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1. Abdelhamid Benmezai and Salima Mechrouk, Positive solution to the nonlinear abstract Hammerstein equation and applications to $\phi-$Laplacian bvp, **Nonlinear Differential Equations and Applications** $20(2013), 489510$.

2. Abdelhamid Benmezai and Salima Mechrouk and Sotiris Ntouyas, Global curve of positive solutions for $\varphi-$Laplacian Dirichlet bvp with at most one turning point, **Differential Equations and Applications**, Volume 5, Number $1(2013), 3352$.

3. Abdelhamid Benmezai and Salima Mechrouk and Sotiris Ntouyas, Existence of positive unbounded solutions for $\phi-$Laplacian BVPs on the half line, **Submitted.**

Journal web sites

1. Nonlinear Differential Equations and Applications

http ://www.Link.springer.com

2. Differential Equations and Applications

http ://dea.ele-math.com

Introduction

Le but de cette thèse est d'étudier l'existence et la multiplicité de solutions positives pour trois problèmes aux limites associés a l'opérateur ϕ-Laplacien qui a été introduit par Garcia [21] posé sur des intervalles bornés ou non-bornés de la demi-droite réelle positive.

Le ϕ-Laplacien généralise le p-Laplacien, $\phi_p(s) = |s|^{p-2}\, s, p > 1$; ce dernier joue un role important dans la dynamique des fluide, en chimie et en physique (voir [3], [16], [30] [33] et [23]) ; ainsi, d'un point de vue physique les solutions positives correspondent à une température, densité...,etc qui sont utiles dans les lois de la physique.

Cette thèse est organisée comme suit, dans le premier chapitre, nous présentons l'historique et l'origine physique de ce type de problèmes, ainsi qu'une revue sur les problèmes aux limites associés aux opérateurs ϕ et p-Laplacien qui sont classiques ou récents.

Les méthodes utilisées reposent sur l'indice du point fixe sur les c*ô*nes dans des espaces de Banach, le théorème du point fixe de Krasnosel'skii et des arguments de la théorie de point fixe.

Dans le deuxième chapitre, nous montrons l'existence de solutions positives pour le problème aux limites :

$$\begin{cases} -\left(\phi\left(u'\right)\right)'(t) = a\left(t\right) f(t, u(t)), \quad t \in (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

où $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ désigne un homéomorphisme croissant et impair avec $\phi(0) = 0,\ \mathbb{R}^+ = [0, +\infty)$, $a : [0,1] \longrightarrow \mathbb{R}^+$ est une fonction continue ne s'annule pas identiquement sur chaque sous-intervalle de $[0,1]$ et $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ est une fonction continue, de plus

$$\exists \alpha, \beta \in \mathbb{R} \text{ avec } 0 < \alpha < \beta \text{ de sorte que}$$
$$t^\beta \phi(x) \leq \phi(tx) \leq t^\alpha \phi(x) \text{ pour tout } x \geq 0 \text{ et } t \in (0,1)\,.$$

L'existence de solutions positive du problème (1) a déja été traité par les théorèmes du point fixe de Leggett Williams [24] ; dans le cas ou la fonction f est de Carathéodory le problème a été récemment étudié dans plusieurs travaux ([6] [7],[21]) et dans le cas où la fonction f est de Carathéodory et dépend de la dérivée, Bachouche et al. [4] se sont interessé aux questions d'existence de solutions positives en utilisant le théorème de Krasnosel'skii et le fait que ϕ est sous-multiplicative, i.e.

$$\forall \alpha, \beta \in \mathbb{R}^+, \phi(\alpha\beta) \leq \phi(\alpha)\phi(\beta).$$

Pour améliorer le travail obtenu dans [21] et pour généraliser le travail de Benmezai [9] et en suivant le même principe utilisé dans les travaux de [9], [7] et [17]-[20], on a présenté le problème (1) sous forme d'une équation d'Hammerstein (voir [34] pour ce genre d'équations)

$$NFu = u \tag{2}$$

où $N : E \to E$ est un opérateur non linéaire complètement continu positif et croissant, $F : C \to C$ est un opérateur continu positif, borné et C un cône d'ordre d'un espace de Banach E, en utilisant dans les démonstrations les théorèmes du point fixe d'excision et de compression d'un cône. Plus précisément, on s'intéresse aux résultats d'existence et de non-existence pour l'equation (2) dans un cône C avec des conditions faites sur F et N et enfin de prendre le problème (1) comme exemple d'application.
Dans ce cadre, on a introduit deux constantes λ_C^N et θ_C^N qui jouent un rôle

important pour obtenir des résultats d'existence et de non-existence.

On a montré aussi que pour $N^{-1}(0) \cap C = \{0\}$, ces deux constantes sont égales et vérifient $\lambda_N = \lambda_C^N = \theta_C^N$, λ_N est l'unique valeur propre positive de l'opérateur N.

On a associé deux classes de non-linearité de F; dans la première, l'équation (2) n'admet pas de solutions dans P avec $N(C) \subset P \subset C$. Plus précisément, si la nonlinearité de l'opérateur F est située en dessue de la linearité de $\lambda_P^{N-1}u$ ou en dessous de la linearité de $\theta_P^{N-1}u$, l'équation d'Hammerstein n'a pas de solutions positives.

La deuxième classe d'équation peut avoir de solutions dans P, selon par exemple le cas quand il existe des constantes positives α, β, R_1, R_2 qui vérifient $\alpha\theta_P^N < 1 < \beta\lambda_P^N$, $R_1 < R_2$ et la fonction F satisfait :

$$\begin{cases} Fu \leq \alpha u \text{ pour tout } u \in K \cap \partial B(0, R_1) \\ Fu \geq \beta u \text{ pour tout } u \in K \cap \partial B(0, R_2). \end{cases}$$

On a un autre résultat d'existence pour l'équation (2) si l'opérateur N est borné inférieurement dans C (un cône normal avec une constante n) et la fonction F satisfait :

$$Fu \geq \beta u \text{ pour tout } u \in P \cap \partial B(0, r)$$
$$Fu \leq \alpha u + G(u) \text{ pour tout } u \in P.$$

où N_C^+ désigne la constante de la bornitude inférieure de l'opérateur N, r est une constante positive, et $G : C \to C$ est une fonction continue avec $G(u) = \circ(\|u\|)$ à l'infini, r, α, β sont des constantes vérifiant $\beta\lambda_P^N > 1 > \alpha n N_C^+$.

L'orsque l'opérateur N est complètement continu, croissant et positif 1-homogène et la fonction F satisfait pour tout $u \in P^*$:

$$Fu \leq \alpha u + G_1(u)$$
$$\beta u - G_2(u) \leq F(u) \leq \gamma u + G_3(u).$$

avec

$$G_1(u) = \circ\left(\|u\|\right) \text{ en } 0 \text{ et}$$
$$G_i(u) = \circ\left(\|u\|\right) \text{ en } \infty \text{ pour } i = 2, 3$$

où α, β, γ sont des constantes avec $\alpha\theta_P^N < 1 < \beta\lambda_P^N$ et les fonctions $G_i : C \to C$ $i = 1, 2, 3$ sont continues.

On a les mêmes résultats dans le cas où N non nécessairement 1-homogène positif mais on suppose qu'il existe un autre opérateur $A : E \to E$ complètement continu positif 1-homogène et croissant avec $A\left(C\right) \subset P$, borné inférieurement dans P et uniformément continu dans la boule unité de E, tel que

$$Nu = A\left(u\right) + \circ\left(\|u\|\right) \text{ à l'infini}$$

et la fonction F vérifie :

$$Fu \geq \beta u \text{ pour tout } u \in C \cap \partial B(0, r),$$
$$Fu \leq \alpha u + Gu\text{pour tout } u \in C \text{ et}$$
$$\beta\lambda_P^N > 1 > \alpha \max\left(\theta_P^N, \theta_P^A\right),$$

où r, α, β sont des constantes positives, $G : C \to C$ est une fonction continue avec $G\left(u\right) = \circ\left(\|u\|\right)$ à l'infini.

Enfin, on utilise tous ces résultats pour étudier l'existence u la non-existence de solutions pour le problème (1) ; pour cela, on pose :

$$Nh(x) = \int_0^x \psi\left(c\left(h\right) - \int_0^t a(s)\phi\left(h\left(s\right)\right)ds\right)dt, \text{ pour tout } x \in [0, 1]$$

où $c(h)$ est l'unique solution de l'équation

$$\int_0^1 \psi\left(c - \int_0^t a(s)\phi\left(h\left(s\right)\right)ds\right)dt = 0.$$

où C désigne le cône normal de l'espace de Banach $E = C^0([0, 1])$ qui contient les fonctions non-negatives et ψ désigne la fonction réciproque de ϕ.

De plus, $F : C \to C$ est un opérateur continu et borné défini pour $u \in C$ par

$$Fu\,(x) = \psi\,(f(x, u(x))) \text{ pour tout } x \in [0,1]\,.$$

On note :

$$f^0 = \limsup_{u \to 0} \left(\max_{t \in [0,1]} \frac{\psi\,(f(t,u))}{u} \right), \quad f^\infty = \limsup_{u \to +\infty} \left(\max_{t \in [0,1]} \frac{\psi\,(f(t,u))}{u} \right)$$
$$f_0 = \liminf_{u \to 0} \left(\min_{t \in [0,1]} \frac{\psi\,(f(t,u))}{u} \right), \quad f_\infty = \liminf_{u \to +\infty} \left(\min_{t \in [0,1]} \frac{\psi\,(f(t,u))}{u} \right)$$

On présente un résultat d'existence pour le probléme (1) lorsque

$$f^\infty \theta_P^N < 1 < f_0 \lambda_P^N$$

et un résultat de non-existence dans le cas suivant :

$$f^0 < (\lambda_N)^{-1} < f_\infty \leq f^\infty < \infty$$

où ϕ est homogène et la fonction f est telle que :

$$f(t,u) < (\lambda_N)^{-1} u, \ \forall t \in [0,1] \ \ \forall u > 0$$

Dans le chapitre 3, on s'inspire des articles [8] [3], [10] et [31] et on s' intéresse à l'étude de l'existence et la multiplicité de solutions positives pour le problème aux limites :

$$-(\varphi(u'(x)))' = \lambda f(u(x)), x \in (0,1), \tag{3}$$

$$u(0) = u(1) = 0, \tag{4}$$

où $\lambda > 0$ est un paramètre réel, φ est un homeomorphisme croissant impair de $\mathbb{R}$ et $f : \mathbb{R}^+ \to \mathbb{R}^+$ est continue où $\mathbb{R}^+ = [0, +\infty)$ et vérifie :

$$f(u) > 0 \text{ pour tout } u > 0.$$

Dans [12], les auteurs ont traité ce problème dans le cas où la fonction f dépende aussi de x, ils ont utilisé la méthode de sous et sur solution.

La solution du problème (3)-(4) est un coule $(\lambda, u) \in (0, +\infty) \times C^1([0,1])$ où $u \geq 0$ dans $(0,1)$, $u(x_0) > 0$ pour $x_0 \in (0,1)$, et (λ, u) vérifie (3)-(4).

Comme le problème est autonome, alors l'idée dans ce chapitre est d'utiliser la méthode de quadrature ; cette méthode a été déja utilisée dans plusieurs papiers, voir par exemple [14] et [28] dans le cas des équations différentielles du second ordre ; elle est utilisée aussi dans [1], [9] [13] et dans [3] pour des problèmes aux limites associés à l'opérateur du p-Laplacien. On se ramène donc au calcul de $T(\lambda, \rho)$ associé au problème initial

$$\begin{cases} -(\varphi(u'))' = \lambda f(u), \\ u(1/2) = \rho, \quad u'(1/2) = 0. \end{cases} \tag{5}$$

La solution positive du problème (3)-(4) est aussi solution pour le problème (5) avec $T(\lambda, \rho) = 1/2$ et elle appartient à l'ensemble :

$$A^+ = \left\{ u \in C^1([0,1]) : u > 0 \text{ dans } (0,1) \text{ et } u \text{ est symétrique par rapport à } \frac{1}{2} \right\};$$

et dans ce cas on a un résultat d'équivalence entre les deux problèmes.

De plus, l'application T vérifie

$$\lim_{\lambda \to 0} T(\lambda, \rho) = +\infty \; et \; \lim_{\lambda \to +\infty} T(\lambda, \rho) = 0,$$

et l'ensemble de solutions positives forme une courbe contin$\hat{u}$ment différentiable $\rho \to \lambda(\rho)$ definie sur $(0, +\infty)$ tel que

$$f \in C^1\left(\mathbb{R}^+\right) \text{ et } \varphi, \psi \in C^1(\mathbb{R}).$$

On a supposé la régularité imposée sur f, φ et ψ et une addition des hypothèses concernant le comportement du rapport $f(x)/\varphi(x)$ en 0 et $+\infty$, pour démontrer qu'on a

$$\lim_{\rho \to 0} \lambda(\rho) = +\infty \text{ et } \lim_{\rho \to +\infty} \lambda(\rho) = +\infty.$$

$$\lim_{\rho\to 0}\lambda(\rho)=0 \text{ et } \lim_{\rho\to+\infty}\lambda(\rho)=0.$$

Par ces arguments et sous les conditions suivantes :

$$\lim_{x\to 0}\frac{f(x)}{x\varphi'(x)}=\lim_{x\to+\infty}\frac{f(x)}{x\varphi'(x)}=0,$$

on conclut qu'il existe $\lambda^+>0$ telle que le probléme (3)-(4) n'admet pas de solutions positives si $\lambda<\lambda^+$; il admet au moins une solution positive si $\lambda=\lambda^+$ et il a au moins deux solutions positives si $\lambda>\lambda^+$. Ce chapitre se termine d'un exemple d'application.

Le chapitre 4 est consacré à l'étude d'existence de solutions non-bornées positives pour le problème :

$$\begin{cases} -(\phi(u'))'=a(t)f(t,u), \quad t\in(0,+\infty), \\ u(0)=0,\ \lim\limits_{t\to+\infty}u'(t)=0 \end{cases} \tag{6}$$

où $\phi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ est un homéomorphisme croissant avec $\phi(0)=0,\ \mathbb{R}^+=[0,+\infty)$, $a:(0,+\infty)\longrightarrow\mathbb{R}^+$ est une fonction mesurable et $f:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ est une fonction continue.

Toute solution positive du problème (6) est une fonction $u\in C^1(\mathbb{R}^+,\mathbb{R}^+)$ avec $\phi(u')\in C^1((0,+\infty),\mathbb{R})$ et $u(t_0)>0$ pour certain $t_0>0$, vérifiant (6).

On suppose les conditions suivantes :

$$\begin{cases} \text{Il existe } m\in C((0,+\infty),\mathbb{R}^+)\ et\ g\in C(\mathbb{R}^+,\mathbb{R}^+) \\ \text{telles que } f(t,(1+t)w)\le m(t)g(w) \text{ pour tout } t,w\in\mathbb{R}^+ \\ \text{et } \displaystyle\int_0^{+\infty}a(t)m(t)dt<\infty, \end{cases}$$

$$\left\{ \lim_{t\to+\infty}\frac{1}{1+t}\int_0^t\psi\left(\int_s^{+\infty}a(\tau)m(\tau)d\tau\right)ds=0, \right.$$

$$\begin{cases} \lim\limits_{t\to+\infty} t\psi\left(\displaystyle\int_t^{+\infty} a(t)f(t,\lambda)\,dt\right) = +\infty \text{ uniformément par rapport à } \lambda \\ \textit{dans} \text{ un interval compact de } (0,+\infty), \end{cases}$$

et

$$\begin{cases} \text{Il existe } \alpha > 0 \text{ de sorte que pour tout } t \in [0,1] \text{ et } u \in \mathbb{R}^+, \\ \phi(tu) \geq t^\alpha \phi(u), \end{cases}$$

où ψ désigne la fonction réciproque de la fonction ϕ.

Le problème (6) avec $\phi = Id$ est étudié dans la littérature. Plusieurs auteurs ont traité l'existence et la multiplicité de solutions positives pour ce problème ; voir [13], [22], [25], [26] et [10].

Par example, Saifi et Djebali dans [15] ont étudié le problème (6) en utilisant l'indice du point fixe, oú $q : (0,+\infty) \longrightarrow (0,+\infty)$ est une fonction continue et la fonction $f : \mathbb{R}^+ \times (0,+\infty) \longrightarrow \mathbb{R}^+$ est continue et vérifie $\lim\limits_{u\to 0} f(t,u) = +\infty$ i.e. $f(t,u)$ peut avoir une singularité en $u = 0$. Notre motivation vient du papier [29], dans lequel D. O'Regan et al. ont considéré le Problème (6) avec $\phi(u) = u$ et f peut avoir une singularité au point $u = 0$, et oú ils ont montré l'existence d'une solution appartenant a l'espace E qui contient les fonctions $u \in C(\mathbb{R}^+,\mathbb{R}^+)$ satisfait $\lim\limits_{t\to+\infty} \frac{u(t)}{1+t} = 0$.

Notre but dans ce chapitre est d'étendre ces résultats en considérant l'opérateur ϕ-Laplacien et en démontrant que le problème (6) admet une solution $u \in E$. Enfin, le choix de cet espace est lié aux conditions aux bords : On utilisera des outils dont les théorèmes du point fixe de Krasnosel'skii dans un cône et la théorie de l'indice du point fixe sont appliqués. Le seul inconvénient de cet espace est qu'il ne permet pas d'informer sur de la bornitude de la fonction u. Pour illustrer les résultats d'existence obtenus, on fait des hypothèses supplementaires liées au comportement du taux $f(t,u)/\varphi(u)$ en 0 et $+\infty$. Soit $\theta > 1$

fixé, $I_\theta = [1/\theta, \theta]$, on pose :

$$g^0 = \limsup_{w \longrightarrow 0} \frac{g(w)}{\phi(w)}, \qquad g^\infty = \limsup_{w \rightarrow +\infty} \frac{g(w)}{\phi(w)},$$

$$f_0 = \liminf_{w \longrightarrow 0} \left(\min_{t \in I_\theta} \frac{f(t, (1+t)w)}{\phi(w)} \right), \quad f_\infty = \liminf_{w \longrightarrow +\infty} \left(\min_{t \in I_\theta} \frac{f(t, (1+t)w)}{\phi(w)} \right)$$

$$\Gamma = \left(\int_0^{+\infty} a(r) m(r) dr \right)^{-1}, \qquad \Theta(\theta) = (1+\theta)^{2\alpha} \theta^\alpha \left(\int_{\frac{1}{\theta}}^{\theta} a(r) dr \right)^{-1}.$$

Suivant les notations ci dessus, on montre que le problème (6) admet au moins une solution positive non-bornée dans un des deux cas suivants :

$$g^0 < \Gamma, \quad \Theta(\theta) < f_\infty$$

oú

$$g^\infty < \Gamma, \quad \Theta(\theta) < f_0$$

Ce chapitre est se termine par un exemple d'application.

Table des matières

Chapitre 0

Introduction

The aim of this thesis is to present some existence and multiplicity results for some ϕ-Laplacian boundary value problems associated with two classes of second-order differential equations subject to limit conditions and posed on bounded or unbounded intervals of the real line. The first one is

$$-\left(\phi\left(u'\right)\right)'(t) = a\left(t\right) f(t, u(t))$$

with various boundary conditions ; the second one is

$$-(\phi(u'(t)))' = \lambda f(u(t)),\ 0 < t < 1,$$

with Dirichlet boundary conditions $u(0) = u(1) = 0$.

The ϕ-Laplacian operator generalizes the well-known p-Laplacian operator $\phi_p(s) = |s|^{p-2}\, s, p > 1$, the latter operator has been investigated in the literature and it arises in modeling different physical and natural phenomena, non-linear elasticity, combustion theory and fluid dynamics (see [3], [16], [30] [33] and [23]).

Mathematically, problems related to the p-Laplacian have been studied since early $60's$ (1964) by Serrin [32] and the difficulty is that operator ϕ_p

is linear only for $p = 2$.

This thesis is organized as follows. In the first chapter, we present the history and the origin of ϕ-Laplacian operator and give some recent results of ϕ-Laplacian boundary value problems on the bounded or unbounded intervals of the real line. The methods used to study these problems depend mainly on the topological degree method, the upper and lower solution technique and some methods based on fixed point index theory on cones of Banach space.

In chapter 2, we are concerned with the existence of positive solutions to the following boundary value problem :

$$\begin{cases} -\left(\phi\left(u'\right)\right)'(t) = a\left(t\right)f(t,u(t)), \quad t \in (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

where $f : [0,1]\times\mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $a : [0,1] \to \mathbb{R}^+$ is continuous and does not vanish identically on any sub-interval of $[0,1]$ and ϕ is an odd increasing homeomorphism of $\mathbb{R}$, satisfying :

$$\begin{aligned} &\exists\alpha,\beta \in \mathbb{R} \text{ with } 0 < \alpha < \beta \text{ such that} \\ &t^\beta\phi(x) \le \phi(tx) \le t^\alpha\phi(x), \quad \text{for all } x \ge 0 \text{ and } t \in (0,1)\,. \end{aligned} \tag{2}$$

Existence results of positive solutions to the ϕ-Laplacian bvp (1) with Dirichlet boundary conditions were obtained in [6], [7] via the Krasnosel'skii fixed point Theorem and in [17] by the degree theory. The case where the nonlinearity depends on the first derivative is well investigated in Bachouche et al. [4] for a class of sub-multiplicative nonlinear operator ϕ, that is satisfying :

$$\forall\alpha,\beta \in \mathbb{R}^+, \phi(\alpha\beta) \le \phi(\alpha)\phi(\beta).$$

Two existence results of positive solutions are provided, using the Krasnosel'skii fixed point theorem of a cone in some Banach spaces.

Their results and all those in [6], [10], [21] and [17]-[20], are obtained by mean of a fixed point formulation having the form :

$$NFu = u. \tag{3}$$

Equation (3) is known as the abstract Hammerstein equation (see chapter 7 in [34]). If the nonnegativity is a requirement for a solution of (3) to exist then we are naturally led to use methods of nonlinear analysis in some ordred Banach spaces.

We suppose that $N : E \to E$ is a completely continuous and increasing operator which is not necessarily linear, $F : K \to K$ is a continuous and bounded map, E is a Banach space, and K is a cone in E.

Under appropriate conditions on the operators F and N, we are interested in existence and nonexistence results for solutions to (3) in the cone K using some basic properties of cones expansion and extension together with the fixed point index theory. To this end, we introduce two constants λ_C^N and θ_C^N which play an important role in the statement of the obtained existence and nonexistence results. We also present a situation where the constants λ_C^N and λ_C^N meet the unique positive eigenvalue of N.

Roughly speaking with any cone P such that $N(K) \subset P \subset K$, we will associate two classes of nonlinearities F, the first class consists on nonlinearities for which equation (3) has no solution in any cone P such that $N(K) \subset P \subset K$. More precisely equation (3) with a nonlinearity F lying below the linearity $\left(\theta_P^N\right)^{-1} u$ or above the linearity $\left(\lambda_P^N\right)^{-1} u$, admits no solution in P^*. In other words, equation (3) may have a solution in the cone P if the nonlinearity F crosses at least once the linearity $\left(\theta_P^N\right)^{-1} u$ or the linearity $\left(\lambda_P^N\right)^{-1} u$. The second one is formed by nonlinearities for which equation (3) may have a solution in P;

this is the case where there exist positive real numbers α, β, R_1, R_2 with $\alpha\theta_P^N < 1 < \beta\lambda_P^N$ and $R_1 < R_2$ such that the following situation : either

$$\begin{cases} Fu \leq \alpha u \text{ for all } u \in K \cap \partial B(0, R_1), \text{ or} \\ Fu \geq \beta u \text{ for all } u \in K \cap \partial B(0, R_2) \end{cases} \tag{4}$$

holds true. Another existence results of positive solutions are provided if N is an increasing positively 1-homogeneous completely continuous operator, C is normal and there exist three nonnegative real numbers α, β, γ and continuous functions $G_i : C \to C$ $i = 1, 2, 3$ such that $\alpha\theta_P^N < 1 < \beta\lambda_P^N$ and for all $u \in P^*$

$$Fu \leq \alpha u + G_1(u)$$
$$\beta u - G_2(u) \leq F(u) \leq \gamma u + G_3(u).$$

with

$$\begin{aligned} G_1(u) &= \circ(\|u\|) \text{ at } 0 \text{ and} \\ G_i(u) &= \circ(\|u\|) \text{ at } \infty \text{ for } i = 2, 3. \end{aligned} \tag{5}$$

In case when N is not necessarily positively 1-homogeneous, we assume that $A : E \to E$ is a positively 1-homogeneous completely continuous increasing operator and lower bounded on P, such that $A(C) \subset P$ which is uniformly continuous on the unit ball of E,

$$Nu = A(u) + \circ(\|u\|) \text{ near infinity.}$$

and the function F satisfies the following conditions :

$$\begin{aligned} &Fu \geq \beta u \text{ for all } u \in C \cap \partial B(0, r), \\ &Fu \leq \alpha u + Gu \text{ for all } u \in C \text{ and} \\ &\beta\lambda_P^N > 1 > \alpha \max\left(\theta_P^N, \theta_P^A\right), \end{aligned} \tag{6}$$

where r, α, β are three positive real numbers and $G : C \to C$ is a continuous function with $G(u) = \circ(\|u\|)$ at ∞.

As application, the results obtained are used to prove some existence results for positive solution to ϕ-Laplacian boundary value problems (1).
Consider an operator N defined for all $h \in E$ by :

$$Nh(x) = \int_0^x \psi \left(c\,(h) - \int_0^t a(s)\phi\,(h\,(s))\,ds \right) dt, \text{ for all } x \in [0,1]$$

and $c(h)$ is the unique solution of

$$\int_0^1 \psi \left(c - \int_0^t a(s)\phi\,(h\,(s))\,ds \right) dt = 0.$$

where ψ denotes the inverse function of ϕ, E is the Banach space of all continuous functions defined on $[0,1]$, C is the normal cone of nonnegative functions in E and $F : C \to C$ be the operator defined for $u \in C$ by

$$Fu\,(x) = \psi\,(f(x,u(x))), \text{ for all } x \in [0,1]\,.$$

F is continuous and bounded. Let :

$$\begin{array}{ll} f^0 = \limsup\limits_{u\to 0} \left(\max\limits_{t\in[0,1]} \dfrac{\psi\,(f(t,u))}{u} \right), & f^\infty = \limsup\limits_{u\to +\infty} \left(\max\limits_{t\in[0,1]} \dfrac{\psi\,(f(t,u))}{u} \right) \\ f_0 = \liminf\limits_{u\to 0} \left(\min\limits_{t\in[0,1]} \dfrac{\psi\,(f(t,u))}{u} \right), & f_\infty = \liminf\limits_{u\to +\infty} \left(\min\limits_{t\in[0,1]} \dfrac{\psi\,(f(t,u))}{u} \right). \end{array}$$

One existence result of positive solution is provided when

$$f^\infty \theta_P^N < 1 < f_0 \lambda_P^N$$

and a nonexistence result is obtained in the following case

$$f^\infty < (\lambda_N)^{-1} < f_0 \le f^0 < \infty,$$

where ϕ is homogeneous and f satisfies :

$$f(t,u) < (\lambda_N)^{-1}\,u,\ \forall t \in [0,1]\,, \forall u > 0$$

λ_N is the unique positive eigenvalue of N.

In Chapter 3, in the same spirit as that in papers [3], [8], [10] and [31], we investigate the existence and the exact number of positive solutions to the second order bvp

$$-(\phi(u'(x)))' = \lambda f(u(x)),\ x \in (0,1), \tag{7}$$

$$u(0) = u(1) = 0, \tag{8}$$

where $\lambda > 0$ is a real parameter, ϕ is an odd increasing homeomorphism of $\mathbb{R}$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with

$$f(u) > 0 \ for\ all\ u > 0.$$

In [12], H. Dang and al. studied the problem (7)-(8) where the function f also depends on x; the upper and lower solution technique is employed to prove existence of solutions.

By a positive solution to problem (7)-(8), we mean a pair $(\lambda, u) \in (0, +\infty) \times C^1([0,1])$ such that $u \geq 0$ in $(0,1)$, $u(x_0) > 0$ for some $x_0 \in (0,1)$, and (λ, u) satisfies (7)-(8).

Because of the autonomous character of the problem, the main tool used in this chapter is the time mapping approach. This method have been used in many papers where several classes of problems related to second order differential equation are studied (see, e.g., [14] [1], [9] and [13] for the existence of solutions for semi-linear second order bvps and [3] for the existence of solutions for second order bvps involving the one dimensional p-Laplacian.

