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Par
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Thème

Méthode d'inférence pour quelques modèles de séries chronologiques à valeurs entières périodiques

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Inference method for some periodic integer-valued time series models

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Abstract

The objective of this thesis is the inference method for some periodic integer-valued time series models, namely, the periodic self exciting threshold integer valued autoregressive PSETINAR$(2;1)$ model, the periodic negative binomial self-exciting threshold integer-valued autoregressive PNBSETINAR$(2;1)$ model, which are more dynamic and well-suited to capture a wide range of empirical characteristics seen in count time series, such as hidden periodicity in autocovariance structures and structural regime changes, and the periodic negative binomial integer-valued generalized autoregressive conditional heteroskedastic PNBINGARCH$(1,1)$ model. For each of the proposed models, we enhance the study of some probabilistic and statistical properties and the development of different estimation methods. In addition, the simulation studies are considered to illustrate some theoretical properties with applications on real datasets.

Keywords and phrases: Integer-valued time series models; Periodic SETINAR model; periodic NBSETINAR model; periodic negative binomial INGARCH$(1,1)$ model; integer-valued process; periodic stationarity in mean and variance.

Résumé

L'objectif de cette thèse est la méthode d'inférence pour certains modèles de séries temporelles à valeurs entières périodiques, à savoir, le modèle auto-excitant autorégressive à seuil à valeurs entières périodique PSETINAR$(2;1)$, et le modèle binomial negatif auto-excitant autorégressive à seuil à valeurs entières périodique PNBSETINAR$(2;1)$, qui sont plus flexibles et aptes à capturer plusieurs caractéristiques empiriques des séries chronologiques de comptages, en particulier la périodicité cachée dans les structures d'autocovariance et les changements de régime structurels, ainsi que le modèle binomial negatif hétéroscédastique conditionnel autorégressif généralisé à valeur entière periodique PNBINGARCH$(1,1)$. Pour chacun des modèles proposés, nous avons amélioré l'étude de certaines propriétés probabilistes et statistiques, et le développement de différentes méthodes d'estimation. En outre, les études de simulation sont considérées pour illustrer certaines propriétés théoriques avec des applications sur des données réelles.

Mots-clés et phrases: Modèle SETINAR périodique; modèle NBSETINAR périodique; modèle binomial négatif INGARCH$(1,1)$ périodique; processus à valeur entière; stationnarité périodique en moyenne et en variance.

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Dedicate

This thesis work is dedicated to

*my great **parents**,*

*my beloved **brothers**,*

*my beloved **sisters**,*

and those who are dear to me.

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List of abbreviations and notations

ARCH	Autoregressive Conditional Heteroskedastic.
ACF	Autocorrelation Function.
ARIMA	Autoregressive Integrated Moving Average.
ARMA	Autoregressive Moving Average.
BVNB	Bivariate Negative Binomial.
CLS	Conditional Least Squares.
CML	Conditional Maximum Likelihood.
D-NeSS	Doubly Nested Sub-Sample Search.
GARCH	Generalized Autoregressive Conditional Heteroskedastic.
INAR	Integer Valued Autoregressive.
INARMA	Integer Valued Autoregressive Moving Average.
INGARCH	Integer-Valued Generalized Autoregressive Conditional Heteroskedastic.
MINARCH	Mixture Integer Valued Autoregressive Conditional Heteroskedastic.
NBINGARCH	Negative Binomial Integer Valued Generalized Autoregressive Conditional Heteroskedastic.
NBSETINAR	Negative Binomial Self Exciting Integer Valued Autoregressive.
NeSS	Nested Sub-Sample Search.
PAR	Periodic Autoregressive.
PACF	Partial Autocorrelation Function.
PARMA	Periodic Autoregressive Moving Average.
PINAR	Periodic Integer Valued Autoregressive.
PINARMA	Periodic Integer Valued Autoregressive Moving Average.
PNBINGARCH	Periodic Negative Binomial Integer-Valued Generalized Autoregressive Conditional Heteroskedastic.
PNBSETINAR	Periodic Negative Binomial Self Exciting Integer Valued Autoregressive.
PSETINAR	Periodic Self Exciting Threshold Integer Valued Autoregressive.
PSETINARMA	Periodic Self Exciting Threshold Integer Valued Autoregressive Moving Average.
RMS	Root Mean Squares.
RMSE	Root Mean Squares Error.
SETINAR	Self Exciting Threshold Integer Valued Autoregressive.

SSE	Sum of squared errors.
TAR	Threshold Autoregressive.
YW	Yule-Walker.
percent.	Percentage.
med.	Median.
a.s.	Almost surely.
a.e.	Almost everywhere.
$\overset{\mathcal{L}}{\rightsquigarrow}$	Convergence in distribution (law).
$\mathcal{P}\left(.\right)$	Poisson distribution.
$\mathcal{NB}\left(.\right)$	Negative binomial distribution.
$\chi^2_{\alpha,n}$	α quantile of a chi-squared distribution with n degrees of freedom.

عَلَى بَرَكَةِ اللهِ

Upon the blessing of Allah

Chapter 1

General Introduction

1.1 Introduction

Over the last three decades, the integer-valued time series model has attracted much attention and interest, with a considerable number of studies appearing in the literature. Indeed, the class of integer-valued autoregressive $INAR$ models, based on the binomial thinning operator, introduced by McKenzie (1985), and by Al-Osh and Alzaid (1987), was widely used in many applications, such as statistical quality control (Weiß 2007) and assurance (Zhu and Joe 2006). In addition, several models have been proposed to describe certain phenomena often observed in integer-valued time series, for example, excess of zeros, changes in volatility over time, multimodality, low counts, overdispersion, and so on, which are encountered in several areas such as epidemiology (e.g., number of infected individuals), economic (e.g., discrete transaction price movements on financial markets), environmental (e.g., number of forest fires in a certain country), criminology (e.g., counts of a certain type of crimes), and many others. Among the models that are proposed in order to capture the above phenomena, we quote the integer-valued $INAR(1)$ processes with zero-inflated Poisson innovations introduced by Jazi *et al.* (2012), the integer-valued generalized autoregressive conditional heteroskedastic $GARCH$, and the integer-valued autoregressive conditional heteroskedastic $ARCH$ models were proposed by Ferland *et al.* (2006), and Zhu *et al.* (2010) who extended the class of integer-valued $ARCH$ model to the mixture integer-valued $ARCH$ model, noted in short by $MINARCH$, which has advantages over the extended one due to its ability to handle multimodality and non-stationary components. However, it seems that the proposed classes of models, which are suitable for integer-valued time series with time-invariant parameters, cannot correctly report and describe the periodicity characteristic of counting time series. Therefore, many researchers have suggested other extensions of these models that take into account the periodicity feature. As my thesis deals with this topic, it seems useful to mention, among many other achievements, and without pretending to be exhaustive, the work of Monteiro *et al.* (2010) on the periodic autoregressive model with integer values of order $PINAR(1)$, driven by a periodic sequence of independent Poisson distributed random variables, Morinã *et al.* (2011) present a model based on two-order integer-valued autoregressive time series to analyze the number of hospital emergency service arrivals, and most recently, Bentarzi and Bentarzi (2017*a*) gave some probabilistic and statistical properties of a periodic integer-valued diagonal bilinear model, even more for the periodic integer-valued $GARCH(1,1)$ model by Bentarzi and Bentarzi (2017*b*), and Ouzzani and Bentarzi (2019) studied the mixture periodic integer-valued autoregressive conditionally heteroskedastic models, also some particular models of the class of periodic integer-valued autoregressive moving average, $PINARMA(p,q)$ models were duly addressed by Bentarzi and Aries (2020*a*), and many other works.

In fact, this improvement seen in the literature by many researchers, cited above, can be seen as motivation and aspiration to propose, in this thesis, some integer-valued time series models, more flexible to capture and describe the periodicity feature. Although the self-exciting threshold integer-valued autoregressive model, with time-invariant coefficients, presented by Thyregod *et al.* (1999) and Monteiro *et al.* (2012), has the ability to deal with time series of counts exhibiting piecewise-type patterns, even so, it remains unable to handle the periodicity hidden in the autocovariance function. That is a good enough reason to give a more general model with a periodic coefficient to model the phenomenon at hand, see Manaa and Bentarzi (2021*a*). Another achievement on the periodic negative binomial self-exciting threshold integer valued autoregressive model, with the negative binomial thinning operator, see Manaa and Bentarzi (2021*b*), which extended, in the time-invariant model, the work of Yang *et al.* (2018*b*). In addition, we have developed the negative binomial integer-valued $GARCH\,(1,1)$ model, defined by Zhu (2011), in the more general framework with periodic coefficients, as an extension of the periodic integer valued $GARCH(1;1)$ model, introduced by Bentarzi and Bentarzi (2017*b*), which is more appropriate and has more flexibility to model such periodically correlated processes. This manuscript consists mainly of working on *"Inference method for some periodic integer-valued time series models"*, and it is thematically, according to the results mentioned previously, divided into Three Chapters:

1.2 Presentation and contributions of the thesis

Chapter 2 : Periodic SETINAR Model

In this Chapter, we introduce a periodic self-exciting threshold integer-valued autoregressive $PSETINAR$ model without knowing the threshold parameters. In the first place, we provide the definitions, basic notations and main assumptions concerning the proposed model. Thereafter, we establish the first and the second moment periodically stationary conditions, under these conditions, the closed forms of these moments are derived. Besides, the existence of high moment and the strict periodic stationarity, are studied. The autocovariance structure is also acquired. Then, we apply the conditional least squares (CLS) and the conditional maximum likelihood (CML) methods to estimate the underling parameters. Furthermore, the unknown threshold parameters are estimated by using the periodic adaptation of Nested Sub-Sample Search ($NeSS$) algorithm. Finally, the performance of the proposed estimation methods is shown via a simulation study, and an application on real data set was provided.

Chapter 3 : Periodic Negative Binomial SETINAR Model

The aim of this Chapter is to present the periodic negative binomial self exciting integer valued autoregressive $PNBSETINAR$ model. Beforehand, we provide the basic notations, definitions and main assumptions concerning the desired model. Then, we established the first and the second moment periodically stationary conditions, while establishing, under these conditions, their closed forms. In addition, the existence of high moment and the strict periodic stationarity, are studied, and the autocovariance structure is also acquired. After that, the unknown periodic parameters of our models are estimated via the conditional least squares (CLS) and the conditional maximum likelihood (CML) estimation methods. Furthermore, the unknown periodic threshold parameters are estimated by using the periodic adaptation of ($NeSS$) algorithm. Finally, the performance of the proposed estimation methods is shown via a simulation study and an application on real data set was provided.

Chapter 4 : Periodic Negative Binomial $INGARCH(1,1)$ Model

This Chapter aims at presenting the periodic negative binomial integer-valued generalized autoregressive conditional heteroskedastic model, noted in short by $PNBINGARCH\,(1,1)$. Initially, we provide the basic notations and definitions concerning the proposed model. After that, we study the periodically stationary problem of the proposed model. The necessary and sufficient conditions for the periodically stationary in mean and in variance are established. Furthermore, the closed-form expressions for both the mean and the variance are, under these conditions, obtained. Then, the existence of higher moments and their calculations are considered. Moreover, the study of the autocovariance structure of the underlying periodic model while providing the explicit expression of the autocorrelation function. The unknown periodic parameters are estimated via the Yule-Walker (YW), the conditional least squares (CLS), and the conditional maximum likelihood (CML) methods. Furthermore, a simulation study and an application on real data set are provided.

Chapter 5 : Conclusion and Perspectives

This chapter highlights the key results from each chapter while also setting them in the perspective of future research.

The following research articles are used in part to support this thesis

Chapter 2 : Manaa, A., and Bentarzi, M. (2021*a*). On a periodic SETINAR model. *Communications in Statistics-Simulation and Computation*, 1-25. (Published).

Chapter 3 : Manaa, A., and Bentarzi, M. (2021*b*). On a periodic negative binomial SETINAR model. *Communications in Statistics-Simulation and Computation*, 1-27. (Published).

Chapter 4 : Manaa, A., and Bentarzi, M. (2021*c*) On periodic negative binomial INGARCH(1,1) model. *Communications in Statistics-Simulation and Computation*.

Chapter 2

Periodic SETINAR Model

2.1 Introduction

It is well known, nowadays, that the analysis of the discrete-time series has received growing attention in linear and non-linear non-negative integer-valued time series models (see, Al-Osh and Alzaid 1987, Alzaid and Al-Osh 1990, Du and Li 1991, Silva and Oliveira 2000, Zhu *et al.* 2010, Zhu 2011, and many others). Moreover, it seems that the class of nonlinear models which is the most theoretically studied and practically employed, in the analysis of the discrete-time series, is the class of threshold autoregressive. Thyregod *et al.* (1999) introduced, in the analysis of integer time series, a class of self-exciting threshold $INAR$ model to analyze tipping bucket rainfall measurements and to capture the high threshold exceedances appearing in clusters, and the so-called piece-wise phenomenon. As well as, Monteiro *et al.* (2012) presented a particular class of self-exciting threshold integer valued autoregressive $SETINAR(2;1)$ model, driven by independent Poisson distributed random variables. And recently, the negative binomial self-exciting threshold integer-valued autoregressive model, based on the negative binomial thinning operator, was introduced by Yang *et al.* (2018*b*). It is recognized that many economic, financial and environmental integer-valued time series, which have encountered in practice, reveal the periodicity feature in their autocovariance structures (e.g., *number of Campylobacteriosis infections time series* studied by Ferland *et al.* 2006, *number of hospital emergency service arrivals,* studied by Morinã *et al.* 2011 and *the daytime road accidents in Schiphol area, in the Netherlands for the year* 2001, studied by Pedeli and Karlis 2011). However, from the works done in the literature, it seems that the study of periodic integer-valued time series models has not received much attention. Indeed, as far as we know, Monteiro *et al.* (2010) and Morinã *et al.* (2011) were the first who worked on the modeling of the periodically correlated integer-valued process, in the Gladyshev's sense (1961), to capture and describe the periodicity feature exhibited by the autocovariance structure and which cannot be accounted by the classical time-invariant parameters time series models. Furthermore, the research has made progress on the periodically correlated integer-valued models see as examples, Bentarzi and Bentarzi (2017*a*), (2017*b*), Ouzzani and Bentarzi (2019), Sadoun and Bentarzi (2019) and among others. The main concern behind this chapter is to study the class of periodic integer-valued threshold autoregressive $PSETINAR\,(2;1)$ model, which was presented by Pereira *et al.* (2015), in the special case where the periodic threshold parameters are unknown.

The rest of this chapter is organized as follows. In Section 2.2, we provide the basic notations, definitions and main assumptions concerning the periodic $SETINAR\,(2;1)$ model. In Section 2.3, we discuss its basic probabilistic and statistical properties. Indeed, the first and second moment periodically stationary conditions are established. The closed-forms of these moments are, under these conditions, derived. Besides, the existence of high moment and the strict periodic stationarity, are studied. The autocovariance structure is also acquired. The Conditional Least Squares (CLS) and Conditional Maximum Likelihood (CML) parameter estimations problem are considered in Section 2.4. Furthermore, the threshold parameters are estimated by using the periodic adaptation Nested Sub-Sample Search ($NeSS$) algorithm. In Section 2.5, the performance of the proposed estimation methods is shown via a simulation study and an application on real data set was provided.

2.2 Notations, definitions and main assumptions

Recall that the self-exciting threshold integer-valued autoregressive $SETINAR\,(2;1)$ model of order one with two regimes, takes the form :

$$y_t = (\alpha_1 \circ y_{t-1})\, I_{t-1}^{(1)}(c) + (\alpha_2 \circ y_{t-1})\, I_{t-1}^{(2)}(c) + \varepsilon_t, \quad t \in \mathbb{Z}. \tag{2.2.1}$$

The integer-valued stochastic process $\{y_t;\ t \in \mathbb{Z}\}$ is said to follow a periodic self-exciting threshold integer-valued autoregressive model of order one with two regimes and period S $(S \geq 2)$, if it is a solution of the following non-linear difference stochastic equation :

$$y_t = (\alpha_{1,t} \circ y_{t-1})\, I_{t-1}^{(1)}(c_t) + (\alpha_{2,t} \circ y_{t-1})\, I_{t-1}^{(2)}(c_t) + \varepsilon_t, \quad t \in \mathbb{Z}, \tag{2.2.2}$$

with $I_{t-1}^{(1)}(c_t)$ is a sequence of Bernoulli random variables defined by :

$$I_{t-1}^{(1)}(c_t) = \begin{cases} 1 & \text{if} \quad y_{t-1} \leq c_t \\ 0 & \text{if} \quad y_{t-1} > c_t, \end{cases} \quad \text{and } I_{t-1}^{(2)}(c_t) = 1 - I_{t-1}^{(1)}(c_t), \tag{2.2.3}$$

where the periodic threshold parameters c_t are assumed to be unknown. The underlying non-negative integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$, is a periodically correlated, in the sense of Gladyshev (1961), with a period S $(S \geq 2)$, and the innovation process $\{\varepsilon_t;\ t \in \mathbb{Z}\}$ is a sequence of independent non-negative integer-valued random variables with Poisson distribution $\mathcal{P}(\lambda_{\varepsilon,t})$. The symbol " $\circ$ " stands, as usual, for the thinning *Steutel-van Harn* (1979) operator, which is defined, for the non-negative integer-valued stochastic process y_{t-1} and any counting periodic sequences of independent non-negative integer-valued random variables $\left\{Z_{j,t}^{(i)}, t \in \mathbb{Z}, j \in \mathbb{N}\right\}$, as follows :

$$\alpha_{i,t} \circ y_{t-1} = \begin{cases} \sum_{j=1}^{y_{t-1}} Z_{j,t}^{(i)} & \text{if} \quad y_{t-1} > 0, \\ 0 & \text{if} \quad y_{t-1} = 0, \end{cases}$$

with $Z_{j,t}^{(i)}$, while $i = 1, 2$, and for a fixed t and j, is a sequence of independent and identically distributed Bernoulli random variables independent of y_{t-1}, such that $P\left(Z_{j,t}^{(i)} = 1\right) = 1 - P\left(Z_{j,t}^{(i)} = 0\right) = \alpha_{i,t} \in [0, 1]$, for $i = 1, 2$, $t \in \mathbb{Z}$, and all the unknown parameters $\alpha_{1,t}$, $\alpha_{2,t}$, $\lambda_{\varepsilon,t}$, and c_t are periodic in time with period S, i.e., $\alpha_{1,t+kS} = \alpha_{1,t}$, $\alpha_{2,t+kS} = \alpha_{2,t}$ and $\lambda_{\varepsilon,t+kS} = \lambda_{\varepsilon,t}$. The innovation process $\{\varepsilon_t;\ t \in \mathbb{Z}\}$ is assumed to be independent of y_{t-1} and $\phi_t \circ y_{t-1}$. One can rewrite the periodic self exciting threshold integer valued autoregressive model given by (2.2.2) in the equivalent form :

$$y_t = \phi_t \circ y_{t-1} + \varepsilon_t, \quad t \in \mathbb{Z}, \tag{2.2.4}$$

with $\phi_t = \alpha_{1,t} I_{t-1}^{(1)}(c_t) + \alpha_{2,t} I_{t-1}^{(2)}(c_t)$. Throughout the chapter, we will omit (c_t) in $I_{t-1}^{(k)}(c_t)$, $k = 1, 2$, to make the notation easy without ambiguity. It is worth noting that the periodic self-exciting threshold integer-valued autoregressive $PSETINAR_S(2;1)$ model given by (2.2.2) or equivalently by (2.2.4), was presented, for the first time, by Pereira *et al.*

(2015). These authors attempted to study the probabilistic and the statistical properties of the scalar model (2.2.2), written in its form (2.2.4), by studying the multivariate model arising from the seasonal decomposition approach (one can see Pagano 1978 and Tiao and Grupe 1980).

2.3 Basic properties of the PSETINAR model

In this paragraph, we provide the conditions on the parameters of the underlying integer-valued process to be periodically stationary in the first and the second order. Furthermore, under these conditions, the closed-forms of the periodic mean and the periodic variance are acquired. Moreover, the existence of the unconditional *m-th* moment and the strict periodic stationarity of the process $\{y_t,\ t \in \mathbb{Z}\}$ are established. In addition, the autocovariance structure is also obtained.

2.3.1 Periodic stationarity in the two first moments

The results given in the following propositions establish the necessary and sufficient conditions, for the process $\{y_t;\ t \in \mathbb{Z}\}$ satisfying (2.2.2) to be periodically stationary with respect to the first two order moments. The closed-forms of these moments are then, under these conditions, obtained.

Proposition 2.3.1 *The process $\{y_t,\ t \in \mathbb{Z}\}$, satisfying (2.2.2), is periodically stationary in the first order if and only if, the periodic parameters $\alpha_{1,t}$ and $\alpha_{2,t}$ satisfy the periodically stationary conditions $\prod_{i=1}^{S} \alpha_{1,i} < 1$ and $\prod_{i=1}^{S} \alpha_{2,i} < 1$, respectively, then, the unconditional periodic mean is, under these conditions, given by*

$$\mu_{y,s} = \mathbb{E}\left(y_{s+\tau S}\right) = \left(1 - \prod_{i=1}^{S} \alpha_{2,i}\right)^{-1} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \alpha_{2,s-i+1}\right) \Phi_{1,s-j+1},$$

or equivalently,

$$\mu_{y,s} = \mathbb{E}\left(y_{s+\tau S}\right) = \left(1 - \prod_{i=1}^{S} \alpha_{1,i}\right)^{-1} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \alpha_{1,s-i+1}\right) \Phi_{2,s-j+1},$$

where, $\Phi_{1,s} = \left(\alpha_{1,s} - \alpha_{2,s}\right) \mu_{y,s-1}^{(1,1)} + \lambda_{\varepsilon,s}$, $\Phi_{2,s} = \left(\alpha_{2,s} - \alpha_{1,s}\right) \mu_{y,s-1}^{(2,1)} + \lambda_{\varepsilon,s}$, $p_{1,s} = P\left(y_{s-1+\tau S} \leq c_s\right)$, $p_{2,s} = 1 - p_{1,s}$ *and* $\mu_{y,s}^{(i,m)} = \mathbb{E}\left(I_{s,\tau}^{(i)} y_{s+\tau S}^{m}\right)$, *for* $i = 1, 2$ *and* $m \in \mathbb{N}$.

In the time-invariant $SETINAR$ model, i.e., $S = 1$, the results in Proposition 2.3.1 can be represented by the following corollary.

Corollary 2.3.1 *The process* $\{y_t,\ t \in \mathbb{Z}\}$*, satisfying the* $SETINAR\,(2;1,1)$*, given in* (2.2.1)*, is stationary in the first moment if and only if* $\alpha_1 < 1$ *and* $\alpha_2 < 1$ *then,*

$$\mu_y = \mathbb{E}\,(y_t) = (1-\alpha_2)^{-1}\ \left((\alpha_1-\alpha_2)\,\mu_y^{(1,1)} + \lambda_\varepsilon\right),$$

or equivalently,

$$\mu_y = \mathbb{E}\,(y_t) = (1-\alpha_1)^{-1}\ \left((\alpha_2-\alpha_1)\,\mu_y^{(2,1)} + \lambda_\varepsilon\right).$$

Proof of Proposition *2.3.1*. In the first place, the mean of the process $\{y_t,\ t \in \mathbb{Z}\}$, defined in (2.2.2) denoted by $\mu_{y,t} = \mathbb{E}\,(y_t)$, is calculated as follows

$$\begin{aligned}\mathbb{E}\,(y_t) &= \mathbb{E}\left((\alpha_{1,t}\circ y_{t-1})\,I_{t-1}^{(1)}\right) + \mathbb{E}\left((\alpha_{2,t}\circ y_{t-1})\,I_{t-1}^{(2)}\right) + \mathbb{E}\,(\varepsilon_t)\,,\\ &= \alpha_{1,t}\mathbb{E}\left(y_{t-1}I_{t-1}^{(1)}\right) + \alpha_{2,t}\mathbb{E}\left(y_{t-1}\left(1-I_{t-1}^{(1)}\right)\right) + \lambda_{\varepsilon,t},\\ \mu_{y,t} &= \alpha_{2,t}\mu_{y,t-1} + (\alpha_{1,t}-\alpha_{2,t})\,\mu_{y,s-1}^{(1,1)} + \lambda_{\varepsilon,t},\end{aligned}$$

then, we have

$$\mu_{y,t} = \alpha_{2,t}\mu_{y,t-1} + \Phi_{1,t}, \tag{2.3.1}$$

with, $\Phi_{1,t} = (\alpha_{1,t}-\alpha_{2,t})\,\mu_{y,t-1}^{(1,1)} + \lambda_{\varepsilon,t}$, by iterating the first-order difference equation (2.3.1), m times, while letting $t = s+\tau S$, $s = 1,...,S$, $\tau \in \mathbb{Z}$ and putting, in the expression, $m = S$ then, taking account of the periodicity of the parameters, we obtain, under the condition $\prod_{i=1}^{S}\alpha_{2,i} < 1$, the closed-form of the mean which is given as it is reported in Proposition 2.3.1. It can be also obtained the stationarity condition in the first order : $\prod_{i=1}^{S}\alpha_{1,i} < 1$, while replacing $I_{t-1}^{(1)}$ by $1 - I_{t-1}^{(2)}$. ∎

The following proposition establishes the necessary and sufficient conditions for the model (2.2.2) to be periodically stationary in the second order.

Proposition 2.3.2 *The process* $\{y_t,\ t \in \mathbb{Z}\}$*, satisfying* (2.2.2)*, is periodically stationary in the second order if and only if the periodic parameters* $\alpha_{1,t}$ *and* $\alpha_{2,t}$ *satisfy the periodically stationary conditions* $\prod_{i=1}^{S}\alpha_{1,i} < 1$ *and* $\prod_{i=1}^{S}\alpha_{2,i} < 1$*, respectively, then, the periodic variance is, under these conditions, given by*

$$\sigma_{y,s}^2 = Var\,(y_{s+\tau S}) = \left(1-\prod_{i=1}^{S}\alpha_{2,s-i+1}^2\right)^{-1}\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\alpha_{2,s-i+1}^2\right)\Theta_{1,s-j+1},$$

or equivalently,

$$\sigma^2_{y,s} = Var\left(y_{s+\tau S}\right) = \left(1 - \prod_{i=1}^{S} \alpha^2_{1,s-i+1}\right)^{-1} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \alpha^2_{1,s-i+1}\right) \Theta_{2,s-j+1},$$

with,

$$\begin{aligned}\Theta_{1,s} = \sigma^{2,(1)}_{y,s-1}\left(\alpha^2_{1,s} - \alpha^2_{2,s}\right) + \left(\alpha_{1,s}\left(1-\alpha_{1,s}\right) - \alpha_{2,s}\left(1-\alpha_{2,s}\right)\right)\mu^{(1,1)}_{y,s-1} + \alpha_{2,s}\left(1-\alpha_{2,s}\right)\mu_{y,s-1} + \\ 2\alpha_{2,s}\left(\alpha_{2,s}-\alpha_{1,s}\right)\left(\mu_{y,s-1} - \mu^{(1,1)}_{y,s-1}\right)\mu^{(1,1)}_{y,s-1} + \lambda_{\varepsilon,s}, \\ \Theta_{2,s} = \sigma^{2,(2)}_{y,s-1}\left(\alpha^2_{2,s} - \alpha^2_{1,s}\right) + \left(\alpha_{2,s}\left(1-\alpha_{2,s}\right) - \alpha_{1,s}\left(1-\alpha_{1,s}\right)\right)\mu^{(2,1)}_{y,s-1} + \alpha_{1,s}\left(1-\alpha_{1,s}\right)\mu_{y,s-1} + \\ 2\alpha_{1,s}\left(\alpha_{1,s}-\alpha_{2,s}\right)\left(\mu_{y,s-1} - \mu^{(2,1)}_{y,s-1}\right)\mu^{(2,1)}_{y,s-1} + \lambda_{\varepsilon,s},\end{aligned}$$

where,

$$p_{1,s} = P\left(y_{s-1+\tau S} \leq c_s\right), p_{2,s} = 1 - p_{1,s}, \mu^{(i,m)}_{y,s} = \mathbb{E}\left(I^{(i)}_{s,\tau} y^m_{s+\tau S}\right) \text{ and } \sigma^{2,(i)}_{y,s} = Var\left(I^{(i)}_{s,\tau} y_{s+\tau S}\right). \tag{2.3.2}$$

In the time-invariant $SETINAR$ model, i.e., $S = 1$, the results of Proposition 2.3.2 can be represented by the following corollary.

Corollary 2.3.2 *The process $\{y_t,\ t \in \mathbb{Z}\}$, satisfying the $SETINAR\,(2;1)$, given in (2.2.1), is stationary in the variance if and only if $\alpha_1 < 1$ and $\alpha_2 < 1$ then,*

$$\sigma^2_y = Var\left(y_t\right) = \left(1 - \alpha^2_2\right)^{-1} \Theta_1, \text{ or equivalently, } \sigma^2_y = Var\left(y_t\right) = \left(1 - \alpha^2_1\right)^{-1} \Theta_2,$$

where,

$$\begin{aligned}\Theta_1 &= \sigma^{2,(1)}_y\left(\alpha^2_1 - \alpha^2_2\right) + \left(\alpha_1\left(1-\alpha_1\right) - \alpha_2\left(1-\alpha_2\right)\right)\mu^{(1,1)}_y + \alpha_2\left(1-\alpha_2\right)\mu_y + \\ &\quad 2\alpha_2\left(\alpha_2-\alpha_1\right)\left(\mu_y - \mu^{(1,1)}_y\right)\mu^{(1,1)}_y + \lambda_\varepsilon, \\ \Theta_2 &= \sigma^{2,(2)}_y\left(\alpha^2_2 - \alpha^2_1\right) + \left(\alpha_2\left(1-\alpha_2\right) - \alpha_1\left(1-\alpha_1\right)\right)\mu^{(2,1)}_y + \alpha_1\left(1-\alpha_1\right)\mu_y + \\ &\quad 2\alpha_1\left(\alpha_1-\alpha_2\right)\left(\mu_y - \mu^{(2,1)}_y\right)\mu^{(2,1)}_y + \lambda_\varepsilon.\end{aligned}$$

Proof of Proposition *2.3.2*. The variance of y_t is obtained as follows :

$$\begin{aligned}\sigma^2_{y,t} &= Var\left(y_t\right) = Var\left(\left(\alpha_{1,t} \circ y_{t-1}\right) I^{(1)}_{t-1}\right) + Var\left(\left(\alpha_{2,t} \circ y_{t-1}\right) I^{(2)}_{t-1}\right) + \\ &\quad 2Cov\left(\left(\alpha_{1,t} \circ y_{t-1}\right) I^{(1)}_{t-1}; \left(\alpha_{2,t} \circ y_{t-1}\right) I^{(2)}_{t-1}\right) + \lambda_{\varepsilon,t}.\end{aligned} \tag{2.3.3}$$

The first term, on the right hand side, can be easily calculated as follows :

$$Var\left(\left(\alpha_{1,t} \circ y_{t-1}\right) I^{(1)}_{t-1}\right) = \alpha^2_{1,t} Var\left(I^{(1)}_{t-1} y_{t-1}\right) + \alpha_{1,t}\left(1-\alpha_{1,t}\right)\mathbb{E}\left(I^{(1)}_{t-1} y_{t-1}\right),$$

then taking account of (2.3.2), we obtain :

$$Var\left(\alpha_{1,t}\circ y_{t-1}I_{t-1}^{(1)}\right)=\alpha_{1,t}^{2}\sigma_{y,t-1}^{2,(1)}+\alpha_{1,t}\left(1-\alpha_{1,t}\right)\mu_{y,t-1}^{(1,1)}. \tag{2.3.4}$$

The second term in (2.3.3), can be easily calculated as follows :

$$\begin{aligned}Var\left(\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(2)}\right)&=Var\left(\left(\alpha_{2,t}\circ y_{t-1}\right)\left(1-I_{t-1}^{(1)}\right)\right),\\&=Var\left(\alpha_{2,t}\circ y_{t-1}\right)+Var\left(\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(1)}\right)-2Cov\left(\left(\alpha_{2,t}\circ y_{t-1}\right);\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(1)}\right),\end{aligned}$$

replacing $\alpha_{1,t}$, in (2.3.4), by $\alpha_{2,t}$, one can easily, obtain the term $Var\left(\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(1)}\right)$, indeed, we have :

$$Var\left(\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(1)}\right)=\alpha_{2,t}^{2}\sigma_{y,t-1}^{2,(1)}+\alpha_{2,t}\left(1-\alpha_{2,t}\right)\mu_{y,t-1}^{(1,1)}, \tag{2.3.5}$$

The variance $Var\left(\alpha_{2,t}\circ y_{t-1}\right)$ and the covariance $Cov\left(\alpha_{2,t}\circ y_{t-1};\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(1)}\right)$, are given as follows :

$$Var\left(\alpha_{2,t}\circ y_{t-1}\right)=\alpha_{2,t}^{2}\sigma_{y,t-1}^{2}+\alpha_{2,t}\left(1-\alpha_{2,t}\right)\mu_{y,t-1}, \tag{2.3.6}$$

$$\begin{aligned}Cov\left(\left(\alpha_{2,t}\circ y_{t-1}\right);\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(1)}\right)&=\mathbb{E}\left(\left(\alpha_{2,t}\circ y_{t-1}\right)^{2}I_{t-1}^{(1)}\right)\\&\quad-\mathbb{E}\left(\left(\alpha_{2,t}\circ y_{t-1}\right)\right)\mathbb{E}\left(\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(1)}\right),\\&=Var\left(\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(1)}\right)+\alpha_{2,t}^{2}\mathbb{E}\left(y_{t-1}I_{t-1}^{(1)}\right)\left(\mathbb{E}\left(y_{t-1}I_{t-1}^{(1)}\right)-\mathbb{E}\left(y_{t-1}\right)\right),\\&=Var\left(\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(1)}\right)-\alpha_{2,t}^{2}\mu_{y,t-1}^{(1,1)}\left(\mu_{y,t-1}-\mu_{y,t-1}^{(1,1)}\right),\end{aligned} \tag{2.3.7}$$

where, $Var\left(\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(1)}\right)$ is given in (2.3.5). Similarly, by taking account of (2.3.2) and from (2.3.5), (2.3.6) and (2.3.7), we have :

$$\begin{aligned}Var\left(\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(2)}\right)&=\alpha_{2,t}^{2}\sigma_{y,t-1}^{2}+\alpha_{2,t}\left(1-\alpha_{2,t}\right)\mu_{y,t-1}-Var\left(\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(1)}\right)\\&\quad+2\alpha_{2,t}^{2}\mu_{y,t-1}^{(1,1)}\left(\mu_{y,t-1}-\mu_{y,t-1}^{(1,1)}\right).\end{aligned} \tag{2.3.8}$$

The last term of (2.3.3) can be calculated as follows :

$$\begin{aligned}Cov\left(\left(\alpha_{1,t}\circ y_{t-1}\right)I_{t-1}^{(1)};\left(\alpha_{2,t}\circ y_{t-1}\right)I_{t-1}^{(2)}\right)&=-\alpha_{1,t}\alpha_{2,t}\mathbb{E}\left(y_{t-1}I_{t-1}^{(1)}\right)\left(\mathbb{E}\left(y_{t-1}\right)-\mathbb{E}\left(y_{t-1}I_{t-1}^{(1)}\right)\right),\\&=-\alpha_{1,t}\alpha_{2,t}\mu_{y,t-1}^{(1,1)}\left(\mu_{y,t-1}-\mu_{y,t-1}^{(1,1)}\right).\end{aligned} \tag{2.3.9}$$

The unconditional variance $\sigma^2_{y,t}$, can be written, while using (2.3.4), (2.3.8) and (2.3.9), in the following first order difference equation :

$$\sigma^2_{y,t} = \alpha^2_{2,t}\,\sigma^2_{y,t-1} + \Theta_{1,t}, \tag{2.3.10}$$

where,

$$\Theta_{1,t} = \left(\alpha^2_{1,t} - \alpha^2_{2,t}\right)\sigma^{2,(1)}_{y,t-1} + \left(\alpha_{1,t}\left(1-\alpha_{1,t}\right) - \alpha_{2,t}\left(1-\alpha_{2,t}\right)\right)\mu^{(1,1)}_{y,t-1} + \alpha_{2,t}\left(1-\alpha_{2,t}\right)\mu_{y,t-1} + 2\alpha_{2,t}\left(\alpha_{2,t}-\alpha_{1,t}\right)\left(\mu_{y,t-1} - \mu^{(1,1)}_{y,t-1}\right)\mu^{(1,1)}_{y,t-1} + \lambda_{\varepsilon,t},$$

by iterating the equation (2.3.10) m times, we obtain

$$\sigma^2_{y,t} = \left(\prod_{i=1}^{m}\alpha^2_{2,t-i+1}\right)\sigma^2_{y,t-m} + \sum_{j=1}^{m}\left(\prod_{i=1}^{j-1}\alpha^2_{2,t-i+1}\right)\Theta_{1,t-j+1}.$$

Letting $t = s+\tau S$, $s = 1,2,...,S$, $\tau\in\mathbb{Z}$ and putting $m = S$, we obtain, while taking account of the periodicity of the parameters :

$$\sigma^2_{y,s} = \left(\prod_{i=1}^{S}\alpha^2_{2,s-i+1}\right)\sigma^2_{y,s} + \sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\alpha^2_{2,s-i+1}\right)\Theta_{1,s-j+1},$$

then, the unconditional variance, for a fixed s, is given, under the periodically stationary condition, namely, $\prod_{i=1}^{S}\alpha^2_{2,i} < 1$, by the following expression :

$$\sigma^2_{y,s} = \left(1 - \prod_{i=1}^{S}\alpha^2_{2,s-i+1}\right)^{-1}\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\alpha^2_{2,s-i+1}\right)\Theta_{1,s-j+1}.$$

It can be also obtained the stationarity condition in the second moment : $\prod_{i=1}^{S}\alpha_{1,i} < 1$, while replacing $I^{(1)}_{t-1}$ by $1 - I^{(2)}_{t-1}$. The unconditional periodic mean and variance and the periodic probabilities $\mu^{(1,1)}_{y,t-1}$, $\sigma^{2,(1)}_{y,t-1}$, $p_{1,t}$ and $p_{2,t}$, respectively, to be estimated empirically. ■

2.3.2 Existence of higher moments

The next proposition ensures the existence of the unconditional m-*th* moment of $\{y_t,\ t\in\mathbb{Z}\}$ and gives their general formula.

Proposition 2.3.3 *The unconditional m-th moment $\mathbb{E}\left(y_t^m\right)$ of the periodically correlated process $\{y_t,\ t\in\mathbb{Z}\}$, defined by (2.2.2), exists, if the unconditional m-th moment $\mathbb{E}\left(y_0^m\right)$ exists. The general formula for $\mathbb{E}\left(y_t^m\right)$ is given by :*

$$\mathbb{E}\left(y_t^m\right) = \mathbb{E}\left(\sum_{j=0}^{m}\binom{m}{j}\left(\phi_t\circ y_{t-1}\right)^j\varepsilon_t^{m-j}\right) = \sum_{j=0}^{m}\binom{m}{j}\mathbb{E}\left(\left(\phi_t\circ y_{t-1}\right)^j\right)\mathbb{E}\left(\varepsilon_t^{m-j}\right). \tag{2.3.11}$$

Proof of Proposition *2.3.3*. It is well known that the m th moment $\mathbb{E}\left(\varepsilon_t^m\right)$ (Poisson process) is given, and to show $\mathbb{E}\left(y_t^m\right)$ is finite, we use the proof by induction. First of all, we start by $m = 1$, so we have :

$$\begin{aligned}\mathbb{E}\left(y_t\right) &\leq \psi_t \mathbb{E}\left(y_{t-1}\right) + \lambda_{\varepsilon,t} \leq \psi_t\left(\psi_{t-1}\mathbb{E}\left(y_{t-2}\right) + \lambda_{\varepsilon,t-1}\right) + \lambda_{\varepsilon,t}, \\ &\leq \left(\prod_{i=1}^{t}\psi_{t-i+1}\right)\mathbb{E}\left(y_0\right) + \sum_{j=0}^{t-1}\left(\prod_{i=1}^{j}\psi_{t-i+1}\right)\lambda_{\varepsilon,t-j} < \infty,\end{aligned} \tag{2.3.12}$$

with, $\psi_t = \max\left(\alpha_{1,t}, \alpha_{2,t}\right)$, we also have for $m = 2$:

$$\mathbb{E}(y_t^2) \leq \psi_t^2 \mathbb{E}\left(y_{t-1}^2\right) + \psi_t\left(1-\psi_t\right)\mathbb{E}\left(y_{t-1}\right) + 2\psi_t \mathbb{E}\left(y_{t-1}\right)\mathbb{E}\left(\varepsilon_t\right) + \mathbb{E}\left(\varepsilon_t^2\right),$$

and,

$$\begin{aligned}\mathbb{E}(y_t^2) &\leq \psi_t^2\left(\psi_{t-1}^2\mathbb{E}\left(y_{t-2}^2\right) + \psi_{t-1}\left(1-\psi_{t-1}\right)\mathbb{E}\left(y_{t-2}\right) + 2\psi_{t-1}\mathbb{E}\left(y_{t-2}\right)\mathbb{E}\left(\varepsilon_{t-1}\right) + \mathbb{E}\left(\varepsilon_{t-1}^2\right)\right) + \\ &\quad \psi_t\left(1-\psi_t\right)\mathbb{E}\left(y_{t-1}\right) + 2\psi_t\mathbb{E}\left(y_{t-1}\right)\mathbb{E}\left(\varepsilon_t\right) + \mathbb{E}\left(\varepsilon_t^2\right), \\ &\vdots \\ &\leq \left(\prod_{i=1}^{t}\psi_{t-i}^2\right)\mathbb{E}\left(y_0^2\right) + \sum_{j=1}^{t}\left(\prod_{i=1}^{j-1}\psi_{t-i+1}^2\right)\psi_{t-j+1}\left(1-\psi_{t-j+1} + 2\mathbb{E}\left(\varepsilon_{t-j+1}\right)\right)\mathbb{E}\left(y_{t-j+1}\right) + \\ &\quad \sum_{j=1}^{t}\left(\prod_{i=1}^{j-1}\psi_{t-i+1}^2\right)\mathbb{E}\left(\varepsilon_{t-j+1}^2\right),\end{aligned}$$

we can write the last one in this form

$$\mathbb{E}(y_t^2) \leq \left(\prod_{i=0}^{t-1}\psi_{t-i}^2\right)\mathbb{E}\left(y_0^2\right) + \Delta_t,$$

where,

$$\begin{aligned}\Delta_t &= \sum_{j=1}^{t}\left(\prod_{i=1}^{j-1}\psi_{t-i+1}^2\right)\psi_{t-j+1}\left(1-\psi_{t-j+1} + 2\mathbb{E}\left(\varepsilon_{t-j+1}\right)\right)\mathbb{E}\left(y_{t-j+1}\right) \\ &\quad + \sum_{j=1}^{t}\left(\prod_{i=1}^{j-1}\psi_{t-i+1}^2\right)\mathbb{E}\left(\varepsilon_{t-j+1}^2\right) < \infty,\end{aligned}$$

since, $\mathbb{E}\left(y_t\right)$ is finite by using (2.3.12) and the assumption the unconditional m-th moment $\mathbb{E}\left(\varepsilon_{1,t}^m\right)$ and $\mathbb{E}\left(\varepsilon_{2,t}^m\right)$ exist and are finite. Thus, the second moment $\mathbb{E}(y_t^2)$ exists. After that, we assume that $\mathbb{E}\left(y_t^{m-1}\right) < \infty$ for $m-1$, and we show that for m. It's easy to see :

$$\mathbb{E}\left(y_t^m\right) \leq \left(\prod_{i=1}^{t}\psi_{t-i+1}^m\right)\mathbb{E}\left(y_0^m\right) + \Delta_t' \tag{2.3.13}$$

where Δ_t' denote the combination of finite k-th moment of process y_t and l th moment of process ε_t where $k \in \{1, 2, .., m-1\}$ and $l \in \{1, 2, .., m\}$ (i.e. induction hypothesis and the assumption). From (2.3.13), one can check that $\mathbb{E}\left(y_t^m\right) < \infty$, $m \geq 1$. ■

2.3.3 Strict periodic stationarity

The following proposition establishes a sufficient condition for the existence of the strict periodic stationarity property of the process $\{y_t,\ t \in \mathbb{Z}\}$ defined in (2.2.4).

