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Sujet

Quelques problèmes d'homogénéisation en élasticité linéarisée

Soutenue le 01/04/2007, devant le jury composé de :

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A la mémoire de ma mère

A mon père

A ma femme qui m'a compris, tant aidé et supporté

A mes soeurs et mes frères

A mes enfants Lotfi et Meriem

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Résumé

Dans cette thèse nous étudions, dans le cadre de la H-convergence, quelques problèmes d'homogénéisation du système de l'élasticité linéarisée dans des milieux composites et/ou perforés et quelques questions liées. La thèse est constituée d'une introduction, de trois parties faisant chacune l'objet d'un article [29, 20, 30] et d'une conclusion.

Dans l'introduction (Chapitre 1), nous donnons d'abord la définition de la H-convergence, notion introduite par L. Tartar et F. Murat [39] et [46] (voir aussi, dans le cas symétrique, la notion de G-convergence dûe à S. Spagnolo [45]) et sa généralisation au système de l'élasticité linéarisée (notée dans cette thèse H_e-convergence), étudiée par G. A. Francfort et F. Murat [26]. Nous présentons ensuite les correspondantes notions de convergence dans le cas de domaines perforés : la H^0-convergence introduite par M. Briane, A. Damlamian et P. Donato [8] et la H_e^0-convergence développée par P. Donato et M. El Hajji [24], dont nous rappelons aussi les propriétés principales. Nous présentons ensuite les résultats principaux de cette thèse.

Dans la première partie (Chapitre 2), nous donnons une méthode itérative générale pour l'homogénéisation du système de l'élasticité linéarisée dans des domaines perforés, avec des petits trous présentant un nombre quelconque d'échelles de périodicité et une condition de traction sur le bord des trous. Notre méthode est basée sur un résultat plus général, dans lequel nous supposons que la plus grande échelle est périodique et dans les autres échelles nous ne supposons que la H_e^0-convergence. Dans le cas d'un nombre fini d'échelles périodiques nous donnons des formules explicites pour l'opérateur homogénéisé en termes de la cascade des problèmes cellulaires. Les hypothèses géométriques sont assez générales, puisqu'on ne suppose pas de condition de bonne séparation d'échelles et la fonction caractéristique du domaine perforé n'est pas, en général, un produit de fonctions caractéristiques dans les différentes échelles.

Dans la deuxième partie (Chapitre 3), nous montrons quelques estimations sur la différence des limites de deux suites H_e-convergentes. Nous donnons deux estimations, l'une de type L^1 et l'autre ponctuelle. Les démonstrations reposent de manière essentielle sur un résultat de régularité L^p, généralisant l'estimation bien connue de N. G. Meyers pour les équations elliptiques, que l'on établit pour le système de l'élasticité linéarisée. Nous montrons ensuite des estimations similaires dans le cadre de la H_e^0-convergence. L'intérêt porté à ce genre d'estimations est dû à leurs utilisations dans l'étude de la stabilité par passage à la H-limite (où G-limite). Ces résultats peuvent aussi servir à l'étude du comportement asymptotique des solutions de problèmes approximants.

Dans la troisième partie (Chapitre 4), nous montrons que l'on peut obtenir la H_e^0-convergence comme cas limite de la H_e-convergence. Plus précisément, soient Ω_ε un domaine perforé avec des petits trous T_ε (e-admissibles) et χ_ε sa fonction caractéristique. Nous montrons que, si $(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0$, alors on peut retrouver la H_e^0-limite comme limite, dans un sens convenable, de la double suite $A_\delta^\varepsilon = (\chi_\varepsilon + \delta(1-\chi_\varepsilon))A^\varepsilon$ quand $(\varepsilon, \delta) \to (0,0)$. En particulier, le problème de l'élasticité linéarisée avec trous peut-être regardé comme cas limite de problèmes sans trous dans des milieux ayant des tenseurs d'élasticité qui deviennent de plus en plus petit dans les trous quand $\delta \to 0$.

Enfin, dans la conclusion, nous présentons quelques commentaires sur les résultats obtenus et donnons quelques perspectives.

Mots clés : Homogénéisation, H-convergence, système de l'élasticité linéarisée, domaines perforés et composites, estimation de Meyers.

Classification A.M.S. : 35B27, 35B40, 74B05, 35B45, 74B05.

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Chapitre 1

Introduction et présentation des résultats

1.1 L'élasticité linéarisée

Soit Ω un domaine borné de $\mathbb{R}^n$ tel que $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ et $\Gamma_1 \cap \Gamma_2 = \emptyset$. On appelle système de l'élasticité linéarisée le système d'équations aux dérivées partielles linéaires suivant :

$$\begin{cases} -div\ \sigma(u) = f \quad \text{dans } \Omega, \\ \sigma(u) = Ae(u), \\ e(u) = \frac{1}{2}(\nabla u +^t \nabla u), \\ \sigma(u)\nu = g \quad \text{sur } \Gamma_1, \\ u = \varphi \quad \text{sur } \Gamma_2. \end{cases}$$

où $A = (A_{ijkl})_{1 \leq i,j,k,l \leq n}$ est un tenseur d'ordre 4 qui vérifie

$$\begin{cases} i)\ A_{ijlk} \in L^\infty(\Omega), \quad \text{pour tout } i,j,l,k = 1,...,n, \\ ii)\ A_{ijlk} = A_{jilk} = A_{lkij}, \quad \text{pour tout } i,j,l,k = 1,...,n, \\ iii)\ \alpha|\eta|^2 \leq A\eta.\eta, \quad \text{pour toute matrice symétrique } \eta \in \mathbb{R}_S^{n\times n}, \\ iv)\ |A\eta| \leq \beta|\eta|, \quad \text{pour toute matrice } \eta \in \mathbb{R}^{n\times n}, \end{cases} \tag{1.1}$$

avec $0 < \alpha < \beta$ et $|\eta| = \left(\sum_{i,j=1}^{n} \eta_{ij}^2 \right)^{\frac{1}{2}}$.

Remarquons que si les propriétés de symétrie (1.1)ii) sont satisfaites par un tenseur A, alors les inégalités (1.1)iii) et (1.1)iv) peuvent être regroupées sous la forme

$$\alpha|\eta|^2 \leq A\eta\eta \leq \beta|\eta|^2, \quad \text{pour toute matrice symétrique } \eta.$$

Dans la théorie de l'élasticité, Ω est le domaine occupé par le matériau élastique en supposant qu'il y'ait des petites déformations. Le champ vectoriel u décrit le déplacement du matériau et son gradient symétrisé $e(u)$ est le tenseur des déformations linéarisé. La fonction f représente les forces extérieures appliquées à Ω. La condition imposée sur Γ_1 est une condition de déplacement (c'est à dire une condition de Dirichlet pour le problème d'élasticité) et celle imposée à Γ_2 est une condition de traction (c'est à dire une condition de Neumann pour le problème d'élasticité).

1.2 La H-convergence et la H_e-convergence

La théorie de l'homogénéisation date des années 1960-1970 même si les physiciens et les ingénieurs s'intéressaient au problème bien avant, à cause de ses multiples applications à beaucoup de phénomènes physiques et mécaniques dans les milieux composites, les milieux perforés et les milieux poreux ainsi que dans d'autres situations similaires. Deux échelles caractérisent le matériau, l'une microscopique, décrivant les hétérogénéités, et l'autre macroscopique, décrivant le comportement globale. Le but de l'homogénéisation est de déterminer le comportement macroscopique des systèmes qui sont microscopiquement hétérogènes, afin de décrire certaines caractéristiques comme le transfert de la chaleur, la conduction électrique, les déformations élastiques et l'écoulement. Ce qui signifie le remplacement du matériau hétérogène par un autre homogène dont la majorité des caractéristiques sont de bonnes approximations de celles qui caractérisent le matériau d'origine.

Du point de vue mathématique, il s'agit d'une équation aux dérivées partielles dépendante d'un petit paramètre qui décrit les hétérogénéités, qui sont petites par rapport à la taille globale du matériau, où quand ce paramètre est très proche de 0, les coefficients de l'équation oscillent rapidement. Cette rapidité rend les calculs numériques coûteux et difficiles. La méthode de l'homogénéisation consiste à étudier la "convergence" de la solution du problème initial vers la solution d'un autre problème limite appelé problème homogénéisé plus facile à résoudre numériquement. En particulier, dans les cas périodiques, les coefficients du problème homogénéisé sont constants et donc les calculs numériques sont très aisés. Parmi les méthodes de l'homogénéisation périodique, on cite la méthode des développements asymptotiques (calculs formels), la méthode des fonctions oscillantes dûe à L. Tartar, la méthode de la convergence à deux échelles de G. Allaire [2] et sa généralisation appelée la convergence multi-échelles de G. Allaire et M. Briane [3]. Nous citons aussi la méthode récente dite d'éclatement périodique, donnée par D. Cioranescu - A. Damlamian -G. Griso [11], qui a été généralisé aux domaines perforés par D. Cioranescu, P. Donato et R. Zaki [13]. La théorie de la H-convergence et ses développements ultérieurs permet de traiter aussi les cas non périodiques.
Pour une bibliographie plus complète nous référons aux livres de A. Bensoussan, J. L. Lions et G. Papanicolaou [5], E. Sanchez-Palancia [43], A. Oleinik, A. S. Shamaev et G. A. Yosifian [42], N. Bakhvalov et G. Panasenko [4] et D. Cioranescu - P. Donato [12] et les références qui s'y trouvent.

Dans cette thèse, nous travaillons dans le cadre de la H-convergence. Pour introduire cette notion de convergence prenons l'exemple suivant :

$$\begin{cases} -div(A^\varepsilon \nabla u^\varepsilon) = f \quad dans\ \Omega, \\ u^\varepsilon = 0\ sur\ \partial\Omega, \end{cases} \tag{1.2}$$

où Ω est un domaine borné de $\mathbb{R}^n$, $f \in H^{-1}(\Omega)$, et $A^\varepsilon \in M(\alpha,\beta,\Omega)$, où l'on a posé

$$M(\alpha,\beta,\Omega) = \{A \in L^\infty(\Omega)^n \text{ t. q. } \alpha|\eta|^2 \leq A\eta\eta \; et \; |A\eta| \leq \beta|\eta| \text{ pour tout } \eta \in \mathbb{R}^n\},$$

avec $0 < \alpha < \beta$. Il est clair que la solution u^ε est bornée dans $H_0^1(\Omega)$, donc elle converge faiblement, pour une sous-suite, dans cet espace vers une limite u^0. La question qu'on se pose alors en homogénéisation est la suivante : quant est ce que toute la suite $\{u^\varepsilon\}$ converge ? et dans ce cas peut-on trouver une matrice A^0 qui satisfait des conditions de bornitude et d'éllipticité similaires à celles satisfaites par A^ε, telle que la limite u^0 soit solution du problème

$$\begin{cases} -div\;(A^0\nabla u^0) = f & \text{dans } \Omega, \\ u^0 = 0 \quad \text{sur } \partial\Omega. \end{cases} \tag{1.3}$$

Cette question a été abordée par F. Murat et L. Tartar dans les années 1970-1980 dans [39] et [46], en introduisant et en développant une notion générale de convergence de suites de champs de matrices appelée la H-convergence, qui généralise celle de la G-convergence introduite en 1968 par S. Spagnolo pour le cas symétrique [45].

Définition 1 *Soit $A^\varepsilon \in M(\alpha,\beta,\Omega)$. Alors, on dit que A^ε H-converge vers une matrice A^0 de $M(\alpha',\beta',\Omega)$ et on écrit $A^\varepsilon \overset{H}{\rightharpoonup} A^0$ si, pour tout $f \in H^{-1}(\Omega)$, la solution u^ε de 1.2 satisfait les convergences faibles suivantes :*

$$\begin{cases} i) \quad u^\varepsilon \rightharpoonup u \quad \text{faiblement dans } H_0^1(\Omega), \\ ii) \quad A^\varepsilon\nabla u^\varepsilon \rightharpoonup A^0\nabla u^0 \quad \text{faiblement dans } L^2(\Omega)^n, \end{cases} \tag{1.4}$$

où u^0 est la solution de (1.3)

L'intérêt de cette convergence repose sur le fait qu'elle possède des propriétés de compacité et de localité. Elle a été généralisée par la suite à l'élasticité linéarisée en 1986 par G. A. Francfort et F. Murat dans [26] (cette convergence sera notée dans cette thèse H_e-convergence). Posons, pour $0 < \alpha < \beta$,

$$M_e(\alpha,\beta,\Omega) = \{A = (A_{ijkl})_{1\leq i,j,k,l\leq n} \text{ tq } A_{ijkl} \in L^\infty(\Omega) \; et \; A \text{ satisfait (1.1)}\}.$$

Définition 2 *([26]) Soit $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$. Alors, on dit que A^ε H_e-converge vers $A^0 \in M_e(\alpha',\beta',\Omega)$ et on écrit $A^\varepsilon \overset{H_e}{\rightharpoonup} A^0$ si, pour tout $f \in H^{-1}(\Omega)^n$, la solution u^ε du système*

$$\begin{cases} -div(A^\varepsilon e(u^\varepsilon)) = f \quad dans \; \Omega, \\ u^\varepsilon = 0 \; sur \; \partial\Omega, \end{cases}$$

satisfait les convergences faibles suivantes :

$$\begin{cases} i) \quad u^\varepsilon \rightharpoonup u \quad \text{faiblement dans } H_0^1(\Omega)^n, \\ ii) \quad A^\varepsilon e(u^\varepsilon) \rightharpoonup A^0 e(u^0) \quad \text{faiblement dans } L^2(\Omega)^{n\times n}, \end{cases}$$

où u^0 est la solution de

$$\begin{cases} -div\;(A^0 e(u^0)) = f \quad dans \; \Omega, \\ u^0 = 0 \quad sur \; \partial\Omega. \end{cases}$$

1.3 La H^0-convergence et la H^0_e-convergence

La notion de la H-convergence a été généralisée aux problèmes dans les domaines perforés, avec une condition de Neumann homogène sur le bord des trous, par M. Briane, A. Damlamian et P. Donato dans [8] (H^0-convergence). On rappelle ici la définition.

Soit Ω un ouvert borné de $\mathbb{R}^n$ dans lequel on considère une suite de compacts (trous) T_ε et soit Ω_ε le domaine perforé défini par

$$\Omega_\varepsilon = \Omega \setminus T_\varepsilon.$$

On dénote par $\tilde{\cdot}$ le prolongement par 0 de Ω_ε à Ω et on pose $\chi^\varepsilon \doteq \chi_{\Omega_\varepsilon}$. Dans la suite ν dénotera le vecteur unitaire normal extérieur à la frontière de Ω_ε. On pose

$$W_\varepsilon = \left\{ v \in H^1(\Omega_\varepsilon) \ \text{ t. q. } \ v = 0 \ \text{ sur } \partial\Omega \right\}$$

Définition 3 *([8]) Une suite de compacts $\{T_\varepsilon\}$ est dite admissible dans Ω si*
i) toute limite faible $\star$ de $\{\chi^\varepsilon\}$ dans $L^\infty(\Omega)$ est strictement positive presque partout dans Ω,
ii) il existe une constante $C > 0$ indépendante de ε et une suite $\{Q_\varepsilon\}$ d'opérateurs de prolongement linéaires telles que, pour tout $\varepsilon > 0$, on a

$$\begin{cases} Q_\varepsilon \in \mathcal{L}(W_\varepsilon, H^1_0(\Omega)), \\ (Q_\varepsilon v)_{|\Omega_\varepsilon} = v, \quad \forall v \in W_\varepsilon, \\ \|\nabla (Q_\varepsilon v)\|_{0,\Omega} \leq C \|\nabla(v)\|_{0,\Omega_\varepsilon}, \quad \forall v \in W_\varepsilon. \end{cases}$$

Notons $Q^\star_\varepsilon$ l'opérateur adjoint de Q_ε, définit de $H^{-1}(\Omega)$ vers W'_ε par

$$\forall f \in H^{-1}(\Omega),\ \forall v \in W\varepsilon,\ \langle Q^\star_\varepsilon f, v\rangle_{W'_\varepsilon, W_\varepsilon} = \langle f, Q_\varepsilon v\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}.$$

Définition 4 *[8] Soient $A^\varepsilon \in M_e(\alpha, \beta, \Omega)$ et $\{T_\varepsilon\}$ une suite de trous admissible dans Ω. On dit que $(A^\varepsilon, T_\varepsilon)$ H^0-converge vers un tenseur $A^0 \in M_e(\alpha', \beta, \Omega)$ et on écrit $(A^\varepsilon, T_\varepsilon) \stackrel{H^0}{\rightharpoonup} A^0$ si, pour tout $f \in H^{-1}(\Omega)$, la solution u^ε du problème*

$$\begin{cases} -div\ (A^\varepsilon \nabla u^\varepsilon) = Q^\star_\varepsilon f \quad \text{dans } \Omega_\varepsilon, \\ (A^\varepsilon \nabla u^\varepsilon)\nu = 0 \quad \text{sur } \partial T_\varepsilon, \\ u^\varepsilon = 0 \quad \text{sur } \partial\Omega, \end{cases}$$

satisfait les convergences faibles suivantes :

$$\begin{cases} i) \quad Q_\varepsilon(u^\varepsilon) \rightharpoonup u^0 \quad \text{faiblement dans } H^1_0(\Omega), \\ ii) \quad A^\varepsilon \widetilde{\nabla u^\varepsilon} \rightharpoonup A^0 \nabla(u^0) \quad \text{faiblement dans } L^2(\Omega)^n, \end{cases}$$

où u^0 est l'unique solution du problème

$$\begin{cases} -div\ (A^0 \nabla u^0) = f \quad \text{dans } \Omega, \\ u^0 = 0 \quad \text{sur } \partial\Omega. \end{cases}$$

La H^0-convergence admet des propriétés de compacité et de localité semblables à celles de la H-convergence. Elle a été généralisée à son tour à l'élasticité linéarisée par P. Donato et M. El Hajji dans [24] (H_e^0-convergence). Nous rappelons maintenant la définition et les propriétés les plus importantes de la H_e^0-convergence.
On pose

$$V_\varepsilon = \left\{ v \in H^1(\Omega_\varepsilon)^n \ \text{ t. q. } \ v = 0 \ \text{ sur } \partial\Omega \right\}.$$

Définition 5 *([24]) Une suite de compacts $\{T_\varepsilon\}$ est dite admissible dans Ω pour l'élasticité linéarisée (où simplement e-admissible) si*
i) toute limite faible $\star$ de $\{\chi^\varepsilon\}$ dans $L^\infty(\Omega)$ est strictement positive presque partout dans Ω,
ii) il existe une constante $C > 0$ indépendante de ε et une suite $\{P_\varepsilon\}$ d'opérateurs de prolongement linéaires telles que, pour tout $\varepsilon > 0$, on a

$$\begin{cases} P_\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega)^n), \\ (P_\varepsilon v)_{|\Omega_\varepsilon} = v, \quad \forall v \in V_\varepsilon, \\ \|e\,(P_\varepsilon v)\|_{0,\Omega} \leq C\,\|e(v)\|_{0,\Omega_\varepsilon}, \quad \forall v \in V_\varepsilon. \end{cases} \tag{1.5}$$

Remarque 6 *1) Les trous inclus strictement dans Ω obtenus par répartition ε-périodique d'un trou de référence de taille ε sont e-admissibles [24].*
2) L'inégalité de Korn reste vraie dans V_ε avec une constante indépendante de ε.

Définition 7 *([24]) Soient $A^\varepsilon \in M_e(\alpha, \beta, \Omega)$ et $\{T_\varepsilon\}$ une suite de trous e-admissible dans Ω. On dit que $(A^\varepsilon, T_\varepsilon)$ H_e^0-converge vers un tenseur $A^0 \in M_e(\alpha', \beta', \Omega)$ et on écrit $(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0$ si, pour tout $f \in H^{-1}(\Omega)^n$, la solution u^ε du système*

$$\begin{cases} -div\,(A^\varepsilon e\,(u^\varepsilon)) = P_\varepsilon^\star f \quad \text{dans } \Omega_\varepsilon, \\ (A^\varepsilon e\,(u^\varepsilon))\nu = 0 \quad \text{sur } \partial T_\varepsilon, \\ u^\varepsilon = 0 \quad \text{sur } \partial\Omega, \end{cases} \tag{1.6}$$

satisfait les convergences faibles suivantes :

$$\begin{cases} i) \quad P_\varepsilon(u^\varepsilon) \rightharpoonup u^0 \quad \text{faiblement dans } H_0^1(\Omega)^n, \\ ii) \quad A^\varepsilon \widetilde{e\,(u^\varepsilon)} \rightharpoonup A^0 e(u^0) \quad \text{faiblement dans } L^2(\Omega)^{n\times n}, \end{cases} \tag{1.7}$$

où u^0 est l'unique solution du système

$$\begin{cases} -div\,(A^0 e(u^0)) = f \quad \text{dans } \Omega, \\ u^0 = 0 \quad \text{sur } \partial\Omega. \end{cases} \tag{1.8}$$

Remarque 8 *Dans le cas où $T_\varepsilon = \emptyset$, cette définition se réduit à celle de la H_e-convergence [26].*

Cette définition est indépendante du choix de la suite d'opérateurs $\{P_\varepsilon\}_\varepsilon$ grâce à la proposition suivant :

Proposition 9 *([24]) Soient T_ε une suite de trous e-admissible dans Ω, $\{P_\varepsilon\}$ et $\{R_\varepsilon\}$ deux familles d'opérateurs de prolongement satisfaisant (1.5). Alors,*

$$\left(v^\varepsilon \rightharpoonup v \quad \textit{faiblement dans } H_0^1(\Omega)^n\right) \Longrightarrow \left(P_\varepsilon(v^\varepsilon_{|\Omega_\varepsilon}) \rightharpoonup v \quad \textit{faiblement dans } H_0^1(\Omega)^n\right).$$

Si $v^\varepsilon \in V_\varepsilon$ et $P_\varepsilon v^\varepsilon \rightharpoonup v$ faiblement dans $H_0^1(\Omega)^n$, alors

$$\begin{cases} R_\varepsilon v^\varepsilon \rightharpoonup v \quad \textit{faiblement dans } H_0^1(\Omega)^n, \\ \forall \phi \in D(\Omega), \quad P_\varepsilon(\phi_{|\Omega_\varepsilon} v^\varepsilon) \rightharpoonup \phi v \quad \textit{faiblement dans } H_0^1(\Omega)^n. \end{cases}$$

Théorème 10 *([24]) Soient $A^\varepsilon \in M_e(\alpha, \beta, \Omega)$ et $\{T_\varepsilon\}$ une suite de trous e-admissible dans Ω. Alors les propositions suivantes sont équivalentes :*

a) $(A^\varepsilon, T_\varepsilon) \overset{H_e^0}{\rightharpoonup} A^0$.

b) *Pour toute suite de fonctions f^ε de $H^{-1}(\Omega)^n$ telle que $f^\varepsilon \to f$ dans $H^{-1}(\Omega)^n$, la solution v^ε du problème*

$$\begin{cases} -div\,(A^\varepsilon e(v^\varepsilon)) = P_\varepsilon^\star f^\varepsilon \quad \textit{dans } \Omega_\varepsilon, \\ (A^\varepsilon e(v^\varepsilon))\nu = 0 \quad \textit{sur } \partial T_\varepsilon, \\ v^\varepsilon = 0 \quad \textit{sur } \partial\Omega, \end{cases}$$

satisfait les convergences faibles suivantes :

$$\begin{cases} i) \quad P_\varepsilon(v^\varepsilon) \rightharpoonup v^0 \quad \textit{faiblement dans } H_0^1(\Omega)^n, \\ ii) \quad A^\varepsilon \widetilde{e(v^\varepsilon)} \rightharpoonup A^0 e(v^0) \quad \textit{faiblement dans } L^2(\Omega)^{n\times n}, \end{cases}$$

où v^0 est l'unique solution du système

$$\begin{cases} -div\,(A^0 e(v^0)) = f \quad \textit{dans } \Omega, \\ v^0 = 0 \quad \textit{sur } \partial\Omega. \end{cases}$$

c) *Pour toute suite de fonctions f^ε de $L^2(\Omega)^n$ telle que $\widetilde{f^\varepsilon} \to f$ dans $L^2(\Omega)^n$ fort, la solution w^ε du problème*

$$\begin{cases} -div\,(A^\varepsilon e(w^\varepsilon)) = f^\varepsilon \quad \textit{dans } \Omega_\varepsilon, \\ (A^\varepsilon e(w^\varepsilon))\nu = 0 \quad \textit{sur } \partial T_\varepsilon, \\ w^\varepsilon = 0 \quad \textit{sur } \partial\Omega, \end{cases}$$

satisfait les convergences faibles suivantes :

$$\begin{cases} i) \quad P_\varepsilon(w^\varepsilon) \rightharpoonup w^0 \quad \textit{faiblement dans } H_0^1(\Omega)^n, \\ ii) \quad A^\varepsilon \widetilde{e(w^\varepsilon)} \rightharpoonup A^0 e(w^0) \quad \textit{faiblement dans } L^2(\Omega)^{n\times n}, \end{cases}$$

où w^0 est l'unique solution du système

$$\begin{cases} -div\ (A^0 e(w^0)) = f \quad \textit{dans } \Omega, \\ w^0 = 0 \quad \textit{sur } \partial\Omega. \end{cases}$$

d) *Pour toute fonction f de $L^2(\Omega)^n$ et pour toute sous-suite $\{\varepsilon'\}$ de $\{\varepsilon\}$ telle que $\chi^{\varepsilon'} \rightharpoonup \chi^0$ dans $L^\infty(\Omega)$ faible $\star$, la solution $\widehat{w}^\varepsilon$ du problème*

$$\begin{cases} -div\ (A^\varepsilon e(\widehat{w}^\varepsilon)) = f \quad \textit{dans } \Omega_\varepsilon, \\ (A^\varepsilon e(\widehat{w}^\varepsilon))\nu = 0 \quad \textit{sur } \partial T_\varepsilon, \\ \widehat{w}^\varepsilon = 0 \quad \textit{sur } \partial\Omega, \end{cases}$$

satisfait les convergences faibles suivantes :

$$\begin{cases} i) \quad P_{\varepsilon'}(\widehat{w}^{\varepsilon'}) \rightharpoonup \widehat{w}^0 \quad \textit{faiblement dans } H_0^1(\Omega)^n, \\ ii) \quad A^{\varepsilon'} \widetilde{e(\widehat{w}^{\varepsilon'})} \rightharpoonup A^0 e(\widehat{w}^0) \quad \textit{faiblement dans } L^2(\Omega)^{n\times n}, \end{cases}$$

où $\widehat{w}^0$ est l'unique solution du système

$$\begin{cases} -div\ (A^0 e(\widehat{w}^0)) = \chi^0 f \quad \textit{dans } \Omega, \\ \widehat{w}^0 = 0 \quad \textit{sur } \partial\Omega. \end{cases}$$

L'intérêt de cette notion de convergence vient des résultats de compacité et de localité suivants :

Théorème 11 *([24]) Soient $A^\varepsilon \in M_e(\alpha, \beta, \Omega)$ et $\{T_\varepsilon\}$ une suite de compacts e-admissible dans Ω. Alors, il existe une sous-suite de $\{\varepsilon\}$ (notée toujours par $\{\varepsilon\}$) et un tenseur $A^0 \in M_e(\frac{\alpha}{C^2}, \beta, \Omega)$ tels que la suite $\{(A^\varepsilon, T_\varepsilon)\}$ H_e^0-converge vers A^0.*

Remarque 12 *Le fait que A^0 appartient à $M_e\left(\frac{\alpha}{C^2}, \beta, \Omega\right)$, pas explicitement énoncé dans le théorème de compacité donné dans [24], est obtenu par les mêmes arguments utilisés pour le cas elliptique sans trous.*

Theorem 13 *([24]) Pour $j \in \{1,2\}$, soient O_j un ouvert borné de $\mathbb{R}^n$, C_j^ε un tenseur d'ordre 4 de $M_e(\alpha, \beta, O_j)$ et $\{T_\varepsilon^j\}$ e-admissible dans O_j telle que $(C_j^\varepsilon, T_\varepsilon^j) \overset{H_e^0}{\rightharpoonup} C_j^0$ dans O_j. Alors, pour tout ouvert ω relativement compact dans $O_1 \cap O_2$, on a*

$$\left(\omega \cap T_\varepsilon^1 = \omega \cap T_\varepsilon^2 \quad \textit{et} \quad C_1^\varepsilon = C_2^\varepsilon \ \ \textit{sur} \ \ \omega \setminus T_\varepsilon^1\right) \Rightarrow \left(C_1^0 = C_2^0 \quad \textit{sur } \omega\right).$$

On rappelle enfin le résultat suivant du type div-rot qui joue un rôle essentiel dans les différents passages à la limite :

Théorème 14 *([24]) Soient $\{T_\varepsilon\}$ une suite de trous e-admissible dans Ω et $\{\xi^\varepsilon\}$ une suite de tenseurs de $L^2(\Omega_\varepsilon)^{n\times n}$ tels que*

$$\begin{cases} \widetilde{\xi^\varepsilon} \ \ \textit{est borné dans } L^2(\Omega)^{n\times n}, \\ -div\ \xi^\varepsilon = P_\varepsilon^\star f^\varepsilon \quad \textit{dans } \Omega_\varepsilon, \\ \xi^\varepsilon \nu = 0 \qquad \textit{sur } \partial T_\varepsilon, \end{cases}$$

où la suite $\{f^\varepsilon\}$ est dans un compact de $H^{-1}(\Omega)^n$. Alors,

i) $div\xi^\varepsilon$ est dans un compact de $H^{-1}(\Omega)^n$.
ii) Si $\widetilde{\xi^\varepsilon}$ converge faiblement vers un certain ξ^0 dans $L^2(\Omega)^{n\times n}$, alors f^ε converge fortement vers $f^0 = -div\ \xi^0$ dans $H^{-1}(\Omega)^n$. De plus si $\eta^\varepsilon \in H^1(\Omega)^n$ est une suite qui converge faiblement vers η^0 dans $H^1(\Omega)^n$, alors $\widetilde{\xi^\varepsilon}.e(\eta^\varepsilon)$ converge vers $\xi^0.e(\eta^0)$ dans $D'(\Omega)$.

1.4 Présentation des résultats

Nous présentons les résultats de cette thèse dans les chapitres 2 à 4.

Chapitre 2 : Homogénéisation itérée pour l'élasticité linéarisée par H_e^0-convergence
La notion de l'homogénéisation réitérée a été introduite par A. Bensoussan - J.L. Lions - G. Papanicolaou dans [5] pour des opérateurs linéaires dans des domaines fixes. Le cas non linéaire correspondant a été étudié par A. Braides - D. Lukkassen [7], J. L. Lions - D. Lukkassen - L. E. Persson - P. Wall [32] et D. Lukkassen - G. Nguetseng [35].

L'homogénéisation des problèmes elliptiques dans les domaines perforés avec double périodicité ont été traités par plusieurs auteurs. Avec une condition de Dirichlet homogène, d'abord par T. Levy [34] pour l'équation de Stokes, puis par P. Donato - J. Saint Jean Paulin [21] pour les equations de Laplace et de Stokes et par T. Mekkaoui - C. Picard ([37]) pour le p-laplacien. Le cas de condition de Neumann homogène sur le bord des trous a été étudié par I.A. Ene [25] par la méthode de convergence trois-échelles, qui est une généralisation de la convergence à deux-échelles introduite par G. Nguetseng [41] et développée par G. Allaire [2].

Le cas multi-échelles générale avec condition de Neumann a été étudié pour les échelles multiples périodiques d'abord par G. Allaire - M. Briane dans [3] avec une condition de "bonne séparation" d'échelles en utilisant la convergence multi-échelles, puis par A. Damlamian - P. Donato dans [17] par la H^0-convergence [8].

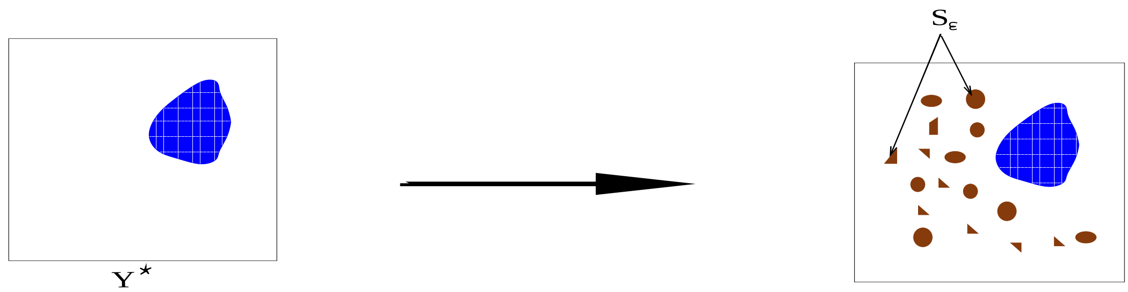
L'homogénéisation des systèmes d'élasticité linéarisée dans les domaines perforés périodiquement avec des trous de taille δ_ε (où $\{\delta_\varepsilon\}$ est une suite de nombres réels positifs telle que $\delta_\varepsilon \leq \varepsilon$) a été d'abord traité par F. Léné dans [31] pour le cas $\delta_\varepsilon = \varepsilon$, par C. Georgelin dans [27] pour le cas $\delta_\varepsilon << \varepsilon$. Nous citons aussi le travail de J. Casado-Diaz et M. Luna-Layneza [9] pour les structures réticulées minces et celui de M. El Hajji [23] qui traite un cas très particulier de la double périodicité.

Dans ce chapitre nous donnons une méthode itérative générale pour l'homogénéisation des systèmes de l'élasticité linéarisée dans des domaines perforés avec des petits trous présentant un nombre quelconque d'échelles de périodicité. Les hypothèses géométriques sont assez générales, puisqu'on ne suppose pas que les échelles soient "bien séparés" (voir [3] pour la définition) et la fonction caractéristique du domaine perforé n'est pas, en général, un produit de fonction caractéristiques dans les différentes échelles. Notre méthode est basé sur un résultat plus générale, dans lequel nous supposons que la plus grande échelle microscopique est périodique et dans les autres échelles nous ne supposons qu'une H^0_e-convergence. Des résultats analogues ont été données par A. Damlamian - P. Donato dans [17] pour les équations elliptiques d'ordre 2.

Plus précisément, nous considérons le domaine perforé Ω_ε et la suite de tenseurs A^ε définis comme suit.

Le domaine perforé :

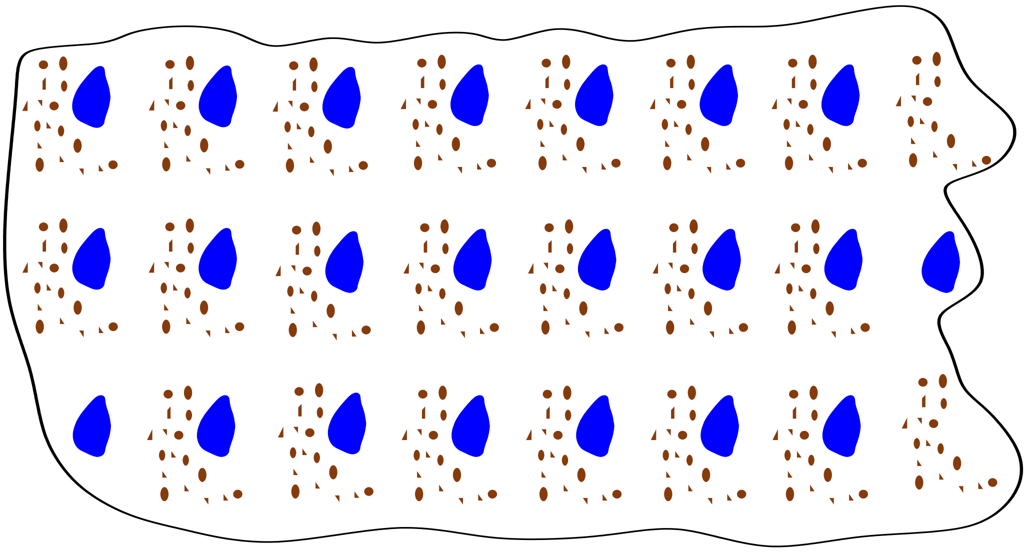
Soit Y un ouvert connexe de $\mathbb{R}^n$ qui possède la propriété du pavage (par rapport à une base $\{b_i\}_{1\leq i\leq n}$) et qui représente la cellule de référence, dans lequel on considère un trou compact lipschitzien $\mathcal{T}^\star$. On pose $Y^\star \doteq Y\backslash\mathcal{T}^\star$ et on considère dans $Y^\star$ une suite de trous $\{\mathcal{S}_\varepsilon\}$, compacts et de frontière lipschitzienne. On suppose aussi que les trous sont admissibles au sens de $H^1(Y^\star)^n$ (définition un peu plus restrictive que celle de la e-admissibilité, voir définition 26). Les trous ne sont pas dûs forcément à une répartition périodique d'un trou de référence.



Nous définissons les trous dans Ω par $T_\varepsilon = T^\star_\varepsilon \cup S_\varepsilon$, avec

$$\begin{cases} T^\star_\varepsilon = \{\cup\, \varepsilon\, \{\mathcal{T}^\star + k_l b_l\} \;\; tq \quad k\in\mathbb{Z}^n \text{ et } \varepsilon\,\{\mathcal{T}^\star + k_l b_l\}\subset\Omega\}\,, \\ S_\varepsilon = \{\cup\, \varepsilon\,\{\mathcal{S}_\varepsilon + k_l b_l\} \;\; tq \quad k\in\mathbb{Z}^n \text{ et } \varepsilon\,\{\mathcal{S}_\varepsilon + k_l b_l\}\subset\Omega\}\,. \end{cases}$$

On remarque donc que les trous S_ε sont obtenus par un changement d'échelles d'ordre ε à partir des trous non périodiques $\mathcal{S}_\varepsilon$, et occupent des zones ε-périodiques, tandis que les trous $T_\varepsilon^\star$ sont ε-périodiques et de taille ε.



Nous montrons que la suite de trous T_ε ainsi définie est e-admissible dans Ω, en utilisant le prolongement associé à $\mathcal{S}_\varepsilon$ et l'opérateur de prolongement classique appartenant à $\mathcal{L}(H^1(Y^\star)^n, H^1(Y)^n)$ donné pour l'élasticité [42].
<u>La suite de tenseurs :</u>
Considérons $B^\varepsilon \in M_e(\alpha, \beta, Y)$ (prolongé par Y-périodicité à $\mathbb{R}^n$) tel que

$$(B^\varepsilon, \mathcal{S}_\varepsilon)\ H_e^0\text{-converge dans } Y^\star \text{ vers un certain } B^0.$$

Posons

$$A^\varepsilon(x) = B^\varepsilon(\frac{x}{\varepsilon}) \quad \text{p.p. dans } \Omega.$$

Le cas périodique classique correspond à $B^\varepsilon = B$ et $\mathcal{S}_\varepsilon = \emptyset$.

