une racine simple du polynôme d'Alexander $\Delta_K(t)$. En utilisant le théorème 1.1 de [HPS01], nous montrons dans la proposition 4.1 que la représentation $r_d \circ \rho_\lambda^z$ est la limite de représentations irréductibles.

Enfin, nous montrons dans le théorème 4.1 que si $\lambda^2$ est une racine simple du polynôme d'Alexander $\Delta_K(t)$ et si $\Delta_K(\lambda^{2k}) \neq 0$ pour $2 \leq k \leq d-1$ alors la représentation $r_d \circ \rho_\lambda^z$ est un point lisse de la variété des représentations $R_d(\Gamma_K) := R\big(\Gamma_K, \mathrm{SL}(d,\mathbb{C})\big)$ du groupe de nœuds $\Gamma_K$ dans $\mathrm{SL}(d,\mathbb{C})$; elle est contenue dans une unique composante $R_{\lambda,d} \subset R_d\big(\Gamma_K\big)$ de dimension $(d-1)(d+2)$.

# Introduction

The Knot Theory is a branch of mathematical discipline called Algebraic Topology. This discipline was invented by the French mathematician Henri Poincaré in the late nineteenth century and focuses on the properties of geometric objects that are invariant under continuous deformations without tearing. In short, the specialist in knot theory is interested in the shape of the knot. Like any other human activity, the mathematician needs tools. In knot theory, the most effective tool is the notion of invariant. An invariant is a quantity ; which can be an integer, a real number, a polynomial, group or any other mathematical object ; that does not change when the knot is subjected to a continuous deformation without tearing. Roughly speaking, we can say that the invariants have especially negative answer to the problem knots. Specifically, suppose that there is an invariant. We can then say that two knots are not equivalent when evaluating the invariant on these two knots does not give the same result. In contrast, if the two knots have the same invariant, so we can not conclude anything. You must either change the invariant or show directly that these are the same knots. All computable invariants are known to be incomplete, that is to say that there are actually different knots with the same invariant.

It turned out that many topological invariants can be derived from the fundamental group. In 1985 A. Casson constructed an invariant for integer homology spheres. This invariant involves the space $R(\Gamma, G)$ where $\Gamma = \pi_1(M)$ is the fundamental group of a rational homology sphere $M$ and $G = SU(2)$. The Casson invariant of $M$ is an integer that counts algebraically the conjugacy classes of representations of the fundamental group $\Gamma$ in $SU(2)$. This invariant for homology spheres was generalized by C. Curtis to groups $G = SO(3)$, $U(2)$, $Spin(4)$ and $SO(2)$, [Cur94]. For more examples concerning the the link between the representation theory of fundamental groups and geometry and topology of 3-manifolds see [CS83] and [BZ85].

In what follows, we are interested in the case where $\Gamma$ is the fundamental group of the complement of a knot $K$ in the three dimensional sphere $\mathbb{S}^3$. In 1967, G. Burde and G. de Rham [Bur67], [deR67], proved, independently, that when $\lambda^2$ is a root of Alexander polynomial $\Delta_K(t)$ then there exists a reducible, metabelian, non-abelian, representation of the knot group into $\mathrm{SL}(2,\mathbb{C})$. Let us recall this result: let $K$ be a knot in $\mathbb{S}^3$, $X = \overline{\mathbb{S}^3 \setminus V(K)}$ its complement, where $V(K)$ is a tubular neighborhood of $K$. Moreover, let $\Gamma_K = \pi_1(X)$ denote the fundamental group of $X$. Let $\varphi\colon \pi_1(X) \to \mathbb{Z}$ denote the canonical surjection which maps the meridian $\mu$ of $K$ to 1 i.e. $\varphi(\gamma) = \mathrm{lk}(\gamma, K)$. Let $\Delta_K(t) \in \mathbb{C}[t^{\pm 1}]$ denote the Alexander polynomial of $K$. We associate to a nonzero complex number $\alpha \in \mathbb{C}$ a homomorphism

$$\alpha^\varphi\colon \Gamma_K \to \mathbb{C}^*, \quad \gamma \mapsto \alpha^{\varphi(\gamma)}.$$

Note that $\alpha^\varphi$ maps the meridian $\mu$ of $K$ to $\alpha$. We define $\mathbb{C}_\alpha$ to be the $\Gamma_K$-module $\mathbb{C}$ with the action induced by $\alpha^\varphi$, i.e. $\gamma \cdot x = \alpha^{\varphi(\gamma)} x$ for all $\gamma \in \Gamma_K$ and all $x \in \mathbb{C}$. The trivial $\Gamma_K$-module $\mathbb{C}_1$ is simply denoted $\mathbb{C}$. For each nonzero complex $\lambda \in \mathbb{C}^*$ there exists a diagonal representation $\rho_\lambda\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C})$ given by

$$\rho_\lambda(\gamma) = \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}.$$

Let $\rho_\lambda^z$ be the application

$$\rho_\lambda^z\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C}), \quad \rho_\lambda^z(\gamma) = \begin{pmatrix} 1 & z(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix}. \tag{0.1}$$

The application $\rho_\lambda^z$ is a homomorphism if and only if the map $z\colon \Gamma_K \to \mathbb{C}_{\lambda^2}$ is a cocycle i.e. $z(\gamma_1\gamma_2) = z(\gamma_1) + \lambda^{2\varphi(\gamma_1)} z(\gamma_2)$. Note also that $\rho_\lambda^z$ is abelian if $\lambda = \pm 1$. If $\lambda^2 \neq 1$ then $\rho_\lambda^z$ is abelian if and only if $z$ is a coboundary i.e. there exists an element $x_0 \in \mathbb{C}$ such that $z(\gamma) = (\lambda^{2\varphi(\gamma)} - 1)x_0$. Burde and de Rham proved that there exist a metabelian, non-abelian representation $\rho_\lambda^z$ if and only if $\lambda^2$ is a root of Alexander polynomial $\Delta_K(t)$.

A generalization of Burde's and de Rham's result is established by H. Jebali in [Jeb08] where the author considers certain reducible, metabelian, non-abelian, representations of $\Gamma_K$ into $\mathrm{GL}(n,\mathbb{C})$. It turns out that the structure of the complex Alexander module $H_1(X_\infty;\mathbb{C})$ is completely determined by these representations.

The question whether or not the representation $\rho_\lambda^z$ is a limit of irreducible representations of $\Gamma_K$ into $\mathrm{SL}(2,\mathbb{C})$ was studied in [HPS01]. Theorem 1.1 of [HPS01] states that a metabelian, non-abelian representation $\rho_\lambda^z\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C})$ is the limit of irreducible representations if $\lambda^2$ is a simple root of Alexander polynomial $\Delta_K(t)$. Moreover, in this case the representation $\rho_\lambda^z$ is a smooth point of the representation variety $R\big(\Gamma_K,\mathrm{SL}(2,\mathbb{C})\big)$; it is contained in a unique 4-dimensional component $R_\lambda \subset R\big(\Gamma_K,\mathrm{SL}(2,\mathbb{C})\big)$.

The problem of deformations of abelian and metabelian representations into $\mathrm{SL}(2,\mathbb{C})$ or $SU(2)$ that correspond to a root of the Alexander polynomial was studies in the literature (see [FK91, Her97, Ben98, HK98, Ben00, BL02, HPS01, HP05, BHJ10]). The result of [FK91] is generalized in [Her97] and [HK98] by replacing the condition of the simple root by a condition on the signature operator. L. Ben Abdelghani and all studied in [BHJ10] the case where $\lambda^2$ is a multiple root of $\Delta(t)$.

In this thesis, we study deformations of certain non abelian, metabelian, reducible representation of the knot group $\pi_1(X)$ into $\mathrm{SL}(d,\mathbb{C})$. More precisely, we are interested in the behavior of the representations $\rho_\lambda^z\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C})$ under the composition with the $d$-dimensional irreducible rational representation $r_d\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(d,\mathbb{C})$.

In order to describe the representation $r_d$, we let $\mathrm{SL}(2,\mathbb{C})$ act as a group of automorphism on the polynomial algebra $R=\mathbb{C}[X,Y]$. If $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C})$ then there is a unique automorphism $r\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ of $R$ given by

$$r\begin{pmatrix} a & b\\ c & d\end{pmatrix}(X) = d\,X - b\,Y \quad\text{and}\quad r\begin{pmatrix} a & b\\ c & d\end{pmatrix}(Y) = -c\,X + a\,Y\,.$$

We let $R_{d-1}\subset R$ denote the $d$-dimensional subspace of homogeneous polynomials of degree $d-1$. The monomials $e_l^{(d-1)} = X^{l-1}Y^{d-l}$, $1\le l\le d$, form a basis of $R_{d-1}$ and $r\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ leaves $R_{d-1}$ invariant. In what follows we will identify $R_{d-1}$ and $\mathbb{C}^d$ by fixing the basis $(e_1^{(d-1)},\dots,e_d^{(d-1)})$ of $R_{d-1}$. It follows that we obtain an $d$-dimensional representation

$r_d\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(R_{d-1}) \cong \mathrm{GL}(d,\mathbb{C})$ defined by

$$\begin{aligned} r_d : \mathrm{SL}(2,\mathbb{C}) &\to \mathrm{GL}(R_{d-1}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto r_d \begin{pmatrix} a & b \\ c & d \end{pmatrix} : R_{d-1} \to R_{d-1} \\ & \qquad e_l^{(d-1)} \mapsto r_d \begin{pmatrix} a & b \\ c & d \end{pmatrix} (e_l^{(d-1)}) = (dX - bY)^{l-1}(-cX + aY)^{d-l} \,. \end{aligned} \tag{0.2}$$

The representation $r_d$ is *rational*, i.e. the coefficients of the matrix coordinates of $r_d\big(A\big)$ are polynomials in the matrix coordinates of $A$. We remark that the image of $r_d$ is contained in $\mathrm{SL}(R_{d-1}) \cong \mathrm{SL}(d,\mathbb{C}) \subset \mathrm{GL}(d,\mathbb{C})$. Moreover, the image of the composition $r_d \circ \rho_\lambda^z$ is contained in the Borel subgroup $B_d \subset \mathrm{SL}(d,\mathbb{C})$ of upper triangular matrices.

Note that any rational irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ is equivalent to some $r_d$ (see Proposition 2.2). Hence the study of the representation $r_d \circ \rho_\lambda^z$ is not restrictive.

Recall that under the hypothesis that $\lambda^2$ is a simple root of $\Delta_K(t)$ the representation $\rho_\lambda^z \in R\big(\Gamma_K, \mathrm{SL}(2,\mathbb{C})\big)$ is a smooth point of the representation variety. It is contained in a unique irreducible 4-dimensional component $R_\lambda \subset R\big(\Gamma_K, \mathrm{SL}(2,\mathbb{C})\big)$ (see [HPS01, Theorem 1.1]). In particular, it is the limit of irreducible representations. Note that *generically* a representation $\rho \in R_\lambda$ is irreducible.

Our first result is the following:

**Proposition 4.1.** *Let $K \in \mathbb{S}^3$ be a knot, $\lambda^2 \in \mathbb{C}$ a simple root of Alexander polynomial $\Delta_K(t)$ and let $z \in Z^1(\Gamma_K, \mathbb{C}_{\lambda^2})$ be a cocycle representing a generator of $H^1(\Gamma_K, \mathbb{C}_{\lambda^2})$.*

*Then the representation $\rho_{\lambda,d}^z = r_d \circ \rho_\lambda^z \colon \Gamma_K \to B_d$ is the limit of irreducible representations in $R(\Gamma_K, \mathrm{SL}(d,\mathbb{C}))$. More precisely, generically a representation $\rho_d = r_d \circ \rho$, $\rho \in R_\lambda$ is irreducible.*

Here, a property of an irreducible algebraic variety $Y$ is said to be true *generically* if it holds except on a proper Zariski-closed subset of $Y$, in other words, if it holds on a non-empty Zariski-open subset.

Firstly, we show in Lemma 1.3 that if for a representation $\rho\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C})$ the image $\rho(\Gamma_k) \subset \mathrm{SL}(2,\mathbb{C})$ is Zariski-dense then the representation $r_d \circ \rho$ is irreducible. This applies in particular to any irreducible representation $\rho$ contained in a neighborhood

$U = U(\rho^z_\lambda) \subset R(\Gamma_K, \mathrm{SL}(2,\mathbb{C}))$. Hence, from Lemma 1.3 it follows that if $\lambda^2$ is a simple root of $\Delta_K(t)$ the representation $\rho^z_{\lambda,d}$ is the limit of irreducible representations in $R_d(\Gamma_K) := R\big(\Gamma_K, \mathrm{SL}(d,\mathbb{C})\big)$.

Our main result is the following:

**Theorem 4.1.** *If $\lambda^2$ is a simple root of $\Delta_K(t)$ and if $\Delta_K(\lambda^{2k}) \neq 0$ for $2 \leq k \leq d-1$ then the reducible metabelian representation $\rho^z_{\lambda,d} := r_d \circ \rho^z_\lambda$ is a limit of irreducible representations. More precisely, $\rho^z_{\lambda,d}$ is a smooth point of $R_d(\Gamma_K)$; it is contained in a unique $(d+2)(d-1)$-dimensional component $R_{\lambda,d} \subset R_d(\Gamma_K)$.*

It follows directly from Proposition 4.1 that $\rho^z_{\lambda,d} := r_d \circ \rho^z_\lambda$ is the limit of irreducible representations.

The Lie algebra $\mathfrak{sl}_d(\mathbb{C})$ of $\mathrm{SL}(d,\mathbb{C})$ turns into an $\mathrm{SL}(2,\mathbb{C})$-module via $\mathrm{Ad} \circ r_d$ where $\mathrm{Ad}\colon \mathrm{SL}(d,\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}_d(\mathbb{C}))$ denotes the adjoint representation and $r_d$ the Representation (0.2). For this action we have the following equivalence (see Chapter 2)

$$\mathrm{Ad} \circ r_d \cong \bigoplus_{i=1}^{d-1} r_{2i+1} \quad \text{which gives} \quad \mathfrak{sl}_d(\mathbb{C})_{r_d} \cong \bigoplus_{i=1}^{d-1} R_{2i}\,. \tag{0.3}$$

Therefore, to calculate dimensions of cohomology groups $H^*(\Gamma_K, \mathfrak{sl}_d(\mathbb{C})_{\rho^z_{\lambda,d}})$ we have to calculate dimensions of $H^*(\Gamma_K; R_{2i})$, $1 \leq i \leq d-1$.

We denote by $\mathbb{C}_{\chi_i}$ the $B_2$-module $\mathbb{C}$ via the action $r_{d-1}\left(\begin{smallmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{smallmatrix}\right)x = \lambda^i x$. Using the short exact sequences of $B_2$-modules

$$0 \to \mathbb{C}_{\chi_{d-1}} \to R_{d-1} \to \bar{R}_{d-1} \to 1$$

and

$$0 \to R_{d-3} \xrightarrow{\phi_{d-3}} \bar{R}_{d-1} \to \mathbb{C}_{\chi_{-d+1}} \to 0,$$

where $\bar{R}_{d-1}$ denotes the quotient $R_{d-1}/\langle e_1^{(d-1)} \rangle$, we prove the following:

**Lemma 4.3.** *Let $\lambda \in \mathbb{C}^*$, $\lambda \neq 1$, and $d \geq 4$ be given. If $\Delta_K(\lambda^{d-1}) \neq 0$ and if $\lambda^{d-1} \neq 1$ then*

$$H^*\big(\Gamma; R_{d-1}\big) \cong H^*\big(\Gamma; R_{d-3}\big)\,.$$

It follows from Lemma 4.3 that If $\lambda^2 \neq 1$ is a simple root of Alexander polynomial and $\Delta_K(\lambda^{2i}) \neq 0$ and $\lambda^{2i} \neq 1$, for $2 \leq i \leq d-1$, we have

$$\dim H^*\big(\Gamma_K; R_{2i}\big) = \dim H^*\big(\Gamma_K; R_2\big), \quad \forall 1 \leq i \leq d-1$$

and Equivalence (0.3) imply that

$$\dim H^*\big(\Gamma; \mathfrak{sl}_d(\mathbb{C})_{\rho^z_{\lambda,d}}\big) = (d-1) \dim H^*\big(\Gamma; R_2\big) .$$

Since hypothesis $\Delta_K(\lambda^{2i}) \neq 0$ for $2 \leq i \leq d-1$ implies that $\lambda^k \neq 1$, for all $k \in \mathbb{Z}$, we conclude:

**Proposition 4.3.** *Let $K \subset S^3$ be a knot and let $\lambda \in \mathbb{C}^*$ and $d \geq 3$.*

*Suppose that $\lambda^2$ is a simple root of the Alexander polynomial $\Delta_K(t)$ and let $\rho^z_\lambda \colon \Gamma \to B_2$ the non-abelian Representation* (0.1).

*If $\Delta_K(\lambda^{2i}) \neq 0$ for $2 \leq i \leq d-1$ then for $\rho^z_{\lambda,d} := r_d \circ \rho^z_\lambda \colon \Gamma \to B_d \subset \mathrm{SL}(d,\mathbb{C})$ we have*

$$\dim H^1\big(\Gamma_K; \mathfrak{sl}_d(\mathbb{C})_{\rho^z_{\lambda,d}}\big) = (d-1) \text{ and } H^0\big(\Gamma_K; \mathfrak{sl}_d(\mathbb{C})_{\rho^z_{\lambda,d}}\big) = 0 .$$

It follows directly from Propositions 4.2 and 4.3 that $\rho^z_{\lambda,d} := r_d \circ \rho^z_\lambda$ is a smooth point of $R_d(\Gamma_K)$; it is contained in a unique $(d+2)(d-1)$-dimensional component $R_{\lambda,d} \subset R_d(\Gamma)$.

P. Menal-Ferrer and J. Porti [FP12] showed that the conclusions of Theorem 4.1 hold for hyperbolic knots if $\rho^z_\lambda$ is replaced by a lift of the holonomy, $\widetilde{\mathrm{hol}} \colon \pi_1(\mathbb{S}^3 \smallsetminus K) \to \mathrm{SL}(2,\mathbb{C})$, of the hyperbolic structure of the complement $\mathbb{S}^3 \smallsetminus K$. Note that Theorem 4.1 and Proposition 4.1 do apply to non-hyperbolic knots. Irreducible metabelian representations and their deformations are studied by H. Boden and S. Friedl in a series of articles [BF08, BF11, BF13, BF14]. In particular the deformations of *irreducible* metabelian representations, which are not considered in this thesis, are studied in [BF13].

This thesis is organized as follows: In Chapter 1, we introduce notations and facts. Chapter 2 is devoted to give irreducible rational representations of $\mathrm{SL}(2,\mathbb{C})$. Also, we give decomposition of an arbitrary rational representation of $\mathrm{SL}(2,\mathbb{C})$. We conclude this chapter by an equivalence between $\mathrm{Ad} \circ r_d$ and $r_d \otimes r_d = \sum_{k=1}^{d-1} r_{2k+1}$. Chapter 3 deals representation spaces and cohomology groups. We present Also a review on the deformations of representations with some important results. In Chapter 4, we make some cohomology calculations to study non abelian, metabelian reducible representation of fundamental group $\Gamma_K$ into $\mathrm{SL}(d,\mathbb{C})$ and then we prove Theorem 4.1. We conclude this chapter by some examples, conclusion and perspectives.

# Chapter 1

# Notations and facts

## Summary

The purpose of this chapter is to recall definitions and basic results of representation theory, in taking the opportunity to introduce the terminology and notations used later. We begin by presenting the notion of algebraic variety and we define linear algebraic group. In Paragraph 1.2, we present some definitions and results on Zariski dense sets. Paragraph 1.3 is to recall some properties on representations of group. Paragraph 1.4 is devoted to present some properties of Lie algebra and algebraic group. We conclude this chapter with some results on knot theory.

## 1.1 Affine algebraic variety

In what follows, the general reference is Springer's LNM [Spr77]. In the following, $\mathbb{K}$ is an algebraically closed field with caracteristic zero, $V$ a $\mathbb{K}-$vector space of finite dimension. Let $(e'_i)_{1\leq i\leq d}$ be a basis of $V$, and $f_i : V \to \mathbb{K}$ are linear functions defined by $f_i(\sum_{i=1}^{d} x_i e'_i) = x_i$, for $1 \leq i \leq d$. The $f_i$ generate a subalgebra $\mathbb{K}[V] = S$ of the algebra of all $\mathbb{K}-$valued functions on $V$. The functions of $S$ are called polynomial functions on $V$ with values in $\mathbb{K}$. This definition is independent of the choice of the basis $(e'_i)_{1\leq i\leq d}$.

**Definition 1.1** (Zeros of ideals of $S$)**.** *Let $I$ be an ideal of $S$. Then $v \in V$ is called a zero of $I$ if $f(v) = 0$ for all $f \in I$.*

**Theorem 1.1** (Hilbert's Nullstellensatz)**.** **(i)** *(first form) A proper ideal $I$ of $S$ has a zero;*

**(ii)** *(second form) Let $I$ be an ideal of $S$ and let $f \in S$ be such that $f(v) = 0$ for all zeros $v$ of $I$. Then there is $n \geq 1$ such that $f^n \in I$. See [Lan71, Chap. X, § 2].*

**Definition 1.2** (The Zariski topology on $V$)**.** *If $I$ is an ideal of $S$, let $\mathcal{V}(I)$ be the set of its zeros. We then have the following properties :*

1. $\mathcal{V}(\{0\}) = V, \quad \mathcal{V}(S) = \emptyset$;
2. $I \subset J \Rightarrow \mathcal{V}(I) \supset \mathcal{V}(J)$;
3. $\mathcal{V}(I \cap J) = \mathcal{V}(I) \cup \mathcal{V}(J)$;

4. *If $(I_\alpha)_{\alpha\in A}$ is a set of ideals and $\sum_{\alpha\in A} I_\alpha$ the ideal of the sums $\sum_{\alpha\in A} f_\alpha$, with $f_\alpha \in I_\alpha$ and $f_\alpha = 0$ for all except finitely many $\alpha$, then*

$$\mathcal{V}(\sum_{\alpha\in A} I_\alpha) = \bigcap_{\alpha\in A} \mathcal{V}(I_\alpha).$$

*It follows from (1), (3) and (4) that there is a topology on $V$ whose closed sets are the $\mathcal{V}(I)$, $I$ running through the ideals of $S$. This is the* Zariski topology.

**Definition 1.3.** *If $X$ is a subset of $V$, define the ideal $\mathcal{J}(X)$ of $S$ by*

$$\mathcal{J}(X) = \{f \in S \quad | \quad f(X) = 0\}.$$

*If $I$ is an ideal of $S$, we let $\sqrt{I}$ denote the radical of $I$. It is defined as*

$$\sqrt{I} = \{r \in S \mid r^n \in I \textit{ for some positive integer } n\}.$$

**Proposition 1.1. (i)** $\mathcal{V}(\mathcal{J}(X)) = \bar{X}$, *the Zariski-cloture of $X$ ;*

**(ii)** $\mathcal{J}(\mathcal{V}(I)) = \sqrt{I}$.

**Definitions 1.1.** *Let $X \subset V$ be a closed subset. Such a set is also called an* affine algebraic variety *of $V$. The restrictions to $X$ of functions of $S$ form an algebra $\mathbb{K}-$valued functions on $X$, denoted by $S_X$. It is isomorphic to $S/\mathcal{J}(X)$. Moreover, we have a bijection from $X$ onto the set of $\mathbb{K}-$algebra homomorphisms $S_X \to \mathbb{K}$. If $f_i : V \to \mathbb{K}$ is the dual of $e'_i$, its restriction to $X$ is denoted by : $f_{i_{|X}} = g_i : X \to \mathbb{K}$, then $S_X = \mathbb{K}[g_1, \dots, g_d]$.*

*Let $V'$ be another finite dimension vector space over $\mathbb{K}$ and $X' \subset V'$ a closed subset. We put $S' = \mathbb{K}[V']$.*

*Let $\phi : X \to X'$ a map. If $f'$ is a function defined on $X'$ with values in $\mathbb{K}$, then $\phi^* f'$ defined by $\phi^* f'(v) = f'(\phi(v))$ is a function defined on $X$ with values in $\mathbb{K}$. The application $\phi$ is called a* morphism of affines algebraic varieties *if $\phi^* S'_{X'} \subset S_X$.*

**Remark 1.1.** *Let $E(V) = E$ be the vector space of all $\mathbb{K}-$linear maps on $V$ (via a basis : the vector space of all $d \times d-$matrices). Then, the group*

$$\mathrm{GL}(V) = \{g : V \to V \mid \det(g) \neq 0\}$$

*is a Zariski-open subset of* $E$*, namely the complement of the closed subset given by the equation* $\det(g) = 0$*. We want to view* $\mathrm{GL}(V)$ *as an affine algebraic variety. This can be done by identifying* $\mathrm{GL}(V)$ *with the closed subset of the* $(d^2+1)-$*dimensional vector space* $E \times \mathbb{K}$*, formed by the* $(g, x)$ *with* $x \det(g) = 1$*, i.e.*

$$\mathrm{GL}(V) \equiv \{(g, x) \in E \times \mathbb{K} \mid x \det(g) = 1\}.$$

**Definition 1.4** (Linear algebraic group)**.** *A* linear algebraic group *is a closed subgroup of some* $\mathrm{GL}(V)$*.*

**Example 1.1.** *The group* $\mathrm{SL}(d, \mathbb{K})$ *is a linear algebraic group of* $\mathrm{GL}(d, \mathbb{K})$*, since* $\mathrm{SL}(d, \mathbb{K})$ *is the inverse image of a closed subset by the polynomial function* det*.*

## 1.2 Zariski dense sets

In what follows, we give some definitions and lemmas that are crucial in the following. The general reference for this section is [KP96].

**Definition 1.5** (Zariski-dense subsets )**.** *A subset* $X$ *of a finite dimensional vector space* $V$ *is called Zariski-dense if every function* $f \in \mathbb{K}[V]$ *which vanishes on* $X$ *is the zero function. More generally, a subset* $X \subset Y (\subset V)$ *is called Zariski-dense in* $Y$ *if every function* $f \in \mathbb{K}[V]$ *which vanishes on* $X$ *also vanishes on* $Y$*.*

In other words every polynomial function $f \in \mathbb{K}[V]$ is completely determined by its restriction $f_{|X}$ to a Zariski-dense subset $X \subset V$. Denote by $\mathcal{J}(X)$ the ideal of functions vanishing on $X \subset V$:

$$\mathcal{J}(X) := \{f \in \mathbb{K}[V] \quad | \quad f(a) = 0, \text{ for all } a \in X\}.$$

$\mathcal{J}(X)$ is called the ideal of $X$. Clearly, we have $\mathcal{J}(X) = \cap_{a \in X} m_a$ where $m_a = \mathcal{J}(\{a\})$ is the maximal ideal of functions vanishing in $a$, i.e. the kernel of the evaluation homomorphism $\epsilon_a : \mathbb{K}[V] \longrightarrow \mathbb{K}$, $f \longmapsto f(a)$. It is called the maximal ideal of $a$.