Proposition 2.3.4 *For a fixed value of $s = 1, ...S$, and $\tau \in \mathbb{Z}$, the process $\{y_{s+\tau S}, \tau \in \mathbb{Z}\}$ is an irreducible, aperiodic, and positive recurrent (and hence ergodic) Markov chain. Therefore, there exists strict periodic stationarity for the model (2.2.4).*

Proof of Proposition 2.3.4. It is easy to see that the process $\{y_{s+\tau S};\ \tau \in \mathbb{Z}\}$, in τ for a fixed value of $s = 1, ..., S$, is a periodic Markov chain on $\mathbb{N}_0$ and its transition probabilities is given by :

$$
\begin{aligned}
&P\left(y_{s+\tau S} = x_{s+\tau S} \middle| y_{s-1+\tau S} = x_{s-1+\tau S}\right)\\
&\qquad = \sum_{l=0}^{m^*}\sum_{k=1}^{2} C_l^{x_{s-1+\tau S}} I_{s-1,\tau}^{(k)} \alpha_{k,s}^{l} \left(1-\alpha_{k,s}\right)^{x_{s-1+\tau S}-l} \exp(-\lambda_{\varepsilon,s}) \frac{\lambda_{\varepsilon,s}^{x_{s+\tau S}-l}}{(x_{s+\tau S}-l)!},\\
&\qquad = p\left(x_{s-1+\tau S}, x_{s+\tau S}, \alpha_{1,s} I_{s-1,\tau}^{(1)} + \alpha_{2,s} I_{s-1,\tau}^{(2)}, \lambda_{\varepsilon,s}\right),
\end{aligned}
$$

where, $m^* = \min\left(x_{s-1+\tau S}, x_{s+\tau S}\right)$ and,

$$
p\left(x_{s-1+\tau S}, x_{s+\tau S}, \alpha_{k,s}, \lambda_{\varepsilon,s}\right) =
$$

$$
\sum_{l=0}^{m^*} C_l^{x_{s-1+\tau S}} I_{s-1,\tau}^{(k)} \alpha_{k,s}^{l} \left(1-\alpha_{k,s}\right)^{x_{s-1+\tau S}-l} \exp(-\lambda_{\varepsilon,s}) \frac{\lambda_{\varepsilon,s}^{x_{s+\tau S}-l}}{(x_{s+\tau S}-l)!} > 0,\ \ k = 1, 2.
$$

Since $P\left(y_t = x_t \middle| y_{t-1} = x_{t-1}\right) > 0$ it follows that $\{y_t;\ t \in \mathbb{Z}\}$ is an irreducible, aperiodic chain. Furthermore, to show that our process y_t is positive recurrent, it is sufficient to prove that $\sum_{t=1}^{+\infty} P\left(y_t = 0 \middle| y_1 = 0\right) = +\infty$, By iterating (2.2.4) t times, we have

$$
\begin{aligned}
y_t &= \phi_t \circ \phi_{t-1} \circ \cdots \circ \phi_1 \circ y_0 + \textstyle\sum_{i=1}^{t-1} \phi_{t-1} \circ \phi_{t-2} \circ ... \circ \phi_{i-1} \circ \varepsilon_{t-i} + \varepsilon_t,\\
&= \textstyle\sum_{i=1}^{t-1} \phi_{t-1} \circ \phi_{t-2} \circ ... \circ \phi_{i-1} \circ \varepsilon_{t-i} + \varepsilon_t,\ \ (\text{since } y_0 = 0),
\end{aligned}
$$

which permit us to write

$$
\begin{aligned}
P\left(y_t = 0 \middle| y_1 = 0\right) &= P\left(\textstyle\sum_{i=1}^{t-1} \phi_{t-1} \circ \phi_{t-2} \circ ... \circ \phi_{i-1} \circ \varepsilon_{t-i} + \varepsilon_t = 0 \middle| y_0 = 0\right),\\
&= P\left(\varepsilon_t = 0, \phi_{t-1} \circ \varepsilon_{t-1} = 0, ..., \phi_{t-1} \circ \phi_{t-2} \circ ... \circ \phi_1 \circ \varepsilon_1 = 0 \middle| y_0 = 0\right),\\
&= \textstyle\sum_{i_2=1}^{2} \sum_{i_3=1}^{2} \cdots \sum_{i_t=1}^{2} P\left(\phi_2 = \alpha_{i_2,2}, \phi_3 = \alpha_{i_3,3}, .., \phi_{t-1} = \alpha_{i_{t-1},t-1} \middle| y_0 = 0\right)\\
&\quad \times P\left(\varepsilon_t = 0, \alpha_{i_{t-1},t-1} \circ \varepsilon_{t-1} = 0, ..., \alpha_{i_{t-1},t-1} \circ \alpha_{i_{t-2},t-2} \circ ... \circ \alpha_{i_2,2} \circ \varepsilon_1 = 0 \middle| y_0 = 0\right),
\end{aligned}
$$

it is equal to,

$$
\begin{aligned}
&P\left(y_t = 0 \middle| y_1 = 0\right) = \textstyle\sum_{i_2=1}^{2} \sum_{i_3=1}^{2} \cdots \sum_{i_t=1}^{2} P\left(\phi_2 = \alpha_{i_2,2}, \phi_3 = \alpha_{i_3,3}, .., \phi_{t-1} = \alpha_{i_{t-1},t-1} \middle| y_0 = 0\right) \times\\
&\exp\left\{-\lambda_{\varepsilon,s}\left(1 + \alpha_{i_{t-1},t-1} + \alpha_{i_{t-1},t-1}\alpha_{i_{t-2},t-2} + \cdots + \alpha_{i_{t-1},t-1}\alpha_{i_{t-2},t-2} \cdots \alpha_{i_2,2}\right)\right\}.
\end{aligned}
\tag{2.3.14}
$$

From the expression (2.3.14) it is easy to see that each factor is strictly monotonically decreasing with respect to $\alpha_{i_j,j}$ with $1 \leq j \leq t$. Therefore, by scaling (2.3.14) t times, and we take $t = s + \tau S$, with , $s = 1, ..., S$ and $\tau \in \mathbb{Z}$, and while s being fixed in $\{1, \ldots, S\}$ we get :

$$P\left(y_{s+\tau S} = 0 \middle| \, y_1 = 0\right) \geq \psi\left(\alpha_{\max}, s + \tau S\right),$$

where $\alpha_{\max} = \max_{1 \leq s \leq S}\left\{\alpha_{1,s}, \alpha_{2,s}\right\}$ and the function $\psi(\alpha, t) = \exp\left\{-\lambda_{\varepsilon,s}\left(1 - \alpha^t\right)/\left(1 - \alpha\right)\right\}$. Since $\psi\left(\alpha_{\max}, s + \tau S\right) \nrightarrow 0$ when $\tau \longrightarrow \infty$, it follows easily that $\sum_{\tau=0}^{+\infty} P\left(y_{s+\tau S} = 0 \middle| \, y_1 = 0\right) = +\infty$ for a fixed value of $s = 1, ..., S$, by using the comparison criterion for series convergence. This proves that $\{y_{s+\tau S};\ \tau \in \mathbb{Z}\}$ is a positive recurrent Markov chain and hence ergodic. ■

Remark 2.3.1 *The result given in Proposition 2.3.4 reduces, in the time invariant case, to Proposition* 2.1, *presented by Monteiro et al.* (2012) .

2.3.4 Autocovariance structure

The autocovariance structure of the process $\{y_t,\ t \in \mathbb{Z}\}$ is defined by the proposition below, when we replace $I_{t-1}^{(2)}$ by $1 - I_{t-1}^{(1)}$, it is easy to obtain also the equivalent form while replacing $I_{t-1}^{(1)}$ by $1 - I_{t-1}^{(2)}$, as in Section 2.3. First we give the following notations: $\gamma^{(t)}(h) = Cov\left(y_t; y_{t-h}\right) = \mathbb{E}\left[\left(y_t - \mu_{y,t}\right)\left(y_{t-h} - \mu_{y,t-h}\right)\right]$, $\gamma_k^{(t)}(h) = Cov\left(I_t^{(k)} y_t; y_{t-h}\right)$, for $k = 1, 2$.

Proposition 2.3.5 *The autocovariance structure of the periodically correlated integer-valued processes* $\{y_t,\ t \in \mathbb{Z}\}$, *satisfying the model* (2.2.2), *is given as follows :*

$$\gamma^{(s)}(h) = \begin{cases} \left(1 - \prod_{i=1}^{S} \alpha_{2,s-i+1}^2\right)^{-1} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \alpha_{2,s-i+1}^2\right) \Theta_{1,s-j+1}. & h = 0, \\ \alpha_{2,s}\gamma^{(s-1)}(h-1) + \left(\alpha_{1,s} - \alpha_{2,s}\right)\gamma_1^{(s-1)}(h-1), & h \geq 1, \end{cases}$$

where, $\Theta_{1,s}$, $s = 1, 2, ..., S$, *are given in Proposition 2.3.2.*

In the time invariant coefficient (classical) model case, defined in (*2.2.1*), the result of this proposition can be represented by the following corollary.

Corollary 2.3.3 *The autocovariance structure of integer-valued processes* $\{y_t,\ t \in \mathbb{Z}\}$, *sat-*

isfying the model (2.2.1), is given as follows :

$$\gamma(h) = \begin{cases} (1-\alpha_2^2)^{-1}\ \Theta_1 & h = 0, \\ \alpha_2\gamma(h-1) + (\alpha_1 - \alpha_2)\gamma_1(h-1), & h \geq 1, \end{cases}$$

where, Θ_1 is given in Corollary 2.3.2.

Proof of Proposition *2.3.5*. For $h = 0$ the autocovariance of $Cov(y_t; y_t) = Var(y_t) = \sigma_{y,t}^2$. Then, for $h = 1$ the autocovariance $\gamma^{(t)}(1)$, can be calculated as follows :

$$\begin{aligned} \gamma^{(t)}(1) = Cov(y_t; y_{t-1}) &= Cov\left((\alpha_{1,t}\circ y_{t-1}) I_{t-1}^{(1)} + (\alpha_{2,t}\circ y_{t-1}) I_{t-1}^{(2)} + \varepsilon_t; y_{t-1}\right), \\ &= Cov\left((\alpha_{1,t}\circ y_{t-1}) I_{t-1}^{(1)}; y_{t-1}\right) + Cov\left((\alpha_{2,t}\circ y_{t-1}) I_{t-1}^{(2)}; y_{t-1}\right), \end{aligned}$$

where,

$$Cov\left((\alpha_{1,t}\circ y_{t-1}) I_{t-1}^{(1)}; y_{t-1}\right) = \alpha_{1,t} Cov\left(y_{t-1} I_{t-1}^{(1)}; y_{t-1}\right) = \alpha_{1,t}\gamma_1^{(t-1)}(0),$$

and

$$\begin{aligned} Cov\left((\alpha_{2,t}\circ y_{t-1}) I_{t-1}^{(2)}; y_{t-1}\right) &= \mathbb{E}\left((\alpha_{2,t}\circ y_{t-1}) I_{t-1}^{(2)} y_{t-1}\right) - \mathbb{E}\left((\alpha_{2,t}\circ y_{t-1}) I_{t-1}^{(2)}\right)\mathbb{E}(y_{t-1}), \\ &= \alpha_{2,t}\left(\mathbb{E}\left(y_{t-1}^2\left(1 - I_{t-1}^{(1)}\right)\right) - \mathbb{E}\left(y_{t-1}\left(1 - I_{t-1}^{(1)}\right)\right)\mathbb{E}(y_{t-1})\right), \\ &= \alpha_{2,t}\gamma^{(t-1)}(0) - \alpha_{2,t}\gamma_1^{(t-1)}(0). \end{aligned}$$

Accordingly,

$$\gamma^{(t)}(1) = \alpha_{2,t}\gamma^{(t-1)}(0) + (\alpha_{1,t} - \alpha_{2,t})\gamma_1^{(t-1)}(0).$$

More generally, calculate now the autocovariance $\gamma^{(t)}(h)$, for $h \geq 1$:

$$\begin{aligned} \gamma^{(t)}(h) = Cov(y_t, y_{t-h}) &= Cov\left((\alpha_{1,t}\circ y_{t-1}) I_{t-1}^{(1)} + (\alpha_{2,t}\circ y_{t-1}) I_{t-1}^{(2)} + \varepsilon_t; y_{t-h}\right), \\ &= Cov\left((\alpha_{1,t}\circ y_{t-1}) I_{t-1}^{(1)}; y_{t-1}\right) + Cov\left((\alpha_{2,t}\circ y_{t-1}) I_{t-1}^{(2)}; y_{t-h}\right), \end{aligned}$$

where,

$$\begin{aligned} Cov\left((\alpha_{1,t}\circ y_{t-1}) I_{t-1}^{(1)}; y_{t-h}\right) &= \mathbb{E}\left((\alpha_{1,t}\circ y_{t-1}) I_{t-1}^{(1)} y_{t-h}\right) - \mathbb{E}\left((\alpha_{1,t}\circ y_{t-1}) I_{t-1}^{(1)}\right)\mathbb{E}(y_{t-h}), \\ &= \alpha_{1,t} Cov\left(y_{t-1} I_{t-1}^{(1)}; y_{t-h}\right) = \alpha_{1,t}\gamma_1^{(t-1)}(h-1), \end{aligned}$$

and

$$\begin{aligned} Cov\left((\alpha_{2,t}\circ y_{t-1}) I_{t-1}^{(2)}; y_{t-h}\right) &= \mathbb{E}\left((\alpha_{2,t}\circ y_{t-1}) I_{t-1}^{(2)} y_{t-h}\right) - \mathbb{E}\left((\alpha_{2,t}\circ y_{t-1}) I_{t-1}^{(2)}\right)\mathbb{E}(y_{t-h}), \\ &= \alpha_{2,t}\left(\mathbb{E}\left(y_{t-1} y_{t-h}\left(1 - I_{t-1}^{(1)}\right)\right) - \mathbb{E}\left(y_{t-1}\left(1 - I_{t-1}^{(1)}\right)\right)\mathbb{E}(y_{t-h})\right), \\ &= \alpha_{2,t}\gamma^{(t-1)}(h-1) - \alpha_{2,t}\gamma_1^{(t-1)}(h-1). \end{aligned}$$

Therefore,

$$\gamma^{(t)}(h) = \alpha_{2,t}\gamma^{(t-1)}(h-1) + (\alpha_{1,t} - \alpha_{2,t})\gamma_1^{(t-1)}(h-1).$$

■

2.4 Parameters estimation

In this Section, we estimate the parameters of the $PSETINAR_S\,(2;1)$ model, assuming that we have the observations $\{y_t;\ t \in \mathbb{Z}\}$ satisfying (2.2.2). Let $\underline{\theta}_s = (\theta_{1,s}, \theta_{2,s}, \theta_{3,s})' = (\alpha_{1,s}, \alpha_{2,s}, \lambda_{\varepsilon,s})'$, for a fixed $s = 1, ..., S$, be the vector of unknown parameters to be estimated via two methods, namely, the Conditional Least Squares (CLS) and the Conditional Maximum Likelihood (CML), also, we propose the periodic adaptation Nested Sub-Sample Search ($NeSS$) algorithm to estimate the periodic threshold parameters c_t. Initially and without loss of generality, these parameters c_t are considered to be known until Section 2.4.3 where we will discuss how to estimate them in the unknown case.

2.4.1 Conditional least square estimators

Recall that the CLS-estimations $\widehat{\underline{\theta}}_{s,CLS} = \left(\widehat{\alpha}_{1,s}, \widehat{\alpha}_{2,s}, \widehat{\lambda}_{\varepsilon,s}\right)'$ of $\underline{\theta}_s$ are obtained by minimizing the following sum of squared deviations about the conditional expectation :

$$Q(\underline{\theta}_t; \underline{Y}) = \sum_{t=2}^{n} \left(y_t - g\left(\underline{\theta}_t, y_{t-1}\right)\right)^2,$$

where,

$$g\left(\underline{\theta}_t, y_{t-1}\right) = \mathbb{E}\left(y_t \middle| y_{t-1}\right) = \alpha_{1,t} y_{t-1} I_{t-1}^{(1)} + \alpha_{2,t} y_{t-1} I_{t-1}^{(2)} + \lambda_{\varepsilon,t}. \tag{2.4.1}$$

Suppose that the size of observed time series n is a multiple of S (i.e. $n = NS$), and while replacing t by $s + \tau S$, with , $s = 1, ..., S$ and $\tau \in \mathbb{Z}$, one can rewrite the last expression in the form

$$\begin{aligned} Q(\underline{\theta}_s; \underline{Y}) &= \sum_{s=1}^{S} \sum_{\tau=0}^{N-1} \left(y_{s+\tau S} - \sum_{k=1}^{2} \alpha_{k,s} y_{s-1+\tau S} I_{s-1,\tau}^{(k)} - \lambda_{\varepsilon,s}\right)^2, \\ &= \sum_{s=1}^{S} \sum_{\tau=0}^{N-1} \left(y_{s+\tau S} - g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right)^2 = \sum_{s=1}^{S} \sum_{\tau=0}^{N-1} U_{s+\tau S}\left(\underline{\theta}_s\right), \end{aligned}$$

where,

$$I_{s-1,\tau}^{(1)} = \begin{cases} 1 & \text{if} \quad y_{s-1+\tau S} \leq c_s, \\ 0 & \text{if} \quad y_{s-1+\tau S} > c_s, \end{cases} \quad \text{and } I_{s-1,\tau}^{(2)} = 1 - I_{s-1,\tau}^{(1)},$$

and,

$$U_{s+\tau S}\left(\underline{\theta}_s\right) = \left(y_{s+\tau S} - g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right)^2.$$

The CLS-vector estimator $\widehat{\underline{\theta}}_{s,CLS}$ can be obtained by solving the system below :

$$\begin{aligned} \frac{\partial Q}{\partial \alpha_{k,s}} &= \sum_{\tau=0}^{N-1} \left[\left(y_{s-1+\tau S} y_{s+\tau S} - y_{s-1+\tau S}^2 \alpha_{k,s} - \lambda_{\varepsilon,s} y_{s-1+\tau S} \right) I_{s-1,\tau}^{(k)} \right] = 0, \;\; k = 1, 2, \\ \frac{\partial Q}{\partial \lambda_{\varepsilon,s}} &= \sum_{\tau=0}^{N-1} \left[y_{s+\tau S} - \alpha_{1,s} y_{s-1+\tau S} I_{s-1,\tau}^{(1)} - \alpha_{2,s} y_{s-1+\tau S} I_{s-1,\tau}^{(2)} - \lambda_{\varepsilon,s} \right] = 0, \end{aligned} \tag{2.4.2}$$

the solution of this system (2.4.2) is given by

$$\hat{\underline{\theta}}_{s,CLS} = \mathbf{A}_s^{-1} \underline{\mathbf{b}}_s, \qquad s = 1, ..., S,$$

with

$$\mathbf{A}_s = \begin{pmatrix} \sum_{\tau=0}^{N-1} y_{s-1+\tau S}^2 I_{s-1,\tau}^{(1)} & 0 & \sum_{\tau=0}^{N-1} y_{s-1+\tau S} I_{s-1,\tau}^{(1)} \\ 0 & \sum_{\tau=0}^{N-1} y_{s-1+\tau S}^2 I_{s-1,\tau}^{(2)} & \sum_{\tau=0}^{N-1} y_{s-1+\tau S} I_{s-1,\tau}^{(2)} \\ \sum_{\tau=0}^{N-1} y_{s-1+\tau S} I_{s-1,\tau}^{(1)} & \sum_{\tau=0}^{N-1} y_{s-1+\tau S} I_{s-1,\tau}^{(2)} & N \end{pmatrix},$$

and

$$\underline{\mathbf{b}}_s = \begin{pmatrix} \sum_{\tau=0}^{N-1} y_{s-1+\tau S} y_{s+\tau S} I_{s-1,\tau}^{(1)} & \sum_{\tau=0}^{N-1} y_{s-1+\tau S} y_{s+\tau S} I_{s-1,\tau}^{(2)} & \sum_{\tau=0}^{N-1} y_{s+\tau S} \end{pmatrix}'.$$

The next result gives the asymptotic property for our CLS-vector estimator.

Theorem 2.4.1 *The CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$ are strongly consistent and asymptotically normally distributed i.e.,*

$$\sqrt{N}\left(\widehat{\underline{\theta}}_{s,CLS} - \underline{\theta}_s\right) \overset{\mathcal{L}}{\rightsquigarrow} \mathcal{N}\left(0, \Sigma_s^{-1}\Omega_s\Sigma_s^{-1}\right),$$

where Σ_s and Ω_s, for $s = 1, ..., S$, are square matrices of order 3 with elements

$$(\Sigma_s)_{i,j} = \mathbb{E}\left[\frac{\partial}{\partial \theta_{i,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right) \frac{\partial}{\partial \theta_{j,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right],$$

and

$$(\Omega_s)_{i,j} = \mathbb{E}\left[U_{s+\tau S}\left(\underline{\theta}_s\right) \frac{\partial}{\partial \theta_{i,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right) \frac{\partial}{\partial \theta_{j,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right],$$

respectively.

Proof of Theorem *2.4.1*. It is easy check that the function g, given in (2.4.1), $\partial g/\partial\theta_{i,s}$, $\partial^2 g/\partial\theta_{i,s}\partial\theta_{j,s}$, and $\partial^3 g/\partial\theta_{i,s}\partial\theta_{j,s}\partial\theta_{k,s}$, for $i, j, k \in \{1, 2, 3\}$, $s = 1, ..., S$, satisfy all the regularity conditions proposed by Klimko and Nelson (1978). Consequently, by the theorem 3.1 in Klimko and Nelson (1978), we conclude that the CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$ are strongly consistent. Then we have to prove the three conditions (A), (B) and (C) hold

(A). $\mathbb{E}\left(y_t | y_{t-1}, y_{t-2}, ..., y_0\right) = \mathbb{E}\left(y_t | y_{t-1}\right)$, $t \geq 1$ a.e.

(B). $\mathbb{E}\left(U_{s+\tau S}\left(\underline{\theta}_s\right)\left|\frac{\partial g(\underline{\theta}_s,y_{s-1+\tau S})}{\partial\theta_{i,s}}\frac{\partial g(\underline{\theta}_s,y_{s-1+\tau S})}{\partial\theta_{j,s}}\right|\right)<\infty$, $i,j=1,2,3$, where $U_{s+\tau S}\left(\underline{\theta}_s\right)=(y_{s+\tau S}$ $-\ g\left(\underline{\theta}_s,y_{s-1+\tau S}\right))^2$.

(C). Σ_s is non singular.

Condition (A) is satisfied since $y_{s+\tau S}$ is a first-order Markov chain while s being fixed in $\{1,\ldots,S\}$. In order to prove condition (B) we check that the components of the matrix Ω_s are all finite.

$$\begin{aligned}
\left(\Omega_s\right)_{1,1} &= \mathbb{E}\left(U^2_{s+\tau S}\left(\underline{\theta}_s\right)\left(\frac{\partial}{\partial\alpha_{1,s}}g\left(\underline{\theta}_s,y_{s-1+\tau S}\right)\right)^2\right)=\mathbb{E}\left(y^2_{s-1+\tau S}I^{(1)}_{s-1,\tau}U^2_{s+\tau S}\left(\underline{\theta}_s\right)\right),\\
&= \mathbb{E}\left(y^2_{s-1+\tau S}I^{(1)}_{s-1,\tau}\mathbb{E}\left(U^2_{s+\tau S}\left(\underline{\theta}_s\right)\middle|\, y_{s-1+\tau S}\right)\right),\\
&= \alpha_{1,s}\left(1-\alpha_{1,s}\right)\mathbb{E}\left(y^3_{s-1+\tau S}I^{(1)}_{s-1,\tau}\right)+\lambda_{\varepsilon,s}\mathbb{E}\left(y^2_{s-1+\tau S}I^{(1)}_{s-1,\tau}\right),\\
&= \alpha_{1,s}\left(1-\alpha_{1,s}\right)\mu^{(1,3)}_{y,s-1}+\lambda_{\varepsilon,s}\mu^{(1,2)}_{y,s-1}<\infty.
\end{aligned}$$

Similarly, we have

$$\left(\Omega_s\right)_{2,2}=\mathbb{E}\left(U^2_{s+\tau S}\left(\underline{\theta}_s\right)\left(\frac{\partial}{\partial\alpha_{2,s}}g\left(\underline{\theta}_s,y_{s-1+\tau S}\right)\right)^2\right)=\alpha_{2,s}\left(1-\alpha_{2,s}\right)\mu^{(2,3)}_{y,s-1}+\lambda_{\varepsilon,s}\mu^{(2,2)}_{y,s-1}<\infty.$$

$$\begin{aligned}
\left(\Omega_s\right)_{3,3} &= \mathbb{E}\left(U^2_{s+\tau S}\left(\underline{\theta}_s\right)\left(\frac{\partial}{\partial\lambda_{\varepsilon,s}}g\left(\underline{\theta}_s,y_{s-1+\tau S}\right)\right)^2\right)=\mathbb{E}\left(\mathbb{E}\left(U^2_{s+\tau S}\left(\underline{\theta}_s\right)\middle|\, y_{s-1+\tau S}\right)\right),\\
&= \alpha_{1,s}\left(1-\alpha_{1,s}\right)\mathbb{E}\left(y_{s-1+\tau S}I^{(1)}_{s-1,\tau}\right)+\alpha_{2,s}\left(1-\alpha_{2,s}\right)\mathbb{E}\left(y_{s-1+\tau S}I^{(2)}_{s-1,\tau}\right)+\lambda_{\varepsilon,s},\\
&= \alpha_{1,s}\left(1-\alpha_{1,s}\right)\mu^{(1,1)}_{y,s-1}+\alpha_{2,s}\left(1-\alpha_{2,s}\right)\mu^{(2,1)}_{y,s-1}+\lambda_{\varepsilon,s}<\infty.\\
\left(\Omega_s\right)_{1,3} &= \left(\Omega_s\right)_{3,1}=\mathbb{E}\left(U^2_{s+\tau S}\left(\underline{\theta}_s\right)\frac{\partial}{\partial\alpha_{1,s}}g\left(\underline{\theta}_s,y_{s-1+\tau S}\right)\frac{\partial}{\partial\lambda_{\varepsilon,s}}g\left(\underline{\theta}_s,y_{s-1+\tau S}\right)\right),\\
&= \mathbb{E}\left(y_{s-1+\tau S}I^{(1)}_{s-1,\tau}\left(c_s\right)\mathbb{E}\left(U^2_{s+\tau S}\left(\underline{\theta}_s\right)\middle|\, y_{s-1+\tau S}\right)\right),\\
&= \alpha_{1,s}\left(1-\alpha_{1,s}\right)\mu^{(1,2)}_{y,s-1}+\lambda_{\varepsilon,s}\mu^{(1,1)}_{y,s-1}<\infty.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\left(\Omega_s\right)_{2,3} &= \left(\Omega_s\right)_{3,2}=\mathbb{E}\left(U^2_{s+\tau S}\left(\underline{\theta}_s\right)\frac{\partial}{\partial\alpha_{2,s}}g\left(\underline{\theta}_s,y_{s-1+\tau S}\right)\frac{\partial}{\partial\lambda_{\varepsilon,s}}g\left(\underline{\theta}_s,y_{s-1+\tau S}\right)\right),\\
&= \alpha_{2,s}\left(1-\alpha_{2,s}\right)\mu^{(2,2)}_{y,s-1}+\lambda_{\varepsilon,s}\mu^{(2,1)}_{y,s-1}<\infty.\\
\left(\Omega_s\right)_{1,2} &= \left(\Omega_s\right)_{2,1}=\mathbb{E}\left(U^2_{s+\tau S}\left(\underline{\theta}_s\right)\tfrac{\partial}{\partial\alpha_{1,s}}g\left(\underline{\theta}_s,y_{s-1+\tau S}\right)\tfrac{\partial}{\partial\alpha_{2,s}}g\left(\underline{\theta}_s,y_{s-1+\tau S}\right)\right)=0.
\end{aligned}$$

Therefore, the condition (B) is also satisfied, then, the elements of the matrix Σ_s are given by :

$$(\Sigma_s)_{1,1} = \mathbb{E}\left(\left(\frac{\partial}{\partial \alpha_{1,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right)^2\right) = \mathbb{E}\left(y^2_{s-1+\tau S} I^{(1)}_{s-1,\tau}\right) = \mu^{(1,2)}_{y,s-1},$$

$$(\Sigma_s)_{2,2} = \mathbb{E}\left(\left(\frac{\partial}{\partial \alpha_{2,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right)^2\right) = \mathbb{E}\left(y^2_{s-1+\tau S} I^{(2)}_{s-1,\tau}\right) = \mu^{(2,2)}_{y,s-1}.$$

$$(\Sigma_s)_{3,3} = \mathbb{E}\left(\left(\frac{\partial}{\partial \lambda_{\varepsilon,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right)^2\right) = 1.$$

$$(\Sigma_s)_{1,3} = (\Sigma_s)_{3,1} = \mathbb{E}\left(\frac{\partial}{\partial \alpha_{1,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right) \frac{\partial}{\partial \lambda_{\varepsilon,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right) = \mathbb{E}\left(y_{s-1+\tau S} I^{(1)}_{s,\tau}\right) = \mu^{(1,1)}_{y,s-1}.$$

$$(\Sigma_s)_{2,3} = (\Sigma_s)_{3,2} = \mathbb{E}\left(\frac{\partial}{\partial \alpha_{2,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right) \frac{\partial}{\partial \lambda_{\varepsilon,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right) = \mathbb{E}\left(y_{s-1+\tau S} I^{(2)}_{s-1,\tau}\right) = \mu^{(2,1)}_{y,s-1}.$$

$$(\Sigma_s)_{1,2} = (\Sigma_s)_{2,1} = \mathbb{E}\left(\frac{\partial}{\partial \alpha_{1,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right) \frac{\partial}{\partial \alpha_{2,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right) = 0,$$

when the calculation of matrix's (Σ_s) determinant, we get

$$\det(\Sigma_s) = \mu^{(2,2)}_{y,s-1}\left(\mu^{(1,2)}_{y,s-1} - \left(\mu^{(1,1)}_{y,s-1}\right)^2\right) - \mu^{(1,2)}_{y,s-1}\left(\mu^{(2,1)}_{y,s-1}\right)^2 > 0.$$

Which lead us to conclude that the matrix Σ_s is invertible. Thus, the condition (C) is also satisfied. Finally, by Theorem 3.2 of Klimko and Nelson (1978), the CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$ are asymptotically normally distributed. ■

2.4.2 Conditional maximum likelihood estimators

The CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$ of the vector parameters $\widehat{\underline{\theta}}_s$ are given by maximizing the conditional likelihood function, which is given for a size $n = NS$, while taking $t = s + \tau S$, $s = 1, ..., S$ and $\tau \in \mathbb{Z}$, by :

$$L\left(\underline{\theta}_s; \underline{Y}\right) = \prod_{s=1}^{S} \prod_{\tau=0}^{N-1} p\left(x_{s-1+\tau S}, x_{s+\tau S}, \alpha_{1,s} I^{(1)}_{s-1,\tau} + \alpha_{2,s} I^{(2)}_{s-1,\tau}, \lambda_{\varepsilon,s}\right),$$

or equivalently which maximizes the log conditional likelihood function $\mathcal{L}\left(\underline{\theta}_s; \underline{Y}\right)$, of these $n = NS$ observations

$$\mathcal{L}\left(\underline{\theta}_s; \underline{Y}\right) = \sum_{s=1}^{S} \sum_{\tau=0}^{N-1} \log p\left(x_{s-1+\tau S}, x_{s+\tau S}, \alpha_{1,s} I^{(1)}_{s-1,\tau} + \alpha_{2,s} I^{(2)}_{s-1,\tau}, \lambda_{\varepsilon,s}\right).$$

The partial derivatives of $\mathcal{L}\left(\underline{\theta}_s\right)$ with respect to $\alpha_{1,s}$, $\alpha_{2,s}$ and $\lambda_{\varepsilon,s}$, $s=1,...,S$, are given for $k=1,2$, respectively, by

$$\begin{cases} \frac{1}{\alpha_{k,s}\left(1-\alpha_{k,s}\right)}\sum_{\tau=0}^{N-1} I_{s-1,\tau}^{(k)}\left(\left(x_{s+\tau S}-\alpha_{k,s}x_{s-1+\tau S}\right)-\lambda_{\varepsilon,s}\dfrac{p\left(x_{s-1+\tau S},x_{s+\tau S}-1,\alpha_{k,s},\lambda_{\varepsilon,s}\right)}{p\left(x_{s-1+\tau S},x_{s+\tau S},\alpha_{k,s},\lambda_{\varepsilon,s}\right)}\right)=0, \\ \sum_{\tau=0}^{N-1}\sum_{i=1}^{2}\dfrac{p\left(x_{s-1+\tau S},x_{s+\tau S}-1,\alpha_{i,s},\lambda_{\varepsilon,s}\right)}{p\left(x_{s-1+\tau S},x_{s+\tau S},\alpha_{i,s},\lambda_{\varepsilon,s}\right)}I_{s-1,\tau}^{(i)}-N=0. \end{cases} \tag{2.4.3}$$

This system can not be explicitly solved, therefore numerical methods of nonlinear optimization have been used. The next results give the consistency and the asymptotic property for our CML-vector estimator.

Theorem 2.4.2 *Let $\{y_{s+\tau S},\ \tau\in\mathbb{Z}\}$ be the process defined by (2.2.2), satisfying the conditions (C_1)-(C_6), while s being fixed in $\{1,...,S\}$, of the Lemma 22.3 in Franke and Seligmann (1993). Then, there exists a consistent solution $\widehat{\underline{\theta}}_{s,CML}$ of (2.4.3) which is a local maximum of $\mathcal{L}\left(\underline{\theta}_s;\underline{Y}\right)$ with probability going to 1. Moreover, any other consistent solution of (2.4.3) coincides with $\widehat{\underline{\theta}}_{s,CML}$ with probability going to 1.*

Theorem 2.4.3 *The CLS-vector estimators $\widehat{\underline{\theta}}_{s,CML}$ are, under the assumption of the theorem 2.4.2 and while s being fixed in $\{1,...,S\}$, asymptotically normally distributed, i.e.*

$$\sqrt{N}\left(\widehat{\underline{\theta}}_{s,CML}-\underline{\theta}_s\right)\overset{\mathcal{L}}{\rightsquigarrow}\mathcal{N}\left(0,I_s\left(\underline{\theta}_s\right)^{-1}\right),$$

where $I_s\left(\underline{\theta}_s\right)$, for $s=1,...,S$, is the fisher information matrix, with elements :

$$I_s\left(\underline{\theta}_s\right)_{k,l}=-\mathbb{E}\left[\frac{\partial^2}{\partial\theta_{k,s}\partial\theta_{l,s}}\mathcal{L}\left(\underline{\theta}_s\right)\right],\ \textit{for } k,l=1,2,3,$$

with, for $k = l$,

$$I_s\left(\underline{\theta}_s\right)_{k,k} = \mathbb{E}\left[\frac{\partial^2 \mathcal{L}\left(\underline{\theta}_s\right)}{\partial \alpha_{k,s}^2}\right], \text{ for } k = 1, 2,\ s = 1, ..., S,$$
$$= \frac{N}{\alpha_{k,s}^2\left(1-\alpha_{k,s}\right)^2} \sum_{x_s}^{+\infty} \sum_{x_{s-1}}^{+\infty} P\left(y_{s-1} = x_{s-1}\right) I_{s-1,\tau}^{(k)} \left\{\left(\left(2\alpha_{k,s} - 1\right) x_s - \alpha_{k,s}^2 x_{s-1}\right) p\left(x_{s-1}, x_s, \alpha_{k,s}, \lambda_{\varepsilon,s}\right)\right.$$
$$\left. + \lambda_{\varepsilon,s} p\left(x_{s-1}, x_s - 1, \alpha_{k,s}, \lambda_{\varepsilon,s}\right) + \lambda_{\varepsilon,s}^2 p\left(x_{s-1}, x_s - 2, \alpha_{k,s}, \lambda_{\varepsilon,s}\right) - \lambda_{\varepsilon,s}^2 \frac{p\left(x_{s-1}, x_s - 1, \alpha_{k,s}, \lambda_{\varepsilon,s}\right)^2}{p\left(x_{s-1}, x_s, \alpha_{k,s}, \lambda_{\varepsilon,s}\right)}\right\},$$
$$I_s\left(\underline{\theta}_s\right)_{k,k} = \mathbb{E}\left[\frac{\partial^2 \mathcal{L}\left(\underline{\theta}_s\right)}{\partial \lambda_{\varepsilon,s}^2}\right], \text{ for } k = 3,\ s = 1, ..., S,$$
$$= N \sum_{x_s}^{+\infty} \sum_{x_{s-1}}^{+\infty} P\left(y_{s-1} = x_{s-1}\right) \left\{\sum_{i=1}^{2} I_{s-1,\tau}^{(i)} p\left(x_{s-1}, x_s - 2, \alpha_{i,s}, \lambda_{\varepsilon,s}\right) - I_{s-1,\tau}^{(i)} \frac{p(x_{s-1}, x_s - 1, \alpha_{i,s}, \lambda_{\varepsilon,s})^2}{p(x_{s-1}, x_s, \alpha_{i,s}, \lambda_{\varepsilon,s})}\right\},$$
$$I_s\left(\underline{\theta}_s\right)_{1,2} = I_s\left(\underline{\theta}_s\right)_{2,1} = \mathbb{E}\left[\frac{\partial^2 \mathcal{L}\left(\underline{\theta}_s\right)}{\partial \alpha_{1,s} \partial \alpha_{2,s}}\right] = 0,$$

and, for $k \neq j$,

$$I_s\left(\underline{\theta}_s\right)_{k,l} = I_s\left(\underline{\theta}_s\right)_{j,k} = \mathbb{E}\left[\frac{\partial^2 \mathcal{L}\left(\underline{\theta}_s\right)}{\partial \alpha_{k,s} \partial \lambda_{\varepsilon,s}}\right], \text{ for } l = 3, \text{ and } k = 1, 2,$$
$$= -\frac{N}{\alpha_{k,s}\left(1-\alpha_{k,s}\right)} \sum_{x_s}^{+\infty} \sum_{x_{s-1}}^{+\infty} P\left(y_{s-1} = x_{s-1}\right) I_{s-1,\tau}^{(k)} \left\{p\left(x_{s-1}, x_s - 1, \alpha_{k,s}, \lambda_{\varepsilon,s}\right) - \lambda_{\varepsilon,s} \frac{p\left(x_{s-1}, x_s - 1, \alpha_{k,s}, \lambda_{\varepsilon,s}\right)^2}{p\left(x_{s-1}, x_s, \alpha_{k,s}, \lambda_{\varepsilon,s}\right)}\right.$$
$$\left. + \lambda_{\varepsilon,s} p\left(x_{s-1}, x_s - 2, \alpha_{k,s}, \lambda_{\varepsilon,s}\right)\right\}.$$

Proof of Theorem 2.4.2 and 2.4.3. The proof follows easily as a generalization for the periodic case of $SETINAR\,(2;1)$ given by Monteiro *et al.* (2012). Since, each regime of $PSETINAR\,(2;1)$ falls, when s being fixed in $\{1,\ldots,S\}$, into the $INAR$ structure considered by Franke and Seligmann (1993), so the both Theorems are special cases of Theorems 2.1 and 2.2 of Billingsley (1961). Then, we have only to check that the conditions $(C1)$-$(C6)$, given in Franke and Seligmann (1993), hold, which imply the conditions of those general. These authors showed that for the Poisson distribution, as the distribution of innovations, the following set of conditions hold.

$(C1)$. The set $\left\{x; P\left(\varepsilon_{s+\tau S} = x\right) = f\left(x, \lambda_{\varepsilon,s}\right) = \exp\left(-\lambda_{\varepsilon,s}\right) \frac{\lambda_{\varepsilon,s}^x}{x!} > 0\right\}$ doesn't depend on $\lambda_{\varepsilon,s}$.

$(C2)$. $\mathbb{E}\left(\varepsilon_{s+\tau S}^3\right) = \lambda_{\varepsilon,s}^3 + 2\lambda_{\varepsilon,s}^2 + \lambda_{\varepsilon,s} < \infty$.

$(C3)$. $P\left(\varepsilon_{s+\tau S} = k\right)$ is three times continuously differentiable on $\lambda_{\varepsilon,s}$.

$(C4)$. For any $\lambda'_{\varepsilon,s} \in B$ (B is an open subset of $\mathbb{R}$), there exists a neighborhood V of $\lambda'_{\varepsilon,s}$ such that

1. $\sum_{x=0}^{\infty} \sup_{\lambda_{\varepsilon,s} \in V} f\left(x, \lambda_{\varepsilon,s}\right) < \infty$. 2. $\sum_{x=0}^{\infty} \sup_{\lambda_{\varepsilon,s} \in V} \dfrac{\partial f\left(x, \lambda_{\varepsilon,s}\right)}{\partial \lambda_{\varepsilon,s}^{(k)}} < \infty$, for $k = 1, ..., r$.

3. $\sum_{x=0}^{\infty} \sup_{\lambda_{\varepsilon,s}\in V} \dfrac{\partial^2 f(x,\lambda_{\varepsilon,s})}{\partial\lambda_{\varepsilon,s}^{(k)}\partial\lambda_{\varepsilon,s}^{(l)}} < \infty,$ for $k,l = 1,...,r.$

$(C5)$. For any $\lambda'_{\varepsilon,s} \in B$, there exists a neighborhood V of $\lambda'_{\varepsilon,s}$ and increasing sequences, $\psi_1(n) = const.n, \psi_{11}(n) = const.n^2$ and $\psi_{111}(n) = const.n^3, n \geq 0$ such that for all $\lambda_{\varepsilon,s} \in V$ and all $x \leq n$ with non vanishing $f(x,\lambda_{\varepsilon,s})$

$$\left|\partial f(x,\lambda_{\varepsilon,s})/\partial\lambda_{\varepsilon,s}^{(k)}\right| = \psi_1(n) f(x,\lambda_{\varepsilon,s}), \quad \left|\partial^2 f(x,\lambda_{\varepsilon,s})/\partial^2\lambda_{\varepsilon,s}^{(k)}\right| = \psi_{11}(n) f(x,\lambda_{\varepsilon,s}),$$
$$\left|\partial^3 f(x,\lambda_{\varepsilon,s})/\partial^3\lambda_{\varepsilon,s}^{(k)}\right| = \psi_{111}(n) f(x,\lambda_{\varepsilon,s}),$$

and with respect to the periodic stationary distribution of the process y_t

$$\mathbb{E}\left(\psi_1^3(y_s)\right) < \infty,\ \mathbb{E}\left(y_s\psi_{11}(y_{s+1})\right) < \infty,\ \mathbb{E}\left(\psi_1(y_s)\psi_{11}(y_{s+1})\right) < \infty,\ \mathbb{E}\left(\psi_{111}(y_s)\right) < \infty.$$

$(C6)$. The Fisher information matrix $I_s(\underline{\theta}_s)$ is non singular, which guarantees that the parameters of $PSETINAR(2;1)$ process are not redundant. We show that, the determinant of $I_s(\underline{\theta}_s)$ matrix is given by :

$$\det(I_s(\underline{\theta}_s)) = \sum_{k=1}^{2} P(I_{s-1,0}^{(3-k)} = 1)P(I_{s-1,0}^{(k)} = 1)\mathbb{E}\left(\left(\frac{\partial p(y_s,y_{s+1},\alpha_{k,s},\lambda_{\varepsilon,s})}{\partial\alpha_{k,s}}\right)^2 \middle| I_{s-1,0}^{(k)} = 1\right)\det(A_{s,3-k}),$$

we can write the last expression as

$$\det(I_s(\underline{\theta}_s)) = (1-p_s)p_s\sum_{k=1}^{2}\mathbb{E}\left(\left(\frac{\partial p(y_s,y_{s+1},\alpha_{k,s},\lambda_{\varepsilon,s})}{\partial\alpha_{k,s}}\right)^2 \middle| I_{s-1,0}^{(k)} = 1\right)\det(A_{s,3-k}),$$

with

$$(A_{s,k})_{1,1} = \mathbb{E}\left(\left(\frac{\partial p(y_s,y_{s+1},\alpha_{k,s},\lambda_{\varepsilon,s})}{\partial\alpha_{k,s}}\right)^2 \middle| I_{s-1,0}^{(k)} = 1\right).$$

$$(A_{s,k})_{1,2} = \mathbb{E}\left(\frac{\partial p(y_s,y_{s+1},\alpha_{k,s},\lambda_{\varepsilon,s})}{\partial\alpha_{k,s}}\frac{\partial p(y_s,y_{s+1},\alpha_{k,s},\lambda_{\varepsilon,s})}{\partial\lambda_{\varepsilon,s}} \middle| I_{s-1,0}^{(k)} = 1\right) = (A_{s,1})_{2,1}.$$

$$(A_{s,k})_{2,2} = \mathbb{E}\left(\left(\frac{\partial p(y_s,y_{s+1},\alpha_{k,s},\lambda_{\varepsilon,s})}{\partial\lambda_{\varepsilon,s}}\right)^2 \middle| I_{s-1,0}^{(k)} = 1\right), \text{ for } k = 1,2.$$

When s being fixed in $\{1,\ldots,S\}$, we can see that the matrix $A_{s,k}$, $k = 1,2$ has the same structure as the Fisher information matrix analyzed by Franke and Seligmann (1993), so the same arguments can be used to prove that $A_{s,k}$ has positive determinant. The matrix $A_{s,k}$, $k = 1,2$ is e.g., satisfied if the matrix with entries

$$\sum_{m_{k,s}}=\begin{pmatrix}\mathbb{E}\left(\left(\frac{\partial p\left(y_s,y_{s+1},\alpha_{k,s},\lambda_{\varepsilon,s}\right)}{\partial\alpha_{k,s}}\right)^2\middle| y_s=m_{s,k}\right) & \mathbb{E}\left(\begin{array}{c}\frac{\partial p\left(y_s,y_{s+1},\alpha_{k,s},\lambda_{\varepsilon,s}\right)}{\partial\alpha_{k,s}}\times\\ \frac{\partial p\left(y_s,y_{s+1},\alpha_{k,s},\lambda_{\varepsilon,s}\right)}{\partial\lambda_{\varepsilon,s}}\end{array}\middle| y_s=m_{s,k}\right)\\ \mathbb{E}\left(\begin{array}{c}\frac{\partial p\left(y_s,y_{s+1},\alpha_{k,s},\lambda_{\varepsilon,s}\right)}{\partial\alpha_{k,s}}\times\\ \frac{\partial p\left(y_s,y_{s+1},\alpha_{k,s},\lambda_{\varepsilon,s}\right)}{\partial\lambda_{\varepsilon,s}}\end{array}\middle| y_s=m_{s,k}\right) & \mathbb{E}\left(\left(\frac{\partial p\left(y_s,y_{s+1},\alpha_{k,s},\lambda_{\varepsilon,s}\right)}{\partial\lambda_{\varepsilon,s}}\right)^2\middle| y_s=m_{s,k}\right)\end{pmatrix},$$

is non singular for set of $m_{s,k}$, $m_{s,1}\leq c_s$ and $m_{s,2}>c_s$, with positive measure under the periodic stationary distribution. They also proved that

$$\begin{aligned}p\left(m,n,\alpha_{k,s},\lambda_{\varepsilon,s}\right) &= \alpha_{k,s}p\left(m-1,n-1,\alpha_{k,s},\lambda_{\varepsilon,s}\right)+\left(1-\alpha_{k,s}\right)p\left(m-1,n,\alpha_{k,s},\lambda_{\varepsilon,s}\right),\\ \frac{\partial p\left(m,n,\alpha_{k,s},\lambda_{\varepsilon,s}\right)}{\partial\alpha_{k,s}} &= \frac{m}{1-\alpha_{k,s}}\left[p\left(m-1,n-1,\alpha_{k,s},\lambda_{\varepsilon,s}\right)-p\left(m,n,\alpha_{k,s},\lambda_{\varepsilon,s}\right)\right],\\ \frac{\partial p\left(m,n,\alpha_{k,s},\lambda_{\varepsilon,s}\right)}{\partial\lambda_{\varepsilon,s}} &= \left(\frac{n}{\lambda_{\varepsilon,s}}-1\right)p\left(m,n,\alpha_{k,s},\lambda_{\varepsilon,s}\right)-\left(\frac{m\alpha_{k,s}}{\lambda_{\varepsilon,s}}\right)p\left(m-1,n-1,\alpha_{k,s},\lambda_{\varepsilon,s}\right),\end{aligned}$$

by using these expressions and the lemma 22.3(b) in Franke and Seligmann (1993), we get

$$\begin{aligned}&\frac{\lambda_{\varepsilon,s}\left(1-\alpha_{k,s}\right)^2}{m_k^2}\det\sum_{m_{k,s}}=Var\left[\left(D\left(m_{s,k},y_{s+1},\alpha_{k,s}\right)-1\right)\left(y_{s+1}-\lambda_{\varepsilon,s}-m_{s,k}\alpha_{k,s}\right)D\left(m_{s,k},y_{s+1},\alpha_{k,s}\right)\right]-\\ &-Cov\left[\left(D\left(m_{s,k},y_{s+1},\alpha_{k,s}\right)-1\right)^2;\left(y_{s+1}-\lambda_{\varepsilon,s}-m_{s,k}\alpha_{k,s}\right)D\left(m_{s,k},y_{s+1},\alpha_{k,s}\right)^2\right],\end{aligned}\tag{2.4.4}$$

with

$$D\left(m,n,\alpha_{k,s}\right)=p\left(m-1,n-1,\alpha_{k,s},\lambda_{\varepsilon,s}\right)/\,p\left(m,n,\alpha_{k,s},\lambda_{\varepsilon,s}\right).$$

As a result, the expression (2.4.4) is positive for a set of m_k, $m_1\leq c_s$ and $m_2>c_s$, with positive measure under the periodic stationary distribution. Therefore (C_6) is satisfied for the Poisson innovation law. In term of derivatives of log-likelihood, each regime of the $PSETINAR_S\,(2;1)$ model falls (when the period s being fixed) into $INAR$ considered by Franke and Seligmann (1993). and in accordance with these authors, the conditions (C_1)-(C_6) imply the conditions (A) and (B) theorems 2.1 and 2.2 in Billingsley (1961) (these two conditions are also given in Monteiro *et al.* (2012)) and the results of Theorems *2.4.2* and *2.4.3* are also valid for the $PSETINAR_S\,(2;1)$ model. ∎

2.4.3 Estimation of the periodic threshold parameter c_t

In this paragraph, we discuss how to estimate the periodic threshold parameters c_t using the periodic adaptation $(NeSS)$ algorithm proposed by Li and Tong (2016) . Using standard least squares, we get the sum of squared errors function as follows :

$$S_N(c_s) = \sum_{\tau=0}^{N-1} \left(y_{s+\tau S} - \sum_{k=1}^{2} \frac{\sum_{\tau_1=0}^{N-1} \left(y_{s+\tau_1 S} - \lambda_{\varepsilon,s} \right) y_{s-1+\tau_1 S} I_{s-1,\tau_1}^{(k)}}{\sum_{\tau_1=0}^{N-1} y_{s+\tau_1 S}^2 I_{s-1,\tau_1}^{(k)}} y_{s-1+\tau S} I_{s-1,\tau}^{(k)} - \lambda_{\varepsilon,s} \right)^2,$$

with $I_{s,\tau}^{(k)}$, $k = 1, 2$ are defined in (2.2.3), from which c_s $s = 1, ..., S$, can be estimated as $\widehat{c}_s = \arg\min_{c_s \in [\underline{c}_s, \overline{c}_s]} S_N(c_s)$. Following Li and Tong (2016), let $J_N(c_s) = S_N - S_N(c_s)$, where

$$S_N = \sum_{\tau=0}^{N-1} \left(y_{s+\tau S} - \frac{\sum_{\tau_1=0}^{N-1} \left(y_{s+\tau_1 S} - \lambda_{\varepsilon,s} \right) y_{s-1+\tau_1 S}}{\sum_{\tau_2=0}^{N-1} y_{s+\tau_2 S}^2} y_{s-1+\tau S} - \lambda_{\varepsilon,s} \right)^2.$$

Then, the periodic threshold c_s can be estimated by maximizing the function $J_N(c_s)$, i.e., $\widehat{c}_s = \arg\max_{c_s \in [\underline{c}_s, \overline{c}_s]} J_N(c_s)$, where $\underline{c}_s$ and $\overline{c}_s$ can be selected as the minimum and maximum values of the samples, respectively, and $\lambda_{\varepsilon,s}$ can take any positive integer value.