This leads us to calculate the time $T(\lambda, \rho)$ required by a solution of the initial value problem (IVP for short)

$$\begin{cases} -(\phi(u'))' = \lambda f(u), \\ u\,(1/2) = \rho, \quad u'\,(1/2) = 0 \end{cases} \tag{9}$$

to reach the value 0, starting from an extremal value ρ. We show that if u is a solution of (9) with $T(\lambda, \rho) = 1/2$, then (λ, u) is a positive solution to (7)-(8). Conversely, if (λ, u) is a positive solution to (7)-(8), then u is a solution to (9) with $\rho = \|u\|$ and $T\,(\lambda, \|u\|) = 1/2$.

Moreover, the function T satisfies for fixed $\rho > 0$

$$\lim_{\lambda \to 0} T\,(\lambda, \rho) = +\infty \textit{ and } \lim_{\lambda \to +\infty} T(\lambda, \rho) = 0.$$

Under regularity conditions on the functions φ and f

$$f \in C^1\left(\mathbb{R}^+\right), \varphi, \psi \in C^1\left(\mathbb{R}\right),$$

By means of the implicit function theorem, we obtain that the set of positive solutions to (7)-(8) reduces to a continuous curve $\lambda : (0, +\infty) \to (0, +\infty)$.
Under additional assumptions on the behavior of the ratio $f(u)/\phi(u)$ at 0 and $+\infty$, we get

$$\lim_{\rho \to 0} \lambda\,(\rho) = +\infty \text{ and } \lim_{\rho \to +\infty} \lambda\,(\rho) = +\infty.$$

or

$$\lim_{\rho \to 0} \lambda\,(\rho) = 0 \text{ and } \lim_{\rho \to +\infty} \lambda\,(\rho) = 0.$$

Using the above results and the following conditions

$$\lim_{x \to 0} \frac{f(x)}{x\varphi'\,(x)} = \lim_{x \to +\infty} \frac{f(x)}{x\varphi'\,(x)} = 0$$

we conclude that there exists $\lambda^+ > 0$ such that : problem (7)-(8) admits no positive solution if $\lambda < \lambda^+$, at least one positive solution if $\lambda = \lambda^+$ and it has at least two positive solutions if $\lambda > \lambda^+$. We end this chapter with an example of application.

In Chapter 4, we are concerned with the existence of positive unbounded solutions to the ϕ-Laplacian boundary value problem

$$\begin{cases} -\left(\phi\left(u'\right)\right)' = a(t)f(t,u), \quad t \in (0,+\infty), \\ u\left(0\right) = 0, \ \lim\limits_{t\to+\infty} u'(t) = 0, \end{cases} \tag{10}$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is an increasing homeomorphism with $\phi(0) = 0$, $a : (0,+\infty) \longrightarrow \mathbb{R}^+$ is a measurable function and $f : \mathbb{R}^+\times\mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function.

By a positive solution to Problem (10) it is meant a function $u \in C^1\left(\mathbb{R}^+, \mathbb{R}^+\right)$ such that $\phi\left(u'\right) \in C^1\left((0,+\infty), \mathbb{R}\right)$ and $u(t_0) > 0$ for some $t_0 > 0$, satisfying (10).

The our main result will be obtained under the following assumptions :

$$\begin{cases} \text{There exist } m \in C\left((0,+\infty), \mathbb{R}^+\right) \text{ and } g \in C\left(\mathbb{R}^+, \mathbb{R}^+\right) \\ \text{ such that } f(t,(1+t)w) \leq m(t)g(w) \text{ for all } t, w \in \mathbb{R}^+ \\ \text{and } \displaystyle\int_0^{+\infty} a(t)m(t)dt < \infty, \end{cases}$$

$$\left\{ \lim_{t\to+\infty} \frac{1}{1+t} \int_0^t \psi\left(\int_s^{+\infty} a(\tau)m(\tau)d\tau\right) ds = 0, \right.$$

$$\begin{cases} \lim\limits_{t\to+\infty} t\psi\left(\displaystyle\int_t^{+\infty} a(t)f\left(t,\lambda\right)dt\right) = +\infty \text{ uniformly for } \lambda \\ \text{in compact intervals of } (0,+\infty), \end{cases}$$

and

$$\begin{cases} \text{there exists } \alpha > 0 \text{ such that for all } t \in [0,1] \text{ and } u \in \mathbb{R}^+, \\ \\ \phi(tu) \geq t^{\alpha}\phi(u), \end{cases}$$

where ψ denotes the inverse function of ϕ.

Problem (10) with $\phi = Id$ has been extensively studied in the literature; we can find in many papers conditions which guarantee existence and multiplicity of bounded positive solutions for problem (10), see ([13], [22], [25], [26] and [10]). In [15] the authors proved existence of positive solution to the boundary value problem (10), where the function $q : (0,+\infty) \longrightarrow (0,+\infty)$ is continuous and the function $f : \mathbb{R}^+ \times (0,+\infty) \longrightarrow \mathbb{R}^+$ is continuous and satisfies $\lim\limits_{u \to 0} f(t,u) = +\infty$. This work is motivated by [29], where D. O'Regan and al. considered Problem (10) with $\phi(u) = u$ and f may be singular at $u = 0$. They obtained some existence and multiplicity results for positive solutions in the functional space of functions $u \in C(\mathbb{R}^+,\mathbb{R}^+)$ satisfying $\lim_{t\to+\infty} u(t)/(1+t) = 0$ endowed with the norm $\|u\| = \sup_{t\in\mathbb{R}^+} \dfrac{|u(t)|}{1+t}$. It is clear that the choice of this space is motivated by the boundary condition in Problem (10), $u'(+\infty) = 0$ and fortunately, this space provides a good framework where the fixed point index theory in Banach spaces can be used. The unique disadvantage of this functional framework is that we know nothing about the boundedness of the obtained positive solutions.

To provide existence results for positive unbounded solutions under additional assumptions on the behavior of the ratio $f(t,u)/\phi(u)$ at 0 and $+\infty$, let $\theta > 1$ be fixed and set $I_\theta = [1/\theta, \theta]$,

$$g^0 = \limsup_{w \longrightarrow 0} \frac{g(w)}{\phi(w)}, \quad g^\infty = \limsup_{w \to +\infty} \frac{g(w)}{\phi(w)},$$

$$f_0 = \liminf_{w \longrightarrow 0} \left(\min_{t \in I_\theta} \frac{f\left(t, (1+t)\, w\right)}{\phi(w)} \right), \quad f_\infty = \liminf_{w \longrightarrow +\infty} \left(\min_{t \in I_\theta} \frac{f\left(t, (1+t)\, w\right)}{\phi(w)} \right)$$

$$\Gamma = \left(\int_0^{+\infty} a(r) m(r) dr \right)^{-1}, \qquad \Theta\left(\theta\right) = \left(1+\theta\right)^{2\alpha} \theta^{\alpha} \left(\int_{\frac{1}{\theta}}^{\theta} a(r) dr \right)^{-1}.$$

Under the conditions

$$g^0 < \Gamma, \quad \Theta\left(\theta\right) < f_\infty \tag{11}$$

and

$$g^\infty < \Gamma, \quad \Theta\left(\theta\right) < f_0, \tag{12}$$

we prove that Problem (10) has at least one unbounded positive solution. The existence result is illustrated by means of an example of application.

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Chapitre 1

Some recent results for ϕ-Laplacian BVPs

Many problems in physics, glaciology and chemistry are governed by boundary value problems with one dimensional p-Laplacian operator, with $\phi_p(s) = |s|^{p-2}\, s, p > 1$, which arises in modeling of different physical and natural phenomena, non linear elasticity, combustion theory, fluid dynamics, non-Newtonian fluid theory, and non-Newtonian filtration (see [6], [20], [22], [25]), the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudo plastics ; if $p = 2$ they are Newtonian fluid.

To solve more practical problems, Garcia and al. [12] generalized the p-Laplacian to an operator called ϕ-Laplacian where ϕ is an increasing homeomorphism.

The p-Laplacian equations have been studied since 1964 by Serrin [19] and the difficulty is that operator ϕ_p is linear only for $p = 2$.

In the last fifteen years, nonlinear boundary value problems containing the operator $(\phi(u'))'$ have been studied with increasing interest by a series of authors (see for example : [13], [8], [9], [17], [18],[21]). We present here a variety of boundary value problems associated with ϕ-Laplacian operator posed on finite intervals with various assumptions on f and the boundary conditions, investigated the case where $\phi = \phi_p$ by means the upper and lower solutions techniques and the fixed point index theory and use the fixed point theorems on cones of Banach spaces.

In [25], Zengji and al. have interested in the existence of the solutions to the following two point boundary value problem :

$$\begin{cases} (\phi_p(u'))' + \lambda f(t,u) = 0 \text{ in } (a,b) \\ \alpha_1 u(a) - \alpha_2 u'(a) = 0, \beta_1 u(b) - \beta_2 u'(b) = 0, \end{cases} \tag{1.1}$$

where λ is a positive parameter and $\phi_p(s) = |s|^{p-2} s, p > 1, a, b \in \mathbb{R}, a < b$.

$(H_1) : f \in C([a,b] \times \mathbb{R}, \mathbb{R})$.

$(H_2)\, \alpha_i \geq 0, \beta_i \geq 0, i = 1,2$ and $\alpha_1 + \alpha_2 > 0, \beta_1 + \beta_2 > 0, \nabla = \alpha_1\beta_1(b-a) + \alpha_1\beta_2 + \alpha_2\beta_1 > 0$.Let

$$M = \left(\frac{\alpha_1}{\alpha_1(b-a) + k\alpha_2}\right)^p + \left(\frac{\beta_1}{\beta_1(b-a) + k\beta_2}\right)^p,$$

where $k > 2$ is a positive constant. Let the real function g defined by

$$g(t,\xi) = \int_0^\xi f(t,u)\,du \quad \text{for all} \quad (t,\xi) \in [a,b] \times \mathbb{R}.$$

The main tool of this work is based on Leggett-William fixed point theorem.

Theorem 1.0.1 *Let c, d, μ, s be given constants with $s < p$, $c < (M(b-a))^{\frac{1}{p}} d$, and assume the function g satisfies the following conditions :*

i) $\quad g(t,\xi) \geq 0 \quad$ *for all* $\quad (t,\xi) \in \left[a, a + \frac{(b-a)}{k}\right] \cup \left[b - \frac{(b-a)}{k}, b\right] \times [0,d]$,

ii) $\quad (b-a) \max\limits_{(t,\xi)\in[a,b]\times[-c,c]} g(t,\xi) < \frac{1}{2}\left(\frac{c}{d}\right)^p \int_{a+\frac{b-a}{k}}^{b-\frac{b-a}{k}} g(t,d)dt,$

$$iii) \quad g(t,\xi) \leq \mu \ (1+|\xi|^s) \quad \textit{for all} \quad (t,\xi) \in [a,b]\times\mathbb{R}.$$

Then, the boundary value problem (1.1) has at least a positive solution.

In [14], Galanis and al. proved the existence of positive solutions for the following three-point singular boundary value problem :

$$\begin{cases} -\left(\phi_p\left(u'\right)\right)' = q(t)f\left(t,u\right) \text{ in } (0,1), \\ u\left(0\right) = g(u'(0)) = 0, \ u\left(1\right) - \beta u(\eta) = 0, \end{cases} \tag{1.2}$$

where, $\phi_p(s) = |s|^{p-1}s, p>1$, $0<\eta<1, 0<\alpha,\beta<1, a_i \geq 0$, and assume the functions f and q satisfy the following conditions :

$(H_1): f \in C\left([0,1]\times(0,+\infty),(0,+\infty)\right)$.

$(H_2): q \in C\left([0,1]\right) \cap L^1[0,1]$ with $q(t)>0$ is non decreasing on $(0,1)$.

$(H_3): \int_0^L \max_{t\in[0,1]} f(t,u)du < \infty$, for any fixed $L>0$.

$(H_4): g \in C(\mathbb{R},\mathbb{R})$ is a nondecreasing with $ug(u)>0, u\neq 0$ and assume that

$$\begin{aligned} f_0 &= \lim_{u\longrightarrow 0} \max_{t\in[0,1]} \frac{f(t,u)}{u} = +\infty \\ g_0 &= \lim_{u\longrightarrow 0} \frac{g^{-1}(u)}{u} = \ell \in \left[0,\frac{1}{2}\right) \end{aligned}$$

Using the nonlinear Alternative of Leray-Schauder, they proved the following result :

Theorem 1.0.2 *Assume* $(H_1)-(H_4)$ *hold and*

$$\sup_{c>0,t\in[0,1]} \frac{c^p}{\int_0^c f(t,u)du} > \frac{p}{p-1}\left[\frac{1}{1-\beta}\int_\eta^1 [q(t)]^{\frac{1}{p}}\,dt + \int_0^\eta [q(t)]^{\frac{1}{p}}\,dt\right]^p.$$

Then, the three boundary value problem (1.2) has at least a positive solution.

In [5], Chunmei Miao and al. have considered the four point singular boundary value problem :

$$\begin{cases} -\left(\phi_p\left(u'\right)\right)' = q(t)f\left(t,u,u'\right) \text{ in } (0,1) \\ u'\left(0\right) - \alpha u(\xi) = 0, u'\left(1\right) + \beta u(\eta) = 0, \end{cases} \tag{1.3}$$

where, $\phi_p(s) = |s|^{p-1} s, p > 1$,

(H_1) $0 < \xi < \eta < 1, 0 < \alpha, \beta, q \in C([0,1]), q(t) > 0, t \in [0,1]$;

(H_2) $f \in C([0,1] \times (0,+\infty) \times \mathbb{R}, (0,+\infty))$ may be singular at $u = 0$.

(H_3) Assume, there are functions f_1, f_2 and h such that :

$$0 < f(t,y,z) \leq h(z)[f_1(y) + f_2(y)] \qquad \text{on} \quad (0,1) \times (0,+\infty) \times \mathbb{R}$$

where f_1 is continuous, positive and non-increasing on $(0,+\infty)$ and such that :

$$\int_0^r f_1(s)\,ds < \infty, \text{ for all } \; r > 0$$

f_2 is continuous, nonnegative and nondecreasing on $[0,+\infty)$ and h is continuous, positive and nondecreasing on $\mathbb{R}$.

(H_4) For given $H > 0$ and $L > 0$, there are a function $\psi_{H,L}$ and a constant $\theta \in [0,1)$ such that $\psi_{H,L}$ is continuous on $[0,1]$, positive on $(0,1)$ and the inequality :

$$f(t,y,z) \geq \psi_{H,L}(t) (\phi_p(|z|))^\theta$$

holds on $[0,1] \times (0,H] \times [-L,L]$;

$(H_5) I_1(x) = \int_0^x \frac{\phi_p^{-1}(u)}{h(\phi_p^{-1}(u))} du < \infty, x > 0$.

Using the Leary-Schauder degree, they proved the following result :

Theorem 1.0.3 *Suppose that $(H_1) - (H_5)$ hold, and*

$$\sup_{0<c<+\infty} \frac{c}{k\phi_p^{-1}(I_1^{-1}(|q|_0 f_2(c)c + |q|_0 \int_0^c f_1(s)ds))} > 1$$

where $k = \max\left\{1 + \frac{1}{\alpha}, 1 + \frac{1}{\beta}\right\}$, then Problem (1.3) has a positive solution u.

In [4], Bo Sun Wang and Weigao studied the existence of positive solutions for two classes of second order three -point p-Laplacian boundary value problem :

$$\begin{cases} -(\phi_p(u'))' = a(t) f(t,u,u') \text{ in } (0,1), \\ u(0) = 0, \; u(1) = u(\eta), \end{cases} \tag{1.4}$$

$$\begin{cases} -\left(\phi_p\left(u'\right)\right)' = a(t)f\left(t,u\right) \text{ in } (0,1)\,, \\ u\left(0\right)=0,\ u(1)=u(\eta), \end{cases} \tag{1.5}$$

They assumed that : $a(t)$ is a nonnegative continuous function defined on $(0,1)$, and $a(t)$ not identically zero on any sub- interval of $(0,1)$, in addition :

$$\int_0^1 a(t)dt < \infty, \int_0^{+\infty} \phi_p^{-1}\left(\int_s^{+\infty} a(t)dt\right) ds < \infty$$

(H) : $\phi_p(s) = |s|^{p-2}\, s, p > 1$.

For convenience, let $k \geq \max\left\{\frac{1}{\eta}, \frac{2}{1-\eta}\right\}$ such that :

$$\begin{aligned}
B &= \min\left\{ \int_{\frac{1}{k}}^{\frac{1+\eta}{2}} \phi_p^{-1}(\int_s^{\frac{1+\eta}{2}} a(t)dt)ds, \int_{\frac{1}{k}}^{\eta} \phi_p^{-1}(\int_s^{\eta} a(t)dt)ds + \int_{\frac{1+\eta}{2}}^{1-\frac{1}{k}} \phi_p^{-1}(\int_{\frac{1+\eta}{2}}^{s} a(t)dt)ds \right\} \\
A_1 &= \int_0^1 \phi_p^{-1}(\int_s^1 a(t)dt)ds \\
A_2 &= \phi_p^{-1}(\int_0^1 a(t)dt) \\
A &= \sqrt{2}\max\left\{A_1, A_2\right\} = \sqrt{2}A_2.
\end{aligned}$$

The proof is based on the monotone iterative method.

Theorem 1.0.4 *Suppose that* (H) *hold, and there exist* $a > 0$, *such that :*

(H_1) : $f \in C\left([0,1] \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+\right)$

(H_2) : $f(t,x_1,y_1) \leq f(t,x_2,y_2)$ *for* $t \in [0,1], 0 \leq x_1 \leq x_2 \leq a,\ 0 \leq |y_1| \leq |y_2| \leq a$;

(H_3) $\max_{0\leq t\leq 1} f(t,a,a) \leq \phi_p\left(\frac{a}{A}\right)$

(H_4) $f(t,0,0) \neq 0$ *for* $t \in [0,1]$.

Then Problem (1.4) has at least one positive solution.

Theorem 1.0.5 *Suppose that* (H) *hold, and there exist an* $0 < b < a$*, such that :*

$(H_1) : f \in C\left([0,1] \times \mathbb{R}^+, \mathbb{R}^+\right)$

$(H_2) : f(t, x_1) \le f(t, x_2)$ *for* $t \in [0,1], 0 \le x_1 \le x_1 \le a,$

$(H_3) \max_{0 \le t \le 1} f(t,a) \le \phi_p\left(\frac{a}{A_1}\right)$

$(H_4) \min_{\frac{1}{k} \le t \le 1-\frac{1}{k}} \quad f(t, \frac{b}{k}) \ge \phi_p\left(\frac{b}{B}\right)$ *for* $t \in [0,1].$

Then the problem (1.5) has at least one positive solution.

In [23], Youwei Zhang considered the following boundary value problem :

$$\begin{cases} -\left(\phi_p\left(u'\right)\right)' = q(t) f\left(t, u, u'\right) \text{ in } (0,1) \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} u(\xi_i), \quad cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{1.6}$$

where, $\phi_p(s) = |s|^{p-1} s, p,$ $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1 (m \ge 3)$,

(H_1) $a, b, c, d \in (0, +\infty), a_i, b_i \in (0, +\infty)\,,$

$$a_i \le b_i, (i = 1, 2, ..., m-2), 0 \le \sum_{i=1}^{m-2} a_i < a, d < \sum_{i=1}^{m-2} b_i < c.$$

(H_2) $f \in C\left([0,1] \times [0, +\infty) \times \mathbb{R}, [0, +\infty)\right).$

(H_3) $q \in L^1\left([0,1]\right)$ is a nonnegative function on $[0,1]$ and $0 < \int_0^1 q(t)dt < \infty.$

$$\begin{aligned}
M &= \int_\sigma^{1-\sigma} \phi_p^{-1}\left(k \int_0^1 q(t)dt - \int_s^1 q(t)dt\right) ds \\
L &= \int_0^1 q(t)dt \\
N &= \frac{a + b + \sum_{i=1}^{m-2} a_i \xi_i - \sum_{i=1}^{m-2} a_i}{a - \sum_{i=1}^{m-2} a_i} \\
\ell_1 &= \frac{(c - \sum_{i=1}^{m-2} b_i)(a - \sum_{i=1}^{m-2} a_i \xi_i^2) + (a - \sum_{i=1}^{m-2} a_i)(c + 2d - \sum_{i=1}^{m-2} b_i \xi_i^2)}{(b + \sum_{i=1}^{m-2} a_i \xi_i)(c - \sum_{i=1}^{m-2} b_i) + (a - \sum_{i=1}^{m-2} a_i)(c + d - \sum_{i=1}^{m-2} b_i \xi_i)} \\
\ell_2 &= \frac{(b + \sum_{i=1}^{m-2} a_i \xi_i)(c + 2d - \sum_{i=1}^{m-2} b_i \xi_i^2) - (c + d - \sum_{i=1}^{m-2} b_i \xi_i)(\sum_{i=1}^{m-2} a_i \xi_i^2)}{(b + \sum_{i=1}^{m-2} a_i \xi_i)(c - \sum_{i=1}^{m-2} b_i) + (a - \sum_{i=1}^{m-2} a_i)(c + d - \sum_{i=1}^{m-2} b_i \xi_i)}
\end{aligned}$$

They have based on the nonlinear alternative of Leray-Schauder to prove the following results :

Theorem 1.0.6 *Suppose that* $(H_1)-(H_3)$ *hold, and* $f(t,0,0) \neq 0$ *and constants* $0 < m_1 < m_2 \leq \frac{\tau_1^2(\ell_1^2+4\ell_2)}{4\max\{\ell_1,|2-\ell_1|\}} m_4$ *such that the following conditions are satisfied :*

$$
\begin{aligned}
(C_1) f(t,u,v) &\leq \frac{1}{L}\phi_p(m_4) \text{ for } (t,u,v) \in [0,1] \times [0,\tau_2 m_4] \times [-m_4, m_4] \\
(C_2) f(t,u,v) &> \phi_p\left(\frac{m_2}{\tau_1 M}\right) \text{ for } (t,u,v) \in [\sigma, 1-\sigma] \times [m_2, \tau_2^{-1} m_2] \times [-m_4, m_4] \\
(C_3) f(t,u,v) &\leq \frac{1}{L}\phi_p\left(\frac{m_1}{N}\right) \text{ for } (t,u,v) \in [0,1] \times [0,m_1] \times [-m_4, m_4]
\end{aligned}
$$

then Problem (1.6) has at least three positive solutions u_1, u_2 *and* u_3*, which satisfy :*

$$
\begin{aligned}
\max_{0\leq t\leq 1} |u_i'(t)| &\leq m_4, i = 1;2;3 \\
\min_{\sigma\leq t\leq 1-\sigma} \{u_2(t)\} &< m_2 < \min_{\sigma\leq t\leq 1-\sigma} \{u_1(t)\}, \max_{0\leq t\leq 1}\{u_2(t)\} > m_1, \\
\max_{0\leq t\leq 1}\{u_3(t)\} &< m_1.
\end{aligned}
$$

In [11], Zhao and al. have been interested to the singular multi-point of fourth order ordinary differential equation with p-Laplacian :

$$
\begin{cases}
(\phi_p(u''))'' = f(t,u,-u'') \text{ in } (0,1), \\
u(1) = \alpha_0 u(\eta), u'(0) = 0, \\
(\phi_p(u''))'(0) = 0, u''(1) = \phi_p^{-1}(\alpha_1) u''(\eta).
\end{cases}
\tag{1.7}
$$

They assumed the following assumptions :

(H_1) : $\phi_p(s) = |s|^{p-2} s, p > 1, \eta \in (0,1), 0 < \alpha_i < 1, i = 0,1.$

(H_2) : $f \in C\left((0,1) \times (\mathbb{R}^+)^2, \mathbb{R}^+\right)$ with $f(t, \frac{1-\alpha_i\eta}{1-\alpha_i} - t, 1) \neq 0, i = 0,1, t \in (0,1)$.

(H_3) : There exist constants $\lambda_i, \ell_i \geq 0, i = 1,2, (0 < \lambda_1 \leq \ell_1, 0 \leq \lambda_2 \leq \ell_2 <$

$p-1, p-1<\lambda_1+\lambda_2)$ such that for all $(t,x,y)\in[0,1]\times\mathbb{R}_+^2$.

$$\begin{aligned} c^{\ell_1}f(t,x,y) &\leq f(t,cx,y)\leq c^{\lambda_1}f(t,x,y) \quad \text{if } 0<c\leq 1;\\ c^{\ell_2}f(t,x,y) &\leq f(t,x,cy)\leq c^{\lambda_2}f(t,x,y) \quad \text{if } 0<c\leq 1. \end{aligned}$$

Denote for $t\in(0,1), i=0,1$

$$\begin{aligned} \psi(t,\alpha_i) &= \frac{1-\alpha_i\eta}{1-\alpha_i}-t\\ F_1(t) &= f(t,\psi(t,\alpha_0),1). \end{aligned}$$

They proved the following result :

Theorem 1.0.7 *Suppose that conditions* $(H_1)-(H_3)$ *hold. Then a necessary and sufficient condition for problem (1.7)to have at least one* $C^2[0,1]$ *positive solution is*

$$0<\int_0^1(1-s)F_1(s)ds<\infty.$$

The proof rely on a recent version of the Krasnosel'skii fixed point theorem of cone.

In [12], Garcia, Manaseviech and Zanolin, considered the following ϕ-Laplacian boundary value Problem :

$$\begin{cases} (\phi(u'))'(x)+f(x,u(x))=0 \text{ in } (0,1)\\ u(0)=u(1)=0. \end{cases} \tag{1.8}$$

They have introduced the notion of upper and lower σ-condition on ϕ, namely

$$\limsup_{s\to\infty}\frac{\phi(\sigma s)}{s}<\infty, \lim_{s\to\infty}\inf\frac{\phi(\sigma s)}{s}>1, \forall\sigma>1.$$

Making use of the time-mapping approach, they were interested in the spectral study of the nonlinear eigenvalue Problem :

$$\begin{cases} (\phi(u'))'(x)+\lambda\phi(u(x))=0 \text{ in } (0,1)\\ u(0)=u(1)=0. \end{cases}$$

Roughly speaking, the asymptotic behavior of the nonlinear function f is then compared with the spectrum of this problem in order to solve Problem (1.8). Notice that in [1] and [3], the bvp (1.8) have been considered when the function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$ and where $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continous or a Carathéodory function. Using the fixed point theorems, nonlinear continuation method of Leray-Schauder and applying Krasnolsel'kii fixed point on cones of Banach spaces, Benmezai and al. proved that this problem has at least one positive solution.

Theorem 1.0.8 *[1] Suppose that* $f : [0,1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is a continuous function and satisfies :*

$(a) : \exists M_1 > 0$ *such that* $f(x,u) \leq M_1$ *for* $x \in [0,1]$ *and* $0 \leq u \leq \psi(M_1) = \eta$.

$(b) : \exists M_2 > 0$ *such that* $f(x,u) \geq M_2$ *for* $x \in [\sigma, 1-\sigma]$ *and* $\sigma\mu \leq u \leq \mu$ *with* $\mu \neq \eta$, $\mu = \frac{1}{2}\sigma D_1^\star$, $D_1^\star = \min_{\sigma \leq x \leq 1-\sigma} D_1(x)$ *and the function* D_1 *is defined by*

$$D_1(x) = \psi(M_2(x-\sigma)) + \psi(M_2(1-x-\sigma)).$$

Then, problem (1.8) has at least one solution u *such that :*

$$\min(\ \eta, \mu) \leq \|u\| \leq \max(\eta, \mu).$$

Theorem 1.0.9 *[1] Suppose that :*

(a) : *there exist* $F_1, F_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$: F_1 *is non increasing, strictly positive,* $\frac{F_2}{F_1}$ *is non decreasing and*

$$\exists r_0 > 0, (1 + \frac{F_2(r_0)}{F_1(r_0)} \int_0^1 F_1(r_0 p(s))ds \leq \phi(r_0).$$

such that $0 \leq f(x,u) \leq F_1(u) + F_2(u)$, *for any* $x \in [0,1]$ *and* $0 \leq u \leq r_0$.

(b) : *There exist* $G_1, G_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$: G_1 *is nonincreasing, strictly positive,*

$\frac{G_2}{G_1}$ *is nondecreasing and satisfy :*

$$\exists R_0 > 0, r_0 \neq R_0 \quad \textit{such that } \sigma D_2^\star \geq 2R_0$$

such that, $f(x,u) \geq G_1(u) + G_2(u)$, *for any* $x \in [0,1]$ *and* $0 \leq u \leq R_0$. *Here*

$$\begin{aligned} D_2(x) &= \psi\left(G_1(R_0)\left(1 + \frac{G_2(\sigma R_0)}{G_1(\sigma R_0)}\right)(x-\sigma)\right) \\ &+ \psi\left(G_1(R_0)\left(1 + \frac{G_2(\sigma R_0)}{G_1(\sigma R_0)}\right)(1-x-\sigma)\right) \end{aligned}$$

and $D_2^\star = \min_{\sigma \leq x \leq 1-\sigma} D_2(x)$.