Nous montrons que si $|\partial\Omega| = 0$, alors la suite $\{(A^\varepsilon, T_\varepsilon)\}$ H_e^0-converge dans Ω vers le tenseur A^0 définie par

$$A^0\Lambda = \mathcal{M}_Y(B^0 \widetilde{e\,(W_\Lambda)}^Y),$$

pour toute $\Lambda \in \mathbb{R}_S^{n\times n}$, où W_Λ est l'unique solution de

$$\begin{cases} \text{Trouver } W_\Lambda \text{ dans } \left\{\Lambda y + H^1_{per}(Y^\star)^n\right\} \text{ tel que} \\ \mathcal{M}_{Y^\star}(W_\Lambda) = 0, \\ \int_{Y^\star} B^0 e\,(W_\Lambda)\,.e\,(\varphi)\ dy = 0, \qquad \forall \varphi \in H^1_{per}(Y^\star)^n, \end{cases} \tag{1.9}$$

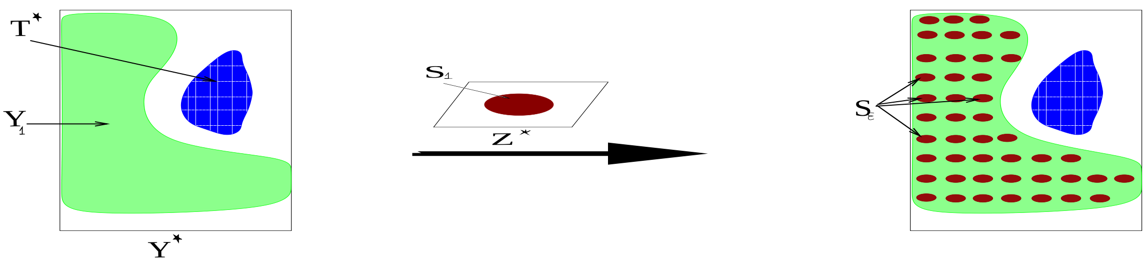
et où $\widetilde{\bullet}^Y$ désigne le prolongement par 0 de $Y^\star$ à Y.

Ceci montre que l'homogénéisation dans le domaine perforé peut se faire séparément en commençant par la plus petite échelle. En d'autre termes, on retrouve le même problème homogénéisé si on remplace $(B^\varepsilon, \mathcal{S}_\varepsilon)$ par sa H_e^0-limite dans $Y^\star$, puis on applique le processus d'homogénéisation périodique classique dans $\Omega \backslash T_\varepsilon^\star$. Ce processus d'homogénéisation itéré du système de l'élasticité linéarisée s'applique en particulier aux domaines perforés présentant plusieurs échelles de périodicité sans imposer une condition de bonne séparation des échelles. Dans ce cas, le système homogénéisé est obtenu par des formules explicites en termes de la cascade des systèmes cellulaires.

Pour illustrer cette façon d'homogénéiser nous traitons dans la deuxième partie de ce chapitre le cas de la double périodicité, c'est à dire le cas où $\mathcal{S}_\varepsilon$ est obtenu par une distribution périodique partielle d'un trou fixe de référence. Plus précisément, considérons un trous $\mathcal{S}_1$ compact régulier dans un ouvert connexe Z de $\mathbb{R}^n$ qui possède la propriété du pavage (par rapport à une base $\{b_i'\}_{1\leq i\leq n}$) et $Y_1 \subset Y^\star$ un ouvert non vide de $\mathbb{R}^n$. On définie la suite de trous $\mathcal{S}_\varepsilon$ dans $Y^\star$ par (distribution périodique partielle)

$$\mathcal{S}_\varepsilon = \{\cup \delta_\varepsilon\{\mathcal{S}_1 + k_l b_l'\};\ k \in \mathbb{Z}^n, \delta_\varepsilon\{\mathcal{S}_1 + k_l b_l'\} \subset Y_1\}.$$

où δ_ε est une suite de nombres réels positifs qui tend vers 0 avec ε.



D'autre part, on défini la suite de tenseurs B^ε par

$$B^\varepsilon(y) = \begin{cases} D(\frac{y}{\delta_\varepsilon}) & \text{pour } y \in Y_1, \\ B(y) & \text{pour } y \in Y \setminus Y_1, \end{cases}$$

où $D \in M_e(\alpha, \beta, Z)$ et $B \in M_e(\alpha, \beta, Y)$.
Dans ce cas, en utilisant la propriété de localité de la H_e^0-convergence nous montrons que si $|\partial Y_1| = 0$, alors $(A^\varepsilon, T_\varepsilon)$ H_e^0-converge dans Ω vers le tenseur A^0 défini, pour tout $\Lambda \in \mathbb{R}_S^{n\times n}$, par

$$A^0\Lambda = \mathcal{M}_Y(B^0 \widetilde{e(W_\Lambda)}^{Y})$$

où W_Λ est l'unique solution de (1.9) avec

$$B^0(y) = \begin{cases} D^0 & \text{pour } y \in Y_1, \\ B(y) & \text{pour } y \in Y \setminus Y_1, \end{cases}$$

où D^0 est défini par

$$D^0\Lambda = \mathcal{M}_Z(\widetilde{De(\hat{w}_\Lambda)}^Z),$$

et $\hat{w}_\Lambda$ est l'unique solution de

$$\begin{cases} \text{Trouver } \hat{w}_\Lambda \text{ dans } \left\{\Lambda z + H^1_{per}(Z^\star)^n\right\} \text{ tel que} \\ \mathcal{M}_{Z^\star}(\hat{w}_\Lambda) = 0, \\ \int_{Z^\star} De(\hat{w}_\Lambda)\ e(\varphi)\,dz = 0, \quad \forall \varphi \in H^1_{per}(Z^\star)^n, \end{cases}$$

où $Z^\star = Z \setminus \mathcal{S}_1$ et $\widetilde{\bullet}^Z$ dénote le prolongement par 0 de $Z^\star$ à Z.

Chapitre 3 : Estimations de type Meyers en élasticité et applications à la H-convergence

Dans ce chapitre, nous montrons quelques estimations de la différence de deux H_e-limites et de deux H_e^0-limites en se basant sur des estimations de type Meyers.
Les premiers résultats dans cette direction concernent les équations elliptiques de deuxième ordre dans les domaines sans trous et ont été établit par F. Colombini - S. Spagnolo dans [15] pour la G-convergence, puis par L. Boccardo - F. Murat dans [6] et par P. Donato dans [19] pour la H-convergence. L'intérêt porté à ce genre d'estimations est dû à leurs applications dans les questions de stabilité par rapport à d'éventuels paramètres par passage à la H-limite (ou G-limite). Ces résultats peuvent aussi servir à l'étude du comportement asymptotique des solutions de différents problèmes qui ne sont pas nécessairement périodiques et qui peuvent être dépendants du temps ou non linéaires (voir [15], [6], [24]).
Dans la première partie de ce chapitre, en considérant deux suite de tenseurs d'ordre 4, $\{A^\varepsilon\} \subset \mathcal{M}_e(\alpha, \beta, \Omega)$ et $\{B^\varepsilon\} \subset \mathcal{M}_e(\alpha', \beta', \Omega)$ tels que

$$\begin{cases} A^\varepsilon \xrightarrow{H_e} A^0, \\ B^\varepsilon \xrightarrow{H_e} B^0, \end{cases}$$

nous établissons deux estimations de la différence $A^0 - B^0$. La première estimation est donnée par :

$$\|A^0 - B^0\|_{L^1(\Omega)} \le C_1 \liminf_{\varepsilon} \|A^\varepsilon - B^\varepsilon\|_{L^s(\Omega)} \le C_2 \liminf_{\varepsilon} \|A^\varepsilon - B^\varepsilon\|^{\frac{1}{s}}_{L^1(\Omega)},$$

où $s \in]2, +\infty[$, C_1 et C_2 sont deux constantes positives indépendantes de ε.
La deuxième estimation de cette différence est par contre ponctuelle. En effet, Si

$$\begin{cases} |A^\varepsilon(x) - B^\varepsilon(x)| \le h^\varepsilon(x) \text{ p.p. dans } \Omega, \\ h^\varepsilon \longrightarrow h^0 \quad \text{fortement dans } L^1(\Omega), \end{cases}$$

alors

$$|A^0(x) - B^0(x)| \le \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}\, h^0(x) \quad \text{p.p dans } \Omega.$$

Ces deux estimations montrent qu'en particulier si

$$A^\varepsilon - B^\varepsilon \longrightarrow 0 \text{ p.p dans } \Omega,$$

alors (grâce au théorème de convergence dominée) $A^0 = B^0$ p.p dans Ω, ce qui signifie que les systèmes de l'élasticité linéarisée avec une condition de déplacement homogène associés à ces tenseurs admettent le même problème limite.

Les deux outils principaux pour démontrer nos estimations sont le lemme div-rot (version élasticité linéarisée) et un résultat de régularité de type Meyers que nous établissons pour l'élasticité linéarisée. Ce résultat de régularité établit que si Ω est un ouvert borné de $\mathbb{R}^n$ de classe $\mathcal{C}^2$, $A \in M_e(\alpha, \beta, \Omega)$, $g \in L^2(\Omega)^{n\times n}$ et u est la solution de

$$\begin{cases} -div(Ae(u)) = div\ g \quad dans\ \Omega, \\ u \in H_0^1(\Omega)^n, \end{cases}$$

alors il existe $\bar{p} > 2$, qui dépend seulement de α, β, Ω et n tel que, pour tout $p \in [2, \bar{p}[$, si $g \in L_S^p(\Omega)^{n\times n}$ alors u appartient à $W_0^{1,p}(\Omega)^n$ et satisfait

$$\|u\|_{W_0^{1,p}(\Omega)^n} \leq c\|g\|_{L^p(\Omega)^{n\times n}},$$

où c dépend seulement de α, β, Ω, n et p.

Pour démontrer ce résultat de régularité nous adaptons à l'élasticité linéarisée la démonstration donnée dans [5] pour les équations elliptiques. La plus grande difficulté provient du fait que le système de l'élasticité linéairisé ne satisfait la condition d'ellipticité forte suivante :

$$A\Lambda.\Lambda \geq \alpha|\Lambda|^2$$

que pour les matrices symétriques $\Lambda \in \mathbb{R}_S^{n\times n}$. La démonstration repose sur

- L'introduction de normes convenables dans $W_0^{1,p}$ et $W^{-1,p}$ (différentes de celles du cas scalaire).
- L'utilisation de l'opérateur de Lamé $\mathbb{E}(\cdot) \doteq -div\ e(\cdot)$ et l'invariance de son indice par rapport à p dans des espaces de Sobolev de type $W^{1,p}$. Dans l'annexe on montre que ce résultat est une application des résultats de G. Geymonat [28] concernant les systèmes elliptiques.
- L'utilisation de l'interpolation de Riesz-Thorin.

Dans la deuxième partie de ce chapitre, sous une hypothèse de régularité de type Meyers, nous montrons pour le cas des domaines perforés des estimations similaires à celles données dans la partie précédente. Nous établissons que si $\omega \subset\subset \Omega$, $A^\varepsilon \in \mathcal{M}_e(\alpha, \beta, \Omega)$, $B^\varepsilon \in \mathcal{M}_e(\alpha', \beta', \Omega)$, et $\{T_\varepsilon^1\}$ et $\{T_\varepsilon^2\}$ sont deux suites de trous (non nécessairement identiques) e-admissibles dans Ω telles que

$$\begin{cases} (A^\varepsilon, T_\varepsilon^1) \xrightarrow{H_e^0} A^0, \\ (B^\varepsilon, T_\varepsilon^2) \xrightarrow{H_e^0} B^0, \end{cases}$$

alors

$$\|A^0 - B^0\|_{L^1(\omega)} \leq c \liminf_{\varepsilon} \left(|(T_\varepsilon^1 \Delta T_\varepsilon^2) \cap \omega| + \|A^\varepsilon - B^\varepsilon\|_{L^1(\omega)}\right)^\tau, \tag{1.10}$$

avec $c > 0$ $\tau > 0$ indépendants de ε.
De plus si on note par χ_i^ε la fonction caractéristique de $\Omega \setminus T_\varepsilon^i$, $i = 1,\ 2$ et

$$\begin{cases} i)\ \chi_2^\varepsilon|A^\varepsilon(x) - B^\varepsilon(x)| \leq h^\varepsilon(x) \longrightarrow h^0 \quad fortement\ dans\ L^1(\omega), \\ ii)\ \chi_1^\varepsilon - \chi_2^\varepsilon \to 0 \quad fortement\ dans\ L^1(\omega), \end{cases} \tag{1.11}$$

alors nous montrons

$$|A^0 - B^0| \leq \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}\; h^0 \quad a.\,e.\; in\; \omega. \tag{1.12}$$

Pour démontrer les deux résultats nous avons besoin de supposer une estimation uniforme de type L^p pour un correcteur associé à $(A^\varepsilon, T_\varepsilon^1)$ ou à $(B^\varepsilon, T_\varepsilon^2)$. Les difficultés principales pour démontrer la première et surtout la deuxième estimation résident dans le fait que l'hypothèse de régularité est faite pour un seul couple et que les deux suites de trous considérées peuvent être différentes (par exemple les deux suites de trous définies dans la section 3 de [8]).

La première estimation, à savoir (1.10), étend à l'élasticité linéarisée celle donnée dans [8] pour le cas scalaire elliptique. Par contre, l'estimation ponctuelle (1.12) est originale en particulier dans le cas scalaire. Nous donnons dans la conclusion une variante de cette estimation ponctuelle dans le cas où l'hypothèse (1.11)ii) est remplacée par

$$\chi_1^\varepsilon - \chi_2^\varepsilon \to \chi^0 \quad \text{fortement dans} \quad L^1(\omega).$$

Chapitre 4 : Une propriété de la H-convergence pour l'élasticité dans les domaines perforés

Dans ce chapitre, nous étudions la relation entre la H-convergence du système de l'élasticité linéarisé dans les milieux perforés et celle dans les milieux composites. Nous montrons que la H-convergence dans les domaines perforés avec une condition de traction sur les trous peut être obtenue comme cas limite de la H-convergence dans les domaines sans trous. En d'autre termes, si $\{T_\varepsilon\}$ est une suite de trous e-admissible dans un ouvert borné Ω de $\mathbb{R}^n$ et $A^\varepsilon \in \mathcal{M}_e(\alpha, \beta, \Omega)$ tels que $(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0$, alors nous montrons que l'on peut retrouver la H_e^0-limite A^0 comme "limite" de la double suite

$$A_\delta^\varepsilon \doteq (\chi_\varepsilon + \delta(1-\chi_\varepsilon))A^\varepsilon \quad \text{p.p. dans } \Omega,$$

définie pour tout $\varepsilon > 0$ et $\delta > 0$. Ceci peut être représenter sous le schéma commutatif suivant :

$$\begin{array}{ccc} A_\delta^\varepsilon & \xrightarrow{H_e} & A_\delta \\ \downarrow & \searrow & \downarrow \\ (A^\varepsilon, T_\varepsilon) & \xrightarrow{H_e^0} & A^0 \end{array}$$

Plus précisément, soient $f,\ f^\varepsilon \in H^{-1}(\Omega)^n$ tel que $f^\varepsilon \to f$ fortement dans $H^{-1}(\Omega)^n$ et u^ε et soient u_δ^ε les solutions respectivement de

$$\begin{cases} -div\ (A^\varepsilon e\,(u^\varepsilon)) = P_\varepsilon^\star f^\varepsilon & \text{dans } \Omega_\varepsilon, \\ (A^\varepsilon e\,(u^\varepsilon))\nu = 0 & \text{sur } \partial T_\varepsilon, \\ u^\varepsilon = 0 \quad \text{sur } \partial\Omega \end{cases}$$

et

$$\begin{cases} -div\ (A_\delta^\varepsilon e(u_\delta^\varepsilon)) = f^\varepsilon & \text{dans } \Omega, \\ u_\delta^\varepsilon = 0 \quad \text{sur } \partial\Omega. \end{cases}$$

Nous montrons que, si A_δ^ε H_e-converge vers un certain A_δ pour une sous-suite de $\{\varepsilon\}$ (toujours vrai pour une sous-suite grâce à la compacité de la H_e-convergence), alors A_δ converge vers A^0 fortement dans $L^p(\Omega)$ pour tout $p \geq 1$, et faiblement $\star$ dans $L^\infty(\Omega)$ et la solution u_δ du système

$$\begin{cases} -div\,(A_\delta e(u_\delta)) = f \;\; in \;\; \Omega, \\ \\ u_\delta = 0 \;\; on \;\; \partial\Omega \end{cases}$$

satisfait les convergences suivantes :

$$\begin{cases} u_\delta \to u^0 \text{ fortement dans } H_0^1(\Omega)^n, \\ \\ A_\delta e(u_\delta) \to A^0 e(u^0) \text{ fortement dans } L^2(\Omega)^{n\times n}. \end{cases}$$

où u^0 est la solution de (1.8).
De plus, si

$$\forall \varepsilon > 0, \;\; \langle f^\varepsilon, v\rangle = 0, \quad \forall v \in H_0^1(\Omega)^n, \; v = 0 \text{ sur } \Omega_\varepsilon,$$

alors $A_\delta^\varepsilon \to (A^\varepsilon, T_\varepsilon)$ quand $\delta \to 0$ dans le sens des convergences suivantes :

$$\begin{cases} u_\delta^\varepsilon \rightharpoonup u^\varepsilon \text{ fortement dans } H^1(\Omega_\varepsilon)^n, \\ \\ A_\delta^\varepsilon e(u_\delta^\varepsilon) \rightharpoonup A^\varepsilon \widetilde{e(u^\varepsilon)} \text{ fortement dans } L^2(\Omega)^{n\times n} \end{cases}$$

et $A_\delta^\varepsilon \to A^0$ quand $(\varepsilon,\delta) \to (0,0)$ dans le sens des convergences suivantes :

$$\begin{cases} u_\delta^\varepsilon \to u^0 \text{ faiblement dans } H_0^1(\Omega)^n, \\ \\ A_\delta^\varepsilon e(u_\delta^\varepsilon) \to A^0 e(u^0) \text{ faiblement dans } L^2(\Omega)^{n\times n}. \end{cases}$$

Les démonstrations de ces convergences se basent sur les estimations a priori suivantes :

$$\begin{cases} \|P_\varepsilon(u^\varepsilon_{\delta_{|\Omega_\varepsilon}}) - P_\varepsilon u^\varepsilon\|_{H_0^1(\Omega)^n} \leq c\left(\delta^{\frac{1}{2}} + |\langle f^\varepsilon, P_\varepsilon(u^\varepsilon_{\delta_{|\Omega_\varepsilon}}) - u_\delta^\varepsilon\rangle|^{\frac{1}{2}}\right), \\ \\ \|e(u_\delta^\varepsilon)\|_{L^2(T_\varepsilon)^{n\times n}} \leq c\left(1 + \delta^{-\frac{1}{2}}|\langle f^\varepsilon, P_\varepsilon(u^\varepsilon_{\delta_{|\Omega_\varepsilon}}) - u_\delta^\varepsilon\rangle|^{\frac{1}{2}}\right), \\ \\ \|A_\delta^\varepsilon e(u_\delta^\varepsilon) - A^\varepsilon \widetilde{e(u^\varepsilon)}\;\|_{L^2(\Omega)^{n\times n}} \leq c\left(\delta^{\frac{1}{2}} + |\langle f^\varepsilon, P_\varepsilon(u^\varepsilon_{\delta_{|\Omega_\varepsilon}}) - u_\delta^\varepsilon\rangle|^{\frac{1}{2}}\right). \end{cases} \tag{1.13}$$

Ces résultats généralisent à l'élasticité linéarisée ceux obtenus par D. Cioranescu - A. Damlamian - P. Donato - L. Mascarenhas [10] pour les équations de diffusion.

Chapitre 2

Iterated homogenization for the linearized elasticity by H_e^0-convergence

Published in [29]

Abstract

In this paper we show a homogenization result for the linearized elasticity system in perforated domains with finitely many periodic scales, by using the H_e^0-convergence [24]. It extends to the elasticity case a results given by A. Damlamian and P. Donato in [17] for elliptic problems.

2.1 Introduction

In this chapter, we give a general iterative method for the homogenization of the linearized elasticity system in perforated domain with finitely many periodic scales. We use the H_e^0-convergence given by P. Donato and M. El Hajji [24], which extends to the linearized elasticity system the H^0-convergence introduced by M. Briane, A. Damlamian and P. Donato in [8]. This notion extends to the case of perforated domains that introduced by S. Spagnolo [45] for the symmetric case (G-convergence), and by F. Murat and L. Tartar [39], [40], [46] for the non-symmetric case (H-convergence). The H-convergence for the elasticity has been studied by G. A. Francfort and F. Murat in [26].

In Section 2 we recall the principal results concerning the H_e^0-convergence and establish a sufficient condition for the sequence $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ to be H_e^0-convergent. It generalizes to our case a similar result given in [17] for the H^0-convergence and shows that it is enough to have $n\frac{n+1}{2}$ suitable sequence of test functions for constructing H_e^0-limit.

In Section 3 we state our main result (Theorem 29), that we describe briefly here. Let Y a reference cell, $Y^\star \equiv Y \backslash \mathcal{T}^\star$, where $\mathcal{T}^\star$ is a compact subset of Y and let $\{(B^\varepsilon, \mathcal{S}_\varepsilon)\}_\varepsilon$ a sequence which H_e^0-converges to some B^0 in $Y^\star$. Then, if $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ is a sequence constructed by εY-periodicity from $\{(B^\varepsilon, \mathcal{S}_\varepsilon)\}_\varepsilon$, one has

$$(A^\varepsilon, T_\varepsilon) \overset{H_e^0}{\rightharpoonup} A^0,$$

with A^0 defined by

$$A^0\Lambda = \mathcal{M}_Y(B^0\widetilde{e\,(W_\Lambda)}),$$

for every symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$, where $\widetilde{\cdot}$ denotes the extension by zero and W_Λ is the unique solution of

$$\begin{cases} \text{Find } W_\Lambda \text{ in } \left\{\Lambda y + H^1_{per}(Y^\star)^n\right\} \text{ such that} \\ \mathcal{M}_{Y^\star}(W_\Lambda) = 0, \\ \int_{Y^\star} B^0 e\,(W_\Lambda)\, e\,(\varphi)\; dy = 0, \qquad \forall \varphi \in H^1_{per}(Y^\star)^n. \end{cases}$$

Section 4 is devoted to the proof of this result. To do that, we first prove some preliminary results, concerning some convergence properties of the solution of a periodic auxiliary problem posed in $Y \setminus S_\varepsilon$.

In Section 5, by using Theorem 29, we give an iterative method for the homogenization of the linearized elasticity system in perforated domain with finitely many periodic scales.

The homogenization for linearized elasticity in domains perforated by holes of size δ_ε (where $\{\delta_\varepsilon\}$ is a positive sequence such that $\delta_\varepsilon \leq \varepsilon$) was studied by many authors. It was firstly treated by F. Léné in [31] for case $\delta_\varepsilon = \varepsilon$, and then by C. Georgelin in [27] for case $\delta_\varepsilon << \varepsilon$. Iterated homogenization (without holes) was mathematically developed and justified by A. Bensoussan, J.L. Lions and G. Papanicolaou in [5].

Let us also mention that elliptic problems in perforated domains with double periodicity have been considered by many authors, for Dirichlet conditions first by T. Levy [34], then by P. Donato and J. Saint Jean Paulin [21] for the homogenization of the Laplace and Stokes equations and by T. Mekkaoui and C. Picard [37] for the p-Laplace operator.

The homogeneous Neumann conditions on the boundary of the holes was studied by I.A. Ene [25] by a three-scale method. A general multi-scale result was independently given by G. Allaire and M. Briane in [2] for periodic and "well-separated" scale.

We also refer for the periodic multi-scale case by the periodic unfolding method to D. Cioranescu, A. Damlamian and G. Griso [11].

2.2 A sufficient condition for H_e^0-convergence

In this section we give a sufficient condition for a sequence to be H_e^0-converging. To do that, we first recall its definition as well as the main properties, given by P. Donato and M. El Hajji in [24].

2.2.1 Notations

- Ω is a bounded domain of $\mathbb{R}^n$,
- $\{\varepsilon\}$ denote a strictly decreasing sequence converging to zero,

• if $\zeta = (\zeta_{ij})_{1\le i,j\le n}$ and $\xi = (\xi_{ij})_{1\le i,j\le n}$ are two square matrix, we set

$$\zeta.\xi = \sum_{i,j=1}^{n} \zeta_{ij}\xi_{ij} \quad \text{and} \quad |\xi| = \left(\sum_{i,j=1}^{n} \xi_{ij}^2\right)^{\frac{1}{2}},$$

• if $v = (v_1,\ldots, v_n)$ is a vector valued function and $\zeta = (\zeta_{ij})_{1\le i,j\le n}$ is a second order tensor of variable $x = (x_1,\ldots, x_n)$, we set

$$\begin{cases} (\nabla v)_{ij} = \dfrac{\partial v_i}{\partial x_j}, \\[2ex] e(v) = \frac{1}{2}(\nabla v +^t \nabla v), \\[2ex] (div\xi)_i = \dfrac{\partial \xi_{ij}}{\partial x_j}, \end{cases}$$

• for two real numbers α and β such that $0 < \alpha < \beta$ and an open set $\mathcal{O}$ of $\mathbb{R}^n$, $M_e(\alpha, \beta, \mathcal{O})$ is the set of the tensors $A = (A_{ijlk})_{1\le i,j,l,k\le n}$ defined on $\mathcal{O}$ such that a.e. on $\mathcal{O}$, we have

$$\begin{cases} \text{i) } A_{ijlk} \in L^\infty(\mathcal{O}), \quad \text{for any } i,j,l,k = 1,\ldots,n, \\[1ex] \text{ii) } A_{ijlk} = A_{jilk} = A_{lkij}, \quad \text{for any } i,j,l,k = 1,\ldots,n, \\[1ex] \text{iii) } \alpha|\eta|^2 \le A\eta\eta, \quad \text{for any symmetric matrix } \eta, \\[1ex] \text{iv) } |A\eta| \le \beta|\eta|, \quad \text{for any matrix } \eta, \end{cases} \tag{2.1}$$

• χ_E denotes the characteristic function of a subset E of $\mathbb{R}^n$,
• $|E|$ denotes the Lebesgue measure of any measurable subset E of $\mathbb{R}^n$,
• $\mathcal{M}_E(v) = \frac{1}{|E|}\int_E v \ dx$, for every measurable subset of $\mathbb{R}^n$ with $|E| > 0$,
• $\|v\|_{0,\mathcal{O}} = \left(\int_{\mathcal{O}} |v|^2 \ dx\right)^{\frac{1}{2}}, \quad \|v\|_{1,\mathcal{O}} = \left(\|v\|_{0,\mathcal{O}}^2 + \|\nabla v\|_{0,\mathcal{O}}^2\right)^{\frac{1}{2}}$.

Along this chapter, we denote by C different constants independent of ε.

Convention : We adopt the Einstein summation convention, i.e. we sum over repeated indices.

Remark 15 *If $A = (A_{ijlk})_{1\le i,j,l,k\le n}$ is a fourth order tensor and $\Lambda = (\Lambda_{ij})_{1\le i,j\le n}$ is a square matrix, we have*

$$A\Lambda = \left((A\Lambda)_{ij}\right)_{1\le i,j\le n} = (A_{ijlk}\Lambda_{lk})_{1\le i,j\le n}.$$

Moreover, if A satisfies (2.1)ii), one has

$$A\Lambda = A\left(\frac{1}{2}(\Lambda +^t \Lambda)\right).$$

2.2.2 Preliminary results

We introduce the perforated domain

$$\Omega_\varepsilon = \Omega \setminus T_\varepsilon,$$

where T_ε is a sequence of compacts subsets of Ω and set

$$V_\varepsilon = \left\{ v \in H^1(\Omega_\varepsilon)^n \ \text{ s. t. } \ v = 0 \ \text{ on } \partial\Omega \right\}.$$

In the following, ν denotes the outward normal unit vector on the boundary of Ω_ε. Further, we denote by $\widetilde{\bullet}$ the extension by 0 from Ω_ε to Ω.

Definition 16 *([24]) The set T_ε is said to be admissible (in Ω) for the linearized elasticity (or e-admissible), if and only if*
i) every L^∞ weak $\star$-limit point of $\left\{\chi_{\Omega_\varepsilon}\right\}_\varepsilon$ is positive a.e. in Ω,
ii) there exists a positive real C, independent of ε, and a sequence $\{P_\varepsilon\}_\varepsilon$ of linear extension operators such that for each ε

$$\begin{cases} P_\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega)^n), \\ (P_\varepsilon v)_{|\Omega_\varepsilon} = v, \quad \forall v \in V_\varepsilon, \\ \left\| e\left(P_\varepsilon v\right)\right\|_{0,\Omega} \leq C \left\| e(v) \right\|_{0,\Omega_\varepsilon}, \quad \forall v \in V_\varepsilon. \end{cases} \tag{2.2}$$

Observe that (2.2) implies that the Korn inequality holds in V_ε with a constant independent of ε, i.e. there exists a positive constant C independent of ε such that

$$\|v\|_{1,\Omega_\varepsilon} \leq C \left\| e(v) \right\|_{0,\Omega_\varepsilon}, \quad \forall v \in V_\varepsilon. \tag{2.3}$$

Definition 17 *([24]) Let $A^\varepsilon \in M_e(\alpha, \beta, \Omega)$, and T_ε be e-admissible in Ω and for every ε, let $P_\varepsilon^\star$ be the adjoint operator of P_ε. We say that the pair $\left\{(A^\varepsilon, T_\varepsilon)\right\}_\varepsilon$ is H_e^0-converge to the tensor $A^0 \in M_e(\alpha', \beta', \Omega)$ (and we write $(A^\varepsilon, T_\varepsilon) \overset{H_e^0}{\rightharpoonup} A^0$) if and only if, for each function f in $H^{-1}(\Omega)^n$, the solution u^ε of*

$$\begin{cases} -div\ (A^\varepsilon e\,(u^\varepsilon)) = P_\varepsilon^\star f \quad in\ \Omega_\varepsilon, \\ (A^\varepsilon e\,(u^\varepsilon))\nu = 0 \quad on\ \partial T_\varepsilon, \\ u^\varepsilon = 0 \quad on\ \partial\Omega, \end{cases} \tag{2.4}$$

satisfies the weak convergence

$$\begin{cases} i) \quad P_\varepsilon(u^\varepsilon) \rightharpoonup u \quad weakly\ in\ H_0^1(\Omega)^n, \\ ii) \quad A^\varepsilon \widetilde{e\,(u^\varepsilon)} \rightharpoonup A^0 e(u) \quad weakly\ in\ L^2(\Omega)^{n\times n}, \end{cases} \tag{2.5}$$

where u is the unique solution of the problem

$$\begin{cases} -div\ (A^0 e\,(u)) = f \quad in\ \Omega, \\ u = 0 \quad on\ \partial\Omega. \end{cases} \tag{2.6}$$

In the theory of elasticity, if v is the displacement vector, $e(v)$ is the strain tensor.

The definition of H_e^0-convergence is independent of the sequence $\{P_\varepsilon\}_\varepsilon$, thanks to the following result :

Lemma 18 *([24]) Let T_ε be e-admissible in Ω, $\{P_\varepsilon\}$ and $\{R_\varepsilon\}$ be two family of extension operators satisfying (2.2). Then,*

$$\left(v^\varepsilon \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega)^n\right) \Longrightarrow \left(P_\varepsilon(v^\varepsilon_{|\Omega_\varepsilon}) \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega)^n\right).$$

Furthermore, if $v^\varepsilon \in V_\varepsilon$ and $P_\varepsilon v^\varepsilon \rightharpoonup v$ weakly in $H_0^1(\Omega)^n$, then

$$\begin{cases} R_\varepsilon v^\varepsilon \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega)^n, \\ \forall \phi \in D(\Omega), \quad P_\varepsilon(\phi_{|\Omega_\varepsilon} v^\varepsilon) \rightharpoonup \phi v \quad \text{weakly in } H_0^1(\Omega)^n. \end{cases}$$

The main properties of the H_e^0-convergence are given by the compactness and locality results below.

Theorem 19 *([24]) Let $A^\varepsilon \in M_e(\alpha, \beta, \Omega)$ and T_ε be e-admissible in Ω. Then, there exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$ and a tensor $A^0 \in M_e\left(\dfrac{\alpha}{C^2}, \dfrac{\beta^2}{\alpha}, \Omega\right)$, such that the sequence $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ H_e^0-converge to A^0.*

Remark 20 *The fact that A^0 belongs to $M_e\left(\dfrac{\alpha}{C^2}, \dfrac{\beta^2}{\alpha}, \Omega\right)$, does not appear explicitly in the statement given in [24], but can be easily deduced from its proof.*

Theorem 21 *([24]) Let O_1 and O_2 be two bounded open sets in $\mathbb{R}^n$, and let T_ε^1 and T_ε^2 be e-admissible in O_1 and O_2 respectively. Let $C_1^\varepsilon \in M_e(\alpha, \beta, O_1)$ and $C_2^\varepsilon \in M_e(\alpha, \beta, O_2)$ be such that*

$$\begin{cases} \{(C_1^\varepsilon, T_\varepsilon^1)\}_\varepsilon \overset{H_e^0}{\rightharpoonup} C_1^0 \quad \text{in } O_1, \\ \{(C_2^\varepsilon, T_\varepsilon^2)\}_\varepsilon \overset{H_e^0}{\rightharpoonup} C_2^0 \quad \text{in } O_2. \end{cases}$$

Then, for any ω relatively compact open set of $O_1 \cap O_2$, one has

$$\left(\omega \cap T_\varepsilon^1 = \omega \cap T_\varepsilon^2 \quad \text{and} \quad C_1^\varepsilon = C_2^\varepsilon \ \text{ on } \ \omega \setminus T_\varepsilon^1\right) \Longrightarrow \left(C_1^0 = C_2^0 \quad \text{on } \omega\right).$$

In the following, we will make use of the div-curl type lemma below.

Theorem 22 *([24]) Let T_ε be e-admissible in Ω and $\{\xi^\varepsilon\} \in L^2(\Omega_\varepsilon)^{n\times n}$ a sequence such that $\widetilde{\xi^\varepsilon}$ is bounded in $L^2(\Omega)^{n\times n}$ and*

$$\begin{cases} -div\ \xi^\varepsilon = P_\varepsilon^\star f^\varepsilon \quad \text{in } \Omega_\varepsilon, \\ \xi^\varepsilon \nu = 0 \quad \text{on } \partial T_\varepsilon, \end{cases}$$

where $\{f^\varepsilon\}$ is in a compact of $H^{-1}(\Omega)^n$. Then,

i) $div\xi^\varepsilon$ is in a compact subset of $H^{-1}(\Omega)^n$.

ii) If $\widetilde{\xi^\varepsilon}$ converges weakly to some ξ^0 in $L^2(\Omega)^{n\times n}$, then f^ε converges strongly to $f^0 = -div\ \xi^0$ in $H^{-1}(\Omega)^n$. Therefore if $\eta^\varepsilon \in H^1(\Omega)^n$ is a vector field which converges weakly to some η^0 in $H^1(\Omega)^n$, then $\widetilde{\xi^\varepsilon}.e(\eta^\varepsilon)$ converges weakly to $\xi^0.e(\eta^0)$ in $D'(\Omega)$.

2.2.3 An H_e^0-convergence result

We can prove now a sufficient condition for the sequence $\{(A^\varepsilon, T_\varepsilon)\}$ to be H_e^0-convergent. It generalizes to the case of elasticity in perforated domains the analogous one given in [17] for the H^0-convergence.

Theorem 23 *Let $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$ and $\{T_\varepsilon\}$ be an e-admissible sequence in Ω. Suppose that, for every symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$, there exists two sequences $\{w_\Lambda^\varepsilon\} \in H^1(\Omega)^n$ and $\{g_\Lambda^\varepsilon\}$ compact in $H^{-1}(\Omega)^n$ such that*

$$\begin{cases} i)\ w_\Lambda^\varepsilon \rightharpoonup \Lambda x \text{ weakly in } H^1(\Omega)^n, \\ ii)\ \begin{cases} -div\,(A^\varepsilon e(w_\Lambda^\varepsilon)) = P_\varepsilon^\star g_\Lambda^\varepsilon \text{ in } \Omega_\varepsilon, \\ (A^\varepsilon e(w_\Lambda^\varepsilon))\nu = 0 \text{ on } \partial T_\varepsilon, \end{cases} \\ iii)\ \chi_{\Omega_\varepsilon} A^\varepsilon e(w_\Lambda^\varepsilon) \text{ is weakly convergent in } L^2(\Omega)^n. \end{cases} \tag{2.7}$$

Then,

$$(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0,$$

where A^0 is the symmetric tensor defined by

$$A^0\Lambda = w\text{-}\lim\left(\chi_{\Omega_\varepsilon} A^\varepsilon e(w_\Lambda^\varepsilon)\right), \tag{2.8}$$

for every symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$, and w-lim denotes the weak limit in $L^2(\Omega)^n$.