2.5 Simulation results and application on real data

In this Section, the purpose is to illustrate the theoretical results given in Section 2.4 and to assess the Conditional Least-Squares (CLS) and Conditional Maximum Likelihood (CML) estimations on two time series, of small, moderate and relatively large sample sizes and also an application on real dataset.

2.5.1 Simulation results

In order to show some empirical estimates properties, we have generated 1000 independent series from $PSETINAR_4(2;1)$ model, with a periodic Poisson innovation distribution, $\mathcal{P}(\lambda_s)$, $s = 1, ..., 4$. The threshold parameters c_s are assumed to be known for Model 1, and unknown for Model 2. The true parameter values of these models are given below :

Model 1 : $\underline{\theta} = (\underline{\theta}_1; \ldots; \underline{\theta}_4)' = ((0.1, 0.7, 3); (0.2, 0.65, 4); (0.6, 0.1, 5); (0.5, 0.8, 2))'$,

$\underline{C} = (c_1, \ldots, c_4) = (6, 9, 13, 11)$, $\varepsilon_t \rightsquigarrow \mathcal{P}(\lambda_{\varepsilon,s})$, $s = 1, ..., 4$, with $\underline{C}$ is known,

Model 2 : $\underline{\theta} = (\underline{\theta}_1; \ldots; \underline{\theta}_4)' = ((0.8, 0.4, 3); (0.15, 0.65, 6); (0.2, 0.7, 5); (0.5, 0.8, 4))'$,

$\underline{C} = (c_1, \ldots, c_4) = (10, 8, 15, 9)$, $\varepsilon_t \rightsquigarrow \mathcal{P}(\lambda_{\varepsilon,s})$, $s = 1, ..., 4$, with $\underline{C}$ is unknown.

For each of the above models, the periodic threshold values, c_s, $s = 1, ..., 4$, were chosen such that the observations in each regime of each period are at least 20% of the sub-series size. As mentioned in Li and Tong (2016), when the proportion of observations in sub-series, i.e., $\{y_{s+\tau S}; \tau \in \mathbb{Z}\}$ while s being fixed in $\{1, ..., 4\}$, of one regime to the whole is less than 5%, the estimated result may not be reliable. The simulation was repeated 1000 times, the mean estimates and their root mean square error $RMSE$ are displayed in Tables 2.1, 2.2 and 2.3. Table 2.1 reports the means, median and root mean square errors ($RMSE$) of the CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$ and CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$, across 1000 replications, for the first model when c_s, $s = 1, ..., 4$ are known. While Table 2.2 reports the performance of the last series with the periodic threshold parameters are considered to be unknown where these threshold parameters are estimated firstly by using the periodic adaptation ($NeSS$) algorithm, the results are reported in Table 2.3.

From Table 2.1, one can easily observe that the adopted estimation method performs better as n increases, i.e., the convergence of all the estimated parameters is guaranteed, which is visible through the box plots in Figure 2.1, namely that they become narrower as the sample size n increases, in the same time, the root mean square error decreasing, see Figure 2.2, which imply that our estimators (CLS- vector estimators $\widehat{\underline{\theta}}_{s,CLS}$, and CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$) are empirically consistent for all the parameters. Moreover, we can notice that the parameters $\alpha_{i,s},\ s = 1, ..., 4,\ i = 1, 2,$ are regularly overestimated, whereas the parameters $\lambda_{\varepsilon,s},\ s = 1, ..., 4,$ are underestimated. Generally, this behavior of the estimates, in terms of the propensity to underestimate or overestimate the parameters, is encountered even in the classical time-invariant model (see, Monteiro *et al.* 2012). Furthermore, it can also be seen that the CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$ have a small root mean square error $RMSE$ compared to one of the CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$, thus implying that the CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$ are much advantage than the CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$, this empirical superiority is noticeable in Figures 2.1 and 2.2.

Moreover, we get the same conclusions from Table 2.2 and Figures 2.3 and 2.4, that the consistency property of the CLS-vector estimators and CML-vector estimators still met, even if the threshold parameters c_s, $s = 1, ..., 4$, are unknown, this encourages us to use confidently the $PSETINAR_4\,(2;1)$ model, when the periodic threshold parameters c_s are unknown, without worrying about to bring a wrong results. The advantage of this scenario is that the estimation behavior, concerning the propensity to underestimate or overestimate the parameters, has disappeared.

Furthermore, from Table 2.3, which reports the means, medians, the percentage, and the root mean square error ($RMSE$) of the periodic threshold estimations $\widehat{c}_s$, $s = 1, ..., 4$, we can find that all the estimation results perform better as n increases, thus implying that the periodic adaptation Nested Sub-Sample Search ($NeSS$) algorithm is empirically consistent. Also, we observe that the median of 1000 repetitions, for the periodic threshold estimations $\widehat{c}_s$, $s = 1, ..., 4$, estimated is generally better than the mean.

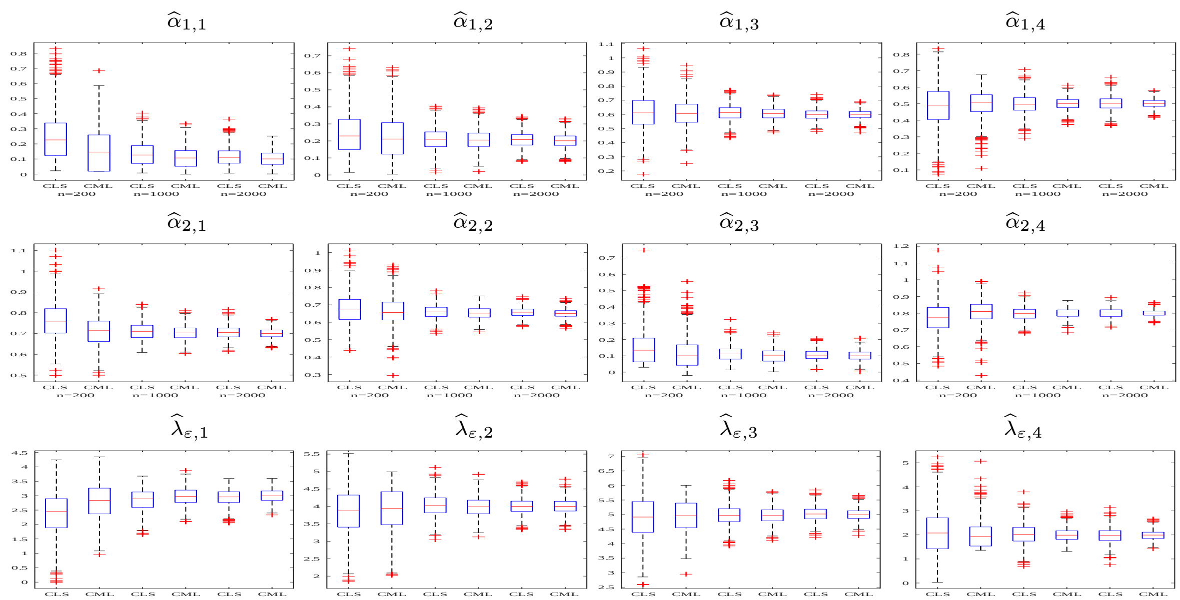

Figure 2.1. Box-plots of the CLS and CML estimation parameters for Model 1.

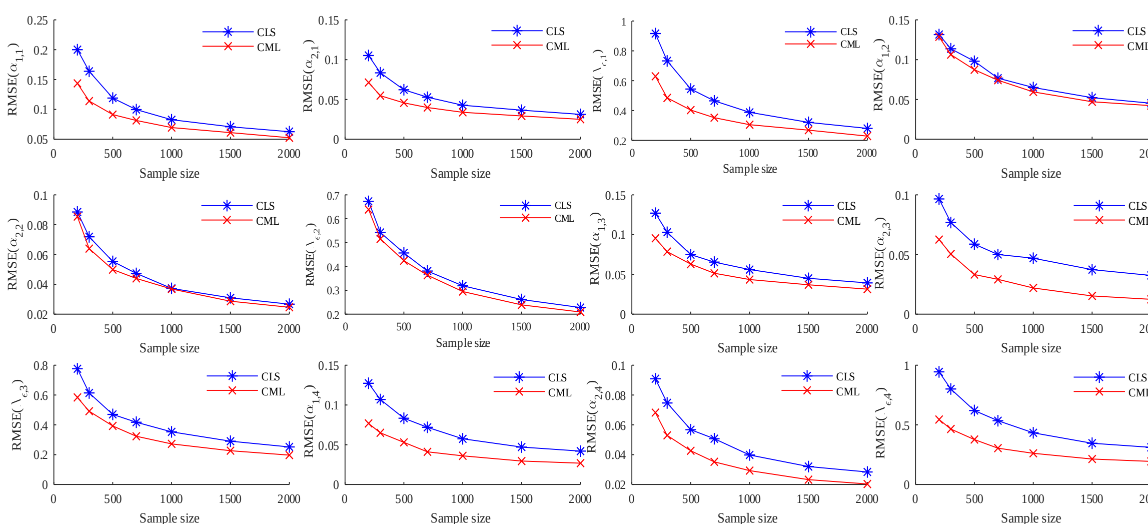

Figure 2.2. RMSE graphics for the parameters of Model 1 for the different methods.

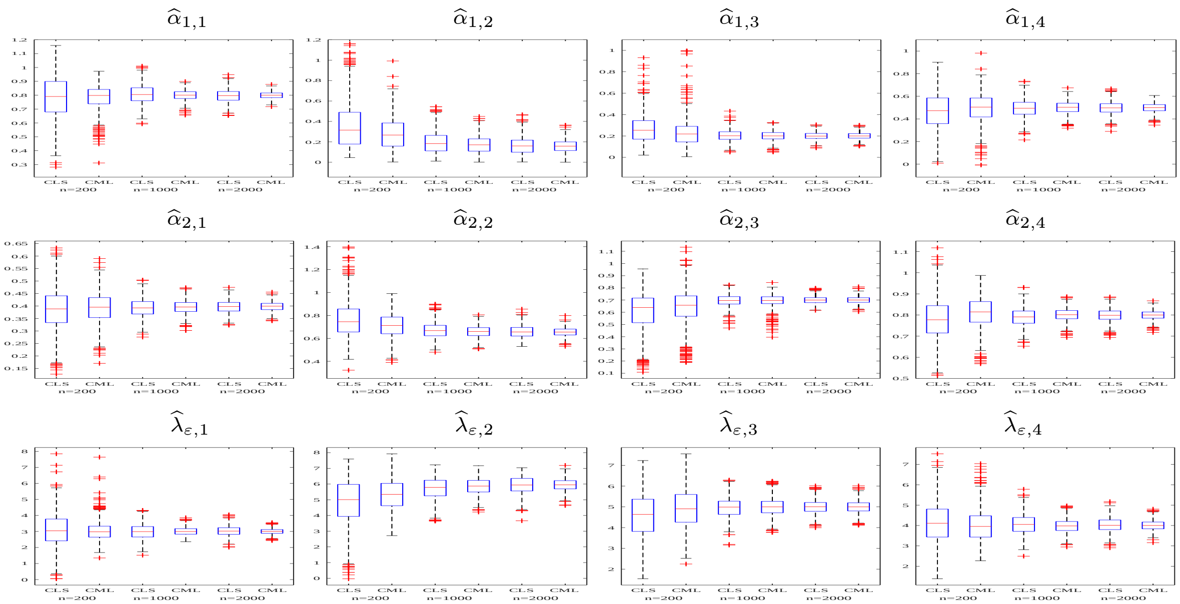

Figure 2.3. Box-plots of the CLS and CML estimation parameters for Model 2.

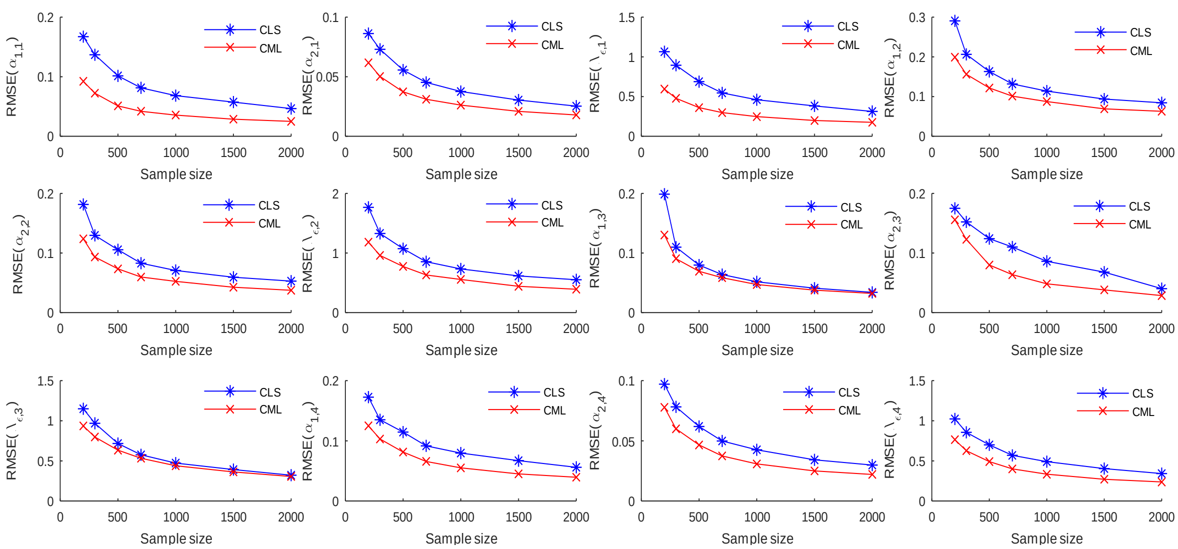

Figure 2.4. RMSE graphics for the parameters of Model 2 for the different methods.

Table 2.1. *Sample mean and root mean square error RMSE (in bracket) for Model 1.*

	T.V	0.1	0.7	3	0.2	0.65	4	0.6	0.1	5	0.5	0.8	2
Size	*Est.*	$\widehat{\alpha}_{1,1}$	$\widehat{\alpha}_{2,1}$	$\widehat{\lambda}_{\varepsilon,1}$	$\widehat{\alpha}_{1,2}$	$\widehat{\alpha}_{2,2}$	$\widehat{\lambda}_{\varepsilon,2}$	$\widehat{\alpha}_{1,3}$	$\widehat{\alpha}_{2,3}$	$\widehat{\lambda}_{\varepsilon,3}$	$\widehat{\alpha}_{1,4}$	$\widehat{\alpha}_{2,4}$	$\widehat{\lambda}_{\varepsilon,4}$
200	*CLS*	0.2264 (0.2001)	0.7562 (0.1052)	2.4500 (0.9162)	0.2291 (0.1316)	0.6707 (0.0885)	3.8732 (0.6729)	0.6153 (0.1270)	0.1347 (0.0966)	4.9136 (0.7758)	0.4915 (0.1273)	0.7758 (0.0909)	2.0792 (0.9439)
	CML	0.1461 (0.1436)	0.7137 (0.0713)	2.8376 (0.6296)	0.2111 (0.1288)	0.6560 (0.0854)	3.9407 (0.6385)	0.6045 (0.0954)	0.1004 (0.0225)	4.9608 (0.5840)	0.5085 (0.0768)	0.8105 (0.0682)	1.9338 (0.5450)
300	*CLS*	0.2039 (0.1638)	0.7390 (0.0835)	2.6113 (0.7328)	0.2185 (0.1137)	0.6678 (0.0719)	3.8771 (0.5423)	0.6186 (0.1030)	0.1295 (0.0768)	4.9200 (0.6137)	0.4921 (0.1069)	0.7819 (0.0747)	2.0717 (0.8009)
	CML	0.1255 (0.1140)	0.7087 (0.0548)	2.8964 (0.4850)	0.2037 (0.1064)	0.6537 (0.0640)	3.9801 (0.5153)	0.6021 (0.0786)	0.1006 (0.0205)	4.9911 (0.4915)	0.5020 (0.0650)	0.8031 (0.0529)	1.9774 (0.4654)
500	*CLS*	0.1579 (0.1190)	0.7237 (0.0622)	2.7616 (0.5443)	0.2166 (0.0981)	0.6670 (0.0554)	3.8795 (0.4567)	0.6182 (0.0748)	0.1183 (0.0587)	4.9498 (0.4703)	0.4935 (0.0833)	0.7879 (0.0567)	2.0526 (0.6205)
	CML	0.1124 (0.0913)	0.7066 (0.0458)	2.9416 (0.4034)	0.2034 (0.0874)	0.6545 (0.0499)	3.9808 (0.4242)	0.5983 (0.0629)	0.1005 (0.0192)	5.0048 (0.3928)	0.4997 (0.0530)	0.8018 (0.0427)	1.9869 (0.3774)
700	*CLS*	0.1398 (0.0997)	0.7172 (0.0529)	2.8263 (0.4652)	0.2125 (0.0768)	0.6613 (0.0473)	4.0250 (0.3809)	0.6113 (0.0655)	0.1147 (0.0501)	4.9533 (0.4175)	0.4950 (0.0719)	0.7960 (0.0507)	2.0454 (0.5348)
	CML	0.1119 (0.0815)	0.7050 (0.0399)	2.9498 (0.3522)	0.2046 (0.0742)	0.6545 (0.0440)	3.9859 (0.3633)	0.6007 (0.0514)	0.1034 (0.0137)	4.9831 (0.3235)	0.5021 (0.0411)	0.8011 (0.0352)	1.9876 (0.3048)
1000	*CLS*	0.1267 (0.0825)	0.7104 (0.0432)	2.8870 (0.3768)	0.2096 (0.0661)	0.6594 (0.0383)	4.0246 (0.3298)	0.6117 (0.0565)	0.1117 (0.0451)	4.9674 (0.3506)	0.4967 (0.0577)	0.7975 (0.0412)	2.0289 (0.4333)
	CML	0.1068 (0.0695)	0.7031 (0.0338)	2.9740 (0.3053)	0.2051 (0.0595)	0.6523 (0.0369)	3.9915 (0.2952)	0.6048 (0.0435)	0.1039 (0.0130)	4.9644 (0.2725)	0.5012 (0.0361)	0.8013 (0.0293)	1.9901 (0.2619)
1500	*CLS*	0.1208 (0.0710)	0.7086 (0.0365)	2.9127 (0.3212)	0.2095 (0.0523)	0.6589 (0.0310)	4.0025 (0.2617)	0.5968 (0.0451)	0.0970 (0.0373)	5.0321 (0.2906)	0.5037 (0.0471)	0.7982 (0.0321)	2.0274 (0.3464)
	CML	0.1024 (0.0611)	0.7010 (0.0292)	2.9904 (0.2684)	0.2006 (0.0471)	0.6508 (0.0287)	3.9938 (0.2390)	0.6023 (0.0370)	0.1022 (0.0124)	4.9822 (0.2269)	0.5021 (0.0295)	0.8010 (0.0233)	1.9936 (0.2139)
2000	*CLS*	0.1108 (0.0627)	0.7046 (0.0313)	2.9556 (0.2802)	0.2080 (0.0452)	0.6576 (0.0267)	4.0024 (0.2280)	0.5992 (0.0394)	0.1042 (0.0325)	5.0178 (0.2522)	0.5025 (0.0420)	0.8011 (0.0284)	1.9741 (0.3118)
	CML	0.1005 (0.0525)	0.7000 (0.0251)	2.9950 (0.2291)	0.2007 (0.0423)	0.6497 (0.0246)	4.0011 (0.2093)	0.6002 (0.0315)	0.1002 (0.0125)	4.9942 (0.1973)	0.5016 (0.0269)	0.8009 (0.0203)	1.9945 (0.1938)

Table 2.2. *Sample mean and root mean square error RMSE (in bracket) for Model 2.*

	T.V	0.8	0.4	3	0.15	0.65	6	0.2	0.7	5	0.5	0.8	4
Size	*Est.*	$\widehat{\alpha}_{1,1}$	$\widehat{\alpha}_{2,1}$	$\widehat{\lambda}_{\varepsilon,1}$	$\widehat{\alpha}_{1,2}$	$\widehat{\alpha}_{2,2}$	$\widehat{\lambda}_{\varepsilon,2}$	$\widehat{\alpha}_{1,3}$	$\widehat{\alpha}_{2,3}$	$\widehat{\lambda}_{\varepsilon,3}$	$\widehat{\alpha}_{1,4}$	$\widehat{\alpha}_{2,4}$	$\widehat{\lambda}_{\varepsilon,4}$
200	*CLS*	0.7907 (0.1672)	0.3887 (0.0862)	3.0434 (1.0650)	0.3154 (0.2907)	0.7453 (0.1815)	5.0109 (1.7674)	0.2529 (0.1988)	0.6392 (0.1751)	4.6341 (1.1500)	0.4719 (0.1726)	0.7775 (0.0971)	4.1135 (1.0235)
	CML	0.7982 (0.0923)	0.3952 (0.0618)	2.9829 (0.5948)	0.2671 (0.1991)	0.7128 (0.1238)	5.3429 (1.1833)	0.2188 (0.1303)	0.6577 (0.1558)	4.9041 (0.9363)	0.5050 (0.1250)	0.8149 (0.0779)	3.9533 (0.7651)
300	*CLS*	0.8062 (0.1370)	0.3876 (0.0731)	2.9737 (0.8945)	0.2556 (0.2064)	0.7098 (0.1296)	5.3478 (1.3288)	0.2242 (0.1099)	0.6723 (0.1522)	4.8215 (0.9696)	0.4883 (0.1350)	0.8133 (0.0782)	4.0813 (0.8563)
	CML	0.8037 (0.0724)	0.3984 (0.0502)	2.9732 (0.4784)	0.2264 (0.1563)	0.6903 (0.0933)	5.5476 (0.9601)	0.2053 (0.0905)	0.6836 (0.1151)	4.9965 (0.7982)	0.5076 (0.1031)	0.8108 (0.0600)	3.9572 (0.6277)
500	*CLS*	0.7942 (0.1015)	0.3897 (0.0556)	3.0079 (0.6884)	0.2243 (0.1634)	0.6931 (0.1056)	5.5309 (1.0730)	0.2150 (0.0797)	0.6919 (0.1242)	4.8800 (0.7172)	0.4917 (0.1146)	0.7863 (0.0620)	4.0791 (0.7005)
	CML	0.7988 (0.0511)	0.3943 (0.0374)	3.0064 (0.3608)	0.1997 (0.1217)	0.6771 (0.0734)	5.6969 (0.7753)	0.2067 (0.0694)	0.6968 (0.0729)	4.9572 (0.6341)	0.5034 (0.0814)	0.8087 (0.0466)	3.9682 (0.4915)
700	*CLS*	0.8054 (0.0814)	0.3918 (0.0453)	3.0013 (0.5468)	0.2020 (0.1319)	0.6814 (0.0829)	5.6640 (0.8543)	0.2031 (0.0638)	0.6941 (0.1101)	4.9774 (0.5757)	0.5112 (0.0917)	0.8102 (0.0499)	3.9455 (0.5696)
	CML	0.8025 (0.0421)	0.3984 (0.0311)	2.9989 (0.2986)	0.1849 (0.1014)	0.6699 (0.0598)	5.7817 (0.6316)	0.2001 (0.0586)	0.6962 (0.0637)	5.0063 (0.5337)	0.5059 (0.0659)	0.8033 (0.0375)	3.9741 (0.4006)
1000	*CLS*	0.8049 (0.0684)	0.3926 (0.0376)	2.9922 (0.4608)	0.1828 (0.1139)	0.6674 (0.0707)	5.7904 (0.7328)	0.2014 (0.0517)	0.6971 (0.0861)	4.9896 (0.4733)	0.4937 (0.0797)	0.7914 (0.0426)	4.0511 (0.4911)
	CML	0.8009 (0.0357)	0.3959 (0.0262)	2.9972 (0.2478)	0.1707 (0.0876)	0.6595 (0.0525)	5.8716 (0.5569)	0.2008 (0.0471)	0.6973 (0.0483)	4.9952 (0.4396)	0.5025 (0.0551)	0.8014 (0.0308)	3.9786 (0.3343)
1500	*CLS*	0.7963 (0.0575)	0.4022 (0.0304)	2.9997 (0.3829)	0.1653 (0.0939)	0.6571 (0.0594)	5.9081 (0.6156)	0.2011 (0.0410)	0.6985 (0.0679)	4.9886 (0.3927)	0.5045 (0.0671)	0.8030 (0.0342)	3.9813 (0.4045)
	CML	0.8011 (0.0287)	0.4002 (0.0209)	2.9858 (0.1991)	0.1621 (0.0691)	0.6551 (0.0426)	5.9289 (0.4415)	0.2004 (0.0377)	0.6994 (0.0382)	4.9961 (0.3629)	0.5028 (0.0451)	0.8013 (0.0250)	3.9958 (0.2716)
2000	*CLS*	0.7966 (0.0466)	0.3977 (0.0253)	3.0233 (0.3133)	0.1603 (0.0845)	0.6557 (0.0529)	5.9375 (0.5519)	0.1992 (0.0340)	0.6995 (0.0402)	5.0067 (0.3223)	0.4975 (0.0561)	0.7987 (0.0299)	4.0140 (0.3420)
	CML	0.7997 (0.0251)	0.3992 (0.0180)	3.0025 (0.1767)	0.1584 (0.0631)	0.6545 (0.0375)	5.9509 (0.3928)	0.2003 (0.0323)	0.7003 (0.0285)	4.9955 (0.3063)	0.4995 (0.0396)	0.7997 (0.0221)	4.0012 (0.2377)

Table 2.3. Simulation results of the threshold parameters for Model 2.

	c_1			c_2			c_3			c_4		
T.V	10			8			15			9		
Size	$\widehat{c}_1$	*percent*	*med*	$\widehat{c}_2$	*percent*	*med*	$\widehat{c}_3$	*percent*	*med*	$\widehat{c}_4$	*percent*	*med*
200	10.1660 (0.8489)	0.7700	10	8.1280 (0.7433)	0.8580	8	14.4940 (2.0484)	0.4050	15	8.9870 (1.0845)	0.6830	9
300	10.0930 (0.5489)	0.8810	10	8.0140 (0.1950)	0.9650	8	14.9470 (1.0536)	0.5910	15	9.0600 (1.0005)	0.7510	9
500	10.0180 (0.2001)	0.9660	10	8.0070 (0.1141)	0.9930	8	15.0220 (0.5022)	0.8110	15	9.0120 (0.5369)	0.8840	9
700	10.0160 (0.1484)	0.9810	10	8.0020 (0.0447)	0.9980	8	15.0040 (0.3348)	0.9020	15	9.0110 (0.3148)	0.9500	9
1000	10.0060 (0.0775)	0.9940	10	8.0000 (0)	1.0000	8	14.9960 (0.1950)	0.9620	15	9.0050 (0.2122)	0.9790	9
1500	10.0010 (0.0316)	0.9990	10	8.0000 (0)	1.0000	8	14.9970 (0.0949)	0.9910	15	9.0000 (0)	1.0000	9
2000	10.0000 (0)	1.0000	10	8.0000 (0)	1.0000	8	15.0010 (0.0316)	0.9990	15	9.0010 (0.0316)	0.9990	9

2.5.2 Real data study

In this paragraph, we consider the dataset of size 365 observations, recorded from births in the Quebec-Canada, for the year 1986. The studied series was, from 01 January 1977 to 31 December 1990, presented by Solanki (2016), who suggested for it the $ARIMA$ model. The data set was collected from `www.datamarket.com`, which is now acquired by `www.qlik.com`. The visualization of the considered time series is shown in Figure 2.5, while Table 2.4 summarizes some descriptive statistics.

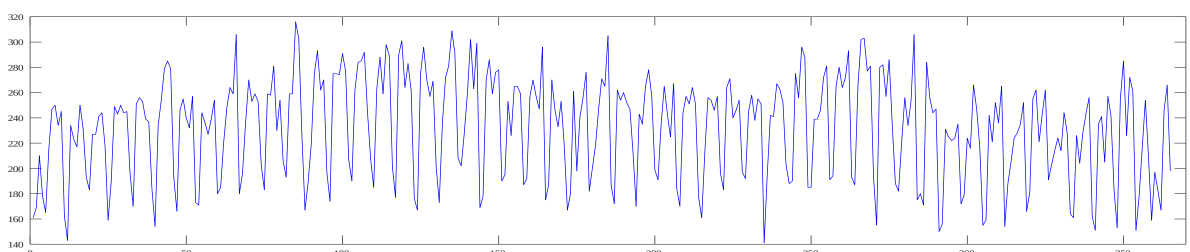

Figure 2.5. Trajectory of the births in Quebec time series.

A look at Table 2.4 explains the reason why we choose the $PSETINAR_7\,(2;1)$ model. Indeed, the fact that the series appears to be overdispersed, indicating that, marginally, Poisson distribution might not be appropriate, but our periodic nonlinear model seems to be adequate for modeling this overdispersed data set.

Table 2.4. Some descriptive statistics for the births in Quebec time series.

Sample size	Minimum	Maximum	Median	Mean	Variance	Skewness	Kurtosis
365	141	316	241	231.8740	1624.0830	−0.2918	2.1038

Basically, the type of periodic model may contribute to describing the overdispersion feature of a periodic integer valued time series. This feature was observed in Bentarzi and Aries (2020*a*) where the authors demonstrated, see Corollary 3.3.1 and Remark 3.3.1 on page 9, that their proposed periodic $INARMA(1,1)$ model is appropriate for modeling an overdispersed data set even with a Poisson innovation distribution. Thereafter, by analyzing the empirical autocorrelation function (ACF) and the empirical partial autocorrelation function ($PACF$) in Figure 2.6, the time series presents periodicity of season $S = 7$, due to the effect of the day. Hence, we are interested in fitting a periodic self-exciting threshold integer-valued autoregressive time series model for this data set.

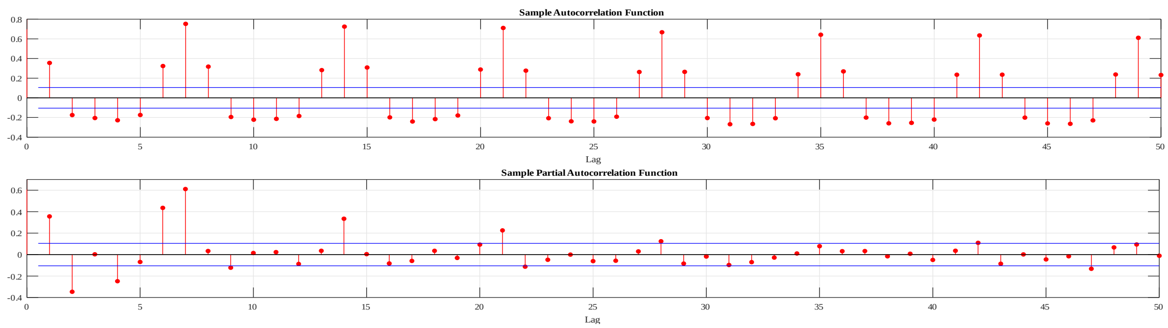

Figure 2.6. Autocorrelations of the births in Quebec time series.

Table 2.5 gives the estimation results of our $PSETINAR_7\,(2;1)$ model. However, the estimated threshold parameters (c_s, $s = 1, ..., 7$) in each period, by using the periodic adaptation ($NeSS$) algorithm, are given by $\underline{C} = (c_1, ..., c_7) = (257, 255, 252, 197, 185, 190, 262)$:

Table 2.5. Estimation results of the $PSETINAR_7\,(2;1,1)$.

s	1	2	3	4	5	6	7
$\alpha_{1,s}$	0.0889	0.3222	0.1961	0.2116	0.0000	0.0041	0.0000
$\alpha_{2,s}$	0.1517	0.3886	0.2526	0.1584	0.0478	0.0000	0.1256
$\lambda_{\varepsilon,s}$	223.0117	165.8242	199.5347	141.4789	173.9122	241.3493	248.7854

Figure 2.7 shows empirically that the residuals of our estimated model do not indicate any statistical significant autocorrelation. So, the adequacy of the model is not statistically rejected. Moreover, one can easily observe that the periodic feature of the residual autocorrelation for the fitted model $PSETINAR_7\,(2;1)$ has been completely disappeared thus the periodic feature has been taken into account by this model. Figure 2.8, exhibits the adjusted trajectory of the births in the Quebec-Canada dataset, such as the series values are shown in blue, while the red line denotes the adjusted series. The fitted values of the

$PSETINAR_7\,(2;1)$ model seem to be suitable for the real dataset values.

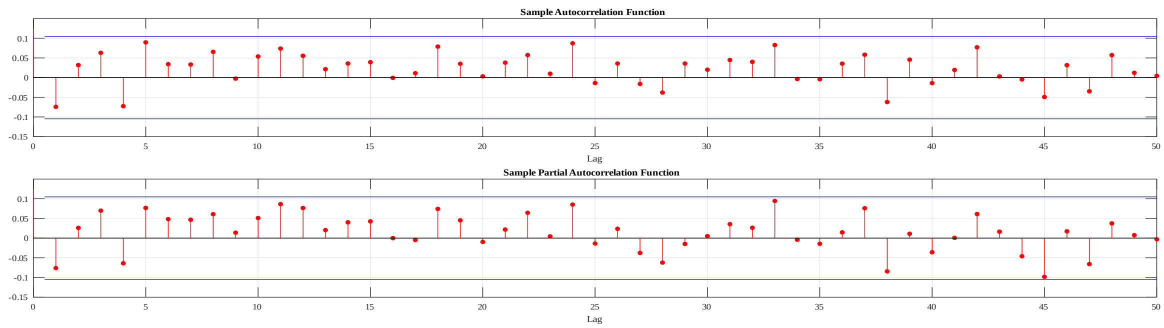

Figure 2.7. Autocorrelations of the residual time series.

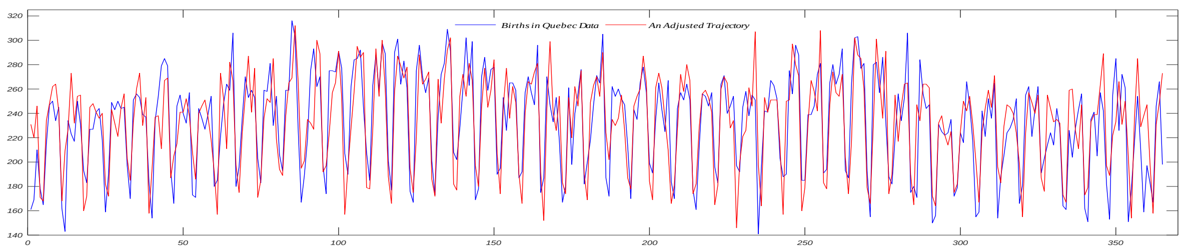

Figure 2.8. An adjusted trajectory proposed to the births in Quebec dataset.

Chapter 3

Periodic Negative Binomial SETINAR Model

3.1 Introduction

In certain situations, it becomes necessary to deal with non-negative integer-valued time series, which take account of the positivity and the discreteness nature of their generator processes, in many practical areas such as medical field (e.g., *the time series consisting on the number of cases of campylobacterosis infections* studied by Ferland *et al.* 2006, Bentarzi and Bentarzi 2017*a* and Ouzzani and Bentarzi 2019), sociological field (e.g., *number of short-term unemployed people in Penamacor County Portugal,* studied by Monteiro *et al.* 2010) and more others. Modeling non-negative integer-valued time series is difficult with the mathematical tools meant for real-valued time series issues, which is not reflective of the discrete nature of the series under study, many researchers have found it helpful to suggest more of time series models. Indeed, considerable attention has been given over the last two decades to modeling and studying the probabilistic and statistical properties in either case, linear or nonlinear non-negative integer-valued time series models, we refer some of their works (see, Al-Osh and Alzaid 1987, Alzaid and Al-Osh 1990, Du and Li 1991, Silva and Oliveira 2000 and many others). Moreover, it seems that the class of nonlinear models, which is the most theoretically studied and practically employed in the analysis of real-time series, is the class of threshold autoregressive (TAR) models, which was introduced, for the first time in the time series literature, by the pioneer Tong (1978) and explored by Tong and Lim (1980), to modeling the high threshold exceedances appearing in clusters, and the so-called piece-wise phenomenon. Furthermore, Lewis and Ray (2002) have introduced and studied the periodic threshold autoregression model as an extension of the threshold autoregressive (TAR) model, to capture and describe the periodicity feature exhibited by the autocovariance structure and which cannot be accounted for by the classical time-invariant parameters time series models. However, for dealing with time series of counts exhibiting a piece-wise type of patterns, Thyregod *et al.* (1999) introduced a class of self-exciting threshold $INAR$ model to analyze tipping bucket rainfall measurements, as well as, Monteiro *et al.* (2012) presented a particular class of self-exciting threshold $INAR$ model driven by independent Poisson distributed random variables.

It is well-known that the Poisson distribution is not always suitable for modeling and studying time series of counts, as was pointed out by Ristić *et al.* (2009). This is due to the equality property of the mean and the variance of a Poisson distribution which is not always verified in the real world. Indeed, many time series encountered in a variety of fields show an overdispersion feature. Consequently, Yang *et al.* (2018*b*) introduced a new integer-valued threshold autoregressive process based on a negative binomial thinning operator $NBSETINAR$. Thereafter, this model is redefined by Wang *et al.* (2019) to give some useful extensions of the original model by removing the assumption of negative-binomial innovations to make the model more flexible. Moreover, many economic, financial and environmental integer-valued time series, which have encountered in practice, reveal the periodicity feature in their autocovariance structures. It is recognized that this periodic feature cannot be adequately accounted for and described by time-invariant parameter integer-valued time series models. Despite of the various advantages and interesting properties satisfied by the $NBSETINAR$ model such as the positivity and the discreteness nature of the realizations, the high threshold exceedances appearing in clusters, and the so-called piece-wise phenomenon, this model still unable to capture the periodicity feature. These facts which concern both the non-periodic $NBSETINAR$ models and the periodicity feature, gave a good reason and motivation to extend this class of time-invariant models to the periodic negative binomial self-exciting threshold integer-valued autoregressive $PNBSETINAR$ models, with time-periodic coefficients. To our knowledge, Monteiro *et al.* (2010) and Morinã *et al.* (2011) were the pioneers on the modeling of the periodically correlated, in the sense of Gladyshev (1961), integer-valued process and over time, the topic has made progress, see for example, Bentarzi and Bentarzi (2017*a*, 2017*b*), Sadoun and Bentarzi (2019) and many others.

In the next section, we provide the basic notations, definitions and main assumptions concerning the class of $PNBSETINAR$ models. In Section 3.3, we discuss its basic probabilistic and statistical properties. Indeed, the first and the second moment periodically stationary conditions are established, while establishing, under these conditions, their-closed forms. In addition, the existence of high moment and the strict periodic stationarity, are studied. The autocovariance structure is also acquired. The unknown periodic parameters of our models are estimated, while using both the Conditional Least Squares (CLS) and the Conditional Maximum Likelihood (CML) estimation methods, in Section 3.4. The unknown periodic threshold parameter is estimated by using a periodic adaptation Nested Sub-Sample Search ($NeSS$) algorithm. In Section 3.5, a simulation study was used to illustrate the performance of the suggested estimate methods, and an application on a real data set was presented.