Then, Problem (1.8) admits at least one solution

$$\min(\ r_0, R_0) \leq \|u\| \leq \max(r_0, R_0).$$

In the sub-linear and super-linear like cases, they supposed further that the operator ϕ satisfies the following condition :

there exist $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$ such that

$$t^\beta \phi(x) \leq \phi(tx) \leq t^\alpha \phi(x) \quad \text{for all } x \geq 0,\ t \in (0,1).$$

They obtained the following results, by the Krasnosel'skii fixed point theorem.

Theorem 1.0.10 *Let*

$$\begin{aligned} \liminf_{x\to+\infty} \min_{t\in[0,1]} \frac{f(t,x)}{\phi(x)} &= l_\infty, \limsup_{x\to0} \max_{t\in[0,1]} \frac{f(t,x)}{\phi(x)} = l^0, \\ \liminf_{x\to0} \min_{t\in[0,1]} \frac{f(t,x)}{\phi(x)} &= l_0, \limsup_{x\to+\infty} \max_{t\in[0,1]} \frac{f(t,x)}{\phi(x)} = l^\infty, \end{aligned}$$

Then, Problem (1.8) has at least one positive nontrivial solution provided one of the following conditions holds true :

either

$$l_0 > \frac{2^\beta}{\sigma^{2\beta}(1-2\sigma)\alpha_\star^\beta} \qquad \textit{and} \qquad l^\infty < 1,$$

or

$$l^0 < 1 \qquad \textit{and} \qquad l_\infty > \frac{2^\beta}{\sigma^{2\beta}(1-\sigma)\alpha_\star^\beta}$$

Here

$$\alpha_\star = \min\left(1, 2\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\right), \ \beta^\star = \min\left(1, 2\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)$$

Theorem 1.0.11 *[3] Let f be L^1 Carathéodory. If one of the following hypotheses is verified, then the boundary value problem (1.8) has at least one solution :*

either

(H_1) : $|f(x,y)| \leq q(x)F(y)$, *for a.e.* $x \in [0,1]$ *and* $y \in \mathbb{R}$ *where the functions* $q \in L^1([0,1], \mathbb{R}^+)$ *and* $F \in C(\mathbb{R}, \mathbb{R}^+)$ *satisfy :*

$$\exists r_0 > 0, \phi(r_0) \geq |q|_1 \max_{|y| \leq r_0} F(y)$$

or

(H_2) : $|f(x,y)| \leq G(x,|y|)$, *for a.e.* $x \in [0,1]$ *and* $y \in \mathbb{R}$ *where the function* $G \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$ *is nondecreasing with respect to the second argument and satisfies :*

$$\exists r_0 > 0, \int_0^1 G(x, r_0)dx \leq \phi(r_0).$$

In [2], Benmezai, Djebali, and Moussaoui considered the separated-variable ϕ-laplacian boundary value problem :

$$\begin{cases} -\left(\phi\left(u'\right)\right)' = q(x)f(u) \ t \in (0,1) \\ u\left(0\right) = u\left(1\right) = 0 \end{cases} \tag{1.9}$$

where $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $q : [0,1] \longrightarrow \mathbb{R}^+$ are continuous and the function ϕ is such that the inverse ϕ^{-1} is super-multiplicative, that is :

$$\forall \alpha, \beta \in \mathbb{R}^+, \phi^{-1}(\alpha\beta) \geq \phi^{-1}(\alpha)\phi^{-1}(\beta).$$

Using the Krasnosel'skii fixed point theorem of cone in Banach space, they have obtained the following multiplicity results :

Theorem 1.0.12 *Assume there are constants* $0 < c_1 < c_2 < c_3 < c_4$, *such that :*
$(a) : f(z) \leq \frac{\phi(c_i)}{|q|_1}$, *for* $z \in [0, c_i]$, $i = 1, 4$
$(b) : f(z) \geq \phi\left(\frac{2c_i}{D(\sigma)}\right)$, *for* $z \in [\sigma c_i, c_i]$, $i = 2, 3$.
Then, the boundary value problem (1.9) admits at least two positive solutions.

Theorem 1.0.13 *Assume there are constants* $0 < a < \sigma b < b \leq \sigma c < c \leq d$, *and assume that the functions* f_1 *and* ϕ *satisfy the following conditions :*
$(a) : f(z) < \frac{\phi(a)}{|q|_1}$, *for all* $z \in [0, a]$.
$(b) : f(z) \leq \frac{\phi(d)}{|q|_1}$, *for all* $z \in [0, d]$.
$(c) : f(z) > \phi\left(\frac{2b}{D(\sigma)}\right)$, *for all* $z \in [\sigma b, c]$.
Then, the boundary value problem (1.9) admits at least three nonnegative solutions u_1, u_2, u_3 *such that :*

$$\|u_1\| < a, \frac{u_2(\sigma) + u_2(1-\sigma)}{2} > b,$$
$$\|u_3\| > a, \frac{u_3(\sigma) + u_3(1-\sigma)}{2} < b.$$

In [10], Young and Yuming considered the following bvp :

$$\begin{cases} (\phi(u'))' + \lambda a(t) f(u) = 0 \text{ in } (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{1.10}$$

where λ is a positive parameter.

The authors made the following assumptions :

$(H_1) : \phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous increasing homeomorphism and there exist two increasing homeomorphisms ψ_1, ψ_2 from $(0, +\infty)$ into $(0, +\infty)$ such that :

$$\psi_1(x)\phi(y) \leq \phi(xy) \leq \psi_2(x)\phi(y) \quad \text{for} \quad x, y \in (0, +\infty),$$

a and f satisfy :

$(H_2) : f \in C(\mathbb{R}^+, \mathbb{R}^+)$.

$(H_3) : a \in C((0,1), \mathbb{R}^+)$ and $\int_0^1 a(s)ds < \infty$. Let us set

$$\begin{aligned}
A &= \frac{2}{\int_0^1 \left(\psi_1^{-1}(\int_s^1 a(t)dt) + \psi_1^{-1}(\int_0^s a(t)dt) \right) ds} \\
q(t) &= \frac{1}{2}\int_\delta^t \psi_2^{-1}(\int_s^t a(t)dt)ds + \frac{1}{2}\int_t^{1-\delta} \psi_2^{-1}(\int_s^t a(t)dt)ds, \delta \in \left(0, \frac{1}{2}\right), \\
L &= \min\{q(t), t \in [\delta, 1-\delta]\}, \\
f^0 &= \limsup_{u \longrightarrow 0} \frac{f(u)}{\phi(u)} \\
f^\infty &= \limsup_{u \longrightarrow \infty} \frac{f(u)}{\phi(u)}
\end{aligned}$$

Using the Legget-William fixed point, they obtained the following results :

Theorem 1.0.14 *Suppose that $(H_1) - (H_3)$ hold and $f^\infty < 1$. Assume there exist numbers $0 < c_1 < c_2$, such that*

$$\begin{aligned}
&(C_1) \quad f(u) \leq \phi(u) \text{ for } 0 \leq u \leq c_1, \\
&(C_2) \quad \text{for } c_2 \leq u \leq \frac{c_2}{\delta}, f(u) \geq \phi(\alpha u) \text{ for some } \alpha > \frac{1}{\delta L \psi_2^{-1}(\psi_1(A))}.
\end{aligned}$$

Then for $\lambda \in (\psi_2(\frac{1}{\alpha\delta L}), \psi_1(A))$, the boundary value problem (1.10) has at least three positive solutions u_1, u_2 and u_3, which satisfy :

$$\begin{aligned}
\|u_1\| &< c_1 < \|u_3\|, \\
\min_{\delta \leq t \leq 1-\delta}\{u_3(t)\} &< c_2 < \min_{\delta \leq t \leq 1-\delta}\{u_2(t)\}.
\end{aligned}$$

In [16], Liu and Zhang established the existence of positive solutions to the following bvp :

$$\begin{cases} -\left(\phi\left(x'\right)\right)' = a(t)f\left(x\right) \text{ in } (0,1); \\ x\left(0\right) - \beta x'\left(0\right) = 0, \\ x\left(1\right) - \delta x'\left(1\right) = 0. \end{cases} \tag{1.11}$$

They have obtained the existence of multiple positive solutions by Krasnoselsel'skii fixed-point theorem.

In [7], Denhong Ji and al. studied the existence of countably many positive solutions for a singular multi-point boundary value problem :

$$\begin{cases} -\left(\phi_p\left(u'\right)\right)'(t) = a(t)f\left(u(t)\right) \text{ in } (0,1) \\ u'(0) - \sum_{i=1}^{m-2}\alpha_i u(\xi_i) = 0, \quad u'(1) + \sum_{i=1}^{m-2}\alpha_i u(\eta_i) = 0, \end{cases} \tag{1.12}$$

where, $\phi_p(s) = |s|^{p-1}s, p > 1$.

They considered the following conditions :

$(H_1) 0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$ and $0 < \eta_1 < \eta_2 < \ldots < \eta_{m-2} < 1$
$\alpha_i > 0, \eta_i > \xi_i$, $\sum_{i=1}^{n-2}\alpha_i\xi_i < 1, \sum_{i=1}^{m-2}\alpha_i(1-\eta_i) < 1, i = 1,..., m-2$ and $f \in C\left([0,+\infty),(0,+\infty)\right)$.

The function $a : [0,1] \longrightarrow \mathbb{R}^+$ is continuous and satisfies

(H_2) There exists a sequence $(t_i)_{i\geq 1}$ such that $t_{i+1} < t_i, t_1 < \frac{1}{2}, \lim_{i\longrightarrow\infty} t_i = t_0, \lim_{t\longrightarrow t_i} a(t) = \infty, i = 1,2...,$ and $a(t) \geq 0$, $0 < \int_0^1 a(t)dt < \infty$. Moreover $a(t)$ does not vanish identically on any subinterval of $[0,1]$.

(H_3) Let $\theta \in \left(0,\frac{1}{2}\right)$, the function

$$A(t) = \int_{t_1}^{t}\phi_p^{-1}\left(\int_s^t a(r)dr\right)ds + \int_t^{1-t_1}\phi_p^{-1}\left(\int_t^s a(r)dr\right)ds, t \in [t_1, 1-t_1]$$

is positive continuous on $[t_1, 1-t_1] \subset [\theta, 1-\theta]$ and has a minimum on $[t_1, 1-t_1]$, so there exists $L > 0$ such that

$$A(t) \geq L, t \in \ [t_1, 1-t_1].$$

Let us set

$$\lambda_1 = \frac{2}{L}, \lambda_2 = \frac{\sum_{i=1}^{m-2} \alpha_i}{\nabla}$$

and

$$\begin{aligned} \nabla &= \phi_p^{-1}(\int_0^1 a(r)dr) + \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \phi_p^{-1}(\int_0^s a(r)dr)ds \\ &+ \sum_{i=1}^{m-2} \alpha_i \int_0^1 \phi_p^{-1}(\int_s^1 a(r)dr)ds \end{aligned}$$

Based on the fixed point index theory and the Legget-Williams fixed point theorem, the authors proved the following results :

Theorem 1.0.15 *Suppose that the conditions $(H_1) - (H_3)$ hold. Let $(\theta_k)_{k\geq 1}$ be such that $\theta_k \in (t_k, t_{k+1})\,(k = 1, 2, ...)$. Let $(r_k)_{k\geq 1}$ and $(R_k)_{k\geq 1}$ be such that*

$$R_{k+1} < \theta_k r_k < r_k < m r_k < R, k = 1, 2, ..., .$$

Furthermore, for each natural number k, they assume that f satisfies :

$$\begin{aligned} (C_1) f(u) &\geq \phi_p\,(m r_k) \text{ for all } u \in [\theta_k r_k, r_k] \\ (C_2) f(u) &\leq \phi_p\,(M R_k) \text{ for all } u \in [0, R_k]\,, \end{aligned}$$

Where, $m \in (\lambda_1, +\infty)\,, M \in (0, \lambda_2)\,.$ Then, Problem (1.12) has infinitely many solutions $(u_k)_{k\geq 1}$, such that :

$$r_k \leq \|u_k\| \leq R_k, = 1, 2, ..., .$$

In [22], Yuji Liu considered the following mixed type multi-point boundary value problem :

$$\begin{cases} -\left(\phi\left(u'\right)\right)' = f\left(t, u, u'\right) \text{ in } (0,1) \\ u\left(0\right) - \alpha u'(0) = \lambda_1 \\ u\left(1\right) - \sum_{i=1}^{m} \beta_i u(\xi_i) = \lambda_2, \end{cases} \tag{1.13}$$

where $(H_1) : \phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a increasing homeomorphism with $\phi(0) = 0$.
$(H_2) : \lambda_1 \geq 0, \lambda_2 \geq 0, \alpha \geq 0, \beta_i \geq 0$ satisfy $\sum_{i=1}^{m} \beta_i < 1$ and $\lambda_1 = \frac{\lambda_2}{1-\sum_{i=1}^{m} \beta_i}$.
$(H_3) : f : [0,1] \times \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is nonnegative and continuous with $f(t, c, 0) \neq 0$ on each sub-interval of $[0,1]$.
New sufficient conditions to guarantee the existence of at least three positive solutions of this problem.
For the notations, given constants r_1, r_2, L_1, L_2 and an integer $k > 2$

$$\begin{aligned}
Q_1 &= \alpha \sum_{i=1}^{m} \beta_i + \lambda_1 \sum_{i=1}^{m} \beta_i + \sum_{i=1}^{m} \beta_i \int_0^1 \phi^{-1}(1-s)\, ds \\
&+ \sum_{i=1}^{m} \beta_i \int_0^1 \phi^{-1}(s)\, ds + \int_0^1 \phi^{-1}(s)\, ds \\
Q_2 &= \alpha + \lambda_1 + \int_0^1 \phi^{-1}(1-s)\, ds \\
Q_3 &= \min \left\{ \lambda_2 + \int_{\frac{1}{2}}^{\frac{k-1}{k}} \phi^{-1}(s - \frac{1}{2}) ds,\ \lambda_1 + \int_{\frac{1}{k}}^{\frac{1}{2}} \phi^{-1}(\frac{1}{2} - s) ds \right\} \\
M_1 &= \phi \left(\frac{r_1}{\max\{Q_1, Q_2\}} \right) \\
M_3 &= \phi \left(\frac{bk}{Q_3} \right). \\
M_2 &= \phi \left(\frac{r_2}{\max\{Q_1, Q_2\}} \right) \\
N_1 &= \phi(L_1) \\
N_3 &= \phi(L_2).
\end{aligned}$$

Applying different fixed point theorems, he obtained the following results :

Theorem 1.0.16 *Suppose that $(H_1)-(H_3)$ hold, and there exist an integer $k > 2$, and constants $0 < r_1 < kb < r_2$ and $0 < L_1 < L_2$. Let $M_1, M_2, M_3, N_1; N_3$ be defined above. If $M_2 < min\{M_1, M_3, N_1; N_3\}$ and if f satisfies the following* *conditions :*

$$\begin{aligned}
(C_1) f(t,u,v) &< \min\{M_1, N_1\} \text{ for } (t,u,v) \in [0,1]\times[0,r_1]\times[-L_1,L_1] \\
(C_2) f(t,u,v) &> M_2 \text{ for } (t,u,v) \in [\frac{1}{k}, 1-\frac{1}{k}]\times[b,kb]\times[-L_2,L_2] \\
(C_3) f(t,u,v) &< \min\{M_3, N_3\} \text{ for } (t,u,v) \in [0,1]\times[0,r_2]\times[-L_2,L_2] \, .
\end{aligned} \tag{1.14}$$

Then, Problem (1.13) has at least three positive solutions u_1, u_2 *and* u_3*, which satisfy :*

$$\begin{aligned}
&\max_{0\le t\le 1}\{u_1(t)\} < r_1, \max_{0\le t\le 1}\{u_1'(t)\} < L_1, \\
&b \le \min_{\frac{1}{k}\le t\le 1-\frac{1}{k}}\{u_2(t)\} \le \max_{0\le t\le 1}\{u_2(t)\} \le r_2, \max_{0\le t\le 1}\{u_2'(t)\} \le L_2, \\
&\min_{\frac{1}{k}\le t\le 1-\frac{1}{k}}\{u_3(t)\} < b, r_1 < \max_{0\le t\le 1}\{u_3(t)\} \le r_2, L_1 < \max_{0\le t\le 1}\{u_3'(t)\} < L_2.
\end{aligned}$$

In [26], Zhiyong and Zhang have considered the following bvp :

$$\begin{cases} -\left(\varphi\left(u'\right)\right)' = a(t) f\left(u, u'\right) \text{ in } (0,1) \\ u'(0) = u(1) = 0, \\ u(0) = u(1) = 0, \end{cases} \tag{1.15}$$

where $\tau > 0$ is positive constant.

They have showed existence and multiplicity of positive solutions under some assumptions by applying Krasnolsel'skii fixed point theorem.

In [24], Yu Yang and Dongmei considered the following m-point boundary value problem

$$\begin{cases} -\left(\phi\left(u'\right)\right)' = q(t) f\left(t, u, u'\right) \text{ in } (0,1) \\ u(0) = \sum_{i=1}^{n-2} \alpha_i u(\xi_i), \quad \phi\left(u'(1)\right) = \sum_{i=1}^{n-2} \beta_i \phi\left(u'(\xi_i)\right), \end{cases} \tag{1.16}$$

They assumed the following assumptions :

(H_1) : $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is an odd increasing homeomorphism and positive homomorphism with $\phi(0) = 0$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \ldots < \xi_{n-2} < 1$ and there exist two increasing homeomorphisms S_1, S_2 from $(0,+\infty)$ into $(0,+\infty)$ such that

$$S_1(x)\phi(y) \leq \phi(xy) \leq S_2(x)\phi(y) \, \text{for} \, x, y \in (0,+\infty)$$

(H_2) :α_i, β_i satisfy $\alpha_i, \beta_i \in (0,+\infty)$, $0 \leq \sum_{i=1}^{n-2} \alpha_i < 1, 0 \leq \sum_{i=1}^{n-2} \beta_i < 1$.

q and f satisfy :

(H_3) : $f \in C\left([0,1] \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+\right)$.

(H_4) : $q : (0,1) \longrightarrow \mathbb{R}^+$ is continuous, not identically zero on any sub- interval of $(0,1)$ such that :

$$0 < \int_0^1 q(t)dt < \infty,$$

(H_5) $q(t)f(t,0,0) \neq 0, f(t,0,0) \geq 0$ for $t \in [0,1]$.

Using the three functionals fixed point theorem, they have studied the existence of multiple positive solution to the boundary value problem (1.16) and they obtained the sufficient conditions for existence of at least three positive solutions.

They made the following notations :

$$\begin{aligned}
C &= S_1^{-1}\left(\frac{\int_0^1 q(t)dt}{1-\sum_{i=1}^{n-2}\beta i}\right) \\
N &= \int_{\frac{1}{k}}^{1-\frac{1}{k}} S_2^{-1}\left(\int_s^{1-\frac{1}{k}} q(t)dt\right) ds \\
M &= \frac{1}{1-\sum_{i=1}^{n-2}\alpha_i}\int_0^1 S_1^{-1}\left(\frac{\int_0^s q(t)dt}{1-\sum_{i=1}^{n-2}\beta i}\right) ds.
\end{aligned}$$

They have proved the following result :

Theorem 1.0.17 *Suppose that $(H_1)-(H_5)$ hold, and there exist $0 < r_1 < b_1 < kb_1 \leq r_2$ and $0 < L_1 \leq L_2$ such that*

$$\frac{kb_1}{N} \leq min\left\{\frac{r_2}{M}, \frac{L_2}{C}\right\}.$$

If f satisfies the following conditions :

$$\begin{aligned}
(C_1)f(t,u,v) &< \min\left\{\phi\left(\frac{r_1}{M}\right),\phi\left(\frac{L_1}{C}\right)\right\} \textit{ for } (t,u,v)\in[0,1]\times[0,r_1]\times[-L_1,L_1] \\
(C_2)f(t,u,v) &> \phi\left(\frac{kb_1}{N}\right) \textit{ for } (t,u,v)\in[\frac{1}{k},1-\frac{1}{k}]\times[b_1,kb_1]\times[-L_2,L_2] \\
(C_3)f(t,u,v) &\leq \min\left\{\phi\left(\frac{r_2}{M}\right),\phi\left(\frac{L_2}{C}\right)\right\} \textit{ for } (t,u,v)\in[0,1]\times[0,r_2]\times[-L_2,L_2] \\
(C_4)f(t,u,v) &\geq 0 \textit{ for } (t,u,v)\in[\frac{1}{k},1-\frac{1}{k}]\times[b_1,r_2]\times[-L_2,L_2]\,,
\end{aligned}$$

then Problem (1.16) has at least three positive solutions u_1, u_2 and u_3, which satisfy

$$\begin{aligned}
&\max_{0\leq t\leq 1}\{u_1(t)\}<r_1, \max_{0\leq t\leq 1}\{u_1'(t)\}<L_1, \\
&b_1<\min_{\frac{1}{k}\leq t\leq 1-\frac{1}{k}}\{u_2(t)\}\leq \max_{0\leq t\leq 1}\{u_2(t)\}\leq r_2, \max_{0\leq t\leq 1}\{u_2'(t)\}\leq L_2, \\
&\min_{\frac{1}{k}\leq t\leq 1-\frac{1}{k}}\{u_3(t)\}<b_1, r_1<\max_{0\leq t\leq 1}\{u_3(t)\}<Kb_1, L_1<\max_{0\leq t\leq 1}\{u_3'(t)\}\leq L_2
\end{aligned}$$

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Chapitre 2

Positive solution to the nonlinear abstract Hammerstein equation and applications to ϕ-Laplacian bvps

Abstract We provide in this Chapter existence results for positive solution to the abstract Hammerstein equation $NFu = u$ where $N : E \to E$ is a completely continuous operator, $F : C \to C$ is a continuous and bounded map and C is a cone in the Banach space E. The obtained results are used to prove existence results for positive solution to $\phi-$Laplacian boundary value problems.

AMS 2010 Subject Classifications : 47H11, 34B15.

Key words : Hammerstein equation, Fixed point index theory, BVPs.

2.1 Introduction

Recently and in an interesting paper, M. Garcia-Huidobro, R. Manasevich and J. R. Ward studied in [10] existence of positive solutions to the $\phi-$Laplace boundary value problem

$$\begin{cases} (\phi(u'))' + f(t,u) = 0, \quad t \in (0,1) \\ \theta(u(0)) = \beta\theta(u'(0)), \ \theta(u(1)) = -\delta\theta(u'(1)) \end{cases}$$

where β, δ are nonnegative real numbers, ϕ, θ are odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous with $f(t,0) = 0$, $f(t,s) > 0$ for all $s > 0$ and $t \in [0,1]$. They obtained existence results under one of the following condition

$$f_0(t) = 0 \text{ and } f_\infty(t) = +\infty$$

or

$$f_0(t) = +\infty \text{ and } f_\infty(t) = 0$$

where for $\nu = 0, +\infty$, $f_\nu(t) = \lim_{s\to\nu} f(t,s)/\phi(s)$.

Their results and all those in [3], [4] and [6]-[9], are obtained by means of a fixed point formulation having the form

$$u = NFu. \tag{2.1}$$

Equation (2.1) is known as the abstract Hammerstein equation (see Chapter 7 in [13]). If nonnegativity is a requirement for a solution to (2.1) then we are naturely led to use methods of nonlinear analysis in ordred Banach spaces. So let E be a Banach space, K a cone in E, $N : E \to E$ is a completely continuous and increasing operator which is not necessarily linear (the case where N is linear has been studied in [1]), $F : K \to K$ is a continuous and bounded map.

We are interested in this Chapter in existence and nonexistence results for solutions to Equation (2.1) in the cone K. Roughly speaking, with any cone P such that $N(K) \subset P \subset K$, we will associate two classes of nonlinearities F, the first class is constituted by nonlinearities for which Equation (2.1) has no solution in P and the second one is concerned with nonlinearities for which Equation (2.1) is able to have a solution in P (see Remark 2.3.3). Since for nonlinearities in the second class we cannot claim that equation (2.1) admits a solution in the cone P we provide by Theorems 2.3.6, 2.3.7, 2.3.8 and 2.3.9 subclasses of nonlinearities F for which equation (2.1) admits a solution in P.

As an application, obtained results are used to prove existence results for positive solution to ϕ-Laplacian boundary value problems.

Throughout this Chapter, for any subset A of a Banach space, we denote $A^* := A \setminus \{0\}$.

2.2 Preliminaries

We will use extensively in this work cones and the fixed point index theory, so let us recall some facts related to these two tools. Let X be a Banach space, a nonempty closed convex subset K of X is said to be an odred cone if

- $(tK) \subset K$ for all $t \geq 0$
- $K \cap (-K) = \{0\}$.

It is well known that an ordred cone K induces a partial order in the Banach space X. We write for all $x, y \in X$: $x \leq y$ if $y - x \in K$, $x < y$ if $y - x \in K$, $y \neq x$ and $x \nleq y$ if $y - x \notin K$. Notations $\geq$, $>$ and $\ngeq$ denote respectively the inverse situations.

A cone K is said to be normal with a constant $n > 0$ if for all u, v in $K,\ u \leq v$ implies $\|u\| \leq n\|v\|$.

A function $f : \Omega \subset X \to X$ is said to be bounded if it maps bounded sets into bounded sets and it is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Now let K be a cone of E and $N : E \to E$ a continuous map.

Définition 2.1 *N is said to be*

- *positive if $N(K) \subset K$,*
- *strongly positive if K has a nonempty interior ($intK \neq \emptyset$) and $N(K^*) \subset intK$,*
- *positively 1-homogeneous if for all $u \in E$ and $t \geq 0,\ N(tu) = tN(u)$,*
- *increasing if for all $u, v \in E,\ u \leq v$ implies $Nu \leq Nv$ and*
- *strictly increasing if for all $u, v \in E,\ u < v$ implies $Nu < Nv$.*

Définition 2.2 *N is said to be upper bounded on K if there exists a positive constant M such that for all $u \in K,\ \|Nu\| \leq M\|u\|$. In such a situation we denote*

$$N_K^+ = \sup\{\|Nu\| / \|u\|,\ u \in K^*\}.$$

Définition 2.3 *N is said to be lower bounded on K if there exists a positive constant m such that for all $u \in K,\ \|Nu\| \geq m\|u\|$. In such a situation we denote*

$$N_K^- = \inf\{\|Nu\| / \|u\|,\ u \in K^*\}.$$

The following result gives a nonlinear version of Krein Rutman theorem,

Theorem 2.2.1 *([5],[12])Let N be an increasing positively 1-homogeneous completely continuous mapping for which there exist $u \in K^*$ and $m > 0$ such that $Nu \geq mu$; then N has a positive eigen-pair (λ_0, u_0).*

Furthermore if N is strongly positive and strictly increasing then

– *if λ is an eigenvalue with nonnegative eigenvector, then $\lambda = \lambda_0$.*

– *λ_0 is geometrically simple.*

– *if λ is a real eigenvalue of N, then $|\lambda| \leq \lambda_0$.*

Let $B(0, R)$ be the ball of radius $R > 0$ centred at the origin and $f : K \to K$ a completely continuous map. The following lemmas provide computations of the fixed point index :

Lemma 2.2.2 *If $fx \ngeq x$ for all $x \in \partial B(0, R) \cap K$ then*

$$i\left(f, B(0, R) \cap K, K\right) = 1.$$

Lemma 2.2.3 *If $fx \nleq x$ for all $x \in \partial B(0, R) \cap K$ then*

$$i\left(f, B(0, R) \cap K, K\right) = 0.$$

Corollary 2.2.4 *If $\|fx\| < \|x\|$ for all $x \in \partial B(0, R) \cap K$ then*

$$i\left(f, B(0, R) \cap K, K\right) = 1.$$

Corollary 2.2.5 *If $\|fx\| > \|x\|$ for all $x \in \partial B(0, R) \cap K$ then*

Corollary 2.2.6 $i(f, B(0, R) \cap K, K) = 0.$

For more details and proofs we refer the reader to [11].

2.3 Existence results

In all this section E is a Banach space, C a cone of E and $N : E \to E$ an increasing operator. For any subset P of C with nonempty set P^* we define the subsets

$$\Lambda_P^N = \{\lambda \geq 0 : \text{ there exist } u \in P^* \text{ such that } Nu \leq \lambda u\},$$

$$\Theta_P^N = \{\theta \geq 0 : \text{ there exist } u \in P^* \text{ such that } Nu \geq \theta u\}.$$

Note that

– $0 \in \Theta_P^N$ and if $\theta \in \Theta_P^N$ then $[0, \theta] \subset \Theta_P^N$.

– If $\lambda \in \Lambda_P^N$ then $[\lambda, +\infty[\subset \Lambda_P^N$.

– if $P \subset K \subset C$ then $\Lambda_P^N \subset \Lambda_K^N \subset \Lambda_C^N$ and $\Theta_P^N \subset \Theta_K^N \subset \Theta_C^N$.