Proof. Observe first that w-lim $\left(\chi_{\Omega_\varepsilon} A^\varepsilon e(w_\Lambda^\varepsilon)\right)$ is a symmetric tensor.
Since $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$, Theorem 19 implies that there exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) such that

$$(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0 \in M_e\left(\frac{\alpha}{C^2}, \frac{\beta^2}{\alpha}, \Omega\right).$$

Then, if $f \in H^{-1}(\Omega)$, the solution $u^\varepsilon \in H_0^1(\Omega)^n$ of (2.4) satisfies (2.5)-(2.6). On the other hand, for every symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$, we have

$$A^\varepsilon \widetilde{e(u^\varepsilon)}.e(w_\Lambda^\varepsilon) = \chi_{\Omega_\varepsilon} A^\varepsilon e(w_\Lambda^\varepsilon).e(P_\varepsilon u^\varepsilon).$$

For computing the limit in $\mathcal{D}'(\Omega)$ of each side of this equation, we apply twice Theorem 22. First, to the left-hand side for $\xi^\varepsilon = A^\varepsilon e(u^\varepsilon)$, $\eta^\varepsilon = w_\Lambda^\varepsilon$ and then to the right-hand side for $\xi^\varepsilon = A^\varepsilon e(w_\Lambda^\varepsilon)$, $\eta^\varepsilon = P_\varepsilon u^\varepsilon$. This gives

$$A^0 e(u^0).\Lambda = \text{w-lim}\left(\chi_{\Omega_\varepsilon} A^\varepsilon e\left(w_\Lambda^\varepsilon\right)\right).e(u^0) \;\text{ a.e. in } \Omega.$$

By virtue to the symmetric properties of A^0 and $\text{w-lim}\left(\chi_{\Omega_\varepsilon} A^\varepsilon e\left(w_\Lambda^\varepsilon\right)\right)$, it comes

$$A^0\Lambda.\nabla u^0 = \text{w-lim}\left(\chi_{\Omega_\varepsilon} A^\varepsilon e\left(w_\Lambda^\varepsilon\right)\right).\nabla u^0 \;\text{ a.e. in } \Omega. \tag{2.9}$$

As the choice of f is arbitrary in $H^{-1}(\Omega)$ and the operator

$$\begin{aligned} \mathcal{U}: \quad H_0^1(\Omega) &\longrightarrow H^{-1}(\Omega), \\ u \quad &\longmapsto \mathcal{U}(u) \equiv -div(A^0 e(u)), \end{aligned}$$

is an isomorphism (since $A^0 \in M_e\left(\dfrac{\alpha}{C^2}, \dfrac{\beta^2}{\alpha}, \Omega\right)$), equality (2.9) is still valid for every $u^0 \in H_0^1(\Omega)$. Hence,

$$A^0\Lambda = \text{w-lim}\left(\chi_{\Omega_\varepsilon} A^\varepsilon e\left(w_\Lambda^\varepsilon\right)\right) \;\text{ a.e. in } \Omega,$$

for every symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$. Moreover A^0 is defined in a single way by (2.8), since its symmetric propriety gives

$$A^0\Lambda = A^0\left(\frac{1}{2}(\Lambda +^t \Lambda\right), \; \forall \Lambda \in \mathbb{R}^{n\times n}.$$

This shows that the whole sequence $\{(A^\varepsilon, T_\varepsilon)\}$ H_e^0-converges to A^0. ■

Corollary 24 *The construction of the H_e^0-limit needs only $\frac{n(n+1)}{2}$ suitable sequences of test functions.*

Proof. If, for every $l, k \in \{1, ..., n\}$, we take in (2.8)

$$\Lambda = E_{lk}, \text{ with } (E_{lk})_{pq} \equiv \frac{1}{2}\left(\delta_{lp}\delta_{kq} + \delta_{lq}\delta_{kp}\right), \forall p, q \in \{1, ..., n\},$$

we obtain

$$\frac{1}{2}(A^0_{ijlk} + A^0_{ijkl}) = \text{w-lim}\left(\chi_{\Omega_\varepsilon} A^\varepsilon_{ijpq} e_{pq}\left(w^\varepsilon_{E_{lk}}\right)\right), \;\; \forall i, j \in \{1, ..., n\}.$$

Then, since $A^0_{ijlk} = A^0_{jikl}$, we have

$$A^0_{ijlk} = \text{w-lim}\left(\chi_{\Omega_\varepsilon} A^\varepsilon_{ijpq} e_{pq}\left(w^\varepsilon_{E_{lk}}\right)\right)$$

and due to the fact that $E_{lk} = E_{kl}$, the construction of A^0_{ijkl} needs only the test functions $w^\varepsilon_{E_{lk}}$ with $l, k \in \{1, ..., n\}$ and $l \leq k$. ■

2.3 Statement of the main result

In this section we state our main result, concerning the iterated homogenization. We prove it in the next section.

Let Y be an connected open set of $\mathbb{R}^n$, with piece-wise smooth boundary, having the "paving property" with respect to the basis $(b_1, ..., b_n)$ (where b_i is a vector of $\mathbb{R}^n$). This means that, setting $Y^k = Y + k_l b_l$, for all $k \equiv (k_i)_{i=1}^{i=n}$, $h \equiv (h_i)_{i=1}^{i=n} \in \mathbb{Z}^n$, we have

$$\begin{cases} \mathbb{R}^n = \underset{k \in \mathbb{Z}^n}{\cup} \overline{Y^k}, \\ \\ Y^k \cap Y^h = \emptyset \text{ if } k \neq h. \end{cases}$$

We recall the following lemma :

Lemma 25 ***([16])*** *Let h_ε be a Y-periodic function in $L^p(Y)$ $(p \in [1, +\infty[)$. Define $\Phi_\varepsilon(x) = h_\varepsilon(\frac{x}{\varepsilon}) \in L^p_{loc}(\mathbb{R}^n)$. Then,*
i) *$\{\Phi_\varepsilon\}$ is bounded in $L^p_{loc}(\mathbb{R}^n)$ if and only if $\{h_\varepsilon\}$ is bounded in $L^p(Y)$.*
ii) *for $1 \ < p < +\infty$, Φ_ε converges weakly in $L^p_{loc}(\mathbb{R}^n)$ if and only if the sequence $\mathcal{M}_Y(h_\varepsilon)$ is convergent. In this case, Φ_ε converges weakly in $L^p_{loc}(\mathbb{R}^n)$ to the limit of $\mathcal{M}_Y(h_\varepsilon)$,*
iii) *for $p = +\infty$, Φ_ε converges weakly $\star$ in $L^\infty(\mathbb{R}^n)$ to the limit of $\mathcal{M}_Y(h_\varepsilon)$.*

Let $\mathcal{T}^\star$ be a compact subset of Y and set

$$Y^\star = Y \setminus \mathcal{T}^\star.$$

As in the elliptic case, we need to extend the notion of admissible holes to functions in H^1.

Definition 26 *Let $\{\mathcal{S}_\varepsilon\}_\varepsilon$ be a sequence of compacts subsets of $Y^\star$ and set*

$$Y_\varepsilon = Y^\star \setminus \mathcal{S}_\varepsilon.$$

One says that $\{\mathcal{S}_\varepsilon\}_\varepsilon$ is $H^1(Y^\star)^n$-admissible for the linearized elasticity, if and only if
i) *every L^∞ weak $\star$-limit point of $\left\{\chi_{Y_\varepsilon}\right\}$ is positive a.e. in $Y^\star$,*
ii) *there exist a positive real C, independent of ε, and a sequence $\{Q^1_\varepsilon\}_\varepsilon$ of linear extension operators such that*

$$\begin{cases} Q^1_\varepsilon \in \mathcal{L}(H^1(Y_\varepsilon)^n, H^1(Y^\star)^n), \\ \\ (Q^1_\varepsilon v)_{|Y_\varepsilon} = v, \quad \forall v \in H^1(Y_\varepsilon)^n, \\ \\ \|Q^1_\varepsilon v\|_{0,Y^\star} \leq C \, \|v\|_{0,Y_\varepsilon}, \quad \forall v \in H^1(Y_\varepsilon)^n, \\ \\ \|e\,(Q^1_\varepsilon v)\|_{0,Y^\star} \leq C \, \|e(v)\|_{0,Y_\varepsilon}, \quad \forall v \in H^1(Y_\varepsilon)^n. \end{cases}$$

Remark 27 *It is clear that if $\{\mathcal{S}_\varepsilon\}_\varepsilon$ is $H^1(Y^\star)^n$-admissible for the linearized elasticity, then $\{\mathcal{S}_\varepsilon\}_\varepsilon$ is admissible in $Y^\star$ for the linearized elasticity in the sense of Definition 16.*

We assume the following regularity and admissibility conditions :

$$\begin{cases} \text{The sets } \mathcal{T}^\star \text{ and } \mathcal{S}_\varepsilon \text{ have a Lipschitz boundary,} \\ \{\mathcal{S}_\varepsilon\}_\varepsilon \text{ is } H^1(Y^\star)^n\text{-admissible for the linearized elasticity,} \end{cases} \tag{2.10}$$

and set

$$T_\varepsilon = T_\varepsilon^\star \cup S_\varepsilon,$$

where

$$\begin{cases} T_\varepsilon^\star = \{\cup\, \varepsilon\, \{\mathcal{T}^\star + k_l b_l\} \quad s.t. \quad k \in \mathbb{Z}^n \text{ and } \varepsilon\, \{\mathcal{T}^\star + k_l b_l\} \subset \Omega\}\,, \\ S_\varepsilon = \{\cup\, \varepsilon\, \{\mathcal{S}_\varepsilon + k_l b_l\} \quad s.t. \quad k \in \mathbb{Z}^n \text{ and } \varepsilon\, \{\mathcal{S}_\varepsilon + k_l b_l\} \subset \Omega\}\,. \end{cases}$$

In the following, we denote by $\widetilde{\bullet}$ (respect. $\widetilde{\bullet}^Y$) the extension by 0 from Y_ε to $Y^\star$ (respect. from $Y^\star$ to Y).

Proposition 28 *The sequence $\{T_\varepsilon\}_\varepsilon$ is admissible in Ω for the linearized elasticity.*

Proof. We have

$$\chi_{\Omega_\varepsilon}(x) \geqslant \chi_{Y_\varepsilon}(\frac{x}{\varepsilon}). \tag{2.11}$$

Indeed, if $\chi_{\Omega_\varepsilon}(x) = 0$, then $\dfrac{x}{\varepsilon}$ belongs to some translated of $\mathcal{T}^\star \cup \mathcal{S}_\varepsilon$, so that $\chi_{Y_\varepsilon}(\dfrac{x}{\varepsilon}) = 0$. The first condition in Definition 16 follows then from (2.10), (2.11) and Lemma 25.

On the other hand since, by assumption, $\partial T^\star$ is Lipschitz continuous, classical extension results (see, for instance, [42]) prove the existence of a linear extension operator Q^2 such that

$$\begin{cases} Q^2 \in \mathcal{L}(H^1(Y^\star)^n, H^1(Y)^n), \\ (Q^2 v)_{|Y^\star} = v, \quad \forall v \in H^1(Y^\star)^n, \\ \|Q^2 v\|_{0,Y} \leq C' \|v\|_{0,Y^\star}\,, \quad \forall v \in H^1(Y^\star)^n, \\ \|e\,(Q^2 v)\|_{0,Y} \leq C' \|e(v)\|_{0,Y^\star}\,, \quad \forall v \in H^1(Y^\star)^n. \end{cases} \tag{2.12}$$

Then, $Q_\varepsilon \equiv Q^2 o Q_\varepsilon^1$ satisfies

$$\begin{cases} Q_\varepsilon \in \mathcal{L}(H^1(Y_\varepsilon)^n, H^1(Y)^n), \\ (Q_\varepsilon v)_{|Y_\varepsilon} = v, \quad \forall v \in H^1(Y_\varepsilon)^n, \\ \|Q_\varepsilon v\|_{0,Y} \leq C \|v\|_{0,Y_\varepsilon}\,, \quad \forall v \in H^1(Y_\varepsilon)^n, \\ \|e\,(Q_\varepsilon v)\|_{0,Y} \leq C \|e(v)\|_{0,Y_\varepsilon}\,, \quad \forall v \in H^1(Y_\varepsilon)^n. \end{cases}$$

If one extends Q_ε^1 and Q^2 by Y-periodicity and sets $v(y) = w(\varepsilon y)$ for every $w \in V_\varepsilon$ and εy in Ω_ε, it is easy to see that the operator P_ε defined by

$$(P_\varepsilon w)(x) = \begin{cases} w(x) \text{ if } \exists k \in \mathbb{Z}^n \text{ s. t. } x \in \varepsilon\{\mathcal{S}_\varepsilon + k_l b_l\} \cap \Omega \text{ and } \varepsilon\{\mathcal{S}_\varepsilon + k_l b_l\} \not\subseteq \Omega, \\ w(x) \text{ if } \exists k \in \mathbb{Z}^n \text{ s. t. } x \in \varepsilon\{\mathcal{T}^\star + k_l b_l\} \cap \Omega \text{ and } \varepsilon\{\mathcal{T}^\star + k_l b_l\} \not\subseteq \Omega, \\ (Q_\varepsilon v)(\frac{x}{\varepsilon}) \text{ elsewhere,} \end{cases}$$

satisfies (ii) of Definition 16. ■

If $\mathcal{S}$ is a compact subset of Y, we denote by $H^1_{per}(Y)^n$ (respect. $H^1_{per}(Y \setminus \mathcal{S})^n$) the space of Y-periodic functions in $H^1_{loc}(\mathbb{R}^n)^n$ (respect. $H^1(Y \setminus \mathcal{S})^n$). We also set

$$W_{per}(Y \setminus \mathcal{S}) = \left\{ v \in H^1_{per}(Y \setminus \mathcal{S})^n;\ \mathcal{M}_{(Y \setminus \mathcal{S})}(v) = 0 \right\}.$$

We give now our main result. It extends to the case of elasticity the result given in [17] (Theorem 3.7) for the H^0-convergence.

Theorem 29 *Assume that $|\partial\Omega| = 0$ and the hypothesis (2.10) holds. Let $\{B^\varepsilon\}_\varepsilon$ be a sequence in $M_e(\alpha, \beta, Y)$ (extended by Y-periodicity to $\mathbb{R}^n$) such that*

$$\{(B^\varepsilon, \mathcal{S}_\varepsilon)\}_\varepsilon\, H_e^0\text{-converges in } Y^\star \text{ to } B^0 \in M_e\left(\frac{\alpha}{C^2}, \frac{\beta^2}{\alpha}, Y\right) \tag{2.13}$$

and set

$$\forall x \in \Omega, \quad A^\varepsilon(x) = B^\varepsilon(\frac{x}{\varepsilon}).$$

Then, the sequence $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ H_e^0-converges in Ω to A^0 defined by

$$A^0\Lambda = \mathcal{M}_Y(\widetilde{B^0 e(W_\Lambda)}^{Y}),$$

for every symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$, where W_Λ is the unique solution of

$$\begin{cases} \text{Find } W_\Lambda \text{ in } \left\{\Lambda y + H^1_{per}(Y^\star)^n\right\} \text{ such that} \\ \mathcal{M}_{Y^\star}(W_\Lambda) = 0, \\ \int_{Y^\star} B^0 e(W_\Lambda).e(\varphi)\, dy = 0, \qquad \forall\varphi \in H^1_{per}(Y^\star)^n. \end{cases} \tag{2.14}$$

The proof of the existence of a solution of (14) will be given in Section 4, together with the proof of this theorem.

2.4 Proof of the main result

In this section, we prove Theorem 29. To do that, we introduce the following auxiliary periodic problem :

$$\begin{cases} \text{Find } W_\Lambda^\varepsilon \text{ in } \left\{\Lambda y + H^1_{per}(Y_\varepsilon)^n\right\} \text{ such that} \\ \mathcal{M}_{Y_\varepsilon}(W_\Lambda^\varepsilon) = 0 \\ \int_{Y_\varepsilon} B^\varepsilon e(W_\Lambda^\varepsilon)\ e(\varphi)\, dy = 0, \qquad \forall\varphi \in H^1_{per}(Y_\varepsilon)^n. \end{cases} \tag{2.15}$$

2.4.1 Some preliminary results

Proposition 30 *For every symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$, problems (14) and (2.15) have a unique solution.*

Proof. Let $\Lambda \in \mathbb{R}^{n\times n}$ be a symmetric matrix and consider the following problem

$$\begin{cases} \text{Find } \varkappa_\Lambda^\varepsilon \text{ in } W_{per}(Y_\varepsilon) \text{ such that} \\ \int_{Y_\varepsilon} B^\varepsilon e\left(\varkappa_\Lambda^\varepsilon\right)\ e\left(\psi\right)\ dy = \int_{Y_\varepsilon} B^\varepsilon \Lambda\, e\left(\psi\right)\ dy, \qquad \forall \psi \in W_{per}(Y_\varepsilon). \end{cases} \tag{2.16}$$

Due to the Korn inequality for the periodic case, the Lax-Milgram theorem shows that problem (2.16) has a unique solution. Then, if we take in (2.16)

$$\psi = \varphi - \frac{1}{|Y_\varepsilon|}\int_{Y_\varepsilon} \varphi\ dy \in W_{per}(Y_\varepsilon),$$

where $\varphi \in H^1_{per}(Y_\varepsilon)^n$, we obtain also the existence and uniqueness of the solution of the following problem

$$\begin{cases} \text{Find } \varkappa_\Lambda^\varepsilon \text{ in } W_{per}(Y_\varepsilon) \text{ such that} \\ \int_{Y_\varepsilon} B^\varepsilon e\left(\varkappa_\Lambda^\varepsilon\right)\ e\left(\varphi\right)\ dy = \int_{Y_\varepsilon} B^\varepsilon \Lambda\, e\left(\varphi\right)\ dy, \qquad \forall \varphi \in H^1_{per}(Y_\varepsilon)^n. \end{cases} \tag{2.17}$$

Since Λ is symmetric, this is equivalent to (2.15), through the relation

$$W_\Lambda^\varepsilon = \Lambda y - \mathcal{M}_{Y_\varepsilon}\left(\Lambda y\right) - \varkappa_\Lambda^\varepsilon. \tag{2.18}$$

Similarly, we can prove that problem (14) has a unique solution. ■

To describe the asymptotic behavior of the solution of (2.15), we need the two results below. The first shows that the Korn inequality for periodic functions is still valid, with a constant independent of ε, for the functions of $H^1_{per}(Y^\star)^n$ having a zero average on Y_ε (instead of $Y^\star$).

Proposition 31 *There exists a positive constant C independent of ε such that*

$$||\varphi||_{1,Y^\star} \leq C||e(\varphi)||_{0,Y^\star}, \tag{2.19}$$

for every φ in $H^1_{per}(Y^\star)^n$ with $\mathcal{M}_{Y_\varepsilon}(\varphi) = 0$.

Proof. Suppose that (2.19) is not true. Then, one can find a subsequence $\{\varepsilon_m\}$ of $\{\varepsilon\}$ and a sequence of functions $\varphi_m \in H^1_{per}(Y^\star)^n$ $(m \in (\mathbb{N}^\star)^n)$ such that

$$\begin{cases} \mathcal{M}_{Y_{\varepsilon_m}}(\varphi_m) = 0 \\ ||\varphi_m||_{1,Y^\star} > m||e(\varphi_m)||_{0,Y^\star}. \end{cases}$$

By setting $\theta_m = \dfrac{\varphi_m}{\|\varphi_m\|_{1,Y_\star}}$, it results

$$\begin{cases} \|\theta_m\|_{1,Y^\star} = 1, \\ \|e(\theta_m)\|_{0,Y^\star} < \dfrac{1}{m}, \\ \mathcal{M}_{Y_{\varepsilon_m}}(\theta_m) = 0. \end{cases}$$

Then, there exists $\theta \in H^1_{per}(Y^\star)^n$ such that (up to a subsequence)

$$\theta_m \rightharpoonup \theta \text{ weakly in } H^1(Y^\star)^n,$$

so that

$$e(\theta_m) \rightharpoonup e(\theta) \text{ weakly in } L^2(Y^\star)^{n\times n}.$$

But

$$e(\theta_m) \to 0 \text{ strongly in } L^2(Y^\star)^{n\times n}.$$

Then, $e(\theta) = 0$, which is equivalent to say that there exists a skew-symmetric matrix L of $\mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ such that $\theta = Ly + b$. But the Y-periodicity of θ gives $L = 0$, hence $\theta = b$.
Moreover, since $\mathcal{S}_\varepsilon$ is $H^1(Y^\star)^n$-admissible in $Y^\star$ and the embedding of $H^1(Y^\star)^n$ in $L^2(Y^\star)^n$ is compact, we have (up to a subsequence)

$$\chi_{Y_{\varepsilon_m}} \rightharpoonup \chi^0 \text{ in } L^\infty(Y^\star) \text{ weak}\star, \text{ with } \chi^0 > 0 \text{ a.e. in } Y^\star$$

and

$$\theta_m \longrightarrow b \text{ strongly in } L^2(Y^\star)^n.$$

Then,

$$0 = \int_{Y_{\varepsilon_m}} \theta_m \, dx = \int_{Y^\star} \theta_m \chi_{Y_{\varepsilon_m}} \, dx \rightharpoonup \int_{Y^\star} b\chi^0 \, dx,$$

so that $\theta = b = 0$.
On the other hand, we have (see, for instance, [42])

$$\|\theta_m\|_{1,Y^\star} \leq C \left(\|\theta_m\|_{0,Y^\star} + \left(\int_{Y^\star} |e(\theta_m)|^2 \, dx \right)^{\frac{1}{2}} \right).$$

Then, by using the previous convergences, we get

$$\lim_{m\to\infty} \|\theta_m\|_{1,Y^\star} = 0.$$

This contradicts the fact that $\|\theta_m\|_{1,Y^\star} = 1$. ■

Lemma 32 *Let $\psi^0 \in H_0^1(Y^\star)^n$ and suppose that (2.13) holds. Then, the solution ψ^ε of*

$$\begin{cases} -div\ \left(B^\varepsilon e\left(\psi^\varepsilon\right)\right) = -(Q_1^\varepsilon)^\star \left(div\left(B^0 e\left(\psi^0\right)\right)\right) \quad in\ Y^\varepsilon, \\ \left(B^\varepsilon e\left(\psi^\varepsilon\right)\right)\nu = 0 \quad on\ \partial\mathcal{S}_\varepsilon, \\ \psi^\varepsilon = 0, \quad on\ \partial Y^\star, \end{cases} \tag{2.20}$$

satisfies

$$\begin{cases} i) \quad Q_1^\varepsilon\psi^\varepsilon \rightharpoonup \psi^0 \quad weakly\ in\ H_0^1(Y^\star)^n, \\ ii) \quad B^\varepsilon \widetilde{e\left(\psi^\varepsilon\right)} \rightharpoonup B^0 e\left(\psi^0\right) \quad weakly\ in\ L^2(Y^\star)^{n\times n}. \end{cases} \tag{2.21}$$

Proof. By assumption, $(B^\varepsilon, \mathcal{S}_\varepsilon)$ H_e^0-converges in $Y^\star$ to B^0. Then

$$\begin{cases} \text{i)} \quad Q_1^\varepsilon\psi^\varepsilon \rightharpoonup \theta^0 \quad \text{weakly in } H_0^1(Y^\star)^n, \\ \text{ii)} \quad B^\varepsilon \widetilde{e\left(\psi^\varepsilon\right)} \rightharpoonup B^0 e\left(\theta^0\right) \quad \text{weakly in } L^2(Y^\star)^{n\times n}, \end{cases} \tag{2.22}$$

where $\theta^0 \in H_0^1(Y^\star)^n$ satisfies

$$\begin{cases} -div\ \left(B^0 e\left(\theta^0\right)\right) = -\left(div\left(B^0 e\left(\psi^0\right)\right)\right) \quad \text{in } Y^\star, \\ \theta^0 = 0 \quad \text{on } \partial Y^\star. \end{cases}$$

Since $\psi^0 \in H_0^1(Y^\star)^n$ and $B^0 \in M_e\left(\frac{\alpha}{C^2}, \frac{\beta^2}{\alpha}, Y\right)$, the uniqueness of solution of this problem implies that $\theta^0 = \psi^0$ a.e.in $Y^\star$. This, together with (2.22), gives (2.21). ■

The following result gives a priori estimate for the solution of (2.15).

Proposition 33 *For each symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$, $\|W_\Lambda^\varepsilon\|_{1,Y_\varepsilon}$ is bounded independently of ε.*

Proof. Taking $\varphi = \varkappa_\Lambda^\varepsilon$ in (2.17) and using the fact $\{B^\varepsilon\}_\varepsilon \in M_e\left(\alpha, \beta, Y\right)$, we find

$$\begin{aligned} \alpha \left\|e\left(\varkappa_\Lambda^\varepsilon\right)\right\|_{0,Y_\varepsilon}^2 &\le \int_{Y_\varepsilon} B^\varepsilon e\left(\varkappa_\Lambda^\varepsilon\right)\ e\left(\varkappa_\Lambda^\varepsilon\right)\ dy \\ &= \int_{Y_\varepsilon} B^\varepsilon \Lambda\ e\left(\varkappa_\Lambda^\varepsilon\right)\ dy \\ &\le \beta \left\|e\left(\varkappa_\Lambda^\varepsilon\right)\right\|_{0,Y_\varepsilon} |\Lambda|\ |Y_\varepsilon|^{\frac{1}{2}} \\ &\le \beta\, |\Lambda|\ |Y|^{\frac{1}{2}} \left\|e\left(\varkappa_\Lambda^\varepsilon\right)\right\|_{0,Y_\varepsilon}. \end{aligned}$$

Then

$$\left\|e\left(\varkappa_\Lambda^\varepsilon\right)\right\|_{0,Y_\varepsilon} \le \frac{\beta}{\alpha} |\Lambda|\ |Y|^{\frac{1}{2}}. \tag{2.23}$$

Since $Q_\varepsilon^1\varkappa_\Lambda^\varepsilon \in H_{per}^1(Y^\star)$ and $\mathcal{M}_{Y_\varepsilon}\left(Q_\varepsilon^1\varkappa_\Lambda^\varepsilon\right) = 0$ (due to the fact that $\varkappa_\Lambda^\varepsilon \in W_{per}(Y_\varepsilon)$), Proposition 31 gives

$$\left\|\varkappa_\Lambda^\varepsilon\right\|_{1,Y_\varepsilon} \le \left\|Q_\varepsilon^1\varkappa_\Lambda^\varepsilon\right\|_{1,Y^\star} \le C \left\|e\left(Q_\varepsilon^1\varkappa_\Lambda^\varepsilon\right)\right\|_{0,Y^\star} \le C^2 \left\|e\left(\varkappa_\Lambda^\varepsilon\right)\right\|_{0,Y_\varepsilon}.$$

So, by using (2.23), one gets

$$\|\varkappa_\Lambda^\varepsilon\|_{1,Y_\varepsilon} \leq \left\|Q_\varepsilon^1 \varkappa_\Lambda^\varepsilon\right\|_{1,Y^\star} \leq C^2 \frac{\beta}{\alpha} |\Lambda| \, |Y|^{\frac{1}{2}}, \tag{2.24}$$

which implies, by virtue of (2.18), that W_Λ^ε is bounded in $H^1(Y_\varepsilon)^n$ independently of ε. ■

The next result describes the asymptotic behavior of W_Λ^ε.

Proposition 34 *Let W_Λ, W_Λ^ε and $\varkappa_\Lambda^\varepsilon$ be the solutions of problems (2.14) (2.15) and (2.17) respectively and L_Λ^ε be given by $L_\Lambda^\varepsilon = \Lambda y - Q_\varepsilon^1 \varkappa_\Lambda^\varepsilon - \mathcal{M}_{Y^\star}\left(\Lambda y - Q_\varepsilon^1 \varkappa_\Lambda^\varepsilon\right)$. We have*

$$\begin{cases} i)\ L_\Lambda^\varepsilon \rightharpoonup W_\Lambda \quad \text{weakly in } H^1(Y^\star)^n, \\ \\ ii)\ B^\varepsilon \widetilde{e\left(W_\Lambda^\varepsilon\right)} \rightharpoonup B^0 e\left(W_\Lambda\right) \quad \text{weakly in } L^2(Y^\star)^{n\times n}. \end{cases} \tag{2.25}$$

Proof. Proposition 33 shows that L_Λ^ε is bounded in $H^1(Y^\star)^n$ and $\widetilde{\xi_\Lambda^\varepsilon} \equiv B^\varepsilon \widetilde{e\left(W_\Lambda^\varepsilon\right)}$ is bounded in $L^2(Y^\star)^{n\times n}$ (since $\{B^\varepsilon\}_\varepsilon \in M_e(\alpha, \beta, Y)$).
Then, up to a subsequence, there exist two functions $W \in H^1(Y^\star)^n$ and $\xi \in L^2(Y^\star)^{n\times n}$ such that

$$\begin{cases} \text{i)} \quad L_\Lambda^\varepsilon \rightharpoonup W \quad \text{weakly in } H^1(Y^\star)^n, \\ \\ \text{ii)} \quad B^\varepsilon \widetilde{e\left(W_\Lambda^\varepsilon\right)} \rightharpoonup \xi \quad \text{weakly in } L^2(Y^\star)^{n\times n}. \end{cases} \tag{2.26}$$

To obtain (2.25)i) and (2.25)ii), it is necessary and sufficient to show that

$$W = W_\Lambda, \qquad \xi = B^0 e\left(W_\Lambda^\varepsilon\right) \quad \text{in } Y^\star.$$

We will show that in two steps. In the first step we prove some properties of W and ξ and in the second one we establish the relationship between them and W_Λ.
Step 1. It is clear that L_Λ^ε is in $\{\Lambda y + H_{per}^1(Y^\star)^n\}$ and $\mathcal{M}_{Y^\star}\left(L_\Lambda^\varepsilon\right) = 0$, so that the same holds for W, since the set $\left\{v \in \left\{\Lambda y + H_{per}^1(Y^\star)^n\right\}, \mathcal{M}_{Y^\star}(v) = 0\right\}$ is a closed subspace of $\{\Lambda y + H_{per}^1(Y^\star)^n\}$.
On the other hand, by virtue of the symmetry properties of B^ε, ξ is symmetric. Moreover, passing to the limit in (2.15), one obtains

$$\int_{Y^\star} \xi \, e\left(\varphi\right) \, dy = 0, \quad \forall \varphi \in H_{per}^1(Y^\star)^n. \tag{2.27}$$

Step 2. Let $\psi^0 \in H_0^1(Y^\star)^n$ and ψ^ε the solution of (2.20). Then

$$\xi_\Lambda^\varepsilon . e\left(\psi^\varepsilon\right) = B^\varepsilon e\left(\psi^\varepsilon\right) . e\left(W_\Lambda^\varepsilon\right), \quad \text{in } Y_\varepsilon.$$

But $e(L_\Lambda^\varepsilon)_{|Y_\varepsilon} = e\left(W_\Lambda^\varepsilon\right)$, hence

$$\widetilde{\xi_\Lambda^\varepsilon} . e\left(Q_1^\varepsilon \psi^\varepsilon\right) = B^\varepsilon \widetilde{e\left(\psi^\varepsilon\right)} . e\left(L_\Lambda^\varepsilon\right) \quad \text{in } Y^\star. \tag{2.28}$$

For computing the limit in $\mathcal{D}'(Y^\star)$ of each side of (2.26), we use Theorem 22 in $Y^\star$. First we choose here $\xi^\varepsilon = \xi^\varepsilon_\Lambda$, $f^\varepsilon = 0$ and $\eta^\varepsilon = Q_1^\varepsilon \psi^\varepsilon$. From (2.21)i), (2.15) and (2.26)ii), we have

$$\widetilde{\xi^\varepsilon_\Lambda}.e\left(Q_1^\varepsilon \psi^\varepsilon\right) \rightharpoonup \xi.e\left(\psi^0\right) \quad \text{in } \mathcal{D}'(Y^\star). \tag{2.29}$$

To pass to the limit in the right-hand side of (2.28), we use here Theorem 22 written in $Y^\star$ for $\xi^\varepsilon = B^\varepsilon e\left(Q_1\psi^\varepsilon\right)$, $f^\varepsilon = -div\left(B^0 e\left(\psi^0\right)\right)$ and $\eta^\varepsilon = L^\varepsilon_\Lambda$. From (2.26)i), (2.20) and (2.21)ii), we have

$$B^\varepsilon \widetilde{e\left(\psi^\varepsilon\right)}.e\left(L^\varepsilon_\Lambda\right) \rightharpoonup B^0 e\left(\psi^0\right).e\left(W\right) \quad \text{in } \mathcal{D}'(Y^\star). \tag{2.30}$$

Combining (2.28), (2.29) and (2.30), we obtain

$$\xi.e\left(\psi^0\right) = B^0 e\left(\psi^0\right).e\left(W\right) \qquad \text{a.e. in } Y^\star.$$

Taking into account the symmetries of B^0, this equality becomes

$$\xi.\nabla\psi^0 = B^0 e\left(W\right).\nabla\psi^0 \qquad \text{a.e. in } Y^\star,$$

for every $\psi^0 \in H^1_0(Y^\star)^n$. Hence

$$\xi = B^0 e\left(W\right) \quad \text{a.e. in } Y^\star.$$

Then, using (2.27), we conclude that

$$\begin{cases} W \in \{\Lambda y + H^1_{per}(Y^\star)^n\}, \ \mathcal{M}_{Y^\star}\left(W\right) = 0, \\ \int_{Y^\star} B^0 e\left(W\right)\ e\left(\varphi\right)\ dy = 0, \qquad \forall \varphi \in H^1_{per}(Y^\star). \end{cases}$$

From the uniqueness of solution of this problem, given by Proposition 30, we deduce $W = W_\Lambda$ and $\xi = B^0 e\left(W_\Lambda\right)$. ■

Before proving Theorem 29, we prove the following lemmas :

Lemma 35 *Let $C_\varepsilon(\partial\Omega)$ be the union of all the cells which intersect $\partial\Omega$ and let $N^\varepsilon(\partial\Omega)$ their number. Then, one has*

$$(|\partial\Omega| = 0) \Longleftrightarrow (\varepsilon^n N^\varepsilon(\partial\Omega) \longrightarrow 0).$$

Proof. Suppose that $|\partial\Omega| = 0$. Denote by $diam(Y)$ the diameter of Y and let $\{l_\varepsilon\}$ be a sequence of positive numbers which tends to zero, such that $l_\varepsilon > \varepsilon\ diam(Y)$, for every ε. Set

$$V_{l_\varepsilon}(\partial\Omega) = \{x \in \mathbb{R}^n / \ dist(x, \partial\Omega) \leq l_\varepsilon\}.$$

Since $V_{l_\varepsilon}(\partial\Omega)$ is a sequence of sets decreasing to $\partial\Omega$,

$$|V_{l_\varepsilon}(\partial\Omega)| \longrightarrow |\partial\Omega| = 0.$$

On the other hand, one has

$$C_\varepsilon(\partial\Omega) \subset V_{l_\varepsilon}(\partial\Omega),$$

hence

$$|C_\varepsilon(\partial\Omega)| \leq |V_{l_\varepsilon}(\partial\Omega)|.$$

But

$$|C_\varepsilon(\partial\Omega)| = |\varepsilon Y| N^\varepsilon(\partial\Omega) = \varepsilon^n |Y| N^\varepsilon(\partial\Omega). \tag{2.31}$$

Then (since $|Y| \neq 0$),

$$\varepsilon^n N^\varepsilon(\partial\Omega) \longrightarrow 0.$$

Conversely, if $\varepsilon^n N^\varepsilon(\partial\Omega) \to 0$, then from (2.31), $C_\varepsilon(\partial\Omega) \to 0$. Since

$$|\partial\Omega| \leq |C_\varepsilon(\partial\Omega)|,$$

passing to the limit in this inequality, one deduces that $|\partial\Omega| = 0$. ■

Lemma 36 *For every $\varphi \in L^2_{per}(Y)^n$, set $\varphi^\varepsilon(x) = \varphi(\frac{x}{\varepsilon})$. We have*

$$\|\varphi^\varepsilon\|^2_{0,\Omega} \leq \frac{|\Omega|}{|Y|}(1 + o(1))\|\varphi\|^2_{0,Y}.$$

Proof. By virtue of the "paving property" of Y, Ω can be covered by $N(\varepsilon)$ cells $\left(\overline{Y_i^\varepsilon}\right)_{i=1}^{i=N(\varepsilon)}$ of type εY, i.e.

$$\Omega \subset \overset{N(\varepsilon)}{\underset{i=1}{\cup}} \overline{Y_i^\varepsilon},$$

where $N(\varepsilon) = \varepsilon^{-n}\frac{|\Omega|}{|Y|}(1 + o(1))$ and each $\overline{Y_i^\varepsilon}$ is a translated set of $\varepsilon\overline{Y}$. Then,

$$\begin{aligned}
\|\varphi^\varepsilon\|^2_{0,\Omega} &= \int_\Omega \left|\varphi\left(\frac{x}{\varepsilon}\right)\right|^2 dx \\
&\leq \sum_{i=1}^{N(\varepsilon)} \int_{\overline{Y_i^\varepsilon}} \left|\varphi\left(\frac{x}{\varepsilon}\right)\right|^2 dx \\
&= \sum_{i=1}^{N(\varepsilon)} \int_{Y_i^\varepsilon} \left|\varphi\left(\frac{x}{\varepsilon}\right)\right|^2 dx.
\end{aligned}$$

Using the change of scale $y = \frac{x}{\varepsilon}$ and the Y-periodicity of φ, we obtain

$$\begin{aligned}
\|\varphi^\varepsilon\|^2_{0,\Omega} &\leq \varepsilon^n \int_Y |\varphi(y)|^2\, dy \sum_{i=1}^{N(\varepsilon)} 1 \\
&= \frac{|\Omega|}{|Y|}(1 + o(1))\|\varphi\|^2_{0,Y}.
\end{aligned}$$

■

2.4.2 Proof of Theorem 29

We are able now to prove Theorem 29. First observe that $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$. Let the sequence $\{w_\Lambda^\varepsilon\}$ be defined by

$$w_\Lambda^\varepsilon = \Lambda x - \varepsilon\left(Q_\varepsilon \varkappa_\Lambda^\varepsilon\right)\left(\frac{x}{\varepsilon}\right), \qquad \text{for } x \text{ a.e. in } \Omega,$$

where $Q_\varepsilon \equiv Q^2 o Q_\varepsilon^1$.
Let us show that the sequence $\{w_\Lambda^\varepsilon\}$ satisfies the assumptions of Theorem 23. We proceed in three steps.

Step 1. Let us establish first (2.7)i), i.e.

$$w_\Lambda^\varepsilon \rightharpoonup \Lambda x \text{ weakly in } H^1(\Omega)^n.$$

By virtue of Lemma 36, we have

$$\begin{aligned}\left\|\varepsilon(Q_\varepsilon \varkappa_\Lambda^\varepsilon\left(\frac{x}{\varepsilon}\right))\right\|_{1,\Omega}^2 &= \varepsilon^2\int_\Omega\left|Q_\varepsilon\varkappa_\Lambda^\varepsilon\left(\frac{x}{\varepsilon}\right)\right|^2 dx + \int_\Omega\left|\nabla_y(Q_\varepsilon\varkappa_\Lambda^\varepsilon)\left(\frac{x}{\varepsilon}\right)\right|^2 dx \\ &\leq \varepsilon^2\frac{|\Omega|}{|Y|}(1+o(1))\left\|Q_\varepsilon\varkappa_\Lambda^\varepsilon\right\|_{0,Y}^2 + \frac{|\Omega|}{|Y|}(1+o(1))\left\|\nabla_y\left(Q_\varepsilon\varkappa_\Lambda^\varepsilon\right)\right\|_{0,Y}^2.\end{aligned}$$

Hence, estimates (2.12) and (2.24) imply that $\varepsilon\left(Q_\varepsilon\varkappa_\Lambda^\varepsilon\right)\left(\frac{x}{\varepsilon}\right)$ is bounded in $H^1(\Omega)^n$ independently of ε, which provides the existence of a subsequence of ε (still denoted ε) and θ in $H^1(\Omega)^n$ such that

$$\varepsilon(Q_\varepsilon\varkappa_\Lambda^\varepsilon\left(\frac{x}{\varepsilon}\right)) \rightharpoonup \theta \text{ weakly in } H^1(\Omega)^n.$$

Then

$$\varepsilon(Q_\varepsilon\varkappa_\Lambda^\varepsilon\left(\frac{x}{\varepsilon}\right)) \rightharpoonup \theta \text{ weakly in } L^2(\Omega)^n.$$

But, estimate (2.12) and (2.24) give also

$$\left\|\varepsilon Q_\varepsilon\varkappa_\Lambda^\varepsilon(\frac{x}{\varepsilon})\right\|_{0,\Omega} \leq \varepsilon C^4\frac{\beta}{\alpha}|\Lambda|\sqrt{|\Omega|(1+o(1))},$$

hence $\varepsilon Q_\varepsilon\varkappa_\Lambda^\varepsilon(\frac{x}{\varepsilon})$ converges strongly (and then weakly) to 0 in $L^2(\Omega)^n$. Then, by virtue of the uniqueness of the weak limit, $\theta = 0$, so that (2.7)i) holds.