3.2 Notations, definitions and main assumptions

Recall that the integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$ is said to satisfy a negative binomial self-exciting threshold integer-valued autoregressive model, if it is given by:

$$y_t = (\alpha_1 * y_{t-1} + \varepsilon_{1,t})\, I_{t-1}^{(1)}(c) + (\alpha_2 * y_{t-1} + \varepsilon_{2,t})\, I_{t-1}^{(2)}(c)\,,\ t \in \mathbb{Z}. \tag{3.2.1}$$

A stochastic process $\{y_t;\ t \in \mathbb{Z}\}$ is said to follow a periodic negative binomial self-exciting threshold integer-valued autoregressive model, of order one with two regimes and period S ($S \geq 2$), if it is a solution of the following non-linear difference stochastic equation :

$$y_t = (\alpha_{1,t} * y_{t-1} + \varepsilon_{1,t})\, I_{t-1}^{(1)}(c_t) + (\alpha_{2,t} * y_{t-1} + \varepsilon_{2,t})\, I_{t-1}^{(2)}(c_t)\,,\ t \in \mathbb{Z}, \tag{3.2.2}$$

with $I_{t-1}^{(1)}(c_t)$ is a sequence of independent Bernoulli random variables defined by :

$$I_{t-1}^{(1)}(c_t) = \begin{cases} 1 & if \quad y_{t-1} \leq c_t, \\ 0 & if \quad y_{t-1} > c_t, \end{cases} \text{ and } I_{t-1}^{(2)}(c_t) = 1 - I_{t-1}^{(1)}(c_t)\,, \tag{3.2.3}$$

where the threshold parameters c_t are assumed to be unknown. The underlying non-negative integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$ is a periodically correlated, in the Gladyshev's sense

(1961), with period S ($S \geq 2$) and the innovation process $\{\varepsilon_{i,t};\ t \in \mathbb{Z}\}$, $i = 1, 2$, is a periodic sequence of independent and identically non-negative integer-valued random variables, with a negative binomial distribution $\mathcal{NB}(r_t, \alpha_{i,t}/(1+\alpha_{i,t}))$, where for a fixed t and i ($i = 1, 2$), the probability mass function is given below :

$$P(\varepsilon_{i,t} = n) = \frac{\Gamma(n + r_t)}{\Gamma(r_t)\Gamma(n+1)} \frac{\alpha_{i,t}^n}{(1+\alpha_{i,t})^{r_t+n}},\ n \in \mathbb{N}.$$

The symbol "$*$" stands for the negative binomial thinning operator (Ristic *et al.* 2009) which is defined, for the non-negative integer-valued stochastic process y_{t-1}, by :

$$\alpha_{i,t} * y_{t-1} = \begin{cases} \sum_{j=1}^{y_{t-1}} Z_{j,t}^{(i)}, & \text{if } y_{t-1} > 0, \\ 0, & \text{if } y_{t-1} = 0, \end{cases}$$

such that $Z_{j,t}^{(i)}$, $i = 1, 2$, is a sequence of independent and identically distributed geometric random variables $\mathcal{G}(\alpha_{i,t}/(1+\alpha_{i,t}))$, with $\alpha_{i,t} \in [0, 1[$, $t \in \mathbb{Z}$ and $i = 1, 2$, where the probability mass function is given, for a fixed t and i ($i = 1, 2$), by:

$$P\left(Z_{j,t}^{(i)} = k\right) = \alpha_{i,t}^k \Big/ (1+\alpha_{i,t})^{k+1},\quad k \in \mathbb{N}.$$

Moreover, all the parameters $\alpha_{1,t}$, $\alpha_{2,t}$, r_t and c_t are periodic, in time, with a period S where, S is the smallest positive integer such that $\alpha_{1,t+kS} = \alpha_{1,t}$, $\alpha_{2,t+kS} = \alpha_{2,t}$, $r_{t+kS} = r_t$ and $c_{t+kS} = c_t$. After that, for a fixed t and i ($i = 1, 2$), the innovation process $\{\varepsilon_{i,t};\ t \in \mathbb{Z}\}$ is assumed to be independent of the counting series $\{\alpha_{i,t-l} * y_{t-l},\ l \geq 1\}$ and $\{y_{t-l},\ l \geq 1\}$ also, the processes $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are assumed to be mutually independent. Assuming that the process $\{y_t;\ t \in \mathbb{Z}\}$ is periodically stationary with S-periodic geometric $\mathcal{G}(\lambda_t/(1+\lambda_t))$ distribution, then for the law of the innovation process $\varepsilon_{i,t}$, $i = 1, 2$, to be well defined, the parameter $\alpha_{i,t}$, for a fixed t and $i = 1, 2$, must take its values, as it has been rigorously shown, in the time-invariant case, by Ristic *et al.* (2009), in the interval $[0, \lambda_t/(1+\lambda_t)]$ and it is, more precisely, distributed as a mixture of two random variables with geometric $\mathcal{G}(\lambda_t/(1+\lambda_t))$ and geometric $\mathcal{G}(\alpha_{i,t}/(1+\alpha_{i,t}))$ distributions. Besides, one can rewrite the periodic negative binomial self-exciting integer-valued threshold autoregressive model, with a period S ($S \geq 2$) given in (3.2.2) in the equivalent form

$$y_t = \varphi_t * y_{t-1} + \varepsilon_t,\quad t \in \mathbb{Z}, \tag{3.2.4}$$

where, $\varphi_t = \alpha_{1,t} I_{t-1}^{(1)}(c_t) + \alpha_{2,t} I_{t-1}^{(2)}(c_t)$ and $\varepsilon_t = \varepsilon_{1,t} I_{t-1}^{(1)}(c_t) + \varepsilon_{2,t} I_{t-1}^{(2)}(c_t)$. Throughout the chapter, we will omit (c_t) in $I_{t-1}^{(k)}(c_t)$, $k = 1, 2$, to make the notation easy without ambiguity.

3.3 Basic properties of the PNBSETINAR model

In this paragraph, we provide the conditions on parameters of the underlying integer-valued process to be periodically stationary in the first and the second order. Furthermore, under these conditions, the closed forms of the periodic mean and the periodic variance are acquired. Moreover, the existence of the unconditional m-th moment and the strict periodic stationarity of the process $\{y_t;\ t \in \mathbb{Z}\}$ are established. In addition, the autocovariance structure is also obtained.

3.3.1 Periodic stationarity in the two first moments

The results given in the following proposition establish the necessary and sufficient conditions, for the process $\{y_t;\ t \in \mathbb{Z}\}$ satisfying (3.2.2) to be periodically stationary with respect to the first two moments. The closed-forms of these moments are then, under these conditions, obtained.

Proposition 3.3.1 *The process $\{y_t;\ t \in \mathbb{Z}\}$, satisfying the model (3.2.2), is periodically stationary in the first moment if and only if, the periodic parameters $\alpha_{1,t}$ and $\alpha_{2,t}$ satisfy the periodically stationary conditions $\prod_{i=1}^{S} \alpha_{1,i} < 1$ and $\prod_{i=1}^{S} \alpha_{2,i} < 1$, respectively, then the unconditional periodic mean is, under these conditions, given by*

$$\mu_{y,s} = \mathbb{E}\left(y_{s+\tau S}\right) = \left(1 - \prod_{i=1}^{S} \alpha_{2,i}\right)^{-1} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \alpha_{2,s-i+1}\right) \Phi_{s-j+1},$$

where, $\Phi_s = (\alpha_{1,s} - \alpha_{2,s})\left(\mu_{y,s-1}^{(1,1)} + p_{1,s} r_s\right) + \alpha_{2,s} r_s$, with $p_{1,s} = P\left(y_{s-1+\tau S} \leq c_s\right)$, $p_{2,s} = 1 - p_{1,s}$, $\mu_{y,s}^{(i,m)} = \mathbb{E}\left(I_{\tau,s}^{(i)} y_{s+\tau S}^{m}\right)$, $i = 1, 2$.

The results of Proposition 3.3.1 can be represented, in the time invariant coefficients case (classical) model, defined in (3.2.1), by the following corollary.

Corollary 3.3.1 *The process $\{y_t;\ t \in \mathbb{Z}\}$, satisfying (3.2.1), is stationary in the first moment if and only if $\alpha_1 < 1$ and $\alpha_2 < 1$, then, we have*

$$\mu_y = \mathbb{E}\left(y_t\right) = \left(1 - \alpha_2\right)^{-1} \left(\left(\alpha_1 - \alpha_2\right)\left(\mu_y^{(1,1)} + p_1 r\right) + \alpha_2 r\right).$$

where, $p_1 = P\left(y_t \leq c\right)$, $p_2 = 1 - p_1$, $\mu_y^{(i,m)} = \mathbb{E}\left(I_t^{(i)} y_t^{m}\right)$, $i = 1, 2$.

Proof of Proposition 3.3.1. In the first place, the mean of the process $\{y_t,\ t \in \mathbb{Z}\}$, defined in (3.2.2) denoted by $\mu_{y,t} = \mathbb{E}(y_t)$, can be calculated as follows :

$$\begin{aligned}\mathbb{E}(y_t) &= \mathbb{E}\left((\alpha_{1,t} * y_{t-1} + \varepsilon_{1,t}) I_{t-1}^{(1)} + (\alpha_{2,t} * y_{t-1} + \varepsilon_{2,t}) I_{t-1}^{(2)}\right), \\ &= \mathbb{E}\left((\alpha_{1,t} * y_{t-1} + \varepsilon_{1,t}) I_{t-1}^{(1)} + (\alpha_{2,t} * y_{t-1} + \varepsilon_{2,t}) \left(1 - I_{t-1}^{(1)}\right)\right), \\ &= \alpha_{2,t}\mu_{y,t-1} + (\alpha_{1,t} - \alpha_{2,t}) \left(\mu_{y,t-1}^{(1,1)} + p_{1,t}r_t\right) + \alpha_{2,t}r_t.\end{aligned}$$

Then, we have

$$\mu_{y,t} = \alpha_{2,t}\mu_{y,t-1} + \Phi_t, \text{ with } \Phi_t = (\alpha_{1,t} - \alpha_{2,t}) \left(\mu_{y,t-1}^{(1,1)} + p_{1,t}r_t\right) + \alpha_{2,t}r_t, \tag{3.3.1}$$

by iterating the first-order difference equation (3.3.1), m times, while letting $t = s + \tau S$, $s = 1, ..., S$, $\tau \in \mathbb{Z}$ and putting, in the expression, $m = S$ then, taking account of the periodicity of the parameters, we obtain, under the condition $\prod_{i=1}^{S} \alpha_{2,i} < 1$, the closed-form of the mean is given as it is reported in Proposition 3.3.1. ■

Remark 3.3.1 *It can also obtained the second condition of stationarity in the first moment* $\prod_{i=1}^{S} \alpha_{1,i} < 1$, *while replacing* $I_{t-1}^{(1)}$ *by* $1 - I_{t-1}^{(2)}$.

The following proposition establishes the necessary and sufficient conditions for the model (3.2.2) to be periodically stationary in the second moment.

Proposition 3.3.2 *The process* $\{y_t;\ t \in \mathbb{Z}\}$, *satisfying the model (3.2.2), is periodically stationary in the variance if and only if the periodic parameters* $\alpha_{1,t}$ *and* $\alpha_{2,t}$ *satisfy the periodically stationary conditions* $\prod_{i=1}^{S} \alpha_{1,i} < 1$ *and* $\prod_{i=1}^{S} \alpha_{2,i} < 1$, *respectively, then the periodic variance is, under these conditions, given by*

$$\sigma_{y,s}^2 = Var(y_{s+\tau S}) = \left(1 - \prod_{i=1}^{S} \alpha_{2,i}^2\right)^{-1} \sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \alpha_{2,s-i+1}^2\right) \Theta_{s-j+1},$$

where,

$$\begin{aligned}\Theta_s = {} & \left(\alpha_{1,s}^2 - \alpha_{2,s}^2\right) \sigma_{y,s-1}^{2,(1)} + 2\alpha_{2,s} \left(\mu_{y,s} - \alpha_{2,s} \left(\mu_{y,s-1} + r_s\right)\right) \mu_{y,s-1}^{(1,1)} + 2p_{2,s}\alpha_{1,s}r_s\mu_{y,s} + \mu_{y,s} \\ & + (\alpha_{1,s} + \alpha_{2,s}) \mu_{y,s} + \alpha_{2,s} \left(2 \left(\mu_{y,s-1} + r_s\right) \alpha_{2,s} - 2 \left(\mu_{y,s} + p_{2,s}\alpha_{1,s}r_s\right)\right) \mu_{y,s-1} - \alpha_{1,s}\mu_{y,s-1} \\ & + \left(\left(\alpha_{2,s}^2 - \alpha_{1,s}^2\right) p_{1,s} - 2\alpha_{1,s}\alpha_{2,s}\right) p_{2,s}r_s^2 - \alpha_{1,s}\alpha_{2,s}r_s,\end{aligned}$$

with $p_{1,s} = P(y_{s-1+\tau S} \leq c_s)$, $p_{2,s} = 1 - p_{1,s}$, $\mu_{y,s}^{(i,m)} = \mathbb{E}\left(I_{\tau,s}^{(i)} y_{s+\tau S}^m\right)$, $\sigma_{y,s}^{2,(i)} = Var\left(I_{\tau,s}^{(i)} y_{s+\tau S}\right)$, $i = 1, 2$.

The results of Proposition 3.3.2 reduces, in the time invariant coefficients case (classical) model presented in (3.2.1), to the following corollary.

Corollary 3.3.2 *The process $\{y_t,\ t\in\mathbb{Z}\}$, satisfying (3.2.1), is stationary in the variance if and only if $\alpha_1<1$ and $\alpha_2<1$ then,*

$$\sigma_y^2 = Var\left(y_t\right) = \left(1-\alpha_2^2\right)^{-1}\ \Theta,$$

where, $\Theta = \left(\alpha_1^2-\alpha_2^2\right)\sigma_y^{2,(1)} + \left(1+\alpha_2+2p_2\alpha_1 r+2\alpha_2\left(\left(\mu_y+r\right)\alpha_2-\left(\mu_y+p_2\alpha_1 r\right)\right)\right)\mu_y + 2\alpha_2\left(\mu_y-\alpha_2\left(\mu_y+r\right)\right)\mu_y^{(1,1)}-\alpha_1\alpha_2 r.$

Proof of Proposition *3.3.2*. At first, it is easy to see that $Var\left(\alpha_{k,t}*y_{t-1}|\,y_{t-1}\right) = \alpha_{k,t}\left(1+\alpha_{k,t}\right)y_{t-1}$ and $Var\left(\varepsilon_{k,t}\right)=\alpha_{k,t}\left(1+\alpha_{k,t}\right)r_t$, for $k=1,2$. Now, the variance $Var\left(y_t\right) = \sigma_{y,t}^2$ is obtained as follows :

$$\begin{aligned}\sigma_{y,t}^2 = {} & Var\left(\left(\alpha_{1,t}*y_{t-1}+\varepsilon_{1,t}\right)I_{t-1}^{(1)}\right)+Var\left(\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)\left(1-I_{t-1}^{(1)}\right)\right)\\ & +2Cov\left(\left(\alpha_{1,t}*y_{t-1}+\varepsilon_{1,t}\right)I_{t-1}^{(1)};\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)\left(1-I_{t-1}^{(1)}\right)\right).\end{aligned} \tag{3.3.2}$$

The first term, on the right hand side, can be easily calculated as follows :

$$\begin{aligned}Var\left(\left(\alpha_{1,t}*y_{t-1}+\varepsilon_{1,t}\right)I_{t-1}^{(1)}\right) &= Var\left(\alpha_{1,t}\left(y_{t-1}+r_t\right)I_{t-1}^{(1)}\right)+\mathbb{E}\left(\alpha_{1,t}\left(1+\alpha_{1,t}\right)\left(y_{t-1}+r_t\right)I_{t-1}^{(1)}\right),\\ &= \alpha_{1,t}^2 Var\left(y_{t-1}I_{t-1}^{(1)}\right)+\alpha_{1,t}^2 Var\left(I_{t-1}^{(1)}r_t\right)+2\alpha_{1,t}^2 Cov\left(\left(\alpha_{1,t}*y_{t-1}\right)I_{t-1}^{(1)};r_t I_{t-1}^{(1)}\right)+\\ &\quad \alpha_{1,t}\left(1+\alpha_{1,t}\right)\mathbb{E}\left(\left(y_{t-1}+r_t\right)I_{t-1}^{(1)}\right),\end{aligned}$$

then taking account the notation given in Proposition 3.3.2, we obtain :

$$\begin{aligned}Var\left(\left(\alpha_{1,t}*y_{t-1}+\varepsilon_{1,t}\right)I_{t-1}^{(1)}\right) = {} & \alpha_{1,t}^2\,\sigma_{y,t-1}^{2,(1)}+\alpha_{1,t}^2\,r_t^2\,p_{1,t}p_{2,t}+2\alpha_{1,t}^2\,r_t\,p_{2,t}\,\mu_{y,t-1}^{(1,1)}+\\ & \alpha_{1,t}\left(1+\alpha_{1,t}\right)\left(\mu_{y,t-1}^{(1,1)}+r_t p_{1,t}\right).\end{aligned} \tag{3.3.3}$$

The second term in (3.3.2), can be easily calculated as follows :

$$\begin{aligned}Var\left(\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)\left(1-I_{t-1}^{(1)}\right)\right) = {} & Var\left(\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)I_{t-1}^{(1)}\right)\\ & +Var\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)-2Cov\left(\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right);\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)I_{t-1}^{(1)}\right).\end{aligned} \tag{3.3.4}$$

The term $Var\left(\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)I_{t-1}^{(1)}\right)$ can be obtained, where we replace $\alpha_{1,t}$ in (3.3.3) by $\alpha_{2,t}$, indeed, we have :

$$\begin{aligned}Var\left(\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)I_{t-1}^{(1)}\right) = {} & \alpha_{2,t}^2\,\sigma_{y,t-1}^{2,(1)}+p_{1,t}p_{2,t}\alpha_{2,t}^2\,r_t^2+2p_{2,t}\alpha_{2,t}^2 r_t\,\mu_{y,t-1}^{(1,1)}+\\ & \alpha_{2,t}\left(1+\alpha_{2,t}\right)\left(\mu_{y,t-1}^{(1,1)}+p_{2,t}r_t\right).\end{aligned} \tag{3.3.5}$$

The variance $Var\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)$ and $Cov\left(\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right);\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)I_{t-1}^{(1)}\right)$, are given as follows :

$$Var\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)=\alpha_{2,t}^2\sigma_{y,t-1}^2+\alpha_{2,t}\left(1+\alpha_{2,t}\right)\mu_{y,t-1}+\alpha_{2,t}\left(1+\alpha_{2,t}\right)r_t,$$

$$\begin{aligned} &Cov\left(\left(\alpha_{2,t} * y_{t-1}+\varepsilon_{2,t}\right);\left(\alpha_{2,t} * y_{t-1}+\varepsilon_{2,t}\right) I_{t-1}^{(1)}\right)=\mathbb{E}\left(\left(\alpha_{2,t} * y_{t-1}+\varepsilon_{2,t}\right)^{2} I_{t-1}^{(1)}\right)- \\ &\qquad \mathbb{E}\left(\alpha_{2,t} * y_{t-1}+\varepsilon_{2,t}\right) \mathbb{E}\left(\left(\alpha_{2,t} * y_{t-1}+\varepsilon_{2,t}\right) I_{t-1}^{(1)}\right), \\ &=Var\left(\left(\alpha_{2,t} * y_{t-1}+\varepsilon_{2,t}\right) I_{t-1}^{(1)}\right)+\alpha_{2,t}\left(\mathbb{E}\left(y_{t-1} I_{t-1}^{(1)}\right)+p_{1,t} r_{t}\right) \times \\ &\qquad \left(\alpha_{2,t}\left(\mathbb{E}\left(y_{t-1} I_{t-1}^{(1)}\right)-\mathbb{E}\left(y_{t-1}\right)\right)-p_{2,t} \alpha_{2,t} r_{t}\right), \\ &=Var\left(\left(\alpha_{2,t} * y_{t-1}+\varepsilon_{2,t}\right) I_{t-1}^{(1)}\right)-\alpha_{2,t}^{2}\left(\mu_{y,t-1}^{(1,1)}+p_{1,t} r_{t}\right)\left(\mu_{y,t-1}-\mu_{y,t-1}^{(1,1)}+p_{2,t} r_{t}\right), \end{aligned} \tag{3.3.6}$$

where, $Var\left(\left(\alpha_{2,t} * y_{t-1}+\varepsilon_{2,t}\right) I_{t-1}^{(1)}\right)$ is given in (3.3.5). Similarly, by taking account the notation given in Proposition 3.3.2 and from (3.3.5)-(3.3.6), we have :

$$\begin{aligned} &Var\left(\left(\alpha_{2,t} * y_{t-1}+\varepsilon_{2,t}\right)\left(1-I_{t-1}^{(1)}\right)\right)=\alpha_{2,t}^{2} \sigma_{y,t-1}^{2}-p_{2,t} \alpha_{2,t}^{2} r_{t}\left(p_{1,t} r_{t}+2\, \mu_{y,t-1}^{(1,1)}\right) \\ &\quad -\alpha_{2,t}^{2} \sigma_{y,t-1}^{2,(1)}+\alpha_{2,t}\left(1+\alpha_{2,t}\right)\left(\mu_{y,t-1}+r_{t}\right)-\alpha_{2,t}\left(1+\alpha_{2,t}\right)\left(\mu_{y,t-1}^{(1,1)}+p_{1,t} r_{t}\right) \\ &\quad +2 \alpha_{2,t}^{2}\left(\mu_{y,t-1}^{(1,1)}+p_{1,t} r_{t}\right)\left(\left(\mu_{y,t-1}-\mu_{y,t-1}^{(1,1)}\right)+p_{2,t} r_{t}\right). \end{aligned} \tag{3.3.7}$$

The last term of (3.3.2) can be calculated as follows :

$$\begin{aligned} &Cov\left(\left(\alpha_{1,t} * y_{t-1}+\varepsilon_{1,t}\right) I_{t-1}^{(1)} ;\left(\alpha_{2,t} * y_{t-1}+\varepsilon_{2,t}\right)\left(1-I_{t-1}^{(1)}\right)\right) \\ &=-\left(\alpha_{1,t} \mathbb{E}\left(y_{t-1} I_{t-1}^{(1)}\right)+p_{1,t} r_{t} \alpha_{1,t}\right)\left(\alpha_{2,t}\left(\mathbb{E}\left(y_{t-1}\right)-\mathbb{E}\left(y_{t-1} I_{t-1}^{(1)}\right)\right)+p_{2,t} r_{t} \alpha_{2,t}\right), \\ &=-\alpha_{1,t} \alpha_{2,t}\left(\mu_{y,t-1}^{(1,1)}+p_{1,t} r_{t}\right)\left(\left(\mu_{y,t-1}-\mu_{y,t-1}^{(1,1)}\right)+p_{2,t} r_{t}\right). \end{aligned} \tag{3.3.8}$$

The unconditional variance $\sigma_{y,t}^{2}$, can be written, while using (3.3.3)-(3.3.8), in the following first order difference equation :

$$\sigma_{y,t}^{2}=\alpha_{2,t}^{2}\, \sigma_{y,t-1}^{2}+\Theta_{t}, \tag{3.3.9}$$

where,

$$\begin{aligned} \Theta_{t}= &\left(\alpha_{1,t}^{2}-\alpha_{2,t}^{2}\right) \sigma_{y,t-1}^{2,(1)}+2 \alpha_{2,t}\left(\mu_{y,t}-\alpha_{2,t}\left(\mu_{y,t-1}+r_{t}\right)\right) \mu_{y,t-1}^{(1,1)}+\mu_{y,t}-\alpha_{1,t} \alpha_{2,t} r_{t}+ \\ &2 \alpha_{2,t}\left(\left(\mu_{y,t-1}+r_{t}\right) \alpha_{2,t}-\left(\mu_{y,t}+p_{2,t} \alpha_{1,t} r_{t}\right)-\alpha_{1,t}\right) \mu_{y,t-1}+ \\ &\left(\left(\alpha_{2,t}^{2}-\alpha_{1,t}^{2}\right) p_{1,t}-2 \alpha_{1,t} \alpha_{2,t}\right) p_{2,t} r_{t}^{2}+\left(\left(\alpha_{1,t}+\alpha_{2,t}\right)+2 p_{2,t} \alpha_{1,t} r_{t}\right) \mu_{y,t}, \end{aligned}$$

by iterating the equation (3.3.9) m times, letting $t=s+\tau S$, $s=1,...,S$, $\tau \in \mathbb{Z}$ and putting $m=S$, while taking account of the periodicity of the parameters, we obtain :

$$\sigma_{y,s}^{2}=\left(\prod_{i=1}^{S} \alpha_{2,i}^{2}\right) \sigma_{y,s}^{2}+\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1} \alpha_{2,s-i+1}^{2}\right) \Theta_{s-j+1},$$

then, the unconditional variance, for a fixed s, is given, under the periodically stationary condition, namely, $\prod_{i=1}^{S} \alpha_{2,i}^{2}<1$, (hence $\prod_{i=1}^{S} \alpha_{2,i}<1$) by the following expression :

$$\sigma_{y,s}^{2}=\left(1-\prod_{i=1}^{S} \alpha_{2,i}^{2}\right)^{-1} \sum_{j=1}^{S}\left(\prod_{i=1}^{j-1} \alpha_{2,s-i+1}^{2}\right) \Theta_{s-j+1}. \qquad \blacksquare$$

Remark 3.3.2 *It can also obtained, in Proposition 3.3.2, the second condition of stationarity in the second order, i.e.,* $\prod_{i=1}^{S} \alpha_{1,i} < 1$, *while replacing* $I_{t-1}^{(1)}$ *by* $1 - I_{t-1}^{(2)}$. *The unconditional periodic mean and variance and the periodic probabilities* $\mu_{y,t-1}^{(1,1)}$, $\sigma_{y,t-1}^{2,(1)}$, $p_{1,t}$ *and* $p_{2,t}$, *respectively, to be estimated empirically.*

3.3.2 Existence of higher moments

The next proposition ensures the existence of the unconditional m-*th* moment of $\{y_t,\ t \in \mathbb{Z}\}$ and gives their general formula.

Proposition 3.3.3 *The unconditional* m-*th moment* $\mathbb{E}\left(y_t^m\right)$ *of the periodically correlated process* $\{y_t,\ t \in \mathbb{Z}\}$, *defined by* (3.2.2), *exists, if the unconditional* m-*th moment* $\mathbb{E}\left(y_0^m\right)$ *exists. The general formula for* $\mathbb{E}\left(y_t^m\right)$ *is given by :*

$$\mathbb{E}\left(y_t^m\right) = \sum_{k=1}^{2} \sum_{j=0}^{m} \binom{m}{j} \mathbb{E}\left(\left(\alpha_{k,t} * y_{t-1}\right)^j I_{t-1}^{(k)}\right) \mathbb{E}\left(\varepsilon_{k,t}^{m-j} I_{t-1}^{(k)}\right).$$

Proof of Proposition 3.3.3. It is well known that the m-*th* moment of $\varepsilon_{1,t}^m$ and $\varepsilon_{2,t}^m$ (Negative binomial processes) are given, and to ensure the existence of $\mathbb{E}\left(y_t^m\right)$, we use the proof by induction. First of all, we start by $m = 1$, so we have

$$\begin{aligned} \mathbb{E}\left(y_t\right) &\leq \psi_t \mathbb{E}\left(y_{t-1}\right) + \mathbb{E}\left(\varepsilon_{\cdot,t}\right) \leq \psi_t \left(\psi_{t-1} \mathbb{E}\left(y_{t-2}\right) + \mathbb{E}\left(\varepsilon_{\cdot,t-1}\right)\right) + \mathbb{E}\left(\varepsilon_{\cdot,t}\right), \\ &\ \vdots \\ &\leq \left(\prod_{i=1}^{t} \psi_{t-i+1}\right) \mathbb{E}\left(y_0\right) + \sum_{j=0}^{t-1} \left(\prod_{i=0}^{j} \psi_{t-i}\right) \mathbb{E}\left(\varepsilon_{\cdot,t-j}\right) < \infty, \end{aligned} \tag{3.3.10}$$

with, $\psi_t = \max\left(\alpha_{1,t}, \alpha_{2,t}\right)$, and $\mathbb{E}\left(\varepsilon_{\cdot,t}\right) = \psi_t r_t$. Similarly, we have, for $m = 2$,

$$\begin{aligned} \mathbb{E}(y_t^2) &\leq \psi_t^2 \mathbb{E}\left(y_{t-1}^2\right) + \left(\psi_t + \psi_t^2\right) \mathbb{E}\left(y_{t-1}\right) + 2\psi_t \mathbb{E}\left(y_{t-1}\right) \mathbb{E}\left(\varepsilon_{\cdot,t}\right) + \mathbb{E}\left(\varepsilon_{\cdot,t}^2\right), \\ &\leq \psi_t^2 \left(\psi_{t-1}^2 \mathbb{E}\left(y_{t-2}^2\right) + \left(\psi_{t-1} + \psi_{t-1}^2\right) \mathbb{E}\left(y_{t-2}\right) + 2\psi_{t-1} \mathbb{E}\left(y_{t-2}\right) \mathbb{E}\left(\varepsilon_{\cdot,t-1}\right) + \mathbb{E}\left(\varepsilon_{\cdot,t-1}^2\right)\right) + \\ &\quad + \psi_t \left(1 + \psi_t\right) \mathbb{E}\left(y_{t-1}\right) + 2\psi_t \mathbb{E}\left(y_{t-1}\right) \mathbb{E}\left(\varepsilon_{\cdot,t}\right) + \mathbb{E}\left(\varepsilon_{\cdot,t}^2\right), \\ &\ \vdots \\ &\leq \left(\prod_{i=1}^{t} \psi_{t-i}^2\right) \mathbb{E}\left(y_0^2\right) + \sum_{j=1}^{t} \left(\prod_{i=1}^{j-1} \psi_{t-i+1}^2\right) \psi_{t-j+1} \left(1 + \psi_{t-j+1} + 2\mathbb{E}\left(\varepsilon_{\cdot,t-j+1}\right)\right) \mathbb{E}\left(y_{t-j+1}\right) \\ &\quad + \sum_{j=1}^{t} \left(\prod_{i=1}^{j-1} \psi_{t-i+1}^2\right) \mathbb{E}\left(\varepsilon_{\cdot,t-j+1}^2\right), \end{aligned}$$

with, $\mathbb{E}\left(\varepsilon_{\cdot,t}^2\right) = \psi_t \left(1 + \psi_t + \psi_t r_t\right) r_t$. The last inequality can be written in the form :

$$\mathbb{E}(y_t^2) \leq \left(\prod_{i=1}^{t} \psi_{t-i}^2\right) \mathbb{E}\left(y_0^2\right) + \Delta_t, \text{ where,}$$

$$\Delta_t = \sum_{j=1}^{t} \left(\prod_{i=1}^{j-1} \psi_{t-i+1}^2\right) \left(\psi_{t-j+1} \left(1 + \psi_{t-j+1} + 2\mathbb{E}\left(\varepsilon_{\cdot,t-j+1}\right)\right) \mathbb{E}\left(y_{t-j+1}\right) + \mathbb{E}\left(\varepsilon_{\cdot,t-j+1}^2\right)\right) < \infty,$$

since, $\mathbb{E}(y_t)$ is finite by using (3.3.10) and our assumption that the unconditional m th moments $\mathbb{E}\left(\varepsilon_{1,t}^m\right)$ and $\mathbb{E}\left(\varepsilon_{2,t}^m\right)$ exist and are finite. Thus, the second moment $\mathbb{E}(y_t^2)$ exists. After that, we assume $\mathbb{E}\left(y_t^{m-1}\right) < \infty$ exists for $m-1$, and we show that for m. It's easy to see

$$\mathbb{E}\left(y_t^m\right) \leq \left(\prod_{i=1}^{t} \psi_{t-i+1}^m\right) \mathbb{E}\left(y_0^m\right) + \Delta_t', \tag{3.3.11}$$

where, Δ_t' denote the combination of finite k-*th* moment of process y_t and l *th* moment of process ε_t, with $k \in \{1, 2, .., m-1\}$ and $l \in \{1, 2, .., m\}$ (i.e., induction hypothesis and the assumption). From (3.3.11), one can check that $\mathbb{E}\left(y_t^m\right) < \infty$, $m \geq 1$. ■

3.3.3 Strict periodic stationarity

The following proposition establishes the existence of the periodic strict stationarity property of the process $\{y_t;\ t \in \mathbb{Z}\}$ defined in (3.2.4).

Proposition 3.3.4 *For a fixed value of $s = 1, ...S$, the process $\{y_{s+\tau S};\ \tau \in \mathbb{Z}\}$ is an irreducible, aperiodic, and positive recurrent (and hence ergodic) Markov chain. Therefore, there exists a strict periodic stationarity for the process $\{y_{s+\tau S};\ \tau \in \mathbb{Z}\}$ defined in (3.2.4).*

Proof of Proposition 3.3.4. We can see that the process $\{y_{s+\tau S};\ \tau \in \mathbb{Z}\}$, for a fixed s in $\{1, ..., S\}$, is a periodic Markov chain on $\mathbb{N}_0$ where its transition probabilities are given by

$$P\left(y_t = y|\, y_{t-1} = x\right) = \frac{\Gamma\left(y + x + r_t\right)}{\Gamma\left(x + r_t\right) \Gamma\left(y + 1\right)} \left(\frac{\alpha_{1,t}^y I_{t-1}^{(1)}}{\left(1 + \alpha_{1,t}\right)^{r_t + y + x}} + \frac{\alpha_{2,t}^y I_{t-1}^{(2)}}{\left(1 + \alpha_{2,t}\right)^{r_t + y + x}} \right),$$

since $P\left(y_t = y|\, y_{t-1} = x\right) > 0$ it follows that $\{y_t;\ t \in \mathbb{Z}\}$ is an irreducible, aperiodic chain. Furthermore, to show that our process $\{y_t;\ t \in \mathbb{Z}\}$ is positive recurrent, it is sufficient to prove that $\sum_{t=1}^{+\infty} P\left(y_t = 0|\, y_0 = 0\right) = +\infty$, By iterating (**??**) t times, while putting $y_0 = 0$, we have $y_t = \sum_{j=0}^{t-1} \left(\prod_{i=1}^{j} \varphi_{t-i+1}\right) * \varepsilon_{t-j}$, $t \in \mathbb{Z}$, $t \in \mathbb{Z}$. Consequently, we can write :

$$\begin{aligned} P\left(y_t = 0|\, y_0 = 0\right) &= P\left(\sum_{i=1}^{t-1} \varphi_t * \varphi_{t-1} * ... * \varphi_i * \varepsilon_i + \varepsilon_t = 0 \middle|\, y_0 = 0\right), \\ &= \sum_{i_2=1}^{2} \sum_{i_3=1}^{2} ... \sum_{i_t=1}^{2} P\left(\varphi_2 = \alpha_{i_2,2}, \varphi_3 = \alpha_{i_3,3}, .., \varphi_t = \alpha_{i_t,t}|\, y_0 = 0\right) \left(\frac{1}{1 + \alpha_{i_t,t}}\right)^{r_t} \times \\ &\left(\frac{1 + \alpha_{i_t,t}}{1 + \alpha_{i_t,t} + \alpha_{i_t,t}\alpha_{i_{t-1},t-1}}\right)^{r_{t-1}} ... \left(\frac{1 + \alpha_{i_t,t} + ... + \alpha_{i_t,t}\alpha_{i_{t-1},t-1}...\alpha_{i_2,2}}{1 + \alpha_{i_t,t} + ... + \alpha_{i_t,t}\alpha_{i_{t-1},t-1}...\alpha_{i_2,2}\alpha_{i_1,1}}\right)^{r_1}. \end{aligned} \tag{3.3.12}$$

From the expression (3.3.12) it is easy to see that each factor is strictly monotonically decreasing with respect to $\alpha_{i_j,j}$ with $1 \leq j \leq t$. Letting $t = s + \tau S$, $s = 1, ..., S$, $\tau \in \mathbb{Z}$ and taking account of the periodicity of the parameters $\alpha_{1,t}$, $\alpha_{2,t}$ and r_t, we obtain

$$P\left(y_{s+\tau S}=0|\, y_0=0\right) \geq \psi\left(\alpha_{\max}, s+\tau S\right) \text{ where }, \alpha_{\max}=\max_{1\leq s\leq S}\left\{\alpha_{1,s},\alpha_{2,s}\right\},$$
$$\textit{and}\quad \psi\left(\alpha,t\right)=\left(\frac{1}{1+\alpha}\right)^{r_t}\left(\frac{1}{1+\alpha+\alpha^2}\right)^{r_{t-1}}\cdots\left(\frac{1}{1+\alpha+...+\alpha^t}\right)^{r_1}.$$

Since $P\left(y_t=0|\, y_1=0\right)>0$, then by the use, for a fixed s, of the comparison criterion for series convergence, one can see easily that $\sum_{s=1}^{S}\sum_{\tau=0}^{+\infty}P\left(y_{s+\tau S}=0|\, y_0=0\right)=+\infty$. This proves that the process, in τ, $\{y_{s+\tau S};\ \tau\in\mathbb{Z}\}$ is a positive recurrent Markov chain and hence ergodic which ensures the existence of a strict periodic stationarity. ∎

Remark 3.3.3 *The results given in Proposition 3.3.4 reduce, in the time-invariant NB-SETINAR model, to the proposition* 2.1 *presented by Yang et al* (2018*b*).

3.3.4 Autocovariance structure

The following proposition establishes the autocovariance structure of the process $\{y_t,\ t\in\mathbb{Z}\}$, given by $\gamma^{(t)}\left(h\right)=Cov\left(y_t;y_{t-h}\right)=\mathbb{E}\left[\left(y_t-\mu_{y,t}\right)\left(y_{t-h}-\mu_{y,t-h}\right)\right]$, first we give these notations : $\gamma_1^{(t)}\left(h\right)=Cov\left(I_t^{(1)}y_t;y_{t-h}\right)$ and $\gamma_2^{(t)}\left(h\right)=Cov\left(I_t^{(2)}y_t;y_{t-h}\right)$.

Proposition 3.3.5 *The autocovariance structure of the periodically correlated integer-valued process $\{y_t,\ t\in\mathbb{Z}\}$, satisfying the model (3.2.2) is given as follows :*

$$\gamma^{(s)}\left(h\right)=\begin{cases}\left(1-\prod_{i=1}^{S}\alpha_{2,s-i+1}^2\right)^{-1}\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\alpha_{2,s-i+1}^2\right)\Theta_{s-j+1}. & h=0,\\ \alpha_{2,s}\gamma^{(s-1)}\left(h-1\right)+\left(\alpha_{1,s}-\alpha_{2,s}\right)\left(\gamma_1^{(s-1)}\left(h-1\right)+\right. & \\ \qquad\left. r_s\left(\mathbb{E}\left(I_{\tau,s-1}^{(1)}y_{s-h+\tau S}\right)-p_{1,s}\mu_{y,s-h}\right)\right), & h\geq 1.\end{cases}$$

where, Θ_s, $s=1,...,S$, are given in Proposition 3.3.2.

The results of Proposition 3.3.5 can be represented, in the time-invariant model, defined in (3.2.1), by the following corollary.

Corollary 3.3.3 *The autocovariance structure of integer-valued process $\{y_t,\ t\in\mathbb{Z}\}$, satisfying the model (3.2.1), is given as follows :*

$$\gamma\left(h\right)=\begin{cases}\left(1-\alpha_2^2\right)^{-1}\Theta & h=0,\\ \alpha_2\gamma\left(h-1\right)+\left(\alpha_1-\alpha_2\right)\left(\gamma_1\left(h-1\right)+r\left(\mathbb{E}\left(y_{t-h}I_{t-1}^{(1)}\right)-p_1\mu_{y,t-h}\right)\right), & h\geq 1.\end{cases}$$

where, Θ is given in Corollary 3.3.2.

Proof of Proposition 3.3.5. For $h=0$ the autocovariance $Cov\left(y_t;y_t\right)=Var\left(y_t\right)=\sigma^2_{y,t}$.

Then, for $h=1$ the autocovariance $\gamma^{(t)}\left(1\right)$, can be calculated as follows :

$$\begin{aligned}\gamma^{(t)}\left(1\right) &= Cov\left(y_t;y_{t-1}\right) = Cov\left(\left(\alpha_{1,t}*y_{t-1}+\varepsilon_{1,t}\right)I^{(1)}_{t-1}+\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)I^{(2)}_{t-1};y_{t-1}\right),\\ &= Cov\left(\left(\alpha_{1,t}*y_{t-1}+\varepsilon_{1,t}\right)I^{(1)}_{t-1};y_{t-1}\right)+Cov\left(\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)\left(1-I^{(1)}_{t-1}\right);y_{t-1}\right),\end{aligned}$$

hence, we have

$$\begin{aligned}\gamma^{(t)}\left(1\right) &= Cov\left(\left(\alpha_{1,t}*y_{t-1}+\varepsilon_{1,t}\right)I^{(1)}_{t-1};y_{t-1}\right)+Cov\left(\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right);y_{t-1}\right)-\\ &\quad Cov\left(\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)I^{(1)}_{t-1};y_{t-1}\right),\\ &= Cov\left(\left(\alpha_{1,t}*y_{t-1}\right)I^{(1)}_{t-1};y_{t-1}\right)+Cov\left(\varepsilon_{1,t}I^{(1)}_{t-1};y_{t-1}\right)+Cov\left(\alpha_{2,t}*y_{t-1};y_{t-1}\right)-\\ &\quad Cov\left(\left(\alpha_{2,t}*y_{t-1}\right)I^{(1)}_{t-1};y_{t-1}\right)-Cov\left(\varepsilon_{2,t}I^{(1)}_{t-1};y_{t-1}\right),\end{aligned}$$

one can easily verify that :

$$\begin{aligned}&Cov\left(\left(\alpha_{1,t}*y_{t-1}\right)I^{(1)}_{t-1};y_{t-1}\right)=\alpha_{1,t}Cov\left(y_{t-1}I^{(1)}_{t-1};y_{t-1}\right)=\alpha_{1,t}\gamma_1^{(t-1)}\left(0\right),\\ &Cov\left(\varepsilon_{1,t}I^{(1)}_{t-1};y_{t-1}\right)=\mathbb{E}\left(\varepsilon_{1,t}y_{t-1}I^{(1)}_{t-1}\right)-\mathbb{E}\left(\varepsilon_{1,t}I^{(1)}_{t-1}\right)\mathbb{E}\left(y_{t-1}\right)=\alpha_{1,t}r_t\left(\mu^{(1,1)}_{y,t-1}-p_{1,t}\mu_{y,t-1}\right),\\ &Cov\left(\alpha_{2,t}*y_{t-1};y_{t-1}\right)=\alpha_{2,t}Cov\left(y_{t-1};y_{t-1}\right)=\alpha_{2,t}\gamma^{(t-1)}\left(0\right),\\ &Cov\left(\left(\alpha_{2,t}*y_{t-1}\right)I^{(1)}_{t-1};y_{t-1}\right)=\alpha_{2,t}Cov\left(y_{t-1}I^{(1)}_{t-1};y_{t-1}\right)=\alpha_{2,t}\gamma_1^{(t-1)}\left(0\right),\\ &Cov\left(\varepsilon_{2,t}I^{(1)}_{t-1};y_{t-1}\right)=\mathbb{E}\left(\varepsilon_{2,t}y_{t-1}I^{(1)}_{t-1}\right)-\mathbb{E}\left(\varepsilon_{2,t}I^{(1)}_{t-1}\right)\mathbb{E}\left(y_{t-1}\right)=\alpha_{2,t}r_t\left(\mu^{(1,1)}_{y,t-1}-p_{1,t}\mu_{y,t-1}\right).\end{aligned}$$

Accordingly, we have

$$\gamma^{(t)}\left(1\right)=\alpha_{2,t}\gamma^{(t-1)}\left(0\right)+\left(\alpha_{1,t}-\alpha_{2,t}\right)\left(\gamma_1^{(t-1)}\left(0\right)+\left(\mu^{(1,1)}_{y,t-1}-p_{1,t}\mu_{y,t-1}\right)r_t\right).$$

More generally, calculate now the autocovariance $\gamma^{(t)}\left(h\right)$, for $h\geq 1$:

$$\begin{aligned}\gamma^{(t)}\left(h\right) &= Cov\left(y_t,y_{t-h}\right)=Cov\left(\left(\alpha_{1,t}*y_{t-1}+\varepsilon_{1,t}\right)I^{(1)}_{t-1}+\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)I^{(2)}_{t-1};y_{t-h}\right),\\ &= Cov\left(\left(\alpha_{1,t}*y_{t-1}+\varepsilon_{1,t}\right)I^{(1)}_{t-1};y_{t-h}\right)+Cov\left(\left(\alpha_{2,t}*y_{t-1}+\varepsilon_{2,t}\right)\left(1-I^{(1)}_{t-1}\right);y_{t-h}\right),\\ &= Cov\left(\left(\alpha_{1,t}*y_{t-1}\right)I^{(1)}_{t-1};y_{t-h}\right)+Cov\left(\varepsilon_{1,t}I^{(1)}_{t-1};y_{t-h}\right)+Cov\left(\alpha_{2,t}*y_{t-1};y_{t-h}\right)+\\ &\quad Cov\left(\left(\alpha_{2,t}*y_{t-1}\right)I^{(1)}_{t-1};y_{t-h}\right)+Cov\left(\varepsilon_{2,t}I^{(1)}_{t-1};y_{t-h}\right),\end{aligned}$$

where, the four terms of the last expression can be calculated as follows :

$$\begin{aligned}Cov\left(\left(\alpha_{1,t}*y_{t-1}\right)I^{(1)}_{t-1};y_{t-h}\right) &= \alpha_{1,t}\left(\mathbb{E}\left(y_{t-1}y_{t-h}I^{(1)}_{t-1}\right)-\mathbb{E}\left(y_{t-1}I^{(1)}_{t-1}\right)\mathbb{E}\left(y_{t-h}\right)\right),\\ &= \alpha_{1,t}Cov\left(y_{t-1}I^{(1)}_{t-1};y_{t-h}\right)=\alpha_{1,t}\gamma_1^{(t-1)}\left(h-1\right),\\ Cov\left(\left(\alpha_{1,t}*y_{t-1}\right)I^{(1)}_{t-1};y_{t-h}\right) &= \alpha_{1,t}\left(\mathbb{E}\left(y_{t-1}y_{t-h}I^{(1)}_{t-1}\right)-\mathbb{E}\left(y_{t-1}I^{(1)}_{t-1}\right)\mathbb{E}\left(y_{t-h}\right)\right),\\ &= \alpha_{1,t}Cov\left(y_{t-1}I^{(1)}_{t-1};y_{t-h}\right)=\alpha_{1,t}\gamma_1^{(t-1)}\left(h-1\right),\end{aligned}$$

$$Cov\left(\varepsilon_{1,t}I^{(1)}_{t-1};y_{t-h}\right)=\mathbb{E}\left(\varepsilon_{1,t}y_{t-h}I^{(1)}_{t-1}\right)-\mathbb{E}\left(\varepsilon_{1,t}I^{(1)}_{t-1}\right)\mathbb{E}\left(y_{t-h}\right)=\alpha_{1,t}r_t\left(\mathbb{E}\left(y_{t-h}I^{(1)}_{t-1}\right)-p_{1,t}\mu_{y,t-h}\right),$$

$$Cov\left(\alpha_{2,t} * y_{t-1}; y_{t-h}\right) = \alpha_{2,t} Cov\left(y_{t-1}; y_{t-h}\right) = \alpha_{2,t}\gamma^{(t-1)}\left(h-1\right),$$
$$Cov\left(\left(\alpha_{2,t} * y_{t-1}\right) I_{t-1}^{(1)}; y_{t-h}\right) = \mathbb{E}\left(\left(\alpha_{2,t} * y_{t-1}\right) y_{t-h} I_{t-1}^{(1)}\right) - \mathbb{E}\left(\left(\alpha_{2,t} * y_{t-1}\right) I_{t-1}^{(1)}\right)\mathbb{E}\left(y_{t-h}\right),$$
$$= \alpha_{2,t} Cov\left(y_{t-1} I_{t-1}^{(1)}; y_{t-h}\right) = \alpha_{2,t}\gamma_1^{(t-1)}\left(h-1\right).$$

Therefore, for any $h \geq 1$, we have :

$$\gamma^{(t)}\left(h\right) = \alpha_{2,t}\gamma^{(t-1)}\left(h-1\right) + \left(\alpha_{1,t} - \alpha_{2,t}\right)\left(\gamma_1^{(t-1)}\left(h-1\right) + \left(\mathbb{E}\left(I_{t-1}^{(1)} y_{t-h}\right) - p_{1,t}\mu_{y,t-h}\right) r_t\right). \blacksquare$$

3.4 Parameter estimation

In this Section, we estimate the parameters of the periodic negative binomial self-exciting threshold integer-valued autoregressive process $PNBSETINAR_S\left(2;1\right)$ model satisfying (3.2.2). Let $\underline{\theta}_s = \left(\theta_{1,s}, \theta_{2,s}, \theta_{3,s}\right)' = \left(\alpha_{1,s}, \alpha_{2,s}, r_s\right)'$, for a fixed $s = 1, ..., S$, be the vector of unknown parameters to be estimated, while opting for the conditional least squares and the conditional maximum likelihood estimation methods. In addition, the periodic adaptation nested sub-sample search algorithm to estimate the periodic threshold parameters c_t. Initially and without loss of generality, these parameters c_t are considered to be known until Section 3.4.3 where we discuss how to estimate them in the unknown case.