The following constants λ_P^N and θ_P^N will play an important role in the statement of the obtained existence and nonexistence results. When they exist, they are defined for any subset $P \subset C$ with $P^* \neq \varnothing$ by

$$\lambda_P^N = \inf \Lambda_P^N, \; \theta_P^N = \sup \Theta_P^N.$$

The following Lemma 2.3.1 and Lemma 2.3.2 provide suffiscient condition for existence of λ_P^N and θ_P^N respectively.

Lemma 2.3.1 *If C is normal with a constant n and N is upper bounded on C then the subset Θ_C^N is bounded from above.*

Proof. If $\lambda > 0$ and $u \in C^*$ are such that $\lambda u \leq Nu$ then the upper boundeness of N and the normality of C lead to

$$\lambda \|u\| \leq n \|Nu\| \leq n N_C^+ \|u\|$$

then

$$\lambda \leq nN_C^+$$

■

Remark 2.3.2 *We can see that if C is normal and N is upper bounded on C then for all subset $P \subset C$ with P^* nonempty the subset Θ_P^N is bounded from above. Also, arguing as in the proof of Lemma 2.3.1 we can show that for any normal cone P with $N(C) \subset P \subset C$ if N is upper bounded on P then Θ_P^N is bounded from above.*

Lemma 2.3.3 *Suppose that N is completely continuous, P is a cone of E with $N(C) \subset P \subset C$ and N is upper bounded on P; then the subset Λ_P^N is nonempty.*

Proof. Let $\lambda > N_P^+$, $e \in P^*$, and consider the equation

$$u = N_\lambda(u, t), \tag{2.2}$$

where for all $u \in C$ and $t \in [0, 1]$

$$N_\lambda(u, t) = \frac{t}{\lambda} Nu + e.$$

Clearly $N_\lambda (P \times [0, 1]) \subset P$ and Equation (2.2) has no solution in $P \cap \partial B(0, R)$ with $R > \max \left(\frac{\lambda}{\lambda - N_P^+} \|e\|, \|e\| \right)$. Thus by homotopy and normality properties of the fixed point index

$$i\left(N_\lambda(\cdot, 1), B(0, R) \cap P, P\right) = i\left(N_\lambda(\cdot, 0), B(0, R) \cap P, P\right) = 1$$

and Equation (2.2) admits a solution $u \in P^* \cap B(0, R)$. We conclude that $\lambda \in \Lambda_P^N$ ■

Remark 2.3.4 *If P is a normal cone with a constant n_P such that $N(C) \subset P \subset C$ and N is lower bounded on P; then Λ_P^N is bounded from below by $n_p N_p^-$. Indeed if $\lambda > 0$ and $u \in P^*$ are such that $\lambda u \geq Nu$; then $\lambda \|u\| \geq n_P \|Nu\| \geq nN_P^- \|u\|$.*

Now let us consider the nonlinear abstract Hammerstein equation

$$NFu = u \tag{2.3}$$

where $F : C \to C$ is a continuous function.

Before presenting existence results for Equation (2.3) we need to draw attention to the following fact. If N admits an eigenvalue $\lambda \geq 0$ with an eigenvector $u \in C^*$ then $\lambda_C^N \leq \theta_C^N$ and $\lambda \in \left[\lambda_C^N, \theta_C^N\right]$.

Lemma 2.3.5 *Let P be a cone of E with $N(C) \subset P \subset C$. If one of the following situations*

$$Fu \leq \alpha u \textit{ for all } u \in P^* \textit{ with } \alpha\theta_P^N < 1 \tag{2.4}$$

$$Fu \geq \beta u \textit{ for all } u \in P^* \textit{ with } \beta\lambda_P^N > 1 \tag{2.5}$$

holds true, then Equation (2.3) *has no solution in P^*.*

Proof. We present the proof in the case where (2.4) holds; the other case is checked similarly. Assume that there is some $u \in P^*$ such that $NFu = u$; then inequality $Fu \leq \alpha u$ yields $Nv \geq \dfrac{1}{\alpha} v$ where $v = \alpha u \in P^*$ and $\dfrac{1}{\alpha} \leq \theta_P^N$, which contradicts $\alpha\theta_P^N < 1$. The Lemma is proved ■

Remark 2.3.6 *Note that Lemma 2.3.3 claims that Equation* (2.3) *with a nonlinearity F lying below the linearity $\left(\theta_P^N\right)^{-1} u$ or above the linearity $\left(\lambda_P^N\right)^{-1} u$,* *admits no solution in* P^*. *In another fashion, Equation* (2.3) *can have a solution in the cone* P *if the nonlinearity* F *crosses at least once the linearity* $\left(\theta_P^N\right)^{-1} u$ *or the linearity* $\left(\lambda_P^N\right)^{-1} u$.

Lemma 2.3.7 *Let* P *be a cone of* E *with* $N(C) \subset P \subset C$, *assume that* N *is completely continuous, and* F *is bounded. If there exist positive real numbers* $\alpha,\ \beta,\ R_1, R_2$ *with* $\alpha\theta_P^N < 1 < \beta\lambda_P^N$ *and* $R_1 < R_2$ *such that one of the following situations*

$$\begin{cases} Fu \leq \alpha u \text{ for all } u \in P \cap \partial B(0, R_1) \\ Fu \geq \beta u \text{ for all } u \in P \cap \partial B(0, R_2) \end{cases} \tag{2.6}$$

$$\begin{cases} Fu \geq \beta u \text{ for all } u \in P \cap \partial B(0, R_1) \\ Fu \leq \alpha u \text{for all } u \in P \cap \partial B(0, R_2) \end{cases} \tag{2.7}$$

holds true, then Equation (2.3) *admits a positive solution* u *with* $R_1 < \|u\| < R_2$.

Proof. We present the proof in the case where (2.6) holds; the other case is checked similarly. Assume that there exists $u \in P \cap \partial B(0, R_1)$ such that $NFu \geq u$. Inequality $Fu \leq \alpha u$ leads to $N(\alpha u) \geq u$ or $N(v) \geq \frac{1}{\alpha} v$ where $v = \alpha u \in P^*$ that is $\frac{1}{\alpha} \leq \theta_P^N$ which contradicts $\alpha\theta_P^N < 1$. So by Lemma 2.2.1, $i(NF, P \cap B(0, R_1), P) = 1$.

If for some $u \in P \cap \partial B(0, R_2)$, $NFu \leq u$. We get from the inequality $Fu \geq \beta u$ that $N(\beta u) \leq u$ or $N(v) \geq \frac{1}{\beta} v$ where $v = \beta u \in P^*$ that is $\frac{1}{\beta} \geq \lambda_P^N$ which contradicts $\beta\lambda_P^N > 1$. So by Lemma 2.2.2, $i(NF, P \cap B(0, R_2), P) = 0$.

Finally end by the excision and solution properties of the fixed point index we obtain

$$\begin{aligned} i(NF, P \cap \left(B(0, R_2) \setminus \overline{B}(0, R_1)\right), P) &= \\ i(NF, P \cap B(0, R_2), P) - i(NF, P \cap B(0, R_1), P) &= -1 \end{aligned}$$

and Equation (2.3) admits a positive solution u with $R_1 < \|u\| < R_2$ ■

Remark 2.3.8 *Arguing as in the proof of Lemma 2.3.4 one can see that if N is completely continuous and P is a cone of E with $N(C) \subset P \subset C$, then for all $R > 0$*

$$i(N(\alpha\cdot), P \cap B(0,R), P) = \begin{cases} 1 \text{ if } \alpha\theta_P^N < 1 \\ 0 \text{ if } \alpha\lambda_P^N > 1. \end{cases}$$

Theorem 2.3.9 *Let P be a cone with $N(C) \subset P \subset C$ and assume that N is completely continuous, then $\lambda_P^N \leq \theta_P^N$.*

Proof. The case $\lambda_P^N = 0$ is obvious, so assume that $\lambda_P^N > \theta_P^N \geq 0$ and consider the function $F : C \to C$ defined by

$$Au = \frac{\eta u + \mu \|u\| u}{1 + \|u\|}$$

with $0 < \eta < \mu$ and $\eta\lambda_P^N > 1 > \mu\theta_P^N$.

Thus we have

$$\eta u < Au < \mu u \text{ for all } u \in P^*.$$

In one hand, by Lemma 2.3.3, Equation (2.3), admits no positive solution; in the other one, for any $0 < R_1 < R_2$ we have

$$Au \leq \mu u \text{ for all } u \in P \cap \partial B(0, R_1) \text{ with } \mu\theta_P^N < 1$$

and

$$Au \geq \eta u \text{ for all } u \in P \cap \partial B(0, R_2) \text{ with } \eta\lambda_P^N > 1$$

and by Lemma 2.3.4 Equation (2.3) admits a positive solution. This is impossible so $\lambda_P^N \leq \theta_P^N$ ■

The first existence result concerns the case where N is positively 1-homogeneous operator.

Theorem 2.3.10 *Let P be a cone of E with $N(C) \subset P \subset C$ and assume that N is an increasing positively 1-homogeneous completely continuous operator, C is normal and there exist three nonnegative real numbers α, β, γ and continuous functions $G_i : C \to C$ $i = 1, 2, 3$ such that*

$$\alpha\theta_P^N < 1 < \beta\lambda_P^N$$

and for all $u \in P^*$

$$Fu \leq \alpha u + G_1(u)$$
$$\beta u - G_2(u) \leq F(u) \leq \gamma u + G_3(u).$$

Then if one of the following situations

$$\begin{aligned} G_1(u) &= \circ\left(\|u\|\right) \text{ at } 0 \text{ and} \\ G_i(u) &= \circ\left(\|u\|\right) \text{ at } \infty \text{ for } i = 2, 3 \end{aligned} \tag{2.8}$$

$$\begin{aligned} G_1(u) &= \circ\left(\|u\|\right) \text{ at } \infty \text{ and} \\ G_i(u) &= \circ\left(\|u\|\right) \text{ at } 0 \text{ for } i = 2, 3 \end{aligned} \tag{2.9}$$

holds true, Equation (2.3) *admits a positive solution.*

Proof. We present the proof in the case where (2.8) holds ; the other case is checked similarly. We have to prove existence of $0 < r < R$ such that

$$\begin{aligned} i(NF, B(0, r) \cap P, P) &= 1 \\ i(NF, B(0, R) \cap P, P) &= 0, \end{aligned}$$

in such a situation, the excision and solution properties of fixed point index imply that

$$i(NF, \big(B(0, R) \setminus \overline{B}(0, r)\big) \cap P, P) =$$
$$i(NF, B(0, R) \cap P, P) - i(NF, B(0, r) \cap P, P) = -1$$

and Equation (2.3) admits a positive solution u with $r < \|u\| < R$.

Now consider the function $H_1 : [0,1] \times C \to C$ defined by $H_1(t,u) = tNFu + (1-t)\beta Nu$ and let us prove the existence of $R > 0$ large enough such that for all $t \in [0,1]$, Equation $H_1(t,u) = u$ has no solution in $\partial B(0,R) \cap P$. By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial B(0,n) \cap P$ such that

$$u_n = t_n NFu_n + (1-t_n)\beta Nu_n.$$

Note that $v_n = \dfrac{u_n}{\|u_n\|} \in \partial B(0,1) \cap P$ and satisfies

$$v_n = t_n N\left(\frac{Fu_n}{\|u_n\|}\right) + (1-t_n)\beta Nv_n.$$

Thus the normality of the cone C combined with

$$\beta v_n - \frac{G_2 u_n}{\|u_n\|} \leq \frac{Fu_n}{\|u_n\|} \leq \gamma v_n + \frac{G_3 u_n}{\|u_n\|}$$

and $G_i(u_n) = \circ\left(\|u_n\|\right)$ at ∞ for $i = 2,3$ yields that $\dfrac{Fu_n}{\|u_n\|}$ is bounded. Then from the compacity of N, we deduce the existence of a subsequence also denoted (v_n) which converges to $v \in \partial B(0,1) \cap P$ and which satisfies $v \geq \beta Nv.$, that is $\dfrac{1}{\beta} \geq \lambda_P^N$ which contradicts $\beta\lambda_P^N > 1$.

For such a $R > 0$ we deduce from the homotopy property of fixed point index and Lemma 2.2.2 that

$$i(NF, B(0,R) \cap P, P) = i(H_1(1,\cdot), B(0,R) \cap P, P) =$$
$$i(H_1(0,\cdot), B(0,R) \cap P, P) = i(\beta N, B(0,R) \cap P, P) = 0.$$

In a similar way, consider the function $H_2 : [0,1] \times C \to C$ defined by $H_2(t,u) = tNFu + (1-t)\alpha Nu$ and let us prove the existence of $r > 0$ small enough such that for all $t \in [0,1]$, the Equation $H_2(t,u) = u$ has no solution in $\partial B(0,r) \cap P$.

By the contrary, suppose that for all integer $n \geq 1$ there exist $t_n \in [0,1]$ and $u_n \in \partial B(0, \frac{1}{n}) \cap P$ such that

$$u_n = t_n N F u_n + (1 - t_n)\beta N u_n.$$

Note that $v_n = \frac{u_n}{\|u_n\|} \in \partial B(0,1) \cap P$ and satisfies

$$v_n = t_n N \left(\frac{F u_n}{\|u_n\|} \right) + (1 - t_n)\alpha N v_n.$$

Thus the normality of the cone C combined with

$$\frac{F u_n}{\|u_n\|} \leq \alpha v_n + \frac{G_1 u_n}{\|u_n\|}$$

and $G_1(u_n) = \circ\left(\|u_n\|\right)$ at 0 yield that $\frac{F u_n}{\|u_n\|}$ is bounded. Then from the compactness of N we deduce the existence of a subsequence also denoted (v_n) which converges to $v \in \partial B(0,1) \cap P$ and satisfies $v \leq \alpha N v$, namely $\frac{1}{\alpha} \leq \theta_P^N$ which contradicts $\alpha \theta_P^N < 1$.

For such a $r > 0$ we deduce from the homotopy property of fixed point index and Lemma 2.2.1 that

$$i(NF, B(0,r) \cap P, P) = i(H_2(1,\cdot), B(0,r) \cap P, P) =$$
$$i(H_2(0,\cdot), B(0,r) \cap P, P) = i(\alpha N, B(0,r) \cap P, P) = 1.$$

This completes the proof ■

Now we will present some existence results when N is not necessarily positively 1-homogeneous. Assume that N is completely continuous, F is bounded and P is a cone with $N(C) \subset P \subset C$. $A : E \to E$ is a positively 1-homogeneous completely continuous increasing operator such that $A(C) \subset P$ which is uniformly continous on the unit ball of E.

Theorem 2.3.11 *Assume that C is normal with a constant n, N is upper bounded on C and there exist three positive real numbers r, α, β with $\beta\lambda_P^N > 1 > \alpha n N_C^+$ and a continuous function $G : C \to C$ with $G(u) = \circ(\|u\|)$ at ∞ such that*

$$Fu \geq \beta u \text{ for all } u \in P \cap \partial B(0,r)$$

$$Fu \leq \alpha u + G(u) \text{ for all } u \in P,$$

then Equation (2.3) *admits at least one positive solution.*

Proof. As in the proof of Lemma 2.3.4 we have $i(NF, B(0,r) \cap P, P) = 0$; thus it remains to prove that there exists $R > r$ such that $i(NF, B(0,R) \cap P, P) = 1$. In view of Lemma 2.2.2 and on the contrary assume that there is a sequence (u_k) with $u_k \in P \cap \partial B(0,k)$ such that $\|u_k\| \leq \|NFu_k\|$; then we get then

$$\|u_k\| \leq \|NFu_k\| \leq n\|N(\alpha u_k + G(u_k))\| \leq nN_C^+ \|\alpha u_k + G(u_k)\|.$$

Finalarily

$$nN_C^+ \left\| \alpha \frac{u_k}{\|u_k\|} + \frac{G(u_k)}{\|u_k\|} \right\| < 1.$$

Letting $k \to \infty$ we get a contradiction with $1 < nN_C^+\alpha \leq 1$. Hence there exists $R > r$ such that $i(NF, B(0,R) \cap P, P) = 1$ and by the excision property of the fixed point index $i(NF, (B(0,R) \setminus B(0,r)) \cap P, P) = 1$ and thus Equation (2.3) admits at least one solution in $\left(B(0,R) \setminus \overline{B}(0,r)\right) \cap P$ ■

Theorem 2.3.12 *Assume that A is lower bounded on P and*

$$Nu = A(u) + \circ(\|u\|) \text{ at infinity.}$$

If there exist three positive real numbers r, α, β and a continuous function $G : C \to C$ with $G(u) = \circ(\|u\|)$ at ∞ such that the following condition

$$\begin{aligned} &Fu \geq \beta u \text{ for all } u \in C \cap \partial B(0,r), \\ &Fu \leq \alpha u + Gu \text{ for all } u \in C \text{ and} \\ &\beta\lambda_P^N > 1 > \alpha \max\left(\theta_P^N, \theta_P^A\right), \end{aligned} \tag{2.10}$$

is satisfied, then Equation (2.3) *admits at least one positive solution.*

Proof. As in the proof of Lemma 2.3.4 we have that $i\left(NF, B(0,r)\cap P, P\right)=0$; thus it remains to prove the existence of $R>r$ such that $i\left(NF, B(0,R)\cap P, P\right)=1$. Consider the function $H_3:[0,1]\times C\to C$ defined by $H_3(t,u)=tNFu+(1-t)N(\alpha u)$ and we claim that there is $R>0$ large enough such that for all $t\in[0,1]$, Equation $H_3(t,u)=u$ has no solution in $\partial B(0,R)\cap P$. If it is not the case then for all integer $n\geq 1$ there exist $t_n\in[0,1]$ and $u_n\in\partial B(0,n)\cap P$ such that

$$u_n=t_nNFu_n+(1-t_n)N\left(\alpha u_n\right).$$

This implies

$$\begin{aligned}
&A\left(\frac{u_n}{\|u_n\|}\right)\\
&=A\left(t_n\frac{N\left(Fu_n\right)}{\|u_n\|}+(1-t_n)\alpha\frac{N\left(\alpha u_n\right)}{\|\alpha u_n\|}\right)\\
&\leq A\left(t_n\frac{N\left(\alpha u_n+Gu_n\right)}{\|u_n\|}+(1-t_n)\alpha\frac{N\left(\alpha u_n\right)}{\|\alpha u_n\|}\right)\\
&=A\left(t_n\frac{\|\alpha u_n+Gu_n\|}{\|u_n\|}A\left(\frac{\alpha u_n+Gu_n}{\|\alpha u_n+Gu_n\|}\right)+(1-t_n)\alpha A\left(\frac{u_n}{\|u_n\|}\right)+\circ\left(1\right)\right)
\end{aligned}$$

From the compactness and the uniform continuity of the operator A, we deduce the existence of a subsequence $\left(\dfrac{u_{n_k}}{\|u_{n_k}\|}\right)$ such that

$$\lim A\left(\frac{u_{n_k}}{\|u_{n_k}\|}\right)=\lim A\left(\frac{\alpha u_n+Gu_n}{\|\alpha u_n+Gu_n\|}\right)=v\in P\text{ and }v\leq A\left(\alpha v\right).$$

Furthermore and since $\dfrac{u_{n_k}}{\|u_{n_k}\|}\in P$ and A is lower bounded on P,

$$\|v\|=\lim\left\|A\left(\frac{u_{n_k}}{\|u_{n_k}\|}\right)\right\|\geq\lim A_P^-\left\|\frac{u_{n_k}}{\|u_{n_k}\|}\right\|=A_P^->0$$

then we have that $w=\alpha v\in C^*$ satisfies

$$Aw\geq\frac{w}{\alpha}$$

and $\dfrac{1}{\alpha} \leq \theta_P^A$ which contradicts $\alpha\theta_P^A < 1$.

So there exists $R > r$ such that the Equation $H_3(t,u) = u$ has no solution in

$$[0,1] \times (\partial B(0,R) \cap P)\,;$$

by the homotopy property of the fixed point index

$$i\,(NF, P \cap B(0,R), P) = i\,(N\,(\alpha\cdot)\,, P \cap B(0,R), P) = 1.$$

Finally, by the excision property of the fixed point index

$$i\,(NF, (B(0,R) \setminus B(0,r)) \cap P, P) = 1$$

and Equation (2.3) admits at least one solution in $\left(B(0,R) \setminus \overline{B}(0,r)\right) \cap P$ ■

Theorem 2.3.13 *Assume that C is normal, A is lower bounded on P and*

$$Nu = A\,(u) + \circ\,(\|u\|) \text{ at infinity.}$$

If there exist four positive real numbers r, α, β, δ and continuous functions $G_1, G_2 : C \to C$ with $G_i\,(u) = \circ\,(\|u\|)$ at ∞ for $i = 1, 2$ such that the following condition

$$\begin{aligned} &Fu \leq \alpha u \text{ for all } u \in C \cap \partial B(0,r), \\ &\beta u - G_1 u \leq Fu \leq \delta u + G_2 u \text{ for all } u \in C \text{ and} \\ &\beta \min\left(\lambda_P^A, \lambda_P^N\right) > 1 > \alpha\theta_P^N \end{aligned} \tag{2.11}$$

is satisfied, then Equation (2.3) *admits at least one positive solution.*

Proof. As in proof of Lemma 2.3.4, we have $i\,(NF, B(0,r) \cap P, P) = 1$; thus it remains to prove that there exists $R > r$ such that $i\,(NF, B(0,R) \cap P, P) = 0$. Consider the function $H_4 : [0,1] \times C \to C$ defined by $H_4(t,u) = tNFu +$

$(1-t)N(\beta u)$ and we claim that there is $R>0$ large enough such that for all $t\in[0,1]$, the Equation $H_4(t,u)=u$ has no solution in $\partial B(0,R)\cap P$. If it is not the case then for all integer $n\geq 1$ there exist $t_n\in[0,1]$ and $u_n\in\partial B(0,n)\cap P$ such that

$$u_n = t_n N F u_n + (1-t_n) N\left(\beta u_n\right).$$

This implies

$$\begin{aligned} A\left(\frac{u_n}{\|u_n\|}\right) &= A\left(t_n\beta\frac{N\left(Fu_n\right)}{\|\beta u_n\|} + (1-t_n)\beta\frac{N\left(\beta u_n\right)}{\|\beta u_n\|}\right) \\ &= A\left(t_n\beta A\left(\frac{Fu_n}{\|\beta u_n\|}\right) + (1-t_n)\beta A\left(\frac{u_n}{\|u_n\|}\right) + \circ\left(1\right)\right). \end{aligned}$$

Since C is normal and

$$\frac{Fu_n}{\|\beta u_n\|} \leq \frac{\delta}{\beta}\frac{u_n}{\|u_n\|} + \frac{G_2 u_n}{\|\beta u_n\|},$$

$\dfrac{Fu_n}{\|\beta u_n\|}$ is bounded and from the compactness of A we deduce existence of a subsequence $\left(\dfrac{u_{n_k}}{\|u_{n_k}\|}\right)$ such that

$$\lim A\left(\frac{u_{n_k}}{\|u_{n_k}\|}\right) = v,\ \lim A\left(\frac{Fu_n}{\|\beta u_n\|}\right) = w.$$

Furthermore, since $\dfrac{u_{n_k}}{\|u_{n_k}\|}\in P$ and A is lower bounded on P, then

$$\|v\| = \lim\left\|A\left(\frac{u_{n_k}}{\|u_{n_k}\|}\right)\right\| \geq \lim A_P^- \left\|\frac{u_{n_k}}{\|u_{n_k}\|}\right\| = A_P^- > 0.$$

Also, from the uniform continuity of the operator A

$$w = \lim A\left(\frac{Fu_n}{\|\beta u_n\|}\right) \geq \lim A\left(\frac{u_n}{\|u_n\|} - \frac{G_1 u_n}{\|\beta u_n\|}\right) \geq v.$$

From all the above we deduce that $\varpi=\beta v$ satisfies

$$A\left(\varpi\right) \leq \frac{\varpi}{\beta}$$

and $\dfrac{1}{\beta} \geq \lambda_P^A$, which contradicts $\beta\lambda_P^A > 1$.

So there exists $R > r$ such that, the Equation $H_4(t,u) = u$ has no solution in $[0,1] \times (\partial B(0,R) \cap P)$; by homotopy property of the fixed point index

$$i\left(NF, P \cap B(0,R), P\right) = i\left(N\left(\beta\cdot\right), P \cap B(0,R), P\right) = 0.$$

Finally, by the excision property of the fixed point index

$$i\left(NF, \left(B(0,R) \setminus B(0,r)\right) \cap P, P\right) = -1$$

and Equation (2.3) admits at least one solution in $\big(B(0,R) \setminus \overline{B}(0,r)\big) \cap P$ ■

Remark 2.3.14 *If $P \subset K \subset C$ are two cones, then $\left[\lambda_P^N, \theta_P^N\right] \subset \left[\lambda_K^N, \theta_K^N\right] \subset \left[\lambda_C^N, \theta_C^N\right]$. This means that the interval $\left[\left(\theta_P^N\right)^{-1}, \left(\lambda_P^N\right)^{-1}\right]$ becomes the smallest if the cone P is the smallest one containing $N\left(C\right)$ and in such a case the above existence and nonexistence results are more accurate. Moreover if $N^{-1}\left(0\right) \cap C = \{0\}$, then for every cone P with $N(C) \subset P \subset C$, $\Lambda_P^N = \Lambda_C^N$, $\Theta_P^N = \Theta_C^N$ and so $\lambda_P^N = \lambda_C^N$, $\theta_P^N = \theta_C^N$. Indeed if $\lambda > 0$ and $u \in P^*$ are such that $Nu \leq \lambda u$ (resp. $Nu \geq \lambda u$) then $w = \dfrac{Nu}{\lambda}$ satisfies $Nw \leq \lambda w$ (resp. $Nw \geq \lambda w$).*

In the end of this section we present a situation where the constants λ_C^N and λ_C^N meet the unique positive eigenvalue of N.

Theorem 2.3.15 *Let E and X be two real Banach spaces with $X \subset E$, C a normal cone in E such that $C_X = X \cap C$ is a cone in X with $int_X\left(C_X\right) \neq \emptyset$ and $N : E \to E$ an increasing positively 1-homogeneous completely continuous operator such that $N\left(E\right) \subset X$, $N^{-1}\left(0\right) \cap C = \{0\}$ and $N_X : X \to X$ is a strongly positive, strictly increasing and completely continuous operator. If $\lambda_C^N > 0$ then $\lambda_C^N = \theta_C^N$ and $\lambda_N = \lambda_C^N = \theta_C^N$ is the unique positive eigenvalue of N.*

Proof. First, taking in account Remark 2.3.10 and Theorem 2.3.5, the conditions $N^{-1}(0) \cap C = \{0\}$ and $\lambda_C^N > 0$ imply that $\theta_{C_X}^N > \lambda_{C_X}^N > 0$ then there exists $m > 0$ and $v_m \in C_X^*$ such that $Nv_m \geq mv_m$.Thus, it follows from Theorem 2.2.1 that N_X admits a unique positive eigenvalue λ_N and clearly

$$0 < \lambda_C^N = \lambda_{C_X}^N \leq \lambda_N \leq \theta_{C_X}^N = \theta_C^N.$$

Let

$$\lambda_{int_X(C_X)}^N = \inf\{\lambda \geq 0, \exists u \in int_X(C_X) \text{ with } Nu \leq \lambda u\}$$

and let us prove that $\lambda_C^N = \lambda_{C_X}^N = \lambda_{int_X(C_X)}^N$. To this aim let us prove that $\Lambda_{C_X}^N \subset \Lambda_{int_X(C_X)}^N$. If $\lambda \in \Lambda_{C_X}^N$ then there exists $u \in C_X^*$ such that $Nu \leq \lambda u$ and we distinguish two cases :

- $u \in int_X(C_X)$ in this case $\lambda \in \Lambda_{int_X(C_X)}^N$
- $u \in \partial_X C_X$ in this case $U = Nu \in int_x C_X$ and satisfies $NU \leq \lambda U$. This again implies $\lambda \in \Lambda_{int_X(C_X)}^N$.

We claim that $\theta_{C_X}^N \leq \lambda_{int_X(C_X)}^N$.