**Lemma 1.1.** *Let* $h \in \mathbb{K}[V]$ *be a non-zero function and define* $V_h := \{v \in V \mid h(v) \neq 0\}$*. Then* $V_h$ *is Zariski-dense in* $V$*.*

*Proof.* Let $f \in \mathbb{K}[V]$ such that $f_{|V_h} = 0$. Then:

- $fh = 0$ on $V_h$ by hypothesis.

- $fh = 0$ on the complementary of $V_h$ in $V$.

Therefore, $fh = 0$ on $V$, but $h \neq 0$ on $V$ then $f = 0$ on $V$. □

**Example 1.2.** *A typical example of a Zariski-dense subset is the linear general group $GL_n(\mathbb{C})$ of $M_n(\mathbb{C})$, since:*

$$GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid det(A) \neq 0\}.$$

**Definition 1.6** (Generic properties)**.** *A property of an irreducible algebraic variety is said to* be true generically *if it holds except on a proper Zariski-closed subset of $Y$, in other words, if it holds on a non-empty Zariski-open subset.*

## 1.3 Group representation

In this section, we present definition of rational representation and then we present definition of irreducible and reducible representation of a group. At the end, we give some results related to representations of group that will be used thereafter. The general reference for this section is Spinger's LNM [Spr77].

Let $G$ be a group acting on a set $X$ and let $F(X) = \{f : X \to \mathbb{K}\}$ be the space of functions on $X$ with values in $\mathbb{K}$. Then $F(X)$ is also equipped with a linear action of $G$ given by

$$(g \cdot f)(x) = f(g^{-1}x), \quad g \in G,\ f \in F(X),\ x \in X.$$

This new action, in a sense, contains as much information as the old, but has the advantage of using the techniques of linear algebra. This is why we are especially interested in the study of linear actions of groups i.e., their representations.

### 1.3.1 Rational representations of group

**Definition 1.7** (Rational representation ). *If $G$ is any group, a* representation *of $G$ in a finite dimensional vector space $W$ over $\mathbb{K}$ is a homomorphism $\rho : G \to \mathrm{GL}(W)$.*

*If $G \subset \mathrm{GL}(V)$ is a linear algebraic group, a* rational or polynomial representation *of $G$ in $W$ is a homomorphism $\rho : G \to \mathrm{GL}(W)$ which is at the same time a morphism of affine algebraic varieties.*

**Remark 1.2.**
1. *This means that, introducing the bases of $V$ and $W$, the matrix coordinates $\rho(g)$ are polynomials in the $d^2$ matrix coordinates of $G$ and of $1/\det(g)$ (if $\dim V = d$).*

2. *Given a representation $\rho\colon G \to \mathrm{GL}(W)$ the vector space $W$ turns into a $G-$module.*

### 1.3.2 Reducible- irreducible representations

**Definition 1.8** (Reducible- irreducible representations). *A representation $\rho : G \to \mathrm{GL}(V)$ is called* reducible *if there exists a subspace $\{0\} \neq W \subsetneq V$ such that for each $g \in G$, $\rho(g)W = W$. Otherwise, it is called* irreducible. *In other words, a representation $\rho : G \to \mathrm{GL}(V)$ is irreducible if the only subspaces of $V$ stable by the action of $G$ are $\{0\}$ and $V$.*

**Remark 1.3.** *If $\rho\colon G \to \mathrm{GL}(V)$ is an irreducible representation, then $V$ is a simple $G-$module.*

**Lemma 1.2** (Schur's Lemma). *Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation. If $t$ is a linear transformation of $V$ which commutes with all $\rho(g)$, $g \in G$, then $t$ is a scalar multiplication.*

*Proof.* Let $a$ be an eigenvalue of $t$ and we put $W = \{v \in V \mid t(v) = av\}$. Then the proper space $W$ of $V$ is not empty, since it contains a proper vector $v$ of $V$, and it is $G-$stable by the action of $G$. In fact, let $v \in W$ and $g \in G$, then we have

$$t\rho(g)v = \rho(g)tv = \rho(g)av = a\rho(g)v.$$

This implies that $W$ is $G-$stable. But, $V$ is an irreducible $G-$module, i.e. admits no proper $G-$subspace, then $W = V$. This implies that for each $v \in V$, $tv = av$. Hence $t$ is the multiplication by a scalar. $\square$

**Definition 1.9** (Homomorphism of representations)**.** *Let $(E_1, \rho_1)$ and $(E_2, \rho_2)$ be two representations of a group $G$. We call homomorphism of representations $\rho_1$ and $\rho_2$ a linear application $T : E_1 \to E_2$ which commutes with $\rho_1$ and $\rho_2$, i.e. for each $g \in G$, $T \circ \rho_1(g) = \rho_2(g) \circ T$.*

**Definition 1.10** (Equivalent representations )**.** *Let $(E_1, \rho_1)$ and $(E_2, \rho_2)$ be two representations of a group $G$. Representations $\rho_1$ and $\rho_2$ are called equivalent if there exists a homomorphism of representations $T : E_1 \to E_2$ which is bijective.*

**Lemma 1.3.** *Let $G$ be a linear algebraic group, $\rho : \Gamma \longrightarrow G$ a representation and $r \colon G \to \mathrm{GL}(W)$ an irreducible rational representation.*

*If $\rho(\Gamma) \subset G$ is Zariski-dense then $r \circ \rho \colon \Gamma \to \mathrm{GL}(W)$ is irreducible.*

*Proof.* We assume that there exists a subspace $V \subset W$ such that $r(\rho(\gamma))V \subset V$, for each $\gamma \in \Gamma$. Let $\{e_1, \cdots, e_d\}$ be a basis of $W$ and let $\{v_1, \cdots, v_k\}$ be a basis of $V$. We put for each $1 \leq i \leq k$, $v_i = \sum_{j=1}^{d} \alpha_{ji} e_j$ and for each $A \in G$, $r(A)v_l = \sum_{i=1}^{d} \beta_{jl} e_j$. For $1 \leq i \leq k$ fixed, we define the matrix $M_i$ by

$$M_i = \begin{pmatrix} \alpha_{1\,1} & \cdots & \alpha_{1\,(i-1)} & \alpha_{1\,i} & \beta_{1\,i} & \alpha_{1\,(i+1)} & \alpha_{1\,k} \\ \alpha_{2\,1} & \cdots & \alpha_{2\,(i-1)} & \alpha_{2\,i} & \beta_{2\,i} & \alpha_{2\,(i+1)} & \alpha_{2\,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{d\,1} & \cdots & \alpha_{d\,(i-1)} & \alpha_{d\,i} & \beta_{d\,i} & \alpha_{d\,(i+1)} & \alpha_{d\,k} \end{pmatrix}.$$

Since $G$ is a linear algebraic group, it is a subgroup of some $GL_n(\mathbb{C})$. Let $F_i \colon M_n(\mathbb{C}) \longrightarrow \mathbb{C}^{\binom{d}{k+1}}$ be an application such that $F_i(A)$ is a vector whose the components are the $\binom{d}{k+1}$ determinants of $(k+1) \times (k+1)$ sub-matrices of $M_i$. Since $\dim V = k$ and $r(\rho(\gamma))V \subset V$, for each $\gamma \in \Gamma$, then $F_i(\rho(\gamma)) = (0, \cdots, 0)$ for each $\gamma \in \Gamma$. Since $\rho(\Gamma)$ is Zariski-dense, $F_i$ is zero on all $G$. The restriction of $F_i$ to $G$ being zero, the components of $F_i(A)$, for $A \in G$, are vanishing. Consequently $r(A)v_i \in V$, for each $1 \leq i \leq k$ and all $A \in G$. It follows that $V \subset W$ is stable by the action of $r$. On the other hand, $r$ is irreducible and hence $V = \{0\}$ or $V = W$. Hence $r \circ \rho$ is irreducible. $\square$

### 1.3.3 Abelian-metabelian representations

Abelian representation being completely determined by the data of the image of a meridian of a knot, the study of such representations provides little information about knot. By cons, the case of metabelian representations provides more information about knot and it presents a domain of study that has interested many authors whose include [Har79], [Fri04], [BF11], [HPS01] and [HKL08].

**Definition 1.11** (Derived group). *If $G$ is a group, we call the nth derived group of $G$ ($n$ is a positive integer), and it is denoted by $D^nG$, the subgroup of $G$ defined inductively as follows : $D^0G \geq D^1G \geq \ldots$ such that*

$$\begin{cases} D^0G = G, \\ D^{n+1}G \qquad \textit{is the group of commutators of } \; D^nG, \; n \in \mathbb{N} \end{cases}$$

*Group $D^1G$ is the group of commutators of $G$ and that we denote $G'$. The group $D^2G$ is denoted $G''$.*

**Definition 1.12** (Abelian-metabelian representations). *Let $k \geq 1$ and let $G$ be a group. A representation $\rho : \pi \to G$ is called $k-$metabelian if the restriction of $\rho$ to $k$th derived group of $\pi$, noted $D^k\pi$, is trivial. A representation $2-$metabelian is called simply metabelian. A representation $1-$metabelian is abelian.*

## 1.4 Some properties of Lie algebra and algebraic group

In this section, we introduce the Lie algebra of an algebraic group. Great interest to combine a Lie algebra with an algebraic group $G$ is that it automatically provides a rational representation of $G$. But before this, we give some useful definitions. The general reference for this section is [Gob09, Sec. 4.3]. Let $A$ a $\mathbb{K}-$ algebra.

### 1.4.1 Derivations

**Definition 1.13** (Derivations). *A derivation of $A$ is a mapping $\mathbb{K}-$linear $D : A \longrightarrow A$ satisfactory for $a, b \in A$,*

$$D(ab) = aD(b) + D(a)b.$$

We denote $Der_{\mathbb{K}}(A)$ the set of derivations of $A$.

One checks by direct calculation that if $D, F$ are two derivations, $D \circ F - F \circ D$ is still a derivation. This endows the set of derivations of a structure of Lie algebra for the bracket $[D, F] = D \circ F - F \circ D$. In the case where $G$ is an algebraic group, we are interested the set of the derivations for $A = \mathbb{K}[G]$. For each $g \in G$, we can define a mapping $\mathbb{K}[G] \longrightarrow \mathbb{K}[G]$ providing an action of $g$ on $\mathbb{K}[G]$ by $(g \cdot f)(x) = f(g^{-1}x)$, where $f \in \mathbb{K}[G]$, $x \in G$.

**Definition 1.14** (Lie algebra of an algebraic group)**.** The Lie algebra *of an algebraic group $G$ is the algebra, noted by $\mathfrak{g}$, given by*

$$\mathfrak{g} = \{D \in Der_{\mathbb{K}}(\mathbb{K}[G]) \mid Dg = gD, \forall g \in G\}.$$

Note that $\mathfrak{g}$ is a subalgebra of $Der_{\mathbb{K}}(\mathbb{K}[G])$.

**Remark 1.4.** *We can define the Lie algebra of an algebraic group as vector space tangent to the neutral element $e$ as follows:*

$$T_e(G) := \Big\{\delta : \mathbb{K}[G] \longrightarrow \mathbb{K} \textit{ linéaire} \mid \delta(fg) = f(e)\delta(g) + \delta(f)g(e), \forall f, g \in \mathbb{K}[G]\Big\}.$$

*The both descriptions give $\mathbb{K}-$isomorphic vector spaces and we can thus transport the structure of Lie algebra obtained above on the tangent space.*

The case we are interested in the following is which of $G = \mathrm{SL}(d, \mathbb{K})$. The Lie algebra of $\mathrm{SL}(d, \mathbb{K})$ is isomorphic to $\mathfrak{sl}_d(\mathbb{K})$, i.e. the vector space constitute of all matrices of trace zero.

Now, we define the notion of differential of a morphism of algebraic groups, which ensures that the Lie algebra of an algebraic group $G$ provides a rational representation.

### 1.4.2 Differential of a morphism and adjoint representation

Let $G$, $H$ two linear algebraic groups and let $\phi : G \longrightarrow H$ be a morphism of algebraic groups. The differential $d\phi$ of $\phi$ is the application $\mathbb{K}-$linear $d\phi : T_e(G) \longrightarrow T_{\phi(e)}(H)$ defined by $d\phi(\delta) = \delta \circ \phi^{\star}$.

The differential behave in functorial way, i.e. for two morphisms $\phi\colon G_1 \longrightarrow G_2$, $\psi\colon G_2 \longrightarrow G_3$, we have $d(\psi \circ \phi) = d\psi \circ d\phi$. Moreover, we can see that $d\phi\colon \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ is a morphism of Lie algebras. Given $x \in G_1$, we define an automorphism of $G_1$ by $y \longmapsto xyx^{-1}$, noted $Int_x$ which is a regular application $\Big($Given two affine algebraic varieties $V \subset \mathbb{K}^n$ and $W \subset \mathbb{K}^m$, we say that $\phi\colon V \longrightarrow W$ is a regular application if there exists polynomials $f_1, \cdots, f_m \in \mathbb{K}[X_1, \cdots, X_n]$ such that for $x \in V$, we may have $\phi(x) = \big(f_1(x), \cdots, f_m(x)\big)\Big)$. Thanks to the functorial behavior of differential, we have

$$d(id) = d(Int_x Int_x^{-1}) = d(Int_x) \circ d(Int_x^{-1}) = id,$$

which ensures that $d(Int_x) \in GL(\mathfrak{g}_1)$. We have thus defined a rational representation of $G_1$

$$\begin{aligned} Ad\colon G_1 &\longrightarrow \mathrm{GL}(\mathfrak{g}_1) \\ x &\longmapsto Ad(x) := d(Int_x). \end{aligned}$$

This representation is called *adjoint representation* of $G_1$.

### 1.4.3 Regular elements

**Definition 1.15.** *An element $A \in \mathrm{SL}(d, \mathbb{C})$ is called* regular element *if its centralizer $Z(A)$ is of minimal dimension among all centralizers of* $\mathrm{SL}(d, \mathbb{C})$ *, i.e.* $\dim Z(A) = d - 1$.

The minimum dimension mentioned above is precisely the *rank* of $\mathrm{SL}(d, \mathbb{C})$, i.e. the dimension of a maximal torus of $\mathrm{SL}(d, \mathbb{C})$.

**Proposition 1.2.** *Regular elements of $\mathrm{SL}(d, \mathbb{C})$ form a dense open in $\mathrm{SL}(d, \mathbb{C})$.*

**Proposition 1.3.**

1. *A semi-simple element of $\mathrm{SL}(d, \mathbb{C})$ or $\mathrm{GL}(d, \mathbb{C})$ is regular if and only if its eigenvalues are distinct in pairs two by two.*
2. *A nilpotent element of $\mathrm{SL}(d, \mathbb{C})$ or $\mathrm{GL}(d, \mathbb{C})$ is regular if and only if there is a unique bloc in the Jordan-Holder form.*
3. *Following propositions are equivalents in $\mathrm{SL}(d, \mathbb{C})$ or in $\mathrm{GL}(d, \mathbb{C})$:*
   - *$x$ is a regular element .*

- *The minimal polynomial of $x$ is of degree $d$, i.e. the minimal polynomial = characteristic polynomial.*
- *$Z(x)$ is abelian.*
- *$\mathbb{C}^d$ is cyclic as $\mathbb{C}[X]-$module.*

For more detail see [Ste74, sec. 3.5].

### 1.4.4 Examples

In what follows, we give some examples of representations of groups that we will need later.

**Example 1.3.** *We let* $\mathrm{SL}(2,\mathbb{C})$ *act as a group of automorphisms on the polynomial algebra in 2 variables* $R = \mathbb{C}[X,Y]$. *If* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C})$ *then there is a unique automorphism* $r\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *of $R$ given by*

$$r\begin{pmatrix} a & b \\ c & d \end{pmatrix}(X) = d\,X - b\,Y \quad \text{and} \quad r(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(Y) = -c\,X + a\,Y\,.$$

*We let $R_{d-1} \subset R$ denote the $d$-dimensional subspace of homogeneous polynomials of degree $d-1$. The monomials $e_l^{(d-1)} = X^{l-1}Y^{d-l}$, $1 \le l \le d$, form a basis of $R_{d-1}$ and $r\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ leaves $R_{d-1}$ invariant. In what follows we will identify $R_{d-1}$ and $\mathbb{C}^d$ by fixing the basis $\{e_1^{(d-1)}, \ldots, e_d^{(d-1)}\}$ of $R_{d-1}$. It follows that we obtain an $d$-dimensional representation* $r_d \colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(R_{d-1}) \cong \mathrm{GL}(d,\mathbb{C})$.

*The representation $r_d$ is* rational *i.e. the coefficients of the matrix coordinates of $r_d(A)$ are polynomials in the matrix coordinates of $A$.*

*Representation $r_d$ maps an unipotent matrix $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ onto an unipotent element of* $\mathrm{SL}(R_{d-1})$ *since for each $1 \le l \le d$*

$$r_d \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} e_l^{(d-1)} = (X - bY)^{l-1} Y^{d-l} = \sum_{k=0}^{l-1} \binom{l-1}{k} (-b)^k e_{l-k}^{(d-1)}$$

*and*

$$r_d \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} e_l^{(d-1)} = X^{l-1}(-cX + Y)^{d-l} = \sum_{k=0}^{d-l} \binom{d-l}{k} (-c)^k e_{l+k}^{(d-1)}.$$

*Moreover, for* $1 \le l \le d$ *we have*

$$r_d\Big(\mathrm{diag}(a,a^{-1})\Big)e_l^{(d-1)} = r_d\Big(\mathrm{diag}(a,a^{-1})\Big)X^{l-1}Y^{d-l} = a^{d-2l+1}e_l^{(d-1)}. \qquad (1.1)$$

*This shows that the image by* $r_d$ *of a diagonal matrix is the diagonal matrix*

$$r_d\Big(\mathrm{diag}(a,a^{-1})\Big) = \mathrm{diag}(a^{d-1},a^{d-3},\dots,a^{-d+3},a^{-d+1}).$$

*Hence the image of* $r_d$ *is contained in* $\mathrm{SL}(R_{d-1}) \cong \mathrm{SL}(d,\mathbb{C})$.

**Example 1.4.**

1. *The representation* $r_1\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(1,\mathbb{C}) = \{1\}$ *is the trivial representation.*

2. *The representation* $r_2\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C})$ *is equivalent to identity*

$$r_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

3. *Let* $\mathfrak{sl}_2(\mathbb{C}) = \big\langle e_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), e_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), e_3 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\big\rangle$, *and* $R_2 = \langle e_1' = Y^2, e_2' = YX, e_3' = X^2\rangle$. *The adjoint representation* $\mathrm{Ad}\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C}))$ *is equivalent to* $r_3$ *if there exists a linear isomorphism* $T\colon \mathfrak{sl}_2(\mathbb{C}) \to R_2$ *such that the following diagram commutes*

$$\begin{array}{ccc} \mathfrak{sl}_2(\mathbb{C}) & \xrightarrow{T} & R_2 \\ {\scriptstyle \mathrm{Ad}_g}\uparrow & & \uparrow{\scriptstyle r_3(g)} \\ \mathfrak{sl}_2(\mathbb{C}) & \xrightarrow{T} & R_2\,. \end{array}$$

*Then we have :*

- *For* $g = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)$

$$M(\mathrm{Ad}_g) = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}, \quad M(r_3(g)) = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}.$$

- *For* $g = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$

$$M(\mathrm{Ad}_g) = \begin{pmatrix} 1 & -2a & -a^2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}, \quad M(r_3(g)) = \begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix}.$$

- *For* $g = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$

$$M(\mathrm{Ad}_g) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad M(r_3(g)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

*We remark that the operator* $T$ *defined by* $T(e_1) = e'_1$, $T(e_2) = 2e'_2$ *and* $T(e_3) = -e'_3$ *verifies* $T \circ \mathrm{Ad}_g = r_3(g) \circ T$, $\forall g \in \mathrm{SL}(2, \mathbb{C})$.

## 1.5 Some results on knot theory

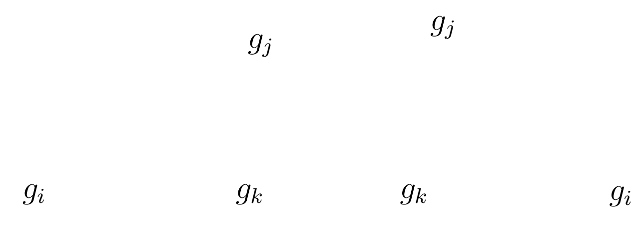
This section is devoted to a review on knot theory that will be used thereafter. In the Subsection 1.5.1, we define a knot group. In the second part 1.5.2 we present the Alexander module and the Alexander polynomial and then some results.

A *polygonal knot* is one which is the union of a finite number of closed straight-line segments called edges, whose endpoints are the vertices of the knot. A *knot* is *tame* if it is equivalent to a polygonal knot; otherwise it is *wild*. Equivalently, a knot is tame if it is the image by a differentiable embedding of a circle $\mathbb{S}^1$ into the sphere $\mathbb{S}^3$. In knot theory and 3-manifold theory, often the adjective "tame" is omitted. In our thesis we are interested in tame knots. The general reference for this section is [BZH13].

### 1.5.1 Knot group

Let $K$ be a knot in $\mathbb{S}^3$ and $V(K)$ is a tubular neighborhood of $K$. We denoted $X = \overline{\mathbb{S}^3 \setminus V(K)}$ the complement space of a knot. The fundamental group $\Gamma_K = \pi_1(X)$ is called *knot group*. Knot group is a group that admits a finite presentation quite special called the *Wirtinger presentation*, and which can be easily read from a regular projection of the knot.

To compute the Wirtinger presentation of a knot, we consider an oriented diagram of a knot, where we read relations on each crossing using the following convention:




$g_j = g_k g_i g_k^{-1}$ $\qquad\qquad$ $g_j = g_k^{-1} g_i g_k$

FIG. 1– Relations of Wirtinger presentation

If the knot has $n$ crossings, we have $n$ generators that represent strands or arcs and $n$ relations identified on the crossings of the knot. The $n$ generators and the $n$ relations constitute the Wirtinger presentation of the knot. Note that one of these relations is a consequence of the others. An example is given for calculating Wirtinger presentation of trefoil knot:


FIG. 2– Diagram of the trefoil knot

**Example 1.5.** *Applying the method, described above, on the diagram of the trefoil knot given above, we have the following relations:*

$$\begin{cases} S_2 S_1 S_3^{-1} S_1^{-1} = 1, \\ S_1 S_3 S_2^{-1} S_3^{-1} = 1, \\ S_3 S_2 S_1^{-1} S_2^{-1} = 1. \end{cases}$$

*And since one relation is a consequence of the others, we can eliminate one of them to have the system:*

$$\begin{cases} S_2 = S_1 S_3 S_1^{-1}, \\ S_1 = S_3 S_2 S_3^{-1}, \end{cases}$$

*which gives* $S_1S_3S_1 = S_3S_1S_3$. *Then a Wirtinger presentation of trefoil knot is the following :*

$$\begin{aligned} \pi_1(X) &= \langle S_1, S_2, S_3 : S_2S_1S_3^{-1}S_1^{-1} = S_1S_3S_2^{-1}S_3^{-1} = S_3S_2S_1^{-1}S_2^{-1} = 1 \rangle \\ &\cong \langle S_1, S_3 : S_1S_3S_1 = S_3S_1S_3 \rangle . \end{aligned} \tag{1.2}$$

One of the most important properties of group of a knot $K$ in $\mathbb{S}^3$ is that given by the following theorem of Papakyriapoulos:

**Theorem 1.2** (Asphericity of the knot complement)**.** *For* $n \neq 1$, *we have* $\pi_n(X) = 0$. *In other words, the complementary space* $X$ *is an* Eilenberg-Mac Lane space $K(\pi, 1)$.

Recall that a *meridian* of a knot $K$ is a simple curve $\mu$ on $\partial V(K)$ such that $[\mu] = 0$ in $\pi_1(V(K))$, and $[\mu] \neq 0$ in $\pi_1(\partial V(K))$ and a *longitude* $l$ of $K$ is a simple curve in $\partial V(K)$ which represents a generator of $\pi_1(V(K))$ and whose equivalence class in the homology group of the complementary space of the knot is trivial. Then, $[\mu]$ and $[l]$ form a basis of $H_1(\partial V(K)) \simeq \mathbb{Z} \oplus \mathbb{Z}$. As elements of the knot group the pair $(\mu, l)$ is unique up to conjugation.

### 1.5.2 Alexander module-Alexander polynomial

Let $K$ be a knot of $\mathbb{S}^3$, $X = \overline{\mathbb{S}^3 \setminus V(K)}$, where $V(K)$ denotes a tubular neighborhood of $K$ and $\Gamma_K = \pi_1(X)$ the fundamental group of $X$. Let $\Gamma_K/\Gamma_K' \simeq \mathbb{Z} = \langle t : - \rangle$ the quotient group cyclic infinite generated by the image $t$ of meridian $\mu$ and $X^\infty$ the covering of $X$ corresponding to the group of commutators $\Gamma_K' = [\Gamma_K, \Gamma_K]$. Let $\Lambda := \mathbb{C}[t^{\pm}]$ the ring of Laurent polynomials with complex coefficients. Since the group of automorphisms of recover $X^\infty$, $Aut(X^\infty) \simeq \Gamma_K/\Gamma_K' \simeq \mathbb{Z}$, the recover $X^\infty$ is called infinite cyclic covering of $X$. The homology groups $H_\star(X^\infty, \mathbb{C})$ are endowed with a structure of $\Lambda$-module. These modules are of finite type and of torsion. Only $H_1$ plays a significant role. Indeed, we have:

$$\begin{aligned}
H_0(X^\infty;\mathbb{C}) &\cong \mathbb{C}, \\
H_2(X^\infty,\partial X^\infty;\mathbb{C}) &\cong \mathbb{C}, \\
H_1(X^\infty,\partial X^\infty;\mathbb{C}) \cong H_1(X^\infty;\mathbb{C}) &\cong \pi/\pi' \otimes \mathbb{C}, \qquad \text{[BZH13]} \\
H_m(X^\infty,\partial X^\infty;\mathbb{C}) &= 0 \qquad \forall m \geq 3, \\
H_m(X^\infty;\mathbb{C}) &= 0 \qquad \forall m \geq 2.
\end{aligned}$$

Moreover, the group of covering transformations acts trivially on $H_2(X^\infty,\partial X^\infty;\mathbb{C})$ and the isomorphism $H_2(X^\infty,\partial X^\infty;\mathbb{C}) \cong \mathbb{C}$ depends only from the orientation of the knot.