Step 2. Let us show now(2.7)ii). By construction,

$$\begin{aligned}A^\varepsilon(x)e\left(w_\Lambda^\varepsilon\right)(x) &= B^\varepsilon(\frac{x}{\varepsilon})e\left(\Lambda x - \varepsilon\left(Q_\varepsilon\varkappa_\Lambda^\varepsilon\right)\left(\frac{x}{\varepsilon}\right)\right) \\ &= B^\varepsilon(\frac{x}{\varepsilon})\left(\Lambda - e\left(Q_\varepsilon\varkappa_\Lambda^\varepsilon\right)\left(\frac{x}{\varepsilon}\right)\right) \quad \text{a.e. in } \Omega.\end{aligned}$$

Then, if we set

$$\theta_\Lambda^\varepsilon = \Lambda - e\left(Q_\varepsilon\varkappa_\Lambda^\varepsilon\right),$$

one has

$$A^\varepsilon e(w_\Lambda^\varepsilon)(x) = (B^\varepsilon\theta_\Lambda^\varepsilon)\left(\frac{x}{\varepsilon}\right) \qquad \text{a.e. in } \Omega_\varepsilon.$$

Setting

$$\Sigma_\Lambda^\varepsilon(x) = (B^\varepsilon \theta_\Lambda^\varepsilon)\,(\frac{x}{\varepsilon})$$

and using (2.15), this gives that

$$\begin{cases} -div\,[A^\varepsilon e\,(w_\Lambda^\varepsilon)] = -div\Sigma_\Lambda^\varepsilon \quad \text{in } \Omega_\varepsilon, \\ (A^\varepsilon e\,(w_\Lambda^\varepsilon))\;\;\nu = 0 \text{ on } \partial T_\varepsilon. \end{cases}$$

We need to show that for a suitable g_Λ^ε compact in $H^{-1}(\Omega)^n$ we have

$$-div\Sigma_\Lambda^\varepsilon = P_\varepsilon^\star g_\Lambda^\varepsilon \quad \text{in } \Omega_\varepsilon.$$

To do that, we denote by $C^\varepsilon(\partial\Omega)$ the union of all the cells which intersect $\partial\Omega$ and by $N^\varepsilon(\partial\Omega)$ their number. We also denote $T^\varepsilon(\partial\Omega)$ the holes of $\mathbb{R}^n$ contained in $C^\varepsilon(\partial\Omega)$. Then, for every $\varphi \in V_\varepsilon$, we have, by symmetry

$$\begin{aligned} \langle -div\Sigma_\Lambda^\varepsilon, \varphi\rangle_{V_\varepsilon', V_\varepsilon} &= \int_{\Omega_\varepsilon} (B^\varepsilon\theta_\Lambda^\varepsilon)(\frac{x}{\varepsilon})\; e(\varphi)(x)\; dx \\ &= \int_{\Omega_\varepsilon \setminus T^\varepsilon(\partial\Omega)} (B^\varepsilon\theta_\Lambda^\varepsilon)(\frac{x}{\varepsilon})\; e(\varphi)(x)\; dx \\ &+ \int_{\Omega\cap T^\varepsilon(\partial\Omega)} (B^\varepsilon\theta_\Lambda^\varepsilon)(\frac{x}{\varepsilon})\; e(\varphi)(x)\; dx. \end{aligned}$$

Since $\theta_\Lambda^\varepsilon = e(W_\Lambda^\varepsilon)$ in $\Omega_\varepsilon \setminus T^\varepsilon(\partial\Omega)$ and $\Omega_\varepsilon \setminus T^\varepsilon(\partial\Omega)$ is a union of translated cells of $\varepsilon Y_\varepsilon$, one has

$$\int_{\Omega_\varepsilon \setminus T^\varepsilon(\partial\Omega)} (B^\varepsilon\theta_\Lambda^\varepsilon)(\frac{x}{\varepsilon})\; e(\varphi)(x)\; dx = 0.$$

Then,

$$\begin{aligned} \langle -div\;\Sigma_\Lambda^\varepsilon, \varphi\rangle_{V_\varepsilon', V_\varepsilon} &= \int_{\Omega\cap T^\varepsilon(\partial\Omega)} (B^\varepsilon\theta_\Lambda^\varepsilon)(\frac{x}{\varepsilon})\; e(\varphi)(x)\; dx \\ &= \int_\Omega \chi_{T^\varepsilon(\partial\Omega)} (B^\varepsilon\theta_\Lambda^\varepsilon)(\frac{x}{\varepsilon})\nabla(P_\varepsilon\varphi)(x)\; dx. \end{aligned}$$

Hence, if we set $g_\Lambda^\varepsilon(x) = -div(\chi_{T^\varepsilon(\partial\Omega)}(B^\varepsilon\theta_\Lambda^\varepsilon)(\frac{x}{\varepsilon}))$, one obtains

$$\langle -div\;\Sigma_\Lambda^\varepsilon, \varphi\rangle_{V_\varepsilon', V_\varepsilon} = \langle P^\star g_\Lambda^\varepsilon, \varphi\rangle_{V_\varepsilon', V_\varepsilon}.$$

Now let us show that $g_\Lambda^\varepsilon \longrightarrow 0$ strongly in $H^{-1}(\Omega)^n$. For every $v \in H_0^1(\Omega)^n$, one has

$$\begin{aligned} |\langle g_\Lambda^\varepsilon, v\rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n}| &= |\int_{\Omega\cap T^\varepsilon(\partial\Omega)} (B^\varepsilon\theta_\Lambda^\varepsilon)(\frac{x}{\varepsilon})\; e(v)(x)\; dx| \\ &\le \beta\left(\int_{T^\varepsilon(\partial\Omega)} |\theta_\Lambda^\varepsilon(\frac{x}{\varepsilon})|^2\; dx\right)^{\frac{1}{2}} \|\nabla v\|_{0,\Omega}. \end{aligned}$$

But

$$\int_{\Omega\cap T^\varepsilon(\partial\Omega)} |\theta_\Lambda^\varepsilon(\frac{x}{\varepsilon})|^2\; dx \le \int_{C_\varepsilon(\partial\Omega)} |\theta_\Lambda^\varepsilon(\frac{x}{\varepsilon})|^2\; dx,$$

which implies by using the change of scale $y = \dfrac{x}{\varepsilon}$

$$\begin{aligned}\int_{\Omega\cap T^\varepsilon(\partial\Omega)} |\theta^\varepsilon_\Lambda(\frac{x}{\varepsilon})|^2\, dx &\leq \varepsilon^n N^\varepsilon(\partial\Omega)\int_Y |\theta^\varepsilon_\Lambda(y)|^2\, dy\\ &\leq \varepsilon^n N^\varepsilon(\partial\Omega)\|\theta^\varepsilon_\Lambda\|^2_{0,Y}\\ &\leq \varepsilon^n N^\varepsilon(\partial\Omega)\|\Lambda - e\left(Q_\varepsilon \varkappa^\varepsilon_\Lambda\right)\|^2_{0,Y}\\ &\leq C\varepsilon^n N^\varepsilon(\partial\Omega).\end{aligned}$$

So

$$|\langle g^\varepsilon_\Lambda, v\rangle_{H^{-1}(\Omega)^n, H^1_0(\Omega)^n}| \leq \beta\left(C\varepsilon^n N^\varepsilon(\partial\Omega)\right)^{\frac{1}{2}}\|\nabla v\|_{0,\Omega}.$$

Thanks to Lemma 35, one deduces

$$\underset{\|v\|_{H^1_0(\Omega)^n}=1}{sup}|\langle g^\varepsilon_\Lambda, v\rangle_{H^{-1}(\Omega)^n, H^1_0(\Omega)^n}| \longrightarrow 0,$$

so that

$$g^\varepsilon_\Lambda \longrightarrow 0 \text{ strongly in } H^{-1}(\Omega)^n.$$

Hence, in particular, (2.7)ii) holds for $\{g^\varepsilon_\Lambda\}$ compact in $H^{-1}(\Omega)^n$.

Step 3. It remains now to show (2.7)iii) and identify the weak limit of $\chi_{\Omega_\varepsilon}A^\varepsilon e\left(w^\varepsilon_\Lambda\right)$. For every $\varphi \in L^2(\Omega)$, one has

$$\begin{aligned}\int_\Omega \chi_{\Omega_\varepsilon}A^\varepsilon e\left(w^\varepsilon_\Lambda\right)\varphi dx &= \int_{\Omega_\varepsilon\setminus T^\varepsilon(\partial\Omega)}(B^\varepsilon\theta^\varepsilon_\Lambda)(\frac{x}{\varepsilon})\,\varphi(x)\, dx\\ &+ \int_{\Omega\cap T^\varepsilon(\partial\Omega)}(B^\varepsilon\theta^\varepsilon_\Lambda)(\frac{x}{\varepsilon})\,\varphi(x)\, dx\end{aligned}$$

Since $\theta^\varepsilon_\Lambda = e(W^\varepsilon_\Lambda)$ in $\Omega_\varepsilon\setminus T^\varepsilon(\partial\Omega)$, one deduces

$$\begin{aligned}\int_\Omega \chi_{\Omega_\varepsilon}A^\varepsilon e\left(w^\varepsilon_\Lambda\right)\varphi dx &= \int_{\Omega_\varepsilon\setminus T^\varepsilon(\partial\Omega)}(B^\varepsilon e(W^\varepsilon_\Lambda))(\frac{x}{\varepsilon})\,\varphi(x)\, dx\\ &+ \int_{\Omega\cap T^\varepsilon(\partial\Omega)}(B^\varepsilon\theta^\varepsilon_\Lambda)(\frac{x}{\varepsilon})\,\varphi(x)\, dx\\ &= \int_\Omega (\widetilde{B^\varepsilon e\left(W^\varepsilon_\Lambda\right)}^{Y})(\frac{x}{\varepsilon})\,\varphi(x)\, dx\\ &+ \int_{\Omega\cap T^\varepsilon(\partial\Omega)}(B^\varepsilon\theta^\varepsilon_\Lambda)(\frac{x}{\varepsilon})\,\varphi(x)\, dx.\end{aligned}$$

In first hand, using the same arguments as in the second step, one obtains

$$\int_{\Omega\cap T^\varepsilon(\partial\Omega)}(B^\varepsilon\theta^\varepsilon_\Lambda)(\frac{x}{\varepsilon})\,\varphi(x)\, dx \to 0.$$

On the other hand from (2.25)ii), it comes

$$\begin{aligned}\mathcal{M}_Y\left(B^\varepsilon \widetilde{\widetilde{e(W_\Lambda^\varepsilon)}}^Y\right) &= \frac{1}{|Y|}\int_Y B^\varepsilon \widetilde{\widetilde{e(W_\Lambda^\varepsilon)}}^Y \, dy \\ &= \frac{1}{|Y|}\int_{Y^\star} B^\varepsilon \widetilde{e(W_\Lambda^\varepsilon)}\, dy \to \frac{1}{|Y|}\int_{Y^\star} B^0 e(W_\Lambda)\, dy \\ &= \mathcal{M}_Y\left(B^0 \widetilde{e(W_\Lambda)}^Y\right).\end{aligned}$$

This implies, by virtue of Lemma 25 written for $h_\varepsilon = B^\varepsilon \widetilde{\widetilde{e(W_\Lambda^\varepsilon)}}^Y$, that

$$\int_\Omega (B^\varepsilon \widetilde{\widetilde{e(W_\Lambda^\varepsilon)}}^Y)(\frac{x}{\varepsilon})\, \varphi(x)\, dx \to \int_\Omega \mathcal{M}_Y\left(B^0 \widetilde{e(W_\Lambda)}^Y\right) \varphi(x)\, dx.$$

Then,

$$\chi_{\Omega_\varepsilon} A^\varepsilon e(w_\Lambda^\varepsilon) \rightharpoonup \mathcal{M}_Y\left(B^0 \widetilde{e(W_\Lambda)}^Y\right) \quad \text{weakly in } L^2(\Omega)^{n\times n}. \tag{2.32}$$

In conclusion, the sequence $\{w_\Lambda^\varepsilon\}$ satisfies all the assumptions of Theorem 23, so that from (2.32), it comes

$$A^0\Lambda = \mathcal{M}_Y(B^0 \widetilde{e(W_\Lambda)}^Y).$$

This ends the proof of Theorem 29.

2.5 The case of multiple periodic scales

Under the notations of the previous section, we consider here the particular case where the holes and the coefficients are periodic at any scale. In this situation, if we suppose that $|\partial\Omega| = 0$ and $\mathcal{T}^\star$ has a Lipschitz boundary, one can describe precisely the homogenized tensor A^0 of Theorem 29.

2.5.1 Homogenization with one periodic scales

Observe first that the classical periodic case studied by F. Léné [31] can be easily obtained as a particular case of Theorem 29.
Indeed, let A be a tensor of $M(\alpha, \beta, Y)$ and set

$$A^\varepsilon(x) \equiv A(\frac{x}{\varepsilon}), \qquad \text{for x in } \Omega.$$

Taking in Theorem 29 $B^\varepsilon = A$, $\mathcal{S}_\varepsilon = \emptyset$ and remarking that (A,$\emptyset$) $\stackrel{H_e^0}{\rightharpoonup}$ A in $Y^\star$, one obtains

$$(A^\varepsilon, T_\varepsilon) \stackrel{H_e^0}{\rightharpoonup} A^0,$$

where A^0 is defined by

$$A^0\Lambda = \mathcal{M}_Y(A \widetilde{e(W_\Lambda)}^Y),$$

for every symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$ and W_Λ is the unique solution of

$$\begin{cases} \text{Find } W_\Lambda \text{ in } \left\{\Lambda y + H^1_{per}(Y^\star)^n\right\} \text{ such that} \\ \mathcal{M}_{Y^\star}\left(W_\Lambda\right) = 0, \\ \int_{Y^\star} Ae\left(W_\Lambda\right)\ e\left(\varphi\right) dz = 0, \qquad \forall \varphi \in H^1_{per}(Y^\star)^n. \end{cases}$$

2.5.2 Homogenization with double periodic scales

We use here the same framework of [17]. Suppose that $\mathcal{S}_\varepsilon$ itself arises from a partially Z-periodic distribution in $Y^\star$. More precisely, assume that Z is a domain with piece-wise smooth boundary which having the paving property in $\mathbb{R}^n$, with respect to a given basis $(b'_1, ..., b'_n)$ and that $\mathcal{S}_1$ is compact smooth subset of Z. Let Y_1 be a nonempty open subset of $Y^\star$ and $\left\{\delta_\varepsilon\right\}_\varepsilon$ be a positive sequence going to zero with ε. Define $\mathcal{S}_\varepsilon \subset Y^\star$ by

$$\mathcal{S}_\varepsilon = \left\{\cup \delta_\varepsilon \left\{\mathcal{S}_1 + k_l b'_l\right\}, k \in \mathbb{Z}^n, \delta_\varepsilon \left\{\mathcal{S}_1 + k_l b'_l\right\} \subset Y_1\right\}.$$

Let D a Z-periodic tensor in $M(\alpha, \beta, Z)$ and B be a tensor in $M(\alpha, \beta, Y)$. Set then,

$$D^\varepsilon(y) = D\left(\frac{y}{\delta_\varepsilon}\right) \quad \text{for } y \in Y_1,$$

and

$$B^\varepsilon(y) = \begin{cases} D^\varepsilon(y) & \text{for } y \in Y_1, \\ B(y) & \text{for } y \in Y \setminus Y_1, \end{cases}$$

which belongs to $M_e\left(\alpha, \beta, Y\right)$ and it is extended by Y-periodicity. Finally denote by $\widetilde{\bullet}^Z$ the extension by 0 from $Z^\star \equiv Z \setminus \mathcal{S}$ to Z.

Proposition 37 *Assume that $|\partial Y_1| = 0$. Then, the pair $(A^\varepsilon, T_\varepsilon)_\varepsilon$ H^0_e-converges in Ω to A^0 defined by*

$$A^0\Lambda = \mathcal{M}_Y(\widetilde{B^0 e\left(W_\Lambda\right)}^Y).$$

for every symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$, and where W_Λ is the unique solution of (2.14) with

$$B^0(y) = \begin{cases} D^0 & \text{in } Y_1, \\ B(y) & \text{for } y \in Y \setminus Y_1, \end{cases} \tag{2.33}$$

with,

$$D^0\Lambda = \mathcal{M}_Z(\widetilde{De\left(\hat{w}_\Lambda\right)}^Z), \tag{2.34}$$

and $\hat{w}_\Lambda$ is the unique solution of

$$\begin{cases} \text{Find } \hat{w}_\Lambda \text{ in } \left\{\Lambda z + H^1_{per}(Z^\star)^n\right\} \text{ such that} \\ \mathcal{M}_{Z^\star}\left(\hat{w}_\Lambda\right) = 0, \\ \int_{Z^\star} De\left(\hat{w}_\Lambda\right)\ e\left(\varphi\right) dz = 0, \qquad \forall \varphi \in H^1_{per}(Z^\star)^n. \end{cases} \tag{2.35}$$

Proof. According to Theorem 29, it suffices to show that

$$\{(B^\varepsilon, \mathcal{S}_\varepsilon)\}_\varepsilon \ H_e^0\text{-converges in } Y^\star \text{ to } B^0, \tag{2.36}$$

where B^0 is defined by (2.33)-(2.35). We do that in three steps.

Step 1. Admissibility of $\mathcal{S}_\varepsilon$
The holes of $\mathcal{S}_\varepsilon$ have size δ_ε, are $\delta_\varepsilon Z-$periodicity in Y_1 and do not touch ∂Y_1. Hence $\{\mathcal{S}_\varepsilon\}_\varepsilon$ is $H^1(Y_1)^n$-admissible for the linearized elasticity (see, for instance, [42]), which implies that it is also $H^1(Y^\star)^n$-admissible for the linearized elasticity. Then, $\{\mathcal{S}_\varepsilon\}_\varepsilon$ is also e-admissible in Y_1 and $Y^\star$ (in the sense of Definition 2).

Step 2. The $\mathbf{H}_e^0$-limits of $\{(B^\varepsilon,\mathcal{S}_\varepsilon)\}_\varepsilon$, $\{(D^\varepsilon, \mathcal{S}_\varepsilon)\}_\varepsilon$ and $\{(B, \emptyset)\}$
Since $\{B^\varepsilon\}_\varepsilon \in M_e(\alpha, \beta, Y)$, the Compactness Theorem 19 implies the existence of a subsequence (still denoted $\{\varepsilon\}$) such that $(B^\varepsilon, \mathcal{S}_\varepsilon)$ H_e^0-converges in $Y^\star$ to some $B^0 \in M_e\left(\frac{\alpha}{C^2}, \frac{\beta^2}{\alpha}, Y\right)$.
By virtue of the results of F. Léné [31], $\{(D^\varepsilon, \mathcal{S}_\varepsilon)\}_\varepsilon$ H_e^0-converges in the open set Y_1 to D^0 defined by (2.34)-(2.35).

On the other hand $\{(B, \emptyset)\}$ H_e^0-converges in the open set $\left(Y^\star \setminus \overline{Y_1}\right)$ to B.

Step 3. Relationship between the $\mathbf{H}_e^0$-limits
In order to prove (2.36), we use the locality property for the H_e^0 -limit given by Theorem 21, applied as follows.

First, choose in Theorem 21

$$\begin{cases} O_1 = Y^\star, \quad ,O_2 = Y_1, \\ T_\varepsilon^1 = \mathcal{S}_\varepsilon,\ T_\varepsilon^2 = \mathcal{S}_\varepsilon, \\ C_1^\varepsilon = B^\varepsilon, \quad C_2^\varepsilon = D^\varepsilon. \end{cases}$$

Then,

$$\begin{cases} \{(B^\varepsilon, \mathcal{S}_\varepsilon)\}_\varepsilon \xrightarrow{H_e^0} B^0 \quad \text{in } Y^\star, \\ \{(D^\varepsilon, \mathcal{S}_\varepsilon)\}_\varepsilon \xrightarrow{H_e^0} D^0 \quad \text{in } Y_1. \end{cases}$$

For every relatively compact open subset ω of $O_1 \cap O_2 = Y_1$, we have

$$\omega \cap T_\varepsilon^1 = \omega \cap T_\varepsilon^2 \quad \text{and} \quad C_1^\varepsilon = C_2^\varepsilon \text{ on } Y_1 \text{ (hence on } \omega \setminus T_\varepsilon^1),$$

so that

$$B^0 = D^0,$$

on every relatively compact open subset ω of Y_1, hence in all Y_1 (because the later is open).

Choose now in Theorem 21

$$\begin{cases} O_1 = Y^\star, \quad , O_2 = Y^\star \backslash \overline{Y_1}, \\ T_\varepsilon^1 = \mathcal{S}_\varepsilon \quad \text{and} \quad T_\varepsilon^2 = \emptyset, \\ C_1^\varepsilon = B^\varepsilon, \quad C_2^\varepsilon = B. \end{cases}$$

We have

$$\begin{cases} \{(B^\varepsilon, \mathcal{S}_\varepsilon)\}_\varepsilon \overset{H_\varepsilon^0}{\rightharpoonup} B^0 \quad \text{in } Y^\star, \\ \{(B, \emptyset)\}_\varepsilon \overset{H_\varepsilon^0}{\rightharpoonup} B \quad \text{in } Y^\star \setminus \overline{Y_1}, \end{cases}$$

and for every relatively compact open subset ω of $O_1 \cap O_2 = Y^\star \setminus \overline{Y_1}$, we have

$$\omega \cap T_\varepsilon^1 = \emptyset = \omega \cap T_\varepsilon^2 \quad \text{and} \quad C_1^\varepsilon = C_2^\varepsilon \text{ on } Y^\star \backslash \overline{Y_1} \text{ (hence on } \omega = \omega \setminus T_\varepsilon^1).$$

Hence

$$B^0 = B$$

on every relatively compact open subset ω of $Y^\star \backslash \overline{Y_1}$, so in all $Y^\star \backslash \overline{Y_1}$ (because the later is open). One then can conclude, that $B^0 = B$ a.e. on $Y^\star \backslash Y_1$, since $|\partial Y_1| = 0$. ■

As observed in the elliptic case, one can iterate this procedure finitely many times. This allows to treat the case of an arbitrary number of partially periodic scales. We also remark that no separation of scales is needed, since we do not suppose that the characteristic function of Ω_ε is product of characteristic functions. For example, in the case where one has three scales (see Proposition 37) and the holes do not touch the boundary of Ω, the characteristic function of Ω_ε can written as below

$$\chi_{\Omega_\varepsilon}(x) = \chi_{Y^\star \backslash Y^1}(\frac{x}{\varepsilon}) + \chi_{Y^1}(\frac{x}{\varepsilon})\chi_{Z^\star}(\frac{x}{\varepsilon \delta_\varepsilon}), \quad \text{a.e. in } \Omega.$$

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Chapitre 3

Meyers type estimates in elasticity and applications to H-convergence

With P. Donato[1], published in [20]

Abstract

In this paper, we prove some estimates on the difference of the limits of two H_e-converging sequences. This convergence extends the H-convergence of Murat-Tartar to the linearized elasticity system . We give both L^1 and pointwise estimates. This is achieved by proving an L^p regularity result, extending the well known Meyers estimate for elliptic equations to the elasticity system. In a second part, we show similar estimates in the framework of the H_e^0-convergence, the generalization of the H_e-convergence to the case of domains with small holes.

3.1 Introduction

The classical notion of G-convergence has been introduced by S. Spagnolo [45] for second-order elliptic symmetric operators and extended to the non symmetric case, called H-convergence, by F. Murat and L. Tartar [39], [40], [46].

A generalization, to the case of perforated domains with a Neumann condition on the boundary of the holes, denoted H^0-convergence, has been done by M. Briane - A. Damlamian - P. Donato in [8].

The H-convergence was extended to the linearized elasticity system by G. A. Francfort - F. Murat in [26]. We denote it here H_e-convergence. In the case of perforated domains the H^0-convergence was extended to the linearized elasticity system with traction conditions on the holes by P. Donato - M. El Hajji in [24] and called H_e^0-convergence.

In this paper, we prove some estimates on the difference of the limit of two sequences which are either H_e-converging or H_e^0-converging. This kind of estimates can

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be applied for computing the effective problem in several homogenization problems (see for instance [6], [19] for the elliptic case).

In the first part of this paper we consider the case of a fixed domain. We prove two main results, stated in Section 3. The first (Theorem 43) provides an estimate of the L^1-norm of the difference of two H_e-limits and states that if $\{A^\varepsilon\} \in \mathcal{M}_e(\alpha, \beta, \Omega)$ and $\{B^\varepsilon\} \in \mathcal{M}_e(\alpha', \beta', \Omega)$ are two sequences of fourth-order tensors which H_e-converge to A^0 and B^0 respectively, then

$$\|A^0 - B^0\|_{L^1(\Omega)} \leq C \liminf_{\varepsilon} \|A^\varepsilon - B^\varepsilon\|^{\frac{1}{s}}_{L^1(\Omega)}.$$

where $s \in]2, +\infty[$ and C is a positive constant independent of ε.

The second result (Theorem 44) gives a pointwise estimate of the difference of two H_e-limits. It states that if $\{A^\varepsilon\}$ and $\{B^\varepsilon\}$ H_e-converge to A^0 and B^0 respectively and

$$|A^\varepsilon(x) - B^\varepsilon(x)| \leq h^\varepsilon(x) \longrightarrow h^0 \quad \textit{strongly in } L^1(\Omega),$$

then

$$|A^0(x) - B^0(x)| \leq \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}\, h^0(x) \quad \textit{a.e. in } \Omega.$$

For analogous pointwise estimates in the elliptic case we refer to P. F. Colombini - S. Spagnolo [15] and L. Boccardo - F. Murat [6] for G-convergence and H-convergence respectively and to P. Donato in [19] for L^1-estimates for the H-convergence.

The main tool for proving both results is an L^p regularity result for the linearized elasticity system which extends the well known estimate for second-order elliptic equations due to N. G. Meyers [36].

Originally, we thought that this result was also known in the case of elasticity systems. We were unable to find reference and, inquiring from colleagues well versed in elasticity theory, it appeared that such estimates were not even obvious. After several discussions, in particular with Luc Tartar, such a result appeared plausible, so, after obtaining a complete proof, we decided to incorporate it in the present paper (Theorem 45). The proof, somewhat technical, follows the lines of that given in [5] for the elliptic case and makes use of the invariance of the index of some operators associated to elliptic systems in Sobolev spaces of type $W^{1,p}$ with respect to p (see G. Geymonat [28]).

In a second part we prove similar estimates for the case of domains with small holes. We prove that (Theorem 63) if $\omega \subset\subset \Omega$, $\{A^\varepsilon\} \in \mathcal{M}_e(\alpha, \beta, \Omega)$, $\{B^\varepsilon\} \in \mathcal{M}_e(\alpha', \beta', \Omega)$ and $\{T^1_\varepsilon\}$ and $\{T^2_\varepsilon\}$ are two sequences of holes e-admissibles in Ω such that $(A^\varepsilon, T^1_\varepsilon) \xrightarrow{H^0_\varepsilon} A^0$ and $(B^\varepsilon, T^2_\varepsilon) \xrightarrow{H^0_\varepsilon} B^0$, then

$$\|A^0 - B^0\|_{L^1(\omega)} \leq c \liminf_{\varepsilon} \left(|(T^1_\varepsilon \Delta T^2_\varepsilon) \cap \omega| + \|A^\varepsilon - B^\varepsilon\|_{L^1(\omega)}\right)^\tau,$$

where Δ denotes the symmetric difference, and $c > 0$ and $\tau > 0$ are independent of ε. This result reduces to that given by M. Briane - A. Damlamian - P. Donato in [8] for the H^0-convergence.

Let χ_i^ε the characteristic function of $\Omega \setminus T_\varepsilon^i$, $i = 1,\ 2$. We also prove (Theorem 65) that if

$$\begin{cases} \chi_2^\varepsilon \, |A^\varepsilon(x) - B^\varepsilon(x)| \leq h^\varepsilon(x) \longrightarrow h^0 \quad \textit{strongly in } L^1(\omega), \\ \chi_1^\varepsilon - \chi_2^\varepsilon \to 0 \quad \textit{strongly in } L^1(\omega), \end{cases}$$

then

$$|A^0 - B^0| \leq \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}\, h^0 \quad \textit{a. e. in } \omega.$$

In both theorems we need to suppose a uniform L^p estimate for one of the correctors associated to our problem (see Assumption (3.59) below). In the case without holes, this estimate is consequence of the Meyers estimate given by Theorem 45 in the first part.

In presence of holes, this estimate has to be assumed by hypothesis. Indeed, a generalization of this estimate in the case of perforated domains remains an open question.

For the elliptic case, an extension of the Meyers estimate to the case of a domain with one reference hole and a Neumann condition has been proved in [8]. This allow to have a uniform Meyers estimate (with respect to ε) at least for periodic functions in periodically perforated domains. Even in this case, the proof given in [8] does not seems to be adaptable to the elasticity system.

Plan :

2. Preliminaries on the H_e-convergence (including notations).
3. Two estimates on the difference of H_e-limits.
4. Proof of a Meyers type estimate.
5. Preliminary results on the H_e^0-convergence.
6. Two estimates on the difference of H_e^0-limits.

3.2 Preliminaries on the H_e-convergence

In this section we recall the definition and some properties of the H-convergence for the linearized elasticity studied by G. A. Francfort and F. Murat in [26]. We will denote it by H_e-convergence.

We use the following notations :

• If $A = (A_{ijkl})_{1\leq i,j,k,l\leq n}$, $B = (B_{ijkl})_{1\leq i,j,k,l\leq n}$ are two fourth-order tensors, $\Lambda, \Upsilon \in \mathbb{R}^{n\times n}$ and $\xi \in \mathbb{R}^n$ we set

$$\begin{cases} (AB)_{ijkl} = \sum\limits_{1\leq p,q\leq n} A_{ijpq} B_{pqkl}, \\ (A\Lambda)_{ij} = \sum\limits_{1\leq k,l\leq n} A_{ijkl}\Lambda_{kl}, \end{cases}$$

and

$$\begin{cases} \Lambda.\Upsilon = \sum\limits_{1\leq i,j\leq n} \Lambda_{ij}\Upsilon_{ij}, \\ |\xi| = \left(\sum\limits_{1\leq i\leq n} (\xi_i)^2\right)^{\frac{1}{2}}, \\ |\Lambda| = \left(\sum\limits_{1\leq i,j\leq n} (\Lambda_{ij})^2\right)^{\frac{1}{2}}, \\ |A| = sup\left\{|A\Lambda.\Upsilon| \;\; t.q. \;\; |\Lambda| = |\Upsilon| = 1 \; and \; \Lambda, \; \Upsilon \in \mathbb{R}^{n\times n}\right\}, \end{cases} \tag{3.1}$$

• if F is a set of matrices fields, $F_S = \{M \in F \; s.\,t. \; M \; is \; symmetric\}$,
• Ω is a domain of $\mathbb{R}^n$,
• $\mathbb{D}'(\Omega)$ is the set of fourth-order tensors with components in $\mathcal{D}'(\Omega)$,
• $\mathbb{L}^p(\Omega)$ is the set of fourth-order tensors with components in $L^p(\Omega)$,
• $\mathbb{W}^{1,p}(\Omega)$ is the set of fourth-order tensors with components in $W^{1,p}(\Omega)$,
• $\mathbb{W}_0^{1,p}(\Omega)$ is the set of fourth-order tensors with components in $W_0^{1,p}(\Omega)$,
• $\{\varepsilon\}$ denotes a strictly decreasing sequence converging to zero,
• if $v = (v_1,\ldots, v_n)$ is a vector valued function and $\zeta = (\zeta_{ij})_{1\leq i,j\leq n}$ is a second order tensor of variable $x = (x_1,\ldots, x_n)$, we set

$$\begin{cases} (\nabla v)_{ij} = \dfrac{\partial v_i}{\partial x_j} \equiv D_{x_j} v_i, \\ e(v) = \frac{1}{2}(\nabla v +^t \nabla v), \\ (div\zeta)_i = \dfrac{\partial \zeta_{ij}}{\partial x_j}, \end{cases}$$

• for two real numbers α and β such that $0 < \alpha < \beta$, $M_e(\alpha, \beta, \Omega)$ is the set of the tensors $A = (A_{ijkl})_{1\leq i,j,k,l\leq n}$ defined on Ω such that a.e. on Ω, we have

$$\begin{cases} \text{i) } A_{ijkl} \in L^\infty(\Omega), \quad \text{for any } i,j,k,l = 1,...,n, \\ \text{ii) } A_{ijkl} = A_{jikl} = A_{klij}, \quad \text{for any } i,j,k,l = 1,...,n, \\ \text{iii) } \alpha|\Lambda|^2 \leq A\Lambda.\Lambda, \quad \text{for any symmetric matrix } \Lambda, \\ \text{iv) } |A\Lambda| \leq \beta|\Lambda|, \quad \text{for any matrix } \Lambda, \end{cases} \tag{3.2}$$

• if A is a fourth-order tensor we set

$$\begin{cases} A_{kl} = (A_{kl})_{1\leq i,j\leq n} = (A_{ijkl})_{1\leq i,j\leq n}, \; \forall k,l \in \{1,...,n\}, \\ A^{ij} = (A^{ij})_{1\leq k,l\leq n} = (A_{ijkl})_{1\leq k,l\leq n}, \; \forall i,j \in \{1,...,n\}, \\ {}^tA = ({}^tA_{ijkl})_{1\leq i,j,k,l\leq n} \equiv (A_{klij})_{1\leq i,j,k,l\leq n}, \end{cases} \tag{3.3}$$

• E denotes the fourth-order tensor defined by

$$E_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \; \forall i,j,k,l \in \{1,2,...,n\}. \tag{3.4}$$

• for $1 \leq p < \infty$, $L^p(\Omega)$, $L^p(\Omega)^{n\times n}$, $W^{1,p}(\Omega)^n$ and $\mathbb{L}^p(\Omega)$ are the Banach spaces equipped by the norms defined by

$$\begin{cases} \|\varphi\|^p_{L^p(\Omega)} = (\int_\Omega |\varphi|^p dx)^{\frac{1}{p}}, \\ \|\Psi\|^p_{L^p(\Omega)^{n\times n}} = (\int_\Omega |\Psi|^p dx)^{\frac{1}{p}}, \\ \|\psi\|^p_{W^{1,p}(\Omega)^n} = (\sum\limits_{1\leq i\leq n} \|\psi_i\|^p_{L^p(\Omega)} + \sum\limits_{1\leq i,j\leq n} \|\frac{\partial \psi_j}{\partial x_i}\|^p_{L^p(\Omega)})^{\frac{1}{p}}, \\ \|A\|_{\mathbb{L}^p(\Omega)^{n\times n}} = \left(\int_\Omega |A|^p dx\right)^{\frac{1}{p}}, \end{cases} \tag{3.5}$$

respectively, for $\varphi \in L^p(\Omega)$, $\Psi \in L^p(\Omega)^{n\times n}$, $\psi \in W^{1,p}(\Omega)^n$ and $A \in \mathbb{L}^p(\Omega)$,
• Δ denotes the symmetric difference of sets.
Convention : We adopt the Einstein summation convention, i.e. we sum over repeated indices.

Remark 38 *A standard linear algebra result on the quadratic forms shows that if A satisfies (3.2)ii), then*

$$\left(\forall \Lambda \in \mathbb{R}^{n\times n}_s, \quad |A\Lambda.\Lambda| \leq \beta' |\Lambda|^2\right) \Leftrightarrow \left(\forall \Lambda \in \mathbb{R}^{n\times n}, \quad |A\Lambda| \leq \beta' |\Lambda|\right).$$

Definition 39 *([26]) Let $A^\varepsilon \in M_e(\alpha, \beta, \Omega)$. We say that the sequence $\{A^\varepsilon\}_\varepsilon$ H_e-converges to the tensor $A^0 \in M_e(\alpha', \beta', \Omega)$ and we write $A^\varepsilon \overset{H_e}{\rightharpoonup} A^0$ if and only if for any function f in $H^{-1}(\Omega)^n$ the solution u^ε of*

$$\begin{cases} -div\ (A^\varepsilon e(u^\varepsilon)) = f \quad in\ \Omega, \\ u^\varepsilon = 0 \quad on\ \partial\Omega, \end{cases} \tag{3.6}$$

satisfies the weak convergences

$$\begin{cases} i) \quad u^\varepsilon \rightharpoonup u \quad weakly\ in\ H^1_0(\Omega)^n, \\ ii) \quad A^\varepsilon e(u^\varepsilon) \rightharpoonup A^0 e(u) \quad weakly\ in\ L^2(\Omega)^{n\times n}, \end{cases} \tag{3.7}$$

where u is the unique solution of the problem

$$\begin{cases} -div\ (A^0 e(u)) = f \quad in\ \Omega, \\ u = 0 \quad on\ \partial\Omega. \end{cases} \tag{3.8}$$

The following compactness result given by G. A. Francfort and F. Murat in [26], state the main property of the H_e-convergence. It extends to the linearized elasticity case the analogous one established for the second-order elliptic equation (G-convergence or H-convergence) by Spagnolo [45], F. Murat and L. Tartar [39], [40], [46] (See also L. Simon [44], V. V. Zhikov, S. M. Kozlov, O.A. Oleinik and K.

T. Ngoan [48]). Let us recall that in Definition 1 the weak convergence (3.7)ii) is not a consequence of the weak convergence (3.7)i), even if A^ε satisfies the symmetry properties (3.2)ii). From this point of view, Definition 1 is a generalization of the definition of H-convergence, which concerns non-symmetric matrices and where the weak convergence of $A^\varepsilon \nabla u^\varepsilon$ has to be supposed in the definition. This is not the case for the G-convergence, where the symmetry of the matrices A^ε provides the convergence of $A^\varepsilon \nabla u^\varepsilon$.

Theorem 40 *(Compactness) ([26]) Let $A^\varepsilon \in M_e(\alpha, \beta, \Omega)$. Then, there exists a subsequences of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and a tensor $A^0 \in M_e(\alpha, \beta, \Omega)$ such that the sequence $\{A^\varepsilon\}_\varepsilon$ H_e-converges to A^0.*

One has also the following locality properties :

Theorem 41 *(Locality) Let $A^\varepsilon \in M_e(\alpha, \beta, \Omega)$ and $B^\varepsilon \in M_e(\alpha, \beta, \Omega)$ be such that*

$$\begin{cases} A^\varepsilon \xrightarrow{H_e} A^0 \ \textit{in } \Omega, \\ B^\varepsilon \xrightarrow{H_e} B^0 \ \textit{in } \Omega. \end{cases}$$

Then, for every open subset ω of Ω, one has

$$(A^\varepsilon = B^\varepsilon \ \textit{on } \omega) \Longrightarrow (A^0 = B^0 \ \textit{on } \omega).$$

In the following, we will make use of the lemma below. It is a variant of the know div-curl lemma [40].