Indeed if $\lambda_{int_X(C_X)}^N < \theta_{C_X}^N$ and $\lambda \in \left(\lambda_{int_X(C_X)}^N, \theta_{C_X}^N\right)$ then for a such λ, there exist $u \in int_X(C_X)$ and $v \in C_X^*$ such that $Nu \leq \lambda u$ and $Nv \geq \lambda v$. The fact $u \in int_X(C_X)$ implies the existence of $t > 0$ small enough such that $u > v_t = tv$. Let $K = \{w \in C, v_t \leq w \leq u\}$, it is clear that K is a convex bounded closed set in E and $T = \dfrac{N}{\lambda}$ maps K into K. So the Schauder fixed point Theorem guarantee existence of a fixed point $w_\lambda \in K$ of T and λ is an eigenvalue of N. This means that all $\lambda \in \left(\lambda_{int_X(C_X)}^N, \theta_{C_X}^N\right)$ are positive eigenvalues of N; then of N_X $(N(Nw_\lambda) = \lambda Nw_\lambda$ with $Nw_\lambda \in C_X)$ and contradicts uniqueness of λ_N. Finally, we have from all the above

$$0 < \lambda_C^N = \lambda_{C_X}^N \leq \lambda_N \leq \theta_{C_X}^N = \theta_C^N \leq \lambda_{int_X(C_X)}^N = \lambda_C^N = \lambda_{C_X}^N$$

which implies that $\lambda_N = \lambda_C^N = \theta_C^N$ is the unique positive eigenvalue of N ∎

2.4 Application to ϕ-Laplacian bvps

In all this section, E is the Banach space of all continuous functions defined on $[0,1]$ equiped with the sup-norm denoted $\|\cdot\|$, C is the normal cone of nonnegative functions in E and $u \in E$ is said to be positive if $u \in C^*$, $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $a : [0,1] \to \mathbb{R}^+$ is continuous and does not vanishes identicaly on any subinterval of $[0,1]$ and ϕ is an odd increasing homeomorphism of $\mathbb{R}$.

Throughout we assume that

$$\begin{aligned} &\exists \alpha, \beta \in \mathbb{R} \text{ with } 0 < \alpha < \beta \text{ such that} \\ &t^{\beta}\phi(x) \le \phi(tx) \le t^{\alpha}\phi(x) \quad \text{for all } x \ge 0 \text{ and } t \in (0,1)\,. \end{aligned} \tag{2.12}$$

In what follows ψ is the inverse function of ϕ and we have from (2.12)

$$t^{\frac{1}{\alpha}}\psi(x) \le \psi(tx) \le t^{\frac{1}{\beta}}\psi(x) \quad \text{for all } x \ge 0 \text{ and } t \in (0,1)\,. \tag{2.13}$$

Let ψ^+,ψ^- be the function defined on $\mathbb{R}^+$ by

$$\psi^+(x) = \begin{cases} x^{\frac{1}{\beta}} \text{ if } x \le 1 \\ x^{\frac{1}{\alpha}} \text{ if } x \ge 1 \end{cases} \qquad \psi^-(x) = \begin{cases} x^{\frac{1}{\alpha}} \text{ if } x \le 1 \\ x^{\frac{1}{\beta}} \text{ if } x \ge 1. \end{cases}$$

It follows from (2.13) that for all $t \ge 0$ and $x \ge 0$

$$\psi^-(t)\,\psi(x) \le \psi(tx) \le \psi^+(t)\,\psi(x). \tag{2.14}$$

Remark 2.4.1 *Note that if ϕ satisfies (2.12) then ϕ satisfies the conditions of Theorems 1.1 and 1.2 in [10].*

Let $F : C \to C$ be the operator defined for $u \in C$ by

$$Fu\,(x) = \psi\,(f(x, u(x))) \text{ for all } x \in [0,1]\,.$$

It is easy to see that F is continuous and bounded (maps bounded sets into bounded sets). We set

$$
\begin{array}{ll}
f^0 = \limsup\limits_{u\to 0}\left(\max\limits_{t\in[0,1]}\dfrac{\psi\left(f(t,u)\right)}{u}\right) & f^\infty = \limsup\limits_{u\to +\infty}\left(\max\limits_{t\in[0,1]}\dfrac{\psi\left(f(t,u)\right)}{u}\right)\\
f_0 = \liminf\limits_{u\to 0}\left(\min\limits_{t\in[0,1]}\dfrac{\psi\left(f(t,u)\right)}{u}\right) & f_\infty = \liminf\limits_{u\to +\infty}\left(\min\limits_{t\in[0,1]}\dfrac{\psi\left(f(t,u)\right)}{u}\right)
\end{array}
$$

and

$$
\begin{array}{ll}
l^0 = \limsup\limits_{u\to 0}\left(\max\limits_{t\in[0,1]}\dfrac{f(t,u)}{\phi\left(u\right)}\right) & l^\infty = \limsup\limits_{u\to +\infty}\left(\max\limits_{t\in[0,1]}\dfrac{f(t,u)}{\phi\left(u\right)}\right)\\
l_0 = \liminf\limits_{u\to 0}\left(\min\limits_{t\in[0,1]}\dfrac{f(t,u)}{\phi\left(u\right)}\right) & l_\infty = \liminf\limits_{u\to +\infty}\left(\min\limits_{t\in[0,1]}\dfrac{f(t,u)}{\phi\left(u\right)}\right).
\end{array}
$$

Remark 2.4.2 *It is easy to see that if ϕ is homogeneous then $l^0 = f^0,\ l_0 = f_0$.*

2.4.1 A ϕ-Laplacian bvp with Dirichlet boundary conditions

Consider the boundary value problem

$$
\begin{cases}
-\left(\phi\left(u'\right)\right)'(t) = a\left(t\right) f(t,u(t)) \quad t\in(0,1)\\
u(0) = u(1) = 0.
\end{cases}
\tag{2.15}
$$

Existence results for positive solutions to ϕ-Laplacian bvp with Dirichlet boundary conditions have been obtained in [3], [4] by Krasnosel'skii or Leggett-William fixed Theorems and by the degree theory in [10]. We will use here the existence results of Section 3 to obtain existence results for positive solutions to Problem (2.15).

Let

$$X = \left\{u\in C^1\left([0,1]\right), u\left(0\right) = u\left(1\right) = 0\right\}$$

equiped with the C^1-norm denoted $\|\cdot\|_1$ (for $u\in X$, $\|u\|_1 = \max\left(\|u\|, \|u'\|\right)$). From Ascoli Arzela Theorem, the embeding $i : X\to E$ is compact. Let C be the normal cone of nonnegative functions in E and $u \in E$ is said to be positive if $u \in C^*$. It is easy to see that the set

$$P = \{u \in C, u(x) \geq \rho(x) \|u\| \ \ \forall x \in [0,1]\},$$

is a cone in E, where for $x \in [0,1]$, $\rho(x) = \min\{x, 1-x\}$ and $C_X = C \cap X = i^{-1}(C)$ is a cone in X.

Let $N : E \to E$ be the operator defined for $h \in E$ by

$$Nh(x) = \int_0^x \psi\left(c(h) - \int_0^t a(s)\phi(h(s))\,ds\right) dt \text{ for all } x \in [0,1],$$

where $c(h)$ is the unique solution of

$$\int_0^1 \psi\left(c - \int_0^t a(s)\phi(h(s))\,ds\right) dt = 0.$$

Let $N_X : X \to X$ be the restriction of N to X, $\Phi_E : E \to L^1(0,1)$ defined for $h \in E$ by $\Phi_E(h)(x) = \phi(h(x))$, $\Phi_X : X \to L^1(0,1)$ the restriction of Φ_E to X and $N_1 : L^1(0,1) \to X$ the operator defined for $h \in L^1(0,1)$ by

$$N_1 h(x) = \int_0^x \psi\left(c_1(h) - \int_0^t a(s)h(s)\,ds\right) dt \text{ for all } x \in [0,1],$$

where $c_1(h)$ is the unique solution of

$$\int_0^1 \psi\left(c - \int_0^t a(s)h(s)\,ds\right) dt = 0.$$

It is easy to see that $N_X = N_1 \circ \Phi_X$ and $N = i \circ N_1 \circ \Phi_E$ and each of Φ_X and Φ_E are continuous bounded maps. So, we deduce from Lemma 2.2 in [6] (see also Section 3 in [10]) that N and N_X are a completely continuous operators and u is a positive solution to (2.15) if and only if u is a fixed point of the completely continuous operator NF.

Lemma 2.4.3 *The operator N has the following properties :*

1. $N^{-1}(0) = \{0\}$,

2. $N(C) \subset P$,

3. N *is strictly increasing,*

4. N *is upper bounded on* C *and lower bounded on* P.

Proof.

1. Obvious.

2. Note that for any $h \in C,\ v = Nh$ is concave and if $t_0 \in [0,1]$ is such that $\|v\| = v(t_0)$ then we have

   $\cdot$ if $t_0 \in (0,1)$, then

   $$v(x) = v((\frac{x}{t_0})t_0 + (1-(\frac{x}{t_0}))0) \geq (\frac{x}{t_0})v(t_0) + (1-(\frac{x}{t_0}))v(0) \geq \rho(x)\|v\|$$

   for all $x \in [0, t_0]$ and

   $$v(x) = v((\frac{1-x}{1-t_0})t_0 + (\frac{x-t_0}{1-t_0})) \geq (\frac{1-x}{1-t_0})v(t_0) + (\frac{x-t_0}{1-t_0})v(1) \geq \rho(x)\|v\|$$

   for all $x \in [t_0, 1]$, then

   $\cdot$ if $t_0 = 0$ or 1,

   $$v(x) = v(x + (1-x)0) \geq xv(1) + (1-x)v(0) \geq \rho(x)\|v\|$$

   for all $x \in [0,1]$. The above shows that $N(C) \subset P$.

3. Let u,$v \in E$ with $u < v$ and set $U = Nu,\ V = Nv$ and $W = V - U$. Assume that for some $t_0 \in (0,1)$, $W(t_0) < 0$ and let $t_* \in (0,1)$ be such that

   $$W(t_*) = \min_{t\in[0,1]} W(t) \text{ and } W'(t_*) = 0.$$

In one hand, from the boundary conditions, there exist $t_1, t_2 \in (0,1)$ such that

$$t_1 < t_* < t_2 \text{ and } W'(t_1) < W'(t_*) = 0 < W'(t_2). \tag{2.16}$$

In the other hand, we have

$$-\left(\phi(V') - \phi(U')\right)' = a\left(\phi(v) - \phi(u)\right) \geq 0 \text{ on } [0,1]$$

which implies that $\phi(V') - \phi(U')$ is nonincreasing on $[0,1]$. Also we have

$$\left(\phi(V') - \phi(U')\right) W'(t) \leq 0 \text{ for all } t \in [0,1]$$

from which, we deduce

$$\left(\phi(V'(t_2)) - \phi(U'(t_2))\right) \leq 0 \leq \left(\phi(V'(t_1)) - \phi(U'(t_1))\right);$$

then

$$W'(t_2) = V'(t_2)) - U'(t_2) \leq 0 \leq V'(t_1) - U'(t_1) = W'(t_1);$$

which contradicts (2.16).

Now if $U = V$ in $[0,1]$ then we get

$$a(s)\left(\phi(u(s)) - \phi(v(s))\right) = 0 \text{ for all } s \in [0,1]$$

which implies since a does not vanishes on any subinterval of $[0,1]$, that $u(s) = v(s)$ for all $s \in [0,1]$ and contradicts $u < v$.

4. Let $u \in C$ and $t_u \in [0,1]$ be such that $Nu(t_u) = \|Nu\|$, we have

$$\begin{aligned} \|Nu\| = Nu(t_u) &= \int_0^{t_u} \psi\left(\int_t^{t_u} a(s)\,\phi(u(s))ds\right) dt \\ &= \int_{t_u}^{1} \psi\left(\int_{t_u}^{t} a(s)\,\phi(u(s))ds\right) dt \end{aligned}$$

then

$$\|Nu\| \leq \int_0^{t_u} \psi\left((t_u - t)\|a\|\,\phi(\|u\|)\right) dt \leq \frac{\beta}{\beta+1}\psi^+\left(\|a\|\right)(t_u)^{\frac{\beta+1}{\beta}}\|u\| \tag{2.17}$$

and

$$\|Nu\| \leq \int_{t_u}^{1} \psi\left((t - t_u)\|a\|\,\phi(\|u\|)\right) dt \leq \frac{\beta}{\beta+1}\psi^+\left(\|a\|\right)(1-t_u)^{\frac{\beta+1}{\beta}}\|u\|. \tag{2.18}$$

We conclude from (2.17) and (2.18) that for all $u \in C$

$$\|Nu\| \leq \frac{\beta}{\beta+1}\min\left((t_u)^{\frac{\beta+1}{\beta}}, (1-t_u)^{\frac{\beta+1}{\beta}}\right)\psi^+\left(\|a\|\right)\|u\| \leq D_\phi^+\|u\|$$

where

$$D_\phi^+ = \frac{\beta}{\beta+1}\left(\frac{1}{2}\right)^{\frac{\beta+1}{\beta}}\psi^+\left(\|a\|\right).$$

Now if $t_u \leq \frac{1}{2}$ we get

$$\begin{aligned}
\|Nu\| &= Nu(t_u) = \int_{t_u}^{1}\psi\left(\int_{t_u}^{t} a(s)\,\phi(u(s))ds\right)dt \\
&\geq \int_{\frac{1}{2}}^{1}\psi\left(\int_{\frac{1}{2}}^{t} a(s)\,\phi(u(s))ds\right)dt \\
&\geq \|u\|\,\psi^-\left(\|a\|\right)\int_{\frac{1}{2}}^{1}\left(\int_{\frac{1}{2}}^{t}(1-s)^\beta\frac{a(s)}{\|a\|}ds\right)^{\frac{1}{\alpha}}dt
\end{aligned}$$

and if $t_u \geq \frac{1}{2}$

$$\begin{aligned}
\|Nu\| &= Nu(t_u) = \int_0^{t_u}\psi\left(\int_t^{t_u} a(s)\,\phi(u(s))ds\right)dt \\
&\geq \int_0^{\frac{1}{2}}\psi\left(\int_t^{\frac{1}{2}} a(s)\,\phi(u(s))ds\right)dt \\
&\geq \|u\|\,\psi^-\left(\|a\|\right)\int_0^{\frac{1}{2}}\left(\int_t^{\frac{1}{2}} s^\beta\frac{a(s)}{\|a\|}ds\right)^{\frac{1}{\alpha}}dt.
\end{aligned}$$

Thus setting

$$D_{\phi}^{-}=\min\left(\psi^{-}\left(\|a\|\right)\int_{0}^{\frac{1}{2}}\left(\int_{t}^{\frac{1}{2}}s^{\beta}\frac{a\left(s\right)}{\|a\|}ds\right)^{\frac{1}{\alpha}}dt,\psi^{-}\left(\|a\|\right)\int_{\frac{1}{2}}^{1}\left(\int_{\frac{1}{2}}^{t}(1-s)^{\beta}\frac{a\left(s\right)}{\|a\|}ds\right)^{\frac{1}{\alpha}}dt\right),$$

we get from the above that for all $u\in P,\ \|Nu\|\geq D_{\phi}^{-}\|u\|.$

■

Remark 2.4.4 *See that*

$$0<D_{\phi}^{-}\leq N_{P}^{-}\leq\lambda_{P}^{N}\leq\theta_{P}^{N}\leq N_{C}^{+}\leq D_{\phi}^{+}.$$

Lemma 2.4.5 *Let*

$$O=\left\{u\in C_{X}:u'\left(0\right)>0,\ u'(1)<0\ \textit{and}\ u(x)>0\ \forall x\in(0,1)\right\}.$$

Then O is an open set in X and $N_{X}\left(C_{X}^{*}\right)\subset O$.

Proof. We have $O^{c}=F_{1}\cup F_{2}\cup F_{3}$, where

$$\begin{aligned}F_{1}&=\left\{u\in X:\text{there exists }x\in\,]0,1[\ u(x)\leq0\right\},\\F_{2}&=\left\{u\in X:u'(0)\leq0\right\}\ \text{and}\\F_{3}&=\left\{u\in X:u'(1)\geq0\right\}.\end{aligned}$$

It is clear that F_2 and F_3 are closed sets in X; so let $(u_n)\subset F_1$ be a sequence tending to u in X and $(x_n)\subset\,]0,1[$ tending to $\overline{x}\in[0,1]$ with $u_n(x_n)\leq0$. We distinguish between the following cases

– $\overline{x}\in\,]0,1[$ in a such situation $u(\overline{x})=\lim u_n(x_n)\leq0$ and $u\in F_1$,

– $\overline{x}=0$ in this case we obtain

$$u'(0)=\lim\frac{u_n(x_n)}{x_n}\leq0\text{ and }u\in F_2.$$

– $\overline{x} = 1$, in this case we obtain

$$u'(1) = \lim \frac{u_n(x_n)}{x_n - 1} \geq 0 \text{ and } u \in F_3.$$

Now, let us show $N_X(C_X^*) \subset O$. We deduce from Lemma 2.4.1 that for all $h \in C_X^*$

$$\|Nh\| > 0 \text{ and } Nh(x) \geq \rho(x)\|Nh\| > 0 \ \forall x \in (0,1)$$

and since $Nh(1) = Nh(0) = 0$,

$$(Nh)'(0) \geq 0 \text{ and } (Nh)'(1) \leq 0.$$

If $(Nh)'(0) = 0$ then after two integrations we get

$$Nh(x) = -\int_0^x \psi\left(\int_0^t a(s)\phi(h(s))\,ds\right)dt \leq 0$$

which is impossible. $(Nh)'(1) = 0$ leads to the same contradiction. So we have $(Nh)'(0) > 0$, $(Nh)'(1) < 0$ and $Nh \in O$. ■

Taking in account Remark 2.4.3, we deduce immediately from Lemma 2.3.3 the following nonexistence result.

Corollary 2.4.6 *Assume that* (2.12) *and one of the following conditions*

$$\psi(f(t,u)) \geq \alpha u \text{ for all } t \in [0,1] \text{ and } u > 0 \text{ with } \alpha D_\phi^- > 1$$

or

$$\psi(f(t,u)) \leq \alpha u \text{ for all } t \in [0,1] \text{ and } u > 0 \text{ with } \alpha D_\phi^+ < 1$$

holds true; then Problem (2.15) *admits no positive solution.*

Corollary 2.4.7 *Assume that* (2.12) *and the following condition*

$$f^\infty D_\phi^+ < 1 < f_0 D_\phi^- \tag{2.19}$$

hold true; then Problem (2.15) *admits a positive solution.*

Proof. Let $\epsilon > 0$ be such that $(f_0 - \epsilon) D_\phi^- > 1 > D_\phi^+ (f^\infty + \epsilon)$. It follows from the definitions of f_0 and f^∞ that there exist $\delta > 0$ and $C > 0$ such that

$$Fu \geq (f_0 - \epsilon) u \text{ for all } u \in P \cap \partial B(0, \delta)$$

$$Fu \leq (f^\infty + \epsilon) u + G(u) \text{ for all } u \in P$$

where for all $u \in C,\ G(u) = cte$. Since $0 < D_\phi^- \leq \lambda_P^N \leq \theta_P^N \leq D_\phi^+$ we deduce from Theorem 2.3.7 that Problem (2.15) admits a positive solution. ■

Remark 2.4.8 *We can see that Corollary 2.4.5 does not cover the case where $f^0 = 0,\ f_\infty = \infty$. Theorem 1.2 in [10] cover this situation when $a \equiv 1$, $f(t, 0) = 0$ and $f(t, s) > 0$ for all $s > 0$ and $t \in [0, 1]$.*

Also note that Theorems 1.1 and 1.2 in [10] are established for more general boundary conditions.

Corollary 2.4.9 *Assume that* (2.12) *and the following condition*

$$\psi^+ (l^\infty) D_\phi^+ < 1 < \psi^- (l_0) D_\phi^-$$

hold true ; then Problem (2.15) *admits a positive solution.*

Proof. It is easy to see that (2.14) implies

$$\psi^- (s) \leq \frac{\psi (s\phi(u))}{u} \leq \psi^+ (s) \text{ for all } s, u \in (0, +\infty).$$

Thus it follows from definition of l_0 for arbitrary $\epsilon > 0$ there exists $\delta > 0$ such that for all $t \in [0, 1]$ and $u \in [0, \delta]$, $f(t, u) \geq (l_0 - \epsilon) \phi(u)$ and then

$$\frac{\psi (f(t, u))}{u} \geq \frac{\psi ((l_0 - \epsilon) \phi(u))}{u}.$$

Passing to the limit we get

$$f_0 = \liminf_{u \to 0} \left(\min_{t \in [0,1]} \frac{\psi (f(t, u))}{u} \right) \geq \liminf_{u \to 0} \frac{\psi ((l_0 - \epsilon) \phi(u))}{u} \geq \psi^- (l_0 - \epsilon);$$

then $f_0 \geq \psi^- (l_0)$. In similar way we can prove that $f^0 \leq \psi^+ (l^0)$ and $\psi^- (l_\infty) \leq f_\infty \leq f^\infty \leq \psi^+ (l^\infty)$ and Corollary 2.4.7 follows from Corollary 2.4.5 ■

Theorem 2.4.10 *Suppose that ϕ is homogeneous; then N admits a unique positive eigenvalue λ_N. Moreover if one of the following conditions 2.20 and 2.21 is satisfied*

$$\psi\left(f\left(t,u\right)\right) > \mu_N u \;\forall t \in [0,1]\,,\;\forall u > 0 \tag{2.20}$$

$$\psi\left(f\left(t,u\right)\right) < \mu_N u \;\forall t \in [0,1]\,,\;\forall u > 0 \tag{2.21}$$

where $\mu_N = (\lambda_N)^{-1}$; then Problem (2.15) *admits no positive solutions and if one of the following situations*

$$f^0 < \mu_N < f_\infty \le f^\infty < \infty \tag{2.22}$$

$$f^\infty < \mu_N < f_0 \le f^0 < \infty \tag{2.23}$$

holds, then Problem (2.15) *admits a positive solution.*

Proof. First note that the fact that ϕ is homogeneous implies that N is positively 1-homogeneous; then taking in consideration Lemma 2.4.2, we deduce from Theorem 2.3.11 that N admits a unique positive eigenvalue λ_N and $\mu_N = \left(\lambda_C^N\right)^{-1} = \left(\theta_C^N\right)^{-1}$. Nonexistence of solutions follows directly from Lemma 2.3.3.

Moreover $f^0 < \mu_N < f_\infty \le f^\infty < \infty$ (the other case is checked similarly) implies that there exists $\varepsilon > 0$ small enough and positive constants C_1, C_2 such that

$$F(u) \le (\mu_N - \varepsilon)\,u + G(u) \text{ for all } u \in P^* \cap B(0,\delta)$$
$$(\mu_N + \varepsilon)\,u - C_1 \le F(u) \le (f^\infty + \varepsilon)\,u + C_2 \text{ for all } u \in P^*$$

where $Gu(t) = \max\left\{\psi\left(f\left(t,u\right)\right) - f^0 u(t), 0\right\}$. Thus the existence follows from Theorem 2.3.6 ■

Remark 2.4.11 *First note that the condition ϕ is homogeneous implies that there is $p > 1$ such that for all $x \in \mathbb{R}$, $\phi\left(x\right) = \phi\left(1\right)\left|x\right|^{p-2} x$. The case $p = 1$*

and $a = 1$ in Theorem 2.4.8 has been obtained in [2] (see Proposition 3.2 and Corollary 3.7). We can say that Theorem 2.4.8 generalizes this principle of existence and nonexistence of positive solution following the position of the nonlinearity f about the linearity $\mu_p u$ where μ_p is the first eigenvalue of the $p-$ Laplacian.

Remark 2.4.12 *Note that in the case $a \equiv 1$, Theorem 2.4.8, does not cover the situations $f^0 = 0,\ f_\infty = \infty$ and $f_0 = \infty,\ f^\infty = 0$. Theorems 1.1 and 1.2 in [10] answer these situations when $a \equiv 1$, $f(t,0) = 0$ and $f(t,s) > 0$ for all $s > 0$ and $t \in [0,1]$.*

2.4.2 ϕ-Laplacian bvps with mixed boundary conditions

In the above subsection we have found several difficulties to apply Theorems 2.3.8 and 2.3.9. In this section we present applications of these two theorems to a ϕ-Laplacian bvp with mixed boundary conditions.

Consider the ϕ-Laplacian bvp

$$\begin{cases} -\left(\phi\left(u'\right)\right)'(t) = a(t)\, f(t,u(t)) & t \in (0,1) \\ u(0) = u'(1) = 0 \end{cases} \tag{2.24}$$

and for $p > 1$, let μ_p be the first eigenvalue of

$$\begin{cases} -\left(\phi_p\left(u'\right)\right)'(t) = \mu a(t)\, \phi_p\left(u(t)\right) & t \in (0,1) \\ u(0) = u'(1) = 0. \end{cases} \tag{2.25}$$

where $\phi_p(x) = |x|^{p-2}\, x$.

Let

$$Y = \left\{u \in C^1\left([0,1]\right), u(0) = u'(1) = 0.\right\}$$

equiped with C^1-norm denoted $\|\cdot\|_1$ (for $u \in Y, \|u\|_1 = \max(\|u\|, \|u'\|)$). From the Ascoli-Arzela Theorem the embeding $i_Y : Y \to E$ is compact. It is easy to see that

$$P = \{u \in C, u(x) \geq x \|u\| \ \ \forall x \in [0,1]\}$$

is a cone in E where for $x \in [0,1]$, and $C_Y = C \cap Y = i_Y^{-1}(C)$ is a cone in Y.

Let A, $N : E \to E$ be the operators defined for $h \in E$ by

$$Nh(x) = \int_0^x \psi\left(\int_t^1 a(s)\phi(h(s))\, ds\right) dt \text{ for all } x \in [0,1],$$

$$Au(x) = \int_0^x \psi_p\left(\int_t^1 a(s)\phi_p(u(s))\, ds\right) dt \text{ for all } x \in [0,1],$$

and $A_Y,\ N_Y : Y \to Y$ be the restrictions of A and N to Y. Arguing as in the above section we can see that each of N and N_Y is completely continuous operator and u is a positive solution to (2.24) if and only if u is a fixed point of the completely continuous operator NF. Moreover applying Theorem 2.3.11, by proving that $A(C_Y^*) \subset \widetilde{O} \subset C_Y$, where

$$\widetilde{O} = \{u \in C_Y, u(x) > 0 \ \forall x \in (0,1] \text{ and } u'(0) > 0\}$$

is an open set in Y, we get that A admits a unique positive eigenvalue $\lambda_p = \lambda_C^A = \theta_C^A$ and $(\lambda_p)^{-1} = \mu_p$. As in the above section we have

- N is increasing on C,
- N is upper bounded on C and $\|Nu\| \leq M_\phi^+ \|u\|$ for all $u \in C$ where

$$M_\phi^+ = \psi^+(\|a\|) \int_0^1 \left(\int_t^1 \frac{a(s)}{\|a\|} ds\right)^{\frac{1}{\beta}} dt,$$

- N is lower bounded on P and $\|Nu\| \geq M_\phi^- \|u\|$ for all $u \in P$ where

$$M_\phi^- = \psi^-(\|a\|) \int_0^1 \left(\int_t^1 \frac{a(s)s^\beta}{\|a\|} ds\right)^{\frac{1}{\alpha}} dt.$$

– $0 < M_\phi^- \leq N_P^- \leq \lambda_P^N \leq \theta_P^N \leq N_C^+ \leq M_\phi^+$.

It is clear that with Corollaries 2.4.1, 2.4.5, 2.4.7 and Theorem 2.4.8 hold, we have also the following result which is derived from Theorems 2.3.8 and 2.3.9.

Theorem 2.4.13 *Suppose that ϕ satisfies* (2.12) *and there exists $p > 1$ such that*

$$\lim_{x \to +\infty} \frac{\phi(x)}{\phi_p(x)} = 1; \tag{2.26}$$

then if one of the following situations

$$f^\infty \max\left(M_\phi^+, \lambda_p\right) < 1 < f_0 M_\phi^-; \tag{2.27}$$

$$f^0 M_\phi^+ < 1 < f_\infty \min\left(M_\phi^-, \lambda_p\right) \text{ and } f^\infty < \infty \tag{2.28}$$

holds true, then Problem (2.24) *admits a positive solution.*

Proof. It follows from conditions (2.26) that

$$\lim_{x \to +\infty} \frac{\psi(x)}{\psi_p(x)} = 1,$$

where $\psi_p = \phi_p^{-1}$ then

$$Nu = A(u) + \circ(\|u\|), \text{ at } \infty.$$

Moreover $f^0 M_\phi^+ < 1 < f_\infty \min\left(M_\phi^-, \lambda_p\right)$ and $f^\infty < \infty$ (the other case is checked similarly) imply that there exists $\varepsilon > 0$ small enough and positive constants C_1, C_2 such that

$$F(u) \leq (f^0 + \varepsilon) u \text{ for all } u \in C \cap B(0, \delta)$$

$$(f_\infty - \varepsilon) u - C_1 \leq F(u) \leq (f^\infty + \varepsilon) u + C_2 \text{ for all } u \in C^*$$

and

$$\left(f^0 - \varepsilon\right) M_\phi^+ < 1 < (f_\infty - \varepsilon) \min\left(M_\phi^-, \lambda_p\right).$$

Applying Theorems 2.3.8 and 2.3.9, we deduce existence of a positive solution to Problem (2.24) ■

Remark 2.4.14 *Let us return to the Dirichlet bvps, and consider*

$$\begin{cases} -\left(\phi\left(u'\right)\right)'(t) = g(u(t)) & t \in (0,2) \\ u(0) = u(2) = 0 \end{cases} \tag{2.29}$$

where $g \in C\left([0,+\infty[\,,[0,+\infty[\right).$

It is well known that if u *is a positive solution to* (2.29) *then* u *is concave, symmetric about* 1 *and reaches its maximum at* 1. *Hence, we deduce that* u *is a positive solution to* (2.29) *if and only if the restriction of* u *to* $[0,1]$ *satisfies*

$$\begin{cases} -\left(\phi\left(u'\right)\right)'(t) = g(u(t)) & t \in (0,1) \\ u(0) = u'(1) = 0. \end{cases}$$

So we can derive from the above existence and nonexistence results for positive solutions to Problem (2.29).