3.4.1 Conditional least square estimators

Recall that the CLS-estimations $\widehat{\underline{\theta}}_{s,CLS} = \left(\widehat{\alpha}_{1,s}, \widehat{\alpha}_{2,s}, \widehat{r}_s\right)'$ of $\underline{\theta}_s$ are given by minimizing the following sum of squared deviations about the conditional expectation :

$$Q(\underline{\theta}_t; \underline{Y}) = \textstyle\sum_{t=2}^{n}\left(y_t - g\left(\underline{\theta}_t; y_{t-1}\right)\right)^2, \tag{3.4.1}$$

where $g\left(\underline{\theta}_t; y_{t-1}\right) = \mathbb{E}\left(y_t \middle| y_{t-1}\right) = \sum_{k=1}^{2}\alpha_{k,t}\left(y_{t-1} + r_t\right) I_{t-1}^{(k)}$.

Letting, for simplicity of notation, the size n of the observed time series be a multiple of S, (i.e., $n = NS$, $N \in \mathbb{N}^*$) and replacing t by $s + \tau S$, with $s = 1, ..., S$ and $\tau \in \mathbb{Z}$, one can rewrite the last expression in the form :

$$Q(\underline{\theta}_s; \underline{Y}) = \textstyle\sum_{s=1}^{S}\sum_{\tau=0}^{N-1}\left(y_{s+\tau S} - g\left(\underline{\theta}_s; y_{s-1+\tau S}\right)\right)^2 = \sum_{s=1}^{S}\sum_{\tau=0}^{N-1} U_{s+\tau S}\left(\underline{\theta}_s\right),$$
$$= \textstyle\sum_{s=1}^{S}\sum_{\tau=0}^{N-1}\left(y_{s+\tau S} - \sum_{k=1}^{2}\alpha_{k,s}\left(y_{s-1+\tau S} + r_s\right) I_{\tau,s-1}^{(k)}\right)^2.$$

For the purposes that the estimation value of r_s is, in each period, a positive integer, we use the Min-Min algorithm, applied in Yang *et al* (2018*b*) . Hence, our contribution is limited to providing an adaptation of their algorithm to our periodic case. The schema is as follows, for a fixed value $s = 1, ..., S$, and for given values of $\alpha_{1,s}$ and $\alpha_{2,s}$ in $[0, 1]$, we calculate r_s which is the solution of the following normal equation :

$$\frac{\partial Q}{\partial r_s} = \sum_{\tau=0}^{N-1} \sum_{k=1}^{2} \alpha_{k,s} \left(y_{s+\tau S} - \alpha_{k,s} \left(y_{s-1+\tau S} + r_s\right)\right) I_{\tau,s-1}^{(k)} = 0. \tag{3.4.2}$$

Then, we update the values of $\alpha_{1,s}$ and $\alpha_{2,s}$ by replacing r_s obtained previously in the following equations, for $k = 1, 2$:

$$\frac{\partial Q}{\partial \alpha_{k,s}} = \sum_{\tau=0}^{N-1} \left[\left(y_{s+\tau S} \left(y_{s-1+\tau S} + r_s\right) - \alpha_{k,s} \left(y_{s-1+\tau S} + r_s\right)^2\right) I_{\tau,s-1}^{(k)}\right] = 0. \tag{3.4.3}$$

Then the estimated parameters is given by iterating these two equations until the convergence is achieved (the details of this algorithm are given in Section 3.4.4).

In order to study the asymptotic property of our obtained CLS-estimators, we make the following technical assumptions.

Assumption 3.4.1 . *The stochastic process $\{y_t,\ t \in \mathbb{Z}\}$ satisfying the model (3.2.2), with the true vector parameters $\underline{\theta}_{0,s} \in \mathcal{D} \times \mathbb{N}^*$, $s = 1, ..., S$, where $\mathcal{D} = [0, 1] \times [0, 1]$ is a compact subset of $\mathbb{R}^2$.*

Assumption 3.4.2 *The model (3.2.2) is identifiable, i.e., $p_{\underline{\theta}_s} \neq p_{\underline{\theta}_{0,s}}$, if $\underline{\theta}_s \neq \underline{\theta}_{0,s}$, while s being fixed in $\{1, \ldots, S\}$.*

The following proposition establishes, under the above assumptions, the consistency property of the CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$ and thereafter its asymptotic distribution.

Proposition 3.4.1 *The CLS-vector estimators are, under the assumptions 3.4.1 and 3.4.2, strongly consistent, i.e.,* $\widehat{\underline{\theta}}_{s,CLS} \rightsquigarrow \underline{\theta}_{0,s}$, *a.s.* $s = 1, ..., S$.

Proof of Proposition 3.4.1. The proof of this proposition is very similar to the elegant demonstration of theorem 3.1. in Yang *et al* (2018*b*). Consider $h_{s+\tau S}(\underline{\theta}_s) = -U_{s+\tau S}(\underline{\theta}_s)$, the proof will be undertaken in three steps.

Step 1. We are going to prove, while s being fixed in $\{1, \ldots, S\}$, that $\mathbb{E}\left(U_{s+\tau S}(\underline{\theta}_s)\right)$ is continuous in $\underline{\theta}_s$, so that for $\mathbb{E}\left(h_{s+\tau S}(\underline{\theta}_s)\right)$. Since r_s is discrete, we just need to prove the next property. For any $\underline{\theta}_s \in \mathcal{D}$, let $V_\eta(\underline{\theta}_s) = B(\underline{\theta}_s, \eta)$ be an open ball centered at $\underline{\theta}_s$ with radius η, then : $\mathbb{E}\left(\sup_{\underline{\theta}'_s \in V_\eta(\underline{\theta}_s)} \left|U_{s+\tau S}(\underline{\theta}_s) - U_{s+\tau S}(\underline{\theta}'_s)\right|\right) \to 0,$ as $\eta \to 0$. To prove the last expression, we can observe that,

$$\begin{aligned}
|U_{s+\tau S}(\underline{\theta}_s) - U_{s+\tau S}(\underline{\theta}'_s)| &= \left(y_{s+\tau S} - \sum_{k=1}^{2} \alpha_{k,s}(y_{s-1+\tau S}+r_s) I_{\tau,s-1}^{(k)}\right)^2 - \\
&\quad \left(y_{s+\tau S} - \sum_{k=1}^{2} \alpha'_{k,s}(y_{s-1+\tau S}+r_s) I_{\tau,s-1}^{(k)}\right)^2, \\
&= \left|2y_{s+\tau S} - \sum_{k=1}^{2}\left(\alpha_{k,s}+\alpha'_{k,s}\right)(y_{s-1+\tau S}+r_s) I_{\tau,s-1}^{(k)}\right| \left|\sum_{k=1}^{2}\left(\alpha'_{k,s}-\alpha_{k,s}\right)(y_{s-1+\tau S}+r_s) I_{\tau,s-1}^{(k)}\right|, \\
&\leq \eta (y_{s-1+\tau S}+r_s)\left|2y_{s+\tau S} + 2(y_{s-1+\tau S}+r_s)\right| \leq 2\eta\left(y_{s+\tau S}(y_{s-1+\tau S}+r_s) + (y_{s-1+\tau S}+r_s)^2\right).
\end{aligned}$$

After that, we can see

$$\mathbb{E}\left(\sup_{\underline{\theta}'_s \in V_\eta(\underline{\theta}_s)} |U_{s+\tau S}(\underline{\theta}_s) - U_{s+\tau S}(\underline{\theta}'_s)|\right) \leq 2\eta \mathbb{E}\left(y_{s+\tau S}(y_{s-1+\tau S}+r_s) + (y_{s-1+\tau S}+r_s)^2\right) \underset{\eta\to 0}{\longrightarrow} 0.$$

Step 2. We prove now that $\mathbb{E}_{\underline{\theta}_{s,0}}\left(h_{s+\tau S}(\underline{\theta}_s) - h_{s+\tau S}(\underline{\theta}_{s,0})\right) < 0$, which is equivalent to prove $\mathbb{E}_{\underline{\theta}_{s,0}}\left(U_{s+\tau S}(\underline{\theta}_s) - U_{s+\tau S}(\underline{\theta}_{s,0})\right) > 0$, for any $\underline{\theta}_s \neq \underline{\theta}_{s,0}$ with $\underline{\theta}_{s,0}$ is the true value of $\underline{\theta}_s$, then

$$\begin{aligned}
\mathbb{E}_{\underline{\theta}_{s,0}}(U_{s+\tau S}(\underline{\theta}_s)) &= \mathbb{E}_{\underline{\theta}_{s,0}}\left[(y_{s+\tau S} - g(\underline{\theta}_s, y_{s-1+\tau S}))^2\right], \\
&= \mathbb{E}_{\underline{\theta}_{s,0}}\Big[\left(y_{s+\tau S} - g(\underline{\theta}_{s,0}, y_{s-1+\tau S})\right)^2 + 2\left(y_{s+\tau S} - g(\underline{\theta}_{s,0}, y_{s-1+\tau S})\right) \times \\
&\quad \left(g(\underline{\theta}_{s,0}, y_{s-1+\tau S}) - g(\underline{\theta}_s, y_{s-1+\tau S})\right) + \left(g(\underline{\theta}_{s,0}, y_{s-1+\tau S}) - g(\underline{\theta}_s, y_{s-1+\tau S})\right)^2\Big], \\
&= \mathbb{E}_{\underline{\theta}_{s,0}}\left(U_{s+\tau S}(\underline{\theta}_{s,0})\right) + I + II,
\end{aligned} \tag{3.4.4}$$

where, I and II are given, respectively, by

$$\begin{aligned}
I &= 2\mathbb{E}_{\underline{\theta}_{s,0}}\left[\left(y_{s+\tau S} - g(\underline{\theta}_{s,0}, y_{s-1+\tau S})\right)\left(g(\underline{\theta}_{s,0}, y_{s-1+\tau S}) - g(\underline{\theta}_s, y_{s-1+\tau S})\right)\right], \\
&= 0,
\end{aligned} \tag{3.4.5}$$

$$II = \mathbb{E}_{\underline{\theta}_{s,0}}\left[\left(g(\underline{\theta}_{s,0}, y_{s-1+\tau S}) - g(\underline{\theta}_s, y_{s-1+\tau S})\right)^2\right] > 0. \text{ (By Assumption 3.4.2).} \tag{3.4.6}$$

Therefore, by (3.4.4)-(3.4.6), we get $\mathbb{E}_{\underline{\theta}_{s,0}}\left(U_{s+\tau S}(\underline{\theta}_s) - U_{s+\tau S}(\underline{\theta}_{s,0})\right) > 0$.

Step 3. Now, we can prove the consistency for $\widehat{\underline{\theta}}_{s,CLS}$, while s being fixed in $\{1,\ldots,S\}$. Consider an arbitrary open neighborhood $V_{\underline{\theta}_{s,0}}$ of $\underline{\theta}_{s,0}$, then for any $\underline{\theta}_s \in V^c_{\underline{\theta}_{s,0}} \cap \mathcal{D}$, we have $\mathbb{E}(h_{s+\tau S}(\underline{\theta}_s)) < \mathbb{E}\left(h_{s+\tau S}(\underline{\theta}_{s,0})\right)$, since $V^c_{\underline{\theta}_{s,0}} \cap \mathcal{D}$ is compact and $\mathbb{E}(h_{s+\tau S}(\underline{\theta}_s))$ is continuous in $\underline{\theta}_s$, we have $k_s = \mathbb{E}\left(h_{s+\tau S}(\underline{\theta}_{s,0})\right) - \sup_{\underline{\theta}_s \in V^c_{\underline{\theta}_{s,0}} \cap \mathcal{D}} \mathbb{E}(h_{s+\tau S}(\underline{\theta}_s)) > 0$. And for any $\underline{\theta}_s \in V^c_{\underline{\theta}_{s,0}} \cap \mathcal{D}$, there exists $\eta_{\underline{\theta}_s} > 0$ such that $\mathbb{E}\left(\sup_{\widetilde{\underline{\theta}}_s \in V_{\eta_{\underline{\theta}_s}}(\underline{\theta}_s)} h_{s+\tau S}\left(\widetilde{\underline{\theta}}_s\right)\right) < \mathbb{E}(h_{s+\tau S}(\underline{\theta}_s)) + \frac{k_s}{6}$. Also, by the compactness of $V^c_{\underline{\theta}_{s,0}} \cap \mathcal{D}$, there exists a finite open cover $\left\{V_{\eta_{\underline{\theta}_{s,j}}}\left(\underline{\theta}_{s,j}\right);\ j = 1,...,m\right\}$ of $V^c_{\underline{\theta}_{s,0}} \cap \mathcal{D}$. For any $\underline{\theta}_s \in \mathcal{D}$, $n \gg 0$, and $j = 1,\ldots,m$. Suppose that the size of observed time series n is a multiple of S (i.e., $n = NS$), we have

$$\sup_{\underline{\theta}_s \in V_{\eta_{\underline{\theta}_{s,j}}}\left(\underline{\theta}_{s,j}\right)} \sum_{s=1}^{S} \sum_{\tau=0}^{N-1} \tfrac{h_{s+\tau S}(\underline{\theta}_s)}{NS} \leq \sum_{s=1}^{S} \sum_{\tau=0}^{N-1} \sup_{\underline{\theta}_s \in V_{\eta_{\underline{\theta}_{s,j}}}\left(\underline{\theta}_{s,j}\right)} \tfrac{h_{s+\tau S}(\underline{\theta}_s)}{NS} + \tfrac{k_s}{6},$$

$$\leq \mathbb{E}\left(\sup_{\underline{\theta}_s \in V_{\eta_{\underline{\theta}_{s,j}}}\left(\underline{\theta}_{s,j}\right)} h_{s+\tau S}\left(\underline{\theta}_s\right)\right) + \tfrac{k_s}{3} \leq \mathbb{E}\left(h_{s+\tau S}\left(\underline{\theta}_{s,j}\right)\right) + \tfrac{k_s}{2},$$

$$\leq \sup_{\underline{\theta}_s \in V^c_{\underline{\theta}_{s,0}} \cap \mathcal{D}} \mathbb{E}\left(h_{s+\tau S}\left(\underline{\theta}_s\right)\right) + \tfrac{k_s}{2} \leq \mathbb{E}\left(h_{s+\tau S}\left(\underline{\theta}_{s,0}\right)\right) - \tfrac{2k_s}{3},$$

and

$$\sup_{\underline{\theta}_s \in V_{\underline{\theta}_{s,0}}} \sum_{s=1}^{S} \sum_{\tau=0}^{N-1} \tfrac{h_{s+\tau S}(\underline{\theta}_s)}{NS} \geq \sum_{s=1}^{S} \sum_{\tau=0}^{N-1} \tfrac{h_{s+\tau S}\left(\underline{\theta}_{s,0}\right)}{NS} \geq \sum_{s=1}^{S} \sum_{\tau=0}^{N-1} \tfrac{h_{s+\tau S}\left(\underline{\theta}_{s,0}\right)}{NS} - \tfrac{k_s}{6},$$

$$\geq \mathbb{E}\left(h_{s+\tau S}\left(\underline{\theta}_{s,0}\right)\right) - \tfrac{k_s}{3}.$$

Therefore, for any (small) neighborhood $V_{\underline{\theta}_{s,0}}$ of $\underline{\theta}_{s,0}$, for N is much greater and while s being fixed, we have almost surely

$$\sup_{\underline{\theta}_s \in V_{\eta_{\underline{\theta}_{s,j}}}} \textstyle\sum_{s=1}^{S} \sum_{\tau=0}^{N-1} \tfrac{h_{s+\tau S}(\underline{\theta}_s)}{NS} \leq \sup_{\underline{\theta}_s \in V_{\underline{\theta}_{s,0}}} \sum_{s=1}^{S} \sum_{\tau=0}^{N-1} \tfrac{h_{s+\tau S}(\underline{\theta}_s)}{NS},$$

which implies $\widehat{\underline{\theta}}_{s,CLS} \in V_{\underline{\theta}_{s,0}}$. ■

As we mentioned previously the value of r_s is, in each period, a positive integer, thus the consistency of $\widehat{r}_{s,CLS}$ is that $\widehat{r}_{s,CLS} = r_s$. The efficiency of the other estimated parameters with r_s being estimated together is asymptotically the same when r_s is known. Therefore, our interest is about the other parameters $\alpha_{1,s}^{(0)}$ and $\alpha_{2,s}^{(0)}$. Let $\underline{\boldsymbol{\alpha}}_{s,0} = \left(\alpha_{1,s}^{(0)}, \alpha_{2,s}^{(0)}\right)'$ be the true vector values of the vector parameters $\underline{\boldsymbol{\alpha}}_s = (\alpha_{1,s}, \alpha_{2,s})'$, $s = 1, ..., S$. The following proposition establishes the asymptotic distribution of the CLS vector estimators $\widehat{\underline{\boldsymbol{\alpha}}}_{s,CLS}$, of this vector parameters.

Proposition 3.4.2 *The CLS-vector estimator $\widehat{\underline{\boldsymbol{\alpha}}}_{s,CLS} = (\widehat{\alpha}_{1,s}, \widehat{\alpha}_{2,s})'$ is, under the assumptions 3.4.1–3.4.2, asymptotically normally distributed,*

$$\sqrt{N}\left(\widehat{\underline{\boldsymbol{\alpha}}}_{s,CLS} - \underline{\boldsymbol{\alpha}}_{s,0}\right) \overset{\mathcal{L}}{\rightsquigarrow} \mathcal{N}\left(\mathbf{0}, \Gamma_s^{-1}\Omega_s\Gamma_s^{-1}\right), \ \ s = 1, ..., S, \ \ \textit{where,}$$

Γ_s and Ω_s are diagonal square matrices of order two, with elements given respectively by

$$\left(\Gamma_s\right)_{i,i} = \mathbb{E}\left[\frac{\partial}{\partial \alpha_{i,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right) \frac{\partial}{\partial \alpha_{i,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right],$$

and

$$\left(\Omega_s\right)_{i,i} = \mathbb{E}\left[U_{s+\tau S}\left(\underline{\theta}_s\right) \frac{\partial}{\partial \alpha_{i,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right) \frac{\partial}{\partial \alpha_{i,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right], \ \ i = 1, 2.$$

Proof of Proposition 3.4.2. To prove the asymptotic normality, we have to check that the following regularity conditions, in Klimko and Nelson (1978) , hold

(A) . $\mathbb{E}\left(y_t|\, y_{t-1}, y_{t-2}, ..., y_0\right) = \mathbb{E}\left(y_t|\, y_{t-1}\right)$, $t \geq 1$ a.e.

(B) . $\mathbb{E}\left(U_{s+\tau S}\left(\underline{\theta}_s\right) \dfrac{\partial}{\partial \alpha_{i,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right) \dfrac{\partial}{\partial \alpha_{j,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right) < \infty$, $(i, j = 1, 2)$ with, $U_{s+\tau S}\left(\underline{\theta}_s\right)$ $= \left(y_{s+\tau S} - g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right)^2$.

(C) . Γ_s is non singular.

Condition (A) is satisfied since the process, in τ, $y_{s+\tau S}$ is a first-order Markov chain while s being fixed in $\{1, \ldots, S\}$. In order to prove condition (B), we check that the elements $\left(\Omega_s\right)_{i,i}$ for $i = 1, 2$ are all finite

$$\begin{aligned}\left(\Omega_s\right)_{i,i} &= \mathbb{E}\left[U_{s+\tau S}\left(\underline{\theta}_s\right)\left(\tfrac{\partial}{\partial \alpha_{i,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right)^2\right], \\ &= \mathbb{E}\left[I_{\tau,s-1}^{(i)}\left(y_{s-1+\tau S} + r_s\right)^2\left(\textstyle\sum_{k=1}^2 \alpha_{k,s}\left(1+\alpha_{k,s}\right) I_{\tau,s-1}^{(k)}\left(y_{s-1+\tau S} + r_s\right)\right)\right], \\ &= \alpha_{i,s}\left(1+\alpha_{i,s}\right)\left(\mu_{y,s-1}^{(i,3)} + 3 r_s \mu_{y,s-1}^{(i,2)} + 3 r_s^2 \mu_{y,s-1}^{(i,1)} + r_s^3\right) < \infty,\ \ i = 1, 2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left(\Omega_s\right)_{1,2} = \left(\Omega_s\right)_{2,1} &= \mathbb{E}\left[U_{s+\tau S}\left(\underline{\theta}_s\right) \frac{\partial}{\partial \alpha_{1,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right) \frac{\partial}{\partial \alpha_{2,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right], \\ &= \mathbb{E}\left[U_{s+\tau S}\left(\underline{\theta}_s\right) I_{\tau,s-1}^{(1)} I_{\tau,s-1}^{(2)}\left(y_{s-1+\tau S} + r_s\right)^2\right] = 0.\end{aligned}$$

Therefore, condition (B) is also satisfied. Finally, the matrix Γ_s is given, for a fixed s, by

$$\left(\Gamma_s\right)_{i,i} = \mathbb{E}\left[\left(\frac{\partial}{\partial \alpha_{i,s}} g\left(\underline{\theta}_s, y_{s-1+\tau S}\right)\right)^2\right] = p_{i,s}\left[\sigma_{y,s-1}^{2,(i)} + \left(\mu_{y,s-1}^{(i,1)} + r_s\right)^2\right],\ \ i = 1, 2,$$

note that the determinant of the matrix Γ_s is $\det\left(\Gamma_s\right) = \prod_{k=1}^2 p_{k,s}(\sigma_{y,s-1}^{2,(k)} + (\mu_{y,s-1}^{(k,1)} + r_s)^2) > 0$, which lead us to conclude that Γ_s is invertible. Thus, condition (C) is also satisfied. Finally, by Theorem 3.2 of Klimko and Nelson (1978), the CLS-estimators $\widehat{\underline{\theta}}_{s,CLS}$ are asymptotically normal. ■

3.4.2 Conditional maximum likelihood estimators

Denote by $\underline{Y} = (y_1, y_2, ..., y_n)$ a realization of a finite size n of a periodically correlated process $\{y_t, t \in \mathbb{Z}\}$ satisfying the $PNBSETINAR_S\left(2;1\right)$ model given by (3.2.2). The CML-vector estimators $\widehat{\underline{\theta}}_{s,CML} = (\widehat{\alpha}_{1,s}, \widehat{\alpha}_{2,s}, \widehat{r}_s)'$ of the vector parameters $\underline{\theta}_s$ is given by maximizing the conditional likelihood function, which is given for a size $n = NS$, while taking $t = s + tS$, $s = 1, ..., S$ and $\tau \in \mathbb{Z}$, by :

$$L\left(\underline{\theta}_s; \underline{Y}\right) = \prod_{s=1}^{S} \prod_{\tau=0}^{N-1} P\left(y_{s+\tau S} = x_{s+\tau S} \middle| y_{s-1+\tau S} = x_{s-1+\tau S}\right),$$

or equivalently which maximizes the log conditional likelihood function $\mathcal{L}\left(\underline{\theta}_s; \underline{Y}\right)$, of these $n = NS$ observations $\mathcal{L}\left(\underline{\theta}_s; \underline{Y}\right) = \sum_{s=1}^{S} \sum_{\tau=0}^{N-1} q_{s+\tau S}\left(\underline{\theta}_s\right)$, with

$$q_{s+\tau S}\left(\underline{\theta}_s\right) = \log P\left(y_{s+\tau S} = x_{s+\tau S} \middle| y_{s-1+\tau S} = x_{s-1+\tau S}\right) \text{ and,}$$

$$P\left(y_{s+\tau S} = y \middle| y_{s-1+\tau S} = x\right) = \frac{\Gamma\left(y + x + r_s\right)}{\Gamma\left(x + r_s\right)\Gamma\left(y + 1\right)} \left(\frac{\alpha_{1,s}^{y} I_{\tau,s-1}^{(1)}}{\left(1 + \alpha_{1,s}\right)^{r_t + y + x}} + \frac{\alpha_{2,s}^{y} I_{\tau,s-1}^{(2)}}{\left(1 + \alpha_{2,s}\right)^{r_t + y + x}} \right).$$

The estimations of $\alpha_{1,s}$, $\alpha_{2,s}$ and r_s, $s = 1, ..., S$, are obtained by the use of the same algorithm "Min-Min", indeed, we use the following normal equations :

$$\frac{\partial \mathcal{L}\left(\underline{\theta}_s\right)}{\partial r_s} = \sum_{\tau=0}^{N-1} \left(\sum_{j=0}^{x_{s+\tau S}-1} \frac{1}{x_{s-1+\tau S} + r_s + j} - \frac{\sum_{k=1}^{2} \frac{\log\left(1 + \alpha_{k,s}\right) \alpha_{k,s}^{x_{s+\tau S}} I_{\tau,s-1}^{(k)}}{\left(1 + \alpha_{k,s}\right)^{x_{s+\tau S} + x_{s-1+\tau S} + r_s}}}{\sum_{k=1}^{2} \frac{\alpha_{k,s}^{x_{s+\tau S}} I_{\tau,s-1}^{(k)}}{\left(1 + \alpha_{k,s}\right)^{x_{s+\tau S} + x_{s-1+\tau S} + r_s}}} \right) = 0. \tag{3.4.7}$$

$$\begin{aligned} \frac{\partial \mathcal{L}\left(\underline{\theta}_s\right)}{\partial \alpha_{i,s}} &= \sum_{\tau=0}^{N-1} \frac{\alpha_{i,s}^{x_{s+\tau S}-1} \left(x_{s+\tau S} - \alpha_{i,s}\left(x_{s-1+\tau S} + r_s\right)\right) I_{\tau,s-1}^{(i)}}{\left(\sum_{k=1}^{2} \alpha_{k,s}^{x_{s+\tau S}} I_{\tau,s-1}^{(k)} \Big/ \left(1 + \alpha_{k,s}\right)^{x_{s+\tau S} + x_{s-1+\tau S} + r_s} \right) \left(1 + \alpha_{i,s}\right)^{x_{s+\tau S} + x_{s-1+\tau S} + r_s + 1}}, \\ &= 0, \ i = 1, 2. \end{aligned} \tag{3.4.8}$$

The details of this algorithm are in Section 3.4.4. The next results give the consistency and the asymptotic distribution of our CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$.

Proposition 3.4.3 *The CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$ of the vector parameters $\underline{\theta}_s$ are, under the assumptions 3.4.1–3.4.2, strongly consistent, i.e.,*

$$\widehat{\underline{\theta}}_{s,CML} \rightsquigarrow \underline{\theta}_{s,0} \ a.s.$$

Proof of Proposition 3.4.3. We can see that this result generalizes the theorem 3.3 given by Yang *et al* (2018*b*), and it is easy also to see that our process $\{X_{s+\tau S}, \tau \in \mathbb{Z}\}$ is in τ, while s being fixed in $\{1, \dots, S\}$. Then, the proof of Proposition 3.4.3 is similar to the theorem 3.3 in Yang *et al* (2018*b*). First, we have, while s being fixed, $q_{s+\tau S}\left(\underline{\theta}_s\right) = W_{s+\tau S} + I_{\tau,s-1}^{(1)} q_{1,s+\tau S}\left(\underline{\theta}_s\right) + I_{\tau,s-1}^{(2)} q_{2,s+\tau S}\left(\underline{\theta}_s\right)$, where $W_{s+\tau S} = \log\left(\frac{\Gamma\left(x_{s+\tau S} + x_{s-1+\tau S} + r_s\right)}{\Gamma\left(x_{s-1+\tau S} + r_s\right)\Gamma\left(x_{s+\tau S} + 1\right)}\right)$, and $q_{k,s+\tau S}\left(\underline{\theta}_s\right) = y_{s+\tau S} \log(\alpha_{k,s}) - \left(y_{s+\tau S} + y_{s-1+\tau S} + r_s\right) \log(\alpha_{k,s} + 1)$, $k = 1, 2$. The proof is also achieved in three steps.

Step 1. While s being fixed in $\{1,\dots,S\}$, we show that $\mathbb{E}\left(q_{s+\tau S}\left(\underline{\theta}_s\right)\right)$ is continuous in $\underline{\theta}_s$. Since r_s is discrete, we need only to prove the following property. For any $\underline{\theta}_s \in \mathcal{D}$, let $V_\eta\left(\underline{\theta}_s\right) = B\left(\underline{\theta}_s,\eta\right)$ be an open ball centered at $\underline{\theta}_s$ with radius η, then

$$\mathbb{E}\left(\sup_{\underline{\theta}'_s\in V_\eta(\underline{\theta}_s)}\left|q_{s+\tau S}\left(\underline{\theta}_s\right)-q_{s+\tau S}\left(\underline{\theta}'_s\right)\right|\right)\rightarrow 0,\ \text{as } \eta\rightarrow 0.$$

Since

$$\begin{aligned}
\left|q_{s+\tau S}\left(\underline{\theta}_s\right)-q_{s+\tau S}\left(\underline{\theta}'_s\right)\right| &= \left|I^{(1)}_{\tau,s-1}\left(q_{1,s+\tau S}\left(\underline{\theta}_s\right)-q_{1,s+\tau S}\left(\underline{\theta}'_s\right)\right)-I^{(2)}_{\tau,s-1}\left(q_{2,s+\tau S}\left(\underline{\theta}_s\right)-q_{2,s+\tau S}\left(\underline{\theta}'_s\right)\right)\right|,\\
&\le \textstyle\sum_{k=1}^{2}\left|\left(q_{k,s+\tau S}\left(\underline{\theta}_s\right)-q_{k,s+\tau S}\left(\underline{\theta}'_s\right)\right)\right|,\\
&\le \sum_{k=1}^{2}\left|y_{s+\tau S}\left(\log(\alpha_{k,s})-\log(\alpha'_{k,s})\right)-\left(y_{s+\tau S}+y_{s-1+\tau S}+r_s\right)\left(\log(\alpha_{k,s}+1)-\log(\alpha'_{k,s}+1)\right)\right|,\\
&\le \textstyle\sum_{k=1}^{2}\left|y_{s+\tau S}\left(\alpha_{k,s}-\alpha'_{k,s}\right)/\,\xi^{(k)}_{s,1}-\left(y_{s+\tau S}+y_{s-1+\tau S}+r_s\right)\left(\alpha_{k,s}-\alpha'_{k,s}\right)/\,\xi^{(k)}_{s,2}\right|,\\
&\le 2\eta\xi_s\left(2y_{s+\tau S}+y_{s-1+\tau S}+r_s\right),\ \text{with } \xi_s=\max\left\{1\left/\xi^{(1)}_{s,1}\right.,1\left/\xi^{(1)}_{s,2}\right.,1\left/\xi^{(2)}_{s,1}\right.,1\left/\xi^{(2)}_{s,2}\right.\right\},
\end{aligned}$$

$\xi^{(k)}_{s,1} = \dfrac{\alpha_{k,s}-\alpha'_{k,s}}{\log(\alpha_{k,s})-\log(\alpha'_{k,s})}$, and $\xi^{(k)}_{s,2} = \dfrac{\alpha_{k,s}-\alpha'_{k,s}}{\log(\alpha_{k,s}+1)-\log(\alpha'_{k,s}+1)}$, $k=1,2$. So, we obtain

$$\mathbb{E}\left(\sup_{\underline{\theta}'_s\in V_\eta(\underline{\theta}_s)}\left|q_{s+\tau S}\left(\underline{\theta}_s\right)-q_{s+\tau S}\left(\underline{\theta}'_s\right)\right|\right)\le 2\eta\xi_s\mathbb{E}\left[2y_{s+\tau S}+y_{s-1+\tau S}+r_s\right]\rightarrow 0,\ \text{as } \eta\rightarrow 0.$$

Step 2. We prove that $\mathbb{E}_{\underline{\theta}_{s,0}}\left[q_{s+\tau S}\left(\underline{\theta}_s\right)-q_{s+\tau S}\left(\underline{\theta}_{s,0}\right)\right]<0$, for any $\underline{\theta}_s\neq\underline{\theta}_{s,0}$ where, $\underline{\theta}_{s,0}$ is the true value of $\underline{\theta}_s$. By using Jensen inequality, we have :

$$\begin{aligned}
\mathbb{E}_{\underline{\theta}_{s,0}}\left[q_{s+\tau S}\left(\underline{\theta}_s\right)-q_{s+\tau S}\left(\underline{\theta}_{s,0}\right)\right] &= \mathbb{E}_{\underline{\theta}_{s,0}}\left[\mathbb{E}_{\underline{\theta}_{s,0}}\left(\log\frac{P_{\underline{\theta}_s}(y_{s+\tau S}|y_{s-1+\tau S})}{P_{\underline{\theta}_{s,0}}(y_{s+\tau S}|y_{s-1+\tau S})}\right|\mathcal{F}_{s-1+\tau S}\right)\right],\\
&\le \mathbb{E}_{\underline{\theta}_{s,0}}\left[\log\mathbb{E}_{\underline{\theta}_{s,0}}\left(\frac{P_{\underline{\theta}_s}(y_{s+\tau S}|y_{s-1+\tau S})}{P_{\underline{\theta}_{s,0}}(y_{s+\tau S}|y_{s-1+\tau S})}\right|\mathcal{F}_{s-1+\tau S}\right)\right]=\mathbb{E}_{\underline{\theta}_{s,0}}\left[\log(1)\right]=0.
\end{aligned}$$

Step 3. Now, we are ready to prove the consistency for $\widehat{\underline{\theta}}_{s,CLS}$, while s being fixed in $\{1,\dots,S\}$ and with $h_{s+\tau S}\left(\underline{\theta}_s\right)=q_{s+\tau S}\left(\underline{\theta}_s\right)$. Therefore, we will have the same procedure as step 3 in Proposition 3.4.1. This achieves the proof. Removing r_s from $\underline{\theta}_s$ again, and consider a central limit theorem for the conditional maximum likelihood estimator. Note that with a known of r_s, $q_{s+\tau S}\left(\underline{\theta}_s\right)$ is differentiable with respect to $\underline{\boldsymbol{\alpha}}_s=(\alpha_{1,s},\alpha_{2,s})'$. Let

$$\mathbf{G}_s=\mathbb{E}_{\underline{\boldsymbol{\alpha}}_{s,0}}\left[\left(\frac{\partial}{\partial\underline{\boldsymbol{\alpha}}_s}q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right)\left(\frac{\partial}{\partial\underline{\boldsymbol{\alpha}}_s}q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right)'\right],$$

where,

$$\left(\frac{\partial}{\partial\underline{\boldsymbol{\alpha}}_s}q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right)'=\left(\begin{array}{cc}\dfrac{\partial}{\partial\alpha_{1,s}}q_{1,s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)I^{(1)}_{\tau,s-1} & \dfrac{\partial}{\partial\alpha_{2,s}}q_{2,s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)I^{(2)}_{\tau,s-1}\end{array}\right).\quad\blacksquare$$

The following proposition establishes the asymptotic distributional property of our CML-vector *estimators* $\widehat{\underline{\boldsymbol{\alpha}}}_{s,CML} = (\widehat{\alpha}_{1,s}, \widehat{\alpha}_{2,s})'$, of the vector parameters $\underline{\boldsymbol{\alpha}}_s$, when r_s is known.

Proposition 3.4.4 *The CML-vector estimators $\widehat{\underline{\boldsymbol{\alpha}}}_{s,CML} = (\widehat{\alpha}_{1,s}, \widehat{\alpha}_{2,s})'$ are, under the assumptions 3.4.1–3.4.2, asymptotically normally distributed, i.e.,*

$$\sqrt{N}\left(\widehat{\underline{\boldsymbol{\alpha}}}_{s,CML} - \underline{\boldsymbol{\alpha}}_{s,0}\right) \overset{\mathcal{L}}{\rightsquigarrow} \mathcal{N}\left(\mathbf{0}, \mathbf{G}_s^{-1}\right),\ s = 1, ..., S.$$

Furthermore, the matrix $\mathbf{G}_s$, for $s = 1, \ldots, S$, can be estimated consistently by :

$$\widehat{\mathbf{G}}_s = \frac{1}{N}\sum_{\tau=0}^{N-1}\left(\frac{\partial}{\partial \underline{\boldsymbol{\alpha}}_s} q_{s+\tau S}\left(\widehat{\underline{\boldsymbol{\alpha}}}_{s,CML}\right)\right)\left(\frac{\partial}{\partial \underline{\boldsymbol{\alpha}}_s} q_{s+\tau S}\left(\widehat{\underline{\boldsymbol{\alpha}}}_{s,CML}\right)\right)'.$$

Proof of Proposition 3.4.4. By Taylor's expansion, for $j = 1, 2$, there exists $\underline{\boldsymbol{\alpha}}_{s,j}$ between $\underline{\boldsymbol{\alpha}}_{s,0}$ and $\widehat{\underline{\boldsymbol{\alpha}}}_s$ such that

$$\begin{aligned}\tfrac{1}{\sqrt{N}}\textstyle\sum_{\tau=0}^{N-1}\frac{\partial}{\partial \underline{\boldsymbol{\alpha}}_{s,j}} q_{s+\tau S}\left(\widehat{\underline{\boldsymbol{\alpha}}}_{s,CML}\right) &= \tfrac{1}{\sqrt{N}}\textstyle\sum_{\tau=0}^{N-1}\frac{\partial}{\partial \underline{\boldsymbol{\alpha}}_{s,j}} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\\ &+ \tfrac{1}{N}\sum_{\tau=0}^{N-1}\frac{\partial^2}{\partial \underline{\boldsymbol{\alpha}}_{s,j}\partial \underline{\boldsymbol{\alpha}}_s'} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\sqrt{N}\left(\widehat{\underline{\boldsymbol{\alpha}}}_{s,CML} - \underline{\boldsymbol{\alpha}}_{s,0}\right) = 0.\end{aligned}$$

The proposition follows if it can be proved that

$$\tfrac{1}{\sqrt{N}}\textstyle\sum_{\tau=0}^{N-1}\partial q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)/\,\partial \underline{\boldsymbol{\alpha}}_s \overset{\mathcal{L}}{\rightsquigarrow} N\left(\mathbf{0}, \mathbf{G}_s\right), \tag{3.4.9}$$

and

$$\tfrac{1}{N}\textstyle\sum_{\tau=0}^{N-1}\partial^2 q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_s^*\right)/\,\partial \underline{\boldsymbol{\alpha}}_s \partial \underline{\boldsymbol{\alpha}}_s' \overset{P}{\rightsquigarrow} -\mathbf{G}_s, \tag{3.4.10}$$

for all $\underline{\boldsymbol{\alpha}}_s^*$ between $\underline{\boldsymbol{\alpha}}_{s,0}$ and $\widehat{\underline{\boldsymbol{\alpha}}}_{s,CML}$, we now begin to prove (3.4.9)

$$\begin{aligned}\mathbb{E}_{\underline{\boldsymbol{\alpha}}_{s,0}}\left[\left(\frac{\partial}{\partial \alpha_{k,s}} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right)\right] &= \mathbb{E}\left[\mathbb{E}\left(\frac{\partial}{\partial \alpha_{k,s}} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\middle|\mathcal{F}_{s-1+\tau S}\right)\right],\\ &= \mathbb{E}\left[I_{\tau,s-1}^{(k)}\mathbb{E}\left(\frac{y_{s+\tau S}}{\alpha_{k,s}} - \frac{y_{s+\tau S} + y_{s-1+\tau S} + r_s}{1+\alpha_{k,s}}\middle|\mathcal{F}_{s-1+\tau S}\right)\right] = \mathbb{E}\left(0\right) = 0,\ \text{for } k = 1, 2.\end{aligned}$$

Then, we have $\mathbb{E}_{\underline{\boldsymbol{\alpha}}_{s,0}}\left[\left(\frac{\partial}{\partial \underline{\boldsymbol{\alpha}}_s} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right)\right] = \mathbf{0}$, which implies that $\left\{\frac{\partial}{\partial \underline{\boldsymbol{\alpha}}_s} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right\}$ is a martingale difference. Similarly, we have

$$\mathbb{E}_{\underline{\boldsymbol{\alpha}}_{s,0}}\left[\left(\frac{\partial}{\partial \underline{\boldsymbol{\alpha}}_s} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right)^2\right] = \mathbb{E}\left[I_{\tau,s-1}^{(k)}\mathbb{E}\left(\left(\frac{y_{s+\tau S}}{\alpha_{k,s}} - \frac{y_{s+\tau S} + y_{s-1+\tau S} + r_s}{1+\alpha_{k,s}}\right)^2\middle|\mathcal{F}_{s-1+\tau S}\right)\right],$$

we get when developing

$$= \mathbb{E}\left[I_{\tau,s-1}^{(k)}\mathbb{E}\left(\left(\frac{y_{s+\tau S}^2}{\alpha_{k,s}^2} - 2\frac{y_{s+\tau S}^2 + y_{s+\tau S}\left(y_{s-1+\tau S} + r_s\right)}{\alpha_{k,s}\left(1+\alpha_{k,s}\right)} + \frac{\left(y_{s+\tau S} + y_{s-1+\tau S} + r_s\right)^2}{\left(1+\alpha_{k,s}\right)^2}\right)\middle|\mathcal{F}_{s-1+\tau S}\right)\right],$$

$$= \mathbb{E}\left[I^{(k)}_{\tau,s-1}\left(\tfrac{\alpha^2_{k,s}(y_{s-1+\tau S}+r_s)^2+\alpha_{k,s}\left(1+\alpha_{k,s}\right)(y_{s-1+\tau S}+r_s)}{\alpha^2_{k,s}} - \right.\right.$$
$$\left.\left. 2\tfrac{\alpha_{k,s}\left(1+\alpha_{k,s}\right)(y_{s-1+\tau S}+r_s)^2+\alpha_{k,s}\left(1+\alpha_{k,s}\right)(y_{s-1+\tau S}+r_s)}{\alpha_{k,s}\left(1+\alpha_{k,s}\right)} + \tfrac{\left(1+\alpha_{k,s}\right)^2(y_{s-1+\tau S}+r_s)^2+\alpha_{k,s}\left(1+\alpha_{k,s}\right)(y_{s-1+\tau S}+r_s)}{\left(1+\alpha_{k,s}\right)^2}\right)\right],$$
$$= \mathbb{E}\left[I^{(k)}_{\tau,s-1}\left(\tfrac{\alpha_{k,s}}{1+\alpha_{k,s}} + \tfrac{1+\alpha_{k,s}}{\alpha_{k,s}} - 2\right)(y_{s-1+\tau S}+r_s)\right],$$
$$= \left(\tfrac{\alpha_{k,s}}{1+\alpha_{k,s}} + \tfrac{1+\alpha_{k,s}}{\alpha_{k,s}} - 2\right)\mathbb{E}\left(I^{(k)}_{\tau,s-1}\left(y_{s-1+\tau S}+r_s\right)\right) > 0, \text{ for } k = 1, 2.$$

Also, It is easier to see that:

$$\mathbb{E}_{\underline{\boldsymbol{\alpha}}_{s,0}}\left[\left(\partial q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)/\,\partial\alpha_{s,i}\right)\left(\partial q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)/\,\partial\alpha_{j,s}\right)\right] = 0, \text{ for } i,j = 1,2, \text{ and } j \neq i.$$

Consequently, we have

$$Var_{\underline{\boldsymbol{\alpha}}_{s,0}}\left[\tfrac{\partial}{\partial\underline{\boldsymbol{\alpha}}_s} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right] = \mathbb{E}_{\underline{\boldsymbol{\alpha}}_{s,0}}\left[\left(\tfrac{\partial}{\partial\underline{\boldsymbol{\alpha}}_s} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right)\left(\tfrac{\partial}{\partial\underline{\boldsymbol{\alpha}}_s} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right)'\right] = \mathbf{G}_s,$$

where $\mathbf{G}_s$ is regular matrix. Since $\left\{\partial q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)/\partial\underline{\boldsymbol{\alpha}}_s\right\}$ is a martingale difference, by the Cramér-Wold device and the central limit theorem in Theorem 18.3 of Billingsley (1999) we have the weak convergence,

$$\frac{1}{\sqrt{N}}\sum_{\tau=0}^{N-1} \partial q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)/\,\partial\underline{\boldsymbol{\alpha}}_s \overset{\mathcal{L}}{\rightsquigarrow} N\left(\mathbf{0},\mathbf{G}_s\right),$$

Now, we prove (3.4.10) . under the direct calculation, we get:

$$\mathbb{E}_{\underline{\boldsymbol{\alpha}}_{s,0}}\left[\left(\frac{\partial^2}{\partial\alpha^2_{k,s}} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right)\right] = \mathbb{E}\left[I^{(k)}_{\tau,s-1}\mathbb{E}\left(-\frac{y_{s+\tau S}}{\alpha^2_{k,s}} + \frac{y_{s+\tau S}+y_{s-1+\tau S}+r_s}{\left(1+\alpha_{k,s}\right)^2}\middle|\,\mathcal{F}_{s-1+\tau S}\right)\right],$$
$$= -\left(\frac{\alpha_{k,s}}{1+\alpha_{k,s}} + \frac{1+\alpha_{k,s}}{\alpha_{k,s}} - 2\right)\mathbb{E}\left(I^{(k)}_{\tau,s-1}\left(y_{s-1+\tau S}+r_s\right)\right), \text{ for } k = 1, 2,$$

and

$$\mathbb{E}_{\underline{\boldsymbol{\alpha}}_{s,0}}\left[\partial^2 q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)/\,\partial\alpha_{k,s}\partial\alpha_{l,s}\right] = 0, \; l,k = 1,2 \text{ and } l \neq k.$$

Then, we have:

$$\mathbb{E}_{\underline{\boldsymbol{\alpha}}_{s,0}}\left[\partial^2 q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)/\,\partial\underline{\boldsymbol{\alpha}}_s\partial\underline{\boldsymbol{\alpha}}'_s\right] = -\mathbf{G}_s.$$

Similarly, we have

$$\mathbb{E}_{\underline{\boldsymbol{\alpha}}_s}\left[\left(\tfrac{\partial^3}{\partial\alpha^3_{k,s}} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right)\right] = \mathbb{E}\left[I^{(k)}_{\tau,s-1}\mathbb{E}\left(-\tfrac{2y_{s+\tau S}}{\alpha^3_{k,s}} + \tfrac{2(y_{s+\tau S}+y_{s-1+\tau S}+r_s)}{\left(1+\alpha_{k,s}\right)^3}\middle|\,\mathcal{F}_{s-1+\tau S}\right)\right],$$
$$= \left(\tfrac{1}{\alpha^2_{k,s}} - \tfrac{1}{\left(1+\alpha_{k,s}\right)^2}\right)\mathbb{E}\left(I^{(k)}_{\tau,s-1}\left(y_{s-1+\tau S}+r_s\right)\right) < \infty, \; k = 1, 2,$$

and $\mathbb{E}_{\underline{\boldsymbol{\alpha}}_s}\left[\left(\tfrac{\partial^3}{\partial\alpha_{j,s}\partial\alpha_{k,s}\partial\alpha_{l,s}} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right)\right] = 0$, for $j,k,l \in \{1,2\}$ and the product $jkl \neq 1, 8$.