Proposition 42 *Let $\xi^\varepsilon \in L^2(\Omega)^{n\times n}$ and $\eta^\varepsilon \in H^1(\Omega)^n$ two sequence such that*

$$\begin{cases} \xi^\varepsilon \rightharpoonup \xi^0 \ \textit{ weakly in } L^2(\Omega)^{n\times n}, \\ \textit{div } \xi^\varepsilon \textit{ is compact in } H^{-1}(\Omega)^n \end{cases}$$

and

$$\eta^\varepsilon \rightharpoonup \eta^0 \ \textit{weakly in } H^1(\Omega)^n.$$

Then,

$$\xi^\varepsilon.e(\eta^\varepsilon) \rightharpoonup \xi^0.e(\eta^0) \ \textit{in } D'(\Omega).$$

For the proofs of Theorem 41 and Proposition 42, we refer to [24], where they are given in the more general framework of perforated domains.

3.3 Two estimates on the difference of H_e-limits

The first result is an estimate of the L^1-norm of the difference of two H_e-limits. It generalizes to the linearized elasticity the analogous one given for the G-convergence by P. F. Colombini - S. Spagnolo in [15] and for the H-convergence by P. Donato in [19].

Theorem 43 *Let $\{A^\varepsilon\} \in \mathcal{M}_e(\alpha,\beta,\Omega)$ and $\{B^\varepsilon\} \in \mathcal{M}_e(\alpha',\beta',\Omega)$ such that*

$$\begin{cases} A^\varepsilon \xrightarrow{H_e} A^0, \\ B^\varepsilon \xrightarrow{H_e} B^0. \end{cases}$$

Then, there exist $s \in]2,+\infty[$ and two strictly positive constants C_1 and C_2 which depend on α, α', β, β', n and Ω such that

$$\|A^0 - B^0\|_{\mathbb{L}^1(\Omega)} \leq C_1 \liminf_{\varepsilon} \|A^\varepsilon - B^\varepsilon\|_{\mathbb{L}^s(\Omega)} \leq C_2 \liminf_{\varepsilon} \|A^\varepsilon - B^\varepsilon\|_{\mathbb{L}^1(\Omega)}^{\frac{1}{s}}. \tag{3.9}$$

The second result is a pointwise estimate of the difference of two H_e-limits, which is of the same type as that given for G-convergence by P. F. Colombini - S. Spagnolo in [15] and for H-convergence by L. Boccardo - F. Murat in [6].

Theorem 44 *Let $\{A^\varepsilon\} \in \mathcal{M}_e(\alpha,\beta,\Omega)$ and $\{B^\varepsilon\} \in \mathcal{M}_e(\alpha',\beta',\Omega)$ such that*

$$\begin{cases} A^\varepsilon \xrightarrow{H_e} A^0, \\ B^\varepsilon \xrightarrow{H_e} B^0. \end{cases}$$

Assume that there exists two functions h^ε, $h^0 \in L^1(\Omega)$ such that

$$\begin{cases} |A^\varepsilon(x) - B^\varepsilon(x)| \leq h^\varepsilon(x) \ \text{ a.e. in } \Omega, \\ \\ h^\varepsilon \longrightarrow h^0 \quad \text{strongly in } L^1(\Omega). \end{cases} \tag{3.10}$$

Then,

$$|A^0(x) - B^0(x)| \leq \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}\, h^0(x) \ \text{ a.e. in } \Omega. \tag{3.11}$$

Theorem 43 and Theorem 44 will be proved at the end of this section. The main tool for their proofs is the following Meyers type estimate. It generalizes the similar one proved in [36] for elliptic equation to the linearized elasticity system .

Theorem 45 *Let Ω be a bounded open set of $\mathbb{R}^n$ of class $\mathcal{C}^2$, $A \in M_e(\alpha,\beta,\Omega)$, $g \in L^2(\Omega)^{n\times n}$ and u be the solution of*

$$\begin{cases} -div(Ae(u)) = div\ g, \\ \\ u \in H_0^1(\Omega)^n. \end{cases} \tag{3.12}$$

There exists $\bar{p} > 2$, which depends only on α, β, Ω and n such that, for all $p \in [2,\bar{p}[$, if $g \in L^p_S(\Omega)^{n\times n}$ then u belongs to $W_0^{1,p}(\Omega)^n$ and satisfies

$$\|u\|_{W_0^{1,p}(\Omega)^n} \leq c\|g\|_{L^p(\Omega)^{n\times n}}, \tag{3.13}$$

where c depends only on α, β, Ω, n and p.

Theorem 45 is proved in Section 4. To prove Theorem 43 and Theorem 44 we need the following result :

Proposition 46 *Let $\{A^\varepsilon\} \in \mathcal{M}_e(\alpha,\beta,\Omega)$ be a sequence which H_e-converges to A^0. Then, for every $\Lambda \in \mathbb{R}_S^{n\times n}$, there exists a function $w_\Lambda^\varepsilon \in H^1(\Omega)$ such that*

$$\begin{cases} i)\ w_\Lambda^\varepsilon \rightharpoonup \Lambda x \ \ \text{weakly in } H^1(\Omega), \\ ii)\ A^\varepsilon e(w_\Lambda^\varepsilon) \rightharpoonup A^0\Lambda \ \ \text{weakly in } L^2(\Omega)^{n\times n}, \\ iii)\ div(A^\varepsilon e(w_\Lambda^\varepsilon)) \text{ is compact in } H^{-1}(\Omega)^n. \end{cases} \tag{3.14}$$

Moreover, if M is the fourth-order tensor corrector defined in Ω by

$$M^\varepsilon \Upsilon = e(w^\varepsilon_{\frac{1}{2}(\Upsilon + {}^t\Upsilon)}), \qquad \forall \Upsilon \in \mathbb{R}^{n\times n}, \tag{3.15}$$

then there exists $q>2$ and $c>0$ independent of ε such that, for all $p\in[2,q[$,

$$\|M^\varepsilon\|_{\mathbb{L}^p(\Omega)} \le c. \tag{3.16}$$

Proof. Let $\Lambda \in \mathbb{R}_S^{n\times n}$.
Step 1. Let us prove that there exists a function $w_\Lambda^\varepsilon \in H^1(\Omega)$ which satisfies (3.14).
Let Ω_1 be a bounded set of $\mathbb{R}^n$ of class $\mathcal{C}^2$ such that $\Omega \subset\subset \Omega_1$ and set

$$C^\varepsilon = \begin{cases} A^\varepsilon & \text{in } \Omega, \\ \alpha E & \text{in } \Omega_1 \setminus \Omega. \end{cases}$$

It is clear that $C^\varepsilon \in \mathcal{M}_e(\alpha,\beta,\Omega_1)$. From Theorem 40, it follows that (up to a subsequence) C^ε H_e-converges to some $C^0 \in M_e(\alpha,\beta,\Omega_1)$ in Ω_1.
Then, for every $\varphi \in D(\Omega_1)$ such that $\varphi = 1$ in Ω, let w_Λ^ε the solution of the following problem

$$\begin{cases} -div(C^\varepsilon(x)e(w_\Lambda^\varepsilon(x))) = -div(C^0(x)e(\varphi(x)\Lambda x)) \text{ in } \Omega_1, \\ w_\Lambda^\varepsilon \in H_0^1(\Omega_1). \end{cases} \tag{3.17}$$

Then,

$$\begin{cases} w_\Lambda^\varepsilon \rightharpoonup \varphi\,\Lambda x \text{ weakly in } H_0^1(\Omega_1)^n, \\ C^\varepsilon e(w_\Lambda^\varepsilon) \rightharpoonup C^0 e(\varphi\,\Lambda x) \text{ weakly in } L^2(\Omega_1)^{n\times n}. \end{cases}$$

Using the fact that $\varphi = 1$ in Ω and Theorem 41, we deduce that $C^0 = A^0$ on Ω and obtain (3.14)i) and (3.14)ii).
Finally, statement (3.14)iii) is a consequence immediate of (3.17), where the right-hand side of the system is independent of ε.
Step 2. By virtue of (3.17) and Theorem 45, there exists $q>2$ which depends only on α, β, Ω_1 and n (hence independent of ε) such that for every $p\in[2,q[$, we have

$$\|w_\Lambda^\varepsilon\|_{W_0^{1,p}(\Omega_1)^n} \le c\,\|C^0(x)e(\varphi(x)\Lambda x)\|_{L^p(\Omega_1)^{n\times n}} \le c_1|\Lambda|, \tag{3.18}$$

where $c > 0$ and $c_1 > 0$ are independent of ε.
On the other hand (see (3.1) and (3.5), we have

$$\begin{aligned}\|M^\varepsilon\|^p_{\mathbb{L}^p(\Omega)} &\doteq \int_\Omega |M^\varepsilon|^p dx \leq \sup_{|\Upsilon|=1} \int_\Omega |M^\varepsilon \Upsilon|^p dx \\ &= \sup_{|\Upsilon|=1} \int_\Omega |e(w^\varepsilon_{\frac{1}{2}(\Upsilon + {}^t\Upsilon)})|^p dx \leq c' \sup_{|\Upsilon|=1} \|w^\varepsilon_{\frac{1}{2}(\Upsilon+{}^t\Upsilon)}\|^p_{W_0^{1,p}(\Omega_1)^n} \leq c' c_1,\end{aligned}$$

where we used (3.18). ■

Remark 47 *The tensor M^ε is a corrector for A^ε in the sense that*

$$\lim_{\varepsilon \to 0} \|e(u^\varepsilon) - M^\varepsilon e(u^0)\|_{L^1(\Omega)^{n\times n}} = 0, \tag{3.19}$$

for every $f \in H^{-1}(\Omega)^n$, where u^ε and u^0 are the solutions of (6) and (8) respectively. For a proof of (3.19) see for instance [24], where the result is given for the general case of a perforated domain.

Proof of Theorem 43. The proof adapt some ideas of [19] to the tensor case and to the norm defined by (3.5). Let $\{A^\varepsilon\} \in \mathcal{M}_e(\alpha, \beta, \Omega)$ and $\{B^\varepsilon\} \in \mathcal{M}_e(\alpha', \beta', \Omega)$ such that

$$\begin{cases} A^\varepsilon \xrightarrow{H_e} A^0, \\ B^\varepsilon \xrightarrow{H_e} B^0. \end{cases}$$

Let v^ε_Λ (respect. w^ε_Λ) be the vector valued function and M^ε (respect. N^ε) the corrector tensor given by Proposition 46 for A^ε (respect. B^ε). Then, using (3.3), we can write

$$\begin{cases} ({}^tN^\varepsilon A^\varepsilon M^\varepsilon)_{ijkl} = ({}^tN^\varepsilon)^{ij}.\,(A^\varepsilon M^\varepsilon_{kl}) = N^\varepsilon_{ij}.\,(A^\varepsilon M^\varepsilon_{kl})\,, \\ \\ ({}^tN^\varepsilon B^\varepsilon M^\varepsilon)_{ijkl} = ({}^tN^\varepsilon)^{ij}.\,(B^\varepsilon M^\varepsilon_{kl}) = N^\varepsilon_{ij}.\,(B^\varepsilon M^\varepsilon_{kl})\,. \end{cases}$$

Using the symmetry proprieties of B^ε, this gives

$$\begin{cases} ({}^tN^\varepsilon A^\varepsilon M^\varepsilon)_{ijkl} = (A^\varepsilon M^\varepsilon_{kl})\,.N^\varepsilon_{ij}, \\ \\ ({}^tN^\varepsilon B^\varepsilon M^\varepsilon)_{ijkl} = \left(B^\varepsilon N^\varepsilon_{ij}\right).M^\varepsilon_{kl}. \end{cases}$$

But, from (3.3), (3.4) and (3.15), we have

$$\begin{cases} M^\varepsilon_{kl} = M^\varepsilon E_{kl} = e(v^\varepsilon_{E_{kl}}), \\ \\ N^\varepsilon_{ij} = N^\varepsilon E_{ij} = e(w^\varepsilon_{E_{ij}}), \end{cases}$$

so that

$$\begin{cases} ({}^tN^\varepsilon A^\varepsilon M^\varepsilon)_{ijkl} = \left(A^\varepsilon e(v^\varepsilon_{E_{kl}})\right).e(w^\varepsilon_{E_{ij}}), \\ \\ ({}^tN^\varepsilon B^\varepsilon M^\varepsilon)_{ijkl} = \left(B^\varepsilon e(w^\varepsilon_{E_{ij}})\right).e(v^\varepsilon_{E_{kl}}). \end{cases}$$

Using (3.14) and Proposition 42 written first for $\xi^\varepsilon = A^\varepsilon e(v^\varepsilon_{E_{kl}})$, $\eta^\varepsilon = w^\varepsilon_{E_{ij}}$ and then for $\xi^\varepsilon = B^\varepsilon e(w^\varepsilon_{E_{ij}})$, $\eta^\varepsilon = v^\varepsilon_{E_{kl}}$, one can pass to the limit in the last relations to obtain

$$\begin{cases} ({}^tN^\varepsilon A^\varepsilon M^\varepsilon)_{ijkl} \to A^0 E_{kl}\cdot E_{ij} = A^0_{ijkl} \quad \text{in} \quad \mathcal{D}'(\Omega), \\ ({}^tN^\varepsilon B^\varepsilon M^\varepsilon)_{ijkl} \to B^0 E_{ij}\cdot E_{kl} = B^0_{klij} = B^0_{ijkl} \quad \text{in} \quad \mathcal{D}'(\Omega). \end{cases}$$

This implies that

$${}^tN^\varepsilon(A^\varepsilon - B^\varepsilon)M^\varepsilon \to A^0 - B^0 \quad \text{in} \quad \mathbb{D}'(\Omega).$$

The weak lower semi-continuity of $L^1(\Omega)$-norm gives

$$\|A^0 - B^0\|_{\mathbb{L}^1(\Omega)} \leq \liminf_\varepsilon \left\|{}^tN^\varepsilon(A^\varepsilon - B^\varepsilon)M^\varepsilon\right\|_{\mathbb{L}^1(\Omega)}, \tag{3.20}$$

since by (3.5), for every $1 \leq r < \infty$,

$$\|D\|_{\mathbb{L}^r(\Omega)} = \int_\Omega |D|^r dx, \quad \forall D \in \mathbb{L}^r(\Omega).$$

On the other hand, by virtue of Proposition 46, there exists $p > 2$ and $c > 0$ independent of ε such that

$$\begin{cases} \|M^\varepsilon\|_{\mathbb{L}^p(\Omega)} \leq c, \\ \|N^\varepsilon\|_{\mathbb{L}^2(\Omega)} \leq c. \end{cases}$$

Hence

$$\begin{aligned} \|{}^tN^\varepsilon(A^\varepsilon - B^\varepsilon)M^\varepsilon\|_{\mathbb{L}^1(\Omega)} &\leq c_1\|{}^tN^\varepsilon\|_{\mathbb{L}^p(\Omega)}\,\|A^\varepsilon - B^\varepsilon\|_{\mathbb{L}^s(\Omega)}\,\|M^\varepsilon\|_{\mathbb{L}^2(\Omega)} \\ &\leq c_2\|A^\varepsilon - B^\varepsilon\|_{\mathbb{L}^s(\Omega)}, \end{aligned}$$

with $c_1 > 0$ and $c_2 > 0$ independent of ε and $\frac{1}{2} + \frac{1}{p} + \frac{1}{s} = 1$. Observe that this implies that $s > 2$.
Combining this inequality with (3.20), we get

$$\|A^0 - B^0\|_{\mathbb{L}^1(\Omega)} \leq c_1 \liminf_\varepsilon \|A^\varepsilon - B^\varepsilon\|_{\mathbb{L}^s(\Omega)}. \tag{3.21}$$

Finally, observe that

$$\liminf_\varepsilon \|A^\varepsilon - B^\varepsilon\|_{\mathbb{L}^s(\Omega)} \leq \liminf_\varepsilon \left(\|A^\varepsilon - B^\varepsilon\|^{\frac{1}{s}}_{\mathbb{L}^1(\Omega)}\|A^\varepsilon - B^\varepsilon\|^{1-\frac{1}{s}}_{\mathbb{L}^\infty(\Omega)}\right),$$

since $\|A^\varepsilon - B^\varepsilon\|_{L^\infty(\Omega)} \leq \beta + \beta'$, which together with (3.21) gives the required inequalities. ■

Proof of Theorem 44. We use a similar argument as the one used in [6] for the scalar case. For every Λ, $\Upsilon \in \mathbb{R}^{n\times n}_S$, let v^ε_Λ (respect. w^ε_Υ) the vector valued function given by Proposition 46 for A^ε (respect. B^ε) and let $\psi \in \mathcal{D}(\Omega)$ such that $\psi \geq 0$. Set

$$\begin{cases} I^\varepsilon_1 = |\int_\Omega \psi(A^\varepsilon - B^\varepsilon)\, e(v^\varepsilon_\Lambda).e(w^\varepsilon_\Upsilon)dx|, \\ \bar{h}^\varepsilon = \inf\{h^\varepsilon, \beta + \beta'\}, \\ \bar{h}^0 = \inf\{h^0, \beta + \beta'\}. \end{cases} \tag{3.22}$$

Then, by (3.14) and Proposition 42 applied first for $\xi^\varepsilon = A^\varepsilon e(v^\varepsilon_\Lambda)$, $\eta^\varepsilon = w^\varepsilon_\Upsilon$ and then for $\xi^\varepsilon = B^\varepsilon e(w^\varepsilon_\Upsilon)$, $\eta^\varepsilon = v^\varepsilon_\Lambda$, it follows

$$\begin{cases} A^\varepsilon e(v^\varepsilon_\Lambda).e(w^\varepsilon_\Upsilon) \longrightarrow A^0\Lambda.\Upsilon & in\ D'(\Omega), \\ B^\varepsilon e(w^\Upsilon_\Lambda).e(v^\varepsilon_\Lambda) \longrightarrow B^0\Lambda.\Upsilon & in\ D'(\Omega). \end{cases}$$

This gives

$$\lim_{\varepsilon\to 0} I_1^\varepsilon = |\int_\Omega \psi(A^0 - B^0)\Lambda.\Upsilon dx|. \tag{3.23}$$

On the other hand, from (3.10) and the fact that $\|A^\varepsilon - B^\varepsilon\|_{L^\infty(\Omega)} \leq \beta + \beta'$, we have

$$\begin{aligned} I_1^\varepsilon &\leq \int_\Omega \psi |A^\varepsilon - B^\varepsilon|\ |e(v^\varepsilon_\Lambda)||e(w^\varepsilon_\Upsilon)|dx \\ &\leq \int_\Omega \psi \bar{h}^\varepsilon\ |e(v^\varepsilon_\Lambda)||e(w^\varepsilon_\Upsilon)|dx \\ &= \int_\Omega \left(\psi \bar{h}^\varepsilon |e(v^\varepsilon_\Lambda)|^2\right)^{\frac{1}{2}} \left(\psi \bar{h}^\varepsilon |e(w^\varepsilon_\Upsilon)|^2\right)^{\frac{1}{2}} dx. \end{aligned}$$

Using the Hölder inequality and the fact that $\{A^\varepsilon\} \in \mathcal{M}_e(\alpha, \beta, \Omega)$ and $\{B^\varepsilon\} \in \mathcal{M}_e(\alpha', \beta', \Omega)$, this implies

$$I_1^\varepsilon \leq \frac{1}{\sqrt{\alpha\alpha'}} \left(\int_\Omega \psi \bar{h}^\varepsilon A^\varepsilon e(v^\varepsilon_\Lambda).e(v^\varepsilon_\Lambda)dx\right)^{\frac{1}{2}} \left(\int_\Omega \psi \bar{h}^\varepsilon B^\varepsilon e(w^\varepsilon_\Upsilon).e(w^\varepsilon_\Upsilon)dx\right)^{\frac{1}{2}}. \tag{3.24}$$

From (3.14) and Proposition 42 written first for $\xi^\varepsilon = A^\varepsilon e(v^\varepsilon_\Lambda)$, $\eta^\varepsilon = v^\varepsilon_\Lambda$ and then for $\xi^\varepsilon = B^\varepsilon e(w^\varepsilon_\Upsilon)$, $\eta^\varepsilon = w^\varepsilon_\Upsilon$, we obtain

$$\begin{cases} A^\varepsilon e(v^\varepsilon_\Lambda).e(v^\varepsilon_\Lambda) \longrightarrow A^0\Lambda.\Lambda & in\ D'(\Omega), \\ B^\varepsilon e(w^\varepsilon_\Upsilon).e(w^\varepsilon_\Upsilon) \longrightarrow B^0\Upsilon.\Upsilon & in\ D'(\Omega). \end{cases}$$

On the other hand, Theorem 45 implies that there exists $p > 2$ such that $A^\varepsilon e(v^\varepsilon_\Lambda).e(v^\varepsilon_\Lambda)$ (respect. $B^\varepsilon e(w^\varepsilon_\Upsilon).e(w^\varepsilon_\Upsilon)$) is bounded in $L^{\frac{p}{2}}(\Omega)$. Therefore, it converges weakly (up to subsequence) in this space. Hence

$$\begin{cases} A^\varepsilon e(v^\varepsilon_\Lambda).e(v^\varepsilon_\Lambda) \rightharpoonup A^0\Lambda.\Lambda & \text{weakly in } L^{\frac{p}{2}}(\Omega), \\ B^\varepsilon e(w^\varepsilon_\Upsilon).e(w^\varepsilon_\Upsilon) \rightharpoonup B^0\Upsilon.\Upsilon & \text{weakly in } L^{\frac{p}{2}}(\Omega). \end{cases} \tag{3.25}$$

Since, in view of (3.10) and (3.22), one has

$$\begin{cases} h^\varepsilon \longrightarrow h^0 \quad strongly\ \ in\ L^1(\Omega), \\ \bar{h}^\varepsilon \leq \beta + \beta', \end{cases}$$

the dominated convergence theorem implies

$$\bar{h}^\varepsilon \longrightarrow \bar{h}^0 \quad strongly\ in\ L^q(\Omega),\ \forall q \in [1, +\infty[.$$

Hence, from this convergence and (3.25), we deduce that

$$\begin{cases} \lim\limits_{\varepsilon\to 0}\left(\int_\Omega \psi \bar{h^\varepsilon} A^\varepsilon e(v^\varepsilon_\Lambda).e(v^\varepsilon_\Lambda)dx\right)^{\frac{1}{2}} = \left(\int_\Omega \psi \bar{h^0} A^0\Lambda.\Lambda dx\right)^{\frac{1}{2}} \leq \sqrt{\beta}|\Lambda|\left(\int_\Omega \psi \bar{h^0} dx\right)^{\frac{1}{2}}, \\ \lim\limits_{\varepsilon\to 0}\left(\int_\Omega \psi \bar{h^\varepsilon} B^\varepsilon e(w^\varepsilon_\Upsilon).e(v^\varepsilon_\Upsilon)dx\right)^{\frac{1}{2}} = \left(\int_\Omega \psi \bar{h^0} B^0\Upsilon.\Upsilon dx\right)^{\frac{1}{2}} \leq \sqrt{\beta'}|\Upsilon|\left(\int_\Omega \psi \bar{h^0} dx\right)^{\frac{1}{2}}. \end{cases}$$

Then, passing to the limit in (3.24), we obtain

$$|\int_\Omega \psi(A^0 - B^0)\Lambda.\Upsilon dx| \leq \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}|\Lambda||\Upsilon|\int_\Omega \psi \bar{h^0} dx.$$

Since the choice of $\psi \geq 0$ is arbitrary in $\mathcal{D}(\Omega)$, we deduce

$$|(A^0 - B^0)\Lambda.\Upsilon| \leq \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}|\Lambda||\Upsilon|\bar{h^0},$$

for every $\Lambda,\ \Upsilon \in \mathbb{R}^{n\times n}_S$. By virtue of the symmetry properties of A^0 and B^0, this inequality is still valid for every $\Lambda,\ \Upsilon \in \mathbb{R}^{n\times n}$, which gives (3.11). ■

3.4 Proof of a Meyers type estimate

In this section, we prove Theorem 45. We will use on $W^{1,p}_0(\Omega)^n$, for $1 < p < \infty$, the following norm

$$|||u|||_{W^{1,p}_0(\Omega)^n} \equiv \|e(u)\|_{L^p(\Omega)^{n\times n}}, \tag{3.26}$$

which from the Korn inequality is equivalent to the usual one. Denote the corresponding norms on $W^{-1,p}(\Omega)^n$ and $\mathcal{L}\left(W^{1,p}_0(\Omega)^n, W^{1,p}_0(\Omega)^n\right)$ by

$$\begin{cases} |||f|||_{W^{-1,p}(\Omega)^n} = \sup\limits_{\substack{v\in W^{1,p'}_0(\Omega)^n \\ v\neq 0}} \dfrac{|\langle f,v\rangle_{W^{-1,p}(\Omega)^n, W^{1,p'}_0(\Omega)^n}|}{|||v|||_{W^{1,p'}_0(\Omega)^n}}, \quad 1 = \frac{1}{p} + \frac{1}{p'}, \\ |||F|||_{\mathcal{L}\left(W^{1,p}_0(\Omega)^n, W^{1,p}_0(\Omega)^n\right)} = \sup\limits_{\substack{v\in W^{1,p}_0(\Omega)^n \\ v\neq 0}} \dfrac{|||F(v)|||_{W^{1,p}_0(\Omega)^n}}{|||v|||_{W^{1,p}_0(\Omega)^n}}, \end{cases} \tag{3.27}$$

for every $f \in W^{-1,p}(\Omega)^n$ and $F \in \mathcal{L}\left(W^{1,p}_0(\Omega)^n, W^{1,p}_0(\Omega)^n\right)$.

The following result is a consequence of known results :

Proposition 48 *The Lamé Operator $\mathbb{E}$ defined by*

$$u \longmapsto \mathbb{E}u \doteq -div\ e(u),$$

is an isomorphism from $W^{1,p}_0(\Omega)^n$ to $W^{-1,p}(\Omega)^n$ for every $p \in [2, \infty[$.

Proof. The result is well know for $p = 2$. This fact, together with Theorem 3.5 of G. Geymonat [28], shows that $\mathbb{E}$ is an isomorphism for $p \geq 2$ from $W^{2,p}(\Omega)^n \cap W_0^{1,p}(\Omega)^n$ onto $L^p(\Omega)^n$. To prove that $\mathbb{E}$ maps $W_0^{1,p}(\Omega)^n$ onto $W^{-1,p}(\Omega)^n$, one can follows exactly the same argument used by J. Lions et E. Magenes in the proof of Theorem 8.1 of [33] for the scalar case, which is still valid in our case. It remains to prove the injectivity of $\mathbb{E}$. Since $p \geq 2$, $W_0^{1,p}(\Omega)^n \subset H_0^1(\Omega)^n$. Then, if $\mathbb{E}u = 0$, the uniqueness in $H_0^1(\Omega)^n$ gives $u = 0$. This concludes the proof. ■

Corollary 49 *For every $p \geq 2$, the mapping $g \mapsto \frac{1}{2}div\ (g +^t g)$ is onto from $L^p(\Omega)^{n\times n}$ to $W^{-1,p}(\Omega)^n$.*

Proof. For every $f \in W^{-1,p}(\Omega)^n$, consider the problem

$$\begin{cases} -div\ e(w) = f, \\ w \in W_0^{1,p}(\Omega)^n. \end{cases}$$

In view of Proposition 48, this problem has a unique solution. The result follows immediately by taking $g = -\nabla w$. ■

Lemma 50 *Let $p \geq 2$ and set, for every $f \in W^{-1,p}(\Omega)^n$,*

$$\|f\|^\star_{W^{-1,p}(\Omega)^n} = \inf\{\|g\|_{L^p(\Omega)^{n\times n}}\ s.t.\ \frac{1}{2}div(g +^t g) = f \ and\ g \in L^p(\Omega)^{n\times n}\}. \quad (3.28)$$

Then,

$$|||f|||_{W^{-1,p}(\Omega)^n} \leq \|f\|^\star_{W^{-1,p}(\Omega)^n}. \quad (3.29)$$

Moreover $\|\cdot\|^\star_{W^{-1,p}(\Omega)^n}$ defines a norm on $W^{-1,p}(\Omega)^n$ and

$$\|div\ g\|^\star_{W^{-1,p}(\Omega)^n} \leq \|g\|_{L^p(\Omega)^{n\times n}}, \quad \forall g \in L^p_S(\Omega)^{n\times n}. \quad (3.30)$$

Proof. We proceed in three steps.
Step 1. Let us show (3.29). Let $f \in W^{-1,p}(\Omega)^n$. According to Corollary 49, there exists $g \in L^p(\Omega)^{n\times n}$ such that $f = \frac{1}{2}div(g +^t g)$. Hence

$$\begin{cases} |||f|||_{W^{-1,p}(\Omega)^n} \doteq \sup\limits_{\substack{w\in W_0^{1,p'}(\Omega)^n \\ w\neq 0}} \dfrac{|\langle f,w\rangle_{W^{-1,p}(\Omega)^n,W_0^{1,p'}(\Omega)^n}|}{|||w|||_{W_0^{1,p'}(\Omega)^n}} \\ \qquad = \sup\limits_{\substack{w\in W_0^{1,p'}(\Omega)^n \\ w\neq 0}} \dfrac{|\langle \frac{1}{2}div(g+^t g),w\rangle_{W^{-1,p}(\Omega)^n,W_0^{1,p'}(\Omega)^n}|}{|||w|||_{W_0^{1,p'}(\Omega)^n}} \\ \qquad = \sup\limits_{\substack{w\in W_0^{1,p'}(\Omega)^n \\ w\neq 0}} \dfrac{|\int_\Omega \frac{1}{2}(g+^t g)\cdot\nabla w\ dx|}{|||w|||_{W_0^{1,p'}(\Omega)^n}}, \end{cases} \quad (3.31)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. But

$$(g +^t g) \cdot \nabla w = (g \cdot (\nabla w) + g \cdot ({}^t\nabla w)) = 2\ g \cdot e(w).$$

Then, from (3.31)

$$\begin{aligned} |||f|||_{W^{-1,p}(\Omega)^n} &= \sup_{\substack{w \in W_0^{1,p'}(\Omega)^n \\ w \neq 0}} \frac{|\int_\Omega g \cdot e(w)\, dx|}{|||w|||_{W_0^{1,p'}(\Omega)^n}} \\ &\leq \sup_{\substack{w \in W_0^{1,p'}(\Omega)^n \\ w \neq 0}} \frac{\|g\|_{L^p(\Omega)^{n\times n}} \|e(w)\|_{L^{p'}(\Omega)^{n\times n}}}{|||w|||_{W_0^{1,p'}(\Omega)^n}} \\ &= \|g\|_{L^p(\Omega)^{n\times n}}. \end{aligned}$$

Since g is an arbitrary function in $L^p(\Omega)^{n\times n}$ such that $f = \frac{1}{2} div(g +^t g)$, this implies (3.29).

Step 2. Let us prove that $\|\cdot\|^\star_{W^{-1,p}(\Omega)^n}$ is a norm on $W^{-1,p}(\Omega)^n$.

First : Let $f \in W^{-1,p}(\Omega)^n$. Then, if $\|f\|^\star_{W^{-1,p}(\Omega)^n} = 0$, in view of (3.29), this implies $f = 0$.

Second : Let $f,\ \widehat{f} \in W^{-1,p}(\Omega)^n$. Then, there exists $g,\ \widehat{g} \in L^2(\Omega)^{n\times n}$ such that

$$\begin{cases} f = \frac{1}{2} div(g +^t g), \\ \widehat{f} = \frac{1}{2} div(\widehat{g} +^t \widehat{g}), \end{cases} \tag{3.32}$$

which implies that $f + \widehat{f} = \frac{1}{2} div((g + \widehat{g}) +^t (g + \widehat{g}))$. Hence

$$\begin{aligned} \|f + \widehat{f}\|^\star_{W^{-1,p}(\Omega)^n} &\doteq \inf\{\|\psi\|_{L^p(\Omega)^{n\times n}}\ s.\ t.\ \frac{1}{2} div(\psi +^t \psi) = f + \widehat{f}\ ,\ \psi \in L^p(\Omega)^{n\times n}\} \\ &\leq \|g + \widehat{g}\|_{L^p(\Omega)^{n\times n}} \leq \|g\|_{L^p(\Omega)^{n\times n}} + \|\widehat{g}\|_{L^p(\Omega)^{n\times n}}. \end{aligned}$$

This is valid for all $g,\ \widehat{g} \in L^2(\Omega)^{n\times n}$ satisfying (3.32), hence

$$\|f + \overline{f}\|^\star_{W^{-1,p}(\Omega)^n} \leq \|f\|^\star_{W^{-1,p}(\Omega)^n} + \|\widehat{f}\|^\star_{W^{-1,p}(\Omega)^n}.$$

Third : Let $f \in W^{-1,p}(\Omega)^n$ and $\lambda \in \mathbb{R}$. Observing that

$$\{g \in L^p(\Omega)^{n\times n}\ s.t.\ \frac{1}{2} div(g +^t g) = \lambda f\} = \{\lambda g\ s.t.\ g \in L^p(\Omega)^{n\times n},\ \frac{1}{2} div(g +^t g) = f\},$$

we obtain

$$\|\lambda f\|^\star_{W^{-1,p}(\Omega)^n} = |\lambda| \|f\|^\star_{W^{-1,p}(\Omega)^n}.$$

Step 3. Let us prove (3.30). If $g \in L^p_S(\Omega)^{n\times n}$, then

$$div\ g = \frac{1}{2} div(g +^t g).$$

Hence $\|div\ g\|^\star_{W^{-1,p}(\Omega)^n} \leq \|g\|_{L^p(\Omega)^{n\times n}}$. ■

In the rest of this section we also adopt the following notations :

- for $F \in \mathcal{L}\left(W_0^{1,p}(\Omega)^n, W^{-1,p}(\Omega)^n\right)$ and $G \in \mathcal{L}\left(W^{-1,p}(\Omega)^n, W_0^{1,p}(\Omega)^n\right)$ we set

$$\begin{cases} \|F\|^\star_{\mathcal{L}\left(W_0^{1,p}(\Omega)^n, W^{-1,p}(\Omega)^n\right)} = \sup\limits_{\substack{v \in W_0^{1,p}(\Omega)^n \\ v \neq 0}} \dfrac{\|F(v)\|^\star_{W^{-1,p}(\Omega)^n}}{|||v|||_{W_0^{1,p}(\Omega)^n}}, \\ \\ \|G\|^\star_{\mathcal{L}\left(W^{-1,p}(\Omega)^n, W_0^{1,p}(\Omega)^n\right)} = \sup\limits_{\substack{\phi \in W^{-1,p}(\Omega)^n \\ \phi \neq 0}} \dfrac{|||G(\phi)|||_{W_0^{1,p}(\Omega)^n}}{\|\phi\|^\star_{W^{-1,p}(\Omega)^n}}. \end{cases} \tag{3.33}$$

To prove Theorem 45 we will use the following two lemmas :

Lemma 51 *With the notations of (3.26)-(3.27)-(3.33), set for every* $p \geq 2$, $H = (\mathbb{E})^{-1}$, $\mathbb{E} : u \in W_0^{1,p}(\Omega)^n \longmapsto -div\ e(u) \in W^{-1,p}(\Omega)^n$ *and*

$$h(p) = \|H\|^\star_{\mathcal{L}\left(W^{-1,p}(\Omega)^n, W_0^{1,p}(\Omega)^n\right)}. \tag{3.34}$$

Then, for every $p_0 > 2$, *there exists a function* $\widehat{h}$ *continuous on* $[2, p_0]$ *such that*

$$\begin{cases} h(p) \leq \widehat{h}(p), \\ \\ \widehat{h}(2) = 1. \end{cases} \tag{3.35}$$

Proof. Let $p_0 > 2$ be fixed. For $p \in [2, p_0]$, let $\mathcal{M}$ be the mapping

$$\mathcal{M} :\ \psi \in L^p(\Omega)^{n\times n} \longrightarrow e(w) \in L^p(\Omega)^{n\times n},$$

where w is the solution of

$$\begin{cases} -div\ e(w) = \frac{1}{2} div(\psi +^t \psi), \\ \\ w \in W_0^{1,p}(\Omega)^n \end{cases} \tag{3.36}$$

and set

$$N(p) = \|\mathcal{M}\|_{\mathcal{L}(L^p(\Omega)^{n\times n}, L^p(\Omega)^{n\times n})}.$$

Let $\phi \in W^{-1,p}(\Omega)^n$. Then, for every $\psi \in L^p(\Omega)^{n\times n}$ such that

$$\phi = \frac{1}{2} div(\psi +^t \psi) \tag{3.37}$$

(which exists from Corollary 49), one has

$$H\phi = \frac{1}{2} H(div(\psi +^t \psi)) = w,$$

where w is the solution of (3.36). This implies

$$\begin{aligned} |||H\phi|||_{W_0^{1,p}(\Omega)^n} &= |||w|||_{W_0^{1,p}(\Omega)^n} = \|e(w)\|_{L^p(\Omega)^{n\times n}} \\ &= \|\mathcal{M}(\psi)\|_{L^p(\Omega)^{n\times n}} \leq N(p)\ \|\psi\|_{L^p(\Omega)^{n\times n}}. \end{aligned}$$

Hence, since ψ is an arbitrary function of $L^p(\Omega)^{n\times n}$ satisfying (3.37), from (3.28) one has

$$|||H\phi|||_{W_0^{1,p}(\Omega)^n} \leq N(p) \inf_{\substack{\psi\in L^p(\Omega)^{n\times n} \\ \frac{1}{2}div(\psi+^t\psi)=\phi}} \|\psi\|_{L^p(\Omega)^{n\times n}} = N(p)\|\phi\|^\star_{W^{-1,p}(\Omega)^n},$$

for every ϕ in $W^{-1,p}(\Omega)^n$. Using (3.33) and (3.34), this gives

$$h(p) \leq N(p).$$

By the Riesz-Thorin interpolation theorem, we have

$$N(p) \leq N(p_0)^{1-\theta(p)}\ N(2)^{\theta(p)},$$

with

$$\frac{1}{p} = \frac{1-\theta(p)}{p_0} + \frac{\theta(p)}{2}. \tag{3.38}$$

Hence,

$$h(p) \leq N(p_0)^{1-\theta(p)}\ N(2)^{\theta(p)}. \tag{3.39}$$

On the other hand, let ψ be in $L^2(\Omega)^{n\times n}$ and take w as a test function in the variational formulation of (3.36) written for $p=2$. We obtain

$$\begin{aligned}
\|e(w)\|^2_{L^2(\Omega)^{n\times n}} &= |\langle\frac{1}{2}(div(\psi+^t\psi),w\rangle_{H^{-1}(\Omega)^n,H_0^1(\Omega)^n}| \\
&= |\int_\Omega \frac{1}{2}(\psi+^t\psi)\cdot\nabla w\ dx| \\
&= |\int_\Omega \psi\cdot e(w)dx| \leq \|\psi\|_{L^2(\Omega)^{n\times n}}\|e(w)\|_{L^2(\Omega)^{n\times n}},
\end{aligned}$$

which implies that

$$\|\mathcal{M}(\psi)\|_{L^2(\Omega)^{n\times n}} = \|e(w)\|_{L^2(\Omega)^{n\times n}} \leq \|\psi\|_{L^2(\Omega)^{n\times n}},$$

for every ψ in $L^2(\Omega)^{n\times n}$. Consequently,

$$N(2) \leq 1.$$

This inequality, together with (3.39), gives

$$h(p) \leq N(p_0)^{1-\theta(p)}. \tag{3.40}$$

But, by virtue of (3.38), $\theta(p) = \frac{2(p_0-p)}{p(p_0-2)}$, hence the function $\widehat{h}$ defined by

$$\widehat{h}(p) \doteq N(p_0)^{1-\theta(p)}$$

is continuous on $[2,p_0]$ and $\widehat{h}(2)=1$, which, together with (3.40), gives (3.35). ■