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Chapitre 3

Global curve of positive solutions for $\varphi-$Laplacian Dirichlet bvp with at most one turning point

Abstract Under suitable conditions, we prove that the set of positive solutions to the $\varphi-$Laplacian boundary value problem

$$-(\varphi(u'))' = \lambda f(u) \text{ in } (0,1);\ u(0) = u(1) = 0$$

where $\lambda > 0$ is a real parameter, φ is an odd increasing homeomorphism of $\mathbb{R}$ and $f \in C([0,+\infty),[0,+\infty))$, consists on a curve $\|u\| \to \lambda(\|u\|)$.

AMS 2010 Subject Classifications : 34B15, 34B16.

Key words : Second order bvp, Positive solution, Global curve.

3.1 Introduction

The study of existence of positive solutions to classes of semilinear boundary value problems (bvp for short), known as positone problems, has been undertaken by several authors over the last forty years (see for example [6], [13], [15], [27], [29], [32], and references therein). Such a study was initiated by Keller and Cohen [25].

Positive solutions for φ-Laplacian Equations with Dirichlet boundary conditions were studied by Benmezai [7], Benmezai *et al.* [9], [10], de-Coster [12], Dang *et al.* [14], Garcia-Huidobro *et al.* [17], Huang [22], Kaper *et al.* [24], Manásevich *et al.* [30], Rynne [34] and Ubilla [35].

We investigate in this Chapter the existence and the exact number of positive solutions to the second order bvp

$$-(\varphi(u'(x)))' = \lambda f(u(x)), \quad x \in (0,1), \tag{3.1}$$

$$u(0) = u(1) = 0, \tag{3.2}$$

$\lambda > 0$ is a real parameter, φ is an odd increasing homeomorphism of $\mathbb{R}$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous where $\mathbb{R}^+ = [0, +\infty)$. In all this Chapter we assume that

$$f(u) > 0 \; for \; all \; u > 0.$$

By *a positive solution* to problem (3.1)-(3.2), we mean a pair $(\lambda, u) \in (0, +\infty) \times C^1([0,1])$ such that $u \geq 0$ in $(0,1)$, $u(x_0) > 0$ for some $x_0 \in (0,1)$, and (λ, u) satisfies (3.1)-(3.2).

Because of the autonomous character of our problem, the main tool of this Chapter will be the time mapping approach. This method have been used in many papers where several classes of problems related to second order differential Equation are studied. For example this method have been used in [16] and

[33] to prove existence of periodic solutions for some classes of second order differential equations. It has been also used in [1], [8] [15] and [29] to study existence of solutions for semi-linear second order bvps and in [2], [3], [4] and [31] to study existence of solutions for second order BVPs involving the one dimensional $p - Laplacian$.

Roughly speaking, this method consists in calculating the time $T(\lambda, \rho)$ required by a solution of the initial value problem (IVP for short)

$$\begin{cases} -(\varphi(u'))' = \lambda f(u), \\ u(1/2) = \rho, \quad u'(1/2) = 0 \end{cases}$$

to reach the value 0, starting from an extremal value ρ. Clearly, positive solutions of $(3.1) - (3.2)$ are those of the above IVP satisfying $T(\lambda, \rho) = 1/2$.

In the same spirit as that of the papers [2], [11], [27]-[29] and [34], under regularity conditions on the functions φ and f, we obtain by means of the implicit function theorem that the set of positive solutions to (3.1)-(3.2) is reduced to a continuous curve $\lambda : (0, +\infty) \to (0, +\infty)$. Namely, for $\rho > 0$, the pair $(\lambda(\rho), u)$ is a positive solution to (3.1)-(3.2).

In all this Chapter, we understand by $\| \cdot \|$ the sup norm and for a continuously differentiable function u defined on a compact interval, $\|u\|_1 = \|u\| + \|u'\|$.

3.2 Preliminaries

We begin this section by introducing some notations. Let φ and f be as mentioned in the introduction.

– ψ denotes the inverse function of φ,

– for all $x \in \mathbb{R}$, $\Phi(x) = \int_0^x \varphi(s)ds$, $\Psi(x) = \int_0^x \psi(s)ds$, $W(x) = \Psi \circ \varphi(x) = x\varphi(x) - \Phi(x)$.

– Γ is the inverse function of the restriction of W to $\mathbb{R}^+$ and

– for all $x \geq 0$, $F(x) = \displaystyle\int_0^x f(s)\, ds$.

Let

$$A^+ = \left\{ u \in C^1([0,1]) : u > 0 \text{ in } (0,1) \text{ and } u \text{ is symmetrical about } \frac{1}{2} \right\}.$$

Lemma 3.2.1 *If (λ, u) is a solution to (3.1) with $u : (\alpha, \beta) \to \mathbb{R}$ then there exists a real constant C such that*

$$W(u'(x)) + \lambda F(u(x)) = C, \quad \text{for all } x \in (\alpha, \beta). \tag{3.3}$$

Proof. Differentiating the function $x \to W(u'(x)) + \lambda F(u(x))$ over $(0,1)$ we get

$$\psi(\varphi(u'(x)))(\varphi(u'(x)))' + f(u(x))u'(x) = [((\varphi(u'(x)))' + \lambda f(u(x)))]u'(x) = 0.$$

■

Remark 3.2.2 *In fact Lemma 3.2.1 holds even f is not positive on $(0, +\infty)$.*

Now, consider for $\lambda > 0$ and $\rho > 0$ the IVP

$$\begin{cases} -(\varphi(u'))' = \lambda f(u), \\ u(1/2) = \rho, \quad u'(1/2) = 0. \end{cases} \tag{3.4}$$

Setting $\upsilon = \varphi(u')$ the IVP (3.4) is reduced to the first order IVP

$$\begin{cases} u' = \psi(\upsilon) \\ \upsilon' = -\lambda f(u) \\ u(1/2) = \rho \\ \upsilon(1/2) = 0. \end{cases} \tag{3.5}$$

Lemma 3.2.3 *For all $\lambda, \rho > 0$ there exists a unique $T(\lambda,\rho) > 0$ such that the IVP (3.4) admits a unique solution u defined on $[1/2 - T(\lambda,\rho), 1/2 + T(\lambda,\rho)]$. Moreover we have :*

i) $u(1/2 - T(\lambda,\rho)) = u(1/2 + T(\lambda,\rho)) = 0$ *and* $u(t) > 0$ *for all* $t \in (1/2 - T(\lambda,\rho), 1/2 + T(\lambda,\rho))$,

ii) $u'(t) > 0$ *for all* $t \in [1/2 - T(\lambda,\rho), 1/2)$, $u'(t) < 0$ *for all* $t \in (1/2, 1/2 + T(\lambda,\rho)]$ *and* $\|u\| = u(1/2) = \rho$,

iii) u *is symmetrical about* $1/2$ *and*

iv) *for all* $t \in [1/2 - T(\lambda,\rho), 1/2 + T(\lambda,\rho)]$, $u(t) \geq p(t)\rho$, *where*

$$p(t) = \min\left(1 - \frac{1}{T(\lambda,\rho)}(1/2 - t), 1 + \frac{1}{T(\lambda,\rho)}(1/2 - t)\right).$$

Proof. Let u be a maximal solution of (3.4) defined on some interval, say (α,β) where α and β can be infinite. The positivity of f implies that u' is decreasing on (α,β) and u is positive and concave on (α,β). More precisely, $u'(t) > 0$ for all $t \in (\alpha, 1/2)$ and $u'(t) < 0$ for all $t \in (1/2, \beta)$. Thus the limits $\lim_{t\to\beta} u(t)$ and $\lim_{t\to\beta} u'(t)$ exit and are finite. Applying Theorem I.3.2 in [23] on the IVP (3.5) we get $(\lim_{t\to\beta} u(t), \lim_{t\to\beta} u'(t)) \in \partial(\mathbb{R}^+ \times \mathbb{R}) = \{0\} \times \mathbb{R}$, that is $\lim_{t\to\beta} u(t) = 0$. Similarly we have $\lim_{t\to\alpha} u(t) = 0$.

Now let us prove that $-\infty < \alpha < \beta < +\infty$. By the contrary suppose that $\beta = +\infty$ (the case $\alpha > -\infty$ can be checked similarly) and set $\lim_{t\to+\infty} u'(t) = l \leq 0$. Then it follows from (3.3) that $l = -\Gamma(\lambda F(\rho)) < 0$. Thus, L'Hôpital's rule leads to the contradiction

$$0 = \lim_{x\to\to+\infty} \frac{u(x)}{x} = \lim_{x\to+\infty} u'(x) = l < 0.$$

Now let ϑ and ω be respectively the inverse functions of the restrictions of u to $(\alpha, 1/2)$ and $(1/2, \beta)$. We have

$$\vartheta'(u(t)) = \frac{1}{u'(t)}, \quad \text{for all } t \in (\alpha, 1/2)$$

and

$$\omega'(u(t)) = \frac{1}{u'(t)}, \quad \text{for all } t \in (1/2, \beta).$$

Then from (3.3) follows

$$\vartheta'(u(t)) = \frac{1}{\Gamma(\lambda(F(\rho) - F(u(t)))}, \quad \text{for all } t \in (\alpha, 1/2),$$

and

$$\omega'(u(t)) = -\frac{1}{\Gamma(\lambda(F(\rho) - F(u(t)))}, \quad \text{for all } t \in (1/2, \beta).$$

Integrating we get

$$\begin{aligned} x - \alpha &= \vartheta(u(x)) - \vartheta(0) = \int_0^{u(x)} \vartheta'(u(t))du(t) \\ &= \int_0^{u(x)} \frac{du(t)}{\Gamma(\lambda(F(\rho) - F(u(t)))}, \quad \text{for all } x \in [\alpha, 1/2], \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \beta - x &= \omega(0) - \omega(u(x)) = \int_0^{u(x)} -\omega'(u(t))du(t) \\ &= \int_0^{u(x)} \frac{du(t)}{\Gamma(\lambda(F(\rho) - F(u(t)))}, \quad \text{for all } x \in [1/2, \beta]. \end{aligned} \tag{3.7}$$

In particular we have for $x = 1/2$

$$\frac{1}{2} - \alpha = \beta - \frac{1}{2} = \int_0^{\rho} \frac{ds}{\Gamma(\lambda(F(\rho) - F(s))}$$

which implies

$$\alpha = \frac{1}{2} - T(\lambda, \rho) \text{ and } \beta = \frac{1}{2} + T(\lambda, \rho),$$

where

$$T(\lambda, \rho) = \int_0^{\rho} \frac{ds}{\Gamma(\lambda(F(\rho) - F(s))}.$$

At this stage, i) and ii) are proved and let us prove iii). For any $x \in \left[\frac{1}{2} - T(\lambda, \rho), \frac{1}{2}\right]$ the symmetrical point to x relatively to $1/2$ is $y = 1 - x \in$

$\left[\frac{1}{2}, \frac{1}{2} + T(\lambda, \rho)\right]$. Taking in consideration that $\alpha = \frac{1}{2} - T(\lambda, \rho)$ and $\beta = \frac{1}{2} + T(\lambda, \rho)$ we deduce respectively from (3.6) and (3.7) that

$$x - \left(\frac{1}{2} - T(\lambda, \rho)\right) = x + T(\lambda, \rho) - \frac{1}{2} = \int_0^{u(x)} \frac{ds}{\Gamma(\lambda(F(\rho) - F(s))}$$

and

$$\begin{aligned} x + T(\lambda, \rho) - \frac{1}{2} &= \left(\frac{1}{2} + T(\lambda, \rho)\right) - (1 - x) = \left(\frac{1}{2} + T(\lambda, \rho)\right) - y = \\ &= \int_0^{u(y)} \frac{ds}{\Gamma(\lambda(F(\rho) - F(s))} = \int_0^{u(1-x)} \frac{ds}{\Gamma(\lambda(F(\rho) - F(s))}. \end{aligned}$$

Since x is arbitrary, we deduce from the above that for all $x \in \left[\frac{1}{2} - T(\lambda, \rho), \frac{1}{2}\right]$, $u(x) = u(1 - x)$.

At the end, iv) follows from the concavity of u and uniqueness of the solution to (3.4) is due to the fact that ϑ and ω depends only on ρ, λ f and φ. ■

Remark 3.2.4 *Consider the map* $\Pi : (0, +\infty) \times (0, +\infty) \to (0, +\infty) \times C^1([0, 1])$ *where* $\Pi(\lambda, \rho) = (\lambda, u)$,

$$u(x) = \begin{cases} \vartheta^{-1}(x), & \text{if} \quad x \in [1/2 - T(\lambda, \rho), 1/2], \\ \vartheta^{-1}(1 - x), & \text{if} \quad x \in [1/2, 1/2 + T(\lambda, \rho)], \end{cases}$$

and for $\gamma \in [0, \rho]$

$$\vartheta(\gamma) = 1/2 - \int_\gamma^\rho \frac{ds}{\Gamma(\lambda(F(\rho) - F(s))}.$$

In fact Lemma 3.2.2 says that, for λ, ρ *in* $(0, +\infty)$, $\Pi(\lambda, \rho)$ *satisfies (3.4).*

Remark 3.2.5 *We understand from Lemma 3.2.2 and its proof that for any function* $g \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $g(u) > 0$ *for all* $u > 0$ *and all* λ, ρ *in* $(0, +\infty)$,

$$T(\lambda, \rho) = \int_0^\rho \frac{ds}{\Gamma(\lambda(G(\rho) - G(s))} < \infty$$

where $G(u) = \int_0^u g(t)dt$.

Lemma 3.2.6 *For $\rho > 0$ fixed we have*

$$\lim_{\lambda \to 0} T(\lambda, \rho) = +\infty \text{ and } \lim_{\lambda \to +\infty} T(\lambda, \rho) = 0.$$

Proof. We have for $\rho > 0$ fixed,

$$\frac{\rho}{\Gamma(\lambda F(\rho))} \leq T(\lambda, \rho) \leq \frac{1}{2} \frac{\rho}{\Gamma\left(\lambda\left(F(\rho) - F\left(\frac{\rho}{2}\right)\right)\right)} + \int_{1/2}^{1} \frac{\rho}{\Gamma(\lambda(F(\rho) - F(s\rho)))} ds,$$

from which follows immediately that $\lim_{\lambda \to 0} T(\lambda, \rho) = +\infty$.

Let $\lambda_0 > 0$ and $\varkappa = \min_{x \in [\rho/2, \rho]} f(x)$. Then we have

$$\frac{\rho}{\Gamma(\lambda(F(\rho) - F(s\rho)))} \leq \frac{\rho}{\Gamma(\lambda_0 \varkappa (\rho - s\rho))}$$

and from Remark 3.2.3 for $g \equiv \varkappa$, we have

$$\int_{1/2}^{1} \frac{\rho}{\Gamma(\lambda_0 \varkappa (\rho - s\rho))} ds \leq \int_{0}^{\rho} \frac{d\xi}{\Gamma(\lambda_0 \varkappa (\rho - \xi))} < \infty.$$

Thus, by the dominated covergence theorem, we deduce that $\lim_{\lambda \to +\infty} T(\lambda, \rho) = 0$. ■

Lemma 3.2.7 *If (λ, u) is a positive solution to (3.1)-(3.2), then u satisfies (3.4) with $\rho = \|u\|$ and $u \in A^+$.*

Proof. The positivity of f implies that u' is decreasing on $[0, 1]$ and u is concave on $[0, 1]$. More precisely, there exists a unique $\delta \in (0, 1)$ such that $u'(t) > 0$ for all $t \in [0, \delta)$ and $u'(t) < 0$ for all $t \in (\delta, 1]$. Thus arguing as in the proof of Lemma 3.2.2 we prove that $\delta = 1/2$ and u is a solution to (3.4) with $\rho = \|u\|$. Thus we deduce from iii) of Lemma 3.2.2 that u is symmetrical about 1/2. ■

Lemma 3.2.8 *Assume that*

$$\varphi \in C^1(\mathbb{R} \setminus \{0\}) \text{ and there exists } c > 0 \text{ such that } \frac{x^2 \varphi'(x)}{W(x)} \geq c \text{ for all } x > 0. \tag{3.8}$$

Then T is differentiable with respect to λ and $\frac{\partial T}{\partial \lambda}(\lambda, \rho) < 0$ for all $\lambda, \rho > 0$.

Proof. We have $T(\lambda,\rho)=\int_0^1 g(s,\lambda,\rho)\,ds$ where $g(s,\lambda,\rho)=\dfrac{\rho}{\Gamma(\lambda(F(\rho)-F(s\rho)))}$.

Thus

$$\begin{aligned}\frac{\partial g}{\partial\lambda}(s,\lambda,\rho) &= \frac{-\rho}{\lambda}\cdot\frac{\lambda(F(\rho)-F(s\rho))(\Gamma)'(\lambda(F(\rho)-F(s\rho)))}{(\Gamma(\lambda(F(\rho)-F(s\rho))))^2}\\ &= \frac{-\rho}{\lambda}\cdot\frac{W(\Gamma(\lambda(F(\rho)-F(s\rho))))}{(\Gamma(\lambda(F(\rho)-F(s\rho))))^3\varphi'(\Gamma(\lambda(F(\rho)-F(s\rho))))}\end{aligned}$$

and

$$\left|\frac{\partial g}{\partial\lambda}(s,\lambda,\rho)\right|\leq\frac{1}{\lambda c}g(s,\lambda,\rho).$$

For an arbitrary interval $[\alpha,\beta]\subset(0,+\infty)$ we have

$$\int_0^1\left|\frac{\partial g}{\partial\lambda}(s,\lambda,\rho)\right|ds\leq\int_0^1\frac{1}{\alpha c}g(s,\alpha,\rho)\,ds=\frac{T(\alpha,\rho)}{\alpha c}<\infty.$$

So, $\dfrac{\partial T}{\partial\lambda}$ exists and $\dfrac{\partial T}{\partial\lambda}(\lambda,\rho)<0$ for all $\lambda>0$ and $\rho>0$. ■

3.3 Global curve of positive solutions

We deduce from Lemma 3.2.2, if u is a solution of (3.4) with $T(\lambda,\rho)=1/2$, then (λ,u) is a positive solution to (3.1)-(3.2). Conversely and from Lemma 3.2.4, if (λ,u) is a positive solution to (3.1)-(3.2), then u is a solution to (3.4) with $\rho=\|u\|$ and $T(\lambda,\|u\|)=1/2$. Let $S\subset\mathbb{R}\times C^1([0,1])$ be the set of positive solutions to (3.1)-(3.2) and $D=\{(\lambda,\rho)\in(0,+\infty)\times(0,+\infty), T(\lambda,\rho)=1/2\}$. The above equivalence means that the restriction of the map Π, defined in Remark 3.2.2, to D and S is one to one. Therefore, we identify the set S to the set D.

Theorem 3.3.1 *Assume that*

$$f\in C^1(\mathbb{R}^+)\ \ \textit{and}\ \varphi,\psi\in C^1(\mathbb{R}).\tag{3.9}$$

Then the set of positive solutions to (3.1)-(3.2) is reduced to a continuously differentiable curve $\rho \to \lambda(\rho)$ *defined on* $(0, +\infty)$.

Proof. Note that u is a solution to (3.4) if and only if (u, v) is a solution to

$$\begin{cases} u' = \psi(v), \quad v' = -\lambda f(u), \\ u(1/2) = \rho, \quad v(1/2) = 0. \end{cases}$$

Thus, since $f \in C^1(\mathbb{R}^+)$ and $\psi \in C^1(\mathbb{R})$, u is continuously differentiable with respect to all its variables.

Differentiating with the respect to λ the equality

$$u(1/2 + T(\lambda, \rho), \lambda, \rho) = 0,$$

we get

$$u'(1/2 + T(\lambda, \rho), \lambda, \rho) \frac{\partial T}{\partial \lambda} + \frac{\partial u}{\partial \lambda}(1/2 + T(\lambda, \rho), \lambda, \rho) = 0. \tag{3.10}$$

Let us prove that $\frac{\partial u}{\partial \lambda}(1/2 + T(\lambda, \rho), \lambda, \rho) < 0$. Set $z(x, \lambda, \rho) = \frac{\partial u}{\partial \lambda}(x, \lambda, \rho)$. Then z satisfies

$$\begin{cases} -(\varphi'(u') z')' = f(u) + \lambda f'(u) z, \\ z(1/2) = 0, \quad z'(1/2) = 0. \end{cases} \tag{3.11}$$

Multiplying the differential Equation in (3.11) by u', and integrating over $[1/2, x]$, we obtain

$$\varphi'(u'(x)) z'(x) u'(x) + \lambda f(u(x)) z(x) = F(\rho) - F(u(x)). \tag{3.12}$$

We deduce from (3.11) that $(\varphi'(u')z')' < 0$ and $z' < 0$ in a right neightborhood of $1/2$; hence if $z(1/2 + T(\lambda, \rho), \lambda, \rho) \geq 0$, then there exists some $x^* \in (1/2, T(\lambda, \rho))$ such that $z(x^*) = \min_{x \in [1/2, 1/2 + T(\lambda, \rho)]} z(x) < 0$ and $z'(x^*) = 0$. Inserting in (3.12) we arrive to the contradiction

$$0 > \lambda f(u(x^*)) z(x^*) = F(\rho) - F(u(x^*)) > 0.$$

Now, with

$$\frac{\partial u}{\partial \lambda}(1/2 + T(\lambda, \rho), \lambda, \rho) < 0$$

and $u'(1/2 + T(\lambda, \rho), \lambda, \rho) = -\Gamma(\lambda F(\rho)) < 0$, we deduce from (3.10) that

$$\frac{\partial T}{\partial \lambda}(\lambda, \rho) < 0 \text{ for all } \lambda > 0 \text{ and } \rho > 0. \tag{3.13}$$

As in the proof of Theorem 3.3.3, for each $\rho > 0$ there is a unique $\lambda = \lambda(\rho)$ solution to the equation $T(\lambda, \rho) = 1/2$ and since the function $(\lambda, \rho) \to T(\lambda, \rho)$ is continuously differentiable on $(0, +\infty) \times (0, +\infty)$ and $\partial T/\partial \lambda < 0$ the implicit function theorem leads to the assertion of Theorem 3.3.1. ■

Remark 3.3.2 *We can see from the above proof that* $z(x, \lambda, \rho) = \partial u/\partial \lambda(x, \lambda, \rho) < 0$ *for all* $x \in (1/2, 1/2 + T(\lambda, \rho)]$.

It is easy to see that Theorem 3.3.1 does not cover the case $\varphi(x) = |x|^{p-2} x$ where $p \in (1, +\infty)$. The following result is adapted to this case.

Theorem 3.3.3 *Assume that* (3.8) *holds. Then the set of positive solutions to (3.1)-(3.2) is reduced to a continuous curve* $\rho \to \lambda(\rho)$ *defined on* $(0, +\infty)$.

Proof. We deduce from Lemma 3.2.3 and Lemma 3.2.5 that all $\rho > 0$ there exists a unique $\lambda = \lambda(\rho)$ solution of the equation $T(\lambda, \rho) = 1/2$. Moreover, since $\dfrac{\partial T}{\partial \lambda}(\lambda, \rho) < 0$, we deduce from the implicit function theorem that $D = \{(\lambda(\rho), \rho),\ \rho > 0\}$ and $\rho \to \lambda(\rho)$ is continuous. ■

Now set

$$m_\sigma = \liminf_{\rho \to 0} \frac{\varphi(\sigma\rho)}{\varphi(\rho)} \qquad m^\sigma = \limsup_{\rho \to 0} \frac{\varphi(\sigma\rho)}{\varphi(\rho)}$$

$$M_\sigma = \liminf_{\rho \to +\infty} \frac{\varphi(\sigma\rho)}{\varphi(\rho)} \quad M^\sigma = \limsup_{\rho \to +\infty} \frac{\varphi(\sigma\rho)}{\varphi(\rho)}.$$

Proposition 3.3.4 *Assume that* (3.8) *or* (3.9) *holds. We have* *i) if*

$$\begin{array}{c} \lim_{x\to 0} f(x)/\varphi\,(x) = 0 \text{ and } 0 < m_\sigma < \infty \text{ for all } \sigma > 1 \text{ or} \\ \limsup_{x\to 0} f(x)/\varphi\,(x) < \infty \text{ and } m_\sigma = \infty \text{ for all } \sigma > 1, \end{array} \tag{3.14}$$

then $\lim_{\rho\to 0} \lambda\,(\rho) = +\infty$.

ii) if

$$\lim_{x\to 0} f(x)/\varphi\,(x) = +\infty \text{ and } m^\sigma < \infty \text{ for all } \sigma > 1,\ m_\sigma > 0 \text{ for all } \sigma < 1 \tag{3.15}$$

then $\lim_{\rho\to 0} \lambda\,(\rho) = 0$.

iii) if

$$\begin{array}{c} \lim_{x\to +\infty} f(x)/\varphi\,(x) = 0 \text{ and } M_\sigma > 0 \text{ for all } \sigma > 1 \text{ or} \\ \limsup_{x\to +\infty} f(x)/\varphi\,(x) < \infty \text{ and } M_\sigma = \infty \text{ for all } \sigma > 1, \end{array} \tag{3.16}$$

then $\lim_{\rho\to +\infty} \lambda\,(\rho) = +\infty$.

iv) if

$$\lim_{x\to +\infty} f(x)/\varphi\,(x) = +\infty \text{ and } M^\sigma < \infty \text{ for all } \sigma > 1,\ M_\sigma > 0 \text{ for all } \sigma < 1 \tag{3.17}$$

then $\lim_{\rho\to +\infty} \lambda\,(\rho) = 0$.