The module $H_1(X^\infty;\mathbb{C})$ is a finitely generated, torsion $\Lambda$-module which is called the *Alexander module* of a knot $K$. A generator of its order ideal is called the *Alexander polynomial* $\Delta_K(t) \in \mathbb{C}[t^\pm]$ of $K$. The Alexander polynomial is unique up to multiplication with a unit in $\mathbb{C}[t^\pm]$ and it is of even degree with integer coefficients. In addition, it has the following properties:

1. $\Delta_K(t) \doteq \Delta_K(t^{-1})$ (symmetry),

2. $\Delta_K(1) = \pm 1$.

Where "$\doteq$" means is equal to, up to multiplication by a unit.

### 1.5.3 Differential calculus of Fox

The general reference for this paragraph is [BZH13, Chap. 9]. Let $\Gamma$ be a group and we denote $\mathbb{Z}\Gamma$ its group ring.

**Definition 1.16** (Derivations in the sense of Fox)**.** *1. There is a homomorphism $\epsilon : \mathbb{Z}\Gamma \to \mathbb{Z}$ defined by $\epsilon(\sum n_i g_i) = \sum n_i$ with $n_i \in \mathbb{Z}$ and $g_i \in \Gamma$, called* augmentation homomorphism*, its kernel $I\Gamma$ is called* ideal of augmentation*.*

*2. An application $D : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma$ is called* derivation *of $\mathbb{Z}\Gamma$ if*

$$D(\xi + \eta) = D(\xi) + D(\eta) \quad (\textit{linearity}),$$

$$D(\xi \cdot \eta) = D(\xi)\epsilon(\eta) + \xi \cdot D(\eta) \quad (product\ rule),$$

*for* $\xi, \eta \in \mathbb{Z}\Gamma$

**Example 1.6.** $\Delta_\epsilon \colon \mathbb{Z}F_n \to \mathbb{Z}F_n$, $w \mapsto w - \epsilon(w)$, *is a derivation, where* $\epsilon$ *is the homomorphism defined by 1.16.*

Let $F_n$ be the free group of generators $S_1, \ldots, S_n$.

**Definition 1.17** (Partial Derivations ). *The derivations* $\frac{\partial}{\partial S_i} : \mathbb{Z}F_n \to \mathbb{Z}F_n$ *given by*

$$\frac{\partial}{\partial S_i}(S_j) = \begin{cases} 1, & si \quad i = j \\ 0, & sinon, \end{cases}$$

*of the group ring of a free group* $F_n$ *are called* partial derivations.

**Proposition 1.4.** *There exists a unique derivation* $\Delta : \mathbb{Z}F_n \to \mathbb{Z}F_n$*, such that* $\Delta S_i = w_i$*, for arbitrary elements* $w_i \in \mathbb{Z}F_n$*, and such that for each* $w \in F_n$*, we have*

1. $\Delta(w) = \sum_{i=1}^{n} \frac{\partial}{\partial S_i}(w)\Delta(S_i)$.
2. $\Delta_\epsilon(w) = w - \epsilon(w) = \sum_{i=1}^{n} \frac{\partial w}{\partial S_i}(S_i - 1)$ *(fundamental formula).*

### 1.5.4 The Jacobian of a presentation

Let $\Gamma_K$ be the fundamental group of the knot $K$ and $\langle S_1, \ldots, S_n : R_1, \ldots, R_n \rangle$ a Wirtinger presentation of $K$. Let $\phi : \Gamma_K \to \Gamma_K/\Gamma_K'$ be the canonical homomorphism of groups which extends to the homomorphism, noted as well, $\phi : \mathbb{Z}\Gamma_K \to \mathbb{Z}(\Gamma_K/\Gamma_K')$ and $\psi : F_n \to \Gamma_K$ the canonical homomorphism of groups which extends to the homomorphism, noted as well, $\psi : \mathbb{Z}F_n \to \mathbb{Z}\Gamma_K$ given by :

$$\left(\sum n_i g_i\right)^\psi = \sum n_i g_i^\psi$$

for $g_i \in F_n$ and $n_i \in \mathbb{Z}$.

**Definition 1.18** (The Jacobian of a presentation). *We call* Jacobian matrix *of the presentation* $\langle S_1, \ldots, S_n : R_1, \ldots, R_n \rangle$ *the matrix, noted* $J(t) = (J_{ji})_{1 \le i, j \le n}$*, given by*

$$J_{ji} = \left(\frac{\partial R_j}{\partial S_i}\right)^{\phi\psi}$$

**Proposition 1.5.** *With the above notations, we have:*

$$\sum_{i=1}^{n} \Big(\frac{\partial R_j}{\partial S_i}\Big)^{\phi\psi} = 0, \quad 1 \leq j \leq n-1 \text{ in } \mathbb{Z}[t^{\pm}]. \tag{1.3}$$

*Proof.* It follows from the fundamental formula applied to $R_j$:

$$0 = (R_j - 1)^{\phi\psi} = \left[\sum_{i=1}^{n} \Big(\frac{\partial R_j}{\partial S_i}\Big)(S_i - 1)\right]^{\phi\psi} = \sum_{i=1}^{n} \Big(\frac{\partial R_j}{\partial S_i}\Big)^{\phi\psi}(t-1).$$

Since $\mathbb{Z}F_n$ has no divisors of zero, equation (1.3) follows. □

**Example 1.7** (Alexander polynomial of trefoil knot)**.**
*Let* $\Gamma_{3_1} = \big\langle S_1, S_2, S_3 \mid S_1S_2S_3^{-1}S_2^{-1}, S_2S_3S_1^{-1}S_3^{-1}, S_3S_1S_2^{-1}S_1^{-1}\big\rangle$ *be a Wirtinger presentation of trefoil knot. If* $R = S_1S_2S_3^{-1}S_2^{-1}$*, then*

$$\frac{\partial R}{\partial S_1} = 1, \quad \frac{\partial R}{\partial S_2} = S_1 - S_1S_2S_3^{-1}S_2^{-1}, \quad \frac{\partial R}{\partial S_3} = -S_1S_2S_3^{-1}$$

*and*

$$(\frac{\partial R}{\partial S_1})^{\phi\psi} = 1, \quad (\frac{\partial R}{\partial S_2})^{\phi\psi} = t-1, \quad (\frac{\partial R}{\partial S_3})^{\phi\psi} = -t$$

*By similar calculations we obtain the matrix of derivatives and we apply* $\phi\psi$ *to get*

$$J(t) = \begin{pmatrix} 1 & t-1 & -t \\ -t & 1 & t-1 \\ t-1 & -t & 1 \end{pmatrix}.$$

*It is clair that* $J(t)$ *verifies Proposition 1.5. The* $2 \times 2$ *minor* $\Delta_{11} = \begin{pmatrix} 1 & t-1 \\ -t & 1 \end{pmatrix}$*, for example, is a presentation matrix of Alexander module.* $\mid \Delta_{11} \mid = 1 - t + t^2 = \Delta_K(t)$ *is an Alexander polynomial of trefoil knot.*

# Chapter 2

# Rational representations of $\mathrm{SL}(2,\mathbb{K})$

## Summary

The principal references for this chapter are [Spr77] and [Gre67]. The purpose of the Section 2.1 is to prove that the rational representation $r_d$ of $\mathrm{SL}(2,\mathbb{K})$ into $\mathrm{SL}(R_{d-1})$ is an irreducible one. In Section 2.2 we show that any irreducible rational representation of $\mathrm{SL}(2,\mathbb{K})$ is equivalent to some representation $r_d$. Section 2.3 is devoted to give the decomposition of an arbitrary rational representation of $\mathrm{SL}(2,\mathbb{K})$ as direct sum of irreducible rational representations of $\mathrm{SL}(2,\mathbb{K})$ into $\mathrm{SL}(d,\mathbb{K})$. In the last section we prove that the adjoint representation $\mathrm{Ad}\circ r_d$ is equivalent to a direct sum of representations $r_k$.

## 2.1 Representation $r_d$ of $\mathrm{SL}(2,\mathbb{K})$ into $R_{d-1}$

In the following, we assume that $\mathbb{K}$ is an algebraically closed field with characteristic zero. Let $R=\mathbb{K}[X,Y]$ be the polynomial algebra in two variables and let $R_{d-1}\subset R$ denote the $d$-dimensional subspace of homogeneous polynomials of degree $d-1$, $d\geq 1$. The monomials $e_l^{(d-1)}=X^{l-1}Y^{d-l}$, $1\leq l\leq d$, form a basis of $R_{d-1}$. To shorten notation we write $e_l$ instead of $e_l^{(d-1)}$ if no confusion can arise.

In this chapter we will use the following notation: let $G=\mathrm{SL}(2,\mathbb{K})$ and

$$T=\left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\ \Big|\ a\in\mathbb{K}^\star\right\},\qquad U=\left\{\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\ \Big|\ a\in\mathbb{K}\right\},\tag{2.1}$$

be respectively the subgroups of diagonal and unipotent matrices of $G$. Let $w=\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ be an element of $G$. It easy to see that $G$ is generate by $T$, $U$ and $w$.

**Definition 2.1** (Character of a group)**.** *A* character *of a group* $\Gamma$ *into* $\mathbb{K}$ *is a homomorphism* $\chi:\Gamma\to\mathbb{K}^\star$*.*

**Lemma 2.1.** *Let* $r_d$ *be the Representation* (0.2)*, and let* $e_i=X^{i-1}Y^{d-i}$*,* $1\leq i\leq d$*, be the above basis of* $R_{d-1}$*. Let* $\chi_i:T\to\mathbb{K}^\star$ *be the character defined by*

$$\chi_i\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}=a^i.\tag{2.2}$$

*Then, for each* $t\in T$ *and* $1\leq i\leq d$ *we have :*

$$r_d(t)e_i=\chi_{d+1-2i}(t)e_i.\tag{2.3}$$

*Proof.* For each $t = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \in T$ and $1 \leq i \leq d$, we have :

$$r_d(t)e_i = r_d \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} X^{i-1}Y^{d-i} = (a^{-1}X)^{i-1}(aY)^{d-i} = a^{d+1-2i}X^{i-1}Y^{d-i} = \chi_{d+1-2i}(t)e_i.$$

□

**Lemma 2.2.** *Let $V$ be a $\mathbb{K}-$vector space and $\Gamma$ be a group. Being given the distinct characters $\chi_i : \Gamma \to \mathbb{K}^\star$ and elements $v_i \in V$, $1 \leq i \leq n$, we have :*

$$(\star) \quad \sum_{i=1}^{n} v_i\chi_i = 0 \Rightarrow v_i = 0 \quad \forall\, 1 \leq i \leq n.$$

*Proof.* We show $(\star)$ by contradiction. For this, suppose that $\sum\limits_{i=1}^{n} v_i\chi_i = 0$ and $\exists i_0 \in \{1,\ldots,n\}$, $v_{i_0} \neq 0$.
Taking $\sum\limits_{i=1}^{k} v_i\chi_i = 0$ for $k$ as small as possible and such that $v_i$ are all nonzero, we have:

$$\chi_1v_1 + \chi_2v_2 + \ldots + \chi_kv_k = 0, \quad v_i \neq 0 \quad \forall\, 1 \leq i \leq k. \tag{2.4}$$

Since the $\chi_i$ are distinct, there exists $\gamma_0 \in \Gamma$ such that $\chi_1(\gamma_0) \neq \chi_2(\gamma_0)$. Then we have for each $\gamma \in \Gamma$,

$$\sum_{i=1}^{k} \chi_i(\gamma\gamma_0)v_i = 0 \Leftrightarrow \sum_{i=1}^{k} \chi_i(\gamma)\chi_i(\gamma_0)v_i = 0 \Leftrightarrow \sum_{i=1}^{k} \chi_i(\gamma)\frac{\chi_i(\gamma_0)}{\chi_1(\gamma_0)})v_i = 0.$$

Subtracting $\sum\limits_{i=1}^{k} v_i\chi_i(\gamma) = 0$ from the last equation we obtain:

$$\sum_{i=2}^{k} \chi_i(\gamma)\Big(\frac{\chi_i(\gamma_0)}{\chi_1(\gamma_0)} - 1\Big)v_i = 0.$$

Since $\chi_1(\gamma_0) \neq \chi_2(\gamma_0)$, the first coefficient is nonzero. Then the last relation is true and has a length smaller than $k$. This contradicts that $k$ is the smaller integer that verify (2.4), consequently, the $\chi_i$ are free. □

Recall that $G = \mathrm{SL}(2,\mathbb{K})$.

**Proposition 2.1.** *Let $r_d : G \to \mathrm{SL}(R_{d-1})$ be the Representation* (0.2).

*(i) If $W$ is a nonzero $G-$stable subspace of $R_{d-1}$ then $e_1 \in W$ and $e_d \in W$.*

*(ii) The representation $r_d : G \to \mathrm{SL}(R_{d-1})$ is irreducible.*

*Proof.* (i) Let $x = \sum_{i\in I} x_i e_i \in W$ be a nonzero vector with $x_i \neq 0$ and $I \subseteq \{1,\dots,d\}$. As $W$ is $G-$stable, Lemma 2.1 gives for each $t \in T$

$$r_d(t)x = \sum_{i\in I} x_i \chi_{d+1-2i}(t) e_i \in W.$$

So, the vectors $\chi_{d+1-2i}(t)e_i$, $i \in I$ are linearly dependent modulo $W$. The linear independence of the characters $\chi_{d+1-2i}$ (see Lemma 2.2) implies that $e_i \in W$ if $i \in I$. Let $e_i \in W$. Then

$$\begin{aligned} r_d\Big(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\Big)e_i &= r_d\Big(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\Big)X^{i-1}Y^{d-i} \\ &= (X+Y)^{i-1}Y^{d-i} \\ &= \sum_{j=0}^{i-1}\binom{i-1}{j} X^j Y^{i-1-j} Y^{d-i} \\ &= \sum_{j=0}^{i-1}\binom{i-1}{j} X^{(j+1)-1} Y^{d-(1+j)} \\ &= \sum_{j=0}^{i-1}\binom{i-1}{j} e_{j+1} \in W. \end{aligned}$$

Since $\binom{i-1}{0} \neq 0$, by the same argument given before, we have $e_1 \in W$. Similarly, using the matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ instead of the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, we show that $e_d \in W$.

(ii) Since $e_1 \in W$,

$$r_d\Big(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\Big)e_1 = r_d\Big(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\Big)Y^{d-1} = (X+Y)^{d-1} = \sum_{j=0}^{d-1}\binom{d-1}{j} e_{j+1} \in W.$$

As $\binom{d-1}{j} \neq 0$, $\forall j \in \{0,\dots,d-1\}$, we have $e_{j+1} \in W$, $\forall j \in \{0,\dots,d-1\}$. So, $W = R_{d-1}$ ans since $R_{d-1}$ is an irreducible $G-$module, $r_d$ is an irreducible representation.

□

## 2.2 Irreducible rational representations of $\mathrm{SL}(2,\mathbb{K})$

The purpose of this section is to show that any irreducible rational representation of $\mathrm{SL}(2,\mathbb{K})$ is equivalent to some representation $r_d$. But before this, we start by giving some results that allow us to show it.

**Lemma 2.3.** *There exists a non-degenerate bilinear form* $\langle , \rangle$ *on* $R_{d-1}$ *such that*

$$\langle r_d(g)x, r_d(g)y\rangle = \langle x, y\rangle\,, \tag{2.5}$$

*for* $g \in G$, $x, y \in R_{d-1}$. *This form is symmetric if* $d-1$ *is even, skew-symmetric if* $d-1$ *is odd and it is defined by*

$$\langle e_i, e_j\rangle = \binom{d-1}{i}^{-1} (-1)^i \delta_{i-1,d-j}, \tag{2.6}$$

*where* $\delta_{i-1,d-j}$ *is the Kronecker symbol.*

*Proof.* Let $V = \langle e_1, e_2\rangle$ be a $\mathbb{C}-$vector space . Let $V^{\otimes(d-1)}$ be the $(d-1)-$th power tensor of $V$ and let $\vee^{\otimes(d-1)}V$ be the subspace of $V^{\otimes(d-1)}$ of $(d-1)$-th power symmetric tensor defined on $\mathbb{C}-$vector space $V$. From [Gre67, Sec. 7.13] $R_{d-1} \cong \vee^{\otimes(d-1)}V$. So, to show the existence of a bilinear form on $R_{d-1}$ it suffices to show the existence of a bilinear form on $\vee^{\otimes(d-1)}V$. The permutation group $S_{d-1}$ acts on $V^{\otimes(d-1)}$ by

$$\sigma\big(x_1 \otimes \cdots \otimes x_{d-1}\big) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(d-1)},$$

where $x_1 \otimes \cdots \otimes x_{d-1} \in V^{\otimes(d-1)}$ and $\sigma \in S_{d-1}$.
Let $\{e_{i_1} \otimes \cdots \otimes e_{i_{d-1}}\}$ $(i_k = 1, 2)$ be a basis of $V^{\otimes(d-1)}$ and $\pi_s : V^{\otimes(d-1)} \to V^{\otimes(d-1)}$ the symmetrizer given by:

$$\pi_s\big(x_1 \otimes \cdots \otimes x_{d-1}\big) = \frac{1}{(d-1)!}\sum_{\sigma\in S_{d-1}} \sigma\big(x_1 \otimes \cdots \otimes x_{d-1}\big) = \frac{1}{(d-1)!}\sum_{\sigma\in S_{d-1}} \Big(x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(d-1)}\Big),$$

for each $x_1 \otimes \cdots \otimes x_{d-1} \in V^{\otimes(d-1)}$. It is well known that $\pi_s(V^{\otimes(d-1)}) = \vee^{\otimes(d-1)} V$.

Let $B : V \times V \to \mathbb{C}$ be a bilinear form on $V$ given by

$$B(e_1, e_2) = -1 = -B(e_2, e_1), \quad B(e_1, e_1) = B(e_2, e_2) = 0.$$

This bilinear form induces a bilinear form on $V^{\otimes(d-1)}$ given by

$$B^\star\Big(x_1 \otimes \cdots \otimes x_{d-1}, y_1 \otimes \cdots \otimes y_{d-1}\Big) = B(x_1, y_1) \cdots B(x_{d-1}, y_{d-1}),$$

for each $x_1 \otimes \cdots \otimes x_{d-1}$, $y_1 \otimes \cdots \otimes y_{d-1} \in V^{\otimes(d-1)}$. It follows that

$$\begin{aligned}
&B^\star\Big(x_1 \otimes \cdots \otimes x_{d-1}, \pi_s(y_1 \otimes \cdots \otimes y_{d-1})\Big) = \\
&\quad \frac{1}{(d-1)!} \sum_{\sigma \in S_{d-1}} B^\star\Big(x_1 \otimes \cdots \otimes x_{d-1}, y_{\sigma^{-1}(1)} \otimes \cdots \otimes y_{\sigma^{-1}(d-1)}\Big) = \\
&\quad \frac{1}{(d-1)!} \sum_{\sigma \in S_{d-1}} B\big(x_1, y_{\sigma^{-1}(1)}\big) \cdots B\big(x_{d-1}, y_{\sigma^{-1}(d-1)}\big) = \\
&\quad \frac{1}{(d-1)!} \sum_{\sigma \in S_{d-1}} B\big(x_{\sigma\sigma^{-1}(1)}, y_{\sigma^{-1}(1)}\big) \cdots B\big(x_{\sigma\sigma^{-1}(d-1)}, y_{\sigma^{-1}(d-1)}\big) = \\
&\quad \frac{1}{(d-1)!} \sum_{\sigma \in S_{d-1}} B\big(x_{\sigma(j_1)}, y_{j_1}\big) \cdots B\big(x_{\sigma(j_{d-1})}, y_{j_{d-1}}\big)
\end{aligned}$$

By increasing rearrangement on the $j_i$ for $1 \leq i \leq d-1$, we have

$$\begin{aligned}
&B^\star\Big(x_1 \otimes \cdots \otimes x_{d-1}, \pi_s(y_1 \otimes \cdots \otimes y_{d-1})\Big) = \\
&\quad \frac{1}{(d-1)!} \sum_{\sigma \in S_{d-1}} B\big(x_{\sigma(1)}, y_1\big) \cdots B\big(x_{\sigma(d-1)}, y_{d-1}\big) = \\
&\quad \frac{1}{(d-1)!} \sum_{\sigma \in S_{d-1}} B^\star\Big(\sigma^{-1}(x_1 \otimes \cdots \otimes x_{d-1}), y_1 \otimes \cdots \otimes y_{d-1}\Big) = \\
&\quad B^\star\Big(\pi_s(x_1 \otimes \cdots \otimes x_{d-1}), y_1 \otimes \cdots \otimes y_{d-1}\Big).
\end{aligned}$$

Since $\pi_s^2 = \pi_s$, it follows:

$$\begin{aligned}
B^\star\Big(\pi_s(x_1 \otimes \cdots \otimes x_{d-1}), \pi_s(y_1 \otimes \cdots \otimes y_{d-1})\Big) &= B^\star\Big(x_1 \otimes \cdots \otimes x_{d-1}, \pi_s^2(y_1 \otimes \cdots \otimes y_{d-1})\Big) \\
&= B^\star\Big(x_1 \otimes \cdots \otimes x_{d-1}, \pi_s(y_1 \otimes \cdots \otimes y_{d-1})\Big).
\end{aligned}$$

Now for $1 \leq j \leq d$, let $e'_j = e_1^{\otimes j-1} \otimes e_2^{\otimes (d-j)}$ and $e''_j = e_2^{\otimes j-1} \otimes e_1^{\otimes (d-j)}$. Since $B(e_i, e_i) = 0$, we have $B^\star\Big(e'_j, e''_i\Big) = 0$ if $i \neq j$. In the case where $i = j$, we have :

$$\begin{aligned} B^\star\Big(\pi_s(e'_i), \pi_s(e''_i)\Big) &= B^\star\Big(e'_i, \pi_s(e''_i)\Big) \\ &= \frac{1}{(d-1)!} \sum_{\sigma \in S_{d-1}} B^\star\Big(e_1^{\otimes i-1} \otimes e_2^{\otimes (d-i)}, \sigma(e_2^{\otimes i-1} \otimes e_1^{\otimes (d-i)})\Big). \end{aligned}$$

Since $\sigma$ permutes the $e_1$ of $(d-i)!$ choice and the $e_2$ of $(i-1)!$ choice, we have :

$$B^\star\Big(\pi_s(e'_i), \pi_s(e''_i)\Big) = \frac{1}{(d-1)!}(i-1)!(d-i)! B(e_1, e_2)^{i-1} B(e_2, e_1)^{d-i} = \begin{pmatrix} d-1 \\ i-1 \end{pmatrix}^{-1} (-1)^{i-1}.$$

Thus we define the bilinear form $\langle , \rangle$ on $R_{d-1}$ by

$$\langle e_i, e_j \rangle = \begin{pmatrix} d-1 \\ i-1 \end{pmatrix}^{-1} (-1)^{i-1} \delta_{i-1,\, d-j}, \quad 1 \leq i,\, j \leq d.$$

Show that the bilinear form $\langle , \rangle$ on $R_{d-1}$ is symmetric if $d-1$ is even.

This form is symmetric if for each $e_i, e_j$ elements of the basis of $R_{d-1}$, $1 \leq i, j \leq d$, we have $\langle e_i, e_j \rangle = \langle e_j, e_i \rangle$.

- If $i + j = d + 1$

  $$\langle e_i, e_j \rangle = \langle e_j, e_i \rangle \Leftrightarrow \begin{pmatrix} d-1 \\ i-1 \end{pmatrix}^{-1} (-1)^{i-1} \delta_{i-1,\, d-j} = \begin{pmatrix} d-1 \\ j-1 \end{pmatrix}^{-1} (-1)^{j-1} \delta_{j-1,\, d-i}.$$

  Since $i + j = d + 1$, $\begin{pmatrix} d-1 \\ i-1 \end{pmatrix} = \begin{pmatrix} d-1 \\ j-1 \end{pmatrix}$ and $\delta_{i-1,\, d-j} = \delta_{i-1,\, i-1} = \delta_{j-1,\, j-1} = \delta_{j-1,\, d-i}$. Moreover, $d-1$ being even, $(-1)^{i-1} = (-1)^{i-1-d+1} = (-1)^{-j+1}$. Then for each $1 \leq i, j \leq d$, we have :

  $$\langle e_i, e_j \rangle = \langle e_j, e_i \rangle .$$

- If $i + j \neq d + 1$.

  $i + j \neq d + 1 \Leftrightarrow (i - 1 > d - j) \vee (i - 1 < d - j)$.

  In both cases $\delta_{i-1,\, d-j} = \delta_{j-1,\, d-i} = 0$. Hence the equality.

Therefore, the bilinear form $\langle , \rangle$ on $R_{d-1}$ is symmetric if $d-1$ is even.

Show that the bilinear form $\langle , \rangle$ on $R_{d-1}$ is skew-symmetric if $d-1$ is odd.

This form is skew-symmetric if for each $1 \leq i, j \leq d$, we have $\langle e_i, e_j \rangle = -\langle e_j, e_i \rangle$.

- If $i + j = d + 1$

$$\langle e_i, e_j \rangle = -\langle e_j, e_i \rangle \Leftrightarrow \begin{pmatrix} d-1 \\ i-1 \end{pmatrix}^{-1} (-1)^{i-1} \delta_{i-1,\, d-j} = -\begin{pmatrix} d-1 \\ j-1 \end{pmatrix}^{-1} (-1)^{j-1} \delta_{j-1,\, d-i}.$$

Since $i + j = d + 1$, $\begin{pmatrix} d-1 \\ i-1 \end{pmatrix} = \begin{pmatrix} d-1 \\ j-1 \end{pmatrix}$ and $\delta_{i-1,\, d-j} = \delta_{i-1,\, i-1} = \delta_{j-1,\, j-1} = \delta_{j-1,\, d-i}$. Moreover, $d-1$ being odd, $(-1)^{i-1} = -(-1)^{i-1-d+1} = -(-1)^{-j+1}$. Then for each $1 \leq i, j \leq d$, we have :

$$\langle e_i, e_j \rangle = -\langle e_j, e_i \rangle .$$

- If $i + j \neq d + 1$.

  $i + j \neq d + 1 \Leftrightarrow (i - 1 > d - j) \vee (i - 1 < d - j)$.

  In the both cases $\delta_{i-1,\, d-j} = \delta_{j-1,\, d-i} = 0$. Hence the equality.

Then, the bilinear form $\langle , \rangle$ on $R_{d-1}$ is skew-symmetric if $d-1$ is odd.