Consider $V_{\underline{\boldsymbol{\alpha}}_{s,0}}$ a neighborhood of $\underline{\boldsymbol{\alpha}}_{s,0}$, then we have for $j,k,l \in \{1,2\}$:

$$\mathbb{E}_{\underline{\boldsymbol{\alpha}}_s}\left[\sup_{\underline{\boldsymbol{\alpha}}_s \in V_{\underline{\boldsymbol{\alpha}}_{s,0}}}\left|\frac{\partial^3}{\partial\alpha_{j,s}\partial\alpha_{k,s}\partial\alpha_{l,s}} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right|\right] < \infty. \tag{3.4.11}$$

By (3.4.11) and the strong law of numbers, we get

$$\overline{\lim_{N\to\infty}} \sup_{\underline{\boldsymbol{\alpha}}_s \in V_{\underline{\boldsymbol{\alpha}}_{s,0}}}\left\|\frac{1}{N}\sum_{\tau=0}^{N-1}\frac{\partial^3}{\partial\alpha_{j,s}\partial\alpha_{k,s}\partial\alpha_{l,s}} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right)\right\| < \infty. \tag{3.4.12}$$

Now, we are ready to prove (3.4.10). Recall that $\underline{\boldsymbol{\alpha}}_s^*$ lies between $\underline{\boldsymbol{\alpha}}_{s,0}$ and $\underline{\boldsymbol{\alpha}}_{s,CML}$. Consider the Taylor's expansion of the second-order derivatives of $q_{s+\tau S}$ at $\underline{\boldsymbol{\alpha}}_{s,0}$, we have

$$\frac{1}{N}\sum_{\tau=0}^{N-1}\frac{\partial^2}{\partial\alpha_{k,s}\partial\alpha_{l,s}} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_s^*\right) = \frac{1}{N}\sum_{\tau=0}^{N-1}\left[\frac{\partial^2}{\partial\alpha_{k,s}\partial\alpha_{l,s}} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right) + \frac{\partial^3 q_{s+\tau S}\left(\widetilde{\underline{\boldsymbol{\alpha}}}_s\right)\left(\underline{\boldsymbol{\alpha}}_s^* - \underline{\boldsymbol{\alpha}}_{s,0}\right)}{\partial\alpha_{j,s}\partial\alpha_{k,s}\partial\alpha_{l,s}},\right]$$

for some $\widetilde{\underline{\boldsymbol{\alpha}}}_s$ between $\underline{\boldsymbol{\alpha}}_{s,0}$ and $\underline{\boldsymbol{\alpha}}_s^*$, the last expression and (3.4.10), (**??**), the strong law of large numbers and the almost sure convergence of $\widetilde{\underline{\boldsymbol{\alpha}}}_s$ to $\underline{\boldsymbol{\alpha}}_{s,0}$ imply

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\tau=0}^{N-1}\frac{\partial^2}{\partial\alpha_{k,s}\partial\alpha_{l,s}} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_s^*\right) = \frac{1}{N}\sum_{\tau=0}^{N-1}\frac{\partial^2}{\partial\alpha_{k,s}\partial\alpha_{l,s}} q_{s+\tau S}\left(\underline{\boldsymbol{\alpha}}_{s,0}\right) = -\left(\mathbf{G}_s\right)_{k,l}, \text{ a.s.} \quad \blacksquare$$

3.4.3 Estimation of the periodic threshold parameter c_t

In order to estimate the unknown periodic threshold parameters c_s, $s = 1, ..., S$, one can adapt, to our periodic case, one of the many existing methods, in the time-invariant case, including the single grid search algorithm (Tsay 1989 and Yu 2012), the doubly-NeSS (*D-NeSS*) algorithm (Li *et al.* 2018), the NeSS algorithm (Li and Tong 2016) and so on.

Considering the factors of speed and calculation burden, we adapted, in our case, the *NeSS* algorithm proposed by Li and Tong (2016) to estimate the periodic threshold parameters. Indeed, adopting standard least squares criterion, we obtain the following sum of squared errors :

$$S_N\left(c_s\right) = \sum_{\tau=0}^{N-1}\left(y_{s+\tau S} - \sum_{k=1}^{2}\frac{\sum_{\tau_1=0}^{N-1} y_{s+\tau_1 S}\left(y_{s-1+\tau_1 S}+r_s\right) I_{\tau_1,s-1}^{(k)}}{\sum_{\tau_1=0}^{N-1}\left(y_{s-1+\tau_1 S}+r_s\right)^2 I_{\tau_1,s-1}^{(k)}}\left(y_{s-1+\tau S}+r_s\right) I_{\tau,s-1}^{(k)}\right)^2,$$

which leads to the estimation $\widehat{c}_s = \arg\min_{c_s \in [\underline{c}_s, \bar{c}_s]} S_N\left(c_s\right)$, $s = 1, ..., S$. Following Li and Tong (2016), let be $J_N\left(c_s\right) = S_N - S_N\left(c_s\right)$, where,

$$S_N = \sum_{\tau=0}^{N-1}\left(y_{s+\tau S} - \frac{\sum_{\tau_1=0}^{N-1} y_{s+\tau_1 S}\left(y_{s+\tau_1 S-1}+r_s\right)}{\sum_{\tau_1=0}^{N-1}\left(y_{s+\tau_1 S-1}+r_s\right)^2}\left(y_{s+\tau S-1}+r_s\right)\right)^2.$$

Then, the threshold parameter c_s, for $s = 1, ..., S$, can be estimated by maximizing $J_N(c_s)$, i.e., $\widehat{c}_s = \arg\max_{c_s \in [\underline{c}_s, \overline{c}_s]} J_N(c_s)$ where $\underline{c}_s$ and $\overline{c}_s$, $s = 1, ..., S$, can be selected as the minimum and maximum values of the samples, respectively. The application scenarios of the *D-NeSS* algorithm (Li *et al.* 2018) to our periodic case, are similar to those applied in Yang *et al.* (2018*a*) when s being fixed in $\{1, ..., S\}$. In fact, we use the notation $J_N(c_s, r_s)$ instead of $J_N(c_s)$ and we seek to find $\widehat{c}_s(r_s)$ that maximizes $J_N(c_s, r_s)$ for any value of r_s. After that, we search for the final $\widehat{c}_s$ for different values of r_s. The essential question facing us is : "how to handle the unknown quantity r_s, $s = 1, ..., S$,". As argued in Li and Tong (2016), under some regularity conditions, $J_N(c_s)/N$, $s = 1, ..., S$, is unimodal over $[\underline{c}_s, \overline{c}_s]$ with probability tending to one as $N \to \infty$, which greatly helps us to find $\widehat{c}_s$. Then, we have found out, similarly to Yang *et al.* (2018*a*) and Yang *et al.* (2018*b*), for different values of r_s and different sizes N, that the shape of $J_N(c_s, r_s)/N$, $s = 1, ..., S$, is unimodal and the maximum does not depend on the value of r_s, which means we can choose any positive integer value for r_s without worrying about getting a wrong result of $\widehat{c}_s$, $s = 1, ..., S$.

3.4.4 The "Min-Min" algorithm

In what follows, we discuss how to apply the Min-Min algorithm in details (as in Yang *et al* (2018*b*) while providing some comments and minor changes to adapt it to our periodic case). First of all, we assume that the sample $(y_1, y_2, ..., y_n)$ is available and we suppose that the threshold parameters c_s, $s = 1, ..., S$, are known. The iterative Min-Min algorithm for searching the CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$ is given as follows. Let $\theta_s^{(i)} = \left(\alpha_{1,s}^{(i)}, \alpha_{2,s}^{(i)}, r_s^{(i)}\right)'$ denote the value in the i th iteration and then by solving the equations (3.4.2) and (3.4.3) we obtain the updated parameters in $(i+1)$ th iteration by the following formulas, for $s = 1, ..., S$, $k = 1, 2$,

$$\widetilde{r}_s^{(i+1)} = \frac{\sum_{\tau=0}^{N-1}\left(y_{s+\tau S}\left(\alpha_{1,s}^{(i)} I_{\tau,s-1}^{(1)} + \alpha_{2,s}^{(i)} I_{\tau,s-1}^{(2)}\right)\right) - \left(y_{s-1+\tau S}\left(\alpha_{1,s}^{(i)} I_{\tau,s-1}^{(1)} + \alpha_{2,s}^{(i)} I_{\tau,s-1}^{(2)}\right)^2\right)}{\sum_{\tau=0}^{N-1}\left(\alpha_{1,s}^{(i)} I_{\tau,s-1}^{(1)} + \alpha_{2,s}^{(i)} I_{\tau,s-1}^{(2)}\right)^2} \tag{3.4.13}$$

$$\alpha_{k,s}^{(i+1)} = \left(\sum_{\tau=0}^{N-1}\left(y_{s-1+\tau S} + r_s^{(i+1)}\right)^2 I_{\tau,s-1}^{(k)}\right)^{-1} \sum_{\tau=0}^{N-1} y_{s+\tau S}\left(y_{s-1+\tau S} + r_s^{(i+1)}\right) I_{\tau,s-1}^{(k)}, \tag{3.4.14}$$

Summarizing the Min-Min algorithm in the following steps:

1. *Choose the starting point* $\alpha_{k,s}^{(0)}$, *for* $k = 1, 2,\ s = 1, ..., S$, *and set* $i = 0$.

2. *Determine whether* c_s *are known, if unknown, estimate them by* $(NeSS)$ *algorithm.*

3. *Calculate* $\widetilde{r}_s^{(i+1)}$, $s = 1, ..., S$, *by* (3.4.13) , *and set* $r_s^{(i+1)} = \left[\widetilde{r}_s^{(i+1)} + 0.5\right]$, *where* $[a]$ *is the integer part of* a.

4. *Calculate* $\alpha_{k,s}^{(i+1)}$, $k = 1, 2, s = 1, ..., S$, *by* (3.4.14) .

5. *Set* $i = i + 1$ *and go to step* 3 *until the convergence is achieved.*

Now, we give the CML-algorithm for searching the CML-estimators $\widehat{\underline{\theta}}_{s,CML}$. It is clear that the analytical estimates cannot be obtained by solving the system of equations (3.4.7) and (3.4.8). Thus, to solve this system numerical procedures must be employed, such as Newton-Raphson method, Secant Method, False Position Method or Regula Falsi Method (see, Press *et al* 2007, Chapter 9). Taking into account the factors of speed and calculation burden, we choose the one-step False Position Method (Press *et al.* 2007). Let $\overline{\theta}_s^{(i)} = \left(\overline{\alpha}_{1,s}^{(i)}, \overline{\alpha}_{2,s}^{(i)}, \overline{r}_s^{(i)}\right)'$ and $\underline{\theta}_s^{(i)} = \left(\underline{\alpha}_{1,s}^{(i)}, \underline{\alpha}_{2,s}^{(i)}, \underline{r}_s^{(i)}\right)'$denote an upper bound and a lower bound of $\underline{\theta}_s$, respectively. Let $f_{k,s}(\alpha_{1,s}, \alpha_{2,s}, r_s) = \dfrac{\partial}{\partial \alpha_{k,s}} \mathcal{L}(\underline{\theta}_s)$, $(k = 1, 2)$ and $g_s(\alpha_{1,s}, \alpha_{2,s}, r_s) = \dfrac{\partial}{\partial r_s} \mathcal{L}(\underline{\theta}_s)$, $s = 1, ..., S$, then, the roots of the equations (3.4.7) and (3.4.8) can be obtained via the following iterative formulas :

$$\widetilde{r}_s^{(i+1)} = \overline{r}_s^{(i)} - \frac{\overline{r}_s^{(i)} - \underline{r}_s^{(i)}}{\nabla g_s^{(i)}} g_s\left(\alpha_{1,s}^{(i)}, \alpha_{2,s}^{(i)}, \overline{r}_s^{(i)}\right), \tag{3.4.15}$$

$$\alpha_{1,s}^{(i+1)} = \overline{\alpha}_{1,s}^{(i)} - \frac{\overline{\alpha}_{1,s}^{(i)} - \underline{\alpha}_{1,s}^{(i)}}{\nabla f_{1,s}^{(i)}} f_{1,s}\left(\overline{\alpha}_{1,s}^{(i)}, \alpha_{2,s}^{(i)}, r_s^{(i+1)}\right), \tag{3.4.16}$$

hence, we have for $s = 1, ..., S$,

$$\alpha_{2,s}^{(i+1)} = \overline{\alpha}_{2,s}^{(i)} - \frac{\overline{\alpha}_{2,s}^{(i)} - \underline{\alpha}_{2,s}^{(i)}}{\nabla f_{2,s}^{(i)}} f_{2,s}\left(\alpha_{1,s}^{(i)}, \overline{\alpha}_{2,s}^{(i)}, r_s^{(i+1)}\right), \tag{3.4.17}$$

where,

$$\begin{aligned}
\nabla g_s^{(i)} &= g_s\left(\alpha_{1,s}^{(i)}, \alpha_{2,s}^{(i)}, \overline{r}_s^{(i)}\right) - g_s\left(\alpha_{1,s}^{(i)}, \alpha_{2,s}^{(i)}, \underline{r}_s^{(i)}\right), \\
\nabla f_{1,s}^{(i)} &= f_{1,s}\left(\overline{\alpha}_{1,s}^{(i)}, \alpha_{2,s}^{(i)}, r_s^{(i+1)}\right) - f_{1,s}\left(\underline{\alpha}_{1,s}^{(i)}, \alpha_{2,s}^{(i)}, r_s^{(i+1)}\right), \\
\nabla f_{2,s}^{(i)} &= f_{2,s}\left(\alpha_{1,s}^{(i)}, \overline{\alpha}_{2,s}^{(i)}, r_s^{(i+1)}\right) - f_{2,s}\left(\alpha_{1,s}^{(i)}, \underline{\alpha}_{2,s}^{(i)}, r_s^{(i+1)}\right).
\end{aligned}$$

Summarizing the Min-Min algorithm for CML-estimators $\widehat{\underline{\theta}}_{s,CML}$ in the following steps:

1. *Choose the starting point* $\alpha_{k,s}^{(0)}$*, for* $k = 1, 2,\ s = 1, ..., S,$*and set* $i = 0$.

2. *Determine whether* c_s*,* $s = 1, ..., S$*, are known, if unknown, estimate them by* $(NeSS)$ *algorithm.*

3. *Choose a upper bound* $\overline{r}_s^{(i)}$ *a lower bound* $\underline{r}_s^{(i)}$ *,*$s = 1, ..., S$*, such that*

$$g_s\left(\alpha_{1,s}^{(i)}, \alpha_{2,s}^{(i)}, \overline{r}_s^{(i)}\right) g_s\left(\alpha_{1,s}^{(i)}, \alpha_{2,s}^{(i)}, \underline{r}_s^{(i)}\right) \leq 0.$$

4. *Calculate* $\widetilde{r}_s^{(i+1)}$*,* $s = 1, ..., S$*, by* (3.4.15)*, and set* $r_s^{(i+1)} = \left[\widetilde{r}_s^{(i+1)} + 0.5\right]$,

5. *For* $k = 1, 2,\ s = 1, ..., S$*, choose a upper bound* $\overline{\alpha}_{k,s}^{(i)}$ *and a lower bound* $\underline{\alpha}_{k,s}^{(i)}$*, such that*

$$f_{1,s}\left(\overline{\alpha}_{1,s}^{(i)}, \alpha_{2,s}^{(i)}, \overline{r}_s^{(i+1)}\right) f_{1,s}\left(\underline{\alpha}_{1,s}^{(i)}, \alpha_{2,s}^{(i)}, \underline{r}_s^{(i+1)}\right) \leq 0,$$
$$f_{2,s}\left(\alpha_{1,s}^{(i)}, \overline{\alpha}_{2,s}^{(i)}, r_s^{(i+1)}\right) f_{2,s}\left(\alpha_{1,s}^{(i)}, \underline{\alpha}_{2,s}^{(i)}, r_s^{(i+1)}\right) \leq 0.$$

6. *Calculate* $\alpha_{k,s}^{(i+1)}$*,* $k = 1, 2,\ s = 1, ..., S$*, by* (3.4.16) *and* (3.4.17).

7. *Set* $i = i + 1$ *and go to step* 3 *until the convergence is achieved.*

3.5 Simulation results and application on real data

In this Section, we illustrate our obtained results and we assess the CLS and the CML estimations on two time series, for small, moderate, and relatively large sample sizes and also an application on real data set.

3.5.1 Simulation results

In order to show some empirical estimates properties, we have generated 1000 independent series from $PNBSETINAR_4\,(2;1)$ model, where the innovation process $\{\varepsilon_t;\ t \in \mathbb{Z}\}$ is generated from a negative binomial distribution with parameters r_s and $\alpha_{k,s}/(1+\alpha_{k,s})$, i.e., $\varepsilon_{k,s+\tau S} \rightsquigarrow \mathcal{NB}(r_s, \alpha_{k,s}/(1+\alpha_{k,s}))$ for $k = 1, 2$. For Model 1, the periodic threshold parameters c_s, for $s = 1, ..., 4$, are assumed to be known and unknown for Model 2. The true parameter values of these models are given below :

$$\begin{aligned}
\text{Model 1} : \underline{\theta} &= (\underline{\theta}_1; \ldots; \underline{\theta}_4)' = ((0.35, 0.25, 6); (0.5, 0.1, 9); (0.7, 0.2, 4); (0.6, 0.4, 7))', \\
\underline{C} &= (c_1, c_2, c_3, c_4) = (10, 4, 9, 7), \text{ with } \underline{C} \text{ is known}, \\
\text{Model 2} : \underline{\theta} &= (\underline{\theta}_1; \ldots; \underline{\theta}_4)' = ((0.2, 0.7, 10); (0.8, 0.5, 8); (0.5, 0.9, 4); (0.6, 0.15, 7))', \\
\underline{C} &= (c_1, c_2, c_3, c_4) = (6, 9, 15, 13), \text{ with } \underline{C} \text{ is unknown}.
\end{aligned}$$

For each of the above models, the threshold values, c_s for $s = 1, ..., 4$, were chosen such that the observations in each regime of each period are at least 20% of the sub-series size. As mentioned in Li and Tong (2016), when the proportion of observations in sub-series, i.e., $\{y_{s+\tau S};\ \tau \in \mathbb{Z}\}$ while s being fixed in $\{1, ..., 4\}$, of one regime to the whole is less than 5%, the estimated result may not be reliable. For each data-generating process, we consider 1000 replications, the mean estimates and their root mean square error ($RMSE$) are displayed in Tables 3.1 to 3.3. Tables 3.1 reports the means, median and $RMSE$ of the CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$ and CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$, across the 1000 replications, for the first model with a known c_s, for $s = 1, ..., 4$. While Table 3.2 reports the performance of the last generated time series when the periodic threshold parameters are considered unknown, where their estimates, obtained by the use of the adapted $NeSS$ algorithm, are reported in Table 3.3. The simulation programs are written in Matlab 8.5 environment.

From Table 3.1, one can easily observe that the adopted estimation method performs better as n increases. Indeed, the convergence of all the parameter estimators is guaranteed and the root mean square error decreasing, as the sample size n increases (see, Figure 3.1), which imply that our estimators (CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$ and CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$) are empirically consistent for all the parameters. It can also be seen that the median estimates generally, for the periodic integer valued r_s, for $s = 1, ..., 4$, better than the mean. Furthermore, we notice that the CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$ have a small $RMSE$ compared to the one of CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$. Thus, the CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$ are much advantage than the CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$, where this empirical superiority is clearly visible in Figure 3.1.

Moreover, we get the same conclusions from Table 3.2, that the consistency property of the CLS-estimators and CML-estimators still met, even if the threshold parameters $\widehat{c}_s$, for $s = 1, ..., 4$, are unknown, this encourages to use confidently the $PNBSETINAR_4\,(2;1)$ model, when the threshold parameters c_s are unknown, without worrying about to bring a wrong results.

Furthermore, from Table 3.3, which reports the means, medians, the percentage and the $RMSE$ of the periodic threshold estimations $\widehat{c}_s$, for $s = 1, ..., 4$, we can find that all the estimation results perform better as n increases, thus implying that the adapted $NeSS$ algorithm is empirically consistent. In addition, we observe that the median of 1000 repetitions, for the periodic threshold estimations $\widehat{c}_s$, for $s = 1, ..., 4$, estimates much better than the mean.

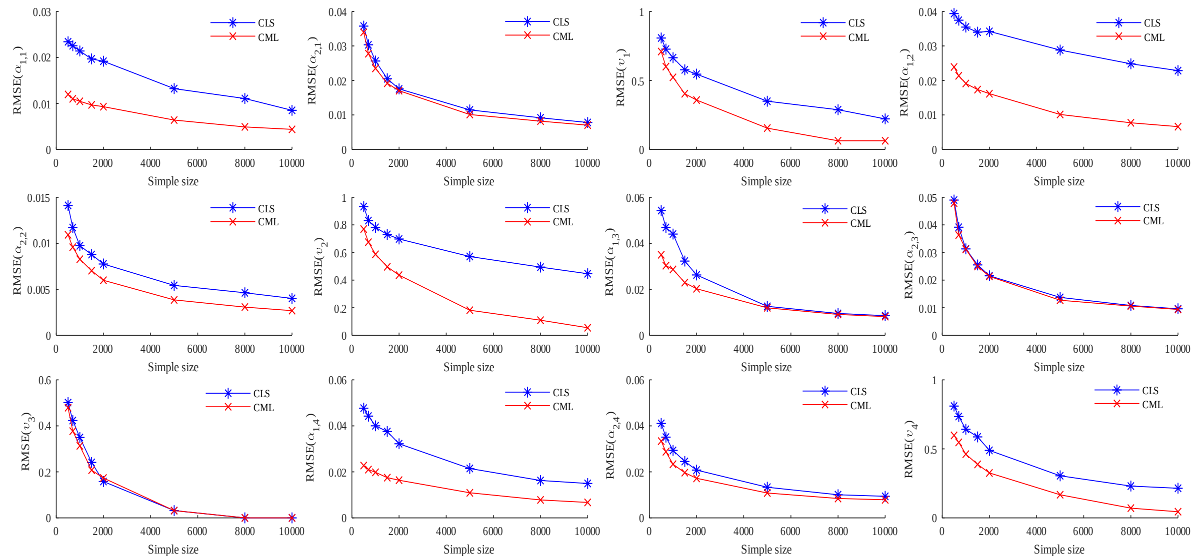

Figure 3.1. RMSE graphics for the parameters of Model 1 for different methods.

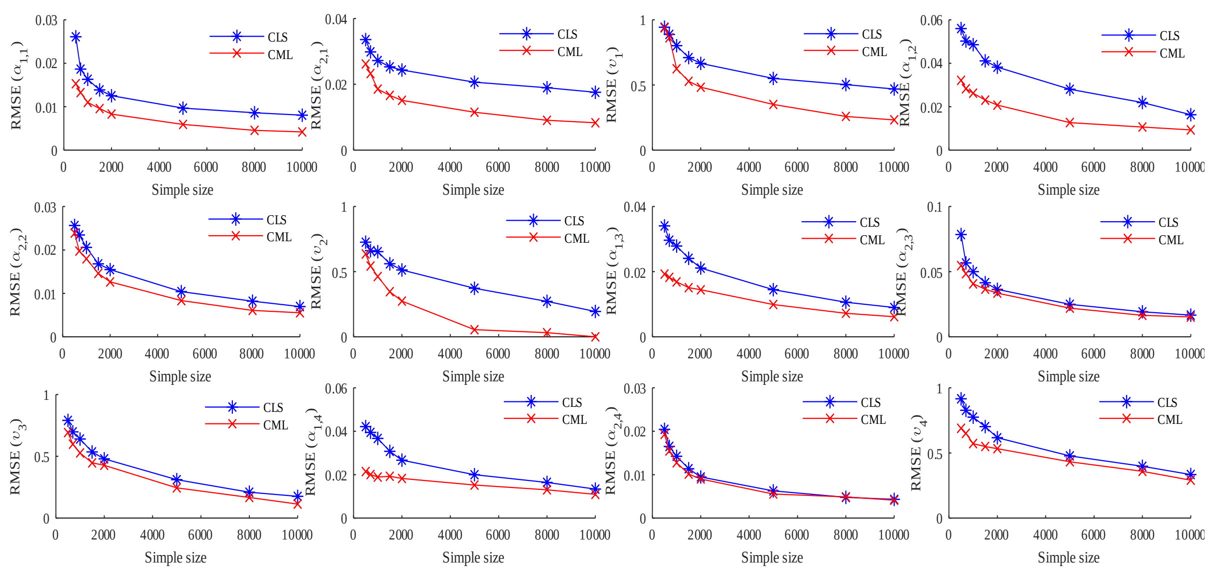

Figure 3.2. RMSE graphics for the parameters of Model 2 for different methods.

Table 3.1. *Sample mean and root mean square error RMSE (in bracket) for Model 1.*

	T.V	0.35	0.25	6		0.5	0.1	9		0.7	0.2	4		0.6	0.4	7	
Size	*Est.*	$\widehat{\alpha}_{1,1}$	$\widehat{\alpha}_{2,1}$	$\widehat{r}_1$	*med*	$\widehat{\alpha}_{1,2}$	$\widehat{\alpha}_{2,2}$	$\widehat{r}_2$	*med*	$\widehat{\alpha}_{1,3}$	$\widehat{\alpha}_{2,3}$	$\widehat{r}_3$	*med*	$\widehat{\alpha}_{1,4}$	$\widehat{\alpha}_{2,4}$	$\widehat{r}_4$	*med*
500	*CLS*	0.3611 (0.0234)	0.2559 (0.0358)	5.6140 (0.8078)	6	0.5252 (0.0394)	0.1038 (0.0141)	8.4730 (0.9325)	9	0.7189 (0.0542)	0.2029 (0.0490)	3.8010 (0.5012)	4	0.6277 (0.0477)	0.4114 (0.0411)	6.5280 (0.8102)	7
	CML	0.3532 (0.0120)	0.2516 (0.0340)	5.9130 (0.7096)	6	0.5040 (0.0239)	0.1015 (0.0109)	8.9400 (0.7698)	9	0.7119 (0.0350)	0.2037 (0.0479)	3.8510 (0.4788)	4	0.5904 (0.0228)	0.3964 (0.0334)	7.1550 (0.5978)	7
700	*CLS*	0.3609 (0.0226)	0.2542 (0.0303)	5.6500 (0.7269)	6	0.5241 (0.0375)	0.1031 (0.0117)	8.5390 (0.8316)	9	0.7128 (0.0469)	0.2039 (0.0392)	3.8450 (0.4233)	4	0.6261 (0.0443)	0.4101 (0.0351)	6.5590 (0.7344)	7
	CML	0.3530 (0.0110)	0.2522 (0.0278)	5.8940 (0.6003)	6	0.5054 (0.0214)	0.1013 (0.0096)	8.8750 (0.6749)	9	0.7082 (0.0303)	0.2016 (0.0363)	3.9120 (0.3770)	4	0.5913 (0.0210)	0.3962 (0.0287)	7.1650 (0.5471)	7
1000	*CLS*	0.3598 (0.0214)	0.2535 (0.0256)	5.6900 (0.6651)	6	0.5216 (0.0355)	0.1029 (0.0097)	8.5330 (0.7819)	9	0.7130 (0.0440)	0.1994 (0.0313)	3.8860 (0.3494)	4	0.6216 (0.0400)	0.4087 (0.0293)	6.6420 (0.6421)	7
	CML	0.3528 (0.0105)	0.2528 (0.0235)	5.8910 (0.5227)	6	0.5037 (0.0191)	0.1009 (0.0083)	8.9160 (0.5868)	9	0.7078 (0.0286)	0.2020 (0.0312)	3.9200 (0.3132)	4	0.5919 (0.0198)	0.3964 (0.0233)	7.1510 (0.4617)	7
1500	*CLS*	0.3589 (0.0197)	0.2541 (0.0205)	5.7170 (0.5773)	6	0.5201 (0.0340)	0.1029 (0.0088)	8.5600 (0.7310)	9	0.7050 (0.0323)	0.2004 (0.0255)	3.9440 (0.2409)	4	0.6185 (0.0375)	0.4070 (0.0245)	6.6760 (0.5867)	7
	CML	0.3527 (0.0097)	0.2518 (0.0192)	5.9270 (0.4039)	6	0.5050 (0.0174)	0.1012 (0.0070)	8.8960 (0.4962)	9	0.7048 (0.0229)	0.2000 (0.0249)	3.9610 (0.2075)	4	0.5935 (0.0175)	0.3967 (0.0196)	7.1380 (0.3875)	7
2000	*CLS*	0.3588 (0.0192)	0.2529 (0.0176)	5.7360 (0.5461)	6	0.5206 (0.0342)	0.1030 (0.0078)	8.5570 (0.6981)	9	0.7025 (0.0262)	0.1990 (0.0216)	3.9750 (0.1582)	4	0.6140 (0.0322)	0.4056 (0.0208)	6.7690 (0.4891)	7
	CML	0.3523 (0.0093)	0.2502 (0.0170)	5.9220 (0.3579)	6	0.5043 (0.0162)	0.1005 (0.0060)	8.9140 (0.4361)	9	0.7035 (0.0202)	0.1995 (0.0213)	3.9700 (0.1733)	4	0.5946 (0.0164)	0.3983 (0.0172)	7.0940 (0.3257)	7
5000	*CLS*	0.3539 (0.0132)	0.2517 (0.0115)	5.8780 (0.3494)	6	0.5150 (0.0288)	0.1019 (0.0054)	8.6780 (0.5712)	9	0.7004 (0.0126)	0.1997 (0.0137)	3.9990 (0.0316)	4	0.6058 (0.0215)	0.4019 (0.0133)	6.9070 (0.3051)	7
	CML	0.3508 (0.0064)	0.2505 (0.0101)	5.9800 (0.1550)	6	0.5017 (0.0101)	0.1000 (0.0038)	8.9790 (0.1817)	9	0.7001 (0.0120)	0.1998 (0.0127)	3.9990 (0.0316)	4	0.5986 (0.0109)	0.3993 (0.0108)	7.0280 (0.1674)	7
8000	*CLS*	0.3526 (0.0111)	0.2513 (0.0092)	5.9190 (0.2882)	6	0.5115 (0.0248)	0.1017 (0.0046)	8.7560 (0.4942)	9	0.7001 (0.0095)	0.1997 (0.0108)	4.0000 (0)	4	0.6030 (0.0163)	0.4015 (0.0101)	6.9470 (0.2303)	7
	CML	0.3501 (0.0049)	0.2500 (0.0082)	5.9960 (0.0633)	6	0.5004 (0.0077)	0.1000 (0.0031)	8.9880 (0.1096)	9	0.6999 (0.0091)	0.2001 (0.0106)	4.0000 (0)	4	0.5996 (0.0078)	0.4000 (0.0084)	7.0050 (0.0707)	7
10000	*CLS*	0.3516 (0.0085)	0.2507 (0.0078)	5.9510 (0.2215)	6	0.5098 (0.0229)	0.1013 (0.0040)	8.8010 (0.4463)	9	0.6999 (0.0086)	0.2002 (0.0096)	4.0000 (0)	4	0.6028 (0.0150)	0.4009 (0.0094)	6.9540 (0.2196)	7
	CML	0.3502 (0.0044)	0.2502 (0.0071)	5.9960 (0.0633)	6	0.5004 (0.0066)	0.1001 (0.0027)	8.9970 (0.0548)	9	0.6999 (0.0081)	0.2000 (0.0094)	4.0000 (0)	4	0.5996 (0.0067)	0.4000 (0.0079)	7.0020 (0.0447)	7

Table 3.2. Sample mean and root mean square error RMSE (in bracket) for Model 2.

	T.V	0.2	0.7	10		0.8	0.5	8		0.5	0.9	4		0.6	0.15	7	
Size	Est.	$\widehat{\alpha}_{1,1}$	$\widehat{\alpha}_{2,1}$	$\widehat{r}_1$	med	$\widehat{\alpha}_{1,2}$	$\widehat{\alpha}_{2,2}$	$\widehat{r}_2$	med	$\widehat{\alpha}_{1,3}$	$\widehat{\alpha}_{2,3}$	$\widehat{r}_3$	med	$\widehat{\alpha}_{1,4}$	$\widehat{\alpha}_{2,4}$	$\widehat{r}_4$	med
500	CLS	0.2059 (0.0261)	0.7110 (0.0336)	9.7010 (0.9433)	10	0.8225 (0.0560)	0.5061 (0.0256)	7.8550 (0.7263)	8	0.5146 (0.0341)	0.9137 (0.0785)	3.7250 (0.7909)	4	0.6295 (0.0421)	0.1537 (0.0204)	6.3710 (0.9162)	7
	CML	0.1971 (0.0153)	0.6901 (0.0262)	10.2620 (0.9385)	10	0.8068 (0.0321)	0.5045 (0.0239)	8.0400 (0.6359)	8	0.5108 (0.0193)	0.8971 (0.0546)	3.7610 (0.6924)	4	0.5858 (0.0215)	0.1482 (0.0193)	7.3110 (0.6895)	7
700	CLS	0.2049 (0.0187)	0.7093 (0.0299)	9.7460 (0.8881)	10	0.8193 (0.0503)	0.5058 (0.0235)	7.8890 (0.6614)	8	0.5115 (0.0297)	0.9097 (0.0567)	3.7410 (0.6982)	4	0.6271 (0.0395)	0.1536 (0.0165)	6.4390 (0.8279)	7
	CML	0.1972 (0.0133)	0.6922 (0.0233)	10.2570 (0.8624)	10	0.8078 (0.0283)	0.5048 (0.0197)	7.9770 (0.5434)	8	0.5103 (0.0183)	0.9026 (0.0485)	3.7840 (0.5986)	4	0.5868 (0.0201)	0.1486 (0.0155)	7.3090 (0.6506)	7
1000	CLS	0.2048 (0.0163)	0.7084 (0.0272)	9.7760 (0.8016)	10	0.8153 (0.0486)	0.5051 (0.0206)	7.8910 (0.6538)	8	0.5095 (0.0279)	0.9126 (0.0501)	3.7880 (0.6406)	4	0.6259 (0.0367)	0.1529 (0.0142)	6.4610 (0.7754)	7
	CML	0.1972 (0.0110)	0.6922 (0.0186)	10.2160 (0.6248)	10	0.8067 (0.0262)	0.5044 (0.0180)	7.9700 (0.4628)	8	0.5093 (0.0169)	0.9064 (0.0407)	3.7920 (0.5275)	4	0.5873 (0.0189)	0.1481 (0.0128)	7.3030 (0.5720)	7
1500	CLS	0.2039 (0.0139)	0.7076 (0.0253)	9.7900 (0.7103)	10	0.8101 (0.0411)	0.5039 (0.0168)	7.9280 (0.5606)	8	0.5077 (0.0241)	0.9081 (0.0415)	3.8350 (0.5360)	4	0.6205 (0.0307)	0.1519 (0.0114)	6.5430 (0.7023)	7
	CML	0.1974 (0.0096)	0.6922 (0.0166)	10.2070 (0.5284)	10	0.8050 (0.0230)	0.5035 (0.0146)	7.9740 (0.3466)	8	0.5076 (0.0151)	0.9056 (0.0366)	3.8180 (0.4474)	4	0.5869 (0.0193)	0.1489 (0.0101)	7.2900 (0.5497)	7
2000	CLS	0.2036 (0.0125)	0.7075 (0.0244)	9.8030 (0.6659)	10	0.8088 (0.0382)	0.5026 (0.0155)	7.9130 (0.5131)	8	0.5062 (0.0211)	0.9068 (0.0364)	3.8560 (0.4798)	4	0.6173 (0.0267)	0.1514 (0.0095)	6.6240 (0.6166)	7
	CML	0.1975 (0.0083)	0.6935 (0.0151)	10.1810 (0.4829)	10	0.8048 (0.0207)	0.5025 (0.0126)	7.9650 (0.2740)	8	0.5066 (0.0144)	0.9071 (0.0335)	3.8310 (0.4280)	4	0.5877 (0.0183)	0.1489 (0.0090)	7.2820 (0.5331)	7
5000	CLS	0.2033 (0.0097)	0.7068 (0.0206)	9.7990 (0.5507)	10	0.8046 (0.0280)	0.5014 (0.0104)	7.9550 (0.3730)	8	0.5034 (0.0145)	0.9038 (0.0250)	3.9170 (0.3116)	4	0.6103 (0.0200)	0.1515 (0.0063)	6.7720 (0.4777)	7
	CML	0.1985 (0.0059)	0.6959 (0.0115)	10.1210 (0.3509)	10	0.8008 (0.0127)	0.5004 (0.0083)	7.9970 (0.0548)	8	0.5030 (0.0099)	0.9029 (0.0220)	3.9400 (0.2451)	4	0.5922 (0.0152)	0.1491 (0.0056)	7.1870 (0.4326)	7
8000	CLS	0.2031 (0.0086)	0.7066 (0.0190)	9.8190 (0.5032)	10	0.8035 (0.0219)	0.5009 (0.0082)	7.9600 (0.2722)	8	0.5020 (0.0106)	0.9021 (0.0192)	3.9580 (0.2099)	4	0.6074 (0.0165)	0.1507 (0.0048)	6.8420 (0.3977)	7
	CML	0.1992 (0.0046)	0.6976 (0.0091)	10.0670 (0.2590)	10	0.7996 (0.0107)	0.5001 (0.0061)	7.9990 (0.0316)	8	0.5010 (0.0072)	0.9004 (0.0165)	3.9720 (0.1674)	4	0.5943 (0.0130)	0.1493 (0.0049)	7.1290 (0.3593)	7
10000	CLS	0.2028 (0.0081)	0.7063 (0.0176)	9.8260 (0.4692)	10	0.8021 (0.0163)	0.5006 (0.0070)	7.9760 (0.1950)	8	0.5015 (0.0090)	0.9015 (0.0167)	3.9710 (0.1762)	4	0.6052 (0.0134)	0.1505 (0.0043)	6.8880 (0.3348)	7
	CML	0.1994 (0.0042)	0.6982 (0.0083)	10.0540 (0.2325)	10	0.8004 (0.0093)	0.5001 (0.0055)	8.0000 (0)	8	0.5008 (0.0062)	0.9007 (0.0152)	3.9870 (0.1141)	4	0.5965 (0.0110)	0.1495 (0.0041)	7.0850 (0.2917)	7

Table 3.3. *Simulation results of the threshold parameters for Model 2.*

		c_1			c_2			c_3			c_4		
	T.V	6			9			15			13		
Size		$\widehat{c}_1$	precent	med	$\widehat{c}_2$	precent	med	$\widehat{c}_3$	precent	med	$\widehat{c}_4$	precent	med
500	CLS	6.0000 (0.2449)	0.9880	6	8.1920 (1.7283)	0.5670	9	15.1270 (0.8574)	0.7570	15	12.8900 (0.4001)	0.8810	13
	CML	6.0000 (0.2490)	0.9860	6	8.2650 (1.6258)	0.5930	9	15.0150 (0.5773)	0.8060	15	12.8810 (0.4303)	0.8670	13
700	CLS	5.9990 (0.0316)	0.9990	6	8.4740 (1.2485)	0.6810	9	15.0690 (0.4917)	0.8660	15	12.9590 (0.2463)	0.9550	13
	CML	6.0100 (0.2865)	0.9980	6	8.4190 (1.3306)	0.6440	9	15.0460 (0.4818)	0.8610	15	12.9490 (0.2600)	0.9380	13
1000	CLS	6.0000 (0)	1	6	8.5790 (1.0594)	0.7340	9	15.0470 (0.3053)	0.9480	15	12.9750 (0.1601)	0.9750	13
	CML	6.0000 (0)	1	6	8.6080 (0.9776)	0.7430	9	15.0340 (0.2513)	0.9580	15	12.9860 (0.1421)	0.9800	13
1500	CLS	6.0000 (0)	1	6	8.7130 (0.8120)	0.8160	9	15.0050 (0.1379)	0.9840	15	12.9970 (0.0549)	0.9970	13
	CML	6.0000 (0)	1	6	8.7250 (0.7878)	0.8150	9	15.0070 (0.1142)	0.9870	15	12.9920 (0.0898)	0.9920	13
2000	CLS	6.0000 (0)	1	6	8.8120	0.8600	9	15.0040 (0.0634)	0.9960	15	13.0000 (0)	1	13
	CML	6.0000 (0)	1	6	8.8340 (0.5381)	0.8690	9	15.0050 (0.0709)	0.9950	15	12.9990 (0.0316)	0.9990	13
5000	CLS	6.0000 (0)	1	6	8.9630 (0.2244)	0.9680	9	15.0000 (0)	1	15	13.0000 (0)	1	13
	CML	6.0000 (0)	1	6	8.9580 (0.2525)	0.9660	9	15.0000 (0)	1	15	13.0000 (0)	1	13
8000	CLS	6.0000 (0)	1	6	8.9960 (0.0634)	0.9960	9	15.0000 (0)	1	15	13.0000 (0)	1	13
	CML	6.0000 (0)	1	6	8.9940 (0.0777)	0.9940	9	15.0000 (0)	1	15	13.0000 (0)	1	13
10000	CLS	6.0000 (0)	1	6	8.9990 (0.0316)	0.9990	9	15.0000 (0)	1	15	13.0000 (0)	1	13
	CML	6.0000 (0)	1	6	8.9990 (0.0316)	0.9990	9	15.0000 (0)	1	15	13.0000 (0)	1	13

3.5.2 Real data study

In this paragraph, we consider the data set of size 365 observations, recorded for the daytime road accidents in the Schiphol area, in the Netherlands for the year 2001. This time series was presented by Pedeli and Karlis (2011) who suggested for it, with the other time series (nighttime road accidents), the bivariate Poisson $INAR(1)$ model and the $INAR(1)$ model with bivariate negative binomial $(BVNB)$ innovations. The visualization of the considered time series is shown in Figure 3.3, while Table 3.4 summarizes some descriptive statistics.