Proof. Let $\rho > 0$ and u be the unique solution of

$$\begin{cases} -(\varphi(u'))' = \lambda(\rho) f(u), \\ u\,(1/2) = \rho, \quad u'\,(1/2) = 0. \end{cases}$$

Integrating twice, we get

$$\rho = \int_{1/2}^{1} \psi\left(\lambda\,(\rho) \int_{1/2}^{t} f(u(s))ds\right) dt. \tag{3.18}$$

i) If $\limsup_{x\to 0} f(x)/\varphi(x) = l$, then for arbitrary $\epsilon > 0$, there exists $\delta > 0$, such that $f(x) \leq (l+\epsilon)\varphi(x)$ for all $x \in [0,\delta]$. Thus, for $\rho \in (0,\delta)$ we have

$$\begin{aligned}
\rho &= \int_{1/2}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{t} f(u(s))ds\right)dt \\
&\leq \int_{1/2}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{1} (l+\epsilon)\,\varphi(u(s))ds\right)dt \\
&\leq \frac{1}{2}\psi\left(\frac{\lambda(\rho)}{2}(l+\epsilon)\,\varphi(\rho)\right),
\end{aligned}$$

which implies

$$\lambda(\rho) \geq \frac{2\varphi(2\rho)}{\varphi(\rho)(l+\epsilon)}.$$

Letting $\rho \to 0$, we get if $m_2 = +\infty$

$$\lim_{\rho\to 0} \lambda(\rho) = +\infty$$

and if $m_2 < \infty$ and $l = 0$

$$\liminf_{\rho\to 0} \lambda(\rho) \geq \frac{2m_2}{\epsilon}.$$

Since ϵ is arbitrary, we have $\lim_{\rho\to 0}\lambda(\rho) = +\infty$.

ii) If $\lim_{x\to 0} f(x)/\varphi(x) = +\infty$, then for arbitrary $K > 0$, there exists $\delta > 0$, such that $f(x) \geq K\varphi(x)$ for all $x \in [0,\delta]$. Thus, for $\rho \in (0,\delta)$ we have from (3.18) and iv) of Lemma 3.2.2

$$\begin{aligned}
\rho &\geq \int_{1/2}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{t} K\varphi(u(s))ds\right)dt \\
&\geq \int_{3/4}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{3/4} K\varphi(u(s))ds\right)dt \\
&\geq \int_{3/4}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{3/4} K\varphi(2\rho(1-s))ds\right)dt \\
&\geq \frac{1}{4}\psi\left(\frac{\lambda(\rho)}{4}K\varphi\left(\frac{\rho}{2}\right)\right),
\end{aligned}$$

which implies

$$\lambda(\rho) \leq \frac{4\varphi(4\rho)}{K\varphi(\rho/2)}.$$

Letting $\rho \to 0$, we get

$$\limsup_{\rho\to 0} \lambda(\rho) \leq \frac{4m^4}{Km_{1/2}}.$$

Since K is arbitrary, we have $\lim_{\rho\to 0} \lambda(\rho) = 0$.

iii) If $\limsup_{x\to+\infty} f(x)/\varphi(x) = l$, then for arbitrary $\epsilon > 0$, there exists $C_\epsilon > 0$, such that $f(x) \leq (l+\epsilon)\varphi(x) + C_\epsilon$ for all $x \geq 0$. Thus, for $\rho > 0$ we have

$$\begin{aligned}
\rho &= \int_{1/2}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{t} f(u(s))ds\right)dt \\
&\leq \int_{1/2}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{1} ((l+\epsilon)\varphi(u(s)) + C_\epsilon)\, ds\right)dt \\
&\leq \frac{1}{2}\psi\left(\frac{\lambda(\rho)}{2}((l+\epsilon)\varphi(\rho) + C_\epsilon)\right)
\end{aligned}$$

which implies

$$\lambda(\rho) \geq \frac{2\varphi(2\rho)}{\varphi(\rho)}\frac{1}{(l+\epsilon) + \frac{C_\epsilon}{\varphi(\rho)}}.$$

As in i), we conclude that $\lim_{\rho\to+\infty} \lambda(\rho) = +\infty$.

iv) If $\lim_{x\to+\infty} f(x)/\varphi(x) = +\infty$, then for arbitrary $K > 0$, there exists $B > 0$, such that $f(x) \geq K\varphi(x)$ for all $x \geq B$. Thus, for $\rho \geq 2B$ as in ii) we have

$$\begin{aligned}
\rho &\geq \int_{3/4}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{3/4} f(u(s))ds\right)dt \\
&\geq \int_{3/4}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{3/4} K\varphi(u(s))ds\right)dt \\
&\geq \int_{3/4}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{3/4} K\varphi(2\rho(1-s))ds\right)dt \\
&\geq \frac{1}{4}\psi\left(\frac{\lambda(\rho)}{4}K\varphi\left(\frac{\rho}{2}\right)\right),
\end{aligned}$$

which implies

$$\lambda(\rho) \leq \frac{4\varphi(4\rho)}{K\varphi(\rho/2)}.$$

Letting $\rho \to +\infty$, we get

$$\lim_{\rho\to+\infty}\sup \lambda(\rho) \leq \frac{4M^4}{KM_{1/2}}.$$

Since K is arbitrary, this means that $\lim_{\rho\to+\infty} \lambda(\rho) = 0$. ■

Remark 3.3.5 *Some conditions on m_σ, m^σ, M_σ and M^σ has been assumed in many papers where φ-Laplacian BVPs are studied, (see for example [9], [10], [14], [17], [18], [19], [20] and [21]). A typical example of a function φ satisfying $0 < m_\sigma, m^\sigma, M_\sigma, M^\sigma < \infty$ is $\varphi(x) = |x|^{p-2}x + |x|^{q-2}x$ with $1 < p < q$.*

Proposition 3.3.6 *Assume that* (3.8) *or* (3.9) *hold. Then we have :*

- *i) if* $f(0) > 0$, *then* $\lim_{\rho\to 0} \lambda(\rho) = 0$,
- *ii) if* $\lim_{x\to 0} F(x)/W(x) = 0$, *then* $\lim_{\rho\to 0} \lambda(\rho) = +\infty$ *and*
- *iii) if* $\lim_{x\to+\infty} F(x)/W(x) = 0$, *then* $\lim_{\rho\to+\infty} \lambda(\rho) = +\infty$.

Proof. Let $\rho > 0$ and u be the unique solution to (3.4) which $\lambda = \lambda(\rho)$ and choose $\eta > 0$ small enough. Since $f(0) > 0$,

$$\delta_\eta = \min_{u\in[0,\eta]} f(u) > 0.$$

Thus, it follows from (3.18) that, if $\rho \in [0, \eta]$ then

$$\begin{aligned} \rho &\geq \int_{3/4}^{1} \psi\left(\lambda(\rho)\int_{1/2}^{3/4} f(u(s))ds\right)dt \\ &\geq \frac{1}{4}\psi\left(\frac{\lambda(\rho)\,\delta_\eta}{4}\right), \end{aligned}$$

leading to

$$\lim_{\rho\to 0}\lambda(\rho) \leq \lim_{\rho\to 0}\frac{4\varphi(4\rho)}{\delta_\eta} = 0,$$

which proves i). We have from (3.3),

$$\rho = \int_0^{1/2} u'(t)dt \leq \int_0^{1/2} u'(0)dt \leq \Gamma\left(\lambda\left(\rho\right) F\left(\rho\right)\right),$$

which implies

$$\lambda\left(\rho\right) \geq \frac{W\left(\rho\right)}{F\left(\rho\right)} = \left(\frac{F\left(\rho\right)}{W\left(\rho\right)}\right)^{-1},$$

leading to assertions ii) and iii) of the proposition. ■

Remark 3.3.7 *By L'Hopital's rule, if* $\varphi \in C^1\left(\mathbb{R} \setminus \{0\}\right)$ *and* $\lim_{x\to 0} f(x)/x\varphi'(x) = 0$ *(respectively* $\lim_{x\to +\infty} f(x)/x\varphi'(x) = 0$*) then* $\lim_{x\to 0} F(x)/W(x) = 0$ *(respectively* $\lim_{x\to +\infty} F(x)/W(x) = 0$*).*

3.4 Existence and multiplicity results

We deduce immediately from Theorem 3.3.3, Proposition 3.3.4, Proposition 3.3.6 and Remark 3.3.7 the following corollaries.

Corollary 3.4.1 *Assume that (3.8) or (3.9) holds. Then Problem (3.1)-(3.2) admits at least one positive solution for all* $\lambda > 0$*, in each of the following situations i)-vi) :*

i) $\lim_{x\to 0} f(x)/x\varphi'(x) = 0$ *and (3.17) holds.*

ii) Hypotheses (3.14) and (3.17) hold.

iii) $f(0) = 0,\ \lim_{x\to +\infty} f(x)/x\varphi'(x) = 0$ *and (3.15) holds.*

iv) $f(0) = 0,$ *(3.15) and (3.16) hold.*

v) $f(0) > 0$ *and* $\lim_{x\to +\infty} f(x)/x\varphi'(x) = 0.$

vi) $f(0) > 0$ *and (3.16) holds.*

Corollary 3.4.2 *Assume that* (3.8) *or* (3.9) *and the following condition*

$$\lim_{x\to 0} \frac{f(x)}{x\varphi'(x)} = \lim_{x\to +\infty} \frac{f(x)}{x\varphi'(x)} = 0$$

hold. Then there exists $\lambda^+ > 0$ *such that :*

i) Problem (3.1)-(3.2) admits no positive solution if $\lambda < \lambda^+$,

ii) Problem (3.1)-(3.2) admits at least one positive solution if $\lambda = \lambda^+$ *and*

iii) Problem (3.1)-(3.2) admits at least two positive solutions if $\lambda > \lambda^+$.

Corollary 3.4.3 *Assume that (3.8), (3.14) and (3.16) or (3.9), (3.14) and (3.16) hold. Then there exists* $\lambda^+ > 0$ *such that :*

i) Problem (3.1)-(3.2) admits no positive solution if $\lambda < \lambda^+$,

ii) Problem (3.1)-(3.2) admits at least one positive solution if $\lambda = \lambda^+$ *and*

iii) Problem (3.1)-(3.2) admits at least two positive solutions if $\lambda > \lambda^+$.

Corollary 3.4.4 *Assume that (3.8), (3.15) and (3.17) or (3.9), (3.15) and (3.17) hold. Then there exists* $\lambda^+ > 0$ *such that :*

i) Problem (3.1)-(3.2) admits no positive solution if $\lambda > \lambda^+$,

ii) Problem (3.1)-(3.2) admits at least one positive solution if $\lambda = \lambda^+$ *and*

iii) Problem (3.1)-(3.2) admits at least two positive solutions if $\lambda < \lambda^+$.

Remark 3.4.5 *We can prove, as in the proof of uniqueness in [7], that, if* $\varphi(u)/u$ *and* $f(u)/u$ *are respectively decreasing and increasing on* $(0, +\infty)$ *in the case i), or* $\varphi(u)/u$ *and* $f(u)/u$ *are respectively increasing and decreasing on* $(0, +\infty)$ *in the case ii), the positive solution obtained in Corollary 3.4.1 is unique.*

Now, with more regularity on φ and f, we will prove that the curve $\rho \to \lambda(\rho)$ admits at most one critical point. To this aim we assume in the following that $f \in C^2(\mathbb{R}^+)$ and $\phi \in C^2(\mathbb{R})$. Note that in this case the unique solution $u(\cdot, \lambda, \rho)$ of (3.4) is twice continuously differentiable with respect to $(x, \lambda, \rho) \in [1/2 - T(\lambda, \rho), 1/2 + T(\lambda, \rho)] \times (0, +\infty) \times (0, +\infty)$ and denote $v = \dfrac{\partial u}{\partial \rho}$ and

$w = \dfrac{\partial^2 u}{\partial \rho^2}$. Then v and w satisfy respectively

$$\begin{cases} -(\varphi'(u')v')' = \lambda f'(u)v, \\ v(1/2) = 1, \quad v'(1/2) = 0, \end{cases} \tag{3.19}$$

and

$$\begin{cases} -\left(\varphi''\left(u'\right)\left(v'\right)^2 + \varphi'\left(u'\right)w'\right)' = \lambda f''\left(u\right)v^2 + \lambda f'\left(u\right)w, \\ w\left(1/2\right) = 0, \quad w'\left(1/2\right) = 0. \end{cases} \tag{3.20}$$

Lemma 3.4.6 *Assume that $\psi \in C^1(\mathbb{R})$. If $u(\cdot, \lambda, \rho)$ is such that (λ, u) is positive solution to problem (3.1)-(3.2) then v has at most one zero in $[1/2, 1]$.*

Proof. First note that if $x_0 \in [1/2, 1]$ is such that $v(x_0) = 0$ then $v'(x_0) \neq 0$. Otherwise, if we set $\theta = \varphi'(u')v'$ then the pair (v, θ) is solution to the IVP

$$\begin{cases} v' = (\varphi'(u'))^{-1}\theta, \quad \theta' = -\lambda f'(u)v, \\ v(x_0) = 0, \quad \theta(x_0) = 0. \end{cases}$$

Note that $(\varphi'(u'))^{-1} = \psi'(\varphi(u'))$ and the right-hand side of the above system is locally Lipschitzian. This makes $v = 0$, which contradicts to $v(1/2) = 1$.

Note that v admits a finite number of zeros, indeed if $(x_n)_{n\geq 1}$ is a sequence of zeros of v and $x^* = \lim_{n\to\infty} x_n$, then

$$v(x^*) = \lim_{n\to\infty} v(x_n) = 0 = v'(x^*) = \lim_{n\to\infty} \frac{v(x_n) - v(x^*)}{x_n - x^*}$$

and for the pair (v, θ) satisfies

$$\begin{cases} v' = (\varphi'(u'))^{-1}\theta, \quad \theta' = -\lambda f'(u)v, \\ v(x^*) = 0, \quad \theta(x^*) = 0. \end{cases}$$

So, we get for the same reasons $v = 0$, which contradicts to $v(1/2) = 1$.

Now multiplying (3.19) by u' and integrating over $[1/2, x]$, we get

$$\varphi'(u'(x))v'(x)\,u'(x) + \lambda f(u(x))\,v(x) = \lambda f(\rho)\,. \tag{3.21}$$

Suppose that v admits more than one zero and let $x_1 < x_2$ be the first two consecutive zeros of v. Then

$$v'(x_1) < 0 \text{ and } v'(x_2) > 0. \tag{3.22}$$

Substituting $x = x_2$ in (3.21), we get

$$\varphi'(u'(x_2))v'(x_2)u'(x_2) = \lambda f(\rho). \tag{3.23}$$

Since $\varphi'(u'(x_2)) > 0$, $u'(x_2) < 0$ and $\lambda f(\rho) > 0$, we deduce from (3.23) that $v'(x_2) < 0$, contradicting (3.22). This completes the proof of the lemma. ■

Lemma 3.4.7 *Assume that $f \in C^2(\mathbb{R}^+)$, $\varphi, \psi \in C^2(\mathbb{R})$ and for all $x > 0$, $f''(x) > 0$ and $\varphi''(x) \leq 0$. Let $u(\cdot, \lambda, \rho)$ be such that (λ, u) is a positive solution to problem (3.1)-(3.2). If $v(1) = 0$, then $w(1) < 0$.*

Proof. Multiplying (3.19) by w and integrating on $(1/2, 1)$ we get

$$-\varphi'(u'(1))\,v'(1)\,w(1) + \int_{1/2}^{1} \varphi'(u')\,v'w' = \lambda \int_{1/2}^{1} f'(u)\,vw. \tag{3.24}$$

Similarly, multiplying (3.20) by v and integrating on $(1/2, 1)$ we get

$$-\varphi'(u'(1))(v'(1))^2 v(1) - \varphi'(u'(1))v(1)w'(1) + \int_{1/2}^{1} (\varphi''(u')(v')^3 + \varphi'(u')v'w')$$
$$= \lambda \int_{1/2}^{1} f''(u)v^3 + \lambda \int_{1/2}^{1} f'(u)vw. \tag{3.25}$$

Subtracting (3.25) and (3.24) and taking in consideration $v(1) = 0$ we get

$$\varphi'(u'(1))\,v'(1)\,w(1) = \lambda \int_{1/2}^{1} f''(u)\,v^3 - \int_{1/2}^{1} \varphi''(u')\,(v')^3\,. \tag{3.26}$$

Note that since $v(1) = 0$, Lemma 3.4.1 leads to $v > 0$ in $[1/2, 1)$ and $v'(1) < 0$. Thus, the convexity of φ and the oddness of φ'' leads to $\varphi''(u') > 0$ in $(1/2, 1]$.

It remains to investigate the sign of v'. We deduce from (3.19)

$$-\varphi''(u')\, u''v' - \varphi'(u')\, v'' = \lambda f'(u)\, v, \quad \text{in } (0,1). \tag{3.27}$$

As $v'(1/2) = 0$, $v(1) = 0$ and $v > 0$ in $[1/2, 1)$ we have :

- either $v' \leq 0$ in $[1/2, 1]$,
- or there exists $x_0 \in (1/2, 1]$ such that $v'(x_0) > 0$.

In fact the second situation does not occur ; indeed if v' changes sign, then it would exist x_1 and x_2 belonging to $(1/2, 1)$ such that $x_1 < x_2$ and at both x_1 and x_2, v reachs respectively a local minimum and a local maximum. In this case substituting respectively $x = x_1$ and $x = x_2$ in (3.27) we get

$$\begin{aligned} -\varphi'(u'(x_1))v''(x_1) &= \lambda f'(u(x_1))v(x_1) \leq 0, \\ -\varphi'(u'(x_2))v''(x_2) &= \lambda f'(u(x_2))v(x_2) \geq 0, \end{aligned}$$

so

$$f'(u(x_1)) \leq 0 \text{ and } f'(u(x_2)) \geq 0.$$

But this is impossible because $u(x_1) > u(x_2)$ and f' is increasing. Hence $v' \leq 0$ in $[1/2, 1]$.

Finally taking in consideration the fact that $v \geq 0$ and $v' \leq 0$ in $[1/2, 1]$, $f''(u) > 0$ and $\varphi''(u') > 0$ in $[1/2, 1]$, $\varphi'(u'(1)) > 0$ and $v'(1) < 0$, we deduce from (3.26) that $w(1) < 0$. This completes the proof. ■

Lemma 3.4.8 *Assume that $f \in C^2(\mathbb{R}^+), \varphi, \psi \in C^2(\mathbb{R})$ and for all $x \in \mathbb{R}^+$, $f''(x) < 0$ and $\varphi''(x) \geq 0$. Let $u(\cdot, \lambda, \rho)$ be such that (λ, u) is a positive solution to Problem (3.1)-(3.2). If $v(1) = 0$ then $w(1) > 0$.*

Proof. First we claim that $f'(x) \geq 0$ for all $x \geq 0$. Indeed, if there exists some $x_0 \geq 0$ such that $f'(x_0) < 0$ then $\lim_{x\to+\infty} f'(x) = 0$, otherwise, if $\lim_{x\to+\infty} f'(x) = l < 0$ then $\lim_{x\to+\infty} f(x)/x = l < 0$ which contradicts the positiveness of f. In this case there exists $\overline{x} > 0$ such that $f'(\overline{x}) = \min_{x\geq 0} f'(x)$ and $f''(\overline{x}) = 0$, which contradicts the positiveness of f''.

Now it is easy to see from (3.19) that if $v(1) = 0$ then $v > 0$ in $[1/2, 1)$ and $v' < 0$ in $(1/2, 1]$. Thus, as in the proof of Lemma 3.4.2 we deduce from (3.26) that $w(1) > 0$. ■

Theorem 3.4.9 *Assume that $f \in C^2(\mathbb{R}^+)$, $\varphi, \psi \in C^2(\mathbb{R})$ and one of the following condition holds,*

$$f''(x) < 0, \text{ and } \varphi''(x) \geq 0 \text{ for all } x > 0, \tag{3.28}$$

$$f''(x) > 0, \text{ and } \varphi''(x) \leq 0 \text{ for all } x > 0. \tag{3.29}$$

Then $\rho \to \lambda(\rho)$ admits on $(0, +\infty)$ at most one critical point.

Proof. We obtain the desired result by proving that if $\lambda'(\rho) = 0$ for some $\rho \in D$ then $\lambda''(\rho) < 0$ or $\lambda''(\rho) > 0$. We have for all $\rho > 0$

$$u(1, \lambda(\rho), \rho) = 0. \tag{3.30}$$

Differentiating in (3.30) with respect to ρ, we get

$$\frac{\partial u}{\partial \lambda}(1, \lambda(\rho), \rho)\lambda'(\rho) + v(1, \lambda(\rho), \rho) = 0. \tag{3.31}$$

and

$$\begin{aligned} &\frac{\partial^2 u}{\partial \lambda^2}(1, \lambda(\rho), \rho)(\lambda'(\rho))^2 + \frac{\partial u}{\partial \lambda}(1, \lambda(\rho), \rho)\lambda''(\rho) \\ &+\frac{\partial v}{\partial \lambda}(1, \lambda(\rho), \rho)\lambda'(\rho) + w(1, \lambda(\rho), \rho) = 0. \end{aligned} \tag{3.32}$$

Suppose that (3.29) holds (the other case can be checked similarly), then if for some $\rho_0 > 0$, $\lambda'(\rho_0) = 0$, then we deduce from (3.31) that $v(1, \lambda(\rho_0), \rho_0) =$

0 and it follows from Lemma 3.4.2, that $w(1, \lambda(\rho_0), \rho_0) < 0$. Thus, we deduce from (3.32) and Remark 3.3.2 that $\lambda''(\rho_0) < 0$. This completes the proof. ■

We deduce from Theorems 3.3.3, 3.3.4 and Proposition 3.3.6 the following corollaries.

Corollary 3.4.10 *Assume that $f \in C^2(\mathbb{R}^+)$, $\varphi, \psi \in C^2(\mathbb{R})$, (3.29), (3.15) and (3.17) hold. Then there exists $\lambda^+ > 0$ such that :*

i) Problem (3.1)-(3.2) admits no positive solution if $\lambda > \lambda^+$,

ii) Problem (3.1)-(3.2) admits exactly one positive solution if $\lambda = \lambda^+$ and

iii) Problem (3.1)-(3.2) admits exactly two positive solutions if $\lambda < \lambda^+$.

Corollary 3.4.11 *Assume that $f \in C^2(\mathbb{R}^+)$, $\varphi, \psi \in C^2(\mathbb{R})$, (3.16) and (3.28) hold. Then there exists $\lambda^+ \geq 0$ such that :*

i) Problem (3.1)-(3.2) admits no positive solution if $\lambda \leq \lambda^+$ and

ii) Problem (3.1)-(3.2) admits exactly one positive solution if $\lambda > \lambda^+$.

Corollary 3.4.12 *Assume that $f \in C^2(\mathbb{R}^+)$, $\varphi, \psi \in C^2(\mathbb{R})$ and (3.28), hold. If $\lim_{x\to+\infty}(f(x)/x\varphi'(x)) = 0$ then there exists $\lambda^+ \geq 0$ such that*

i) Problem (3.1)-(3.2) admits no positive solution if $\lambda \leq \lambda^+$ and

ii) Problem (3.1)-(3.2) admits exactly one positive solution if $\lambda > \lambda^+$.

Remark 3.4.13 *It is clear that in corollaries 3.4.8 and 3.4.9, $\lambda^+ = \lim_{\rho\to 0}\lambda(\rho)$ and because of the inequality $\lambda(\rho) \geq \frac{W(\rho)}{F(\rho)}$ it can happens that $\lambda^+ > 0$.*

Remark 3.4.14 *In corollaries 3.4.8 and 3.4.9, we cannot assume that $\lim_{x\to 0}(f(x)/x\varphi'(x)) = 0$ or $\lim_{x\to 0}(f(x)/\varphi(x)) = 0$, in order to obtain multiplicity results. This difficulty is caused by the fact that $\varphi'(0) > 0$ and $(f(x)/x)$ and $(x/\varphi(x))$ are decreasing functions on $(0, +\infty)$.*

Remark 3.4.15 *In case where $\varphi(x) = x$, many exact multiplicity results have been obtained under a requirement on the convexity of the nonlinear term, (see Theorem 3.2 in [29], Theorem 2 in [5], Theorem 1 and Theorem 2 in [11]). Moreover, note that for $\lambda = 1$, Problem (3.1)-(3.2) admits at most one positive solution. This result has been obtained by Korman and Li in [26].*

3.5 Examples

Example 3.5.1 *Consider the bvp (3.1)-(3.2) with*

$$\varphi(u) = \begin{cases} e^u - 1, & \text{for} \quad u \geq 0, \\ 1 - e^{-u}, & \text{for} \quad u \leq 0, \end{cases}$$

and $f(u) = e^u$.

We have $\varphi, \psi, f \in C^1(\mathbb{R})$ and

$$f(0) > 0 \text{ and } \lim_{x \to +\infty} \frac{f(x)}{x\varphi'(x)} = 0.$$

Thus, we deduce from iii) of Corollary 3.4.1 that for all $\lambda > 0$, the bvp (3.1)-(3.2) admits at least one positive solution.

Note that in this example we have

$$\begin{aligned} \lim_{u \to +\infty} (f(u)/\varphi(u)) &= 1 \text{ and} \\ \lim_{\rho \to +\infty} (\varphi(\sigma\rho)/\varphi(\rho)) &= +\infty \text{ for all } \sigma > 1. \end{aligned}$$

Example 3.5.2 *Consider the bvp (3.1)-(3.2) with*

$$\varphi(u) = \begin{cases} e^u - 1, & \text{for} \quad u \geq 0, \\ 1 - e^{-u}, & \text{for} \quad u \leq 0, \end{cases}$$

and $f(u) = u^2$.

We have

$$\lim_{u\to 0}\frac{f(u)}{u\varphi'(u)} = \lim_{u\to +\infty}\frac{f(u)}{u\varphi'(u)} = 0.$$

Thus, we deduce from Corollary 3.4.2 that there exists $\lambda^+ > 0$ *such that :*

i) Problem (3.1)-(3.2) admits no positive solution if $\lambda < \lambda^+$,

ii) Problem (3.1)-(3.2) admits at least one positive solution if $\lambda = \lambda^+$ *and*

iii) Problem (3.1)-(3.2) admits at least two positive solutions if $\lambda > \lambda^+$.

Example 3.5.3 *Consider the bvp (3.1)-(3.2) with* $\varphi(u) = \sinh(u)$ *and* $f(u) = \sqrt{1+u}$.

Since $\lim_{\rho\to 0}\lambda(\rho) = 0$, *we have from Corollary 3.4.9 that for all* $\lambda > 0$ *Problem (3.1)-(3.2) admits exactly one positive solution.*

Example 3.5.4 *Consider the bvp (3.1)-(3.2) with* $\varphi(u) = u + \dfrac{u}{\sqrt{1+u^2}}$ *and* $f(u) = 1 + u^2$.

We have from Corollary 3.4.7 that there exists $\lambda^+ > 0$ *such that :*

i) Problem (3.1)-(3.2) admits no positive solution if $\lambda > \lambda^+$,

ii) Problem (3.1)-(3.2) admits exactly one positive solution if $\lambda = \lambda^+$ *and*

iii) Problem (3.1)-(3.2) admits exactly two positive solutions if $\lambda < \lambda^+$.

Example 3.5.5 *Consider the bvp (3.1)-(3.2) with* $\varphi(u) = |u|^{p-2}u + |u|^{q-2}u$ *where* $1 < p < q$ *and* $f(u) = e^u$.

We have from Corollary 3.4.4 that there exists $\lambda^+ > 0$ *such that :*

i) Problem (3.1)-(3.2) admits no positive solution if $\lambda > \lambda^+$,

ii) Problem (3.1)-(3.2) admits at least one positive solution if $\lambda = \lambda^+$ *and*

iii) Problem (3.1)-(3.2) admits at least two positive solutions if $\lambda < \lambda^+$.

The case $p = q$ *has been considered in [2] where exactness result is obtained.*

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Chapitre 4

Existence of unbounded positive solution for ϕ-Laplacian BVPs on the half line

Abstract We provide in this work sufficient conditions for existence of positive unbounded solutions to the boundary value problem

$$\begin{cases} -\left(\phi\left(u'\right)\right)' = a(t)f(t,u), \quad t \in (0,+\infty), \\ u\left(0\right) = 0, \ \lim\limits_{t\to+\infty} u'(t) = 0 \end{cases}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing homeomorphism with $\phi\left(0\right) = 0$, $a : (0,+\infty) \to \mathbb{R}^+$ is a measurable function which does not vanish identically on $(0,+\infty)$ and the function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

AMS 2010 Subject Classifications : 34B15, 34B40.

Key words : ϕ-Laplacian BVPs, positive solution, fixed point theorem.

4.1 Introduction

In this Chapter, we are concerned with existence of positive unbounded solutions to the $\phi-$Laplacian boundary value problem

$$\begin{cases} -\left(\phi\left(u'\right)\right)' = a(t)f(t,u), \quad t \in (0,+\infty), \\ u\left(0\right) = 0, \ \lim\limits_{t\to+\infty} u'(t) = 0 \end{cases} \tag{4.1}$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is an increasing homeomorphism with $\phi(0) = 0$, $\mathbb{R}^+ = [0,+\infty)$, $a : (0,+\infty) \longrightarrow \mathbb{R}^+$ is a measurable function and $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function.

By a positive solution to Problem (4.1), we mean a function $u \in C^1\left(\mathbb{R}^+, \mathbb{R}^+\right)$ such that $\phi\left(u'\right) \in C^1\left((0,+\infty), \mathbb{R}\right)$ and $u(t_0) > 0$ for some $t_0 > 0$, satisfying the equation in (4.1).

Our main result will be obtained under the following assumptions :

$$\begin{cases} \text{There exist } m \in C\left((0,+\infty), \mathbb{R}^+\right) \text{ and } g \in C\left(\mathbb{R}^+, \mathbb{R}^+\right) \\ \text{such that } f(t,(1+t)w) \leq m(t)g(w) \text{ for all } t, w \in \mathbb{R}^+ \\ \text{and } \displaystyle\int_0^{+\infty} a(t)m(t)dt < \infty, \end{cases} \tag{4.2}$$

$$\begin{cases} \lim\limits_{t\to+\infty} \dfrac{1}{1+t} \int_0^t \psi\left(\int_s^{+\infty} a(\tau)m(\tau)d\tau\right) ds = 0, \end{cases} \tag{4.3}$$

$$\begin{cases} \lim\limits_{t\to+\infty} t\psi\left(\displaystyle\int_t^{+\infty} a(t)f\left(t,\lambda\right)dt\right) = +\infty \text{ uniformly for } \lambda \\ \text{in compact intervals of } (0,+\infty), \end{cases} \tag{4.4}$$

and

$$\begin{cases} \text{there exists } \alpha > 0 \text{ such that for all } t \in [0,1] \text{ and } u \in \mathbb{R}^+, \\ \phi(tu) \geq t^\alpha \phi(u), \end{cases} \tag{4.5}$$

where ψ denotes the inverse function of ϕ.

Because of their mathematical and physical interest, the study of second order differential equations posed on the half line and subject to various boundary conditions have received a great deal of attention during the latter two decades; see [9]-[14] and references therein.