This bilinear form is non degenerate. Indeed, for all $x = \sum_{i=1}^{d} x_i\, e_i$ and $y = \sum_{i=1}^{d} y_i\, e_i$ of $R_{d-1}$,

$$\langle x,\, y \rangle = \sum_{1 \leq i,\, j \leq d} x_i y_j \langle e_i\, e_j \rangle = \sum_{1 \leq i,\, j \leq d} x_i\, y_j \begin{pmatrix} d-1 \\ i-1 \end{pmatrix}^{-1} (-1)^{i-1} \delta_{i-1,\, d-j}.$$

Since this form is symmetric if $d-1$ is even and it is skew-symmetric if $d-1$ is old it suffices to show that for all $x = \sum_{i=1}^{d} x_i\, e_i$ of $R_{d-1}$, $\langle x,\, y \rangle = 0$ imply that $y_j = 0$, for all $1 \leq j \leq d$.

$\langle x\, , y \rangle = 0$, for all $x \in R_{d-1}$, is equivalent to $\sum_{1 \leq i,\, j \leq d} x_i\, y_j \binom{d-1}{i-1}^{-1} (-1)^{i-1} \delta_{i-1,\, d-j} = 0$ for all $x \in R_{d-1}$. It is clear that $y_j = 0$, for all $1 \leq j \leq d$.

Since $G$ is generate by $T$, $U$ and $w$, defined by (2.1), to show (2.5) it suffices to verify it for the generators $t = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$, $a \in \mathbb{C}^*$, $u = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, $b \in \mathbb{C}$, and $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

For $g = w$, we have $r_d(w)e_i = (-1)^{i-1}e_{d-i+1}$. Then

$$\begin{aligned} \langle r_d(w)e_i,\, r_d(w)e_j\rangle &= (-1)^{i+j}\langle e_{d-i+1},\, e_{d-j+1}\rangle \\ &= (-1)^{i+j}\begin{pmatrix} d-1 \\ d-i \end{pmatrix}^{-1}(-1)^{d-i}\delta_{d-i,\,d-d+j-1} \\ &= (-1)^{d+j}\begin{pmatrix} d-1 \\ d-i \end{pmatrix}^{-1}\delta_{d-i,\,j-1}. \end{aligned}$$

We distinguish two cases :

- If $j-1 = d-i$ then $(-1)^{d+j} = (-1)^{i-1}$ and $\binom{d-1}{d-i} = \binom{d-1}{i-1}$.
- If $j-1 \neq d-i$ then $\delta_{d-i,\,j-1} = \delta_{i-1,\,d-j} = 0$.

Thus, for the generator $g = w$, we have the Equality (2.5).

From Lemma 2.1 we have $r_d(t)e_i = a^{d-2i+1}e_i$. Then

$$\langle r_d(t)e_i, r_d(t)e_j\rangle = a^{2d-2i-2j+2}\langle e_i,\, e_j\rangle .$$

To show equality (2.5) we have to show that $d-i-j+1 = 0$ or $\langle e_i,\, e_j\rangle = 0$.

So, we distinguish two cases :

- It is clear that if $j-1 = d-i$ we have $a^{2d-2i-2j+2} = 0$.
- If $j-1 \neq d-i$ then $\delta_{i-1,\,d-j} = 0$.

Thus, for the generator $g = t$, we have the Equality (2.5).

Similarly, for the generator $g = u$, we have the Equality (2.5). □

**Lemma 2.4.** *Let* $R^\star_{d-1} = \hom(R_{d-1}, \mathbb{K})$ *be the dual vector space of* $R_{d-1}$*. Then*

1. $R_{d-1}$ *and* $R^\star_{d-1}$ *are isomorphic as* $G-$*modules, where the action of* $G$ *on* $R^\star_{d-1}$ *is given by* $(g\cdot f)(x) = f\Big(r_d(g)^{-1}x\Big)$, *for* $g \in G$, $x \in R_{d-1}$ *and* $f \in R^\star_{d-1}$.

2. *$R_{d-1}\otimes R^\star_{d-1}$ is $G-$isomorphic to the space $End(R_{d-1})$ of all linear transformations of $R_{d-1}$, where the action of $G$ on $End(R_{d-1})$ is defined by*

$$g\cdot t = r_d(g)\circ t\circ r_d(g)^{-1},$$

*for $g\in G$, $t\in End(R_{d-1})$.*

*Proof.* 1. $R_{d-1}\cong R^\star_{d-1}$ as $G-$modules if there exists a linear isomorphism $T: R_{d-1}\to R^\star_{d-1}$ which commutes with the action of $G$. Let the operator $T$ defined by :

$$\begin{aligned} T:\ R_{d-1} \to \quad & R^\star_{d-1} \\ x\mapsto \quad & T(x):\quad R_{d-1}\to\mathbb{K} \\ & \qquad\qquad y\mapsto T(x)y=\langle x,y\rangle\,, \end{aligned}$$

where $\langle,\rangle$ is the bilinear form defined by (2.6). So, for each $x,y\in R_{d-1}$ and $g\in G$:

$$\begin{aligned} T\big(g\cdot x\big)(y) &= T\Big(r_d(g)x\Big)(y) \\ &= \langle r_d(g)x,y\rangle \\ &= \big\langle x, r_d(g)^{-1}y\big\rangle \\ &= T(x)\big(r_d(g)^{-1}y\big) \\ &= \big(g\cdot T(x)\big)(y) \end{aligned}$$

Then $Tg=gT$. Hence the isomorphism of $R_{d-1}$ and $R^\star_{d-1}$ as $G-$modules.

2. $R_{d-1}\otimes_{\mathbb{K}} R^\star_{d-1}\cong End(R_{d-1})$ as $G-$modules if there exists a linear isomorphism $T: R_{d-1}\otimes_{\mathbb{K}} R^\star_{d-1}\to End(R_{d-1})$ which commutes with the action of $G$. Let the operator $T$ defined by:

$$\begin{aligned} T: R_{d-1}\otimes_{\mathbb{K}} R^\star_{d-1} \to \quad & End(R_{d-1}) \\ x\otimes_{\mathbb{K}} f\mapsto \quad & T(x\otimes_{\mathbb{K}} f):\quad R_{d-1}\to R_{d-1} \\ & \qquad\qquad w\mapsto T(x\otimes_{\mathbb{K}} f)(w)=f(w)x \end{aligned}$$

Then

$$T\Big(g\cdot(x\otimes_{\mathbb{K}} f)\Big)(w)=T\Big(r_d(g)x\otimes_{\mathbb{K}} r^\star_d(g)f\Big)(w)=r^\star_d(g)f(w)r_d(g)x$$

and

$$\begin{aligned} g\cdot\Big(T(x\otimes_{\mathbb{K}} f)\Big)(w) &= r_d(g)\Big(T(x\otimes_{\mathbb{K}} f)(r_d(g)^{-1}(w))\Big) \\ &= r_d(g) f\Big(r_d(g)^{-1}(w)\Big)x \\ &= f\Big(r_d(g)^{-1}(w)\Big)r_d(g)x \\ &= r_d^{\star}(g)f(w)r_d(g)x. \end{aligned}$$

So, $T$ is the desired isomorphism. □

**Remark 2.1.** *We can view* SL$(2,\mathbb{K})$ *as a closed subset of the vector space* $\mathbb{K}^4$*, consisting of the points* $(\xi,\eta,\zeta,\tau)$ *with* $\xi\tau-\eta\zeta=1$*. Then we have :*

- *The functions on* SL$(2,\mathbb{K})$ *induced by the polynomial functions on* $\mathbb{K}^4$ *form a* $\mathbb{K}-$*algebra* $A$ *called the coordinate algebra of* $G=\mathrm{SL}(2,\mathbb{K})$*, i.e.* $A=\{f:\mathrm{SL}(2,\mathbb{K})\to\mathbb{K}\mid f \textit{ polynomial function}\}$*. Then we have :*

$$A=\mathbb{K}[x,y,z,t]\cong\mathbb{K}[X,Y,Z,T]/(XT-YZ-1),$$

*where* $x,y,z,t$ *are the coordinate functions defined by*

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{K}) & \stackrel{x}{\to} & \mathbb{K} \\ \begin{pmatrix}\xi & \eta\\ \zeta & \tau\end{pmatrix} & \mapsto & \xi, \quad \textit{etc.} \end{array}$$

- *$G$ operates on $A$ by :*

$$\big(g\cdot f\big)(h)=f(g^{-1}h),\quad f\in A,\ g,h\in G.$$

- *For* $d-1\geq 0$*, define* $V_{d-1}$ *to be the subspace of* $A$ *given by :*

$$V_{d-1}=\Big\{f\in A\quad|\quad f(gtu)=\chi_{d-1}(t)f(g),\ g\in G,\ t\in T,\ u\in U\Big\}. \tag{2.7}$$

- *The subspace* $V_{d-1}$ *is* $G-$*stable. Moreover,* $V_{d-1}$ *is the set of* $f\in A$ *such that* $f\begin{pmatrix}x & y\\ z & t\end{pmatrix}$ *depends only of* $x$ *and* $z$ *and for* $\xi\in\mathbb{K}^{\star}$*, we have*

$$f\begin{pmatrix}\xi x & y\\ \xi z & t\end{pmatrix}=\xi^{d-1}f\begin{pmatrix}x & y\\ z & t\end{pmatrix}. \tag{2.8}$$

- $\dim V_{d-1} = d$. *Indeed, from* (2.8), $V_{d-1}$ *is an homogeneous subspace of* $A$ *and since it depends only of* $x$ *and* $z$, *we obtain* $\dim V_{d-1} = \binom{(d-1)+2-1}{d-1} = d$.

**Lemma 2.5.** *The* $G-$*modules* $V_{d-1}$ *and* $R^\star_{d-1}$ *are isomorphic.*

*Proof.* Let $\{e_i = X^{i-1}Y^{d-i}\}_{1\leq i\leq d}$ be the basis of $R_{d-1}$ defined above. Let $\phi$ be the homomorphism of $G-$modules defined by

$$\begin{aligned} \phi : R^\star_{d-1} \to \quad & A \\ l \mapsto \quad & \phi(l) : \; G \to \mathbb{K} \\ & \qquad g \mapsto \phi(l)(g) = l(r_d(g)e_1). \end{aligned}$$

We will show first that $Im\phi \subseteq V_{d-1}$, where $V_{d-1}$ is the subspace defined by (2.7).

Note that for $u = \left(\begin{smallmatrix} 1 & -a \\ 0 & 1 \end{smallmatrix}\right)$,

$$r_d(u)e_1 = r_d \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} Y^{d-1} = Y^{d-1} = e_1. \tag{2.9}$$

So, Relation (2.9) together with Relation (2.3) give

$$\begin{aligned} \phi(l)(gtu) &= l\Big(r_d(gtu)e_1\Big) \\ &= l\Big(r_d(g)r_d(t)r_d(u)e_1\Big) \\ &= l\Big(r_d(g)r_d(t)e_1\Big) \\ &= l\Big(r_d(g)\chi_{d-1}(t)e_1\Big) \\ &= \chi_{d-1}(t)l\Big(r_d(g)e_1\Big) \\ &= \chi_{d-1}(t)\phi(l)(g), \end{aligned}$$

for $g \in G$, $t \in T$ and $u \in U$.

Now, it is easy to see that $\phi$ is a map of $G-$modules.

To prove that $\phi$ is injective, let $l \in R^\star_{d-1}$ such that $\phi(l) = 0$. Then, $\phi(l) = 0$ if and only if $l(r_d(g)e_1) = 0$, $\forall g \in G$. The subspace

$$\Big\{r_d(g)e_1 \;\mid\; g \in G\Big\} \subset R_{d-1}$$

being $G-$stable, the irreducibility of $r_d$ implies that $\{r_d(g)e_1 \mid g \in G\} = R_{d-1}$. Then $l = 0$ on all $R_{d-1}$ and $\phi$ is injective.

Since $\dim V_{d-1} = \dim R^\star_{d-1} = d$, we conclude that $V_{d-1}$ and $R^\star_{d-1}$ are isomorphic. □

**Lemma 2.6.** *Let* $T_n$ *be the subgroup of diagonal matrices of* $\mathrm{SL}(n,\mathbb{K})$*. If* $\rho : T_n \to \mathrm{GL}(W)$ *is a rational representation then there exists a basis* $\{e'_1,\dots,e'_n\}$ *of* $W$ *and characters* $\chi_i : T_n \to \mathbb{K}^\star$ *such that for each* $t \in T_n$

$$\rho(t)e'_i = \chi_i(t)e'_i.$$

*Proof.* Let $t \in T$ be the diagonal matrix $diag(x_1,\dots,x_n)$, with $x_i \neq 0$. By the definition of a rational representation, the matrix elements of $\rho(t)$, with respect to some fixed basis of $W$, are the linear combinations of products $x_1^{a_1}\dots x_n^{a_n}$, with $a_i \in \mathbb{Z}$.

For a fixed n-tuple$(a_1,\dots,a_n)$ the function

$$\begin{array}{rcl} \chi : T_n & \to & \mathbb{K}^\star \\ t = diag(x_1,\dots,x_n) & \mapsto & \chi(t) = x_1^{a_1}\dots x_n^{a_n} \end{array}$$

defines a rational representation of $T_n$ into $\mathbb{K}^\star$. Such a $\chi$ is called a rational character of $T_n$. It follows that we can write :

$$\rho(t) = \sum_{\chi\in S}\chi(t)A_\chi,$$

where $\chi$ runs through a finite set $S$ of rational characters of $T_n$, and $A_\chi$ is a linear transformation of $W$.

Since $\chi$ and $\rho$ are homomorphisms, we have for each $t \in T_n$:

$$\begin{array}{rcl} \rho(tt') = \rho(t)\rho(t') & \Leftrightarrow & \sum_{\chi\in S}\chi(tt')A_\chi = \Big(\sum_{\chi\in S}\chi(t)A_\chi\Big)\Big(\sum_{\chi'\in S}\chi'(t')A_{\chi'}\Big) \\ & \Leftrightarrow & \sum_{\chi\in S}\chi(t)\chi(t')A_\chi = \sum_{\chi,\chi'\in S}\chi(t)\chi'(t')A_\chi A_{\chi'}. \end{array}$$

The linear independence of characters $\chi$ (see Lemma 2.2) implies that

$$\chi(t')A_\chi = \sum_{\chi'\in S}\chi'(t')A_\chi A_{\chi'},$$

which gives :

$$\sum_{\chi'\in S,\chi'\neq\chi}\chi'(t')A_\chi A_{\chi'} + \chi(t')\Big(A_\chi A_\chi - A_\chi\Big) = 0.$$

Once again, the linear independence of characters $\chi$ gives

$$\begin{cases} A_\chi A_{\chi'} = 0, & \text{if} \quad \chi \neq \chi' \\ A_\chi A_\chi = A_\chi. \end{cases} \tag{2.10}$$

Moreover,

$$Id=\rho(1)=\sum_{\chi\in S}\chi(1)A_\chi=\sum_{\chi\in S}A_\chi.$$

So, we have $W=\sum_{\chi\in S}A_\chi W=\sum_{\chi\in S}W_\chi$. To show that this sum is direct we have to show the uniqueness of the writing. We suppose that for $w\in W$ there exist $v,v'\in W$ such that

$$w=\sum_{\chi\in S}A_\chi v=\sum_{\chi\in S}A_\chi v'.$$

Then, for $\chi'\in S$ fixed we have :

$$A_{\chi'}w=\sum_{\chi\in S}A_{\chi'}A_\chi v'=\sum_{\chi\in S}A_{\chi'}A_\chi v.$$

Considering Relations (2.10), we have : $A_{\chi'}v'=A_{\chi'}v$. By definition of $A_{\chi'}$ we obtain the uniqueness of writing of $w$.

Moreover the $W_\chi$ are stable by $\rho(t)$. Indeed, since $\rho(t)=\sum_{\chi\in S}\chi(t)A_\chi$, we have :

$$\begin{aligned}\rho(t)W_{\chi'} &= \sum_{\chi\in S}\chi(t)A_\chi W_{\chi'}\\ &= \sum_{\chi\in S}\chi(t)A_\chi A_{\chi'}W\\ &= \chi'(t)A_{\chi'}A_{\chi'}W\\ &= \chi'(t)A_{\chi'}W\\ &= \chi'(t)W_{\chi'}.\end{aligned}$$

Thus

$$\rho(t)_{|W_\chi}=\chi(t)Id_{|W_\chi}.$$

This proves that there exists a basis such that $\rho(t)$ is diagonal on each $W_\chi$, i.e. there exists a basis such that $\rho(t)$ is diagonal. □

**Proposition 2.2.** *Any irreducible rational representation of* $G=\mathrm{SL}(2,\mathbb{K})$ *is equivalent to some* $r_d$.

*Proof.* Let $\rho:\mathrm{SL}(2,\mathbb{K})\to\mathrm{GL}(V)$ be an irreducible rational representation of $\mathrm{SL}(2,\mathbb{K})$ into a vector space $V$ and let $r_d:\mathrm{SL}(2,\mathbb{K})\to\mathrm{SL}(R_{d-1})$ be the irreducible rational

Representation (0.2). To show that $r_d$ and $\rho$ are equivalent, it suffices to show that $V$ is isomorphic to $R_{d-1}$ as $G-$modules. By Lemma 2.6, there exists a basis $(e'_i)_{1\leq i\leq n}$ of $V$ such that

$$\rho(t)e'_i = \chi_{a_i}(t)e'_i,$$

for all $t \in T$ and $a_i \in \mathbb{Z}$, $1 \leq i \leq n$, as in the proof of Lemma 2.6. Assume the $e'_i$ to be ordered such that $a_1 \leq a_2 \ldots \leq a_n$.

Since $\rho$ is a rational representation, there are polynomials $f_i \in \mathbb{K}[X]$ such that

$$\rho \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} e'_n = \sum_{i=1}^{n} f_i(\xi)e'_i. \tag{2.11}$$

From (2.11) and the equality

$$\begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta^{-1} & 0 \\ 0 & \eta \end{pmatrix} = \begin{pmatrix} 1 & \eta^2\xi \\ 0 & 1 \end{pmatrix}, \ \xi \in \mathbb{K},\ \eta \in \mathbb{K}^\star,$$

we have for $t = \left(\begin{smallmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} 1 & \xi \\ 0 & 1 \end{smallmatrix}\right)$

$$\rho(tut^{-1})e'_n = \sum_{i=1}^{n} f_i(\xi\eta^2)e'_i. \tag{2.12}$$

The fact that $\rho$ is a homomorphism together with Relation (2.3) and Lemma 2.6 imply that

$$\begin{aligned}
\rho(tut^{-1})e'_n &= \rho(t)\rho(u)\rho(t^{-1})e'_n \\
&= \rho(t)\rho(u)\chi_{-a_n}(t)e'_n \\
&= \chi_{-a_n}(t)\rho(t)\sum_{i=1}^{n} f_i(\xi)e'_i \\
&= \chi_{-a_n}(t)\sum_{i=1}^{n} f_i(\xi)\rho(t)e'_i \\
&= \chi_{-a_n}(t)\sum_{i=1}^{n} f_i(\xi)\chi_{a_i}(t)e'_i \\
&= \sum_{i=1}^{n} \chi_{a_i-a_n}(t)f_i(\xi)e'_i.
\end{aligned}$$

Thus

$$\rho(tut^{-1})e'_n = \sum_{i=1}^{n} \chi_{a_i-a_n}(t)f_i(\xi)e'_i. \tag{2.13}$$

Identifying (2.12) with (2.13), we have for each $1 \leq i \leq n$, $\xi \in \mathbb{K}$ and $\eta \in \mathbb{K}^\star$:

$$f_i(\xi\eta^2) = \chi_{a_i - a_n}(t) f_i(\xi) = \eta^{a_i - a_n} f_i(\xi), \tag{2.14}$$

from which we conclude that $f_i$ is constant. More precisely, $f_i = 1$ if $a_i = a_n$ and $f_i = 0$ if $a_i < a_n$. So, for all $t \in T$ and $u \in U$,

$$\rho(tut^{-1})e'_n = e'_n.$$

Firstly, using relation (2.11) and then Lemma 2.6 we obtain :

$$\rho(tu)e'_n = \rho(t)\rho(u)e'_n = \sum_{i=1}^{n} f_i(\xi)\rho(t)e'_i = \sum_{i=1}^{n} f_i(\xi)\chi_{a_i}(t)e'_i.$$

Since $a_i \leq a_n$, we have for all $u \in U$ and $t \in T_n$:

$$\rho(tu)e'_n = \chi_{a_n}(t)e'_n\,. \tag{2.15}$$

Now, we show that $V$ is isomorphic to some $R_{d-1}$. Define the $G-$linear map $\phi$ from the dual $V^\star$ of $V$ into $A$ by

$$\phi(l)(g) = l(\rho(g)e'_n), \quad l \in V^\star,\, g \in G.$$

Since $V$ is irreducible, we have that $\phi$ is injective (as in the proof of Lemma 2.5).

Moreover, $Im(\phi) \subseteq V_{a_n} \subseteq A$, where $V_{a_n}$ is given by (2.7). Indeed, Relation (2.15) allows us to write :

$$\begin{aligned}
\phi(l)(gtu) &= l\big(\rho(gtu)e'_n\big) \\
&= l\big(\rho(g)\rho(tu)e'_n\big) \\
&= l\big(\rho(g)\chi_{a_n}(t)e'_n\big) \\
&= \chi_{a_n}(t)l\big(\rho(g)e'_n\big) \\
&= \chi_{a_n}(t)\phi(l)(g)
\end{aligned}$$

Since $\phi$ maps $V^\star$ into the space $V_{a_n} \subseteq A$ with $V_{a_n} \cong R^\star_{a_n}$ then $\phi : V^\star \to R^\star_{a_n}$ is an injective map of $G-$modules. It follows that $\phi$ determines a surjective homomorphism of $G-$modules $R_{a_n} \to V$. As $R_{a_n}$ is irreducible, it follows that $V$ is isomorphic to $R_{a_n}$, which proves the assertion. □

## 2.3 Decomposition of an arbitrary rational representation of $\mathrm{SL}(2, \mathbb{K})$

In this section, we present the theorem of decomposition of any rational representation of $\mathrm{SL}(2, \mathbb{K})$ as direct sum of irreducible rational representations of $\mathrm{SL}(2, \mathbb{K})$.

**Theorem 2.1.** *Let $\rho : \mathrm{SL}(2, \mathbb{K}) \to \mathrm{GL}(V)$ be an arbitrary rational representation. Then*

1. $V \cong \oplus_{k\geq 0} R_k^{m(k)}$ *and* $\rho \cong \sum_{k\geq 1} m(k) r_k$.
2. *For each* $\xi \in \mathbb{K}^\star$, $tr\rho \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} = \sum\limits_{k\geq 0} m(k) \frac{\xi^k - \xi^{-k}}{\xi - \xi^{-1}}$.
3. *The multiplicities* $m(k)$ *are uniquely determined by* $\rho$.

*Proof.* 1. $(a)$ If $V$ is an irreducible $G-$module, by Proposition 2.2, $V$ is isomorphic to some $R_{k-1}$ and the representation $\rho$ is equivalent to some representation $r_k$.

$(b)$ If $V$ is a reducible $G-$module, Lemma 2.6 implies that $V = \oplus_{k\in I} V_{k-1}$, where $V_{k-1}$ are the irreducible $G-$submodules of $V$ and $I$ is a finite set of indices. By Proposition 2.2, each $V_{k-1}$ is isomorphic to some $R_{k-1}$. Let $m(k)$ be the number of subspaces $V_{k-1}$ isomorphic to some $R_{k-1}$. It follows:

$$V \cong \oplus_{k\geq 1} R_{k-1}^{m(k)} \qquad \text{and} \quad \rho \cong \sum_{k\geq 1} m(k) r_k.$$

2. Since

$$tr\left( r_k \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \right) = \sum_{i=1}^{k} \xi^{k+1-2i} = \frac{\xi^k - \xi^{-k}}{\xi - \xi^{-1}},$$

we have

$$tr\left( \rho \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \right) = \sum_{k\geq 1} m(k) \frac{\xi^k - \xi^{-k}}{\xi - \xi^{-1}}.$$

Indeed, $r_k\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}e_i = \xi^{k+1-2i}e_i$, for each $1 \le i \le k$, then

$$r_k\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix} = \begin{pmatrix}\xi^{k-1} & 0 & \dots & 0\\ 0 & \xi^{k-3} & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & \xi^{-k+1}\end{pmatrix}.$$

3. The multiplicities $m(k)$ are uniquely determined by $\rho$ since the function $\xi \mapsto \xi^N tr\left(r_k\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}\right)$ is a polynomial function on $\mathbb{C}$ for a some $N$ sufficiently large. Then the coefficients of this polynomial are uniquely determined by $\rho$.

□

## 2.4 Equivalence between $Ad \circ r_d$ and $\sum_{k=1}^{d-1} r_{2k+1}$

The following theorem is crucial for the next chapter saw it facilitates the calculation of dimensions of certain cohomology groups.

**Theorem 2.2** (of Clebsch-Gordan formula)**.** *For $d \ge e \ge 1$, we have isomorphism of $G-$modules:*

$$R_{d-1} \otimes_{\mathbb{K}} R_{e-1} \cong R_{d-1+e-1} \oplus R_{d-1+e-1-2} \oplus \dots \oplus R_{d-e}, \tag{2.16}$$

$$R_{d-1} \otimes_{\mathbb{K}} R_{d-1} \cong R_{2d-2} \oplus R_{2d-4} \oplus \dots \oplus R_2 \oplus R_0. \tag{2.17}$$

*Moreover, we have :*

$$Ad \circ r_d \cong r_d \otimes r_d \cong \sum_{k=1}^{d-1} r_{2k+1}. \tag{2.18}$$

*Proof.* 1. To show that $R_{d-1} \otimes_{\mathbb{K}} R_{e-1} \cong R_{d-1+e-1} \oplus R_{d+e-4} \oplus \dots \oplus R_{d-e}$, it suffices to show equivalence of representations of $G$ into $R_{d-1} \otimes_{\mathbb{K}} R_{e-1}$ and into $R_{d+e-2} \oplus R_{d+e-4} \oplus \dots \oplus R_{d-e}$. To do this, we show that these representations have the same trace, i.e.

$$tr\Big(r_d(t) \otimes r_e(t)\Big) = tr\Big(r_{d+e-1}(t) \oplus r_{d+e-3}(t) \oplus \dots \oplus r_{d-e+1}(t)\Big).$$

On one side we have :

$$\begin{aligned} tr\Big(r_d(t)\otimes r_e(t)\Big) &= tr\big(r_d(t)\big)tr\big(r_e(t)\big) \\ &= \Big(\frac{\xi^d-\xi^{-d}}{\xi-\xi^{-1}}\Big)\Big(\frac{\xi^e-\xi^{-e}}{\xi-\xi^{-1}}\Big) \\ &= \frac{\xi^{d+e}+\xi^{-d-e}-\xi^{e-d}-\xi^{d-e}}{(\xi-\xi^{-1})^2}. \end{aligned}$$

On the other side, we have :

$$\begin{aligned} tr\Big(r_{d+e-1}(t)\oplus r_{d+e-3}(t)\oplus\ldots\oplus r_{d-e+1}(t)\Big) &= \sum_{i=0}^{e-1} tr\Big(r_{d+e-1-2i}(t)\Big) \\ &= \sum_{i=0}^{e-1}\frac{\xi^{d+e-1-2i}-\xi^{-d-e+1+2i}}{\xi-\xi^{-1}} \\ &= \frac{\xi^{d+e}+\xi^{-d-e}-\xi^{e-d}-\xi^{d-e}}{(\xi-\xi^{-1})^2}. \end{aligned}$$

This proves the equivalence of the representations and thus the isomorphism of $G-$modules.