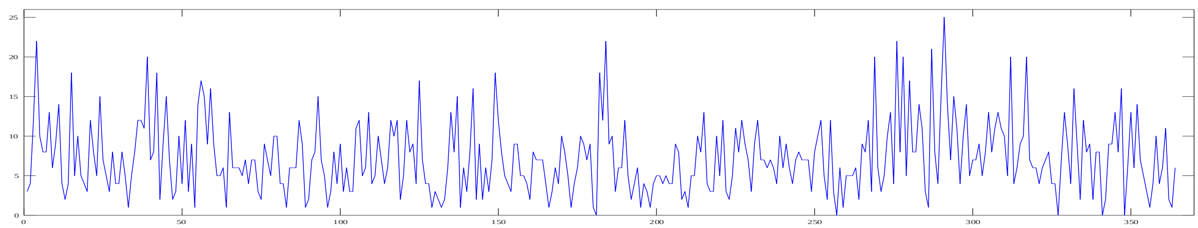

Figure 3.3. Trajectory of the daytime road accidents time series.

Table 3.4. Descriptive statistics for the daytime road accidents time series.

Sample size	Minimum	Maximum	Median	Mean	Variance	Skewness	Kurtosis
365	0	25	6	7.2767	20.9370	1.0005	4.0530

A look at Table 3.4 explains the reason why we choose the $PNBSETINAR_7(2; 1, 1)$ model. Indeed, despite the fact that the series seems to be over-dispersed, which indicates that, marginally, a Poisson distribution might not be appropriate.

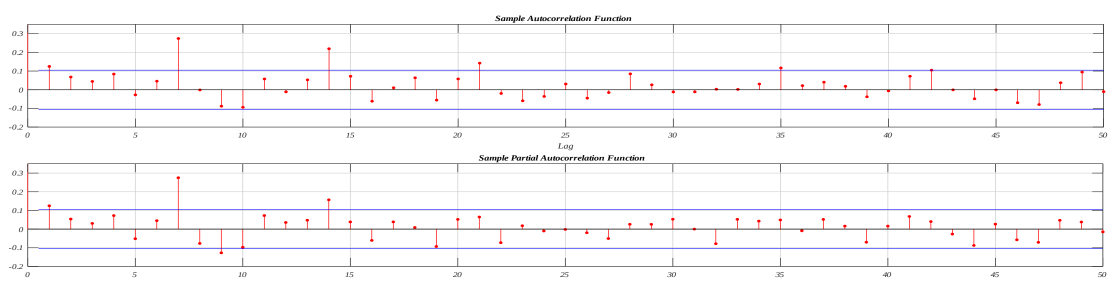

Figure 3.4. Autocorrelations of the road accidents time series.

Furthermore, by analyzing the empirical autocorrelation function (ACF) and the empirical partial autocorrelation function $(PACF)$ in Figure 3.4, the time series presents periodicity of season $S = 7$. Following this example, we discuss in more detail the empirical method of determining the period S, which is much easier and faster compared to the identification problem existing in the literature. Basically, it is about comparing the observations on different days and seeing if there is a certain aspect that repeats in each period, using the empirical autocorrelation function (ACF) and the empirical partial autocorrelation function ($PACF$). Indeed, Figure 3.4 reveals very marked peaks at delays ($S = 7, 14, ...$) because of the daily data effect. Hence, the periodicity is of season $S = 7$. Thereafter, we are interested in fitting a periodic negative binomial self-exciting threshold integer-valued autoregressive time series model for this data set.

Table 3.5. Fitting results of different models.

	para.				CML				RMS
$SETINAR\,(2;1)$	α_1				0.0903				5.2903
	α_2				0.0800				
	λ				6.6727				
	c				8				
$NBSETINAR\,(2;1)$	α_1				0.4776				5.2785
	α_2				0.3464				
	r				10				
	c				8				
$PNBSETINAR_7\,(2;1)$	s	1	2	3	4	5	6	7	4.6466
	$\alpha_{1,s}$	0.3195	0.3856	0.6891	0.6383	0.4364	0.6062	0.8833	
	$\alpha_{2,s}$	0.2589	0.2862	0.3685	0.3265	0.2577	0.2581	0.5793	
	r_s	18	16	10	6	4	5	7	

Table 3.5 gives the fitting results of the $SETINAR$ model (Monteiro *et al.* 2012), the $NBSETINAR$ (Yang *et al.* 2018*b*) and our $PNBSETINAR$ model, and compare the fitting results via the root mean squares (RMS) criterion. The estimated threshold parameters c_s, for $s = 1, ..., 7$, by using the adapted $NeSS$ algorithm, is given by $\underline{C} = (c_1, \ldots, c_7) = (11, 8, 6, 15, 8, 10, 3)$. As can be seen from Table 3.5 that our proposed model exhibits an improvement comparing to the $SETINAR$ model (Monteiro *et al.* 2012) and to the $NBSETINAR$ model (Yang *et al.* 2018*b*), which is confirmed empirically by the RMS selection criterion, where the $PNBSETINAR$ model has the smallest RMS value.

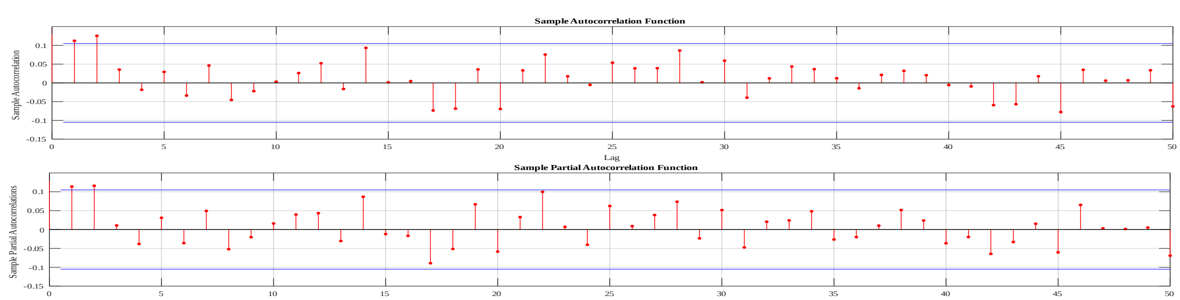

Figure 3.5. ACF and PACF of residual time series.

Moreover, Figure 3.5 shows empirically that the most of the autocorrelations, for the proposed model, are inside significance bounds (except for lags 1 and 2, which are slightly significant). To reinforce the non-significance of the residuals correlations, we use the Ljung-Box test, where the obtained value (24.4738) is less than the critical value $\chi^2_{0.05,20}$ for significance level 0.05 and 20 degree of freedom (31.4104), so we accept the independence of residuals. Therefore, the proposed model provides an adequate fit to the data in terms of no correlation within the residuals. Figure 3.6, indicates the adjusted trajectory of the Daytime road accidents data set, such as the series values are shown in blue, while the red line denotes the adjusted series. The fitted values of the $PNBSETINAR_7\,(2;1)$ model seem to be suitable for the real data set values.

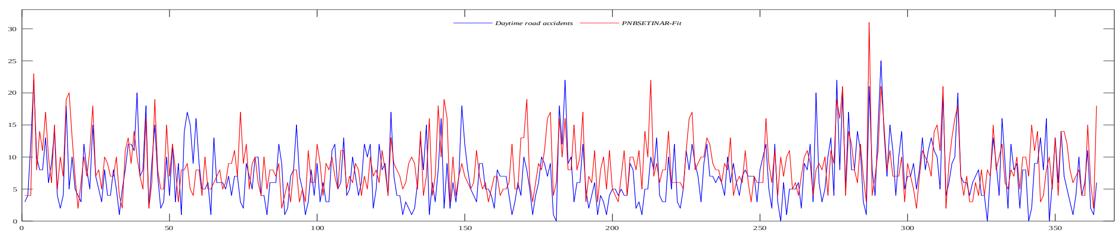

Figure 3.6. An adjusted trajectory proposed to the daytime road accidents data set.

Chapter 4

Periodic Negative Binomial INGARCH(1,1) Model

4.1 Introduction

Different models have been proposed in order to take into account the different features encountered in such integer valued time series, as an example, excess of zeros, volatility changes in time, asymmetric distributions, low counts, overdispersion, etc. (see, Kedem and Fokianos 2005, Costa *et al.* 2016, Ouzzani and Bentarzi 2019, among others). As indicated in Zhu (2011) and Ahmad and Francq (2016), these models can be classified into two classes: models of thinning operators, commonly called integer valued autoregressive moving average ($INARMA$) models, and regression models which deal with a modification of the autoregressive moving average ($ARMA$) approach, they referred as the integer-valued generalized autoregressive conditional heteroskedastic ($INGARCH$) models (Ferland *et al.* 2006, Zhu 2011, Fokianos 2012, Weiß 2018, and many others). Regarding the integer time series of overdispersion, many researchers have turned to overdispersed Poisson and binomial regression models. In addition, a model that has been used for overdispersion count time series is the integer-valued generalized autoregressive conditional heteroskedastic $INGARCH$ with Poisson distribution introduced by Ferland *et al.* (2006) and Fokianos *et al.* (2009), its properties were then studied by Weiß (2009, 2010), Zhu *et al.* (2015, 2016), Li *et al.* (2016) and among others. As it is known, the Poisson distribution is not always suitable for modeling and studying the integer time series, as was pointed out by Ristić *et al.* (2009). This is due to the equality of the mean and the variance which is not always verified in the real-world data set. In fact, many researchers have found it useful to propose other models that address the problem of overdispersion with another distribution and specially the negative binomial distribution. Indeed, Zhu (2011, 2012*b*), introduced a negative binomial $INGARCH$ model to deal with the phenomena of overdispersion and subsequently, its zero-inflated version to treat the zero inflation situation. Furthermore, the generalized Poisson $INGARCH$ model was presented and studied by Zhu (2012*a*), which can explain the overdispersion and underdispersion characteristics. Recently, Mao *et al.* (2020) proposed a more general mixture $INGARCH$ model, which includes negative binomial and generalized Poisson mixture $INGARCH$ models that can deal with multi-modality feature either in the marginal or the conditional distribution.

However, despite the fact that much nonnegative integer-valued time series encountered in several fields as the epidemiology, economic, environmental, criminology, and others reveal the periodicity feature in their autocovariance structures. Regardless of the various advantages and interesting properties satisfied by these models and more precisely by the negative binomial $INGARCH$ model such as the positivity and the discreteness nature of the realizations, the volatility changes in time, this model still unable to capture the periodicity feature, a feature that cannot be adequately accounted and described by time invariant parameter integer-valued time series models. This fact gave us a good reason and motivation to extend this class of time-invariant models to the periodic negative binomial $INGARCH$ model with time-periodic coefficients, an extension of the periodic $INGARCH$ model, introduced by Bentarzi and Bentarzi (2017*b*). To our knowledge, Monteiro *et al.* (2010) and Morinã *et al.* (2011) were the pioneers on the modeling of the periodically correlated, in the sense of Gladyshev (1961), integer-valued process and over time, the topic has made great progress, see, for example, Sadoun and Bentarzi (2019), Bentarzi and Sadoun (2020), Bentarzi and Aries (2020*a*, 2020*b*), Manaa and Bentarzi (2021*a*, 2021*b*) and many others.

In Section 4.2, we provide the basic notations and definitions concerning a periodic negative binomial integer-valued generalized autoregressive conditional heteroskedastic model, noted in short by $PNBINGARCH\,(1,1)$. The periodically stationary problem of the proposed model is investigated in Section 4.3. For the periodic stationarity in mean and variance, the necessary and sufficient conditions are specified. Furthermore, the closed-form expressions for both the mean and variance are, under these conditions, obtained. In Section 4.4, the existence of higher moments and their calculations are considered. The autocovariance structure of the underlying periodic model is studied in Section 4.5, and even the autocorrelation function is explicitly expressed. The Yule-Walker, Conditional Least Squares, and Conditional Maximum Likelihood methods are all explained clearly in Section 4.6. Finally, Section 4.7 includes a simulation study as well as an application using real data set.

4.2 Notations, definitions and main assumptions

Recall that an integer-valued stochastic process $\{y_t;\ t \in \mathbb{Z}\}$ is said to satisfy a Negative Binomial Integer-Valued Generalized Autoregressive Conditional Heteroskedastic model, with orders p and q, noted $NBINGARCH(p,q)$, if it is given by :

$$\begin{array}{c} y_t\,|\mathcal{F}_{t-1} \sim \mathcal{NB}\,(r, p_t)\,, \\ (1-p_t)/\,p_t = \lambda_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i} + \sum_{j=1}^{q} \beta_j \lambda_{t-j}. \end{array} \tag{4.2.1}$$

A periodically correlated integer-valued process, in the Gladyshev's sense (1961), $\{y_t;\ t \in \mathbb{Z}\}$ is said to satisfy a periodic negative binomial integer-valued generalized autoregressive conditional heteroskedastic model, with period S and orders p and q, noted in short by $PNBINGARCH_S\,(p,q)$, if it has the form :

$$\begin{array}{c} y_t\,|\mathcal{F}_{t-1} \sim \mathcal{NB}\,(r_t, p_t)\,, \\ (1-p_t)/\,p_t = \lambda_t = \alpha_{0,t} + \sum_{i=1}^{p} \alpha_{i,t} y_{t-i} + \sum_{j=1}^{q} \beta_{j,t} \lambda_{t-j}. \end{array} \tag{4.2.2}$$

where $\mathcal{F}_{t-1}$ denotes, as usually, the σ-field generated by $\{y_{t-1}, y_{t-2}, \ldots\}$ and r_t is a positive number. The parameters $\alpha_{i,t},\ i = 0, 1, ..., p,$ and $\beta_{j,t},\ j = 1, ..., q,$ are periodic in t, with period S , i.e., $\alpha_{i,t+vS} = \alpha_{i,t},\ \beta_{j,t+vS} = \beta_{j,t}$ and $r_{t+kS} = r_t,\ t, k \in \mathbb{Z}$. To avoid the possibility of zero or negative conditional variances, the following conditions for $\alpha_{i,t}$'s must be imposed : $\alpha_{0,t} > 0,\ \alpha_{i,t} \geq 0,\ i = 1, ..., p$ and $\beta_{j,t} \geq 0,\ j = 1, ..., q,\ t \in \mathbb{Z}$. Particularly, we have, for $p = q = 1$, the periodic model, $PNBINGARCH_S\,(1,1)$, which is the object in this chapter

:

$$\begin{aligned} y_t \,|\mathcal{F}_{t-1} &\sim \mathcal{NB}\left(r_t, p_t\right), \\ (1-p_t)/\,p_t = \lambda_t &= \alpha_{0,t} + \alpha_{1,t} y_{t-1} + \beta_t \lambda_{t-1}, \end{aligned} \tag{4.2.3}$$

where, the parameters $\alpha_{i,t}$, $i = 0, 1$, β_t and r_t are periodic in t, with period S , i.e., $\alpha_{i,t+kS} = \alpha_{i,t}$, $i = 0, 1$, $\beta_{t+kS} = \beta_t$ and $r_{t+kS} = r_t$, $t, k \in \mathbb{Z}$. Moreover, these parameters are such that : $\alpha_{0,t} > 0$, $\alpha_{1,t} \geq 0$, and $\beta_t \geq 0$, $t \in \mathbb{Z}$. Letting $t = s + \tau S$, $s = 1, ..., S$ and $\tau \in \mathbb{Z}$, the last model can be rewritten in the equivalent form

$$\begin{aligned} y_{s+\tau S} \,|\mathcal{F}_{s-1+\tau S} &\sim \mathcal{NB}\left(r_s, p_{s+\tau S}\right), \\ (1-p_{s+\tau S})/\,p_{s+\tau S} = \lambda_{s+\tau S} &= \alpha_{0,s} + \alpha_{1,s} y_{s-1+\tau S} + \beta_s \lambda_{s-1+\tau S}. \end{aligned}$$

The last model extends the following time-invariant $NBINGARCH\,(1,1)$ studied by Zhu (2011) to the time periodic case :

$$\begin{aligned} y_t \,|\mathcal{F}_{t-1} &\sim \mathcal{NB}\left(r, p_t\right), \\ (1-p_t)/\,p_t = \lambda_t &= \alpha_0 + \alpha_1 y_{t-1} + \beta \lambda_{t-1}. \end{aligned} \tag{4.2.4}$$

If $r_1 = \ldots = r_S = 1$, then the Negative Binomial distribution becomes the geometric distribution and the $PNBINGARCH$ model can be, in this case, called the periodic geometric $INGARCH$ model.

4.3 Stationarity conditions

In this paragraph, we provide the conditions on parameters of the underlying integer-valued process to be periodically stationary in the first and second orders. Furthermore, under these conditions, the closed forms of the periodic mean and periodic variance are acquired.

4.3.1 Periodic stationarity in the first order

The results in the following proposition establish the necessary and sufficient condition, for the process $\{y_t;\ t \in \mathbb{Z}\}$ satisfying (4.2.3) to be periodically stationary with respect to the first moment. The closed-forms of the periodic mean is then, under this condition, obtained.

Proposition 4.3.1 *The periodically correlated integer-valued process* $\{y_t;\ t \in \mathbb{Z}\}$*, satisfying the periodic negative binomial* $INGARCH\,(1,1)$ *model* (4.2.3)*, is periodically stationary in the mean, if and only if,* $\prod_{i=1}^{S}\left(r_{i-1}\alpha_{1,i} + \beta_i\right) < 1$*. Furthermore, the closed-form of the mean*

$\mu_{y,s} = \mathbb{E}(y_s) = r_s\mathbb{E}(\lambda_s)$, $s = 1, ..., S$, *of such process is, under this condition, given by :*

$$\mu_{y,s} = r_s \left(1 - \prod_{i=1}^{S}(r_{i-1}\alpha_{1,i} + \beta_i)\right)^{-1} \sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}(r_{s-i}\alpha_{1,s-i+1} + \beta_{s-i+1})\right)\alpha_{0,s-j+1},$$

with the convention $\prod_{i=1}^{j} x_i = 1$ *if* $j < 1$.

In the following, we will consider some special cases of Proposition 4.3.1. Suppose that the process $\{y_t;\ t \in \mathbb{Z}\}$ following a $PNBINARCH(1)$ model, (i.e., $q = 0$), then the following corollary gives the periodic stationarity in mean.

Corollary 4.3.1 *The periodically correlated integer-valued process* $\{y_t;\ t \in \mathbb{Z}\}$*, satisfying the periodic negative binomial* $INARCH(1)$ *model, is periodically stationary in the mean, if and only if,* $\prod_{i=1}^{S} r_{i-1}\alpha_{1,i} < 1$. *Furthermore, the closed-form of the mean* $\mathbb{E}(y_s) = \mu_{y,s}$, $s = 1, ..., S$, *of such process is, under this condition, given by :*

$$\mu_{y,s} = r_s \left(1 - \prod_{i=1}^{S} r_{i-1}\alpha_{1,i}\right)^{-1} \sum_{j=1}^{S}\left(\prod_{i=1}^{j-1} r_{s-i}\alpha_{1,s-i+1}\right)\alpha_{0,s-j+1}.$$

In the time invariant coefficients case (classical) model (4.2.4), i.e. $S = 1$, the results of Proposition 4.3.1 can be presented by the following corollary.

Corollary 4.3.2 *The integer-valued process* $\{y_t;\ t \in \mathbb{Z}\}$*, satisfying the classical* $NBINGARCH(1,1)$ *model (4.2.4), is stationary in the first moment if and only if,* $r\alpha_1 + \beta < 1$. *Furthermore, the closed-forms of the mean* $\mu_y = \mathbb{E}(y_t)$ *of such process is, under this condition given by :* $\mu_y = r(1 - (r\alpha_1 + \beta))^{-1}\alpha_0$.

Proof of Proposition 4.3.1. The unconditional mean of the process $\{y_t;\ t \in \mathbb{Z}\}$, is noted by $\mu_{y,t} = \mathbb{E}(y_t) = r_t\mathbb{E}(\lambda_t)$ where,

$$\begin{aligned} \mu_{\lambda,t} &= \mathbb{E}(\lambda_t) = \mathbb{E}(\alpha_{0,t} + \alpha_{1,t}y_{t-1} + \beta_t\lambda_{t-1}) = \alpha_{0,t} + (r_{t-1}\alpha_{1,s} + \beta_t)\mathbb{E}(\lambda_{t-1}), \\ &= \psi_{1,t}\mu_{\lambda,t-1} + \alpha_{0,t}, \text{ where } \psi_{1,t} = r_{t-1}\alpha_{1,t} + \beta_t. \end{aligned}$$

By iteration m times the last equation, we obtain

$$\mu_{\lambda,t} = \left(\prod_{i=1}^{m}\psi_{1,t-i+1}\right)\mu_{\lambda,t-m} + \sum_{j=1}^{m}\left(\prod_{i=1}^{j-1}\psi_{1,t-i+1}\right)\alpha_{0,t-j+1},$$

replacing m by t, while letting $t = s + \tau S$, and by taking into account the periodicity of parameters, we obtain

$$\begin{aligned}
\mu_{\lambda,s,\tau} &= \left(\prod_{i=1}^{s+\tau S} \psi_{1,s-i+1}\right)\mu_{y,0} + \sum_{j=1}^{s-1+\tau S}\left(\prod_{i=1}^{j-1}\psi_{1,s-i+1}\right)\alpha_{0,s-j+1}, \\
&= \left(\prod_{i=1}^{s+\tau S} \psi_{1,s-i+1}\right)\mu_{y,0} + \sum_{j=1}^{\tau S}\left(\prod_{i=1}^{j-1}\psi_{1,s-i+1}\right)\alpha_{0,s-j+1} + \sum_{j=1}^{s}\left(\prod_{i=1}^{j-1+\tau S}\psi_{1,s-i+1}\right)\alpha_{0,s-j+1}, \\
&= \left(\prod_{i=1}^{S}\psi_{1,i}\right)^{\tau}\left(\prod_{i=1}^{s}\psi_{1,s-i+1}\right)\mu_{y,0} + \sum_{k=0}^{\tau-1}\left(\prod_{i=1}^{S}\psi_{1,i}\right)^{k}\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\psi_{1,s-i+1}\right)\alpha_{0,s-j+1} + \\
&\quad + \left(\prod_{i=1}^{S}\psi_{1,i}\right)^{\tau}\sum_{j=1}^{s}\left(\prod_{i=1}^{j-1}\psi_{1,s-i+1}\right)\alpha_{0,s-j+1},
\end{aligned}$$

$$\begin{aligned}
\mu_{\lambda,s,\tau} &= \frac{1-\left(\prod_{i=1}^{S}\psi_{1,i}\right)^{\tau}}{1-\left(\prod_{i=1}^{S}\psi_{1,i}\right)}\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\psi_{1,s-i+1}\right)\alpha_{0,s-j+1} + \left(\prod_{i=1}^{S}\psi_{1,i}\right)^{\tau}\left[\left(\prod_{i=1}^{s}\psi_{1,s-i+1}\right)\mu_{y,0} + \right. \\
&\quad \left.\sum_{j=1}^{s}\left(\prod_{i=1}^{j-1}\psi_{1,s-i+1}\right)\alpha_{0,s-j+1}\right],
\end{aligned}$$

letting $t \to \infty$, (hence $\tau \to \infty$), then $\mu_{\lambda,s,\tau}$, $s = 1,...,S$, converges, as $\tau \to \infty$, to $\left(1-\left(\prod_{i=1}^{S}\psi_{1,i}\right)\right)^{-1}\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\psi_{1,s-i+1}\right)\alpha_{0,s-j+1}$, if and only if $\prod_{i=1}^{S}\psi_{1,i} < 1$. ∎

4.3.2 Periodic stationarity in the second order

The following proposition establishes a necessary and sufficient condition for the periodic integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$ satisfying (4.4.4) to be stationary with respect to the second order moment. The closed form of this moment is then, under this condition, obtained.

Proposition 4.3.2 *The periodically correlated integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$ satisfying the periodic $PNBINGARCH\,(1,1)$ model (4.4.4) is periodically stationary in the second order if and only if, $\prod_{i=1}^{S}\left(r_{i-1}\alpha_{1,i}^2 + (r_{i-1}\alpha_{1,i}+\beta_i)^2\right) < 1$. Furthermore, the closed-form of the variance $Var\,(y_s) = \gamma_y^{(s)}(0)$, $s = 1,...,S$, of such process and the variance $Var\,(\lambda_s) = \gamma_\lambda^{(s)}(0)$ are, under this condition, given by :*

$$\begin{aligned}
\gamma_y^{(s)}(0) &= \mu_{y,s} + (r_s + r_s^2)\left(1-\left(\prod_{i=1}^{S}\psi_{2,i}\right)\right)^{-1}\sum_{j=0}^{S-1}\left(\prod_{i=1}^{j}\psi_{2,s-i+1}\right)F_{1,s-j+1} + \frac{1}{r_s}\mu_{y,s}^2, \\
\gamma_\lambda^{(s)}(0) &= \left(1-\left(\prod_{i=1}^{S}\psi_{2,i}\right)\right)^{-1}\sum_{j=0}^{S-1}\left(\prod_{i=1}^{j}\psi_{2,s-i+1}\right)F_{1,s-j+1},
\end{aligned}$$

where $\psi_{2,s} = \left(r_{s-1}\alpha_{1,s}^2 + (r_{s-1}\alpha_{1,s}+\beta_s)^2\right)$, $F_{1,s} = \alpha_{1,s}^2\left(\mu_{y,s-1} + \frac{1}{r_{s-1}}\mu_{y,s-1}^2\right)$ *and* $\mu_{y,s} =$

$r_s\mu_{\lambda,s}$ is given in Proposition 4.3.1, with the convention $\prod_{i=1}^{j} x_i = 1$ if $j < 1$.

In what follows, we consider some special cases of Proposition 4.3.2. Suppose that the process $\{y_t;\ t \in \mathbb{Z}\}$ following a $PNBINARCH(1)$ model, (i.e., $q = 0$), then the corollary below gives the periodic stationarity in second order.

Corollary 4.3.3 *The periodically correlated integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$, satisfying the periodic $PNBINARCH\,(1)$ model, is periodically stationary in the second order, if and only if, $\prod_{i=1}^{S} \left(r_{s-i} + r_{s-i}^2\right) \alpha_{1,i}^2 < 1$. Furthermore, the closed-form of the variance $\gamma_y^{(s)}(0) = Var\,(y_s)$, $s = 1, ..., S$, of such process and the variance $\gamma_\lambda^{(s)}(0) = Var\,(\lambda_s)$ are, under this condition, given by :*

$$\gamma_y^{(s)}(0) = \mu_{y,s} + (r_s + r_s^2)\left(1 - \left(\prod_{i=1}^{S}\delta_i\right)\right)^{-1}\sum_{j=0}^{S-1}\left(\prod_{i=1}^{j}\delta_{s-i+1}\right)\digamma_{1,s-j+1} + \frac{1}{r_s}\mu_{y,s}^2,$$
$$\gamma_\lambda^{(s)}(0) = \left(1 - \left(\prod_{i=1}^{S}\delta_i\right)\right)^{-1}\sum_{j=0}^{S-1}\left(\prod_{i=1}^{j}\delta_{s-i+1}\right)\digamma_{1,s-j+1},$$

where $\delta_s = \left(r_{t-1} + r_{t-1}^2\right)\alpha_{1,s}^2$, $\digamma_{1,s} = \alpha_{1,s}^2\left(\mu_{y,s-1} + \dfrac{1}{r_{s-1}}\mu_{y,s-1}^2\right)$ and $\mu_{y,s} = r_s\mu_{\lambda,s}$ is given in Corollary 4.3.1, with the convention $\prod_{i=1}^{j} x_i = 1$ if $j < 1$.

In the time invariant coefficients case (classical) model (4.2.4), i.e. $S = 1$, the results of Proposition 4.3.2 can be presented by the following corollary.

Corollary 4.3.4 *The integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$, satisfying the classical $NBINGARCH\,(1,1)$ model (4.2.4), is stationary in the second order, if and only if, $\left(r\alpha_1^2 + (r\alpha_1 + \beta)^2\right) < 1$. Furthermore, the closed-form of the $Var\,(y_t) = \gamma_y(0)$ of such process and the variance $Var\,(\lambda_t) = \gamma_\lambda(0)$ are, under this condition, given by*

$$\gamma_\lambda(0) = \alpha_1^2\left(\frac{\mu_y^2}{r} + \mu_y\right)\left(1 - \left(r\alpha_1^2 + (r\alpha_1 + \beta)^2\right)\right)^{-1},$$
$$\gamma_y(0) = \left(\frac{\mu_y^2}{r} + \mu_y\right)\frac{1 - (r\alpha_1 + \beta)^2 + r\alpha_1^2}{1 - \left(r\alpha_1^2 + (r\alpha_1 + \beta)^2\right)}.$$

Proof of Proposition 4.3.2. The unconditional second order moment of the process $\{y_t;\ t \in \mathbb{Z}\}$ is given by, $\mathbb{E}\left(y_t^2\right) = (r_t + r_t^2)\,\mathbb{E}\left(\lambda_t^2\right) + r_t\mathbb{E}\left(\lambda_t\right)$. Or equivalently,

$$\gamma_y^{(t)}(0) = \left(r_t + r_t^2\right)\gamma_\lambda^{(t)}(0) + r_t\left(1 + \mathbb{E}\left(\lambda_t\right)\right)\mathbb{E}\left(\lambda_t\right), \tag{4.3.1}$$

with $\mathbb{E}(\lambda_t)$ is calculated previously. Hence, we need to calculate $\gamma_\lambda^{(t)}(0)$ which is given by $\gamma_\lambda^{(t)}(0) = \psi_{2,t}\gamma_\lambda^{(t-1)}(0) + r_t\alpha_{1,t}^2(1+\mathbb{E}(\lambda_{t-1}))\mathbb{E}(\lambda_{t-1})$, with $\psi_{2,t} = \left(r_{t-1}\alpha_{1,t}^2 + (r_{t-1}\alpha_{1,t}+\beta_t)^2\right)$. By iteration m times, we obtain

$$\gamma_\lambda^{(t)}(0) = \left(\prod_{i=1}^m \psi_{2,t-i+1}\right)\gamma_\lambda^{(t-m)}(0) + \sum_{j=1}^m \left(\prod_{i=1}^{j-1}\psi_{2,t-i+1}\right)F_{1,t-j+1},$$

where $F_{1,t} = r_{t-1}\alpha_{1,t}^2\left(1+\mu_{\lambda,t-1}\right)\mu_{\lambda,t-1}$, replacing m by t, while letting $t = s+\tau S$, and by taking into account the periodicity of parameters, we obtain

$$\begin{aligned}
\gamma_y^{(s+\tau S)}(0) &= \left(\prod_{i=1}^{s+\tau S}\psi_{2,s-i+1}\right)\gamma_\lambda^{(0)}(0) + \sum_{j=1}^{s-1+\tau S}\left(\prod_{i=1}^{j-1}\psi_{2,s-i+1}\right)F_{1,s-j+1},\\
&= \left(\prod_{i=1}^{s+\tau S}\psi_{2,s-i+1}\right)\gamma_\lambda^{(0)}(0) + \sum_{j=1}^{\tau S}\left(\prod_{i=1}^{j-1}\psi_{2,s-i+1}\right)F_{1,s-j+1} + \sum_{j=1}^{s}\left(\prod_{i=1}^{j-1+\tau S}\psi_{2,s-i+1}\right)F_{1,s-j+1},\\
&= \left(\prod_{i=1}^{S}\psi_{2,i}\right)^\tau\left(\prod_{i=1}^{s}\psi_{2,s-i+1}\right)\gamma_\lambda^{(0)}(0) + \sum_{k=0}^{\tau-1}\left(\prod_{i=1}^{S}\psi_{2,i}\right)^k\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\psi_{2,s-i+1}\right)F_{1,s-j+1}+\\
&\quad\left(\prod_{i=1}^{S}\psi_{2,i}\right)^\tau\sum_{j=1}^{s}\left(\prod_{i=1}^{j-1}\psi_{2,s-i+1}\right)F_{1,s-j+1},\\
&= \frac{1-\left(\prod_{i=1}^S\psi_{2,i}\right)^\tau}{1-\left(\prod_{i=1}^S\psi_{2,i}\right)}\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\psi_{2,s-i+1}\right)F_{1,s-j+1} + \left(\prod_{i=1}^{S}\psi_{2,i}\right)^\tau\left[\left(\prod_{i=1}^{s}\psi_{2,s-i+1}\right)\gamma_\lambda^{(0)}(0)+\right.\\
&\quad\left.\sum_{j=1}^{s}\left(\prod_{i=1}^{j-1}\psi_{2,s-i+1}\right)F_{1,s-j+1}\right],
\end{aligned}$$

where $\psi_{2,t} = \left(r_{t-1}\alpha_{1,t}^2 + (r_{t-1}\alpha_{1,t}+\beta_t)^2\right)$, $F_{1,s} = r_{s-1}\alpha_{1,s}^2\left(1+\mu_{\lambda,s-1}\right)\mu_{\lambda,s-1}$. From the last expression, we see that $\gamma_\lambda^{(s+\tau S)}(0)$ convergence, as $\tau\to\infty$, to

$$\gamma_\lambda^{(s)}(0) = \left(1-\left(\prod_{i=1}^S\psi_{2,i}\right)\right)^{-1}\sum_{j=0}^{S-1}\left(\prod_{i=1}^{j}\psi_{2,s-i+1}\right)F_{1,s-j+1},$$

if and only if $\prod_{i=1}^S\psi_{2,i} < 1$. The variance of y_t is then given by

$$\gamma_y^{(s)}(0) = (r_s + r_s^2)\left(1-\left(\prod_{i=1}^S\psi_{2,i}\right)\right)^{-1}\sum_{j=0}^{S-1}\left(\prod_{i=1}^{j}\psi_{2,s-i+1}\right)F_{1,s-j+1} + \mu_{y,s} + \frac{1}{r_s}\mu_{y,s}^2. \qquad \blacksquare$$

4.4 Existence of higher moments and their calculations

This Section is devoted to obtaining the existence condition of the m-th order moment $\mathbb{E}(y_t^m)$ and its explicit formula, in terms of the model's parameters. Following that, the skewness and kurtosis coefficients are calculated using the particular cases $\mathbb{E}(y_t^4)$, $\mathbb{E}(y_t^3)$ and $\mathbb{E}(y_t^2)$.

4.4.1 Calculation of higher moments of the process λ_t

In this Section, we obtain the existence condition of the m-th order moment $\mathbb{E}\left(\lambda_t^m\right)$. Let the m-column vector $\underline{\Lambda}_t^{(m)} = \left(\lambda_t^m, \lambda_t^{m-1}, ..., \lambda_t\right)'$.

Proposition 4.4.1 *The unconditional m-th moment $\mathbb{E}\left(\lambda_t^m\right)$ exists and is finite if and only if :*

$$\prod_{i=1}^{S} \psi_{m,i} < 1, \tag{4.4.1}$$

with $x_{(i)} = x(x-1)...(x-i+1)$ *and* $\psi_{m,i} = \sum_{j=0}^{m} \binom{m}{j} \alpha_{1,i}^{j} \beta_i^{m-j} \left(r_{i-1}+j-1\right)_{(j)}$. *The unconditional m-th moment $\mathbb{E}\left(\lambda_t^m\right)$ is, under this condition, given by :*

$$\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\right) = \left(I - \Psi_{s,S}^{(m)}\right)^{-1} \sum_{i=1}^{S} \Psi_{s,i}^{(m)} \underline{\alpha}_{0,s-i+1}^{(m)},$$

where, $\Psi_{s,j}^{(m)} = \prod_{i=1}^{j-1} \Theta_{s-i+1}^{(m)}$ *and* $\underline{\alpha}_{0,t}^{(m)} = \left(\alpha_{0,t}^m, \alpha_{0,t}^{m-1}, ..., \alpha_{0,t}\right)'$ *and where the* $m \times m$ *matrix* $\Theta_s^{(m)}$, $s = 1, ..., S$, *is given by :*

$$\Theta_t^{(m)} = \begin{pmatrix} \psi_{m,t} & \phi_{m-1,t}^{(m)} & \cdots & \cdots & \phi_{2,t}^{(m)} & \phi_{1,t}^{(m)} \\ 0 & \psi_{m-1,t} & \phi_{m-2,t}^{(m-1)} & \cdots & \phi_{2,t}^{(m-1)} & \phi_{1,t}^{(m-1)} \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \psi_{3,t} & \phi_{2,t}^{(3)} & \phi_{1,t}^{(3)} \\ 0 & 0 & \cdots & 0 & \psi_{2,t} & \phi_{1,t}^{(2)} \\ 0 & 0 & \cdots & 0 & 0 & \psi_{1,t} \end{pmatrix}, \tag{4.4.2}$$

with $\phi_{i,t}^{(m)} = \binom{m}{i} \alpha_{0,t}^{m-i} \psi_{i,t} + \sum\limits_{j=i+1}^{m} \sum\limits_{k=j-i}^{j-1} \mathcal{K}_{j,k+1,j-i}^{(m,t)}$, *where,*

$$\mathcal{K}_{i,j,k}^{(m,t)} = \binom{m}{i} \binom{i}{j} \alpha_{0,t}^{m-i} \alpha_{1,t}^{j} \beta_t^{i-j} \left\{ {j \atop j-k} \right\} \left(r_{t-1}+1-(j-k)\right)_{(j-k)}.$$

In the time invariant coefficients case (classical) model (4.2.4), i.e. $S = 1$, the results of this proposition can be presented by the following corollary.

Corollary 4.4.1 *The unconditional m-th moment $\mathbb{E}\left(\lambda_t^m\right)$ exists and is finite if and only if :*

$$\psi_m < 1. \tag{4.4.3}$$

with $x_{(i)} = x(x-1)...(x-i+1)$ *and* $\psi_m = \sum_{i=0}^{m} \binom{m}{i} \alpha_1^{i} \beta^{m-i} \left(r+i-1\right)_{(i)}$. *The unconditional m-th moment $\mathbb{E}\left(\lambda_t^m\right)$ is, under this condition, given by :*

$$\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\right) = \left(I - \Theta^{(m)}\right)^{-1} \underline{\alpha}_0^{(m)},$$

where, $\underline{\alpha}_0^{(m)} = \left(\alpha_0^m, \alpha_0^{m-1}, ..., \alpha_0\right)'$ *and,*

$$\Theta^{(m)} = \begin{pmatrix} \psi_m & \phi_{m-1,m} & \cdots & \cdots & \phi_{2,m} & \phi_{1,m} \\ 0 & \psi_{m-1} & \phi_{m-2,m-1} & \cdots & \phi_{2,m-1} & \phi_{1,m-1} \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \psi_3 & \phi_{2,3} & \phi_{1,3} \\ 0 & 0 & \cdots & 0 & \psi_2 & \phi_{1,2} \\ 0 & 0 & \cdots & 0 & 0 & \psi_1 \end{pmatrix},$$

with $\phi_{i,m} = \binom{m}{i}\alpha_0^{m-i}\psi_i + \sum\limits_{j=i+1}^{m}\sum\limits_{k=j-i}^{j-1}\mathcal{K}_{j,k+1,j-i}^{(m)}$, *where,*

$$\mathcal{K}_{i,j,k}^{(m)} = \binom{m}{i}\binom{i}{j}\alpha_0^{m-i}\alpha_1^j\beta^{i-j}\left\{{j \atop j-k}\right\}(r+1-(j-k))_{(j-k)}.$$

Proof of Proposition 4.4.1. The conditional expectation $\mathbb{E}\left(\lambda_t^m\middle|\mathcal{F}_{t-2}\right)$ is given by

$$\mathbb{E}\left(\lambda_t^m\middle|\mathcal{F}_{t-2}\right) = \sum_{i=0}^m\binom{m}{i}\alpha_{0,t}^{m-i}\sum_{j=0}^i\binom{i}{j}\alpha_{1,t}^j\beta_t^{i-j}\lambda_{t-1}^{i-j}\mathbb{E}\left(y_{t-1}^j\middle|\mathcal{F}_{t-2}\right),$$

It is well known that the j-th moment of a negative binomial variable with mean $p_t = \dfrac{1}{1+\lambda_t}$ is given, while employing the second-kind Stirling number formula, by :

$$\mathbb{E}\left(y_t^j\middle|\mathcal{F}_{t-1}\right) = \sum_{k=1}^j\left\{{j \atop k}\right\}(r_t-k+1)_{(k)}\lambda_t^k,$$

with $x_{(k)} = x(x-1)...(x-k+1)$, then, we have :

$$\mathbb{E}\left(\lambda_t^m\middle|\mathcal{F}_{t-2}\right) = \alpha_{0,t}^m + \psi_{m,t}\lambda_{t-1}^m + \sum_{i=1}^{m-1}\binom{m}{i}\alpha_{0,t}^{m-i}\psi_{i,t}\lambda_{t-1}^i + \sum_{i=1}^{m}\sum_{j=1}^{i}\sum_{k=1}^{j-1}\binom{m}{i}\binom{i}{j}\alpha_{0,t}^{m-i}\alpha_{1,t}^j\beta_t^{i-j}$$
$$\left\{{j \atop k}\right\}(r_{t-1}-k+1)_{(k)}\lambda_{t-1}^{i-(j-k)},$$

which can be written in the form

$$\mathbb{E}\left(\lambda_t^m\middle|\mathcal{F}_{t-2}\right) = \alpha_{0,t}^m + \psi_{m,t}\lambda_{t-1}^m + \sum_{i=1}^{m-1}\binom{m}{i}\alpha_{0,t}^{m-i}\psi_{i,t}\lambda_{t-1}^i + \sum_{i=1}^m\sum_{j=1}^i\sum_{k=1}^{j-1}\mathcal{K}_{i,j,k}^{(m,t)}\lambda_{t-1}^{i-k},$$

where $\mathcal{K}_{i,j,k}^{(m,t)}$ and $\psi_{i,t}$ are given by, $\mathcal{K}_{i,j,k}^{(m,t)} = \binom{m}{i}\binom{i}{j}\alpha_{0,t}^{m-i}\alpha_{1,t}^j\beta_t^{i-j}\left\{{j \atop j-k}\right\}(r_{t-1}+1-(j-k))_{(j-k)}$, $\psi_{i,t} = \sum\limits_{j=0}^{i}\binom{i}{j}\alpha_{1,t}^j\beta_t^{i-j}(r_{t-1}+j-1)_{(j)}$, respectively. The last two sums in the precedent expression can be rearranged as follows

$$\sum_{i=1}^{m-1}\binom{m}{i}\alpha_{0,t}^{m-i}\psi_{i,t}\lambda_{t-1}^i + \sum_{i=1}^m\sum_{j=1}^i\sum_{k=1}^{j-1}\mathcal{K}_{i,j,k}^{(m,t)}\lambda_t^{i-k} = \sum_{i=1}^{m-1}\phi_{i,t}^{(m)}\lambda_{t-1}^i,$$

where, $\phi_{i,t}^{(m)} = \binom{m}{i}\alpha_{0,t}^{m-i}\psi_{i,t} + \sum_{j=i+1}^m\sum_{k=j-i}^{j-1}\mathcal{K}_{j,k+1,j-i}^{(m,t)}$. Hence, we have

$$\mathbb{E}\left(\lambda_t^m\middle|\mathcal{F}_{t-2}\right) = \alpha_{0,t}^m + \psi_{m,t}\lambda_{t-1}^m + \sum_{i=1}^{m-1}\phi_{i,t}^{(m)}\lambda_{t-1}^i,$$

Replacing i par m, $m-1$, $m-2$,..., 3, 2, 1, we obtain the following matrix difference equation

$$\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\middle|\mathcal{F}_{t-2}\right)=\Theta_t^{(m)}\underline{\Lambda}_{t-1}^{(m)}+\underline{\alpha}_{0,t}^{(m)},$$

where the m-column vectors $\left(\lambda_t^m,\lambda_t^{m-1},...,\lambda_t\right)'$ and $\underline{\alpha}_{0,t}^{(m)}=\left(\alpha_{0,t}^m,\alpha_{0,t}^{m-1},...,\alpha_{0,t}\right)'$ and where the $m\times m$-matrix $\Theta_t^{(m)}$ is given by (4.4.2). Iterating the last equation n times, we obtain

$$\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\middle|\mathcal{F}_{t-2-n}\right)=\left(\prod_{i=1}^{n+1}\Theta_{t-i+1}^{(m)}\right)\underline{\Lambda}_{t-(n+1)}^{(m)}+\sum_{j=0}^{n}\left(\prod_{i=1}^{j}\Theta_{t-i+1}^{(m)}\right)\underline{\alpha}_{0,t-j}^{(m)}.$$

Letting $n=kS-2$, then we have, while taking account of the matrix $\Theta_t^{(m)}$ and the column vector $\underline{\alpha}_{0,t}^{(m)}$

$$\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\middle|\mathcal{F}_{t-kS}\right)=\left(\prod_{i=1}^{kS-1}\Theta_{t-i+1}^{(m)}\right)\underline{\Lambda}_{t-(kS-1)}^{(m)}+\sum_{j=0}^{kS-2}\left(\prod_{i=1}^{j}\Theta_{t-i+1}^{(m)}\right)\underline{\alpha}_{0,t-j}^{(m)},$$

which can be written in the form

$$\begin{aligned}\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\middle|\mathcal{F}_{t-kS}\right)&=\textstyle\sum_{l=0}^{k-2}\left(\prod_{i=1}^{S}\Theta_{t-i+1}^{(m)}\right)^l\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\Theta_{t-i+1}^{(m)}\right)\underline{\alpha}_{0,t-(lS+j)+1}^{(m)}+\left(\prod_{i=1}^{S}\Theta_{t-i+1}^{(m)}\right)^{k-1}\\&\textstyle\left(\sum_{j=1}^{S-1}\left(\prod_{i=1}^{j-1}\Theta_{t-i+1}^{(m)}\right)\underline{\alpha}_{0,t-(lS+j)+1}^{(m)}+\left(\prod_{i=1}^{S-1}\Theta_{t-i+1}^{(m)}\right)\underline{\Lambda}_{t-(kS-1)}^{(m)}\right).\end{aligned}$$