Lemma 52 *With the notations of (3.4)-(3.26)-(3.27)-(3.33), let $A \in M_e(\alpha, \beta, \Omega)$ and set, for every $p \in [2, +\infty[$,*

$$\begin{cases} \mathbb{E}(\cdot) = -div(e(\cdot)), \quad \mathbb{A}(\cdot) = -div(Ae(\cdot)), \\ k(p) = \|\mathbb{E} - \frac{1}{\beta}\mathbb{A}\|^{\star}_{\mathcal{L}\left(W_0^{1,p}(\Omega)^n, W^{-1,p}(\Omega)^n\right)}. \end{cases} \tag{3.41}$$

Then,

$$k(p) \leq 1 - \frac{\alpha}{\beta}.$$

Proof. Let $v \in W_0^{1,p}(\Omega)^n$. Then, since $(\mathbb{E} - \frac{1}{\beta}\mathbb{A})v = -div((E - \frac{1}{\beta}A)e(v))$ and $(E - \frac{1}{\beta}A)e(v)$ is symmetric, from Lemma 50 we have

$$(\|(\mathbb{E} - \frac{1}{\beta}\mathbb{A})v\|^{\star}_{W^{-1,p}(\Omega)^n})^p \leq \|(E - \frac{1}{\beta}A)e(v)\|^p_{L^p(\Omega)^{n\times n}},$$

i.e.

$$(\|(\mathbb{E} - \frac{1}{\beta}\mathbb{A})v\|^{\star}_{W^{-1,p}(\Omega)^n})^p \leq \int_\Omega |(E - \frac{1}{\beta}A)e(v)|^p dx. \tag{3.42}$$

Since

$$\forall \Lambda \in \mathbb{R}^{n\times n}_S, \;\; |(E - \frac{1}{\beta}A)\Lambda.\Lambda| \leq (1 - \frac{\alpha}{\beta})|\Lambda|^2,$$

by Remark 38, we have

$$\forall \Lambda \in \mathbb{R}^{n\times n}, \;\; |(E - \frac{1}{\beta}A(x))\Lambda| \leq (1 - \frac{\alpha}{\beta})|\Lambda|.$$

Hence, choosing $\Lambda = e(v)(x)$, we obtain

$$|(E - \frac{1}{\beta}A(x))e(v)(x)| \leq (1 - \frac{\alpha}{\beta})|e(v)(x)|.$$

From this inequality and (3.42), one deduces

$$(\|(\mathbb{E} - \frac{1}{\beta}\mathbb{A})v\|)^{\star}_{W^{-1,p}(\Omega)^n})^p \leq (1 - \frac{\alpha}{\beta})^p \|e(v)\|^p_{L^p(\Omega)^{n\times n}} = (1 - \frac{\alpha}{\beta})^p \||v\||^p_{W_0^{1,p}(\Omega)^n},$$

for every $v \in W_0^{1,p}(\Omega)^n$. Using the fact that

$$k(p) \doteq \sup_{\substack{v \in W_0^{1,p}(\Omega)^n \\ v \neq 0}} \frac{\|(\mathbb{E} - \frac{\alpha}{\beta}\mathbb{A})v\|^{\star}_{W^{-1,p}(\Omega)^n}}{\||v\||_{W_0^{1,p}(\Omega)^n}},$$

we obtain the claimed result. ■

Proof of Theorem 45. Let $p_0 > 2$. We still use the notations of (3.26), (3.27), (3.33), (3.34) and (3.41) and proceed in two steps.
Step 1. By virtue of Lemma 52, we have

$$\forall p \geq 2, \; k(p) \leq 1 - \frac{\alpha}{\beta}. \tag{3.43}$$

Moreover in view of the continuity of $\widehat{h}$ on $[2,p_0]$ and the fact that $\widehat{h}(2)=1$, Lemma 51 shows that

$$\exists\ \delta\in]0,p_0-2]\ s.t.\ \forall p\in[2,2+\delta[,\ h(p)\leq\widehat{h}(p)<1+\frac{\alpha}{\beta}. \tag{3.44}$$

Using (3.43) and (3.44), and setting

$$\bar{p}=2+\delta, \tag{3.45}$$

this gives

$$\exists\bar{p}\in]2,p_0]\ s.t.\ \forall p\in[2,\bar{p}[,\ h(p)\ k(p)<1-(\frac{\alpha}{\beta})^2<1. \tag{3.46}$$

On the other hand, from (3.34) and (3.41), using the norm $\|\cdot\|^{\star}_{W^{-1,p}(\Omega)}$, one has

$$|||H(-\mathbb{E}+\frac{1}{\beta}\mathbb{A})|||_{\mathcal{L}\left(W_0^{1,p}(\Omega)^n,W_0^{1,p}(\Omega)^n\right)}\leq h(p)\ k(p).$$

Then, from (3.46)

$$\exists\bar{p}\in]2,p_0]\ s.t.\ \forall p\in[2,\bar{p}[,\ |||H(-\mathbb{E}+\frac{1}{\beta}\mathbb{A})|||_{\mathcal{L}\left(W_0^{1,p}(\Omega)^n,W_0^{1,p}(\Omega)^n\right)}<1-(\frac{\alpha}{\beta})^2<1. \tag{3.47}$$

This implies that, for any p such that $2\leq p<\bar{p}$, the map $I+H(-\mathbb{E}+\frac{1}{\beta}\mathbb{A})$, where $I=id$, is invertible in $W_0^{1,p}(\Omega)^n$.
Step 2. Equation (3.12) is equivalent to

$$\mathbb{E}u+(-\mathbb{E}+\frac{1}{\beta}\mathbb{A})u=\frac{1}{\beta}div\ g.$$

Hence,

$$(I+H(-\mathbb{E}+\frac{1}{\beta}\mathbb{A}))u=\frac{1}{\beta}H(div\ g). \tag{3.48}$$

Let $\bar{p}$ be defined by (3.45). By assumption, $g\in L^p_S(\Omega)^{n\times n}$ for some $p\in[2,\bar{p}[$. From Step 1, $I+H(-\mathbb{E}+\frac{1}{\beta}\mathbb{A})$ is invertible in $W_0^{1,p}(\Omega)^n$, so that (3.48) implies that $u\in W_0^{1,p}(\Omega)^n$. Moreover from (3.33), (3.34), (3.47) and (3.48)

$$\begin{aligned}|||u|||_{W_0^{1,p}(\Omega)^n} &\leq |||H(-\mathbb{E}+\frac{1}{\beta}\mathbb{A})|||_{\mathcal{L}\left(W_0^{1,p}(\Omega)^n,W_0^{1,p}(\Omega)^n\right)}|||u|||_{W_0^{1,p}(\Omega)^n}\\ &+ \frac{1}{\beta}|||H(div\ g)|||_{W_0^{1,p}(\Omega)^n}\\ &\leq (1-(\frac{\alpha}{\beta})^2)|||u|||_{W_0^{1,p}(\Omega)^n}\\ &+ \frac{1}{\beta}\|H\|^{\star}_{\mathcal{L}\left(W^{-1,p}(\Omega)^n,W_0^{1,p}(\Omega)^n\right)}\|div\ g\|^{\star}_{W^{-1,p}(\Omega)^n}.\end{aligned}$$

This gives, from (3.44),

$$|||u|||_{W_0^{1,p}(\Omega)^n}\leq(1-(\frac{\alpha}{\beta})^2)|||u|||_{W_0^{1,p}(\Omega)^n}+\frac{1}{\beta}(1+\frac{\alpha}{\beta})\|div\ g\|^{\star}_{W^{-1,p}(\Omega)^n}.$$

Hence,

$$(\frac{\alpha}{\beta})^2 |||u|||_{W_0^{1,p}(\Omega)^n} \leq \frac{1}{\beta}(1+\frac{\alpha}{\beta})\|div\ g\|^{\star}_{W^{-1,p}(\Omega)^n},$$

so that, from Lemma 50 and the symmetry of g,

$$|||u|||_{W_0^{1,p}(\Omega)^n} \leq \frac{\alpha+\beta}{\alpha^2}\|g\|_{L^p(\Omega)^{n\times n}}.$$

This gives (3.13), since the two norms $|||\cdot|||_{W_0^{1,p}(\Omega)^n}$ and $\|\cdot\|_{W_0^{1,p}(\Omega)^n}$ on $W_0^{1,p}(\Omega)^n$ are equivalent with constants depending only on p, Ω and n. ■

3.5 Preliminary results on the H_e^0-convergence

We introduce the perforated domain

$$\Omega_\varepsilon = \Omega \setminus T_\varepsilon,$$

where T_ε is a sequence of compacts subsets of Ω and set

$$V_\varepsilon = \left\{v \in H^1(\Omega_\varepsilon)^n \ \text{ s. t. } \ v = 0 \ \text{ on } \partial\Omega\right\}.$$

We denote by $\widetilde{\cdot}$ the extension by 0 from Ω_ε to Ω and set $\chi^\varepsilon = \chi_{\Omega_\varepsilon}$. In the following ν denotes the outward normal unit vector on the boundary of Ω_ε.
To generalize to perforated domains the results given in Section 3 for fixed domains, we recall first the main results concerning the H_e^0-convergence, introduced by P. Donato and M. El Hajji [24].

Definition 53 *([24]) The set T_ε is said to be admissible (in Ω) for the linearized elasticity (or e-admissible), if and only if*
i) every L^∞ weak $\star$-limit point of $\{\chi^\varepsilon\}_\varepsilon$ is positive a.e. in Ω,
ii) there exists a positive real C, independent of ε, and a sequence $\{P_\varepsilon\}_\varepsilon$ of linear extension operators such that for each ε

$$\begin{cases} P_\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega)^n), \\ (P_\varepsilon v)_{|\Omega_\varepsilon} = v, \quad \forall v \in V_\varepsilon, \\ \|e\,(P_\varepsilon v)\|_{L^2(\Omega)^{n\times n}} \leq C\,\|e(v)\|_{L^2(\Omega_\varepsilon)^{n\times n}}, \quad \forall v \in V_\varepsilon. \end{cases} \tag{3.49}$$

We denote by $P_\varepsilon^\star$ the adjoint operator of P_ε, which is defined from $H^{-1}(\Omega)^n$ to V_ε' with $P_\varepsilon^\star$ given for every $f \in H^{-1}(\Omega)^n$ by

$$\forall v \in V_\varepsilon,\ \langle P_\varepsilon^\star f, v\rangle_{V_\varepsilon', V_\varepsilon} = \langle f, P_\varepsilon v\rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n}.$$

Definition 54 *([24]) Let $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$ and T_ε be e-admissible in Ω. We say that the pair $(A^\varepsilon, T_\varepsilon)$ H_e^0-converges to the tensor $A^0 \in M_e(\alpha',\beta',\Omega)$ and we write $(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0$ if and only if for each function f in $H^{-1}(\Omega)^n$, the solution u^ε of*

$$\begin{cases} -div\,(A^\varepsilon e\,(u^\varepsilon)) = P_\varepsilon^\star f \quad in\ \Omega_\varepsilon, \\ (A^\varepsilon e\,(u^\varepsilon))\nu = 0 \quad on\ \partial T_\varepsilon, \\ u^\varepsilon = 0 \quad on\ \partial\Omega, \end{cases} \tag{3.50}$$

satisfies the weak convergence

$$\begin{cases} i) \quad P_\varepsilon(u^\varepsilon) \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega)^n, \\ ii) \quad A^\varepsilon \widetilde{e(u^\varepsilon)} \rightharpoonup A^0 e(u^0) \quad \text{weakly in } L^2(\Omega)^{n\times n}, \end{cases} \tag{3.51}$$

where u^0 is the unique solution of the problem

$$\begin{cases} -div\ (A^0 e(u^0)) = f \quad \text{in } \Omega, \\ u^0 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.52}$$

Remark 55 *1) In the case where $T_\varepsilon = \emptyset$, this definition reduces to the definition of H_e-convergence [26].*
2) The definition of H_e^0-convergence is independent of the choice of the extension operators P^ε (see [24]).

The main properties of the H_e^0-convergence are given by the results below.

Theorem 56 *(compactness) ([24]) Let $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$ and T_ε be e-admissible in Ω. Then, there exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$ and a tensor $A^0 \in M_e\left(\frac{\alpha}{C^2},\beta,\Omega\right)$ such that the sequence $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ H_e^0-converge to A^0.*

Remark 57 *The fact that A^0 belongs to $M_e\left(\frac{\alpha}{C^2},\beta,\Omega\right)$ is not explicitly state in [24], but can be easily deduced with the same arguments as those given for the case of a fixed domain.*

Theorem 58 *(Locality) ([24]) Let O_1 and O_2 be two bounded open sets in $\mathbb{R}^n$, and let T_ε^1 and T_ε^2 be e-admissible in O_1 and O_2 respectively. Let C_1^ε belong to $M_e(\alpha,\beta,O_1)$ and C_2^ε to $M_e(\alpha,\beta,O_2)$ and satisfy*

$$\begin{cases} \{(C_1^\varepsilon, T_\varepsilon^1)\}_\varepsilon \overset{H_e^0}{\rightharpoonup} C_1^0 \quad \text{in } O_1, \\ \{(C_2^\varepsilon, T_\varepsilon^2)\}_\varepsilon \overset{H_e^0}{\rightharpoonup} C_2^0 \quad \text{in } O_2. \end{cases}$$

Then, for any ω relatively compact open subset of $O_1 \cap O_2$, we have

$$\left(\omega \cap T_\varepsilon^1 = \omega \cap T_\varepsilon^2 \quad \text{and} \quad C_1^\varepsilon = C_2^\varepsilon \ \text{ on } \ \omega \setminus T_\varepsilon^1\right) \Longrightarrow \left(C_1^0 = C_2^0 \quad \text{on } \omega\right).$$

Finally, we give the following result of type div-rot which will be used in the following :

Theorem 59 ***([24])*** *Let T_ε be e-admissible in Ω and $\{\xi^\varepsilon\} \in L^2(\Omega_\varepsilon)^{n\times n}$ a sequence such that $\widetilde{\xi^\varepsilon}$ is bounded in $L^2(\Omega)^{n\times n}$ and*

$$\begin{cases} -div\ \xi^\varepsilon = P_\varepsilon^\star f^\varepsilon \quad \text{in } \Omega_\varepsilon, \\ \xi^\varepsilon \nu = 0 \quad \text{on } \partial T_\varepsilon, \end{cases}$$

where $\{f^\varepsilon\}$ *is in a compact of* $H^{-1}(\Omega)^n$*. Then,*

i) div $\widetilde{\xi^\varepsilon}$ *is in a compact subset of* $H^{-1}(\Omega)^n$*.*

ii) If $\widetilde{\xi^\varepsilon}$ *converges weakly to some* ξ^0 *in* $L^2(\Omega)^{n\times n}$*, then* f^ε *converges strongly to* $f^0 = -div\ \xi^0$ *in* $H^{-1}(\Omega)^n$*. Therefore if* $\eta^\varepsilon \in H^1(\Omega)^n$ *is a vector field which converges weakly to some* η^0 *in* $H^1(\Omega)^n$*, then*

$$\widetilde{\xi^\varepsilon}.e(\eta^\varepsilon) \to \xi^0.e(\eta^0) \text{ in } D'(\Omega).$$

Proposition 60 *Assume that* $(A^\varepsilon, T_\varepsilon) \xrightharpoonup{H_e^0} A^0$ *and let* $\omega \subset\subset \Omega$*. Suppose that, for every symmetric matrix* $\Lambda \in \mathbb{R}^{n\times n}$*, there exists a sequence* $\{w_\Lambda^\varepsilon\}$ *bounded in* $H^1(\Omega)^n$ *such that*

$$\begin{cases} i) \begin{cases} -div\ (A^\varepsilon e(w_\Lambda^\varepsilon)) = P_\varepsilon^\star g_\Lambda^\varepsilon \text{ in } \Omega_\varepsilon, \\ \text{with } g_\Lambda^\varepsilon \text{ is in a compact of } H^{-1}(\Omega)^n, \end{cases} \\ ii)\ A^\varepsilon e(w_\Lambda^\varepsilon)\,\nu = 0 \quad \text{on } \partial T_\varepsilon, \\ iii)\ w_\Lambda^\varepsilon \rightharpoonup \Lambda x \text{ weakly in } H^1(\omega)^n. \end{cases} \tag{3.53}$$

Then, w_Λ^ε *and* $\chi^\varepsilon A^\varepsilon e(w_\Lambda^\varepsilon)$ *lie in a weak compact of* $H^1(\Omega)^n$ *and* $L^2(\Omega)^{n\times n}$ *respectively. Moreover, one has*

$$\chi^\varepsilon A^\varepsilon e(w_\Lambda^\varepsilon) \rightharpoonup A^0\Lambda \quad \text{in } L^2(\omega)^{n\times n}. \tag{3.54}$$

Proof. Since $\{w_\Lambda^\varepsilon\}$ is bounded in $H^1(\Omega)^n$ with respect to ε and A^ε belongs to $M_e(\alpha, \beta, \Omega)$, then there exists $w \in H^1(\Omega)$ and $\xi^0 \in L_S^2(\Omega)^{n\times n}$ such that, to up subsequence,

$$\begin{cases} w_\Lambda^\varepsilon \rightharpoonup w \text{ weakly in } H^1(\Omega)^n, \\ \chi^\varepsilon A^\varepsilon e(w_\Lambda^\varepsilon) \rightharpoonup \xi^0 \text{ weakly in } L^2(\Omega)^{n\times n}. \end{cases} \tag{3.55}$$

On the other hand, let $f \in H^{-1}(\Omega)$ and $u^\varepsilon \in H_0^1(\Omega)^n$ the solution of (3.50). Then, for every symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$, we have

$$A^\varepsilon \widetilde{e(u^\varepsilon)}.e(w_\Lambda^\varepsilon) = \chi^\varepsilon A^\varepsilon e(w_\Lambda^\varepsilon).e(P_\varepsilon u^\varepsilon) \text{ a. e. in } \Omega.$$

Passing to the limit in $D'(\Omega)$ in this equation and using Theorem 59, first to the left-hand side for $\xi^\varepsilon = A^\varepsilon e(u^\varepsilon)$, $\eta^\varepsilon = w_\Lambda^\varepsilon$ and then to the right-hand side for $\xi^\varepsilon = A^\varepsilon e(w_\Lambda^\varepsilon)$, $\eta^\varepsilon = P_\varepsilon u^\varepsilon$, we obtain

$$A^0 e(u^0).e(w) = \xi^0.e(u^0) \text{ a.e. in } \Omega,$$

where we used (3.55) and the fact that $(A^\varepsilon, T_\varepsilon) \xrightharpoonup{H_e^0} A^0$.
By virtue of the symmetry properties of A^0 and ξ^0, this is equivalent to

$$A^0 e(w).\nabla u^0 = \xi^0.\nabla u^0 \text{ a.e. in } \Omega. \tag{3.56}$$

Since the choice of f is arbitrary in $H^{-1}(\Omega)$ and the operator

$$\begin{aligned} \mathcal{U}: \quad H_0^1(\Omega) &\longrightarrow H^{-1}(\Omega), \\ u \quad &\longmapsto \mathcal{U}(u) \doteq -div(A^0 e(u)), \end{aligned}$$

is an isomorphism (as $A^0 \in M_e\left(\dfrac{\alpha}{C^2}, \beta, \Omega\right)$), equality (3.56) is still valid for every $u^0 \in H_0^1(\Omega)$. Hence,

$$A^0 e(w) = \xi^0 \text{ a.e. in } \Omega.$$

This, together with (3.55) and (3.53)iii), gives

$$A^0 \Lambda = \xi^0 \text{ a.e. in } \omega.$$

This implies that the whole sequence $\chi^\varepsilon A^\varepsilon e(w_\Lambda^\varepsilon)$ converges weakly to $A^0\Lambda$ in $L^2(\omega)^{n\times n}$. ■

Remark 61 *In Proposition 60 we do not need to make any assumption on the trace of w_Λ^ε on $\partial\Omega$.*

Proposition 62 *Assume that $(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0$. Then for every $\omega \subset\subset \Omega$ and $\Lambda \in \mathbb{R}_S^{n\times n}$, there exists a sequence $\{w_\Lambda^\varepsilon\}$ of valued functions of $H^1(\Omega)^n$ such that*
i) $\{w_\Lambda^\varepsilon\}$ is bounded in $H^1(\Omega)^n$,
ii) $\{w_\Lambda^\varepsilon\}$ satisfies (3.53),
ii) $\exists c > 0$ independent of ε and Λ such that

$$\|e(w_\Lambda^\varepsilon)\|_{L^2(\omega)^{n\times n}} \le c|\Lambda|.$$

Proof. Let v_Λ^ε be the solution of the system

$$\begin{cases} -div(A^\varepsilon(x) e(v_\Lambda^\varepsilon(x))) = P_\varepsilon^\star\left(-div(A^0(x) e(\varphi(x)\Lambda x))\right) \text{ in } \Omega, \\ (A^\varepsilon e(v_\Lambda^\varepsilon))\nu = 0 \quad on\ \partial T_\varepsilon, \\ v_\Lambda^\varepsilon = 0 \quad in\ \partial\Omega, \end{cases} \tag{3.57}$$

with $\varphi \in D(\Omega)$ and $\varphi = 1$ in ω. Using the fact that $(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0$ and $A^0 \in M_e(\frac{\alpha}{C^2}, \beta, \Omega)$, we deduce that $w_\Lambda^\varepsilon = P_\varepsilon v_\Lambda^\varepsilon$ is bounded in $H_0^1(\Omega)^n$ and satisfies (3.53) for ω. Moreover, choosing v_Λ^ε as test function in the variational formulation of (3.57) gives (in view of (3.49))

$$\|e(P_\varepsilon v_\Lambda^\varepsilon)\|_{L^2(\Omega)^{n\times n}} \le c|\Lambda|,$$

with $c > 0$ independent of ε. ■

3.6 Two estimates on the difference of H_e^0-limits

In this section we extend the estimates stated in Section 3 for the difference of two H_e-limits to the difference of two H_e^0-limits and we give again two estimates. The first result (Theorem 63 below) is an estimate of the L^1-norm of the difference

of two H_e^0-limits and generalizes the one established by M. Briane, A. Damlamian and P. Donato [8] for the second-order elliptic equation to the linearized elasticity system. The second one (Theorem 65 below) gives a pointwise estimate.
In both cases, we need to assume an uniform L^p estimates for one corrector (see Assumption (3.59) below). This is not an additional assumption in the case of a fixed domain, since it follows by the Meyers estimate given in Theorem 45. As mentioned in the introduction, the generalization of the Meyers estimate in the case of perforated domains with a traction condition remains an open question, and the proof given for the elliptic case and one reference hole (see [8]) seems not be adaptable to the elasticity system.
Let T_ε^1 (respect. T_ε^2) be e-admissible in Ω and χ_1^ε (respect. χ_2^ε) be the characteristic function of $\Omega \setminus T_\varepsilon^1$ (respect. $\Omega \setminus T_\varepsilon^1$).

Theorem 63 *Let ω be a relatively compact open subset of Ω, , $\{A^\varepsilon\} \in \mathcal{M}_e(\alpha, \beta, \Omega)$ and $\{B^\varepsilon\} \in \mathcal{M}_e(\alpha', \beta', \Omega)$ such that*

$$\begin{cases} (A^\varepsilon, T_\varepsilon^1) \xrightarrow{H_e^0} A^0, \\ (B^\varepsilon, T_\varepsilon^2) \xrightarrow{H_e^0} B^0. \end{cases} \tag{3.58}$$

Assume that $(B^\varepsilon, T_\varepsilon^2)$ satisfies

$$\begin{cases} \exists p > 2 \text{ s.t. } for\ every\ \Lambda \in \mathbb{R}_S^{n\times n},\ there\ exists\ \{w_\Lambda^\varepsilon\}\ bounded \\ in\ H^1(\Omega)^n\ which\ satisfies\ (3.53)\ and \\ \qquad\qquad \|e(w_\Lambda^\varepsilon)\|_{L^p(\omega)^{n\times n}} \leq d\,|\Lambda|, \\ where\ d > 0\ is\ independent\ of\ \varepsilon\ and\ \Lambda. \end{cases} \tag{3.59}$$

Then, there exists $c > 0$ (which depends on α, β, β', d, C, p, n and ω) such that

$$\|A^0 - B^0\|_{\mathbb{L}^1(\omega)} \leq c \liminf_\varepsilon \left(|(T_\varepsilon^1 \Delta T_\varepsilon^2) \cap \omega| + \|A^\varepsilon - B^\varepsilon\|_{\mathbb{L}^1(\omega)}\right)^\tau, \tag{3.60}$$

where $\tau = \frac{2p-2}{2p}$ and C is given by (3.49) for T_ε^1.

Proof. We denote by χ_1^ε (respect. χ_2^ε) the characteristic function of $\Omega \setminus T_\varepsilon^1$ (respect. $\Omega \setminus T_\varepsilon^2$). Let $\Lambda \in \mathbb{R}_S^{n\times n}$ and v_Λ^ε the vector valued function given by Proposition 62 and associated to the pair $(A^\varepsilon, T_\varepsilon^1)$.
Set for x a.e. in Ω

$$\begin{cases} M^\varepsilon \Lambda = e(v_\Lambda^\varepsilon), \quad \forall \Lambda \in \mathbb{R}_S^{n\times n}, \\ N^\varepsilon \Lambda = e(w_\Lambda^\varepsilon) \quad \forall \Lambda \in \mathbb{R}_S^{n\times n}. \end{cases} \tag{3.61}$$

Using Proposition 62 and assumption (3.59) on $(B^\varepsilon, T_\varepsilon^2)$, we get

$$\begin{cases} \|M^\varepsilon\|_{\mathbb{L}^2(\omega)} \leq c, \\ \|N^\varepsilon\|_{\mathbb{L}^p(\omega)} \leq d, \end{cases} \tag{3.62}$$

with $c > 0$ and $d > 0$ are independent of ε.
On the other hand, using (3.61) and arguing as in the proof of Theorem 43, we obtain

$$\left\{ \begin{array}{l} ({}^tN^\varepsilon \chi_1^\varepsilon A^\varepsilon M^\varepsilon)_{ijkl} = \chi_1^\varepsilon A^\varepsilon e(v^\varepsilon_{E_{kl}}).e(w^\varepsilon_{E_{ij}}), \\ \\ ({}^tN^\varepsilon \chi_2^\varepsilon B^\varepsilon M^\varepsilon)_{ijkl} = \chi_2^\varepsilon B^\varepsilon e(w^\varepsilon_{E_{ij}}).e(v^\varepsilon_{E_{kl}}). \end{array} \right. \tag{3.63}$$

Taking into account Proposition 60 and Proposition 62 and applying Theorem 59 first for $\xi^\varepsilon = A^\varepsilon e(v^\varepsilon_{E_{kl}})$, $\eta^\varepsilon = w^\varepsilon_{E_{ij}}$ and then for $\xi^\varepsilon = B^\varepsilon e(w^\varepsilon_{E_{ij}})$, $\eta^\varepsilon = v^\varepsilon_{E_{kl}}$, we obtain

$$\left\{ \begin{array}{l} ({}^tN^\varepsilon \chi_1^\varepsilon A^\varepsilon M^\varepsilon)_{ijkl} \to A^0 E_{kl}.E_{ij} = A^0_{ijkl} \quad \text{in} \quad \mathcal{D}'(\omega), \\ \\ ({}^tN^\varepsilon \chi_2^\varepsilon B^\varepsilon M^\varepsilon)_{ijkl} \to B^0 E_{ij} E_{kl} = B^0_{klij} = B^0_{ijkl} \quad \text{in} \quad \mathcal{D}'(\omega). \end{array} \right.$$

This implies

$${}^tN^\varepsilon \left(\chi_1^\varepsilon A^\varepsilon - \chi_2^\varepsilon B^\varepsilon\right) M^\varepsilon \to A^0 - B^0 \quad in\ \mathbb{D}'(\omega),$$

thus

$${}^tN^\varepsilon \left(\chi_1^\varepsilon A^\varepsilon - \chi_2^\varepsilon B^\varepsilon\right) M^\varepsilon \to A^0 - B^0 \quad in\ \mathbb{D}'(\omega).$$

This gives, from the weak lower semi-continuity of $L^1(\omega)$-norm

$$\|A^0 - B^0\|_{\mathbb{L}^1(\omega)} \le \liminf_\varepsilon \left\| {}^tN^\varepsilon \left(\chi_1^\varepsilon A^\varepsilon - \chi_2^\varepsilon B^\varepsilon\right) M^\varepsilon \right\|_{\mathbb{L}^1(\omega)}.$$

Now, using (3.62) we obtain

$$\begin{aligned} \|A^0 - B^0\|_{L^1(\omega)} &\le c' c\, d \liminf_\varepsilon \|(\chi_1^\varepsilon A^\varepsilon - \chi_2^\varepsilon B^\varepsilon)\|_{\mathbb{L}^q(\omega)} \\ &\le c' c\, d \liminf_\varepsilon \left(\|\chi_1^\varepsilon A^\varepsilon - \chi_2^\varepsilon B^\varepsilon\|^{\frac{1}{q}}_{\mathbb{L}^1(\omega)} \|\chi_1^\varepsilon A^\varepsilon - \chi_2^\varepsilon B^\varepsilon\|^{1-\frac{1}{q}}_{L^\infty(\omega)} \right). \end{aligned}$$

where $c' > 0$ and $\frac{1}{2} + \frac{1}{p} + \frac{1}{q} = 1$. Taking into account the fact that $\|A^\varepsilon\|_{L^\infty(\omega)} \le \beta$ and $\|B^\varepsilon\|_{L^\infty(\omega)} \le \beta'$, we deduce

$$\begin{aligned} \|A^0 - B^0\|_{L^1(\omega)} &\le c_1 \liminf_\varepsilon \|\chi_1^\varepsilon A^\varepsilon - \chi_2^\varepsilon B^\varepsilon\|^{\frac{1}{q}}_{\mathbb{L}^1(\omega)} \\ &= c_1 \liminf_\varepsilon \|\chi_1^\varepsilon \left(A^\varepsilon - B^\varepsilon\right) + (\chi_1^\varepsilon - \chi_2^\varepsilon) B^\varepsilon\|^{\frac{1}{q}}_{\mathbb{L}^1(\omega)} \\ &\le c_1 \liminf_\varepsilon \left(\|\chi_1^\varepsilon \left(A^\varepsilon - B^\varepsilon\right)\|_{\mathbb{L}^1(\omega)} + \|(\chi_1^\varepsilon - \chi_2^\varepsilon) B^\varepsilon\|_{\mathbb{L}^1(\omega)} \right)^{\frac{1}{q}} \\ &\le c_2 \liminf_\varepsilon \left(\|A^\varepsilon - B^\varepsilon\|_{\mathbb{L}^1(\omega)} + \|\chi_1^\varepsilon - \chi_2^\varepsilon\|_{L^1(\omega)} \right)^{\frac{1}{q}}, \end{aligned}$$

with $c_1 = c' c d(\beta + \beta')^{1-\frac{1}{q}}$ and $c_2 = c_1 \max(1, \beta')$. This gives (3.60). ■

Corollary 64 *Let T_ε be e-admissible in Ω and $A^\varepsilon \in \mathcal{M}_e(\alpha, \beta, \Omega)$ such that $A^\varepsilon \xrightarrow{H_e} A^0$. Then, if $|T_\varepsilon|$ goes to 0, one has*

$$(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0. \tag{3.64}$$

Proof. Let T_ε be e-admissible in Ω such that $|T_\varepsilon| \to 0$ and let $A^\varepsilon \in \mathcal{M}_e(\alpha, \beta, \Omega)$ which H_e-converges to A^0. Then,

$$(A^\varepsilon, \emptyset) \xrightarrow{H_e^0} A^0.$$

Moreover, Proposition 46 shows that the pair $(A^\varepsilon, \emptyset)$ satisfies Assumption (3.59) for some $p > 2$. On the other hand, in view of Theorem 56, there exists
$B^0 \in \mathcal{M}_e(\frac{\alpha}{C^2}, \beta, \Omega)$ such that

$$(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} B^0.$$

Also, for every relatively compact open subset ω of Ω,

$$|\omega \cap (T_\varepsilon \bigtriangleup \emptyset)| = |\omega \cap T_\varepsilon| \leq |T_\varepsilon| \to 0.$$

Hence, from Theorem 63, we obtain

$$B^0(x) A^0(x) \quad a.\ e.\ in\ \omega,$$

which gives the result (since the choice of $\omega \subset\subset \Omega$ is arbitrary). ■

The second result is the following pointwise estimate of the difference of two H_e^0-limits :

Theorem 65 *Let ω be a relatively compact open subset of Ω, T_ε^1 and T_ε^2 be e-admissible in Ω, $\{A^\varepsilon\} \in \mathcal{M}_e(\alpha, \beta, \Omega)$ and $\{B^\varepsilon\} \in \mathcal{M}_e(\alpha', \beta', \Omega)$ such that*

$$\begin{cases} (A^\varepsilon, T_\varepsilon^1) \xrightarrow{H_e^0} A^0, \\ (B^\varepsilon, T_\varepsilon^2) \xrightarrow{H_e^0} B^0 \end{cases}$$

and $(B^\varepsilon, T_\varepsilon^2)$ satisfy the estimate (3.59) for some $p > 2$ and $d > 0$. If

$$\begin{cases} \chi_2^\varepsilon |A^\varepsilon(x) - B^\varepsilon(x)| \leq h^\varepsilon(x) \ \ a.e.\ in\ \omega, \\ \\ with\ h^\varepsilon \longrightarrow h^0 \quad strongly\ in\ L^1(\omega) \end{cases} \tag{3.65}$$

and

$$\chi_1^\varepsilon - \chi_2^\varepsilon \to 0 \quad strongly\ in\ L^1(\omega), \tag{3.66}$$

then

$$|A^0 - B^0| \leq \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}\, h^0 \quad a.\ e.\ in\ \omega. \tag{3.67}$$

Remark 66 *Assumption (3.66) is obviously satisfied when $T_\varepsilon^1 = T_\varepsilon^2$ for every ε. If not, we need to suppose the strong convergence (3.66), since one needs some strong convergence in L^1, as already seen in Theorem 44 for the case without holes.*

Proof. Let $\Lambda \in \mathbb{R}_S^{n\times n}\backslash\{0\}$ and v_Λ^ε the vector valued function given by Proposition 62 and associated to the pair $(A^\varepsilon, T_\varepsilon^1)$. Let also $\psi \in \mathcal{D}(\omega)$ such that $\psi \geq 0$. Set

$$\begin{cases} \bar{h}^\varepsilon = \inf\{h^\varepsilon, \beta + \beta'\}, \\ \bar{h}^0 = \inf\{h^0, \beta + \beta'\}. \end{cases}$$

Then, by virtue of the dominated convergence theorem and hypotheses (3.65), we have

$$\bar{h}^\varepsilon \longrightarrow \bar{h}^0 \quad \textit{strongly in } L^q(\omega),\ \forall q : 1 \leq q < +\infty. \tag{3.68}$$

On the other hand, we have

$$D^\varepsilon \doteq \int_\omega \psi(\chi_1^\varepsilon A^\varepsilon - \chi_2^\varepsilon B^\varepsilon)\ e(v_\Lambda^\varepsilon).e(w_\Lambda^\varepsilon)dx = I^\varepsilon + J_\varepsilon,$$

where

$$\begin{cases} I^\varepsilon = \int_\omega \psi\chi_1^\varepsilon\chi_2^\varepsilon(A^\varepsilon - B^\varepsilon)\ e(v_\Lambda^\varepsilon).e(w_\Lambda^\varepsilon)dx, \\ J^\varepsilon = \int_\omega \psi(\chi_1^\varepsilon - \chi_2^\varepsilon)(\chi_1^\varepsilon A^\varepsilon + \chi_2^\varepsilon B^\varepsilon)\ e(v_\Lambda^\varepsilon).e(w_\Lambda^\varepsilon)dx. \end{cases}$$

Then,

$$|D^\varepsilon| \leq |I^\varepsilon| + |J_\varepsilon|. \tag{3.69}$$

Step 1 (Computation of $\lim\limits_{\varepsilon\to 0} D^\varepsilon$*).* By Proposition 60 and Proposition 62, if we apply Theorem 59 first for $\xi^\varepsilon = A^\varepsilon e(v_\Lambda^\varepsilon)$, $\eta^\varepsilon = w_\Lambda^\varepsilon$ and then for $\xi^\varepsilon = B^\varepsilon e(w_\Lambda^\varepsilon)$, $\eta^\varepsilon = v_\Lambda^\varepsilon$, we get

$$\begin{cases} \chi_1^\varepsilon A^\varepsilon e(v_\Lambda^\varepsilon).e(w_\Lambda^\varepsilon) \longrightarrow A^0\Lambda.\Lambda \quad \text{in} \quad D'(\omega), \\ \chi_2^\varepsilon B^\varepsilon e(w_\Lambda^\varepsilon).e(v_\Lambda^\varepsilon) \longrightarrow B^0\Lambda.\Lambda \quad \text{in} \quad D'(\omega). \end{cases} \tag{3.70}$$

This implies

$$\lim_{\varepsilon\to 0} D^\varepsilon = \int_\omega \psi(A^0 - B^0)\ \Lambda.\Lambda dx. \tag{3.71}$$

Step 2 (Estimation of $\limsup\limits_{\varepsilon\to 0}|I^\varepsilon|$*).* Since $\chi_2^\varepsilon|A^\varepsilon - B^\varepsilon| \leq h^\varepsilon$ a.e in ω and $\chi_2^\varepsilon|A^\varepsilon - B^\varepsilon| \leq \beta + \beta'$ a.e. in Ω, then

$$\begin{aligned} |I^\varepsilon| &= \int_\omega \psi\chi_1^\varepsilon\chi_2^\varepsilon\left(\chi_2^\varepsilon(A^\varepsilon - B^\varepsilon)\right)\ e(v_\Lambda^\varepsilon).e(w_\Lambda^\varepsilon)dx \\ &\leq \int_\omega \psi\left(\chi_1^\varepsilon|e(v_\Lambda^\varepsilon)|\right)\left(\bar{h}^\varepsilon\chi_2^\varepsilon|e(w_\Lambda^\varepsilon)|\right)dx. \end{aligned}$$