This work is motivated by that in [12], where D. O'Regan *et al.* have considered Problem (4.1) with $\phi(u) = u$ and f may be singular at $u = 0$. They have obtained existence and multiplicity results for positive solutions in the functional space consisting in the functions $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfying $\lim_{t \to +\infty} u(t)/(1+t) = 0$ endowed with the norm $\|u\| = \sup_{t \in \mathbb{R}^+} \dfrac{|u(t)|}{1+t}$. Clearly, the choice of this space is motivated by the boundary condition in Problem (4.1), $u'(+\infty) = 0$ and fortunately, this space provide a good framework where the fixed point index theory or theorems of cone expansion and compression in a Banach space can be used.

The unique disadvantage of this framework is that we know nothing about the boundeness of the obtained positive solutions.

The main goal of this Chapter is to provide existence results for positive unbounded solutions under additional assumptions on the behavior of the ratio $f(t,u)/\phi(u)$ at 0 and $+\infty$ as those obtained in [1]-[6]. We will make use of a theorem of cone expansion and compression in a Banach space in the same framework as that in [12]. In contrast to [12], here hypothesis (4.4) ensure that the obtained positive solution is unbounded.

4.2 Preliminaries

The main tool of this Chapter is the following theorem of cone expansion and compression in a Banach space.

Theorem 4.2.1 *([8]) Let X be a Banach space and P be a cone of X. Assume Ω_1, Ω_2 are open bounded subsets of X with $0 \in \Omega_1 \subset \bar{\Omega_1} \subset \Omega_2$ and let $A : P \cap \left(\bar{\Omega_2} \backslash \Omega_1\right) \longrightarrow P$ be a completely continuous operator such that either*

i) $\|Au\| \leq \|u\|$ *for all* $u \in P \cap \partial\Omega_1$ *and* $\|Au\| \geq \|u\|$ *for all* $u \in P \cap \partial\Omega_2$,

ii) $\|Au\| \leq \|u\|$ *for all* $u \in P \cap \partial\Omega_2$ *and* $\|Au\| \geq \|u\|$ *for all* $u \in P \cap \partial\Omega_1$.

Then A has at least one fixed point in $P \cap \left(\bar{\Omega_2} \backslash \Omega_1\right)$.

In all this Chapter E denotes the Banach space defined

$$E = \left\{ u \in C\left(\mathbb{R}^+, \mathbb{R}^+\right) : \lim_{t \to +\infty} \frac{u(t)}{1+t} = 0 \right\}$$

equiped with the norm $\|\cdot\|_E$, defined for $u \in E$ by $\|u\|_E = \sup_{t \geq 0} \left| \frac{u(t)}{1+t} \right|$. In order to prove the compacteness of some operator we will make use of the following lemma.

Lemma 4.2.2 *([7], [12], [13]) A nonempty subset M of E is relatively compact if the following conditions hold :*

(a) M is bounded in E,

(b) the functions belonging to $\left\{ u : u(t) = \frac{x(t)}{1+t},\ x \in M \right\}$ are locally equicontinuous on $[0, +\infty)$, that is, equicontinuous on every compact interval of $\mathbb{R}^+$ and

(c) the functions belonging to $\left\{ u : u(t) = \frac{x(t)}{1+t},\ x \in M \right\}$ are equiconvergent at $+\infty$, that is, given $\epsilon > 0$, there corresponds $T(\epsilon) > 0$ such that $|x(t) - x(+\infty)| < \epsilon$ for any $t \geq T(\epsilon)$ and $x \in M$.

Throughout, K is the cone of E given by

$$K = \{u \in E,\ u \geq 0,\ u(0) = 0 \text{ and u is concave in } (0, +\infty)\}. \tag{4.6}$$

Lemma 4.2.3 *Let $u \in K$ and $\theta \in (1, +\infty)$. Then*

$$u(t) \geq \frac{1}{\theta} \|u\|_E\,, \ for\ all\ t \in \left(\frac{1}{\theta}, \theta\right).$$

Proof. First, note that the function $h(t) = \dfrac{u(t)}{1+t}$ is continuous and satisfies $h(0) = h(+\infty) = 0$; so, it achieves its maximum at some $t_0 > 0$.

Then, since u is concave and nondecreasing on $(0, +\infty)$, we have for all $t \in \left[\frac{1}{\theta}, \theta\right]$

$$\begin{aligned} u(t) &\geq \min_{t\in[\frac{1}{\theta},\theta]} u(t) = u\left(\frac{1}{\theta}\right) = u\left(\frac{\theta - 1 + \theta t_0}{\theta + \theta t_0}\frac{1}{\theta - 1 + \theta t_0} + \frac{1}{\theta + \theta t_0} t_0\right) \\ &\geq \frac{\theta - 1 + \theta t_0}{\theta + \theta t_0} u\left(\frac{1}{\theta - 1 + \theta t_0}\right) + \frac{1}{\theta + \theta t_0} u(t_0) \\ &\geq \frac{1}{\theta + \theta t_0} u(t_0) = \frac{u(t_0)}{\theta(1 + t_0)} = \frac{1}{\theta} \|u\|_E\,. \end{aligned}$$

The lemma is proved. ■

Remark 4.2.4 *We have from the above lemma that for all $u \in K$ and $\theta > 1$*

$$\frac{u(t)}{1+t} \geq \frac{1}{\theta(1+\theta)} \|u\|_E \ \ for\ all\ t \in \left(\frac{1}{\theta}, \theta\right).$$

Let ψ be the inverse function of ϕ and consider the operator $T : K \to K$ defined for $u \in K$ by

$$Tu(t) = \int_0^t \psi\left(\int_s^{+\infty} a(r) f(r, u(r))\, dr\right) ds.$$

Lemma 4.2.5 *Assume that* (4.2) *and* (4.3) *hold. Then the operator T is well defined.*

Proof. Let $u \in K$ and $v = Tu$. We have for arbitrary $\delta > 0$,

$$\begin{aligned} v(t) &= \int_0^t \psi\left(\int_s^{+\infty} a(r) f\left(r, u(r)\right) dr\right) ds \\ &\leq \int_0^t \psi\left(\int_s^{+\infty} a(r) m(r) g\left(\frac{u(r)}{1+r}\right) dr\right) ds \\ &\leq \overline{g} \int_0^\delta \psi\left(\int_s^{+\infty} a(r) m(r) dr\right) ds < \infty \ for\ all\ t \in [0, \delta] \end{aligned}$$

and

$$v(t_2) - v(t_1) \leq \overline{g} \int_{t_1}^{t_2} \psi\left(\int_s^{+\infty} a(r) m(r) dr\right) ds$$

for all $t_1, t_2 \in [0, \delta]$ with $t_1 < t_2$, where $\overline{g} = \max\{g(t),\ t \in [0, \|u\|_E]\}$.

Clearly, the above inequalities mean that v is defined and continuous on $[0, \delta]$ and since δ is arbitrary v is continuous on $\mathbb{R}^+$.

We have obviously that $v(0) = 0$ and $(\phi(v'))'(t) = -a(t) f(t, u(t)) \leq 0$. Thus, the facts that ϕ is increasing and $\lim_{t\to+\infty} v'(t) = 0$ imply that v' is nonincreasing and nonnegative on $(0, +\infty)$. That is, v is concave.

Finally we have

$$\begin{aligned} \frac{v(t)}{1+t} &= \frac{1}{1+t} \int_0^t \psi\left(\int_s^{+\infty} a(r) f\left(r, u(r)\right) dr\right) ds \\ &\leq \overline{g} \frac{1}{1+t} \int_0^t \psi\left(\int_s^{+\infty} a(r) m(r) dr\right) ds \end{aligned}$$

which combined with hypothesis (4.3), imply that $\lim\limits_{t\to+\infty} \dfrac{Tu(t)}{1+t} = 0$, and $Tu \in K$. ■

Lemma 4.2.6 *Assume that* (4.2) *and* (4.3) *hold. Then the operator* T *is completely continuous.*

Proof. In order to prove that T is continuous, let $(u_n)_n \subset K$ be such that $\lim u_n = u$ in E and $R > 0$ be such that $\|u_n\|_E \leq R$ for all positive integer n.

We have $\lim_{n\to\infty} f(t, u_n(t)) = f(t, u(t))$ for all $t \geq 0$ and for all $s \geq 0$

$$\begin{aligned}\Big|\int_s^{+\infty} a(t)f(t,u_n(t))\,dt &- \int_s^{+\infty} a(t)f(t,u(t))\,dt\Big| \\ &\leq \int_0^{+\infty} a(t)m(t)\left|\frac{f(t,u_n(t))}{m(t)} - \frac{f(t,u(t))}{m(t)}\right| dt.\end{aligned}$$

Since $\lim_{n\to\infty} f(t, u_n(t)) = f(t, u(t))$ for all $t \geq 0$ and

$$\left|\frac{f(t,u_n(t))}{m(t)} - \frac{f(t,u(t))}{m(t)}\right| \leq 2g_R = 2\max_{\eta\in[0,R]} g(\eta),$$

the sequence $\left(\int_s^{+\infty} a(t)f(t,u_n(t))\,dt\right)$ converge uniformly to $\left(\int_s^{+\infty} a(t)f(t,u(t))\,dt\right)$ by Lebesgue dominated convergence theorem.

Thus, the uniform continuity of ψ on compact intervals of $\mathbb{R}^+$ implies that for all $\epsilon > 0$ there exists a positive integer n_ϵ such that for all $n \geq n_\epsilon$ and all $s \geq 0$

$$\left|\psi\left(\int_s^{+\infty} a(t)f(t,u_n(t))\,dt\right) - \psi\left(\int_s^{+\infty} a(t)f(t,u(t))\,dt\right)\right| \leq \epsilon.$$

Then

$$\begin{aligned}&\|Tu_n - Tu\|_E \\ = &\sup_{t\in\mathbb{R}^+}\left|\frac{\int_0^t \left[\psi\left(\int_s^{+\infty} a(r)f(r,u_n(r))\,dr\right) - \psi\left(\int_s^{+\infty} a(r)f(r,u(r))\,dr\right)\right] ds}{1+t}\right| \\ \leq &\sup_{t\in\mathbb{R}^+}\left[\frac{\int_0^t \epsilon dr}{1+t}\right] = \epsilon.\end{aligned}$$

Let $\delta > 0$ and $B_\delta = \{u \in K : \|u\|_E \leq \delta\}$. We have for all $u \in B_\delta$

$$\begin{aligned}\|Tu\|_E &= \sup_{t\in\mathbb{R}^+}\left[\frac{1}{1+t}\int_0^t \psi\left(\int_s^{+\infty} a(r)f(r,u(r))\,dr\right) ds\right] \\ &\leq \sup_{t\in\mathbb{R}^+}\left[\frac{1}{1+t}\int_0^t \psi\left(\int_s^{+\infty} a(r)m(r)g\left(\frac{u(r)}{1+r}\right) dr\right) ds\right] \\ &\leq g_\delta\Delta,\end{aligned}$$

where $g_\delta = \sup_{\eta \in [0,\delta]} g(\eta)$ and $\Delta = \sup\limits_{t \in \mathbb{R}^+} \left[\frac{1}{1+t} \int_0^t \psi \left(\int_s^{+\infty} a(r)m(r)dr \right) ds \right]$.
This means that $T(B_\delta)$ is uniformly bounded.

Let $A > 0$ and $t_1, t_2 \in [0, A]$. We have :

$$\begin{aligned}
&\left| \frac{Tu(t_1)}{1+t_1} - \frac{Tu(t_2)}{1+t_2} \right| \\
&= \left| \frac{1}{1+t_1} \int_0^{t_1} \psi \left(\int_s^{+\infty} a(r) f(r, u(r))\, dr \right) ds \right. \\
&\quad \left. - \frac{1}{1+t_2} \int_0^{t_2} \psi \left(\int_s^{+\infty} a(r) f(r, u(r))\, dr \right) ds \right| \\
&\leq \left| \frac{1}{1+t_1} \int_{t_2}^{t_1} \psi \left(\int_s^{+\infty} a(r) f(r, u(r))\, dr \right) ds \right| \\
&\quad + \left| \left(\frac{1}{1+t_1} - \frac{1}{1+t_2} \right) \int_0^{t_2} \psi \left(\int_s^{+\infty} a(r) f(r, u(r))\, dr \right) ds \right| \\
&\leq g_\delta \left| \int_{t_2}^{t_1} \psi \left(\int_s^{+\infty} a(r) m(r)\, dr \right) ds \right| + |t_1 - t_2|\, g_\delta \Delta.
\end{aligned}$$

This together the uniform continuity of the function

$$t \to \int_0^t \psi \left(\int_s^{+\infty} a(r) m(r)\, dr \right) ds$$

on the interval $[0, A]$ imply that $T(B_\delta)$ is equicontinuous on $[0, A]$.

At this stage we have for all $u \in B_\delta$

$$\left| \frac{Tu(t)}{1+t} - \lim_{t \to +\infty} \frac{Tu(t)}{1+t} \right| = \frac{Tu(t)}{1+t} \leq g_\delta \frac{1}{1+t} \int_0^t \psi \left(\int_s^{+\infty} a(r)m(r)dr \right) ds.$$

This together with hypothesis (4.3) implies that $T(B_\delta)$ is equiconvergent.

Therefore we deduce from Lemma 4.2.1 that $T(B_\delta)$ is relatively compact in E and the operator T is completely continuous. ■

It is easy to prove the following lemma.

Lemma 4.2.7 *Assume that* (4.2) *and* (4.3) *hold. Then u is a positive solution to Problem (4.1) if and only if u is a positive fixed point of T.*

Lemma 4.2.8 *Assume that* (4.2), (4.3) *and* (4.4) *hold. Then any positive solution u of Problem (4.1) is unbounded (ie : $\lim_{t\to+\infty} u(t) = +\infty$).*

Proof. Let u be a positive solution to Problem (4.1). We have from Lemma 4.2.5 that

$$\begin{aligned} u(t) &= \int_0^t \psi\left(\int_s^{+\infty} a(\tau)f(\tau, u(\tau))\, d\tau\right) ds \\ &\geq t\psi\left(\int_t^{+\infty} a(\tau)f(\tau, u(\tau))\, d\tau\right). \end{aligned}$$

Suppose that u is bounded and let $u_\infty = \lim_{t\to+\infty} u(t) > 0$. Let $\epsilon_0 > 0$ be such that $u_\infty - \epsilon_0 > 0$. There exists $t_\infty > 0$ such that $u(t) \geq u_\infty - \epsilon_0$ for all $t \geq t_\infty$.

Thus we obtain from hypothesis (4.4) and the above inequality the contradiction

$$+\infty > u_\infty = \lim_{t\to+\infty} u(t) \geq \lim_{t\to+\infty} t\psi\left(\int_t^{+\infty} a(\tau)f(\tau, u(\tau))\, d\tau\right) = +\infty.$$

This completes the proof. ■

4.3 Existence result

The statement of the main result of this Chapter needs to introduce the following notations. We have from hypothesis (4.5) that

$$t\psi(x) \geq \psi(t^\alpha x) \text{ for all } t \in [0,1] \text{ and } x \in \mathbb{R}^+ \tag{4.7}$$

or in another manner

$$\psi(tx) \leq t^{1/\alpha}\psi(x) \text{ for all } t \in [0,1] \text{ and } x \in \mathbb{R}^+. \tag{4.8}$$

In fact (4.5) is a technical condition and we often met in ϕ-Laplacian bvps literature conditions looking like it, see [1]-[6] and references therein.

Let $\theta > 1$ be fixed and set $I_\theta = [1/\theta, \theta]$,

$$g^0 = \limsup_{w \longrightarrow 0} \frac{g(w)}{\phi(w)}, \qquad g^\infty = \limsup_{w \to +\infty} \frac{g(w)}{\phi(w)},$$

$$f_0 = \liminf_{w \longrightarrow 0} \left(\min_{t \in I_\theta} \frac{f(t,(1+t)w)}{\phi(w)} \right), \quad f_\infty = \liminf_{w \longrightarrow +\infty} \left(\min_{t \in I_\theta} \frac{f(t,(1+t)w)}{\phi(w)} \right)$$

$$\Gamma = \left(\int_0^{+\infty} a(r)m(r)dr \right)^{-1}, \qquad \Theta(\theta) = (1+\theta)^{2\alpha} \theta^\alpha \left(\int_{\frac{1}{\theta}}^{\theta} a(r)dr \right)^{-1}.$$

Theorem 4.3.1 *Assume that hypotheses* $(4.2)-(4.5)$ *and one of the following conditions*

$$g^0 < \Gamma, \qquad \Theta(\theta) < f_\infty \tag{4.9}$$

and

$$g^\infty < \Gamma, \qquad \Theta(\theta) < f_0 \tag{4.10}$$

hold true. Then Problem (4.1) has at least one unbounded positive solution.

Proof. a). We consider first the case where (4.9) is satisfied. Let $\epsilon > 0$ be such that $(g^0 + \epsilon) < \Gamma$. For a such ϵ, there exists $R_1 > 0$ such that $g(u) \leq (g^0 + \epsilon)\phi(u)$ for all $u \in [0, R_1]$. Thus we have for all $u \in K \cap \partial\Omega_1$, where $\Omega_1 = \{u \in E : \|u\|_E < R_1\}$, that

$$\begin{aligned}
\|Tu\|_E &= \sup_{t \in \mathbb{R}^+} \left[\frac{1}{1+t} \int_0^t \psi \left(\int_s^{+\infty} a(r) f(r, u(r))\, dr \right) ds \right] \\
&\leq \sup_{t \in \mathbb{R}^+} \left[\frac{1}{1+t} \int_0^t \psi \left(\int_s^{+\infty} a(r)m(r) g\left(\frac{u(r)}{1+r} \right) dr \right) ds \right] \\
&\leq \sup_{t \in \mathbb{R}^+} \left[\frac{t}{1+t} \psi \left(\int_0^{+\infty} a(r)m(r)(g^0+\epsilon)\phi\left(\frac{u(r)}{1+r} \right) dr \right) \right] \\
&\leq \psi\left(\phi(\|u\|_E)(g^0+\epsilon)\Gamma^{-1} \right) < \|u\|_E.
\end{aligned}$$

Now, let $\nu > 0$ be such that $(f_\infty - \nu) > \Theta(\theta)$. There exists $R_2 > R_1$ such that $f(r,u) > (f_\infty - \nu)\phi\left(\frac{u}{1+r} \right)$ for all $u \geq R_2$ and all $r \in I_\theta$. Thus we have for all $u \in K \cap \partial\Omega_2$, where $\Omega_2 = \{u \in E : \|u\|_E < \theta(1+\theta)R_2\}$, that

$$\begin{aligned}
\|Tu\|_E &\geq \frac{\theta Tu\,(1/\theta)}{1+\theta} \geq \frac{\theta}{1+\theta}\int_0^{\frac{1}{\theta}} \psi\left(\int_{\frac{1}{\theta}}^{\theta} a(r)f\,(r,u(r))\,dr\right)ds \\
&> \frac{1}{1+\theta}\psi\left(\int_{\frac{1}{\theta}}^{\theta} a(r)(f_\infty-\nu)\phi\left(\frac{u(r)}{1+r}\right)dr\right) \\
&\geq \frac{1}{1+\theta}\psi\left(\int_{\frac{1}{\theta}}^{\theta} a(r)(f_\infty-\nu)\phi\left(\frac{1}{\theta(1+\theta)}\|u\|_E\right)dr\right) \\
&\geq \psi\left((f_\infty-\nu)\frac{1}{(1+\theta)^\alpha}\frac{1}{\theta^\alpha(1+\theta)^\alpha}\left(\int_{\frac{1}{\theta}}^{\theta} a(r)dr\right)\phi(\|u\|_E)\right) \\
&= \psi\left((f_\infty-\nu)\left(\Theta\,(\theta)\right)^{-1}\phi(\|u\|_E)\right) \geq \|u\|_E\,.
\end{aligned}$$

Therfore, we deduce from i) of Theorem 4.2.1 that T admits a fixed point $u \in K$ with $R_1 < \|u\|_E < R_2$ which is, by Lemmas 4.2.5 and 4.2.6, a positive unbounded solution to Problem (4.1).

b). Now we consider the case where condition (4.10) is satisfied. Let $\epsilon > 0$ be such that $(f_0-\epsilon) > \Theta\,(\theta)$. There exists $R_1 > 0$ such that $f\,(r,u) > (f_0 - \nu)\phi\left(\frac{u}{1+r}\right)$ for all $u \leq R_1$ and all $r \in I_\theta$. Thus we have for all $u \in K \cap \partial\Omega_1$, where $\Omega_1 = \{u \in E : \|u\|_E < \theta(1+\theta)R_1\}$, that

$$\begin{aligned}
\|Tu\|_E &\geq \frac{\theta Tu\,(1/\theta)}{1+\theta} \geq \frac{\theta}{1+\theta}\int_0^{\frac{1}{\theta}} \psi\left(\int_{\frac{1}{\theta}}^{\theta} a(t)f\,(r,u(r))\,dr\right)ds \\
&> \frac{1}{1+\theta}\psi\left(\int_{\frac{1}{\theta}}^{\theta} a(r)(f_\infty-\epsilon)\phi\left(\frac{u(r)}{1+r}\right)dr\right) \\
&\geq \frac{1}{1+\theta}\psi\left(\int_{\frac{1}{\theta}}^{\theta} a(r)(f_\infty-\epsilon)\phi\left(\frac{1}{\theta(1+\theta)}\|u\|_E\right)dr\right) \\
&\geq \psi\left((f_\infty-\epsilon)\frac{1}{(1+\theta)^\alpha}\frac{1}{\theta^\alpha(1+\theta)^\alpha}\left(\int_{\frac{1}{\theta}}^{\theta} a(r)dr\right)\phi(\|u\|_E)\right) \\
&= \psi\left((f_\infty-\epsilon)\left(\Theta\,(\theta)\right)^{-1}\phi(\|u\|_E)\right) > \|u\|_E\,.
\end{aligned}$$

Let $\nu > 0$ be such that $(g^\infty + \nu) < \Gamma$. Then there exists $c > 0$ and $R_2 > R_1$

such that

$$g(u) \leq (g^{\infty} + \nu)\phi(u) + c \text{ for all } t, u \geq 0$$

and

$$(g^{\infty} + \nu)\,\Gamma^{-1}\phi\,(R_2) + c\Gamma^{-1} < \phi\,(R_2)\,.$$

Thus we have for all $u \in K \cap \partial\Omega_2$, where $\Omega_2 = \{u \in E : \|u\|_E < R_2\}$, that

$$\begin{aligned} \|Tu\|_E &\leq \psi\left(\int_0^{+\infty} a(r)m(r)g\left(\frac{u(r)}{1+t}\right) dr\right) \\ &\leq \psi\left(\int_0^{+\infty} a(r)m(r)\left((g^{\infty}+\nu)\phi\left(\frac{u(r)}{1+r}\right)+c\right) dr\right) \\ &\leq \psi\left(\Gamma^{-1}(g^{\infty}+\nu)\phi\left(\|u\|\right)_E + c\Gamma^{-1}\right) \leq \|u\|_E\,. \end{aligned}$$

We deduce from ii) of Theorem 4.2.1 that T admits a fixed point $u \in K$ with $R_1 < \|u\|_E < R_2$ which is, by Lemmas 4.2.5 and 4.2.6 a positive unbounded solution to Problem (4.1). This ends the proof. ■

4.4 Example

Example 4.4.1 *Consider the boundary value problem* (4.1) *with* $\phi\,(u) = u^{p-1} + u^{q-1}$, $2 < p < q$, $a(t) = \dfrac{1}{(1+t)^p}$ *and* $f(t,u) = \dfrac{t}{1+t}\dfrac{Au^{p-1}}{(1+t)^{p-1}}$.

Clearly (4.5) is satisfied with $\alpha = q-1$ and (4.2) is satisfied with $m(t) = 1$ and $g\,(w) = Aw^{p-1}$.

Straighforward computations lead to

$$\int_s^{+\infty} a(t)m(t)dt = \int_s^{+\infty} \frac{dt}{(1+t)^p} = \frac{1}{(p-1)(1+s)^{p-1}}.$$

Then we have from (4.8)

$$\psi\left(\int_s^{+\infty} a(t)m(t)dt\right) \leq (1+s)^{-\frac{p-1}{q-1}}\psi\left(\frac{1}{p-1}\right) \textit{ for all } s > 0.$$

Integrating we get

$$\begin{aligned}\int_0^t \psi\left(\int_s^{+\infty} a(t)m(t)dt\right) ds &\leq \frac{q-1}{q-p}\psi\left(\frac{1}{p-1}\right)\left(1-(1+t)^{\frac{p-q}{q-1}}\right)\\ &\leq \frac{q-1}{q-p}\psi\left(\frac{1}{p-1}\right)\end{aligned}$$

leading to

$$\lim_{t\to+\infty}\frac{1}{1+t}\int_0^t \psi\left(\int_s^{+\infty} a(t)m(t)dt\right) ds = 0$$

and (4.3) is satisfied.

Let $\lambda > 0$. We have

$$\begin{aligned}\int_t^{+\infty} a(s)f(s,\lambda)ds &= A\lambda^{p-1}\left(\frac{1}{2(p-1)}\frac{1}{(1+t)^{2(p-1)}} - \frac{1}{2p-1}\frac{1}{(1+t)^{2p-1}}\right)\\ &\geq A\lambda^{p-1}\frac{1}{2(p-1)}\frac{t}{(1+t)^{2(p-1)}}.\end{aligned}$$

Then from (4.8) we get

$$\psi\left(\int_t^{+\infty} a(s)f(s,\lambda)ds\right) \geq \psi\left(\frac{A\lambda^{p-1}}{2(p-1)}\right)\frac{t^{\frac{1}{p-1}}}{(1+t)^2}$$

and

$$\lim_{t\to+\infty} t\psi\left(\int_t^{+\infty} a(s)f(s,\lambda)ds\right) \geq \psi\left(\frac{A\lambda^{p-1}}{2(p-1)}\right)\lim_{t\to+\infty}\frac{t^{\frac{p}{p-1}}}{(1+t)^2} = +\infty,$$

and thus (4.4) is satisfied.

Let $\theta > 1$ be fixed. By simple computations we get

$$g^{\infty} = 0 \quad \text{and} \quad f_0 = \frac{A\theta}{1+\theta}.$$

Thus, we conclude from Theorem 4.3.1 that Problem (4.1) has at least one positive unbounded solution if $A > \left(\frac{1+\theta}{\theta}\right)\Theta(\theta)$.

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Conclusion

This thesis was devoted to the investigation of two classes of differential equations with ϕ-Laplacian subject to limit conditions and posed on bounded or unbounded intervals of the real line. The first one is $-\left(\phi\left(u'\right)\right)'(t) = a(t)\, f(t, u(t))$ with various boundary conditions ; the second one is $-(\phi(u'))'(t) = \lambda f(u(t))$, with boundary conditions $u(0) = u(1) = 0$.

We have obtained some existence, multiplicity results under fairly simple and general conditions on the nonlinearity.

Two methods have been mainly used in this thesis : the time mapping approach and the fixed point index theory on cones with argument from fixed point theory.

Multiplicity is obtained in the third chapter, using the time mapping approach.

As for compactness, we have employed Ascoli-Arzéla theorem on bounded domains as well as Corduneanu's criterion on unbounded domains.

We believe that these work contribute to the study of boundary value problems for a wide class differential equations with ϕ-Laplacian posed on bounded or unbounded intervals.

We hope that these results can be generalized for a wider class of boundary value problems.

Conclusion

Cette thèse a été consacrée à l'étude de deux classes des équations différentielles associées à l'opérateur ϕ-Laplacien posées sur des intervalles bornés ou non bornés de la demi-droite positive. La première est $-\left(\phi\left(u'\right)\right)'(t) = a\left(t\right) f(t, u(t))$ associée à différentes conditions aux bords, la seconde est $-(\phi(u'))'(t) = \lambda f(u(t))$, avec les conditions aux bords de type Dirichlet homogène $u(0) = u(1) = 0$. Nous avons obtenu des résultats d'existence et de multiplicité avec quelques conditions simples et générales sur la non-linéarité. Deux méthodes ont été principalement utilisées dans cette thèse : la méthode de quadrature, la théorie de l'indice de point fixe sur les cônes combinées avec les arguments de la théorie de point fixe.

La multiplicité est obtenue dans le troisième chapitre, en utilisant la méthode de quadrature comme pour la compacité, nous avons employé le théorème d'Ascoli-Arzéla sur des domaines bornés ainsi que le critère de Corduneanu pour les domaines non bornés.

Nous pensons que ces travaux contribuent à l'étude de problèmes aux limites pour une large classe d'équations différentielles avec ϕ-Laplacian posé sur des intervalles bornés ou illimités.

Nous espérons pouvoir généraliser certains de résultats à une classe plus large de problèmes aux limites.