2. From Relation (2.16) it follows for $d=e$ the following equivalence and isomorphism:

$$r_d\otimes r_d\cong r_{2d-1}\oplus r_{2d-3}\oplus\ldots\oplus r_1,$$

$$R_{d-1}\otimes_{\mathbb{K}} R_{d-1}\cong R_{2d-2}\oplus R_{2d-4}\oplus\ldots\oplus R_0.$$

3. By Lemme 2.4, we have :

$$End(R_{d-1})\cong R_{d-1}\otimes_{\mathbb{K}} R^{\star}_{d-1}\cong R_{d-1}\otimes_{\mathbb{K}} R_{d-1}.$$

Since the action of $G$ on $End(R_{d-1})$ is defined by $g\cdot t = r_d(g)\circ t\circ r_d(g)^{-1}$ for $t\in End(R_{d-1})$ and $g\in G$, we have :

$$Ad\circ r_d\cong r_d\otimes r_d\cong\sum_{k=1}^{d-1} r_{2k+1}.$$

□

# Chapter 3

# Representation spaces and cohomology groups

## Summary

In Section 3.1, we present some useful definitions and results. In Paragraph 3.1.1 we define the representation space and we show that it admits a structure of affine variety. In Paragraph 3.1.2 we define the Zariski tangent space and we give a review on cohomology groups. We end this section by some results on cohomology groups of knot group. In Section 3.2, we give a review on the deformations of representations. In the last section, we give some important results.

## 3.1 Representation space and cohomology groups

### 3.1.1 Representation spaces

**Definition 3.1.** *Let* $\Gamma$ *be a finitely presented group and* $G$ *an algebraic group. The* space of representations *of* $\Gamma$ *to* $G$*, denoted* $R(\Gamma, G)$*, is the set of all group homomorphisms from* $\Gamma$ *to* $G$*.*

The set $R(\Gamma, G)$ admits a structure of affine variety (real or complex) not necessarily irreducible [CS83]. Indeed, if $F_n$ is the free group of generators $S_1, \dots, S_n$, the set $R(\Gamma, G)$ is naturally identified with the affine variety $G^n$. If $G$ admits the presentation $\langle S_1, \dots, S_n : R_1, \dots, R_m \rangle$, we can embed $R(\Gamma, G)$ into $G^n$ via the application $f$ given by $f(\rho) = \big(\rho(S_1), \dots, \rho(S_n)\big)$. The application $f$ is injective because the $S_i$ generate $\Gamma$. Let $(\sigma_1, \dots, \sigma_n)$ be an element of $G^n$. If we substitute $\sigma_i$ instead of $S_i$ into $R_j$, then we can consider each word $R_j$ as a polynomial application of $G^n$ into $G^m$. Therefore

$$Imf = \cap\{R_j^{-1}(e) \mid j = 1, \dots, m\} = R^{-1}(e, \dots, e)$$

where $e$ is the unity in $G$ and $R = (R_1, \dots, R_m) : G^n \to G^m$. Thus, we can identify $R(\Gamma, G)$ with $Imf$. This structure is independent of the chosen presentation (see [LM85]).

### 3.1.2 Zariski tangent space and cohomology groups

**Definition 3.2** (Zariski tangent space)**.** *Let* $V$ *be an affine sub-variety of* $\mathbb{C}^d$ *which ideal of definition is* $I(V)$*. Let* $p$ *be a point of* $V$*. The Zariski tangent space on* $V$ *at* $p$*, denoted* *by $T_p^{Zar}(V)$, is given by:*

$$\left\{ \frac{d\gamma(t)}{dt}\Big|_{t=0} \in \mathbb{C}^d \;\Big|\; \gamma \in (\mathbb{C}[t])^d, \gamma(0) = p,\ f \circ \gamma \in t^2\mathbb{C}[t] \text{ for each } f \in I(V) \right\}$$

*is the vector space of derivatives of polynomial germs $\gamma(t)$ defined near the origin, with values in $\mathbb{C}^d$ such that $\gamma(0) = p$ and satisfy the equations of $I(V)$ modulo $t^2$.*

In general, $\dim(T_p^{Zar}(V)) \geq \dim V$ and we have equality if and only if the point $p$ is not singular.

Let $\Gamma = \langle S_1, \ldots, S_n : R_1, \ldots, R_m \rangle$ be a presentation of the group of a knot $K$ in $\mathbb{S}^3$, $G$ a connected algebraic Lie group and $\mathfrak{g}$ its Lie algebra. Let $\rho\colon \Gamma \to G$ be a representation of $\Gamma$ in $G$. The Lie algebra $\mathfrak{g}$ can be equipped with the structure of $\Gamma-$module via the action of the adjoint representation:

$$\begin{aligned} \mathrm{Ad} : \Gamma \times \mathfrak{g} &\to \mathfrak{g} \\ (\gamma, x) &\mapsto Ad \circ \rho(\gamma)(x) \end{aligned}$$

and will be denoted $\mathfrak{g}_\rho$. A cocycle $d \in Z^1(\Gamma; \mathfrak{g}_\rho)$ is a map $d\colon \Gamma \to \mathfrak{g}$ satisfying

$$d(\gamma_1\gamma_2) = d(\gamma_1) + \mathrm{Ad}_{\rho(\gamma_1)} d(\gamma_2), \ \ \forall \gamma_1, \gamma_2 \in \Gamma.$$

It was observed by Weil [Wei64] that there is a natural inclusion $T_\rho^{Zar} R(\Gamma, G) \hookrightarrow Z^1(\Gamma; \mathfrak{g}_\rho)$. Informally speaking, given a smooth curve $\rho_\epsilon$ of representations through $\rho_0 = \rho$ one gets a $1-$cocycle $d : \Gamma \to \mathfrak{g}_\rho$ by defining

$$d(\gamma) := \frac{d\rho_\epsilon(\gamma)}{dt}\Big|_{\epsilon=0} \rho(\gamma)^{-1}, \quad \forall \gamma \in \Gamma.$$

In general, this inclusion is strict and by [HPS01, Sec. 2.2] we have

$$Z^1(\Gamma; \mathfrak{g}_\rho) \cong \Big\{ (X_1, \cdots, X_n) \in \mathfrak{g}^n \quad \Big| \quad \sum_{i=1}^n \partial_i R_j \circ X_i, \text{ for } j = 1, \cdots, m \Big\}.$$

The tangent space to the orbit of $\rho$ by conjugation $\mathcal{O}(\rho) := \{A\rho A^{-1} \mid A \in G\}$ corresponds to the space of $1-$coboundaries $B^1(\Gamma; G)$. Here, $b : \Gamma \to \mathfrak{g}$ is a coboundary if there exists $x \in \mathfrak{g}$ such that

$$b(\gamma) = \mathrm{Ad}_{\rho(\gamma)} x - x.$$

A detailed account can be found in [LM85].

Let $\dim_\rho R(\Gamma, G)$ be the local dimension of $R(\Gamma, G)$ at $\rho$, i.e. the maximal dimension of the irreducible components of $R(\Gamma, G)$ containing $\rho$ (see [Sha77, Ch. II, §1.4]). So, we obtain, for every $\rho \in R(\Gamma, G)$:

$$\dim_\rho R(\Gamma, G) \leq \dim T_\rho^{Zar} R(\Gamma, G) \leq \dim Z^1(\Gamma; \mathfrak{g}_\rho).$$

**Definition 3.3.** *A representation $\rho \in R(\Gamma, G)$ is called* regular *if* $\dim_\rho R(\Gamma, G) = \dim Z^1(\Gamma; \mathfrak{g}_\rho)$.

**Lemma 3.1.** *Let $\rho$ be a representation in $R(\Gamma, G)$. If $\rho$ is regular, then $\rho$ is a smooth point of the representation variety $R(\Gamma, G)$. Moreover, $\rho$ is contained in a unique component of $R(\Gamma, G)$ of dimension* $\dim Z^1\big(\Gamma; \mathfrak{g}_\rho\big)$.

*Proof.* For every $\rho \in R(\Gamma, G)$ we have

$$\dim_\rho R(\Gamma, G) \leq \dim T_\rho^{Zar} R(\Gamma, G) \leq \dim Z^1(\Gamma; \mathfrak{g}_\rho).$$

Since $\dim_\rho R(\Gamma, G) = \dim Z^1\big(\Gamma; \mathfrak{g}_\rho\big)$ we obtain $\dim T_\rho^{Zar} R(\Gamma, G) = \dim Z^1(\Gamma; \mathfrak{g}_\rho)$. Then $\rho$ is a smooth point of the representation variety $R(\Gamma, G)$ and it is contained in a unique component of $\dim Z^1\big(\Gamma; \mathfrak{g}_\rho\big)$. □

**Remark 3.1.** *Note that there are discrete groups and representations $\rho$ which are smooth points of the representation variety without be regular. See [LM85, Example 2.10] for more details.*

### 3.1.3 Some results on cohomology groups of knot group

In this paragraph we give some important results on cohomology groups on knot theory which are crucial in the following.

If $\Gamma$ is a finitely presented group and $M$ is a $\Gamma-$module, we denote by $C^n(\Gamma; M) := \{f \colon \Gamma^n \to M\}$ the space of $n-$cochains. The coboundary operator is denoted by $\partial : C^n(\Gamma; M) \to C^{n+1}(\Gamma; M)$ and is given by :

$$\partial f(\gamma_1, \ldots, \gamma_{n+1}) = \gamma_1 f(\gamma_2, \ldots, \gamma_{n+1}) + \\ \sum_{i=1}^{n} (-1)^i f(\gamma_1, \ldots, \gamma_{i-1}, \gamma_i\gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_{n+1}) + (-1)^{n+1} f(\gamma_1, \ldots, \gamma_n).$$

For example for $n=0$, $C^0(\Gamma;M)=M$ and for each $\gamma\in\Gamma$, $\partial A(\gamma)=\gamma\cdot A-A$. For $n=1$ and $f\in C^1(\Gamma;M)$, $\partial f(\gamma_1,\gamma_2)=\gamma_1 f(\gamma_2)-f(\gamma_1\gamma_2)+f(\gamma_1)$, for each $\gamma_1,\gamma_2\in\Gamma$. We denote by $B^\star(\Gamma;M)$ $\Big($resp. $Z^\star(\Gamma;M)$, $H^\star(\Gamma;M)\Big)$ the sets coboundaries $\big($resp. cocycles, cohomology class$\big)$ of $\Gamma$ with coefficients in $M$. See [Bro82] for more details.

The following lemma presents a classical result which proof can be found in [HP05, Lemma 3.1].

**Lemma 3.2.** *Let $M$ be a connected, compact, orientable $3-$manifold such that $\partial M$ is a torus. Let $A$ be a $\pi_1(M)-$module and let $X$ be any CW-complex with $\pi_1(X)\cong\pi_1(M)$. Then there are natural morphisms $H_i\big(X;A\big)\to H_i\big(\pi_1(M);A\big)$ which are isomorphisms for $i=0,1$ and surjection for $i=2$. In cohomology, there are natural morphisms $H^i\big(\pi_1(M);A\big)\to H^i\big(X;A\big)$ which are isomorphisms for $i=0,1$ and injection for $i=2$.*

**Remark 3.2.** *In the case where $X$ is the complement of a knot, the homomorphisms $H^\star\big(\pi_1(X);A\big)\to H^\star\big(X;A\big)$ and $H^\star\big(\pi_1(\partial X);A\big)\to H^\star\big(\partial X;A\big)$ are isomorphisms. This is a consequence of the asphericity of $X$ and $\partial X$. Moreover, the knot complement $X$ has the homotopy type of a $2-$dimensional $CW-$complex which implies that $H^k\big(\pi_1(X);A\big)=0$ for $k\geq 3$. See [Whi78] for more details.*

Let $K$ be a knot and $\Gamma_K$ its group.The Laurent polynomial ring $\mathbb{C}[t^{\pm1}]$ turns into a $\Gamma_K-$module via the action $\gamma\cdot p(t)=t^{\varphi(\gamma)}p(t)$ for all $\gamma\in\Gamma_K$ and all $p(t)\in\mathbb{C}[t^{\pm1}]$. Recall that there are isomorphisms of $\mathbb{C}[t^{\pm1}]-$modules

$$H_*(\Gamma_K;\mathbb{C}[t^{\pm1}])\cong H_*(X;\mathbb{C}[t^{\pm1}])\cong H_*(X^\infty;\mathbb{C}),$$

where $X^\infty$ denotes the infinite cyclic covering of the knot complement $X$. For more details see [DK01, Chapter 5]. The module $H_1(\Gamma_K;\mathbb{C}[t^{\pm1}])$ is a finitely generated torsion module called the *Alexander module* of $K$. A generator of its order ideal is called the *Alexander polynomial* $\Delta_K(t)\in\mathbb{C}[t^{\pm1}]$ of $K$. The Alexander polynomial is unique up to multiplication with a unit in $\mathbb{C}[t^{\pm1}]$.

For completeness we will state the next lemma which shows that the cohomology groups $H^*(\Gamma_K;\mathbb{C}_\alpha)$ are determined by the Alexander module $H_1(\Gamma_K;\mathbb{C}[t^{\pm1}])$, i.e. $H^1(\Gamma_K;\mathbb{C}_\alpha)\cong\hom_\Lambda\Big(H_1(X^\infty;\mathbb{C});\mathbb{C}_\alpha\Big)$

**Lemma 3.3.** *Let $K \subset \mathbb{S}^3$ be a knot and $\Gamma_K$ its group. Let $\alpha \in \mathbb{C}^*$ be a nonzero complex number and let $\mathbb{C}_\alpha$ denotes the $\Gamma_K$-module given by the action $\gamma \cdot z = \alpha^{\varphi(\gamma)} z$.*

*If $\alpha = 1$ then $\mathbb{C}_\alpha = \mathbb{C}$ is a trivial $\Gamma_K$-module and $H^k(\Gamma_K; \mathbb{C}) \cong \mathbb{C}$ for $k = 0, 1$ and $H^k(\Gamma_K; \mathbb{C}) = 0$ for $k \geq 2$.*

*If $\alpha \neq 1$ then $H^0(\Gamma_K; \mathbb{C}_\alpha) = 0$ and $\dim H^1(\Gamma_K; \mathbb{C}_\alpha) = \dim H^2(\Gamma_K; \mathbb{C}_\alpha)$. Moreover, $H^1(\Gamma_K; \mathbb{C}_\alpha) \neq 0$ if and only if $\Delta_K(\alpha) = 0$.*

*Proof.* From [BZH13, Prop. 8.16] we have

$$H_0(X^\infty; \mathbb{C}) \cong \mathbb{C} \cong \mathbb{C}[t^{\pm 1}]/(t-1), \quad \text{and} \quad H_k(X^\infty; \mathbb{C}) = 0, \quad \text{for } k \geq 2.$$

If $\alpha = 1$ then $H^k(\Gamma_K; \mathbb{C}) \cong \mathbb{C}$ for $k = 0, 1$ and $H^k(\Gamma_K; \mathbb{C}) = 0$ for $k \geq 2$ follows.

Now suppose that $\alpha \in \mathbb{C}^*$, $\alpha \neq 1$, and notice that we have an isomorphism $\mathbb{C}_\alpha \cong \mathbb{C}[t^{\pm 1}]/(t - \alpha)$. The cohomology group $H^0(\Gamma_K; \mathbb{C}_\alpha)$ vanishes, since the $\Gamma_K-$module $\mathbb{C}_\alpha$ has no invariants and $H^k(\Gamma_K; \mathbb{C}_\alpha) = 0$ for $k > 2$ since the knot complement $X$ has the homotopy type of a $2-$complex. Recall that the Alexander module $H_1(\Gamma_K; \mathbb{C}[t^{\pm 1}])$ is finitely generated torsion module and hence a sum of non-free cyclic modules since $\mathbb{C}[t^{\pm 1}]$ is a principal ideal domain. The Alexander polynomial is the order ideal of $H_1(\Gamma_K; \mathbb{C}[t^{\pm 1}])$. Since $\alpha \neq 1$, it follows from the universal coefficient theorem that $H^1(\Gamma_K; \mathbb{C}_\alpha) \cong \hom(H_1(\Gamma_K; \mathbb{C}); \mathbb{C}_\alpha)$. Hence $H^1(\Gamma_K; \mathbb{C}_\alpha) \neq 0$ if and only if the module $H_1(\Gamma_K; \mathbb{C})$ has $(t - \alpha)-$torsion which is equivalent to $\Delta_K(\alpha) = 0$. Finally, $\dim H^1(\Gamma_K; \mathbb{C}_\alpha) = \dim H^2(\Gamma_K; \mathbb{C}_\alpha)$ follows since the Euler characteritic of $X$ vanishes. See [Ben00, Proposition 2.1] for more details. □

Let $M$ be a connected, compact, orientable $3-$manifold with torus boundary. Let $d \geq 2$ and let $\rho \colon \pi_1(M) \to \mathrm{SL}(d, \mathbb{C})$ be a representation from the fundamental group of $M$ into $\mathrm{SL}(d, \mathbb{C})$. The Lie algebra $\mathfrak{sl}_d(\mathbb{C})$ turns into a $\pi_1(M)-$module via the action of the adjoint representation $\mathrm{Ad} \circ \rho$. A detailed account can be found in [Por97].

**Lemma 3.4** (Poincaré duality)**.** *For $i \in \{0, 1\}$, we have :*

1. $H^i\big(M; \mathfrak{sl}_d(\mathbb{C})_\rho\big)^\star \cong H^{3-i}\big(M, \partial M; \mathfrak{sl}_d(\mathbb{C})_\rho\big)$.
2. $H^i\big(\partial M; \mathfrak{sl}_d(\mathbb{C})_\rho\big)^\star \cong H^{2-i}\big(\partial M; \mathfrak{sl}_d(\mathbb{C})_\rho\big)$.

*Accordingly,* $\dim H^1\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho\big) = 2\dim H^0\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho\big)$.

**Lemma 3.5.** *In the exact cohomology sequence of the pair* $(M,\partial M)$ *we have*

$$H^1\big(M;\mathfrak{sl}_d(\mathbb{C})_\rho\big) \xrightarrow{\alpha} H^1\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho\big) \xrightarrow{\beta} H^2\big(M,\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho\big)$$

*with* $\mathrm{rk}(\alpha) = \frac{1}{2}\dim H^1\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho\big)$

*Proof.* Lemma 3.4 implies that $\alpha$ and $\beta$ are dual to each other. Hence $\dim Im(\beta) = \dim Im(\alpha)$ and by $Im(\alpha) = \ker(\beta)$ we obtain

$$\dim H^1(\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho) = \mathrm{rk}(\beta) + \dim(\ker\beta) = 2\mathrm{rk}(\alpha).$$

□

## 3.2 Review on the deformations of representations

The general references for this section are [Bro82] and [HPS01].

In what follows, let $d \geq 2$ be an integer, $\Gamma$ a discrete group and let $C^1(\Gamma;\mathfrak{sl}_d(\mathbb{C})) := \big\{c\colon \Gamma \to \mathfrak{sl}_d(\mathbb{C})\big\}$ denote the 1-cochains of $\Gamma$ with coefficients in $\mathfrak{sl}_d(\mathbb{C})$ and we shall denote by $\delta$ the coboundary operator of $C^\star(\Gamma,\mathfrak{sl}_d^\rho(\mathbb{C}))$ (see [Bro82, p.59]).

**Definition 3.4.** *Let* $\rho\colon \Gamma \to \mathrm{SL}(d,\mathbb{C})$ *be a representation. A formal deformation of* $\rho$ *is a homomorphism* $\rho_\infty\colon \Gamma \to \mathrm{SL}(d,\mathbb{C}[\![t]\!])$ *such that*

$$\rho_\infty(\gamma) = \exp\left(\sum_{i=1}^{\infty} t^i u_i(\gamma)\right)\rho(\gamma)$$

*where* $u_i : \Gamma \to \mathfrak{sl}_d(\mathbb{C})$ *are elements of* $C^1\big(\Gamma,\mathfrak{sl}_d(\mathbb{C})\big)$ *such that* $ev_0 \circ \rho_\infty = \rho$. *We say that* $\rho_\infty$ *is a formal deformation up to order* $k$ *of* $\rho$ *if* $\rho_\infty$ *is a homomorphism modulo* $t^{k+1}$.

Here, $ev_0\colon \mathrm{SL}(d,\mathbb{C}[\![t]\!]) \to \mathrm{SL}(d,\mathbb{C})$ is the evaluation homomorphism at $t = 0$ and we denote by $\mathbb{C}[\![t]\!]$ the ring of formal powers series.

In general, the formal deformations of a representation $\rho\colon \Gamma \to \mathrm{SL}(d,\mathbb{C})$ are determined by an infinite sequence of obstructions. These obstructions were first studied by Kodaira and Spencer in a different context. When the $k-$th obstruction vanishes, the obstruction of order $k+1$ is defined, it lives in $H^2\big(\Gamma; \mathfrak{sl}_d(\mathbb{C})_\rho\big)$. This is what we will present in the following.

For every $k \in \mathbb{Z}$, $k \geqslant 0$, we define the ring $A_k := \mathbb{C}[\![t]\!]/(t^{k+1})$ and $A_\infty := \mathbb{C}[\![t]\!]$. We are interested in the following Lie group $G_k := \mathrm{SL}(d, A_k)$ and in its Lie algebra $\mathfrak{g}_k := \mathfrak{sl}_d(A_k)$. Note that $G_0 = \mathrm{SL}(d,\mathbb{C})$, $\mathfrak{g}_0 = \mathfrak{sl}_d(\mathbb{C})$ and $\mathfrak{g}_k = \Big\{ \sum_{i=0}^{k} t^i X_i \mid X_i \in \mathfrak{sl}_d(\mathbb{C})\Big\}$. For every $k > l$ we have a projection $\pi_{k,l}\colon G_k \to G_l$. The projection $\pi_{k+1,k}$ is denoted by $\pi_k$ and $\pi_{\infty,k}$ is denoted by $p_k$.

Let $\rho \in R(\Gamma)$ and $u_i\colon \Gamma \to \mathfrak{sl}_d(\mathbb{C})$, $i = 1, \cdots, k$, be given. We define a map

$$\tilde\rho_k := \tilde\rho_k^{(\rho,u_1,\cdots,u_k)}\colon \Gamma \to G_\infty$$

by

$$\tilde\rho_k(\gamma) := \exp\Bigg(tu_1(\gamma) + \cdots + t^k u_k(\gamma)\Bigg)\rho(\gamma). \tag{3.1}$$

For all $i \geqslant 0$ we obtain a map $\rho_i\colon \Gamma \to G_i$ given by $\rho_i := \rho_i^{(\rho,u_1,\cdots,u_k)} := p_i \circ \tilde\rho_k$. We define $\tilde U_{k-1} := \tilde U_{k-1}^{(u_1,\cdots,u_k)}\colon \Gamma \to \mathfrak{g}_\infty$ as follows

$$\tilde U_{k-1}(\gamma) := u_1(\gamma) + 2tu_2(\gamma) + \cdots + kt^{k-1}u_k(\gamma). \tag{3.2}$$

Thus, for all $i \geqslant 0$ we obtain a map $U_i\colon \Gamma \to \mathfrak{g}_i$ given by $U_i := U_i^{(u_1,\cdots,u_k)} := p_i \circ \tilde U_{k-1}$.

We fix from now on a representation $\rho \in R(\Gamma)$.