We use here an argument introduced by L. Tartar for the proof of a similar result for the elliptic case and a fixed domain (see [47], Proposition 16). For every $a,\ b \in \mathbb{R}_+^\star$ such that $4ab\alpha\alpha' = 1$, we have

$$XY \leq a\alpha X^2 + b\alpha' Y^2, \quad \forall X,\ Y \in \mathbb{R}.$$

Hence

$$|I^\varepsilon| \leq \underbrace{a\alpha\int_\omega \psi\chi_1^\varepsilon|e(v_\Lambda^\varepsilon)|^2dx}_{I_1^\varepsilon} + \underbrace{b\alpha'\int_\omega \psi(\bar{h}^\varepsilon)^2\chi_2^\varepsilon|e(w_\Lambda^\varepsilon)|^2dx}_{I_2^\varepsilon}. \tag{3.72}$$

As $\{A^\varepsilon\} \in \mathcal{M}_e(\alpha,\beta,\Omega)$ and $\{B^\varepsilon\} \in \mathcal{M}_e(\alpha',\beta',\Omega)$, we get

$$\begin{cases} I_1^\varepsilon \leq a\int_\omega \psi\chi_1^\varepsilon A^\varepsilon e(v_\Lambda^\varepsilon).e(v_\Lambda^\varepsilon)dx, \\ I_2^\varepsilon \leq b\int_\omega \psi(\bar{h}^\varepsilon)^2\chi_2^\varepsilon B^\varepsilon e(w_\Lambda^\varepsilon).e(w_\Lambda^\varepsilon)dx. \end{cases} \tag{3.73}$$

On the other hand, in view of Proposition 60 and Proposition 25, Theorem 59 applied first for $\xi^\varepsilon = A^\varepsilon e(v_\Lambda^\varepsilon)$, $\eta^\varepsilon = v_\Lambda^\varepsilon$ and then for $\xi^\varepsilon = B^\varepsilon e(w_\Lambda^\varepsilon)$, $\eta^\varepsilon = w_\Lambda^\varepsilon$, implies

$$\begin{cases} \chi_1^\varepsilon A^\varepsilon e(v_\Lambda^\varepsilon).e(v_\Lambda^\varepsilon) \longrightarrow A^0\Lambda.\Lambda \quad \text{in} \quad D'(\omega), \\ \chi_2^\varepsilon B^\varepsilon e(w_\Lambda^\varepsilon).e(w_\Lambda^\varepsilon) \longrightarrow B^0\Lambda.\Lambda \quad \text{in} \quad D'(\omega). \end{cases}$$

But by virtue of Assumption (3.59) made for $(B^\varepsilon, T_\varepsilon)$, there exists $p > 2$ such that $\chi_2^\varepsilon B^\varepsilon e(w_\Lambda^\varepsilon).e(w_\Lambda^\varepsilon)$ is bounded independently of ε in $L^{\frac{p}{2}}(\omega)$. Then, it converges weakly (up to subsequence) in this space. Hence

$$\begin{cases} \chi_1^\varepsilon A^\varepsilon e(v_\Lambda^\varepsilon).e(v_\Lambda^\varepsilon) \longrightarrow A^0\Lambda.\Lambda \quad \text{in} \quad D'(\omega), \\ \chi_2^\varepsilon B^\varepsilon e(w_\Lambda^\varepsilon).e(w_\Lambda^\varepsilon) \rightharpoonup B^0\Lambda.\Lambda \quad \text{weakly in } L^{\frac{p}{2}}(\omega). \end{cases}$$

This together with (3.68) and (3.73) gives, passing to the limit as $\varepsilon \to 0$ in (3.72),

$$\limsup_{\varepsilon\to 0}|I^\varepsilon| \leq a\int_\omega \psi A^0\Lambda.\Lambda.dx + b\int_\omega \psi(\bar{h}^0)^2 B^0\Lambda.\Lambda dx.$$

Hence, (in view of the fact that $A^0 \in \mathcal{M}_e(\frac{\alpha}{C^2},\beta,\Omega)$ and $B^0 \in \mathcal{M}_e(\frac{\alpha'}{C^2},\beta',\Omega)$)

$$\limsup_{\varepsilon\to 0}|I^\varepsilon| \leq |\Lambda|^2\int_\omega \psi\left(a\beta + b\beta'(\bar{h}^0)^2\right)dx, \tag{3.74}$$

for every $a,\ b \in \mathbb{R}_+^\star$ such that $4ab\alpha\alpha' = 1$.
Step 3 (Computation of $\lim\limits_{\varepsilon\to 0}|J_\varepsilon|$*).* By Proposition 62 and assumption (3.59), one has

$$\begin{cases} \|e(v_\Lambda^\varepsilon)\|_{L^2(\omega)^{n\times n}} \leq c|\Lambda|, \\ \|e(w_\Lambda^\varepsilon)\|_{L^p(\omega)^{n\times n}} \leq d|\Lambda|, \end{cases}$$

with c and d are two positives constants independent of ε. Hence

$$\begin{aligned} |J^\varepsilon| &\leq \int_\omega \psi|\chi_1^\varepsilon - \chi_2^\varepsilon|(\beta+\beta')|e(v_\Lambda^\varepsilon)||e(w_\Lambda^\varepsilon)|dx \\ &\leq \sup_\omega|\psi|\ \|\chi_1^\varepsilon - \chi_2^\varepsilon\|_{L^r(\omega)}\|e(v_\Lambda^\varepsilon)\|_{L^2(\omega)^{n\times n}}\|e(w_\Lambda^\varepsilon)\|_{L^p(\omega)^{n\times n}} \\ &\leq c'\ \|\chi_1^\varepsilon - \chi_2^\varepsilon\|_{L^r(\omega)}|\Lambda|^2, \end{aligned}$$

with $c' > 0$ independent of ε and $1 = \frac{1}{r} + \frac{1}{2} + \frac{1}{p}$. But, in view of(3.66), we have

$$\begin{aligned} \|\chi_1^\varepsilon - \chi_2^\varepsilon\|_{L^r(\omega)} &= \left(\int_\omega |\chi_1^\varepsilon - \chi_2^\varepsilon|^r dx \right)^{\frac{1}{r}} \\ &= \left(\int_\omega |\chi_1^\varepsilon - \chi_2^\varepsilon| dx \right)^{\frac{1}{r}} \to 0. \end{aligned}$$

Then,

$$\lim_{\varepsilon \to 0} |J^\varepsilon| = 0. \tag{3.75}$$

Step 4 (Conclusion). From (3.69), (3.71), (3.74) and (3.75), it follows

$$|\int_\omega \psi (A^0 - B^0)\Lambda.\Lambda dx| \leq \int_\omega \psi \left(a\beta + b\beta' (\bar{h}^0)^2 \right) |\Lambda|^2 dx.$$

As the choice of $\psi \geq 0$ is arbitrary in $D(\omega)$, we deduce that, for every symmetric matrix $\Lambda \in \mathbb{R}_S^{n\times n} \setminus \{0\}$ and for every $a,\ b \in \mathbb{R}_+^\star$ such that $4ab\alpha\alpha' = 1$, we have

$$|(A^0 - B^0)\Lambda.\Lambda| \leq \left(a\beta + b\beta' (\bar{h}^0)^2 \right) |\Lambda|^2 \quad \textit{a.e in } \omega. \tag{3.76}$$

Let us prove now that for x a.e in ω there exists $a(x),\ b(x) \in \mathbb{R}_+^\star$ such that $4a(x)b(x)\alpha\alpha' = 1$ and for every $\Lambda \in \mathbb{R}_S^{n\times n}$

$$|(A^0(x) - B^0(x))\Lambda.\Lambda| \leq \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}\, \bar{h}^0(x)|\Lambda|^2. \tag{3.77}$$

To do that, let x a.e. in ω.

<u>*First case* $\bar{h}^0(x) \neq 0$:</u> It suffices to choose in (3.76)

$$\begin{cases} a(x) = \frac{1}{2}\beta' \bar{h}^0(x) \frac{1}{\sqrt{\alpha\alpha'\beta\beta'}}, \\ b(x) = \frac{1}{2\beta' \bar{h}^0(x)} \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}. \end{cases}$$

<u>*Second case* $\bar{h}^0(x) = 0$:</u> In this case we have to prove that

$$(A^0(x) - B^0(x))\Lambda.\Lambda = 0. \tag{3.78}$$

We take in (3.76), for $m \in \mathbb{N}^\star$

$$\begin{cases} a = \dfrac{1}{m}, \\ b = m\dfrac{1}{4\alpha\alpha'}, \end{cases}$$

which satisfies $4ab\alpha\alpha' = 1$. Then,

$$|(A^0(x) - B^0(x))\Lambda.\Lambda| \leq \frac{\beta}{m}|\Lambda|^2,$$

which gives (3.78), as $m \to \infty$.
To conclude the proof, observe that from (3.77) and Remark 38, for all matrix $\Lambda \in \mathbb{R}^{n\times n}$, we have

$$|(A^0(x) - B^0(x))\Lambda| \leq \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}\bar{h}^0(x)|\Lambda| \quad a.e. \ in \ \omega,$$

which gives finally (3.67). ■

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Chapitre 4

A property of the H-convergence for elasticity in perforated domains

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Abstract

In this work, we obtain the H_e^0-convergence as a limit case of the H_e-convergence. More precisely, if Ω_ε is a perforated domain with (admissible) holes T_ε and χ_ε denote its characteristic function and if $(A^\varepsilon, T_\varepsilon) \overset{H_e^0}{\rightharpoonup} A^0$, we show how the behavior as $(\varepsilon, \delta) \to (0,0)$ of the double sequence of tensors $A_\delta^\varepsilon = (\chi_\varepsilon + \delta(1-\chi_\varepsilon))A^\varepsilon$ is connected to A^0. These results extend those given by D. Cioranescu, A. Damlamian, P. Donato and L. Mascarenhas in [10] for the H-convergence of the scalar second elliptic operators to the linearized elasticity systems.

4.1 Introduction

The notion of H-convergence has been introduced by F. Murat and L. Tartar [39], [40], [46] for the second-order elliptic operators (non necessary symmetric) and extended to the case of holes by M. Briane, A. Damlamian and P. Donato in [8] and called H^0-convergence. In [10], D. Cioranescu, A. Damlamian, P. Donato and L. Mascarenhas obtain the H^0-convergence as a limit case of the H-convergence with a vanishing coercivity constant in the holes.

In this work, we show that a similar property holds for the linearized elasticity systems, namely between the H_e-convergence studied by G. A. Francfort and F. Murat in [26] and its generalization to the case of holes, denoted by H_e^0-convergence, which has been developed by P. Donato and M. El Hajji in [24]. The H_e-convergence deals with the convergence of the solutions of a system of linearized elasticity whose tensor coefficient $\{A^\varepsilon\}$ are equibounded and uniformly definite positive. The H_e^0-convergence treat the same problem in a perforated domain Ω_ε with a traction condition on the holes for which uniform Korn estimates hold.

Let us briefly describe here the main results of this paper. Let Ω a bounded open subset of $\mathbb{R}^n$, $\{T_\varepsilon\}$ a sequence of (admissible) holes, denote $\Omega_\varepsilon = \Omega \setminus T_\varepsilon$ the perforated domain and χ^ε the characteristic function of Ω_ε. Let also $\{A^\varepsilon\}$ a sequence of linearized elasticity tensors on Ω such that $(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0$. We prove (Theorem 78) that if we set for every $\delta > 0$

$$A_\delta^\varepsilon = (\chi_\varepsilon + \delta(1-\chi_\varepsilon))A^\varepsilon \quad \text{a.e. in} \quad \Omega$$

and if A_δ^ε H_e-converges to a tensor A_δ (for some subsequence), then $A_\delta \to A^0$ strongly in $L^p(\Omega)$ for any $p \geq 1$, and weakly $\star$ in $L^\infty(\Omega)$. Moreover, under suitable assumption (see (4.32) below), we have also (Theorem 79)

$$(A_\delta^\varepsilon \xrightarrow{\delta\to 0} (A^\varepsilon, T_\varepsilon)) \texttt{ in the sense } \begin{cases} u_\delta^\varepsilon \to u^\varepsilon \textit{ strongly in } H^1(\Omega_\varepsilon)^n, \\ \\ A_\delta^\varepsilon e(u_\delta^\varepsilon) \to A^\varepsilon \widetilde{e(u^\varepsilon)} \textit{ strongly in } L^2(\Omega)^{n\times n} \end{cases}$$

and (Theorem 80)

$$(A_\delta^\varepsilon \xrightarrow{(\varepsilon,\delta)\to(0,0)} A^0) \texttt{ in the sense } \begin{cases} u_\delta^\varepsilon \to u^0 \textit{ weakly in } H_0^1(\Omega)^n, \\ \\ A_\delta^\varepsilon e(u_\delta^\varepsilon) \to A^0 e(u^0) \textit{ weakly in } L^2(\Omega)^{n\times n}, \end{cases}$$

where u^ε, u_δ^ε and u^0 are the solutions of (4.2), (4.13) and (4.4) respectively. This results can be resumed by the following commutative schema :

$$\begin{array}{ccc} A_\delta^\varepsilon & \xrightarrow{H_e} & A_\delta \\ \downarrow & \searrow & \downarrow \\ (A^\varepsilon, T_\varepsilon) & \xrightarrow{H_e^0} & A^0. \end{array}$$

The definition and the main properties of the H_e^0-convergence are recalled in Section 2. In Section 3 we give some preliminary results and in Section 4 we state and prove the main results.

4.2 The H_e^0-convergence

We use the following notation :
• If $A = (A_{ijkl})_{1\leq i,j,k,l\leq n}$ is a forth order tensors and $\Lambda \in \mathbb{R}^{n\times n}$, we set

$$\begin{cases} A\Lambda = \sum_{1\leq i,j,k,l\leq n} A_{ijkl}\Lambda_{pq} \\ \\ \Lambda.\Upsilon = \sum_{1\leq i,j\leq n} \Lambda_{ij}\Upsilon_{ij}, \\ \\ |\Lambda| = (\sum_{1\leq i,j\leq n} |\Lambda_{ij}|^2)^{\frac{1}{2}}, \end{cases}$$

• Ω is a domain of $\mathbb{R}^n$,
• if F is a set of matrices fields, $F_S = \{M \in F \textit{ s. t. } M \textit{ is symmetric}\}$,
• $\{\varepsilon\}$ and $\{\delta\}$ denote a two strictly decreasing sequence converging to zero,

• if $v = (v_1, \ldots, v_n)$ is a vector valued function and $\zeta = (\zeta_{ij})_{1 \le i,j \le n}$ is a second order tensor of variable $x = (x_1, \ldots, x_n)$, we set

$$\begin{cases} (\nabla v)_{ij} = \dfrac{\partial v_i}{\partial x_j} \equiv D_{x_j} v_i, \\ e(v) = \frac{1}{2}(\nabla v +^t \nabla v), \\ (div \xi)_i = \dfrac{\partial \xi_{ij}}{\partial x_j}, \end{cases}$$

• for two real numbers α and β such that $0 < \alpha < \beta$, $M_e(\alpha, \beta, \Omega)$ is the set of the tensors $A = (A_{ijkl})_{1 \le i,j,k,l \le n}$ defined on Ω, such that a.e. on Ω we have

$$\begin{cases} \text{i) } A_{ijkl} \in L^\infty(\Omega), \quad \text{for any } i, j, k, l = 1, ..., n, \\ \text{ii) } A_{ijkl} = A_{jikl} = A_{klij}, \quad \text{for any } i, j, k, l = 1, ..., n, \\ \text{iii) } \alpha |\Lambda|^2 \le A\Lambda.\Lambda, \quad \text{for any symmetric matrix } \Lambda, \\ \text{iv) } |A\Lambda| \le \beta |\Lambda|, \quad \text{for any matrix } \Lambda, \end{cases}$$

• if $f \in H^{-1}(\Omega)^n$ and $u \in H_0^1(\Omega)^n$, we set $\langle f, u \rangle = \langle f, u \rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n}$.

Let us recall first the main results concerning the H_e^0-convergence introduced by P. Donato and M. El Hajji [24]. Introduce the perforated domain

$$\Omega_\varepsilon = \Omega \setminus T_\varepsilon,$$

where T_ε is a sequence of compacts subsets of Ω and set

$$V_\varepsilon = \left\{ v \in H^1(\Omega_\varepsilon)^n \ \text{ s. t. } \ v = 0 \ \text{ on } \partial\Omega \right\}.$$

In the following, we denote by ν the outward normal unit vector on the boundary of Ω_ε and $\tilde{\cdot}$ the extension by 0 from Ω_ε to Ω and set $\chi^\varepsilon = \chi_{\Omega_\varepsilon}$.

Definition 67 *([24]) The set T_ε is said to be admissible (in Ω) for the linearized elasticity (or e-admissible), if and only if*
i) every L^∞ weak $*$-limit point of $\{\chi_{\Omega_\varepsilon}\}_\varepsilon$ is positive a.e. in Ω,
ii) there exists a positive real C, independent of ε, and a sequence $\{P_\varepsilon\}_\varepsilon$ of linear extension operators such that for each ε

$$\begin{cases} P_\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega)^n), \\ (P_\varepsilon v)_{|\Omega_\varepsilon} = v, \quad \forall v \in V_\varepsilon, \\ \|e(P_\varepsilon v)\|_{0,\Omega} \le C \|e(v)\|_{0,\Omega_\varepsilon}, \quad \forall v \in V_\varepsilon. \end{cases} \tag{4.1}$$

We denote by $P_\varepsilon^\star$ the adjoint operator of P_ε, which is defined from $H^{-1}(\Omega)^n$ to V_ε' with $P_\varepsilon^\star$ given for every $f \in H^{-1}(\Omega)^n$ by

$$\forall v \in V_\varepsilon,\ \langle P_\varepsilon^\star f, v\rangle_{V_\varepsilon', V_\varepsilon} = \langle f, P_\varepsilon v\rangle_{H^{-1}(\Omega)^n, H_0^1(\Omega)^n}.$$

Definition 68 *([24]) Let $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$, T_ε be e-admissible in Ω. The pair $(A^\varepsilon, T_\varepsilon)$ is said H_e^0-converge to the tensor $A^0 \in M_e(\alpha',\beta',\Omega)$ and denoted $(A^\varepsilon, T_\varepsilon) \xrightarrow{H_e^0} A^0$ if and only if for each function $f^\varepsilon \in H^{-1}(\Omega)^n$ such that $f^\varepsilon \to f$ strongly in $H^{-1}(\Omega)^n$, the solution u^ε of*

$$\begin{cases} -div\,(A^\varepsilon e(u^\varepsilon)) = P_\varepsilon^\star f^\varepsilon & \text{in } \Omega_\varepsilon, \\ (A^\varepsilon e(u^\varepsilon))\nu = 0 & \text{on } \partial T_\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

satisfies the weak convergence

$$\begin{cases} i) & P_\varepsilon(u^\varepsilon) \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega)^n, \\ ii) & \widetilde{A^\varepsilon e(u^\varepsilon)} \rightharpoonup A^0 e(u^0) \quad \text{weakly in } L^2(\Omega)^{n\times n}, \end{cases} \tag{4.3}$$

where u^0 is the unique solution of the problem

$$\begin{cases} -div\,(A^0 e(u^0)) = f & \text{in } \Omega, \\ u^0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Remark 69 *1) In [24] the definition is given for fixed $f^\varepsilon \doteq f$. The two definitions are equivalent in view of Proposition 2 of [24].*
2) In the case where $T_\varepsilon = \emptyset$, this definition reduces to the definition of the H_e-convergence [26].

This notion of convergence makes sense in view of the following compactness theorem :

Theorem 70 ***([24])*** *Let $A^\varepsilon \in M_e(\alpha,\beta,\Omega)$ and T_ε be e-admissible in Ω. Then, there exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and a tensor $A^0 \in M_e\left(\frac{\alpha}{C^2},\beta,\Omega\right)$, such that the sequence $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ H_e^0-converge to A^0.*

Remark 71 *The fact that A^0 belongs to $M_e\left(\frac{\alpha}{C^2},\beta,\Omega\right)$, does not appears explicitly in the statement given in [24], but can be easily deduced with the same arguments as that used in the non perforated case.*

Let us recall also a property recently proved in [20].

Theorem 72 ***([20])*** *Let $\{A^\varepsilon\} \in \mathcal{M}_e(\alpha,\beta,\Omega)$ and $\{B^\varepsilon\} \in \mathcal{M}_e(\alpha',\beta',\Omega)$ such that $A^\varepsilon \xrightarrow{H_e} A^0$ and $B^\varepsilon \xrightarrow{H_e} B^0$. Assume that there exists two functions h^ε, $h^0 \in L^1(\Omega)$ such that*

$$|A^\varepsilon - B^\varepsilon| \le h^\varepsilon \longrightarrow h^0 \quad \text{strongly in } L^1(\Omega).$$

Then,

$$|A^0(x) - B^0(x)| \le \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}\, h^0(x) \ \text{ a.e. in } \Omega.$$

The following proposition complete a result given in [24] :

Proposition 73 *One has*
1) If $\{v^\varepsilon\}$ *is a bounded sequence in* $H_0^1(\Omega)$*, then*

$$\left(v^\varepsilon \rightharpoonup v \quad \textit{weakly in } H_0^1(\Omega)^n\right) \Leftrightarrow \left(P_\varepsilon({v^\varepsilon}_{|\Omega_\varepsilon}) \rightharpoonup v \quad \textit{weakly in } H_0^1(\Omega)^n\right).$$

2) If (ε,δ) *is a sequence of* $\mathbb{R}_+^\star \times \mathbb{R}_+^\star$ *such that* $(\varepsilon,\delta) \to (0,0)$ *and* $\{v_\delta^\varepsilon\}$ *is a sequence in* $H_0^1(\Omega)$ *bounded independently of* ε *and* δ*, then*

$$\left(v_\delta^\varepsilon \rightharpoonup v \quad \textit{weakly in } H_0^1(\Omega)^n\right) \Leftrightarrow \left(P_\varepsilon({v_\delta^\varepsilon}_{|\Omega_\varepsilon}) \rightharpoonup v \quad \textit{weakly in } H_0^1(\Omega)^n\right).$$

Proof. Suppose that

$$P_\varepsilon({v^\varepsilon}_{|\Omega_\varepsilon}) \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega)^n. \tag{4.5}$$

Observe first that

$$v^\varepsilon \chi^\varepsilon = P_\varepsilon({v^\varepsilon}_{|\Omega_\varepsilon})\chi^\varepsilon. \tag{4.6}$$

On the other hand, since $\{v^\varepsilon\}$ is a bounded sequence in $H_0^1(\Omega)$, there exists a $\{\varepsilon'\} \subset \{\varepsilon\}$ and $w \in H_0^1(\Omega)^n$ such that

$$v^{\varepsilon'} \rightharpoonup w \textit{ weakly in } H_0^1(\Omega)^n. \tag{4.7}$$

But $|\chi^{\varepsilon'}| \leq 1$, hence there exists $\chi^0 \in L^\infty(\Omega)$ and $\{\varepsilon''\} \subset \{\varepsilon'\}$ such that

$$\chi^{\varepsilon''} \rightharpoonup \chi^0 \textit{ weakly} \star \textit{ in } L^\infty(\Omega). \tag{4.8}$$

Passing to the limit (in $D'(\Omega)$) in (4.6) by using (4.5), (4.7) and (4.8), we find

$$\chi^0 w = \chi^0 v.$$

Taking now into account the fact that (in view of Definition 67) $\chi^0 > 0$, we obtain $w = v$. This, together with (4.7), implies that the whole sequence $\{v^\varepsilon\}$ converge weakly to v. We refer to [24] for the converse implication.
The proof of (2) follows by the same arguments. ■

4.3 Preliminary results

Along this chapter, $\{A^\varepsilon\}$ is a sequence of fourth-order tensors of $\mathcal{M}_e(\alpha,\beta,\Omega)$ and $\{T_\varepsilon\}$ is a sequence of holes e-admissible in Ω such that

$$A^\varepsilon \overset{H_e^0}{\rightharpoonup} A^0. \tag{4.9}$$

Set, for every $\delta > 0$,

$$A_\delta^\varepsilon = (\chi_\varepsilon + \delta(1-\chi_\varepsilon))A^\varepsilon \quad \textit{a.e. in } \Omega. \tag{4.10}$$

Since, for fixed $\delta > 0$, $A^\varepsilon_\delta \in \mathcal{M}_e(\min(1,\delta)\alpha, \max(1,\delta)\beta, \Omega)$, in view of the compactness properties of H_e-convergence, there exists a subsequence $\{\varepsilon_m\}$ of $\{\varepsilon\}$ and $A_\delta \in \mathcal{M}_e(\min(1,\delta)\alpha, \max(1,\delta)\beta, \Omega)$ such that $A^{\varepsilon_m}_\delta \overset{H_e}{\rightharpoonup} A_\delta$ as $\varepsilon_m \to 0$. Hence, for every $\delta > 0$, the set

$$W_\delta = \{A_\delta;\ \exists \{\varepsilon_m\}_{m\in\mathbb{N}} \subset \{\varepsilon\}\ s.\ t.\ A^{\varepsilon_m}_\delta \overset{H_e}{\rightharpoonup} A_\delta\} \tag{4.11}$$

is not empty.
Let $\{f^\varepsilon\}$ be a sequence in $H^{-1}(\Omega)^n$ such that

$$f^\varepsilon \to f \text{ strongly in } H^{-1}(\Omega)^n \tag{4.12}$$

and let A_δ be in W_δ. Let u^ε_δ and u_δ the solutions of

$$\begin{cases} -div\,(A^\varepsilon_\delta e(u^\varepsilon_\delta)) = f^\varepsilon \ \ in \ \ \Omega, \\ u^\varepsilon_\delta = 0 \ \ on \ \ \partial\Omega \end{cases} \tag{4.13}$$

and

$$\begin{cases} -div\,(A_\delta e(u_\delta)) = f \ \ in \ \ \Omega, \\ u_\delta = 0 \ \ on \ \ \partial\Omega \end{cases} \tag{4.14}$$

respectively. We consider now the following sets

$$\begin{cases} U_\delta = \{u_\delta \quad s.t. \ \ u_\delta \ is\ the\ solution\ of\ (4.14)\ for\ some\ A_\delta \in W_\delta\}, \\ V_\delta = \{The\ set\ of\ weak\ limit\ points\ of\ A^\varepsilon_\delta e(u^\varepsilon_\delta)\ in\ L^2(\Omega)^n\ as\ \varepsilon \to 0\}. \end{cases} \tag{4.15}$$

One has the following result

Lemma 74 *One has*

$$V_\delta = \{A_\delta e(u_\delta)\ \ s.\ t.\ \ A_\delta \in W_\delta\ and\ u_\delta\ is\ the\ solution\ of\ (4.14)\}.$$

Proof. It is clear that, if $A_\delta \in W_\delta$ and u_δ is the solution of (4.14), then $A_\delta e(u_\delta)$ belongs to V_δ. On the other hand, let $v \in V_\delta$. Then, there exists a subsequence $\{\varepsilon_m\}$ of ε such that

$$A^{\varepsilon_m}_\delta e(u^{\varepsilon_m}_\delta) \rightharpoonup v \quad weakly\ in\ L^2(\Omega)^n, \tag{4.16}$$

as $\varepsilon_m \to 0$. But the compactness property of the H_e-convergence shows that there exists a subsequence $\{\varepsilon'_m\}$ of $\{\varepsilon_m\}$ and a forth-order tensor A_δ such that

$$A^{\varepsilon'_m}_\delta \overset{H_e}{\rightharpoonup} A_\delta.$$

This implies in particular

$$A^{\varepsilon'_m}_\delta e(u^{\varepsilon'_m}_\delta) \rightharpoonup A_\delta e(u_\delta) \quad weakly\ in\ L^2(\Omega)^n.$$

This, together with (4.16), gives $v = A_\delta e(u_\delta)$. ■

Remark 75 *Let us show that in view of Theorem 72, there exists* $\{\varepsilon_m\} \subset \{\varepsilon\}$ *and for all* $\delta > 0$ *a tensor* $\widehat{A_\delta}$ *such that*

$$A_\delta^{\varepsilon_m} \xrightarrow{H_e} \widehat{A_\delta}. \tag{4.17}$$

Let us also point out that in Theorem 78 we will consider a more general situation, where for every $\delta > 0$*, there exists* $\{\varepsilon_\delta\}$ *and a tensor* A_δ *such that* $A_\delta^{\varepsilon_\delta} \xrightarrow{H_e} A_\delta$.
Let us prove (4.17). Using the diagonal subsequence procedure and the compactness property of the H_e*-convergence, one extracts a subsequence* $\{\varepsilon_m\}$ *of* $\{\varepsilon\}$ *such that, for every* $\delta \in \mathbb{Q}_+^\star$*, one has*

$$A_\delta^{\varepsilon_m} \ H_e\text{-convergences to a limit } A_\delta. \tag{4.18}$$

Since a.e in Ω *one has*

$$\begin{cases} |A_{\delta_1}^{\varepsilon_m} - A_{\delta_2}^{\varepsilon_m}| \leq \beta|\delta_1 - \delta_2|, \quad \forall \delta_1,\ \delta_2 \in \mathbb{Q}_+^\star, \\ A_{\delta_1}^{\varepsilon_m} \in \mathcal{M}_e(\min(1,\delta_1)\alpha, \max(1,\delta_1)\beta, \Omega), \\ A_{\delta_2}^{\varepsilon_m} \in \mathcal{M}_e(\min(1,\delta_2)\alpha, \max(1,\delta_2)\beta, \Omega). \end{cases}$$

Then, from Theorem 72, it follows

$$|A_{\delta_1} - A_{\delta_2}| \leq \frac{\beta^2}{\alpha} \sqrt{\frac{max(1,\delta_1)max(1,\delta_2)}{min(1,\delta_1)min(1,\delta_2)}}\ |\delta_1 - \delta_2|.$$

This implies that the mapping $\delta \in \mathbb{Q}_+^\star \mapsto A_\delta \in \mathbb{L}^\infty(\Omega)$ *is uniformly continuous. Hence, it can be extended to a mapping (still denoted by* $\delta \mapsto A_\delta$*) defined and uniformly continuous on all* $\mathbb{R}_+^\star$ *(since* $\mathbb{Q}_+^\star$ *is dense in* $\mathbb{R}_+^\star$*).*
Let now δ *be a strictly positive real and* $\{\delta_s\}$ *be a sequence of* $\mathbb{Q}_+^\star$ *which converges to* δ *as* $s \to \infty$*. Then, there exists a sub-subsequence* $\{\varepsilon'_m\}$ *of* $\{\varepsilon_m\}$ *such that*

$$A_\delta^{\varepsilon'_m} \ H_e\text{-converges to some } A. \tag{4.19}$$

In view of Theorem 72 this give, together with (4.18) and the fact that $|A_\delta^{\varepsilon'_m} - A_{\delta_s}^{\varepsilon'_m}| \leq \beta|\delta - \delta_s|$*, the following inequality :*

$$|A - A_{\delta_s}| \leq \frac{\beta^2}{\alpha} \sqrt{\frac{max(1,\delta)max(1,\delta_s)}{min(1,\delta)min(1,\delta_s)}}\ |\delta - \delta_s| \quad a.e.\ in\ \Omega.$$

Using the continuity of the mapping $\delta \mapsto A_\delta$ *on* $\mathbb{R}_+^\star$ *and passing to the limit in this inequality as* $s \to \infty$*, one finds*

$$A = A_\delta, \quad a.e.\ in\ \Omega.$$

The uniqueness of the limit implies then that the whole subsequence $A_\delta^{\varepsilon_m}$ H_e*-converges to* A_δ*, for every* $\delta > 0$.

The following results state some a priori estimates that we will need in the following :

Proposition 76 *Let u^ε and u^ε_δ the solutions of (4.2) and (4.13) respectively. Then, there exists $c > 0$ independent of ε and δ such that*

$$\begin{cases} i)\ \|P_\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - P_\varepsilon u^\varepsilon\|_{H^1_0(\Omega)^n} \le c\left(\delta^{\frac{1}{2}} + |\langle f^\varepsilon, P_\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - u^\varepsilon_\delta\rangle|^{\frac{1}{2}}\right), \\ ii)\ \|e(u^\varepsilon_\delta)\|_{L^2(T_\varepsilon)^{n\times n}} \le c\left(1 + \delta^{-\frac{1}{2}}|\langle f^\varepsilon, P_\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - u^\varepsilon_\delta\rangle|^{\frac{1}{2}}\right), \\ iii)\ \|A^\varepsilon_\delta e(u^\varepsilon_\delta) - \widetilde{A^\varepsilon e(u^\varepsilon)}\|_{L^2(\Omega)^{n\times n}} \le c\left(\delta^{\frac{1}{2}} + |\langle f^\varepsilon, P_\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - u^\varepsilon_\delta\rangle|^{\frac{1}{2}}\right). \end{cases} \tag{4.20}$$

Proof. Observe that the variational formulations of problems (4.2) and (4.13) are

$$\forall w \in H^1_0(\Omega)^n, \quad \int_{\Omega_\varepsilon} A^\varepsilon e(u^\varepsilon).e(w)dx = \langle f^\varepsilon, P_\varepsilon(w_{|\Omega_\varepsilon})\rangle \tag{4.21}$$

and

$$\forall w \in H^1_0(\Omega)^n, \int_{\Omega_\varepsilon} A^\varepsilon e(u^\varepsilon_\delta).e(w)dx + \delta\int_{T_\varepsilon} A^\varepsilon e(u^\varepsilon_\delta).e(w)dx = \langle f^\varepsilon, w\rangle$$

respectively. Then, for every $w \in H^1_0(\Omega)^n$, one has

$$\int_{\Omega_\varepsilon} A^\varepsilon(e(u^\varepsilon_\delta) - e(u^\varepsilon)).e(w)dx + \delta\int_{T_\varepsilon} A^\varepsilon e(u^\varepsilon_\delta).e(w)dx = -\langle f^\varepsilon, P_\varepsilon(w_{|\Omega_\varepsilon}) - w\rangle.$$

In particular, for $w = u^\varepsilon_\delta - P_\varepsilon u^\varepsilon$, this gives

$$\begin{aligned}\int_{\Omega_\varepsilon} A^\varepsilon(e(u^\varepsilon_\delta) - e(u^\varepsilon)).(e(u^\varepsilon_\delta) - e(P_\varepsilon u^\varepsilon))dx + \delta\int_{T_\varepsilon} A^\varepsilon e(u^\varepsilon_\delta).(e(u^\varepsilon_\delta) - e(P_\varepsilon u^\varepsilon))dx = \\ = -\langle f^\varepsilon, P_\varepsilon((u^\varepsilon_\delta - P_\varepsilon u^\varepsilon)_{|\Omega_\varepsilon}) - u^\varepsilon_\delta - P_\varepsilon u^\varepsilon\rangle.\end{aligned}$$

Using the fact that $P_\varepsilon u^\varepsilon{}_{|\Omega_\varepsilon} = u^\varepsilon$, one deduces

$$\begin{aligned}\int_{\Omega_\varepsilon} A^\varepsilon(e(u^\varepsilon_\delta) - e(u^\varepsilon)).(e(u^\varepsilon_\delta) - e(u^\varepsilon))dx + \delta\int_{T_\varepsilon} A^\varepsilon e(u^\varepsilon_\delta).e(u^\varepsilon_\delta)dx = \\ = \delta\int_{T_\varepsilon} A^\varepsilon e(u^\varepsilon_\delta).e(P_\varepsilon u^\varepsilon)dx - \langle f^\varepsilon, P_\varepsilon(u^\varepsilon_\delta)_{|\Omega_\varepsilon} - u^\varepsilon_\delta\rangle.\end{aligned}$$

This gives, in view of the fact that $A^\varepsilon \in M_e(\alpha, \beta, \Omega)$,

$$\begin{cases} \alpha\int_{\Omega_\varepsilon} |e(u^\varepsilon_\delta) - e(u^\varepsilon))|^2dx + \alpha\delta\int_{T_\varepsilon} |e(u^\varepsilon_\delta)|^2dx \le \\ \qquad \le \delta|\int_{T_\varepsilon} A^\varepsilon e(u^\varepsilon_\delta).e(P_\varepsilon u^\varepsilon)dx| + |\langle f^\varepsilon, P_\varepsilon(u^\varepsilon_{\delta|\Omega_\varepsilon}) - u^\varepsilon_\delta\rangle|. \end{cases} \tag{4.22}$$

Using the Young's inequality, one obtains

$$\begin{aligned} |\int_{T_\varepsilon} A^\varepsilon e(u^\varepsilon_\delta).e(P_\varepsilon u^\varepsilon)dx| &\le \beta\int_{T_\varepsilon} |e(u^\varepsilon_\delta)||e(P_\varepsilon u^\varepsilon)|dx \\ &\le \frac{\alpha}{2}\int_{T_\varepsilon} |e(u^\varepsilon_\delta)|^2dx + \frac{\beta^2}{2\alpha}\int_{T_\varepsilon} |e(P_\varepsilon u^\varepsilon)|^2dx \\ &\le \frac{\alpha}{2}\int_{T_\varepsilon} |e(u^\varepsilon_\delta)|^2dx + \frac{\beta^2}{2\alpha}\int_{\Omega} |e(P_\varepsilon u^\varepsilon)|^2dx. \end{aligned}$$

But, taking $w = P_\varepsilon u^\varepsilon$ in (4.21), one finds

$$\int_\Omega |e(P_\varepsilon u^\varepsilon)| dx \leq c_1,$$

with $c_1 > 0$ independent of ε and δ. Then,

$$|\int_{T_\varepsilon} A^\varepsilon e(u^\varepsilon_\delta).e(P_\varepsilon u^\varepsilon)dx| \leq \frac{\alpha}{2}\int_{T_\varepsilon} |e(u^\varepsilon_\delta)|^2 dx + c_2,$$

where $c_2 > 0$ independent of ε and δ. This, together with (4.22), gives

$$\alpha \int_{\Omega_\varepsilon} |e(u^\varepsilon_\delta) - e(u^\varepsilon))|^2 dx + \frac{\alpha\delta}{2}\int_{T_\varepsilon} |e(u^\varepsilon_\delta)|^2 dx \leq c_3(\delta + |\langle f^\varepsilon, P_\varepsilon(u^\varepsilon_{\delta|_{\Omega_\varepsilon}}) - u^\varepsilon_\delta\rangle|). \quad (4.23)$$

with $c_3 > 0$ independent of ε and δ. From this, (4.20)ii) follows immediately. Moreover, since $u^\varepsilon_{\delta|_{\Omega_\varepsilon}} - u^\varepsilon \in H^1_0(\Omega_\varepsilon)^n$, Definition 67 shows that

$$\|e(P_\varepsilon(u^\varepsilon_{\delta|_{\Omega_\varepsilon}} - u^\varepsilon))\|_{0,\Omega} \leq C\|e(u^\varepsilon_\delta - u^\varepsilon)\|_{0,\Omega_\varepsilon}.$$

Hence, by virtue of the Korn inequality, (4.23) gives also (4.20)i).
On the other hand, since $A^\varepsilon_\delta = A^\varepsilon$ a.e. in Ω_ε and $A^\varepsilon_\delta = \delta A^\varepsilon$ a.e. in T_ε, one has

$$\begin{aligned}\int_\Omega |A^\varepsilon_\delta e(u^\varepsilon_\delta) - A^\varepsilon \widetilde{e(u^\varepsilon)}|^2 dx \;&\leq\; \int_{\Omega_\varepsilon} |A^\varepsilon e(u^\varepsilon_\delta) - A^\varepsilon e(u^\varepsilon)|^2 dx + \delta^2 \int_{T_\varepsilon} |A^\varepsilon e(u^\varepsilon_\delta)|^2 dx \\ &\leq\; \beta^2 \int_{\Omega_\varepsilon} |e(u^\varepsilon_\delta) - e(u^\varepsilon)|^2 dx + \beta^2\delta^2 \int_{T_\varepsilon} |e(u^\varepsilon_\delta)|^2 dx,\end{aligned}$$

which, together with (4.20)i) and (4.20)ii), gives (4.20)iii). ■

Proposition 77 *Let u^0 the solution of (4.4). Then,*

$$\begin{cases} i)\ \sup_{u\in U_\delta} \|u - u^0\|_{H^1_0(\Omega)^n} \leq c\,\delta^{\frac{1}{2}}, \\ \\ ii)\ \sup_{v\in V_\delta} \|v - A^0 e(u^0)\|_{L^2(\Omega)^{n\times n}} \leq c\,\delta^{\frac{1}{2}}. \end{cases} \quad (4.24)$$

Proof. Let be u_δ in U_δ. This means that there exists $A_\delta \in W_\delta$ such that u_δ is the solution of (4.14). But the fact that A_δ is in W_δ implies that there exists a subsequence $\{\varepsilon_m\}$ of ε such that $A^{\varepsilon_m}_\delta$ H_e-converges to A_δ. Hence, the solution u^ε_δ of

$$\begin{cases} -div\,(A^{\varepsilon_m}_\delta e(u^{\varepsilon_m}_\delta)) = f^{\varepsilon_m} \ \ in\ \ \Omega, \\ \\ u^{\varepsilon_m}_\delta = 0 \ \ on\ \ \partial\Omega \end{cases} \quad (4.25)$$

satisfies as $\varepsilon_m \to 0$

$$\begin{cases} i)\ u^{\varepsilon_m}_\delta \rightharpoonup u_\delta \quad weakly\ in\ H^1_0(\Omega)^n, \\ \\ ii)\ A^{\varepsilon_m}_\delta e(u^{\varepsilon_m}_\delta) \rightharpoonup A_\delta e(u_\delta) \quad weakly\ in\ L^2(\Omega)^{n\times n}. \end{cases} \quad (4.26)$$

Estimation (4.24)i) : By Lemma 73, (4.26)i) implies that, for every fixed $\delta > 0$,

$$P_{\varepsilon_m}(u_\delta^{\varepsilon_m}{}_{|\Omega_{\varepsilon_m}}) \rightharpoonup u_\delta \quad \textit{weakly in } H_0^1(\Omega)^n. \tag{4.27}$$

Hence, by virtue of (4.12), one has

$$\lim_{\varepsilon_m \to 0} \langle f_{\varepsilon_m}, P_{\varepsilon_m}(u_\delta^{\varepsilon_m}{}_{|\Omega_{\varepsilon_m}}) - u_\delta \rangle = 0. \tag{4.28}$$

From this and (4.20)i), it comes

$$\lim_{\varepsilon_m \to 0} \|P_{\varepsilon_m}(u_\delta^{\varepsilon_m}{}_{|\Omega_{\varepsilon_m}}) - P_{\varepsilon_m} u^{\varepsilon_m}\|_{H_0^1(\Omega)^n} \leq c\delta^{\frac{1}{2}}. \tag{4.29}$$

But, (4.27) and (4.3)i) imply

$$P_{\varepsilon_m}(u_\delta^{\varepsilon_m}{}_{|\Omega_\varepsilon}) - P_{\varepsilon_m}(u^{\varepsilon_m}) \rightharpoonup u_\delta - u^0 \quad \textit{weakly in } H_0^1(\Omega)^n.$$

This gives, by using the weak lower semi-continuity of the H_0^1-norm,

$$\|u_\delta - u^0\|_{H_0^1(\Omega)^n} \leq \lim_{\varepsilon_m \to 0} \|P_{\varepsilon_m}(u_\delta^{\varepsilon_m}{}_{|\Omega_{\varepsilon_m}}) - P_{\varepsilon_m} u^{\varepsilon_m}\|_{H_0^1(\Omega)^n},$$

where u^0 is the solution of (4.4). Hence, (4.29) gives

$$\|u_\delta - u^0\|_{H_0^1(\Omega)^n} \leq c\delta^{\frac{1}{2}}.$$

This is still valid for every $u_\delta \in U_\delta$, which implies (4.24)i).