**Lemma 3.6.** *Let $u_i : \Gamma \to \mathfrak{sl}_d(\mathbb{C})$, $i = 1, \cdots, k+1$ be given and define $\tilde\rho_{k+1}$ (resp. $\tilde U_k$) as in equation* (3.1) *(resp. equation* (3.2)*). Assume that $\rho_k := p_k \circ \tilde\rho_{k+1}\colon \Gamma \to G_k$ is a homomorphism. Then $\rho_{k+1} := p_{k+1} \circ \tilde\rho_{k+1}\colon \Gamma \to G_{k+1}$ is a homomorphism if and only if*

$$U_k := p_k \circ \tilde U_k \in Z^1(\Gamma, \mathfrak{g}_k^{\rho_k})$$

*is a cocycle.*

*Proof.* The map $\rho_{k+1}$ is a homomorphism if and only if

$$\tilde\rho_{k+1}(\gamma_1)\tilde\rho_{k+1}(\gamma_2) \equiv \tilde\rho_{k+1}(\gamma_1\gamma_2) \ mod\ t^{k+2}.$$

If we apply the usual differential operator $\frac{d}{dt}$ to this equation we obtain:

$$\frac{d}{dt}\Big(\tilde\rho_{k+1}(\gamma_1)\Big)\tilde\rho_{k+1}(\gamma_2) + \tilde\rho_{k+1}(\gamma_1)\frac{d}{dt}\Big(\tilde\rho_{k+1}(\gamma_2)\Big) \equiv \frac{d}{dt}\Big(\tilde\rho_{k+1}(\gamma_1\gamma_2)\Big) \ mod\ t^{k+1}.$$

Hence

$$\tilde{U}_k(\gamma_1)\tilde{\rho}_{k+1}(\gamma_1)\tilde{\rho}_{k+1}(\gamma_2)+\tilde{\rho}_{k+1}(\gamma_1)\tilde{U}_k(\gamma_2)\tilde{\rho}_{k+1}(\gamma_2) \equiv \tilde{U}_k(\gamma_1\gamma_2)\tilde{\rho}_{k+1}(\gamma_1\gamma_2)\ mod\ t^{k+1}.$$

Which is equivalent to the following equation

$$\tilde{U}_k(\gamma_1)\tilde{\rho}_{k+1}(\gamma_1)\tilde{\rho}_{k+1}(\gamma_2)+\tilde{\rho}_{k+1}(\gamma_1)\tilde{U}_k(\gamma_2)\tilde{\rho}_{k+1}(\gamma_1)^{-1}\tilde{\rho}_{k+1}(\gamma_1)\tilde{\rho}_{k+1}(\gamma_2) \equiv \tilde{U}_k(\gamma_1\gamma_2)\tilde{\rho}_{k+1}(\gamma_1\gamma_2)\ mod\ t^{k+1}.$$

Since $\rho_k$ is a homomorphism this is equivalent to the following equation in $\mathfrak{g}_k$

$$U_k(\gamma_1)+\rho_k(\gamma_1)U_k(\gamma_2)\rho_k(\gamma_1)^{-1}=U_k(\gamma_1\gamma_2).$$

Hence $U_k \in Z^1(\Gamma, \mathfrak{g}_k^{\rho_k})$ is a cocycle.

if $U_k \in Z^1(\Gamma, \mathfrak{g}_k^{\rho_k})$ is a cocycle , i.e.

$$U_k(\gamma_1)+\rho_k(\gamma_1)U_k(\gamma_2)\rho_k(\gamma_1)^{-1}-U_k(\gamma_1\gamma_2) \equiv 0\ mod\ t^{k+1}$$

we have

$$p_k\circ\tilde{U}_k(\gamma_1)+\tilde{\rho}_{k+1}(\gamma_1)p_k\circ\tilde{U}_k(\gamma_2)\tilde{\rho}_{k+1}(\gamma_1)^{-1}-p_k\circ\tilde{U}_k(\gamma_1\gamma_2) \equiv 0\ mod\ t^{k+1}$$

Multiplying this equation by $\tilde{\rho}_{k+1}(\gamma_1)\tilde{\rho}_{k+1}(\gamma_2)$ and integrating it we obtain

$$\tilde{\rho}_{k+1}(\gamma_1)\tilde{\rho}_{k+1}(\gamma_2)-\tilde{\rho}_{k+1}(\gamma_1\gamma_2) \equiv C\ mod\ t^{k+2},$$

where $C \in M_d(\mathbb{C})$ is a matrix. Evaluating this equation at $t=0$ we obtain $C=0$. Then $\rho_{k+1}$ is a homomorphism. □

The following proposition gives a similar result to that of [HPS01, Proposition 3.1]. It concerns the case of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

**Proposition 3.1.** *Let $u_1,\dots,u_k \in C^1\big(\Gamma;\mathfrak{sl}_d(\mathbb{C})_\rho\big)$ such that*

$$\rho_k(\gamma)=\exp\left(\sum_{i=1}^{k} t^i u_i(\gamma)\right)\rho(\gamma)$$

*is a homomorphism of group $\Gamma$ into $\mathrm{SL}(d,\mathbb{C}[\![t]\!])$ modulo $t^{k+1}$. Then there exists an obstruction class $\zeta_{k+1} := \zeta_{k+1}^{(u_1,\cdots,u_k)} \in H^2(\Gamma;\mathfrak{sl}_d(\mathbb{C})_\rho)$ with the following proprieties:*

*(i) There is a cochain* $u_{k+1}\colon \Gamma \to \mathfrak{sl}_d(\mathbb{C})_\rho$ *such that*

$$\rho_{k+1}(\gamma) = \exp\left(\sum_{i=1}^{k+1} t^i u_i(\gamma)\right)\rho(\gamma)$$

*is a homomorphism modulo* $t^{k+2}$ *if and only if* $\zeta_{k+1} = 0$.

*(ii) The obstruction* $\zeta_{k+1}$ *is natural, i.e. if* $f\colon \Gamma_1 \to \Gamma$ *is a homomorphism, then* $f^\star\rho_k := \rho_k \circ f$ *is also a homomorphism modulo* $t^{k+1}$ *and* $f^\star(\zeta_{k+1}^{(u_1,\ldots,u_k)}) = \zeta_{k+1}^{(f^\star u_1,\cdots,f^\star u_k)} \in H^2\big(\Gamma_1; \mathfrak{sl}_d(\mathbb{C})_{f^\star\rho}\big)$.

As a consequence we obtain:

**Corollary 3.1.** *Let* $\rho \in R(\Gamma)$ *be given. An infinite sequence* $u_i \in C^1\big(\Gamma; \mathfrak{sl}_d(\mathbb{C})_\rho\big)$, $i \in \mathbb{N}$, *defines a representation* $\rho_\infty\colon \Gamma \to \mathrm{SL}(d, \mathbb{C}[\![t]\!])$,

$$\rho_\infty(\gamma) = \exp\left(\sum_{i=1}^{\infty} t^i u_i(\gamma)\right)\rho(\gamma),$$

*if and only if* $u_1 \in Z^1\big(\Gamma; \mathfrak{sl}_d(\mathbb{C})_\rho\big)$ *is a cocycle and* $\zeta_{k+1}^{(u_1,\ldots,u_k)} = 0$ *for all* $k \geqslant 1$.

*Proof.* If $\rho_\infty$ is a homomorphism then $\rho_k := p_k \circ \rho_\infty$ is a homomorphism for all $k$. Since $\rho_k = \rho_k^{(\rho;u_1,\cdots,u_k)}$ and $\rho_{k+1} = \rho_{k+1}^{(\rho;u_1,\cdots,u_{k+1})}$ are homomorphisms we have $\zeta_{k+1}^{(u_1,\cdots,u_k)} = 0$ by Proposition 3.1.

If $u_1 \in Z^1\big(\Gamma; \mathfrak{sl}_d(\mathbb{C})_\rho\big)$ is a cocycle and $\zeta_{k+1}^{(u_1,\ldots,u_k)} = 0$ for all $k \geqslant 1$ then by Proposition 3.1 $\rho_k := \rho_k^{(\rho;u_1,\cdots,u_k)}$ is a homomorphism for all $k \geq 1$ and hence $\rho_\infty$ is a homomorphism. □

Let $\rho_k := \rho_k^{(\rho;u_1,\cdots,u_k)}\colon \Gamma \to G_k$ be a homomorphism. In order to find a cochain $u_{k+1}\colon \Gamma \to \mathfrak{sl}_d(\mathbb{C})$ such that $\rho_{k+1} := \rho_{k+1}^{(\rho;u_1,\cdots,u_{k+1})}$ is a homomorphism we consider the following exact sequence of $\Gamma-$modules

$$0 \to \mathfrak{sl}_d(\mathbb{C})^\rho \xrightarrow{\alpha_k} \mathfrak{g}_k^{\rho_k} \xrightarrow{\pi_{k-1}} \mathfrak{g}_{k-1}^{\rho_{k-1}} \to 0$$

where $\alpha_k(X) = t^k X$ and $\rho_{k-1} = \pi_{k-1} \circ \rho_k$. This sequence gives rise to the following exact sequence in cohomology (see Proposition 6.1 of [Bro82, Chap. III]):

$$H^1(\Gamma, \mathfrak{g}_k^{\rho_k}) \xrightarrow{(\pi_{k-1})_\star} H^1(\Gamma, \mathfrak{g}_{k-1}^{\rho_{k-1}}) \xrightarrow{\beta_{k-1}} H^2(\Gamma, \mathfrak{sl}_d(\mathbb{C})^\rho) \xrightarrow{(\alpha_k)_\star} H^2(\Gamma, \mathfrak{g}_k^{\rho_k}). \qquad (3.3)$$

**Definition 3.5.** *Let $u_i$, $i = 1, \cdots, k$, be given. If $\rho_k := \rho_k^{(\rho; u_1, \cdots, u_k)} \colon \Gamma \to G_k$ is a homomorphism then by Lemma 3.6, $U_{k-1}^{(u_1, \cdots, u_k)} \in Z^1(\Gamma, \mathfrak{g}_{k-1}^{\rho_{k-1}})$ where $\rho_{k-1} = \pi_{k-1} \circ \rho_k$. We define*

$$\zeta_{k+1} = \zeta_{k+1}^{(u_1, \cdots, u_k)} := \beta_{k-1}([U_{k-1}^{(u_1, \cdots, u_k)}]) \in H^2(\Gamma, \mathfrak{sl}_d(\mathbb{C})^\rho).$$

**Example 3.1.** *Let $\rho \colon \Gamma \to \mathrm{SL}(d, \mathbb{C})$ be a representation and let $u_1 \in Z^1(\Gamma, \mathfrak{s}l_d(\mathbb{C})^\rho)$ be given. We have a homomorphism $\rho_1 \colon \Gamma \to G_1$ given by*

$$\rho_1(\gamma) = \Big(1 + tu_1(\gamma)\Big)\rho(\gamma).$$

*We consider $u_1$ as a map $u_1 \colon \Gamma \to \mathfrak{g}_1$. Therefore we have $\delta_1 u_1(\gamma_1, \gamma_2) = u_1(\gamma_1) + \gamma_1 \circ u_1(\gamma_2) - u_1(\gamma_1\gamma_2)$. Since*

$$\begin{aligned} \gamma_1 \circ u_1(\gamma_2) =& \rho_1(\gamma_1) u_1(\gamma_2) \rho_1(\gamma_1)^{-1} \\ =& \Big(1 + tu_1(\gamma_1)\Big)\rho(\gamma_1) u_1(\gamma_2) \Big(1 - tu_1(\gamma_1)\Big)\rho(\gamma_1)^{-1} \end{aligned}$$

*we have*

$$\begin{aligned} \delta_1 u_1(\gamma_1, \gamma_2) =& u_1(\gamma_1) + \rho(\gamma_1) u_1(\gamma_2) \rho(\gamma_1)^{-1} - u_1(\gamma_1\gamma_2) + \\ & tu_1(\gamma_1)\rho(\gamma_1) u_1(\gamma_2)\rho(\gamma_1)^{-1} - t\rho(\gamma_1) u_1(\gamma_2) u_1(\gamma_1) \rho(\gamma_1)^{-1}. \end{aligned}$$

*Since $u_1 \in Z^1(\Gamma, \mathfrak{s}l_d(\mathbb{C})^\rho)$ we have $\delta_1 u_1(\gamma_1\gamma_2) = tu_1(\gamma_1)\rho(\gamma_1)u_1(\gamma_2)\rho(\gamma_1)^{-1} = t(u_1 \smile u_1)(\gamma_1, \gamma_2)$. By definition of $\alpha_1$, we have $\zeta_2 = \zeta_2^{(u_1)} = [u_1 \smile u_1]$. If $\zeta_2 \in H^2(\Gamma, \mathfrak{sl}_d(\mathbb{C})^\rho)$ we can choose $u_2 \in Z^1(\Gamma, \mathfrak{s}l_d(\mathbb{C})^\rho)$ such that $2\delta_1(u_2) + u_1 \smile u_1 = 0$. The map $U_1^{(u_1, u_2)} \in Z^1(\Gamma, \mathfrak{g}_1^{\rho_1})$ is a cocyle and*

$$\rho_2^{(u_1, u_2)} \colon \Gamma \to G_2$$

*is a homomorphism.*

*Proof of Proposition 3.1.* Let $\rho_k := \rho_k^{(\rho; u_1, \cdots, u_k)} \colon \Gamma \to G_k$ be a homomorphism. By Lemma 3.6 we have $U_{k-1}^{(u_1, \cdots, u_k)} \in Z^1(\Gamma, \mathfrak{g}_{k-1}^{\rho_{k-1}})$ where $\rho_{k-1} = \pi_{k-1} \circ \rho_k$.

From the exactness of the sequence (3.3) it follows that $\beta_{k-1}([U_{k-1}^{(u_1, \cdots, u_k)}]) = 0$ if and only if $[U_{k-1}^{(u_1, \cdots, u_k)}] \in Im_{(\pi_{k-1})_\star}$. This is equivalent to the existence of a cocycle $U_k \in Z^1(\Gamma, \mathfrak{g}_k^{\rho_k})$ such that $U_{k-1}^{(u_1, \cdots, u_k)} = \pi_{k-1} \circ U_k$. It follows that $U_k = U_k^{(u_1, \cdots, u_{k+1})}$ for a map $u_{k+1} \colon \Gamma \to \mathfrak{s}l_d(\mathbb{C})$ and $\rho_{k+1}^{(u_1, \cdots, u_{k+1})}$ is a homomorphism by Lemma 3.6.

The naturality assertion follows from the definition of the connection homomorphism (see Proposition 6.1 of [Bro82, Chap. III]). □

We denote by $\mathbb{C}\{t\} \subset \mathbb{C}[\![t]\!]$ the ring of convergent power series. Starting from a formal deformation of $\rho$ we obtain a convergent deformation as follows:

**Proposition 3.2.** *Let $\rho_\infty\colon \Gamma \to \mathrm{SL}(d, \mathbb{C}[\![t]\!])$ be a formal deformation of $\rho \in R(\Gamma)$. Then for every $N \in \mathbb{N}$ there exists a convergent deformation $\hat\rho_\infty\colon \Gamma \to \mathrm{SL}(d, \mathbb{C}\{t\})$ such that $\hat\rho_\infty(\gamma) \equiv \rho_\infty(\gamma) \mod T^N$ for all $\gamma \in \Gamma$.*

*Proof.* Let $\Gamma = \langle S_1, \dots, S_n : R_1, \dots, R_m\rangle$ be a finite presentation. We have

$$R(\Gamma) \subset \mathrm{SL}(d, \mathbb{C})^n$$

and we fix $(A_1, \cdots, A_n) \in \mathrm{SL}(d, \mathbb{C})^n$ such that $\rho(S_i) = A_i$. It is easy to see that we can identify the space $R(\Gamma)$ with the following subset of $\mathbb{C}^{d^2 n}$:

$$\Big\{(Y_1, \cdots, Y_n) \in M_d(\mathbb{C}) \quad | \quad (E+Y_i) \in \mathrm{SL}(d, \mathbb{C}),\ R_j\Big((E+Y_1)A_1, \cdots, (E+Y_n)A_n)\Big) = E\Big\}.$$

Hence there is a system of polynomial equations $\mathbf{F}(\mathbf{y}) = \mathbf{0}$ such that

$$R(\Gamma) \cong V(\mathbf{F}) := \Big\{\mathbf{y} \in \mathbb{C}^{d^2 n} \quad | \quad \mathbf{F}(\mathbf{y}) = \mathbf{0}\Big\}. \tag{3.4}$$

Note that the solution $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ corresponds to the representation $\rho$. A formal deformation of $\rho$ corresponds to a formal solution $\mathbf{y}(t) \in \mathbb{C}[\![t]\!]$, $\mathbf{y}(0) = \mathbf{0}$, of the system $\mathbf{F}(\mathbf{y}(t)) = \mathbf{0}$. By a theorem of Artin (see [Art68]) there is for a given $N \in \mathbb{N}$ a convergent solution $\hat{\mathbf{y}}(t) \equiv \mathbf{y}(t) \mod t^N$. □

**Lemma 3.7.** *Let $\rho \in R(\Gamma)$ be regular and let $u_i \in C^1(\Gamma, \mathfrak{sl}_d(\mathbb{C})_\rho)$ be given such that $\rho^{(\rho;\, u_1, \cdots, u_k)}\colon \Gamma \to G_k$ is a homomorphism. Then there exists for every $v \in Z^1(\Gamma, \mathfrak{sl}_d(\mathbb{C})_\rho)$ a cochain $u_{k+1} \in C^1(\Gamma, \mathfrak{sl}_d(\mathbb{C})_\rho)$ such that $\rho^{(\rho;\, u_1, \cdots, u_k, u_{k+1}+v)}\colon \Gamma \to G_{k+1}$ is a homomorphism.*

*Proof.* Recall that $\rho \in R(\Gamma)$ is regular if and only if $\dim_\rho R(\Gamma) = \dim Z^1(\Gamma, \mathfrak{sl}_d(\mathbb{C})_\rho)$. We have the identification $R(\Gamma) \cong V := V(\mathbf{F}) \subset \mathbb{C}^{d^2 n}$ where the solution $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ corresponds to the representation $\rho$ (see equation (3.4)). The representation $\rho^{(\rho;\, u_1, \cdots, u_k)}\colon \Gamma \to G_k$ corresponds to a polynomial vector $\mathbf{y}_k(t) \in (\mathbb{C}[t])^{d^2 n}$ of degree $k$ such that $\mathbf{F}(\mathbf{y}_k(t)) \equiv$

$\mathbf{0}\ mod\, t^{k+1}$. The element $v \in Z^1(\Gamma, \mathfrak{sl}_d(\mathbb{C})_\rho)$ gives us a vector $\mathbf{v} \in T_{\mathbf{0}}(V)$. It follows from Lemma 3.1 that $\mathbf{0} \in V$ is a smooth point.

It is now easy to see (using the formal implicit function theorem, see [Mum76]) that we can extend $\mathbf{y}_k(t)$, i.e. there is a $\mathbf{w} \in \mathbb{C}^{d^2 n}$ such that $\mathbf{y}_{k+1} := \mathbf{y}_k(t) + t^k(\mathbf{v} + \mathbf{w})$ satisfies

$$\mathbf{F}(\mathbf{y}_{k+1}(t)) \equiv \mathbf{0}\ mod\, t^{k+2}.$$

This gives us the existence of the representation $\rho^{(\rho;\, u_1, \cdots, u_k, u_{k+1}+v)} \colon \Gamma \to G_{k+1}$ claimed in the lemma. □

## 3.3 Some results

The following result gives relation between metabelian irreducible representations and traces of images of meridian $\mu$ and longitude $l$ of the knot. For more details, see [Nag07, Proposition1.1 and Theorem1.1]

**Theorem 3.1.** *If $K \subset \mathbb{S}^3$ is a knot and $X$ its complement space, then any irreducible metabelian representation $\rho \colon \pi_1(X) \to \mathrm{SL}(2, \mathbb{C})$ satisfies*

$$tr\big(\rho(\mu)\big) = 0 \quad and \quad tr\big(\rho(l)\big) = 2,$$

*where $\mu$ and $l$ are respectively a meridian and longitude of a knot $K$. Further, there exist only finitely many conjugacy classes of irreducible metabelian representations of $\pi_1(X)$ into $\mathrm{SL}(2, \mathbb{C})$, and their number equals*

$$\frac{|\Delta(-1)| - 1}{2}.$$

**Theorem 3.2.** *Let $G$ be a simply connected semi-simple algebraic group. let $C(G) = \{(x, y) \in G \times G \quad | \quad xy = yx\}$ and $(x, y) \in C(G)$. Let $N$ be a neighbourhood of $(x, y)$ in $C(G)$. Then there exists a maximal torus $T$ of $G$ such that $N$ meets $T \times T$. Consequently $C(G)$ is irreducible.*

*Proof.* See [Ric79, Theorem C]. □

**Theorem 3.3.** *Let $\mathfrak{g}$ be a non commutative reductive Lie algebra over an algebraically closed field $\mathbb{K}$ of characteristic zero with adjoint group $G$. Let $C = C(G)$ be the commuting variety of $G$,*

$$C = C(G) := \{(x, y) \in G \times G \quad | \quad xy = yx\}.$$

*Then we have*

$$\dim C = \dim G + rk\, G.$$

*Proof.* See [Pop08]. □

**Example 3.2.** *If $G = \mathrm{SL}(d, \mathbb{C})$ and $\rho^z_{\lambda,d} := r_d \circ \rho^z_\lambda$ is the representation given by* (0.1)-(0.2)*, then Theorem 3.2 and Theorem 3.3 imply that*

1. *$C(\mathrm{SL}(d, \mathbb{C})) \cong R(\mathbb{Z} \oplus \mathbb{Z}, \mathrm{SL}(d, \mathbb{C}))$ is an irreducible variety and*

$$\dim C(\mathrm{SL}(d, \mathbb{C})) = \dim \mathrm{SL}(d, \mathbb{C}) + rk(\mathrm{SL}(d, \mathbb{C})) = d^2 + d - 2.$$

2. *From [Pop08, Sec. 2, Equation 8] we have : $\rho^z_{\lambda,d} \circ i_\star \in R(\mathbb{Z} \oplus \mathbb{Z}, \mathrm{SL}(d, \mathbb{C}))$ is a smooth point if and only if $\mathfrak{sl}_d(\mathbb{C})^{\mathbb{Z}\oplus\mathbb{Z}} = H^0(\mathbb{Z}\oplus\mathbb{Z}, \mathfrak{sl}_d(\mathbb{C})_{\rho^z_{\lambda,d}\circ i_\star})$ has dimension $d-1$, where $i_\star \colon \pi_1(\partial X) \hookrightarrow \pi_1(X)$ designs the inclusion.*

# Chapter 4

# Deformations of reducible representations of knot groups into $\mathrm{SL}(d, \mathbb{C})$

## Summary

In this chapter, we study deformations of certain non-abelian, metabelian, reducible representations of the knot group $\Gamma_K := \pi_1(X)$ into $\mathrm{SL}(d,\mathbb{C})$ which are associated to a simple root of the Alexander polynomial. Section 4.1 deals with the deformation of such representations and some related results. In Section 4.2 we give a proof of the main results. In Section 4.3, we make cohomological calculations which allows us to prove results of Propositions 4.3. Finally, we give some examples.

## 4.1 Deforming representations

The aim of the following section is to prove that if $\lambda^2$ is a simple root of Alexander polynomial then the non-abelian, metabelian reducible representation $\rho^z_{\lambda,d} := r_d \circ \rho^z_\lambda$ is the limit of irreducible non-metabelian representations. Here $\rho^z_\lambda$ is the representation given by (0.1) and $r_d\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(d,\mathbb{C})$ denotes the irreducible representation (0.2) (see chapter 2).

Recall that a property of an irreducible algebraic variety $Y$ is said to *be true generically* if it holds except on a proper Zariski-closed subset of $Y$, in other words, if it holds on a non-empty Zariski-open subset (see Definition 1.6).

Let $B_d \subset \mathrm{SL}(d,\mathbb{C})$ denote the Borel subgroup of upper triangular matrices. Let $K \subset \mathbb{S}^3$ be a knot, $\lambda^2 \in \mathbb{C}$ is a simple root of $\Delta_K(t)$ and let $z \in Z^1(\Gamma_K, \mathbb{C}_{\lambda^2})$ be a cocycle representing a generator of $H^1(\Gamma_K, \mathbb{C}_{\lambda^2})$. Following [HPS01, Theorem 1.1] the representation $\rho^z_\lambda \in R_2(\Gamma_K)$ is a smooth point of the representation variety. It is contained in a unique irreducible 4−dimensional component $R_\lambda \subset R_2(\Gamma_K)$. It is the limit of non metabelian irreducible representations. Note that generically a representation $\rho \in R_\lambda$ is irreducible.

**Proposition 4.1.** *Let $K \subset \mathbb{S}^3$ be a knot, $\lambda^2 \in \mathbb{C}$ is a simple root of $\Delta_K(t)$ and let $z \in Z^1(\Gamma_K, \mathbb{C}_{\lambda^2})$ be a cocycle representing a generator of $H^1(\Gamma_K, \mathbb{C}_{\lambda^2})$.*

*Then the representation $\rho^z_{\lambda,d} := r_d \circ \rho^z_\lambda \colon \Gamma_K \to B_d$ is the limit of irreducible representations in $R_d(\Gamma_K)$. More precisely, generically a representation $\rho_d := r_d \circ \rho$, $\rho \in R_\lambda$ is irreducible.*

*Proof.* It follows from [HPS01, Theorem 1.1] that $\rho^z_\lambda \in R_2(\Gamma_K)$ is the limit of irreducible representations. Moreover, $\rho^z_\lambda \in R_2(\Gamma_K)$ is a smooth point which is contained in a unique $4-$dimensional component $R_\lambda \subset R_2(\Gamma_K)$.

Let $\rho\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C})$ be an irreducible representation of $R_\lambda \subset R_2(\Gamma_K)$. If the image $\rho(\Gamma_K) \subset \mathrm{SL}(2,\mathbb{C})$ is Zariski-dense then from Lemma 1.3 the representation $\rho_d := r_d \circ \rho \in R_d(\Gamma_K)$ is irreducible. Hence, in order to prove Proposition 4.1 we show that there is a neighborhood $U = U(\rho^z_\lambda) \subset R_2(\Gamma_K)$ such that $\rho(\Gamma_K) \subset \mathrm{SL}(2,\mathbb{C})$ is Zariski-dense for each irreducible $\rho \in U$. Let now $\rho\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C})$ be any irreducible representation of $U$ and let $G \subset \mathrm{SL}(2,\mathbb{C})$ denote the Zariski-closure of $\rho(\Gamma_K)$. Suppose that $G \neq \mathrm{SL}(2,\mathbb{C})$. Since $\rho$ is irreducible it follows that $G$ is, up conjugation, not a subgroup of upper-triangular matrices of $\mathrm{SL}(2,\mathbb{C})$. Then by [Kov86, Sec. 1.4] and [Kap57, Theorem 4.12] there are, up to conjugation, only two cases left:

- $G$ is a subgroup of the infinite dihedral group
$$D_\infty = \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1}\end{smallmatrix}\right) \,\middle|\, \alpha \in \mathbb{C}^* \right\} \cup \left\{ \left(\begin{smallmatrix} 0 & \alpha \\ -\alpha^{-1} & 0\end{smallmatrix}\right) \,\middle|\, \alpha \in \mathbb{C}^* \right\}.$$
- $G$ is one of the groups $A_4^{\mathrm{SL}(2,\mathbb{C})}$ (the tetrahedral group), $S_4^{\mathrm{SL}(2,\mathbb{C})}$ (the octahedral group) or $A_5^{\mathrm{SL}(2,\mathbb{C})}$ (the icosahedral group). These groups are the preimages in $\mathrm{SL}(2,\mathbb{C})$ of the subgroups $A_4$, $S_4$, $A_5 \subset \mathrm{PSL}(2,\mathbb{C})$.

In the first case it follows directly from [Nag07] that if $\rho$ is an irreducible metabelian representation then the trace of the image of a meridian $\mathrm{tr}(\rho(\mu)) = 0$, i.e. $\rho(\mu)$ is similar to $\left(\begin{smallmatrix} i & 0 \\ 0 & -i\end{smallmatrix}\right)$. Now, $\mathrm{tr}(\rho^z_\lambda(\mu)) \neq 0$ since $\Delta_K(-1) \neq 0$ and $\Delta_K(\lambda^{\pm 2}) = 0$. For the second case there are up to conjugation only finitely many irreducible representations of $\Gamma_K$ onto the subgroups $A_4^{\mathrm{SL}(2,\mathbb{C})}$, $S_4^{\mathrm{SL}(2,\mathbb{C})}$ and $A_5^{\mathrm{SL}(2,\mathbb{C})}$. Note that these finitely many orbits are closed and $3-$dimensional. Hence the irreducible $\rho \in R_\lambda$ such that $r_d \circ \rho$ is reducible is contained in a Zariski-closed subset of $R_\lambda$. Hence generically $r_d \circ \rho$ is irreducible for $\rho \in R_\lambda$. □

**Remark 4.1.** *Recall that a finite group has only finitely many irreducible representations (see [Ser78, FH91])). Hence, the restriction of* $r_d$ *to the groups* $A_4^{\mathrm{SL}(2)}$, $S_4^{\mathrm{SL}(2)}$ *and* $A_5^{\mathrm{SL}(2)}$ *is reducible, for all but finitely many* $d \in \mathbb{N}$.

In order to prove that a certain representation $\rho \in R_d(\Gamma_K)$ is a smooth point of the representation variety we will prove that every cocycle $u \in Z^1\big(\Gamma_K; \mathfrak{sl}_d(\mathbb{C})_\rho\big)$ is integrable.

In order to do this, we use the classical approach, i.e. we first solve the corresponding formal problem and apply then a theorem of [Art68] (see Section 3.1).

The following result streamlines the arguments given in [HP05, BHJ10] :

**Proposition 4.2.** *Let $M$ be a connected, compact, orientable 3-manifold such that $\partial M$ is a torus boundary and let $\rho\colon \pi_1(M) \to \mathrm{SL}(d,\mathbb{C})$ be a representation.*

*If $\dim H^1\big(\pi_1(M); \mathfrak{sl}_d(\mathbb{C})_\rho\big) = d-1$ then $\rho$ is a smooth point of the $\mathrm{SL}(d,\mathbb{C})$-representation variety $R_d(\pi_1(M), \mathrm{SL}(d,\mathbb{C}))$. Moreover, $\rho$ is contained in a unique component of dimension $d^2+d-2-\dim H^0\big(\pi_1(M); \mathfrak{sl}_d(\mathbb{C})_\rho\big)$.*

*Proof.* First, we will show that the map $i^\star\colon H^2\big(\pi_1(M); \mathfrak{sl}_d(\mathbb{C})_\rho\big) \to H^2\big(\pi_1(\partial M); \mathfrak{sl}_d(\mathbb{C})_\rho\big)$, induced by the inclusion $\partial M \hookrightarrow M$, is injective.

Recall that for any $CW-$complex $X$ with $\pi_1(X) \cong \pi_1(M)$ and for any $\pi_1(M)-$module $A$ there are natural morphisms $H^i(\pi_1(M); A) \to H^i(X; A)$ which are isomorphisms for $i = 0, 1$ and an injection for $i = 2$ (see Lemma 3.2). Note also that $\partial M \cong \mathbb{S}^1 \times \mathbb{S}^1$ is aspherical and hence $H^i(\pi_1(\partial M); A) \to H^i(\partial M; A)$ is an isomorphism (see Remark 3.2).

First, we will prove that for every representation $\varrho \in R_d(\mathbb{Z}\oplus\mathbb{Z}, \mathrm{SL}(d,\mathbb{C}))$ we have

$$\dim H^0\big(\mathbb{Z}\oplus\mathbb{Z}; \mathfrak{sl}_d(\mathbb{C})_\varrho\big) = \frac{1}{2}\dim H^1\big(\mathbb{Z}\oplus\mathbb{Z}; \mathfrak{sl}_d(\mathbb{C})_\varrho\big) \geq d-1 \tag{4.1}$$

Moreover, we will prove that $\varrho \in R_d(\mathbb{Z}\oplus\mathbb{Z}, \mathrm{SL}(d,\mathbb{C}))$ is regular if and only if equality holds in (4.1). It follows from Lemma 3.4 that for every $\varrho \in R_d(\mathbb{Z}\oplus\mathbb{Z}, \mathrm{SL}(d,\mathbb{C}))$ we have

$$\dim H^0\big(\partial M; \mathfrak{sl}_d(\mathbb{C})_\varrho\big) = \dim H^2\big(\partial M; \mathfrak{sl}_d(\mathbb{C})_\varrho\big),$$

and since the Euler characteristic of $M$ vanishes we obtain the first equality in (4.1)

$$\dim H^1\big(\partial M; \mathfrak{sl}_d(\mathbb{C})_\varrho\big) = 2\dim H^0\big(\partial M; \mathfrak{sl}_d(\mathbb{C})_\varrho\big) = 2\dim \mathfrak{sl}_d(\mathbb{C})_\varrho^{\mathbb{Z}\oplus\mathbb{Z}}.$$

Now, Example 3.2 show that the representation variety $R_d(\mathbb{Z}\oplus\mathbb{Z}, \mathrm{SL}(d,\mathbb{C}))$ is an irreducible algebraic variety of dimension $(d+2)(d-1)$. Hence we obtain for every $\varrho \in R_d(\mathbb{Z}\oplus\mathbb{Z}, \mathrm{SL}(d,\mathbb{C}))$ that

$$\dim Z^1\big(\partial M; \mathfrak{sl}_d(\mathbb{C})_\varrho\big) \geq (d+2)(d-1) = d^2+d-2 \tag{4.2}$$

where the equality holds if and only if $\varrho$ is regular (see Lemma 3.1 ). At the same time, we have

$$\dim Z^1\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big)=\dim H^1\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big)+\dim B^1\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big)$$

and the exactness $0\to H^0\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big)\to\mathfrak{sl}_d(\mathbb{C})\to B^1\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big)\to 0$ gives

$$\dim B^1\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big)=\dim\mathfrak{sl}_d(\mathbb{C})-\dim H^0\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big).$$

This together with $\dim H^1\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big)=2\dim H^0\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big)$ and (4.2) give for all $\varrho\in R_d(\mathbb{Z}\oplus\mathbb{Z},\mathrm{SL}(d,\mathbb{C}))$:

$$\dim Z^1\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big)=\dim H^0\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big)+d^2-1\geq d-1+(d^2-1).$$

It follows that

$$\dim H^0\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big)\geq d-1,\ \text{ for }\ \text{ all } \varrho\in R_d(\mathbb{Z}\oplus\mathbb{Z},\mathrm{SL}(d,\mathbb{C})), \tag{4.3}$$

and $\varrho\in R_d(\mathbb{Z}\oplus\mathbb{Z},\mathrm{SL}(d,\mathbb{C}))$ is regular if and only if $\dim H^0\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\varrho\big)=d-1$ (see Example 3.2).

Now, the exact cohomology sequence of the pair $(M,\partial M)$

$$0\to\partial M\to M\to(M;\partial M)\to 0$$

gives

$$\begin{aligned}&\to H^1\big(M,\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho\big)\\ &\qquad\to H^1\big(M;\mathfrak{sl}_d(\mathbb{C})_\rho\big)\xrightarrow{\alpha}H^1\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho\big)\xrightarrow{\beta}H^2\big(M,\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho\big)\\ &\qquad\quad\xrightarrow{j}H^2\big(M;\mathfrak{sl}_d(\mathbb{C})_\rho\big)\xrightarrow{i^*}H^2(\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho)\to H^3(M,\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho)\to 0\,.\end{aligned}$$

Poincaré-Lefschetz duality implies that $\alpha$ and $\beta$ are dual to each other (see Lemma 3.5 ). This together with 4.3 gives:

$$d-1=\dim H^1\big(M;\mathfrak{sl}_d(\mathbb{C})_\rho\big)\geq\mathrm{rk}(\alpha)=\frac{1}{2}\dim H^1\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho\big) \tag{4.4}$$

$$=\dim H^0\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho\big)\geq d-1. \tag{4.5}$$

Therefore, $\dim H^0\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho\big) = d-1$ holds in (4.1), and consequently $i^\star\rho = \rho\circ i_\sharp \in R_d(\partial M,\mathrm{SL}(d,\mathbb{C}))$ is regular (here $i\colon \partial M \to M$ is the inclusion). Note also that $\beta$ is surjective:

$$d-1 = \dim H^2(M,\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho) = \dim\ker j+\mathrm{rk}(j) = \dim Im(\beta)+\dim\ker(i^\star) = d-1+\dim\ker(i^\star).$$

Hence

$$i^\star\colon H^2\big(M;\mathfrak{sl}_d(\mathbb{C})_\rho\big) \to H^2\big(\partial M;\mathfrak{sl}_d(\mathbb{C})_\rho\big)$$

is injective. The following commutative diagram shows that $i^\star\colon H^2\big(\pi_1(M);\mathfrak{sl}_d(\mathbb{C})_\rho\big) \to H^2\big(\pi_1(\partial M);\mathfrak{sl}_d(\mathbb{C})_\rho\big)$ is also injective:

$$\begin{array}{ccc} H^2\big(M;\mathfrak{sl}_d(\mathbb{C})_\rho\big) & \xrightarrow{i^\star} & H^2\big(\partial(M);\mathfrak{sl}_d(\mathbb{C})_\rho\big) \\ \uparrow & & \uparrow\cong \\ H^2\big(\pi_1(M);\mathfrak{sl}_d(\mathbb{C})_\rho\big) & \xrightarrow{i^\star} & H^2\big(\pi_1(\partial M);\mathfrak{sl}_d(\mathbb{C})_\rho\big)\,. \end{array}$$

In order to prove that $\rho$ is a smooth point of $R_d(\pi_1(M),\mathrm{SL}(d,\mathbb{C}))$, we show that all cocycles in $Z^1\big(\pi_1(M);\mathfrak{sl}_d(\mathbb{C})_\rho\big)$ are integrable. In what follows we will prove that all obstructions vanish, by using the fact that the obstructions vanish on the boundary. Let $u_1,\dots,u_k\colon \pi_1(M)\to\mathfrak{sl}_d(\mathbb{C})$ be given such that

$$\rho_k(\gamma) = \exp\left(\sum_{i=1}^k t^i u_i(\gamma)\right)\rho(\gamma)$$

is a homomorphism modulo $t^{k+1}$. Then the restriction $i^\star\rho_k\colon \pi_1(\partial M)\to \mathrm{SL}(d,\mathbb{C}[\![t]\!])$ is also a formal deformation of order $k$. Since $i^\star\rho$ is a smooth point of the representation variety $R_d(\mathbb{Z}\oplus\mathbb{Z},\mathrm{SL}(d,\mathbb{C}))$, the formal implicit function theorem gives that $i^\star\rho_k$ extends to a formal deformation of order $k+1$ (see Lemma 3.7). Therefore, we have that

$$0 = \zeta_{k+1}^{(i^\star u_1,\dots,i^\star u_k)} = i^\star\zeta_{k+1}^{(u_1,\dots,u_k)}$$

Now, $i^\star$ is injective and the obstruction $\zeta_{k+1}^{(u_1,\dots,u_k)}$ vanishes.

Hence all cocycles in $Z^1\big(\Gamma;\mathfrak{sl}_d(\mathbb{C})_\rho\big)$ are integrable. By applying Artin's theorem [Art68] we obtain from a formal deformation of $\rho$ a convergent deformation (see Proposition 3.2).

Thus $\rho$ is a regular point of the representation variety $R_d(\pi_1(M),\mathrm{SL}(d,\mathbb{C}))$. Hence, $\dim H^1\big(\pi_1(M);\mathfrak{sl}_d(\mathbb{C})_\rho\big) = d-1$ and the exactness of

$$0 \to H^0\big(\pi_1(M);\mathfrak{sl}_d(\mathbb{C})_\rho\big) \to \mathfrak{sl}_d(\mathbb{C})_\rho \to B^1\big(\pi_1(M);\mathfrak{sl}_d(\mathbb{C})_\rho\big) \to 0$$

implies

$$\dim_\rho R_d(\pi_1(M)) = \dim Z^1\big(\pi_1(M);\mathfrak{sl}_d(\mathbb{C})_\rho\big) = d^2+d-2-\dim H^0\big(\pi_1(M);\mathfrak{sl}_d(\mathbb{C})_\rho\big)\,.$$

Finally, the proposition follows from Lemma 3.1. □

## 4.2 Reducible representations of knot group into $\mathrm{SL}(d,\mathbb{C})$

This section is devoted to present the main result and its proof. But before that, we begin by presenting a smoothness result.

**Proposition 4.3.** *Let $K \subset \mathbb{S}^3$ be a knot, let $\lambda \in \mathbb{C}^\star$ and $d \geq 3$. Suppose that $\lambda^2$ is a simple root of the Alexander polynomial $\Delta_K(t)$ and let $\rho^z_\lambda\colon \Gamma_K \to \mathrm{SL}(2,\mathbb{C})$ be the non-abelian representation given by (0.1).*

*If $\Delta_K(\lambda^{2i}) \neq 0$ for $2 \leq i \leq d-1$ then for $\rho^z_{\lambda,d} := r_d \circ \rho^z_\lambda\colon \Gamma_K \to B_d \subset \mathrm{SL}(d,\mathbb{C})$ we have*

$$\dim H^1\big(\Gamma_K;\mathfrak{sl}_d(\mathbb{C})_{\rho^z_{\lambda,d}}\big) = (d-1) \text{ and } H^0\big(\Gamma_K;\mathfrak{sl}_d(\mathbb{C})_{\rho^z_{\lambda,d}}\big) = 0\,.$$

*Proof.* A proof of the cohomological calculation will be given in Section 4.3. □

**Theorem 4.1.** *If $\lambda^2$ is a simple root of $\Delta_K(t)$ and if $\Delta_K(\lambda^{2k}) \neq 0$ for $2 \leq k \leq d-1$ then the reducible metabelian representation $\rho^z_{\lambda,d} := r_d \circ \rho^z_\lambda$ is a limit of irreducible representations. More precisely, $\rho^z_{\lambda,d}$ is a smooth point of $R_d(\Gamma_K,\mathrm{SL}(d,\mathbb{C}))$; it is contained in a unique $(d+2)(d-1)-$dimensional component $R_{\lambda,d} \subset R_d(\Gamma_K,\mathrm{SL}(d,\mathbb{C}))$.*

*Proof.* It follows directly form Propositions 4.2 and 4.3 that $\rho^z_{\lambda,d} := r_d \circ \rho^z_\lambda$ is a smooth point of $R_d(\Gamma_K,\mathrm{SL}(d,\mathbb{C}))$ which is contained in a unique component $R_{\lambda,d} \subset R_d(\Gamma_K,\mathrm{SL}(d,\mathbb{C}))$, $\dim R_{\lambda,d} = d^2+d-2$.

That $\rho^z_{\lambda,d}$ is the limit of irreducible representations which are contained in the component $R_{\lambda,d}$ follows from Proposition 4.1. □

## 4.3 Cohomological calculations

In this section, using the decomposition of an arbitrary rational representation of $\mathrm{SL}(2,\mathbb{C})$ in direct sum of rational irreducible representations and the equivalence between $\mathrm{Ad}\circ r_d$ and $\sum_{i=1}^{d-1} r_{2i+1}$, we make calculations that prove the Proposition 4.3 (see chapter 2).

The Lie algebra $\mathfrak{sl}_d(\mathbb{C})$ of $\mathrm{SL}(d,\mathbb{C})$ turns into an $\mathrm{SL}(2,\mathbb{C})$-module via $\mathrm{Ad}\circ r_d$ where $\mathrm{Ad}\colon SL_d(\mathbb{C})\to \mathrm{Aut}(\mathfrak{sl}_d(\mathbb{C}))$ denotes the adjoint representation and $r_d$ the Representation (0.2). For this action we have from Theorem (2.2)

$$\mathrm{Ad}\circ r_d \cong \bigoplus_{i=1}^{d-1} r_{2i+1}\,.$$

Let $B_d\subset \mathrm{SL}(d,\mathbb{C})$ denote the Borel subgroup of upper triangular matrices. The vector space $R_{d-1}$ turns into a $B_2-$module via restriction of $r_d$ to $B_2$. For $\left(\begin{smallmatrix}\lambda & \lambda^{-1}b\\ 0 & \lambda^{-1}\end{smallmatrix}\right)\in \mathrm{SL}(2,\mathbb{C})$, we have

$$\begin{aligned}
r_d\left(\begin{pmatrix}\lambda & \lambda^{-1}b\\ 0 & \lambda^{-1}\end{pmatrix}\right)e_l^{(d-1)} &= r_d\left(\begin{pmatrix}\lambda & \lambda^{-1}b\\ 0 & \lambda^{-1}\end{pmatrix}\right)X^{l-1}Y^{d-l}\\
&= \left(\lambda^{-1}X-\lambda^{-1}b\,Y\right)^{l-1}(\lambda\,Y)^{d-l}\\
&= \lambda^{d-2l+1}\sum_{j=0}^{l-1}(-b)^j\binom{l-1}{j}X^{l-1-j}Y^{d+j-l}\\
&= \lambda^{d-2l+1}\sum_{j=0}^{l-1}(-b)^j\binom{l-1}{j}e_{l-j}^{(d-1)}\,.
\end{aligned}$$

Then

$$r_d\left(\begin{pmatrix}\lambda & \lambda^{-1}b\\ 0 & \lambda^{-1}\end{pmatrix}\right)e_l^{(d-1)} = \lambda^{d-2l+1}\sum_{j=0}^{l-1}(-b)^j\binom{l-1}{j}e_{l-j}^{(d-1)}\,. \tag{4.6}$$

Hence $r_d(B_2)$ is contained in $B_d\subset \mathrm{SL}(d,(\mathbb{C}))$ and the one-dimensional vector space $\langle e_1^{(d-1)}\rangle$ is $B_2-$invariant since $r_d\left(\begin{smallmatrix}\lambda & \lambda^{-1}b\\ 0 & \lambda^{-1}\end{smallmatrix}\right)e_1^{(d-1)} = \lambda^{d-1}e_1^{(d-1)}$. For a given integer $i\in\mathbb{Z}$ we let $\chi_i\colon B_2\to\mathbb{C}^* = \mathrm{GL}(1,\mathbb{C})$ denote the rational character given by

$$\chi_i\left(\begin{smallmatrix}\lambda & \lambda^{-1}b\\ 0 & \lambda^{-1}\end{smallmatrix}\right) = \lambda^i\,.$$

Now $\mathbb{C}$ turns into a $B_2$-module via $\chi_i$ i.e. $\left(\begin{smallmatrix}\lambda & \lambda^{-1}b\\ 0 & \lambda^{-1}\end{smallmatrix}\right).z = \lambda^i\,z$ for $z\in\mathbb{C}$. We will denote this $B_2$-module by $\mathbb{C}_{\chi_i}$. It follows that the $B_2$-module $\langle e_1^{(d-1)}\rangle\subset R_{d-1}$ is isomorphic to

$\mathbb{C}_{\chi_{d-1}}$ and we obtain a short exact sequence of $B_2$-modules

$$0 \to \mathbb{C}_{\chi_{d-1}} \to R_{d-1} \to \bar{R}_{d-1} \to 1 \tag{4.7}$$

where $\bar{R}_{d-1}$ denotes the quotient $R_{d-1}/\langle e_1^{(d-1)}\rangle$. For a given element $x \in R_{d-1}$ we let $\bar{x} \in \bar{R}_{d-1}$ denote the class represented by $x$ i.e. $\bar{x} = x + \langle e_1^{(d-1)}\rangle$.

**Lemma 4.1.** *The linear map $\phi_{d-3}\colon R_{d-3} \to \bar{R}_{d-1}$ defined by*

$$\phi_{d-3}(e_l^{(d-3)}) = \frac{1}{l}\,\bar{e}_{l+1}^{(d-1)}, \quad l = 1, \cdots, d-2,$$

*is an injective $B_2$-module morphism i.e. for all $x \in R_{d-3}$ we have*

$$r_d\left(\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1}\end{pmatrix}\right)\phi_{d-3}(x) = \phi_{d-3}\left(r_d\left(\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1}\end{pmatrix}\right)x\right).$$

*Proof.* The linear map $\phi_{d-3}$ is injective since the vectors $\bar{e}_l^{(d-1)}$, $2 \le l \le d$, form a basis of $\bar{R}_{d-1}$. Now, using Equation (4.6) we have

$$\begin{aligned}
r_d\left(\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1}\end{pmatrix}\right)\phi_{d-3}(e_l^{(d-3)}) &= \frac{1}{l} r_d\left(\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1}\end{pmatrix}\right)\bar{e}_l^{(d-1)} \\
&= \frac{1}{l}\lambda^{d-2l-2+1}\sum_{j=0}^{l}(-b)^j\binom{l}{j}\bar{e}_{l-j+1}^{(d-1)} \\
&= \lambda^{d-2l-1}\frac{1}{l}\sum_{j=0}^{l}(-b)^j\binom{l}{j}\bar{e}_{l-j+1}^{(d-1)}.
\end{aligned}$$

Since $\binom{l}{j}(l-j) = l\binom{l-1}{j}$ and $\bar{e}_1^{(d-1)} = 0$ it follows

$$\begin{aligned}
r_d\left(\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1}\end{pmatrix}\right)\phi_{d-3}(e_l^{(d-3)}) &= \lambda^{(d-2)-2l+1}\sum_{j=0}^{l-1}(-b)^j\binom{l-1}{j}\frac{1}{l-j}\bar{e}_{l-j+1}^{(d-1)} \\
&= \phi_{d-3}\left(r_d\left(\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1}\end{pmatrix}\right)e_l^{(d-3)}\right).
\end{aligned}$$

Hence $\phi_{d-3}$ is a $B_2$-module morphism. □

**Lemma 4.2.** *There is a short exact sequence of* $B_2$*-modules*

$$0 \to R_{d-3} \xrightarrow{\phi_{d-3}} \bar{R}_{d-1} \to \mathbb{C}_{\chi_{-d+1}} \to 0. \tag{4.8}$$

*Proof.* Again the lemma follows from Equation (4.6):

$$r_d\left(\begin{pmatrix} \lambda & \lambda^{-1}b \\ 0 & \lambda^{-1} \end{pmatrix}\right)e_d^{(d-1)} = \lambda^{-d+1}\sum_{j=0}^{d-1}(-b)^j\binom{d-1}{j}e_{d-j}^{(d-1)} \equiv \lambda^{-d+1}e_d^{(d-1)} \bmod \langle e_1^{(d-1)},\dots,e_{d-1}^{(d-1)}\rangle\,.$$

□

Let us fix a representation $\rho_\lambda^z\colon \Gamma_K \to B_2$. Then $R_d$ turns into a $\Gamma_K$-module and the exact sequences (4.7) and (4.8) are exact sequences of $\Gamma_K$-modules. Note that $\mathbb{C}_{\chi_k} \cong \mathbb{C}_{\lambda^k}$ since for all $\gamma \in \Gamma_K$ and $k \in \mathbb{Z}$ the equation $\chi_k\big(\rho_\lambda^z(\gamma)\big) = \lambda^{k\varphi(\gamma)}$ holds.

**Lemma 4.3.** *Let* $\lambda \in \mathbb{C}^\star$*,* $\lambda \neq 1$*, and* $d \geq 4$ *be given. If* $\Delta_K(\lambda^{d-1}) \neq 0$ *and if* $\lambda^{d-1} \neq 1$ *then*

$$H^*(\Gamma_K; R_{d-1}) \cong H^*(\Gamma_K; R_{d-3})\,.$$

*Proof.* The assertion of the lemma follows form Lemma 3.3 and the long exact cohomology sequences [Bro82, III. § 6] associated to the short exact sequences (4.7) and (4.8).

Indeed, the long exact cohomology sequences associated to the short exact sequences of $\Gamma_K-$modules (4.7) and (4.8), are given by

$$\begin{aligned} 0 \;\to\;& H^0(\Gamma_K;\mathbb{C}_{\lambda^{d-1}}) \to H^0(\Gamma_K;R_{d-1}) \to H^0(\Gamma_K;\bar{R}_{d-1}) \\ \to\;& H^1(\Gamma_K;\mathbb{C}_{\lambda^{d-1}}) \to H^1(\Gamma_K;R_{d-1}) \to H^1(\Gamma_K;\bar{R}_{d-1}) \\ \to\;& H^2(\Gamma_K;\mathbb{C}_{\lambda^{d-1}}) \to H^2(\Gamma_K;R_{d-1}) \to H^2(\Gamma_K;\bar{R}_{d-1}) \\ \to\;& H^3(\Gamma_K;\mathbb{C}_{\lambda^{d-1}}) \to \cdots\,, \end{aligned}$$

and

$$\begin{aligned} 0 \;\to\;& H^0(\Gamma_K;R_{d-3}) \to H^0(\Gamma_K;\bar{R}_{d-1}) \to H^0(\Gamma_K;\mathbb{C}_{\lambda^{-d+1}}) \\ \to\;& H^1(\Gamma_K;R_{d-3}) \to H^1(\Gamma_K;\bar{R}_{d-1}) \to H^1(\Gamma_K;\mathbb{C}_{\lambda^{-d+1}}) \\ \to\;& H^2(\Gamma_K;R_{d-3}) \to H^2(\Gamma_K;\bar{R}_{d-1}) \to H^2(\Gamma_K;\mathbb{C}_{\lambda^{-d+1}}) \\ \to\;& H^3(\Gamma_K;R_{d-1}) \to \cdots\,. \end{aligned}$$

Now $H^0(\Gamma_K;\mathbb{C}_{\lambda^{d-1}}) = 0$ since $\lambda^{d-1} \neq 1$ and $\dim H^1(\Gamma_K;\mathbb{C}_{\lambda^{d-1}}) = \dim H^2(\Gamma_K;\mathbb{C}_{\lambda^{d-1}}) = 0$ since $\Delta_K(\lambda^{d-1}) \neq 0$ (see Lemma 3.3). Hence

$$H^k(\Gamma_K; R_{d-1}) \xrightarrow{\cong} H^k(\Gamma_K; \bar{R}_{d-1}) \quad \text{for } k = 0,1,2. \tag{4.9}$$

Finally, $H^0(\Gamma_K;\mathbb{C}_{\lambda^{-d+1}}) = 0$ since $\lambda^{-d+1} \neq 1$ and $\dim H^1(\Gamma_K;\mathbb{C}_{\lambda^{-d+1}}) = \dim H^2(\Gamma_K;\mathbb{C}_{\lambda^{-d+1}}) = 0$ since $\Delta_K(\lambda^{-d+1}) \neq 0$ (note that $\Delta_K(t)$ is symmetric). Hence

$$H^k(\Gamma_K; R_{d-3}) \xrightarrow{\cong} H^k(\Gamma_K; \bar{R}_{d-1}) \quad \text{for } k = 0,1,2. \tag{4.10}$$

It follows from 4.9 and 4.10 that

$$H^k(\Gamma_K; R_{d-1}) \cong H^k(\Gamma_K; R_{d-3}) \quad \text{for } k = 0,1,2.$$

□

**Proposition 4.4.** *Let* $\lambda \in \mathbb{C}^\star$, $\Delta_K(\lambda^2) = 0$, $d \geq 3$ *and let* $\rho_\lambda^z\colon \Gamma_K \to B_2$ *be given as in* (0.1). *If* $\Delta_K(\lambda^{2i}) \neq 0$ *and* $\lambda^{2i} \neq 1$ *for* $2 \leq i \leq d-1$ *then for* $\rho_{\lambda,d}^z := r_d \circ \rho_\lambda^z\colon \Gamma_K \to B_d \subset \mathrm{SL}(d,\mathbb{C})$ *we have*

$$\dim H^\star(\Gamma_K; \mathfrak{sl}_d(\mathbb{C})_{\rho_{\lambda,d}^z}) = (d-1)\dim H^\star(\Gamma_K; R_2)\,.$$

*Proof.* It follows from (2.18) that we have an isomorphism of $\Gamma_K$-modules:

$$\mathfrak{sl}_d(\mathbb{C})_{\rho_{\lambda,d}^z} \cong \bigoplus_{i=1}^{d-1} R_{2i}\,.$$

Now Lemma 4.3 implies that $\dim H^\star(\Gamma_K, R_{2i}) = \dim H^\star(\Gamma_K, R_2)$ since $\Delta_K(\lambda^{2i}) \neq 0$ and $\lambda^{2i} \neq 1$ for $2 \leq i \leq d-1$. Hence the assertion of the proposition follows. □

*Proof of Proposition 4.3.* Let $\lambda \in \mathbb{C}^\star$ and $d \in \mathbb{Z}$, $d \geq 3$. Suppose that $\lambda^2$ is a simple root of the Alexander polynomial $\Delta_K(t)$ and let $\rho_\lambda^z\colon \Gamma_K \to B_2$ be a non-abelian Representation as in (0.1).