Estimation (4.24)ii) : From (4.20)iii) and (4.28), it comes

$$\lim_{\varepsilon_m \to 0} \|A_\delta^{\varepsilon_m} e(u_\delta^{\varepsilon_m}) - A^{\varepsilon_m} \widetilde{e(u^{\varepsilon_m})}\,\|_{H_0^1(\Omega)^{n\times n}} \leq c\delta^{\frac{1}{2}}.$$

But (4.3)ii) and (4.26)ii) imply, as $\varepsilon_m \to 0$, that

$$(A_\delta^{\varepsilon_m} e(u_\delta^{\varepsilon_m}) - A^{\varepsilon_m} \widetilde{e(u^{\varepsilon_m})}) \rightharpoonup (A_\delta e(u_\delta) - A^0 e(u^0)) \quad \textit{weakly in } L^2(\Omega)^{n\times n}.$$

In view of the weak lower semi-continuity of the L^2-norm, these two last relations give

$$\|(A_\delta e(u_\delta) - A^0 e(u^0))\|_{L^2(\Omega)^{n\times n}} \leq c\delta^{\frac{1}{2}},$$

for every $u_\delta \in U_\delta$. Hence,

$$\sup_{u_\delta \in U_\delta} \|(A_\delta e(u_\delta) - A^0 e(u^0))\|_{L^2(\Omega)^{n\times n}} \leq c\delta^{\frac{1}{2}},$$

which, together with Lemma 8, gives the claimed result. ■

4.4 Main results

We give here our main results.

Theorem 78 *Let $A_\delta \in W_\delta$. Then, the solution u_δ of (4.14) satisfies as $\delta \to 0$*

$$\begin{cases} i)\ u_\delta \to u^0 \text{ strongly in } H_0^1(\Omega)^n, \\ ii)\ A_\delta e(u_\delta) \to A^0 e(u^0) \text{ strongly in } L^2(\Omega)^{n\times n}, \end{cases} \tag{4.30}$$

where u^0 is the solution of (4.4). Moreover, one has the following convergence :

$$\forall p \in [1, \infty[, \quad A_\delta \to A^0, \tag{4.31}$$

strongly in $L^p(\Omega)$ and weakly $\star$ in $L^\infty(\Omega)$.

Theorem 79 *Let f^ε, f be in $H^{-1}(\Omega)^n$ satisfying (4.12). Suppose that*

$$\forall \varepsilon > 0, \ \langle f^\varepsilon, v\rangle = 0, \quad \forall v \in H_0^1(\Omega)^n, \ v = 0 \text{ on } \Omega_\varepsilon. \tag{4.32}$$

Then, as $\delta \to 0$

$$\begin{cases} i)\ u_\delta^\varepsilon \to u^\varepsilon \text{ strongly in } H^1(\Omega_\varepsilon)^n, \\ ii)\ A_\delta^\varepsilon e(u_\delta^\varepsilon) \to A^\varepsilon \widetilde{e(u^\varepsilon)} \text{ strongly in } L^2(\Omega)^{n\times n}, \end{cases} \tag{4.33}$$

where u^ε and u_δ^ε are the solutions of (4.2) and (4.13) respectively.

Theorem 80 *Let f^ε, $f \in H^{-1}(\Omega)^n$ satisfying (4.12) and (4.32). Then, as $(\varepsilon, \delta) \to (0, 0)$*

$$\begin{cases} i)\ u_\delta^\varepsilon \to u^0 \text{ weakly in } H_0^1(\Omega)^n, \\ ii)\ A_\delta^\varepsilon e(u_\delta^\varepsilon) \to A^0 e(u^0) \text{ weakly in } L^2(\Omega)^{n\times n}, \end{cases} \tag{4.34}$$

where u^0 and u_δ^ε are the solutions of (4.4) and (4.13) respectively.

To prove these results we use similar arguments as that used in [10]. Before giving these proofs, we recall the following lemma :

Lemma 81 *([10]) Let $\{\psi_m\}$ be a sequence of $L^2(\Omega)$. Suppose that there exists ψ, $\phi \in L^2(\Omega)$ such that*

$$\begin{cases} \psi_m \to \psi \text{ strongly in } L^2_{loc}(\Omega), \\ \forall m \in \mathbb{N}, \ |\psi_m| \le \phi \quad a.e. \text{ in } \Omega. \end{cases}$$

Then,

$$\psi_m \to \psi \text{ strongly in } L^2(\Omega).$$

Proof of Theorem 78. Observe first that (4.30) follows immediately from Proposition 77.
On the other hand, taking in (4.2) and (4.13), $f^\varepsilon = f \doteq -div(A^0 e(\varphi\Lambda x))$ with $\varphi \in \mathcal{D}(\Omega)$ and $\Lambda \in \mathbb{R}_S^{n\times n}$, then (4.4) reeds

$$\begin{cases} -div\ (A^0 e(u^0)) = -div(A^0 e(\varphi\Lambda x)) & \text{in } \Omega, \\ u^0 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

This implies, in view of the fact that $A^0 \in \mathcal{M}_e(\frac{\alpha}{C^2}, \beta, \Omega)$, that

$$u^0 = \varphi\Lambda x.$$

This, together with (4.30), gives

$$\begin{cases} u_\delta \to \varphi\Lambda x \quad \textit{strongly in } H_0^1(\Omega)^n, \\ A_\delta e(u_\delta) \to A^0 e(\varphi\Lambda x) \quad \textit{strongly in } L^2(\Omega)^{n\times n}. \end{cases}$$

Taking now $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi = 1$ on ω and where $\omega \subset\subset \Omega$, one obtains

$$\begin{cases} u_\delta \to \Lambda \quad \textit{strongly in } H^1(\omega)^n, \\ A_\delta e(u_\delta) \to A^0\Lambda \quad \textit{strongly in } L^2(\omega)^{n\times n}. \end{cases} \tag{4.35}$$

On the other hand, one has almost everywhere in ω,

$$\begin{aligned} |A_\delta\Lambda - A^0\Lambda| &\leq |A_\delta\Lambda - A_\delta e(u_\delta)| + |A_\delta e(u_\delta) - A^0\Lambda| \\ &\leq \beta|\Lambda - e(u_\delta)| + |A_\delta e(u_\delta) - A^0\Lambda|. \end{aligned}$$

Then, using (4.35), one gets

$$A_\delta\Lambda \to A^0\Lambda \quad \textit{strongly in } L^2(\omega)^{n\times n},$$

for every $\omega \subset\subset \Omega$.
Since $|A_{\delta_m}\Lambda| \leq \beta|\Lambda|$, this gives by Lemma 81 written for $\psi_m = A_{\delta_m}\Lambda$ (with $\delta_m \to 0$),

$$A_\delta\Lambda \to A^0\Lambda \quad \textit{strongly in } L^2(\Omega)^{n\times n}.$$

By virtue of the symmetry properties of A_δ and A^0, this convergence is still valid for every matrix $\Lambda \in \mathbb{R}^{n\times n}$. Thus

$$A_\delta \to A^0 \quad \textit{strongly in } L^2(\Omega)^{n\times n}.$$

From this convergence and the fact that $\|A_\delta\|_{L^\infty(\Omega)} \leq \beta$, one obtains convergence (4.31). ∎

Proof of Theorem 79. From hypothesis (4.32), Proposition 76 and the fact that $P_\varepsilon(u^\varepsilon_{\delta|_{\Omega_\varepsilon}}) - u^\varepsilon_\delta = 0 \textit{ in } \Omega_\varepsilon$, it follows

$$\begin{cases} \|u^\varepsilon_{\delta|_{\Omega_\varepsilon}} - u^\varepsilon\|_{H^1(\Omega_\varepsilon)^n} \leq \|P_\varepsilon(u^\varepsilon_{\delta|_{\Omega_\varepsilon}}) - P_\varepsilon u^\varepsilon\|_{H^1_0(\Omega)^n} \leq c\delta^{\frac{1}{2}}, \\ \|A^\varepsilon_\delta e(u^\varepsilon_\delta) - A^\varepsilon \widetilde{e(u^\varepsilon)}\|_{L^2(\Omega)^{n\times n}} \leq c\delta^{\frac{1}{2}}. \end{cases}$$

Passing to the limit as $\delta \to 0$, one obtains (4.33). ■

Proof of Theorem 80. i) Under hypothesis (4.32), Proposition 76 gives

$$\lim_{(\varepsilon,\delta)\to(0,0)} \|P_\varepsilon(u^\varepsilon_{\delta|_{\Omega_\varepsilon}}) - P_\varepsilon u^\varepsilon\|_{H^1_0(\Omega)^n} = 0$$

and the fact that $A^\varepsilon \xrightarrow{H_e} A^0$ implies

$$P_\varepsilon u^\varepsilon - u^0 \rightharpoonup 0 \quad \textit{weakly in } H^1_0(\Omega).$$

Hence, by passing to the weak limit in $H^1_0(\Omega)$ as $(\varepsilon,\delta) \to (0,0)$ in the following equality

$$P_\varepsilon(u^\varepsilon_{\delta|_{\Omega_\varepsilon}}) - u^0 = (P_\varepsilon(u^\varepsilon_{\delta|_{\Omega_\varepsilon}}) - P_\varepsilon u^\varepsilon) + (P_\varepsilon u^\varepsilon - u^0),$$

one deduces

$$P_\varepsilon(u^\varepsilon_{\delta|_{\Omega_\varepsilon}}) \rightharpoonup u^0 \textit{ weakly in } H^1_0(\Omega)^n. \tag{4.36}$$

On the other hand, using (4.32) and Proposition 76, one gets

$$\begin{cases} \|P_\varepsilon(u^\varepsilon_{\delta|_{\Omega_\varepsilon}}) - P_\varepsilon u^\varepsilon\|_{H^1_0(\Omega)^n} \leq c\delta^{\frac{1}{2}}, \\ \|e(u^\varepsilon_\delta)\|_{L^2(T_\varepsilon)^{n\times n}} \leq c. \end{cases}$$

This implies

$$\begin{cases} \|u^\varepsilon_\delta\|_{H^1_0(\Omega_\varepsilon)^n} \leq \|P_\varepsilon(u^\varepsilon_{\delta|_{\Omega_\varepsilon}})\|_{H^1_0(\Omega)^n} \leq c\delta^{\frac{1}{2}} + \|P_\varepsilon u^\varepsilon\|_{H^1_0(\Omega)^n}, \\ \|e(u^\varepsilon_\delta)\|_{L^2(T_\varepsilon)^{n\times n}} \leq c. \end{cases}$$

Since $P_\varepsilon u^\varepsilon$ is bounded independently of ε in $H^1_0(\Omega)^n$, one deduces that u^ε_δ is bounded independently of ε and δ in $H^1_0(\Omega)^n$. This, together with (4.36) and Proposition 73, gives (4.34)i).

ii) Using Proposition 76, the fact that $A^\varepsilon \xrightarrow{H_e} A^0$ and

$$A^\varepsilon_\delta e(u^\varepsilon_\delta) - A^0 e(u^0) = (A^\varepsilon_\delta e(u^\varepsilon_\delta) - A^\varepsilon \widetilde{e(u^\varepsilon)}) + (A^\varepsilon \widetilde{e(u^\varepsilon)} - A^0 e(u^0)),$$

one obtains the convergence (4.34)ii). ■

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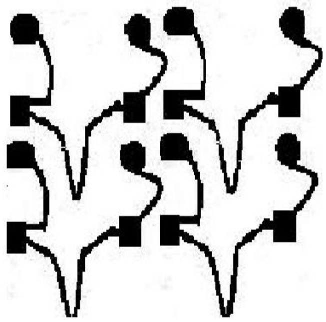
Conclusion

Les résultats présentés dans cette thèse contribuent au développement des techniques de l'homogénéisation pour l'élasticité dans les milieux composites et perforés et peuvent s'appliquer à plusieurs situations. Dans cette section, après quelques commentaires sur les résultats obtenus, nous donnons quelques perspectives d'applications

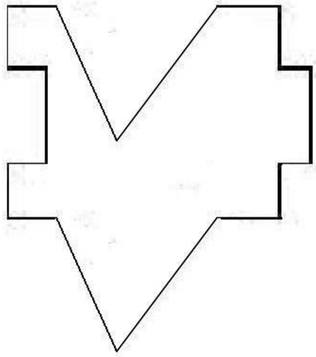
Commentaires

a) Dans le chapitre 2, nous avons rappelé la propriété de compacité de la H_e^0-convergence (théorème 19) avec A^0 appartenant à $M_e\left(\frac{\alpha}{C^2}, \frac{\beta^2}{\alpha}, \Omega\right)$, cette appartenance était déduite de la démonstration de cette propriété [24]. Mais, dans les autres chapitres nous l'avions énoncé avec plus de précision, à savoir, $A^0 \in M_e\left(\frac{\alpha}{C^2}, \beta, \Omega\right)$. Cette information supplémentaire sur la constante de bornitude s'obtient exactement comme dans le cas scalaire symétrique sans trous (voir proposition (13.6) de [12]).

b) Les résultats du chapitre 2 sont assez généraux car nous n'avons pas supposé une bonne séparation d'échelles et nous avons considéré comme cellules de références des ouverts connexes de $\mathbb{R}^n$ qui possèdent la propriété du pavage. Ce qui permet d'envisager une classe de trous plus large que celle qu'on peut obtenir par les pavés. Par exemple, les trous suivants :



sont périodiques pour la cellule suivante (ce qui n'est pas possible par les pavés) :



c) Le résultat d'homogénéisation dans les domaines perforés avec double périodicité donné par la proposition 37 (Chapitre 2) est équivalent au résultat suivant (dont la présentation est plus classique) : Pour tout $f \in L^2(\Omega)^n$, la solution du système

$$\begin{cases} -div\ (A^\varepsilon e(u^\varepsilon)) = f \quad \text{dans } \Omega_\varepsilon, \\ (A^\varepsilon e(u^\varepsilon))\nu = 0 \quad \text{sur } \partial T_\varepsilon, \\ u^\varepsilon = 0 \quad \text{sur } \partial\Omega, \end{cases}$$

satisfait les convergences

$$\begin{cases} P_\varepsilon(u^\varepsilon) \rightharpoonup u^0 \quad \text{faiblement dans } H_0^1(\Omega)^n, \\ A^\varepsilon \widetilde{e(u^\varepsilon)} \rightharpoonup A^0 e(u^0) \quad \text{faiblement dans } L^2(\Omega)^{n\times n}, \end{cases}$$

où u^0 est l'unique solution du système

$$\begin{cases} -div\ (A^0 e(u^0)) = \left(\frac{|Y^\star \backslash Y_1|}{|Y|} + \frac{|Y_1||Z^\star|}{|Y||Z|}\right) f \quad \text{dans } \Omega, \\ u^0 = 0 \quad \text{sur } \partial\Omega. \end{cases}$$

En effet, pour obtenir cette équivalence il suffit d'après la théorème 10 de montrer que

$$\chi_{\Omega_\varepsilon} \rightharpoonup \frac{|Y^\star \backslash Y_1|}{|Y|} + \frac{|Y_1||Z^\star|}{|Y||Z|} \ \ dans\ L^\infty(\Omega)\ \ faible\ \star. \tag{C.1}$$

On rappelle que dans la construction de Ω_ε on a enlevé les trous qui touchent le bord de Ω. Soient donc $C^\varepsilon(\partial\Omega)$ l'union de toutes les cellules qui touchent $\partial\Omega$ et N^ε leur nombre et soit aussi $T_b^\varepsilon(\partial\Omega)$ l'ensemble des trous de $\mathbb{R}^n$ défini par

$$T_b^\varepsilon(\partial\Omega) = T_\varepsilon^\star(\partial\Omega) \cup S_\varepsilon(\partial\Omega),$$

avec

$$\begin{cases} T_\varepsilon^\star(\partial\Omega) = \bigcup_{k\in\mathbb{Z}^n} \{\varepsilon\{\mathcal{T}^\star + k_l b_l\} \text{ tq } \varepsilon\{\mathcal{T}^\star + k_l b_l\} \nsubseteq \Omega \text{ et } \varepsilon\{\mathcal{T}^\star + k_l b_l\} \cap \Omega \neq \emptyset\}, \\ S_\varepsilon(\partial\Omega) = \bigcup_{k\in\mathbb{Z}^n} \varepsilon\{\varepsilon\{\mathcal{S}_\varepsilon + k_l b_l\} \text{ tq } \varepsilon\{\mathcal{S}_\varepsilon + k_l b_l\} \nsubseteq \Omega \text{ et } \varepsilon\{\mathcal{S}_\varepsilon + k_l b_l\} \cap \Omega \neq \emptyset\}. \end{cases}$$

Notons $\chi_{Y^\star\backslash Y_1}$ (resp. χ_{Y_1} , $\chi_{Z^\star}$, $\chi_{T_b^\varepsilon(\partial\Omega)\cap\Omega}$) la fonction caractéristique de $Y^\star\backslash Y_1$ (resp. Y_1, $Z^\star$, $T_b^\varepsilon(\partial\Omega)\cap\Omega$) dans Y (resp. Y , Z, Ω) et prolongeons $\chi_{Y^\star\backslash Y_1}$ et χ_{Y_1} (resp. $\chi_{Z^\star}$) par Y-périodicité (resp. Z-périodicité) à $\mathbb{R}^n$. Alors,

$$\chi_{\Omega_\varepsilon}(x) = \chi_{Y^\star\backslash Y_1}(\frac{x}{\varepsilon}) + \chi_{Y_1}(\frac{x}{\varepsilon})\chi_{Z^\star}(\frac{x}{\varepsilon\delta_\varepsilon}) + \chi_{(T_b^\varepsilon(\partial\Omega)\cap\Omega)}(x), \quad \text{p.p. dans } \Omega. \tag{C.2}$$

D'une part, on a

$$\begin{cases} i) \ \chi_{Y^\star\backslash Y_1}(\frac{x}{\varepsilon}) \rightharpoonup \frac{|Y^\star\backslash Y_1|}{|Y|} \ \ dans\ L^\infty(\mathbb{R}^n)\ faible \star, \\ ii) \ \chi_{Z^\star}(\frac{x}{\delta_\varepsilon}) \rightharpoonup \frac{|Z^\star|}{|Z|} \ \ dans\ L^\infty(\mathbb{R}^n)\ faible \star, \end{cases} \tag{C.3}$$

où l'on a appliqué le lemme 25 respectivement

√ pour la fonction Y-périodique $h_\varepsilon(y) \doteq \chi_{Y^\star\backslash Y_1}(y)$,

√ pour la fonction Z-périodique $h_{\delta_\varepsilon}(z) \doteq \chi_{Z^\star}(z)$.

De plus, on a

$$\begin{aligned} \mathcal{M}_Y\left(\chi_{Y_1}(y)\chi_{Z^\star}(\frac{y}{\delta_\varepsilon})\right) &= \frac{1}{|Y|}\int_Y \chi_{Y_1}(y)\chi_{Z^\star}(\frac{y}{\delta_\varepsilon})\, dy \\ &= \frac{1}{|Y|}\int_{Y_1} \chi_{Z^\star}(\frac{y}{\delta_\varepsilon})\, dy, \end{aligned}$$

ce qui, grâce à (C.3)ii), implique

$$\mathcal{M}_Y\left(\chi_{Y_1}(y)\chi_{Z^\star}(\frac{y}{\delta_\varepsilon})\right) \longrightarrow \frac{|Y_1|}{|Y|}\frac{|Z^\star|}{|Z|}.$$

En appliquant le lemme 25 pour la fonction Y-périodique $h_\varepsilon(y) \doteq \chi_{Y_1}(y)\chi_{Z^\star}(\frac{y}{\delta_\varepsilon})$, il en découle

$$\chi_{Y_1}(\frac{x}{\varepsilon})\chi_{Z^\star}(\frac{x}{\varepsilon\delta_\varepsilon}) \rightharpoonup \frac{|Y_1|}{|Y|}\frac{|Z^\star|}{|Z|} \ \ dans\ L^\infty(\mathbb{R}^n)\ faible\ \star. \tag{C.4}$$

D'autre part, on a pour tout $p \geq 1$

$$\begin{aligned} \int_\Omega \left(\chi_{(T_b^\varepsilon(\partial\Omega)\cap\Omega)}\right)^p dx &= \int_\Omega \chi_{(T_b^\varepsilon(\partial\Omega)\cap\Omega)} dx \\ &= |T_b^\varepsilon(\partial\Omega)\cap\Omega| \\ &\leq |C_\varepsilon(\partial\Omega)| \\ &= |\varepsilon Y| N^\varepsilon(\partial\Omega) \\ &= \varepsilon^n |Y| N^\varepsilon(\partial\Omega). \end{aligned}$$

Puisque $|\partial\Omega| = 0$, on en déduit grâce au lemme 35

$$\chi_{(T_b^\varepsilon(\partial\Omega)\cap\Omega)} \to 0 \ \ dans\ L^p(\Omega) \text{ fort.} \tag{C.5}$$

Ainsi, de (C.2), (C.3)i), (C.4) et (C.5) on obtient (C.1).

d) Dans le théorème 65 (chapitre 3) qui donne une estimation ponctuelle de la difference de deux H_e^0-limites nous avons supposé que

$$\chi_1^\varepsilon - \chi_2^\varepsilon \to 0 \quad \text{fortement dans } L^1(\omega).$$

Si nous remplaçons cette hypothèse par

$$\chi_1^\varepsilon - \chi_2^\varepsilon \to \chi^0 \quad \text{fortement dans } L^1(\omega),$$

alors nous aurons

$$|A^0 - B^0| \leq \sqrt{\frac{\beta\beta'}{\alpha\alpha'}}\, h^0 + (\beta + \beta')\chi^0 \ \text{ p.p. dans } \omega.$$

Pour le montrer il suffit de suivre la même démarche que dans la démonstration du théorème 65, en remplaçant le résultat obtenu dans la troisième étape de la démonstration par

$$\begin{aligned} \limsup_{\varepsilon\to 0} |J^\varepsilon| &\leq \int_\omega \psi\chi^0 A^0 \Lambda.\Lambda.dx + b\int_\omega \psi\chi^0 B^0 \Lambda.\Lambda dx \\ &\leq (\beta + \beta') \int_\omega \psi\chi^0 \Lambda.\Lambda.dx. \end{aligned}$$

Ceci est une conséquence de l'hypothèse de régularité (3.59), le théorème 59 de type div-rot et évidement de la nouvelle hypothèse.

Perspectives

1) Je voudrais utiliser les estimations montrées dans le chapitre 3 pour étudier l'homogénéisation périodique, en milieux composites et perforés, du système de l'élasticité linéarisée dont le tenseur A^ε est donné par

$$A^\varepsilon(x) = A(x, \frac{x}{\varepsilon})$$

avec $A(x, y)$ est Y-périodique en y (comme pour le cas scalaire elliptique sans trous traité dans [24]).

2) Je voudrais étudier l'homogénéisation et les correcteurs pour le système de l'élasticité linéarisée dynamique dans les milieux perforés par des trous e-admissibles.

3) Je voudrais aborder le problème de donner une estimations de type Meyers pour l'élasticité linéarisée avec des conditions mixte déplacement-traction. Une chose qui n'est pas facile à faire car les techniques standards utilisées pour le cas des équations elliptiques du deuxième ordre ne s'adaptent pas pour l'élasticité. Ceci permettrait de traiter l'homogénéisation du système de l'élasticité linéarisée dans des domaines perforés par les trous non répartis périodiquement définis dans [8], c'est à dire des sphères identiques de taille ε centrées sur un réseau non périodique. Ceci permettrait à son tour d'autre applications du théorème 29 donné dans le chapitre 2 pour traiter l'homogénéisation dans certains cas de domaines perforés à plusieurs échelles de trous avec au moins une échelles non périodique. Par exemple, on pourrait envisager le cas des domaines perforés avec deux échelles microscopiques, la plus grande échelle est périodique et la petite est faite par les trous définies dans [8].

4) Enfin, comme la H^0_e-convergence traite les systèmes de l'élasticité dans les domaines perforés par des trous e-admissibles on est conduit à s'intéresser à cette admissibilité. La e-admissibilité est liée à l'existence d'opérateurs de prolongement uniformément bornés. Alors, ça serait intéressant de répondre à une question similaire à celle posé et traitée initialement par P. Donato et A. Damlamian dans [18] pour les équations elliptiques du deuxième ordre, à savoir (dans notre cas), quelles sont les trous qui sont admissibles pour l'homogénéisation périodique avec une condition de traction ? ceci dans le but de donner une présentation générale de l'homogénéisation du système de l'élasticité linéarisée de traction dans des domaines perforés, où la forme du trou de référence varie avec la taille de la période, en considérant les deux cas de l'existence et de la non existence des opérateurs de prolongement bornés uniformément par rapport au paramètre d'homogénéisation.

Annexe : Invariance de l'indice de l'opérateur de Lamé $-div\ e(\cdot)$

Dans cette partie nous démontrons la proposition suivante, qui est une application des résultats montrés par G. Geymonat dans [28].

Proposition 82 *Soient* $1 < p < \infty$ *et* Ω *un domain borné de* $\mathbb{R}^n$ *de classe* $\mathcal{C}^2$. *Alors, l'opérateur de Lamé* $\mathbb{E}\cdot \doteq dive(\cdot)$ *défini de* $W_0^{1,p}(\Omega)^n \cap W^{2,p}(\Omega)^n$ *dans* $L^p(\Omega)^n$ *admet un indice* $\chi(\mathbb{E})$ *et cet indice est indépendant de* p.

Rappelons d'abord quelques définitions et le résultat donnés par G. Geymonat dans [28]. Posons pour tout $p \in \mathbb{N}^n$, $i, k \in \{1, ..., n\}$ et $\vartheta \in \mathbb{C}^n$

$$\begin{cases} \vartheta^p = \vartheta_1^{p_1}...\vartheta_n^{p_n}, \\ D^p = D_{x_1}^{p_1}...D_{x_n}^{p_n}, \\ D_{x_i}^k = \frac{\partial^k}{\partial x_i^k}, \end{cases}$$

et soient Ω un ouvert borné de $\mathbb{R}^n$ de classe $\mathcal{C}^2$ et $\mathbb{A}$ un $(n \times n)$-opérateur différentiel. Supposons qu'il existe $s_1, ...s_n, t_1, ..., t_n \in \mathbb{Z}$ tel que

$$\begin{cases} (\mathbb{A}u)_i \doteq l_{ij}u_j \\ l_{ij}(x, D) = \sum\limits_{|p| \leq s_i + t_j} B_{ijp}D^p, , \quad dans\ \Omega, \quad \forall i = 1, ..., n. \end{cases} \tag{A.1}$$

avec $l_{ij} = 0$ si $s_i + t_j < 0$. Posons maintenant

$$\begin{cases} l^0(x, D) = \left(\sum\limits_{|p| = s_i + t_j} B_{ijp}D^p \right)_{1 \leq i,j \leq n}, \\ L^0(x, \vartheta) = det\ l^0(x, \vartheta), \ \ pour\ tout\ \vartheta \in \mathbb{C}^n. \end{cases}$$

Définition 83 *On dit que* $\mathbb{A}$ *est elliptique si*

$$L^0(x, \xi) \neq 0, \ \forall \xi \in \mathbb{R}^n \backslash \{0\}, \ \forall x \in \Omega.$$

Définition 84 *Supposons que*

$$\sum_{1 \leq i,j \leq n} (s_i + t_j) = 2r \tag{A.2}$$

avec $r > 0$. Alors, on dit que $\mathbb{A}$ est proprement elliptique si, pour tout deux vecteurs $\xi,\ \eta \in \mathbb{R}^n \setminus \{0\}$ linéairement indépendants, le polynôme P en τ défini par

$$P(\tau) \equiv L^0_{ij}(x, \xi + \tau\eta) \tag{A.3}$$

a exactement r racines à parties imaginaires strictement positives.

Remarque 85 *Les définitions précédentes sont plus restrictives que celles données par G. Geymonat dans [28].*

Définition 86 *Soit φ un opérateur linéaire d'un espace vectoriel **E** vers un autre **F** et soit $Coker\varphi$ l'espace quotient $F/Im\varphi$. Alors, si $dim\ Ker\varphi < \infty$ et $dim\ Coker\varphi = cod_F Im\varphi < \infty$, on définit l'indice de φ par*

$$\chi(\varphi) = dim\ Ker\varphi - cod_F Im\varphi.$$

Théorème 87 *Soient $1 < p < \infty$ et Ω un domain borné de $\mathbb{R}^n$ de classe $\mathcal{C}^2$. Si $\mathbb{A} : W_0^{1,p}(\Omega)^n \cap W^{2,p}(\Omega)^n \to L^p(\Omega)^n$ est elliptique et proprement elliptique. Alors, l'indice $\chi(A)$ existe et il est indépendant de p.*

Remarque 88 *Ce théorème est un cas particulier du théorème 3.5 donné par G. Geymonat dans [28].*

Démonstration de la proposition 82. On a pour tout $i = 1, ..., n$

$$\begin{aligned}(\mathbb{E}u)_i &= (-div\ e(u))_i \\ &= -\frac{1}{2}\Delta u_i - \frac{1}{2}div(D_{x_i}u) \\ &= -\frac{1}{2}(D^2_{x_1}u_i + ... + D^2_{x_n}u_i + D_{x_1}D_{x_i}u_1 + ... + D_{x_n}D_{x_i}u_n).\end{aligned}$$

Donc, $(\mathbb{E}u)_i = l_{ij}(x, D)u_j$ avec

$$l_{ij}(x, D) = -\frac{1}{2}(\delta_{ij}D^2_{x_1} + ... + \delta_{ij}D^2_{x_n} + \delta_{1j}D_{x_1}D_{x_i} + ... + \delta_{nj}D_{x_n}D_{x_i}),$$

ce qui implique que $\mathbb{E}$ peut s'écrire sous la forme (A.1) avec $t_i + s_j = 2$ (donc (A.2) est vérifie avec $r = n$). Alors, pour tout $\vartheta \in \mathbb{C}^n$, on a

$$l^0(x, \vartheta) = \frac{1}{2}\begin{pmatrix} |\vartheta|^2 + \vartheta_1^2 & \vartheta_1\vartheta_2 & ... & \vartheta_1\vartheta_n \\ \vartheta_2\vartheta_1 & |\vartheta|^2 + \vartheta_2^2 & ... & \vartheta_2\vartheta_n \\ \vdots & \vdots & & \vdots \\ \vartheta_n\vartheta_1 & \vartheta_n\vartheta_2 & ... & |\vartheta|^2 + \vartheta_n^2 \end{pmatrix}.$$

Ainsi, si $L^0(x, \vartheta) \doteq \det l^0(x, \vartheta) = 0$, il existe $\theta \in \mathbb{C}^n \setminus \{0\}$ tel que $l^0(x, \vartheta)\theta = 0_{\mathbb{R}^n}$. Par conséquent,

$$\theta_i|\vartheta|^2 + \vartheta_i \sum_{1 \le j \le n} \vartheta_j\theta_j = 0, \quad \forall i \in \{1, ..., n\},$$

d'où

$$\exists \theta \in \mathbb{C}^n \setminus \{0\}, \quad |\theta|^2|\vartheta|^2 + \left(\sum_{1\le j\le n} \vartheta_j\theta_j\right)^2 = 0. \tag{A.4}$$

Par ailleurs, si on suppose qu'il existe $\xi,\ \eta \in \mathbb{R}^n \setminus \{0\}$ linéairement indépendants tels que le polynôme P donné par (A.3) admet une racine réel ϱ, alors en vertu de (A.4), il existe $\theta \in \mathbb{C}^n \setminus \{0\}$ tel que

$$|\theta|^2|\xi + \varrho\eta|^2 + \left(\sum_{1\le j\le n} \theta_j(\xi + \varrho\eta)_j\right)^2 = 0.$$

Puisque ϱ est un nombre réel et $\theta \neq 0$, il en découle

$$\xi + \varrho\eta = 0,$$

ce qui contredit le fait que les vecteurs ξ et η sont linéairement indépendants. Ainsi P n'admet pas de racines réels, ce qui implique :
(i) $\mathbb{E}$ est elliptique, puisque $\tau = 0$ n'est pas une racine de P,
(ii) $\mathbb{E}$ est proprement elliptique car $P \in \mathbb{R}_{2n}[X]$, donc P admet exactement n racines à parties imaginaires strictement positives.
Ce qui, grâce au théorème 87, achève la démonstration. ■

Corollaire 89 *Soient Ω un domaine borné de $\mathbb{R}^n$ de classe $\mathcal{C}^2$. Alors, l'opérateur de Lamé $\mathbb{E}$ est un isomorphisme de $W^{2,p}(\Omega)^n \cap W_0^{1,p}(\Omega)^n$ dans $L^p(\Omega)^n$ pour tout $p \in [2,\infty[$.*

Démonstration. Pour qu'il n'y ait pas de confusion, notons dans cette démonstration l'opérateur $\mathbb{E}$: $W_0^{1,p}(\Omega)^n \cap W^{2,p}(\Omega)^n \to L^p(\Omega)^n$ par $\mathbb{E}_p$. Du théorème 82, il vient

$$\chi(\mathbb{E}_p) = \chi(\mathbb{E}_2), \quad \forall p \in [2,\infty[. \tag{A.5}$$

Comme le système

$$\begin{cases} -div\ e(u) = 0 \quad dans\ \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{A.6}$$

admet une unique solution, alors $\chi(\mathbb{E}_2) = 0$, ce qui avec (A.5) donne

$$\chi(\mathbb{E}_p) = 0, \quad \forall p \in [2,+\infty[.$$

Ainsi,

$$dim\ Ker\mathbb{E}_p = cod\ Im\mathbb{E}_p. \tag{A.7}$$

Par ailleurs, si $u \in Ker\mathbb{E}_p$, alors

$$\begin{cases} -div\ e(u) = 0, \quad dans\ \Omega, \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

Or, pour tout $p \in [2,\infty[$, on a $W_0^{1,p}(\Omega) \subset W_0^{1,2}(\Omega)$, donc u satisfait (A.6), d'où $u = 0$. On en déduit $Ker\mathbb{E}_p = \{0\}$, d'où, grâce à (A.7), le résultat. ■

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