In order to apply Proposition 4.4 we have to show that $\lambda^{2i} \neq 1$ for $2 \leq i \leq d-1$. Suppose that there exists $i \in \mathbb{Z}$, $2 \leq i \leq d-1$, such that $\lambda^{2i} = 1$. Next note that $\lambda^{-2} = \lambda^{2i-2}$ is a root of the Alexander polynomial since $\Delta_K(t)$ is symmetric. Therefore the assumption of the proposition implies that $i = 2$ i.e. $\lambda^4 = 1$ and hence $\lambda^2 = \pm 1$. On the other hand, $\pm 1$ is not a root of $\Delta_K(t)$ since $\Delta_K(1) = \pm 1$ and $\Delta_K(-1)$ is an odd integer. This gives a contradiction and hence $\lambda^{2i} \neq 1$ for $2 \leq i \leq d-1$. Therefore, Proposition 4.4 implies that

$$\dim H^\star(\Gamma_K; \mathfrak{sl}_d(\mathbb{C})_{\rho^z_{\lambda,d}}) = (d-1)\dim H^\star(\Gamma_K; R_2)\,. \tag{4.11}$$

Finally, observe that $\mathfrak{sl}_2(\mathbb{C})_{\rho^z_\lambda} \cong R_2$ (see Example 1.4) and $\dim H^1(\Gamma_K; R_2) = 1$ follows from [HP05, Corollary 5.4] or [HPS01, 4.4]. Then Equality (4.11) gives

$$\dim H^1(\Gamma_K; \mathfrak{sl}_d(\mathbb{C})_{\rho^z_{\lambda,d}}) = (d-1).$$

Since $\rho^z_\lambda$ is a non-abelian representation, $\dim B^1(\Gamma_K; R_2) = 3$ and the exactness of

$$0 \to H^0(\Gamma_K; R_2) \to R_2 \to B^1(\Gamma_K; R_2) \to 0$$

implies

$$H^0(\Gamma_K; R_2) = 0.$$

This together with (4.11) gives

$$\dim H^0(\Gamma_K; \mathfrak{sl}_d(\mathbb{C})_{\rho^z_{\lambda,d}}) = 0.$$

□

## 4.4 Examples

Let $K \subset \mathbb{S}^3$ be a knot and $\lambda^2$ a simple root of $\Delta_K(t)$. Theorem 4.1 implies that If $\Delta_K(\lambda^{2k}) \neq 0$ for all $k \in \mathbb{Z}$, $i \neq \pm 1$, then for all $d \geq 2$, $d \in \mathbb{Z}$, the representation space $R_d(\Gamma_K, \mathrm{SL}(d,\mathbb{C}))$ contains a component $R_{\lambda,d}$ of dimension $d^2 + d - 2$. Moreover, Proposition 3.8 of [New78] shows that if component contains an irreducible representation, then generic representations on that component are irreducible.

**Corollary 4.1.** *Let $K \subset \mathbb{S}^3$ be a knot with the Alexander polynomial of the figure-eight knot.*

*Then the representation variety* $R_d(\Gamma_K, \mathrm{SL}(d,\mathbb{C}))$ *contains a* $(d^2+d-2)-$*dimensional component and the irreducible representations form an Zariski-open subset of this component.*

*Proof.* The Alexander polynomial of the figure-eight knot is $\Delta_K(t) = t^2 - 3t + 1$ and its roots are $\lambda^{\pm 2} = 3/2 \pm \sqrt{5}/2$ and no power $\lambda^{\pm 2k}$, $k \neq \pm 1$, is a root of $\Delta_K(t)$. So, Corollary 4.1 follows from Theorem 4.1. $\square$

The situation for the trefoil knot $3_1$ is more complicate since the roots of its Alexander polynomial $\Delta_{3_1}(t) = t^2 - t + 1$ are the primitive $6-$th roots of unity $\lambda^{\pm 2} = e^{\pm i\pi/3}$. Hence $R_d(\Gamma_{3_1}, \mathrm{SL}(d,\mathbb{C}))$ contains a $(d^2+d-2)-$dimensional component $R_{\lambda,d}$ for $d \in \{2,3,4,5\}$ since $e^{\pm i\pi/3}$ is a simple root of $\Delta_{3_1}(t)$ and since $\Delta_{3_1}(e^{\pm ik\pi/3}) \neq 0$ for $k \in \{2,3,4\}$.

Let us study the case $d = 6$: the group $\Gamma_{3_1}$ is free product with amalgamation

$$\Gamma_{3_1} = \langle S, T \mid STS = TST \rangle \cong \left\langle x, y \mid x^2 = y^3 \right\rangle \cong \langle x|-\rangle *_{\langle c|-\rangle} \langle y|-\rangle ,$$

where $x = STS$, $y = TS$, and $c = x^2 = y^3$ generates the center of $\Gamma_{3_1}$. Note that a meridian $\mu$ of $3_1$ is represented by the Wirtinger generator $\mu = S = xy^{-1}$. Let $\rho\colon \Gamma_{3_1} \to \mathrm{SL}(6,\mathbb{C})$ be an irreducible representation. It follows from Schur's lemma that if $\rho$ is irreducible then the generator of the center $x^2 = c = y^3$ has to be mapped into the center of

$$\mathcal{C}_6 := \{\exp(2\pi \frac{k}{6}) I_6 \quad | \quad 1 \leq k \leq 6\} \subset \mathrm{SL}(6,\mathbb{C})$$

of $\mathrm{SL}(6,\mathbb{C})$. Notice that for each element of the center $\mathcal{C}_6$ there are only finitely many square et cube roots up to conjugation in $\mathrm{SL}(6,\mathbb{C})$. This implies that if $R \subset R_6(\Gamma_{3_1}, \mathrm{SL}(d,\mathbb{C}))$ is an irreducible component of the representation variety then the conjugacy classes represented by the elements $\rho(c)$, $\rho(x)$, $\rho(y)$ in $\mathrm{SL}(6,\mathbb{C})$ do not vary with $\rho \in R$. Now let $\lambda = e^{i\pi/6}$ be a primitive $12-$th root of unity. A cohomological non-trivial cocycle $z \in Z^1(\Gamma_{3_1}; \mathbb{C}_{\lambda^2})$ is given by $z(S) = 0$ and $z(T) = 1$. Therefore the representation $\rho_\lambda^z\colon \Gamma_{3_1} \to \mathrm{SL}(2,\mathbb{C})$ is given by

$$\rho_\lambda^z(S) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad \rho_\lambda^z(T) = \begin{pmatrix} \lambda & \lambda^{-1} \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Hence

$$\rho_\lambda^z(x) = \begin{pmatrix} i & \lambda^{-1} \\ 0 & -i \end{pmatrix}, \quad \rho_\lambda^z(y) = \begin{pmatrix} \lambda^2 & \lambda^{-2} \\ 0 & \lambda^{-2} \end{pmatrix} \quad \text{and} \quad \rho_\lambda^z(c) = -I_2.$$

Proposition 4.1 implies that $\rho_{\lambda,6}^z = r_6 \circ \rho_\lambda^z$ is a limit of irreducible representations. Computer supported calculations show that $\dim H^1(\Gamma_{3_1}; R_{10}) = 3$ and Lemma 4.3 implies that $\dim H^1(\Gamma_{3_1}; R_{2k}) = \dim H^1(\Gamma_{3_1}; R_2) = 1$ for $k \in \{2,3,4\}$. Hence Formula (0.3) implies that

$$\dim H^1(\Gamma_{3_1}; \mathfrak{sl}_6(\mathbb{C})_{\rho_{\lambda,6}^z}) = 7 \quad \text{i.e.} \qquad \dim Z^1(\Gamma_{3_1}; \mathfrak{sl}_6(\mathbb{C})_{\rho_{\lambda,6}^z}) = 42.$$

In order to see that $\rho_{\lambda,d}^z$ is contained in a 42−dimensional component of $R_6(\Gamma_{3_1}, \mathrm{SL}(d,\mathbb{C}))$ we proceed as follows: let $A = \rho_{\lambda,6}^z(x)$ and $B = \rho_{\lambda,6}^z(y)$ denote the image of $x$ and $y$ respectively. Notice that the matrices $A$ and $B$ are conjugate to $r_6\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{smallmatrix}\right)$. Hence

$$A \sim \begin{pmatrix} i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix} \quad \text{and} \quad B \sim \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^{-2} \end{pmatrix}.$$

Further note that a choice of eigenspaces $E_A(i)$, $E_A(-i)$, $E_B(-1)$, $E_B(\lambda^2)$, $E_B(\lambda^{-2})$ such that $E_A(i) \oplus E_A(-i) \cong \mathbb{C}^6$ and $E_B(-1) \oplus E_B(\lambda^2) \oplus E_B(\lambda^{-2}) \cong \mathbb{C}^6$ determines a representation $\rho\colon \Gamma_{3_1} \to \mathrm{SL}(6,\mathbb{C})$ completely.

Let $Gr(p,d)$ denote the Grassmannian which parametrize all $p$−dimensional subspaces of $\mathbb{C}^d$. Hence the choice of two elements in $Gr(3,6)$ in generic position determines $A$ and the choice of three elements in $Gr(2,6)$ in generic position determines $B$. The representation will be irreducible if the eigenspaces of $A$ and $B$ are in general position and reducible if not.

It is well known that $\dim Gr(p,d) = p(d-p)$ and hence

$$\dim \big(Gr(3,6) \times Gr(3,6)\big) = 18 \quad \text{and} \quad \dim \big(Gr(2,6) \times Gr(2,6) \times Gr(2,6)\big) = 24.$$

Therefore, we construct a $42-$dimensional component of representations $C \subset R_6(\Gamma_{3_1})$ which also contains $\rho^z_{\lambda,6} = r_6 \circ \rho^z_\lambda$ and which contains irreducible representations. Note that $6^2 + 6 - 2 = 40 < 42$. In conclusion we have:

**Corollary 4.2.** *The representation variety* $R_6(\Gamma_{3_1})$ *contains a* $42-$*dimensional component* $C$*. The generic representation of* $C$ *is irreducible and* $\rho^z_{\lambda,6} \in C \subset R_6(\Gamma_{3_1})$ *is a smooth point.*

*Proof.* Computer supported calculations give that $\dim Z^1(\Gamma_{3_1}; \mathfrak{sl}_6(\mathbb{C})_{\rho^z_{\lambda,6}}) = 42$. Additionally, we constructed a $42-$dimensional component $C$ containing $\rho^z_{\lambda,6}$. Now, the assertion follows from Lemma 3.1. □

# Conclusion et perspectives

À tout nœud $K \subset \mathbb{S}^3$ on peut associer le groupe fondamental $\Gamma_K = \pi_1(\mathbb{S}^3/K)$. Ce dernier est un invariant puissant, il a permit de classifier tous les nœuds toriques. Toutefois, l'étude directe des groupes de nœuds est en général difficile. Des mathématiciens tels que A. Casson et W. Thurston ont inventé, dans les années 1980, une méthode permettant de contourner cette difficulté en examinant l'image homomorphe des groupes de nœuds dans un espace dont la géométrie est mieux connue comme par exemple $SU(2)$. Ceci permet de mieux apprécier la structure globale du groupe d'un nœud.

Pour certains invariants comme le polynôme d'Alexander, la relation avec le groupe fondamental est bien comprise et il est possible de déduire le polynôme d'Alexander, $\Delta_K(t)$, du groupe fondamental du nœud. Pour mieux comprendre les groupes de nœuds, on étudit le lien entre l'existence et la régularité de représentations des groupes de nœuds et les racines du polynôme d'Alexander.

G. Burde et G. de Rham ont montré dans [Bur67] et [deR67] qu'il existe une représentation non abélienne et métabélienne $\rho_\lambda^z$ des groupes de nœuds si et seulement si $\lambda^2$ est une racine du polynôme d'Alexander $\Delta_K(t)$. Cette étude a été suivit par celle de Heusener, Porti et Suàrez dans [HPS01], où les auteurs montrent que cette dernière est un point lisse de la variété des représentations. Le but de notre travail est de généraliser les résultats de [HPS01], concernant $\mathrm{SL}(2,\mathbb{C})$ à $\mathrm{SL}(d,\mathbb{C})$. Plus précisément, nous nous sommes intéressés à l'étude de certaines représentations non-abéliennes, métabéliennes et réductibles des groupes de nœuds dans le groupe des matrices triangulaires supérieures $\mathrm{SL}(d,\mathbb{C})$.

Dans la proposition 4.1, nous avons montré que si $K \subset \mathbb{S}^3$ est un nœud et si $\lambda^2$ est une racine simple du polynôme d'Alexander, la représentation en question est une limite de représentations non-métabéliennes et irréductibles.

Les questions qui se posent:

Dans le cas des racines multiples, a t-on de résultats similaires et quelles sont les hypothèses suffisantes?

Qu'en est il dans le cas des entrelacs?

Nous avons montré dans le théorème 4.1 que si $K \subset \mathbb{S}^3$ est un nœud et si $\lambda^2$ est une racine simple du polynôme d'Alexander telle que $\Delta_K(\lambda^{\pm 2k}) \neq 0$, $k \in \mathbb{Z}/\{\pm 1, 0\}$, la représentation en question est un point lisse de la variété des représentations. Qu'en est il si la multiplicité de $\lambda^2$ est supérieure à 1? La condition sur la racine du polynôme d'Alexander: $\lambda^2$ n'est pas racine de l'unité, est crucial dans notre étude de la régularité de la représentation en question. Que se passe t-il si la racine du polynôme d'Alexander est racine de l'unité?

Tout au long de cette thèse, notre étude concerne les nœuds dans $\mathbb{S}^3$. Les résultats qu'on vient de montrer se généralisent-ils aux nœuds dans les sphères d'homologie rationnelle de dimension 3?

La même question se pose pour les nœuds de dimensions supérieures. Quelles conditions supplémentaires faut-il imposer à ces nœuds pour avoir des résultats analogues aux résultats du Théorème 4.1?

Soit $\rho_{\lambda,d}\colon \Gamma_K \to \mathrm{SL}(d, \mathbb{C})$ la représentation diagonale donnée par $\rho_{\lambda,d} = r_d \circ \rho_\lambda$. Le groupe $\mathrm{SL}(d, \mathbb{C})$ agit sur la variété des représentations $R_d(\Gamma_K, \mathrm{SL}(d, \mathbb{C}))$ par conjugation, et l'orbite $O(\rho_{\lambda,d})$ est contenue dans la clôture $\overline{\mathcal{O}(\rho^z_{\lambda,n})}$ . D'où, $\rho_{\lambda,d}$ et $\rho^z_{\lambda,d}$ se projettent sur le même point $\chi_{\lambda,d}$ de la variété des caracteres

$$X\Big(\Gamma_K, \mathrm{SL}(d, \mathbb{C})\Big) = R\Big(\Gamma_K, \mathrm{SL}(d, \mathbb{C})\Big) /\!\!/ \, \mathrm{SL}(d, \mathbb{C})\,.$$

Ici, $R\Big(\Gamma_K, \mathrm{SL}(d, \mathbb{C})\Big) /\!\!/ \, \mathrm{SL}(d, \mathbb{C})$ dénote le quotient GIT de l'action (voir [New78] pour plus de détails). Rappelons que le quotient GIT paramétrise les orbites fermées sous l'action de $\mathrm{SL}(d, \mathbb{C})$.

Il est possible d'étudier l'image locale de la variété des caracteres en $\chi_{\lambda,d}$ comme dans [HPS01] et [HP05]. Malheureusement, il y a des difficultés techniques supplémentaires, et les calculs nécessaires sont beaucoup plus complexes.

Ces complications sont due au fait que la représentation diagonale $\rho_{\lambda,d}$ est contenue dans $2^{d-1}$ composantes de $R(\Gamma_K, \mathrm{SL}(d, \mathbb{C}))$. Néanmoins, seule la composante $R_{\lambda,d}$ contient des représentations irréductibles. Nous aborderons ce sujet dans un prochain article.

# Bibliography


[Art68] M. Artin, On the solutions of analytic equations, *Invent. Math.,* 5:277-291, 1968.

[Ben00] L. Ben Abdelghani, Espace des représentations du groupe d'un noeud dans un groupe de Lie, *Ann. Inst. Fourier,* 4(50), 1297–1321, 2000.

[Ben98] L. Ben Abdelghani, Espace des représentations du groupe d'un noeud classique dans un groupe de Lie, *Ph. D. Thesis.*, Université de Bourgogne, 1998.

[BF08] H. U. Boden and S. Frield, Metabelian $\mathrm{SL}(n,\mathbb{C})$ representations of knot groups, *Pacific J. Math.*, 238(1): 7–25, 2008.

[BF11] H. U. Boden and S. Frield, Metabelian $\mathrm{SL}(n,\mathbb{C})$ representations of knot groups II: fixed points, *Pacific J. Math.*, 249(1): 1–10, 2011.

[BF13] H. U. Boden and S. Friedl, *Metabelian* $\mathrm{SL}(n,\mathbb{C})$ *representations of knot groups III: deformations* Q. J. Math., 2013.

[BF14] H. U. Boden and S. Frield, Metabelian $\mathrm{SL}(n,\mathbb{C})$ representations of knot groups IV: twisted Alexander polynomials, *Math. Proc. Cambridge Philos. Soc.* , 156(1), 81-97, 2014.

[BHJ10] L. Ben Abdelghani, M. Heusener and H. Jebali, Deformations of metalelian representations of knot groups into $SL_3(\mathbb{C})$, *J. Knot Theory Ramifications,* 3(19), 385–404, 2010.

[Bir74] J. S. Birman, *Braids, links, and mapping class groups,* Annals of mathematics studies, Princeton university press, 1974.

[BL02] L. Ben Abdelghani, D. Lines, Involutions on knot groups and varieties of representations in a Lie group, *J. Knot Theory Ramifications,* 1(11), 81–104, 2002.

[Bla57] R. C Blanchfield, Intersection theory of manifolds with operators with applications to knot theory, *Ann. of Math.* 65(2), 340–356, 1957.

[Bro82] K. S. Brown, *Cohomology of Groups,* Springer-Verlag, New York, 1982.

[Bur67] G. Burde, Darstellungen von Knotengruppen, *Math. Ann.* 173: 24–33, 1967.

[BZ85] G. Burde and H. Zieschang, *Knots,* Berlin: Walter de Gruyter, 1985.

[BZH13] G. Burde, H. Zieschang, and M. Heusener, *Knots,* Berlin: Walter de Gruyter, 3rd fully revised and extended edition, 2013.

[Cas85] A. Casson, *Lectures at MSRI,* 1985.

[CS83] M. Culler and P. B. Shalen, Varieties of groups representations of $3-$manifolds groups, *Ann. Math.* 117(1): 109–146, 1983.

[Cur94] C. L. Curtis, Generalized Casson invariants for SO(3), U(2), Spin(4), and SO(4), *Trans. Amer. Math. Soc.* 343(1): 49–86, 1994.

[deR67] G. de Rham, Introduction aux polynômes d'un noeud, *Enseign. Math.* 2(13): 187–194, 1967.

[DK01] J. F. Davis and P. Kirk, *Lecture notes in algebraic topology,* Graduate Studies in Mathematics. American Mathematical Society, Procidence, RI, 35, 2001.

[FK91] C. D. Frohman and E.P. Klassen, Deforming representations of knot groups in $SU(2)$, *Comment. Math. Helv*, 66, 340–361, 1991.

[FH91] W. Fulton and J. Harris, *Representation theory,* Springer-Verlag New York, 1991.

[FP12] P. M. Ferrer and J. Porti, Twisted cohomology for hyperbolic three manifolds, *Osaka J. Math.*, 49(3):741–769, 2012.

[Fri04] S. Friedl, Eta invariants sliceness obstructions and their relation to Casson-Gordon invariants, *Algebr. Geom. Topol.*, 4: 893–934 (electronic), 2004.

[Gob09] T. Gobet, Vers le théorème de Serre sur la semi-simplicité des produits tensoriels de la représentation des groupes, *Institut de géomètrie, Algèbre et topologie*, 2009.

[Gol84] W. M. Goldman, The symplectic nature of fundamental groups of surfaces, *Adv. in Math.*, 54(2):200–225, 1984.

[Gor78] C. M. Gordon, Some aspects of classical knot theory, *Lecture Notes in Mathematics*, volume 685, 1–60, DOI, 1978.

[Gre67] W. H. Greub, *Multilinear Algebra,* Springer-Verlag Berlin Heidelberg, 1967.

[Har79] R. Hartley, Metabelian representations of knot groups, *Pacific. J. Math.* 82(1) :93–104, 1979.

[Her97] C. M. Herald, Existence of irreducible representations for knot complement with non constant equivariant signature, *Ann. Math.* 1(309) 21–35, 1997.

[HKL08] C. Herald, P. Kirk, and C. Livingston Metabelian representations, twisted Alexander polynomials, knot slicing and mutation, *arXiv:* 0804. 1355.

[HK98] M. Heusener and J. Kroll, Deforming abelian $SU(2)-$representations of knot groups, *Comment. Math. Helv.* 73, 480–498, 1998.

[Hod86] C. D. Hodgson, Degeneration and regeneration of geometry structures on three-manifolds, *Ph. D. Thesis.*, Princeton University, 1986.

[HP05] M. Heusener and J. Porti, Deformations of reducible representations of $3-$manifolds groups into $PSL_2(\mathbb{C})$, *Algebr. Geom. Topol.*, 5 :965–997, 2005.

[HP11] M. Heusener and J. Porti, Infinitesimal projective rigidity under Dhen filling, *Geom. Topol.*, 15 :2017–2071, 2011.

[HPS01] M. Heusener, J. Porti and E. Suàrez Peirò, Deformations of reducible representations of $3-$manifolds groups into $SL(2,\mathbb{C})$, *J. Reine Angew. Math.* 530, 191–227, 2001.

[Jac80] William Jaco, Lectures on three-manifold topology, *American Mathematical Society*, Providence, R. I. , 1980.

[Jeb08] H. Jebali, Module d'Alexander et représentations métabéliennes, *Ann. Fac. Sci. Toulouse Math.* 4(17), 751–764, 2008.

[JM87] D. Johnson and J. Millson, *Deformation spaces associated to compact hyperbolic manifolds,* Birkhäuser Boston, 67, 48–106, 1987.

[Kap57] I. Kaplansky, *An introduction to differential algebra,* Actualités Sci. Ind., No. 1251=Publ. Inst. Math.Univ. Nancago, No. 5. Hermann, Paris 1957.

[Kla91] E. P. Klassen, Representations of knot groups in $SU(2)$, *Transactions of the AMS*, 326(2):795–828, August 1991.

[KP96] H. Kraft and C. Procesi, *Classical invariant theory*, Preliminary version, July 1996.

[Kov86] J. J. Kovacic, An algorithm for solving second order linear homogeneous differential equations, *J. Symbolic Comput.* , 2(1):3–43, 1986.

[Kos06] Y. Kosmann-Schwarzbach, *Groupes et symétries.* Éddition de l'école polytechnique, Deuxième édition, 2006.

[Lan71] S. Lang, *Algebra*, Addition-Wesley Publishing Company, 1971

[Lic97] W.B. Lichorish, *An Introduction to Knot Theory.* Springer-Verlag, New York, 1997.

[LM85] A. Lubotzky and A. R. Magid, Varieties of representations of finitely generated groups, *Mem. Amer. Math. Soc.*, 58(336):xi+117, 1985.

[Mum76] D.Mumford, *Algebraic geometry,* Springer-Verlag, Berlin, Complex projective varieties, reprint the 1976 edition.

[Nag07] F. Nagasato, Finiteness of a section of the $SL(2,\mathbb{C})-$character variety of the knot group, *Kobe J. Math.*, 24(2):125–136, 2007.

[New78] P. E. Newstead, Introduction to moduli problems and orbit spaces, Tata Institute of fundamental Research Lectures on Mathematics and Physics. volume 51, Bombay 1978.

[Pop08] V. L. Popov, Irregular and singular loci of commuting varieties, *Transform.Groups.*, 13(3-4):819–837, 2008.

[Por97] J. Porti, Torsion de Reidemeister pour les variétés hyperboliques. *Mem. Amer. Math. Soc.*, 128(612) :x+139, 1997.

[Rei88] M. Reid, *Undergraduate algebraic geometry,* London Mathematical Society Student Texts, Cambridge University Press, 1988.

[Ric79] R. W. Richardson, Commuting varieties of semisimple Lie algebra and algebraic groups, *Compos. Math.*, 38:311–327, 1979.

[Rol90] D. Rolfsen, *Knots and links.* Mathematics lecture note, series N 7. Publish or Perish Inc, 1990.

[Ser78] J. P. Serre, *Représentations linéaires des groupes finis,* Hermann, Paris,1978.

[Sha77] I. R. Shafarevich, *Basic Algebraic Geometry,* Springer Verlag, Berlin, study edition,1977. Translated from the Russian by K. A. Hirsch, revised printing of Grundlehren der mathematischen, Vol. 213, 1974.

[Sho91] D. J. Shors, Deforming reductible representations of knot groups in $SL(2,\mathbb{C})$, *Ph.D. Thesis*,U.C.L.A, 1991.

[Spr77] T. A. Springer, *Invariant Theory,* Lecture Notes in Mathematics, Vol. 585. Springer-Verlag, Berlin, 1977.

[Ste74] R. Steinberg, Conjugacy Classes in Algebraic Groups, *Lecture Notes in Mathematics*. Vol 366. 1974.

[Wei64] A. Weil, Remarks on the cohomology of groups, *Ann. of Math.*, (2), 80:149–157, 1964.

[Whi78] G. W. Whitehead, *Elements of homotopy theory,* Graduate Texts in Mathematics. Springer-Verlag, New York, 61, 1978.