Replacing t by $s+\tau S$ and taking account of the periodicity, we obtain

$$\begin{aligned}\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\middle|\mathcal{F}_{t-(k-\tau)S}\right)&=\textstyle\sum_{l=0}^{k-2}\left(\prod_{i=1}^{S}\Theta_{s-i+1}^{(m)}\right)^l\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\Theta_{s-i+1}^{(m)}\right)\underline{\alpha}_{0,s-j+1}^{(m)}+\\&\textstyle\left(\prod_{i=1}^{S}\Theta_{s-i+1}^{(m)}\right)^{k-1}\left[\sum_{j=1}^{S-1}\left(\prod_{i=1}^{j-1}\Theta_{s-i+1}^{(m)}\right)\underline{\alpha}_{0,s-j+1}^{(m)}+\left(\prod_{i=1}^{S-1}\Theta_{s-i+1}^{(m)}\right)\underline{\Lambda}_{s-((k-\tau)S-1)}^{(m)}\right].\end{aligned}$$

Since the matrices $\Theta_{s-i+1}^{(m)}$, $i=1,2,...,S$ are upper-triangular with eigenvalues $\psi_{m,s-i+1}$, $\psi_{m-1,s-i+1}$, ..., $\psi_{2,s-i+1}$, $\psi_{1,s-i+1}$, then a sufficient condition for the matrix $\left(\prod_{i=1}^{S}\Theta_{s-i+1}^{(m)}\right)^{k-1}$ to converge, as $k\to\infty$, to the null matrix is that

$$\textstyle\prod_{i=1}^{S}\psi_{m,t}<1,\ \text{ with }\psi_{m,t}=\sum_{j=0}^{m}\binom{m}{j}\alpha_{1,t}^{j}\beta_t^{m-j}\left(r_{t-1}+j-1\right)_{(j)}.$$

Under this condition, we have, the closed-form of the vector

$$\begin{aligned}\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\right)&=\lim_{k\to\infty}\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\middle|\mathcal{F}_{t-(k-\tau)S}\right)=\textstyle\sum_{l=0}^{\infty}\left(\prod_{i=1}^{S}\Theta_{s-i+1}^{(m)}\right)^l\sum_{j=1}^{S}\left(\prod_{i=1}^{j-1}\Theta_{s-i+1}^{(m)}\right)\underline{\alpha}_{0,s-j+1}^{(m)},\\&=\textstyle\left(I-\Psi_{s,S}^{(m)}\right)^{-1}\sum_{i=0}^{S-1}\Psi_{s,i}^{(m)}\underline{\alpha}_{0,s-i}^{(m)},\ \text{ with, }\Psi_{s,j}^{(m)}=\prod_{i=1}^{j-1}\Theta_{s-i+1}^{(m)}.\end{aligned}$$

■

Corollary 4.4.2 *Under the condition (4.4.1) with $m=4$, the fourth unconditional moment, $\mu_{\lambda,t}^{(4)}=\mathbb{E}\left(\lambda_t^4\right)$, exists and its closed-form for the unconditional first four moments, are, under this condition, given by*

$$\mathbb{E}\left(\underline{\Lambda}_t^{(4)}\right)=\left(I-\Psi_{s,S}^{(4)}\right)^{-1}\sum_{i=0}^{S-1}\Psi_{s,i}^{(4)}\underline{\alpha}_{0,s-i}^{(4)},$$

where, $\underline{\alpha}_{0,t}^{(4)}=\left(\alpha_{0,t}^4,\alpha_{0,t}^3,\alpha_{0,t}^2,\alpha_{0,t}\right)'$, $\Psi_{s,i}^{(4)}=\prod_{i=1}^{j-1}\Theta_{s-i+1}^{(4)}$ *and the periodic* 4×4 *matrix* $\Theta_s^{(4)}$, *for* $s=1,...,4,$ *is given by*

$$\Theta_s^{(4)}=\begin{pmatrix}\psi_{4,s} & \alpha_s^{(1,2)} & \alpha_s^{(1,3)} & \alpha_s^{(1,4)}\\ 0 & \psi_{3,s} & \alpha_s^{(2,3)} & \alpha_s^{(2,4)}\\ 0 & 0 & \psi_{2,s} & \alpha_s^{(3,4)}\\ 0 & 0 & 0 & \psi_{1,s}\end{pmatrix},$$

where,

$$\begin{aligned}\psi_{4,s}&=6r_{s-1}\alpha_{1,s}^2\beta_s^2+4\left(r_{s-1}\left(r_{s-1}+2\right)\beta_s+2r_{s-1}^2\beta_s\right)\alpha_{1,s}^3+\left(r_{s-1}\alpha_{1,s}+\beta_s\right)^4+\\&\quad\left(r_{s-1}\left(r_{s-1}+3\right)\left(3r_{s-1}+2\right)+3r_{s-1}^3\right)\alpha_{1,s}^4,\\ \psi_{3,s}&=3r_{s-1}\alpha_{1,s}^2\beta_s+\left(2r_{s-1}\left(r_{s-1}+1\right)+r_{s-1}^2\right)\alpha_{1,s}^3+\left(r_{s-1}\alpha_{1,s}+\beta_s\right)^3,\\ \psi_{2,s}&=r_{s-1}\alpha_{1,s}^2+\left(r_{s-1}\alpha_{1,s}+\beta_s\right)^2,\ \psi_{1,s}=r_{s-1}\alpha_{1,s}+\beta_s,\end{aligned}$$

and,

$$\begin{aligned}\alpha_s^{(1,2)}&=4\alpha_{0,s}\beta_s^3+2\alpha_{1,s}^3\left(r_{s-1}+1\right)\left(3r_{s-1}\left(r_{s-1}+2\right)\alpha_{1,s}+2\left(r_{s-1}\left(r_{s-1}+2\right)\alpha_{0,s}+3r_{s-1}\beta_s\right)\right)\\&\quad+6r_{s-1}\alpha_{1,s}\beta_s\left(\left(\alpha_{1,s}+2\alpha_{0,s}\right)\beta_s+2\left(1+r_{s-1}\right)\alpha_{0,s}\alpha_{1,s}\right),\\ \alpha_s^{(1,3)}&=6\alpha_{0,s}^2\beta_s^2+12r_{s-1}\alpha_{0,s}^2\alpha_{1,s}\beta_s+12r_{s-1}\alpha_{0,s}\alpha_{1,s}^2\beta_s+6r_{s-1}\left(1+r_{s-1}\right)\alpha_{0,s}^2\alpha_{1,s}^2+4r_{s-1}\alpha_{1,s}^3\beta_s\\&\quad+12\alpha_{0,s}\alpha_{1,s}^3r_{s-1}\left(r_{s-1}+1\right)+7r_{s-1}\left(r_{s-1}+1\right)\alpha_{1,s}^4,\\ \alpha_s^{(1,4)}&=r_{s-1}\alpha_{1,s}^2\left(\alpha_{1,s}^2+6r_{s-1}\alpha_{0,s}^2\right)+4\alpha_{0,s}\left(r_{s-1}\alpha_{1,s}^3+\alpha_{0,s}^2\left(r_{s-1}\alpha_{1,s}+\beta_s\right)\right).\\ \alpha_s^{(2,3)}&=3\left(\alpha_{0,s}\beta_s^2+r_{s-1}\alpha_{1,s}^2\left(\left(1+r_{s-1}\right)\alpha_{0,s}+\beta_s\right)+2r_{s-1}\alpha_{0,s}\alpha_{1,s}\beta_s+r_{s-1}\left(r_{s-1}+1\right)\alpha_{1,s}^3\right),\\ \alpha_s^{(2,4)}&=r_{s-1}\alpha_{1,s}^3+3\alpha_{0,s}\left(r_{s-1}\alpha_{1,s}^2+\alpha_{0,t}\left(r_{s-1}\alpha_{1,s}+\beta_s\right)\right),\\ \alpha_s^{(3,4)}&=r_{s-1}\alpha_{1,s}^2+2\alpha_{0,t}\left(r_{s-1}\alpha_{1,s}+\beta_s\right).\end{aligned}$$

4.4.2 Calculation of higher moments of the process y_t

Let be the vectors column of dimension m, $\underline{\mu}_{y,t}^{(m)}=\left(\mathbb{E}\left(y_t^m\right),\mathbb{E}\left(y_t^{m-1}\right),...,\mathbb{E}\left(y_t\right)\right)'$ and $\underline{\Lambda}_t^{(m)}=\left(\lambda_t^m,\lambda_t^{m-1},...,\lambda_t\right)'$ and the matrix $\Omega^{(m)}$ of dimension $m\times m$ is given as follows :

$$\Omega_t^{(m)}=\begin{pmatrix}\omega_{m,t}^{(m)} & \omega_{m-1,t}^{(m)} & \cdots & \omega_{2,t}^{(m)} & \omega_{1,t}^{(m)}\\ 0 & \omega_{m-1,t}^{(m-1)} & \cdots & \omega_{2,t}^{(m-1)} & \omega_{1,t}^{(m-1)}\\ \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & \cdots & 0 & \omega_{2,t}^{(2)} & \omega_{1,t}^{(2)}\\ 0 & 0 & \cdots & 0 & \omega_{1,t}^{(1)}\end{pmatrix},\tag{4.4.4}$$

where, $\omega_{k,t}^{(m)} = \left\{ {m \atop k} \right\} (r_t + k - 1)_{(k)}$ with $x_{(i)} = x(x-1)\ldots(x-i+1)$. Using these notations and definitions, we can state the following lemma establishes an expression of the unconditional vector moments $\underline{\mu}_{y,t}^{(m)}$ as a function of the unconditional vector moments $\mathbb{E}\left(\underline{\Lambda}_t^{(m)}\right)$.

Lemma 4.4.1 *The unconditional m-th moments of the process* $\{y_t;\ t \in \mathbb{Z}\}$, $\mu_{y,s}^{(m)} = \mathbb{E}\left(y_{s+\tau S}^m\right)$ *are, under the condition (4.4.1), given as a function of the unconditional m-th moments* $\mu_{\lambda,s}^{(m)} = \mathbb{E}\left(\lambda_{s+\tau S}^m\right)$*, by the vector form below*

$$\underline{\mu}_{y,s}^{(m)} = \Omega_s^{(m)} \mathbb{E}\left(\underline{\Lambda}_s^{(m)}\right).$$

Proof. The proof is straightforward.

In the time invariant coefficients case (classical) model (4.2.4), i.e. $S = 1$, the results of Lemma 4.4.1 can be presented by the following corollary.

Corollary 4.4.3 *The unconditional m-th moments of the process* $\{y_t;\ t \in \mathbb{Z}\}$*, satisfy the model* (4.2.4), $\mu_y^{(m)} = \mathbb{E}\left(y_t^m\right)$ *are, under the condition (4.4.3), given as a function of the unconditional m-th moments* $\mu_\lambda^{(m)} = \mathbb{E}\left(\lambda_t^m\right)$*, by the vector form below*

$$\underline{\mu}_y^{(m)} = \Omega^{(m)} \mathbb{E}\left(\underline{\Lambda}_t^{(m)}\right).$$

with,

$$\Omega^{(m)} = \begin{pmatrix} \omega_{m,m} & \omega_{m-1,m} & \cdots & \omega_{2,m} & \omega_{1,m} \\ 0 & \omega_{m-1,m-1} & \cdots & \omega_{2,m-1} & \omega_{1,m-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \omega_{2,2} & \omega_{1,2} \\ 0 & 0 & \cdots & 0 & \omega_{1,1} \end{pmatrix},$$

where, $\omega_{k,m} = \left\{ {m \atop k} \right\} (r + k - 1)_{(k)}$ *with* $x_{(i)} = x(x-1)\ldots(x-i+1)$.

Proof. The proof is straightforward.

Corollary 4.4.4 *The unconditional m-th moments of the periodically correlated process,* $\mu_{y,t}^{(m)} = \mathbb{E}\left(y_t^m\right)$ *are, under the condition (4.4.1), given by :*

$$\underline{\mu}_{y,s}^{(m)} = \Omega_s^{(m)} \left(I - \Psi_{s,S}^{(m)}\right)^{-1} \sum_{i=1}^{S} \Psi_{s,i}^{(m)} \underline{\alpha}_{0,s-i+1}^{(m)}.$$

where, $\Psi_{s,j}^{(m)} = \prod_{i=1}^{j-1} \Theta_{s-i+1}^{(m)}$*, with* $\Theta_{s-i+1}^{(m)}$ *and* $\Omega^{(m)}$ *are given by (4.4.2) and (4.4.4) respectively.*

Proof. The proof is straightforward.

Corollary 4.4.5 *The first four unconditional moments of the periodically correlated process are, under the condition (4.4.1) with $m = 4$, given by :*

$$\underline{\mu}_{y,s}^{(4)} = \Omega_s^{(4)} \left(I - \Psi_{s,S}^{(4)} \right)^{-1} \sum_{i=1}^{S} \Psi_{s,i}^{(4)} \underline{\alpha}_{0,s-i+1}^{(4)}.$$

where, $\Psi_{s,j}^{(4)} = \prod_{i=1}^{j-1} \Theta_{s-i+1}^{(4)}$ and the matrix $\Omega_s^{(4)}$ is given by

$$\Omega_s^{(4)} = \begin{pmatrix} (r_s+3)_{(4)} & 6\,(r_s+2)_{(3)} & 7\,(r_s+1)_{(2)} & r_s \\ 0 & (r_s+2)_{(3)} & 3\,(r_s+1)_{(2)} & r_s \\ 0 & 0 & (r_s+1)_{(2)} & r_s \\ 0 & 0 & 0 & r_s \end{pmatrix},$$

and the matrix $\Theta_{s-i+1}^{(4)}$ is given by (4.4.2).

Proof. The proof is straightforward.

4.4.3 Skewness and kurtosis coefficients

For the need of the first fourth moments in the calculation of the skewness and the kurtosis coefficients, we present the following Corollary in which the matrix $\Theta_s^{(4)}$, for a fixed s, is given explicitly.

Corollary 4.4.6 *The skewness and the kurtosis coefficients of the process $\{y_t;\ t \in \mathbb{Z}\}$, are, under the condition (4.4.1), given, for $s = 1, 2, ..., S$, by :*

$$\mathcal{K}ur_s = \mu_{y,s}^{*(4)} / \left(\mu_{y,s}^{*(2)}\right)^2 = \left(\mu_{y,s}^{(4)} - 4\mu_{y,s}\mu_{y,s}^{(3)} + 6\mu_{y,s}^2\mu_{y,s}^{(2)} - 3\mu_{y,s}^4\right) / \left(\mu_{y,s}^{(2)} - \mu_{y,s}^2\right)^2,$$

and

$$\mathcal{S}k_s = \mu_{y,s}^{*(3)} / \left(\mu_{y,s}^{*(2)}\right)^{3/2} = \left(\mu_{y,s}^{(3)} - 3\mu_{y,s}\mu_{y,s}^{(2)} + 2\mu_{y,s}^3\right) / \left(\mu_{y,s}^{(2)} - \mu_{y,s}^2\right)^{3/2},$$

where, $\mu_{y,s}^{(4)}$, $\mu_{y,s}^{(3)}$, $\mu_{y,s}^{(2)}$, and $\mu_{y,s}$ are given, in terms of the parameters, by Corollary 4.4.5.

4.5 Autocovariance structure

The following proposition establishes the autocovariance structure of the process $\{y_t;\ t \in \mathbb{Z}\}$.

Proposition 4.5.1 *The autocovariance structure of the periodically correlated integer-valued processes $\{y_t;\ t \in \mathbb{Z}\}$ and $\{\lambda_t;\ t \in \mathbb{Z}\}$ satisfying the model (4.2.3) are, under the condition*

(4.4.1) *with* $m = 2$, *given as follows :*

$$\gamma_y^{(s)}(0) = \left(r_s + r_s^2\right)\left(1 - \left(\prod_{i=1}^{S}\psi_{2,i}\right)\right)^{-1}\sum_{j=0}^{S-1}\left(\prod_{i=1}^{j}\psi_{2,s-i+1}\right)F_{1,s-j+1} + \frac{1}{r_s}\mu_{y,s}^2 + \mu_{y,s},$$

$$\gamma_y^{(s)}(h) = \left(\prod_{i=1}^{h-1}\psi_{1,s-i+1}\right)\left[\left(\psi_{1,s-h+1} - \frac{\beta_{s-h+1}}{1+r_{s-h}}\right)\frac{r_{s-h+1}\gamma_y^{(s-h)}(0)}{r_{s-h}} + \frac{r_{s-h+1}\beta_{s-h+1}}{1+r_{s-h}}\left(\frac{\mu_{y,s-h}^2}{r_{s-h}} + \mu_{y,s-h}\right)\right],$$

$$\gamma_\lambda^{(s)}(0) = \left(1 - \left(\prod_{i=1}^{S}\psi_{2,i}\right)\right)^{-1}\sum_{j=0}^{S-1}\left(\prod_{i=1}^{j}\psi_{2,s-i+1}\right)F_{1,s-j+1},$$

$$\gamma_\lambda^{(s)}(h) = \left(\prod_{i=1}^{h}\psi_{1,s-i+1}\right)\gamma_\lambda^{(s-h)}(0),$$

with $\psi_{2,s} = \left(r_{s-1}\alpha_{1,s}^2 + \left(r_{s-1}\alpha_{1,s} + \beta_s\right)^2\right)$, $\psi_{1,s} = r_{s-1}\alpha_{1,s} + \beta_s$, $F_{1,s} = \alpha_{1,s}^2\left(\mu_{y,s-1} + \dfrac{1}{r_{s-1}}\mu_{y,s-1}^2\right)$

and the convention $\prod_{i=1}^{j} x_i = 1$, *if* $j < 1$.

Proof of Proposition 4.5.1. The variances $\gamma_\lambda^{(s)}(0)$ and $\gamma_y^{(s)}(0)$ were established in Proposition 4.3.2. The autocovariance $\gamma_y^{(s)}(1)$, can be calculated as follows :

$$\begin{aligned}
\gamma_y^{(s)}(1) &= Cov\left(y_s; y_{s-1}\right) = r_s Cov\left(\lambda_s; y_{s-1}\right) = r_s\alpha_{1,s}\gamma_y^{(s-1)}(0) + r_s r_{s-1}\beta_s\gamma_\lambda^{(s-1)}(0),\\
&= \frac{r_s}{r_{s-1}}\left(\left(r_{s-1}\alpha_{1,s} + \beta_s\right) - \frac{1}{1+r_{s-1}}\beta_s\right)\gamma_y^{(s-1)}(0) - \frac{r_s\beta_s}{1+r_{s-1}}\left(\frac{1}{r_{s-1}}\mu_{y,s-1}^2 + \mu_{y,s-1}\right),\\
&= \frac{r_s}{r_{s-1}}\left(\psi_{1,s} - \frac{1}{1+r_{s-1}}\beta_s\right)\gamma_y^{(s-1)}(0) - \frac{r_s\beta_s}{1+r_{s-1}}\left(\frac{1}{r_{s-1}}\mu_{y,s-1}^2 + \mu_{y,s-1}\right),\\
&= \frac{r_s}{r_{s-1}}\psi_{1,s}\gamma_y^{(s-1)}(0) - \frac{r_s\beta_s}{1+r_{s-1}}\left(\frac{1}{r_{s-1}}\gamma_y^{(s-1)}(0) - \left(\frac{1}{r_{s-1}}\mu_{y,s-1}^2 + \mu_{y,s-1}\right)\right).
\end{aligned}$$

More generally, let us calculate the autocovariance $\gamma_y^{(s)}(h)$, $h \geq 2$,

$$\begin{aligned}
\gamma_y^{(s)}(h) &= Cov\left(y_s; y_{s-h}\right) = r_s Cov\left(\lambda_s; y_{s-h}\right) = r_s Cov\left(\left(\alpha_{1,s}y_{s-1} + \beta_s\lambda_{s-1}\right); y_{s-h}\right),\\
&= r_s\alpha_{1,s}Cov\left(y_{s-1}; y_{s-1-(h-1)}\right) + \frac{r_s}{r_{s-1}}\beta_s Cov\left(r_{s-1}\lambda_{s-1}; y_{s-1-(h-1)}\right),\\
&= \frac{r_s}{r_{s-1}}\left(r_{s-1}\alpha_{1,s} + \beta_s\right)\gamma_y^{(s-1)}(h-1) = \frac{r_s}{r_{s-1}}\psi_{1,s}\gamma_y^{(s-1)}(h-1),\ h \geq 2.
\end{aligned}$$

Iterating the last equation m times and replacing m by $h-1$ then using (4.1), we obtain

$$\gamma_y^{(s)}(h) = \left(\prod_{i=1}^{h-1}\psi_{1,s-i+1}\right)\left(\left(\psi_{1,s-h+1} - \frac{\beta_{s-h+1}}{1+r_{s-h}}\right)\frac{r_{s-h+1}\gamma_y^{(s-h)}(0)}{r_{s-h}} + \frac{r_{s-h+1}\beta_{s-h+1}}{1+r_{s-h}}\left(\frac{\mu_{y,s-h}^2}{r_{s-h}} + \mu_{y,s-h}\right)\right).$$

The autocovariance $\gamma_\lambda^{(s)}(h)$, $h \geq 1$, is given as follows :

$$\begin{aligned}
\gamma_\lambda^{(s)}(h) &= Cov\left(\lambda_s; \lambda_{s-h}\right) = Cov\left(\left(\alpha_{1,s}y_{s-1} + \beta_s\lambda_{s-1}\right); \lambda_{s-h}\right),\\
&= \left(r_{s-1}\alpha_{1,s} + \beta_s\right)Cov\left(\lambda_{s-1}; \lambda_{s-1-(h-1)}\right) = \left(r_{s-1}\alpha_{1,s} + \beta_s\right)\gamma_\lambda^{(s-1)}(h-1).
\end{aligned}$$

Iterating the equation m times and replacing m by h, we obtain

$$\gamma_{\lambda}^{(s)}(h) = \left(\prod_{i=1}^{h} \psi_{1,s-i+1}\right) \gamma_{\lambda}^{(s-h)}(0). \qquad \blacksquare$$

Corollary 4.5.1 *The autocorrelation function of the periodically correlated integer-valued processes $\{y_t;\ t \in \mathbb{Z}\}$ and $\{\lambda_t;\ t \in \mathbb{Z}\}$ satisfying the model (4.2.3) are, under the condition (4.4.1) with $m = 2$, given as follows :*

$$\rho_{y}^{(s)}(\nu + kS) = \left(\prod_{i=1}^{S} \psi_{1,i}\right)^{k} \left(\prod_{i=1}^{\nu} \psi_{1,s-i+1}\right) \sqrt{\gamma_{y}^{(s-\nu)}(0) \Big/ \gamma_{y}^{(s)}(0)} \left(\frac{r_{s-\nu+1}}{r_{s-\nu}} \psi_{1,s-\nu+1} - \frac{r_{s-\nu+1}\beta_{s-\nu+1}}{r_{s-\nu}(1 + r_{s-\nu})} \left(1 - \frac{\mu_{y,s-\nu}^{2} + r_{s-\nu}\mu_{y,s-\nu}}{\gamma_{y}^{(s-\nu)}(0)}\right)\right),\ \nu = 1, 2, ..., S \text{ and } k \in \mathbb{N},$$

$$\rho_{\lambda}^{(s)}(\nu + kS) = \left(\prod_{i=1}^{S} \psi_{1,i}\right)^{k} \left(\prod_{i=1}^{\nu} \psi_{1,s-i+1}\right) \sqrt{\gamma_{\lambda}^{(s-\nu)}(0) \Big/ \gamma_{\lambda}^{(s)}(0)},\ \nu = 1, 2, ..., S \text{ and } k \in \mathbb{N},$$

4.6 Parameters estimation

In the present section, we address the parameter estimation for our model, given in (4.2.3), while considering three methods, Yule-Walker (YW), Conditional Least Squares (CLS), and Conditional Maximum Likelihood (CML).

4.6.1 Yule-Walker estimation

In this paragraph, we give the Yule-Walker (YW) estimations for the vector of parameters which will be used as the starting values for the conditional maximum likelihood (CML) estimation. Indeed, the Yule-Walker estimation proceeds in steps 1 to 3 outlined below:

Step 1 : *Estimation of* $\psi_{1,s} = r_{s-1}\alpha_{1,s} + \beta_s$ *by using,* $\widehat{\psi}_{1,s} = r_{s-1}\widehat{\gamma}_{y}^{(s)}(2) \Big/ r_s\widehat{\gamma}_{y}^{(s-1)}(1)$.

Step 2 : *Estimation of* $\alpha_{0,s}$ *by using,* $\widehat{\alpha}_{0,s} = \frac{1}{r_s}\widehat{\mu}_{y,s} - \frac{1}{r_{s-1}}\widehat{\psi}_{1,s}\widehat{\mu}_{y,s-1}$.

Step 3 : *Estimation of* $\alpha_{1,s}$ *and* β_s*, for* $s = 1, ..., S$*, respectively, by*

$$\widehat{\alpha}_{1,s} = \frac{r_s\widehat{\psi}_{1,s}\left(\widehat{\mu}_{y,s-1}\left(\frac{\widehat{\mu}_{y,s-1}}{r_{s-1}}+1\right)-\widehat{\gamma}_{y}^{(s-1)}(0)\right)+(1+r_{s-1})\widehat{\gamma}_{y}^{(s)}(1)}{r_s\left(\widehat{\gamma}_{y}^{(s-1)}(0)+\widehat{\mu}_{y,s-1}\left(\widehat{\mu}_{y,s-1}+r_{s-1}\right)\right)}, \text{ and } \widehat{\beta}_s = \frac{(1+r_{s-1})\left(r_s\widehat{\psi}_{1,s}\widehat{\gamma}_{y}^{(s-1)}(0)-r_{s-1}\widehat{\gamma}_{y}^{(s)}(1)\right)}{r_s\left(\widehat{\gamma}_{y}^{(s-1)}(0)+\widehat{\mu}_{y,s-1}\left(\widehat{\mu}_{y,s-1}+r_{s-1}\right)\right)}.$$

4.6.2 Conditional least squares estimation

This paragraph focuses on the estimation, adopting the conditional least squares estimation method, of the underlying parameters of the model (4.2.3) which can be written in the form of a $PARMA_S(1,1)$ model $y_t = r_t\alpha_{0,t} + a_t y_{t-1} + \varepsilon_t + b_t\varepsilon_{t-1}$, with $a_t = r_t\psi_{1,t}\big/ r_{t-1}$, $b_t = -r_t\beta_t\big/ r_{t-1}$ where, ε_t verifies $\mathbb{E}(\varepsilon_t) = 0$, $Var(\varepsilon_t) = \gamma_y^{(t)}(0)$ and $Cov(\varepsilon_t;\varepsilon_{t-h}) = 0$, $\forall h \geq 1$. The conditional least squares (CLS) procedure is as follows:

(i). Let $x_{s+\tau S} = y_{s+\tau S} - \frac{1}{N}\sum_{\tau=0}^{N-1} y_{s+\tau S}$, $s = 1,...,S$, fit the data by using a higher-order $PAR(p^*)$ model, then obtain the conditional least squares (CLS) estimators for the periodic autoregressive coefficients $\widehat{a}_{i,t}$ and define $\widehat{\varepsilon}_{s+\tau S} = x_{s+\tau S} - \sum_{i=1}^{p^*}\widehat{a}_{i,s}x_{s-i+\tau S}$.

(ii). The CLS-estimations $\widehat{\underline{\theta}}_{s,CLS} = \left(\widehat{\alpha}_{0,s},\widehat{\alpha}_{1,s},\widehat{\beta}_s\right)'$ of $\underline{\theta}_s$ are obtained by minimizing

$$\sum_{s=1}^{S}\sum_{\tau=0}^{N-1}\left(y_{s+\tau S} - \left(r_s\alpha_{0,s} + a_s y_{s-1+\tau S} + b_s\widehat{\varepsilon}_{s-1+\tau S}\right)\right)^2,$$

where, $\widehat{\beta}_s = -r_{s-1}\widehat{b}_s\Big/ r_s$ and $\widehat{\alpha}_{1,s} = \left(\widehat{a}_s - \widehat{b}_s\right)\Big/ r_s$. In the simulation study, we choose $p^* = \left[\sqrt{N}\right]$, where $[x]$ is the integer part of x.

4.6.3 Conditional maximum likelihood estimation

Let the column vector of parameters $\underline{\theta} = \left(\underline{\alpha}_0', \underline{\alpha}_1', \underline{\beta}'\right)$, where the S-column vectors are $\underline{\alpha}_0' = (\alpha_{0,1},\alpha_{0,2},...,\alpha_{0,S})'$, $\underline{\alpha}_1' = (\alpha_{1,1},\alpha_{1,2},...,\alpha_{1,S})'$ and $\underline{\beta}' = (\beta_1,\beta_2,...,\beta_S)'$. For a simplicity in manipulation, we consider an observed time series of size $n = NS$, $\underline{y}_t = (y_1, y_2, ..., y_n)$. The conditional log-likelihood function is then given by

$$\begin{aligned}\mathcal{L}\left(\underline{\theta}|\,\underline{y}_t\right) &= \log\left(L\left(\underline{\theta}|\,\underline{y}_t\right)\right) = r\sum_{t=1}^{n}\log(p_t) + \sum_{t=1}^{n} y_t\log(1-p_t) + \log\prod_{t=1}^{n}\binom{y_t+r_t-1}{r_t-1},\\ &= \sum_{t=1}^{n} y_t\log(\lambda_t) - \sum_{t=1}^{n}(r_t+y_t)\log(1+\lambda_t) + \log\prod_{t=1}^{n}\binom{y_t+r_t-1}{r_t-1},\\ &= \sum_{t=1}^{n}\left(y_t\log(\lambda_t) - (r_t+y_t)\log(1+\lambda_t) + \sum_{i=1}^{y_t}(i+r_t-1) - \log(y_t!)\right).\end{aligned}$$

Let $t = s+\tau S$, $s = 1,...,S$ and $\tau = 0,1,...,N-1$, then the conditional log-likelihood function can be written in the form :

$$\begin{aligned}\mathcal{L}\left(\underline{\theta}|\,\underline{y}_{s+\tau S}\right) = &\sum_{\tau=0}^{N-1}\sum_{s=1}^{S}\Big[y_{s+\tau S}\log(\lambda_{s+\tau S}) - (r_s+y_{s+\tau S})\log(1+\lambda_{s+\tau S}) - \log(y_{s+\tau S}!) +\\ &\sum_{i=1}^{y_{s+\tau S}}(i+r_s-1)\Big]\end{aligned} \tag{4.6.1}$$

Analytical estimates of this log-likelihood function cannot be found, therefore numerical optimization methods must be employed. In order to obtain the asymptotic standard errors of the CML, we need at first calculate the partial derivatives $\mathcal{L}\left(\underline{\theta}\right|\underline{y}_{s+\tau S}\Big)$ with respect to θ_j, $j = 1, 2, 3$, which has the form :

$$\frac{\partial \mathcal{L}}{\partial \theta_j} = \sum_{\tau=0}^{N-1}\left(\frac{y_{s+\tau S}}{\lambda_{s+\tau S}} - \frac{r_s + y_{s+\tau S}}{1+\lambda_{s+\tau S}}\right)\frac{\partial \lambda_{s+\tau S}}{\partial \theta_j}$$

with

$$\partial\lambda_{s+\tau S}/\,\partial\alpha_{0,s} = 1,\ \ \partial\lambda_{s+\tau S}/\,\partial\alpha_{1,s} = y_{s-1+\tau S},\ \text{and}\ \ \partial\lambda_{s+\tau S}/\,\partial\beta_t = \lambda_{s-1+\tau S},$$

The second partial derivatives are

$$\begin{aligned}\frac{\partial^2 \mathcal{L}}{\partial \theta_{i,s}\partial \theta_{j,s}} &= \sum_{\tau=0}^{N-1}\left(\frac{y_{s+\tau S}}{\lambda_{s+\tau S}} - \frac{r_s + y_{s+\tau S}}{1+\lambda_{s+\tau S}}\right)\frac{\partial^2 \lambda_{s+\tau S}}{\partial \theta_{i,s}\partial \theta_{j,s}} - \left(\frac{y_{s+\tau S}}{\lambda^2_{s+\tau S}} - \frac{r_s + y_{s+\tau S}}{(1+\lambda_{s+\tau S})^2}\right)\frac{\partial \lambda_{s+\tau S}}{\partial \theta_{i,s}}\frac{\partial \lambda_{s+\tau S}}{\partial \theta_{j,s}},\\ &= -\sum_{\tau=0}^{N-1}\left(\frac{y_{s+\tau S}}{\lambda^2_{s+\tau S}} - \frac{r_s + y_{s+\tau S}}{(1+\lambda_{s+\tau S})^2}\right)\frac{\partial \lambda_{s+\tau S}}{\partial \theta_{i,s}}\frac{\partial \lambda_{s+\tau S}}{\partial \theta_{j,s}},\end{aligned}$$

By taking the expectation on both sides of the last equation, we get

$$\mathbb{E}\left(\frac{\partial^2 \mathcal{L}}{\partial \theta_{i,s}\partial \theta_{j,s}}\middle|\, \mathcal{F}_{t-1}\right) = -r_s\sum_{\tau=0}^{N-1}\left(\frac{1}{\lambda_{s+\tau S}} - \frac{1}{1+\lambda_{s+\tau S}}\right)\frac{\partial \lambda_{s+\tau S}}{\partial \theta_{i,s}}\frac{\partial \lambda_{s+\tau S}}{\partial \theta_{j,s}},$$

and

$$\begin{aligned}\mathbb{E}\left(\frac{\partial \mathcal{L}}{\partial \theta_{i,s}}\frac{\partial \mathcal{L}}{\partial \theta_{j,s}}\middle|\, \mathcal{F}_{t-1}\right) &= \sum_{\tau=0}^{N-1}\mathbb{E}\left(\left(\frac{y_{s+\tau S}}{\lambda_{s+\tau S}} - \frac{r_s + y_{s+\tau S}}{1+\lambda_{s+\tau S}}\right)^2\middle|\, \mathcal{F}_{t-1}\right)\frac{\partial \lambda_{s+\tau S}}{\partial \theta_{i,s}}\frac{\partial \lambda_{s+\tau S}}{\partial \theta_j},\\ &= \sum_{\tau=0}^{N-1}\mathbb{E}\left(\left(\frac{(y_{s+\tau S} - r_s\lambda_{s+\tau S})^2}{\lambda^2_{s+\tau S}(1+\lambda_{s+\tau S})^2}\right)\middle|\, \mathcal{F}_{t-1}\right)\frac{\partial \lambda_{s+\tau S}}{\partial \theta_{i,s}}\frac{\partial \lambda_{s+\tau S}}{\partial \theta_j},\\ &= r_s\sum_{\tau=0}^{N-1}\left(\frac{1}{\lambda_{s+\tau S}} - \frac{1}{1+\lambda_{s+\tau S}}\right)\frac{\partial \lambda_{s+\tau S}}{\partial \theta_{i,s}}\frac{\partial \lambda_{s+\tau S}}{\partial \theta_j}.\end{aligned}$$

Then, the model satisfies the information matrix equality:

$$-\mathbb{E}\left(\frac{\partial^2 \mathcal{L}}{\partial \theta_{i,s}\partial \theta_{j,s}}\middle|\, \mathcal{F}_{t-1}\right) = \mathbb{E}\left(\frac{\partial \mathcal{L}}{\partial \theta_{i,s}}\frac{\partial \mathcal{L}}{\partial \theta_{j,s}}\middle|\, \mathcal{F}_{t-1}\right).$$

Asymptotic standard errors of CML can be computed from the following matrix (Ferland *et al.* 2006):

$$\frac{1}{N}\left(\widehat{D}_{s,N}\widehat{S}_{s,N}\widehat{D}_{s,N}\right)^{-1}$$

where,

$$\widehat{S}_{s,N} = \frac{1}{N}\sum_{\tau=0}^{N-1} \frac{\partial \mathcal{L}\left(\underline{\theta} | \, \underline{y}_{s+\tau S}\right)}{\partial \underline{\theta}} \frac{\partial \mathcal{L}\left(\underline{\theta} | \, \underline{y}_{s+\tau S}\right)}{\partial \underline{\theta}'},$$

and

$$\widehat{D}_{s,N} = -\frac{1}{N}\sum_{\tau=0}^{N-1} \frac{\partial^2 \mathcal{L}\left(\underline{\theta} | \, \underline{y}_{s+\tau S}\right)}{\partial \underline{\theta} \partial \underline{\theta}'}.$$

In the case where the parameter r_s, $s = 1, ..., S$, is unknown, their estimation is simple which can take two steps :

Step 1 : *Calculate* $\widehat{\alpha}_{0,s}$, $\widehat{\alpha}_{1,s}$, $\widehat{\beta}_s$, *and* $\widetilde{r}_s$, $s = 1, ..., S$ *that maximize the conditional log-likelihood function* (4.6.1) *.*

Step 2 : *The estimate* $\widehat{r}_s$ *is determined by* $\widehat{r}_s = [\widetilde{r}_s + 0.5]$ *,* $s = 1, ..., S$*, where* $[a]$ *is the integer part of* a*.*

4.7 Simulation results and application on real data

In this Section, we illustrate our obtained results and we assess the Yule-Walker, Conditional Least Squares, and Conditional Maximum Likelihood estimations on time series, for small, moderate, and relatively large sample sizes and also an application on real data set.

4.7.1 Simulation results

In order to show some empirical estimate properties, we have generated 1000 independent series from $PNBINGARCH_4(1,1)$ model. The true values of each model considered are selected so as to verify the second order periodical stationary condition given in Proposition 4.3.2. Indeed, it equals to 0.0157 for the first model, 0.4460 for the second one, and 0.1715 for the last one, where the parameters r_s, $s = 1, ..., 4$, are assumed to be known for the first and the second models, and unknown for the last one. The estimations of these parameters can be done by selecting those which maximize the log conditional likelihood function. For each data-generating process, we consider 1000 replications, the mean estimated and their root mean square error $(RMSE)$ are reported in Tables 4.1 to 4.3. The true parameter values of these models are given below :

$$\text{Model 1} : \underline{\theta} = (\underline{\theta}_1; \ldots; \underline{\theta}_4)' = ((4, 0.2, 0.25); (1, 0.1, 0.15); (2, 0.25, 0.2); (3, 0.15, 0.1))',$$

$$\text{with } \underline{r} = (r_1, \ldots, r_4) = (4, 1, 2, 3).$$

$$\text{Model 2} : \underline{\theta} = (\underline{\theta}_1; \ldots; \underline{\theta}_4)' = ((2, 0.1, 0.3); (1, 0.35, 0.1); (3, 0.1, 0.5); (4, 0.15, 0.4))',$$

$$\text{with } \underline{r} = (r_1, \ldots, r_4) = (2, 3, 4, 5).$$

$$\text{Model 3} : \underline{\theta} = (\underline{\theta}_1; \ldots; \underline{\theta}_4)' = ((3.5, 0.1, 0.15); (1, 0.45, 0.1); (2.5, 0.6, 0.25); (2, 0.25, 0.1))'.$$

$$\text{with } \underline{r} = (r_1, \ldots, r_4) = (2, 1, 2, 3).$$

From Table 4.1, we observe that the adopted estimation methods perform better as n increases. Indeed, the convergence of all the parameter estimators is guaranteed and the root mean square error decreasing, as the sample size n increases, see, Figure 4.1, which imply that our estimators, YW-vector estimators $\widehat{\underline{\theta}}_{s,YW}$, CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$ and CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$, are empirically consistent for all the parameters. Furthermore, we notice that the CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$ have a small $RMSE$ compared to one of the CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$, and YW-vector estimators $\widehat{\underline{\theta}}_{s,YW}$. Thus, the CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$ are much advantageous than the YW-vector estimators $\widehat{\underline{\theta}}_{s,YW}$ and CLS-vector estimators $\widehat{\underline{\theta}}_{s,CLS}$, where this empirical superiority is clearly visible in Figure 4.1. Furthermore, the same conclusions can be obtained from Table 4.2 and Figure 4.2, that the consistency property of the YW-vector estimators $\widehat{\underline{\theta}}_{s,YW}$, CLS-vector estimators and CML-vector estimators still met. From Table 4.3, one can easily observe that the condition maximum likelihood method performs better as n increases. Indeed, the convergence of all the parameter estimations is guaranteed and the root mean square error decreasing, as the sample size n increases, which imply that our estimators (CML-vector estimators $\widehat{\underline{\theta}}_{s,CML}$) are empirically consistent for all the parameters. It can also be seen that the median estimates generally, for the periodic integer valued r_s, $s = 1, ..., 4$, better than the mean.

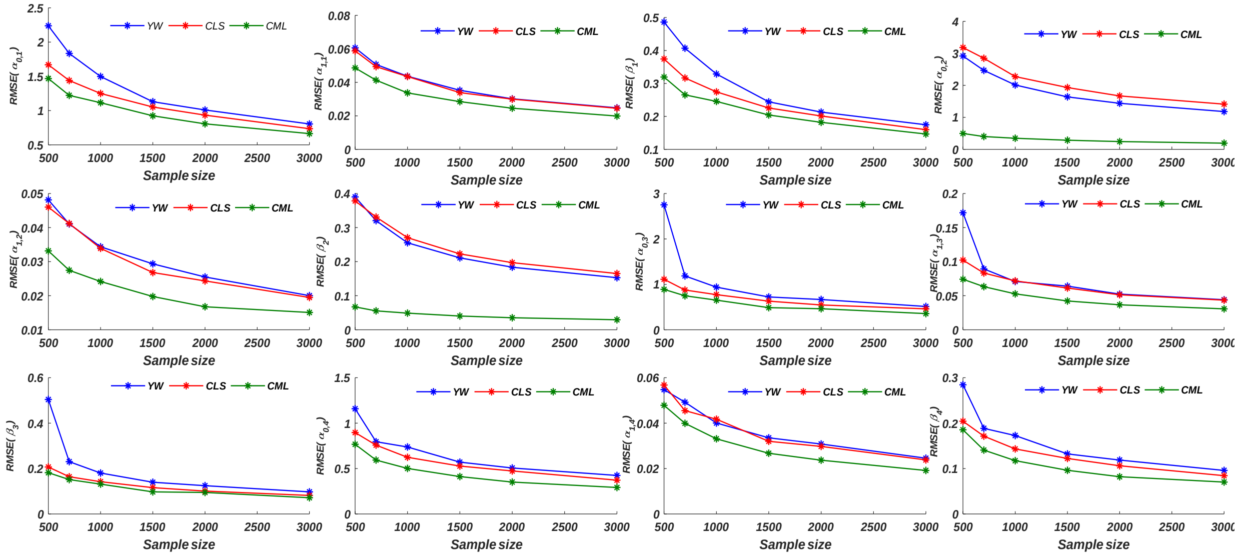

Figure 4.1. *RMSE graphics for the parameters of Model 1 for the different methods.*

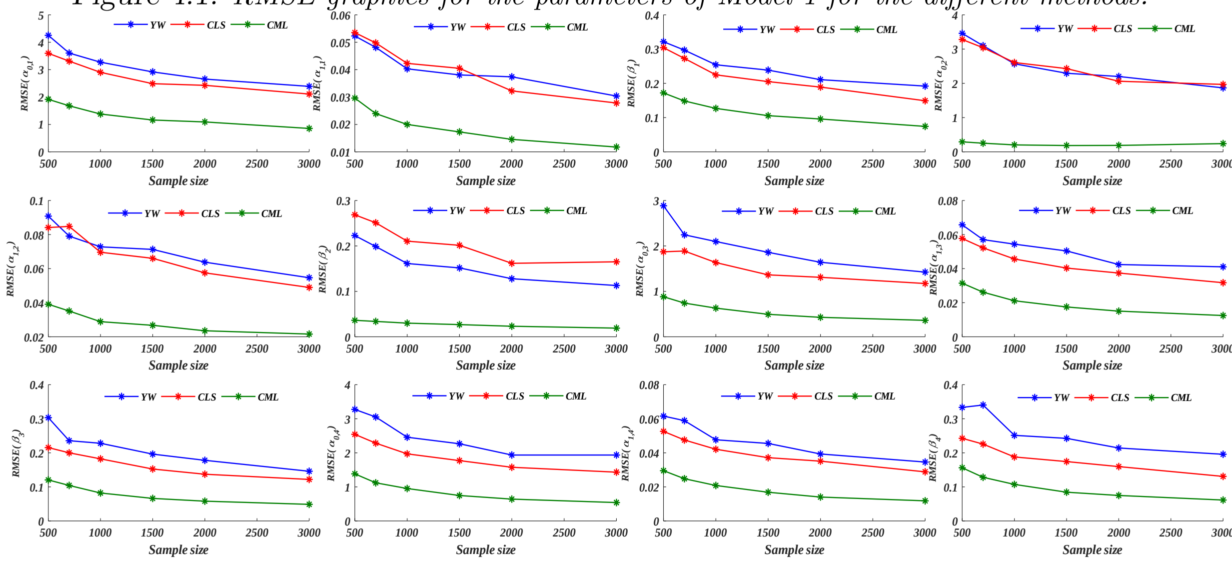

Figure 4.2. *RMSE graphics for the parameters of Model 2 for the different methods.*

Table 4.1. Sample mean and root mean square error RMSE (in bracket) for Model 1.

	T.V	4	0.2	0.25	1	0.1	0.15	2	0.25	0.2	3	0.15	0.1
Size	*Est.*	$\widehat{\alpha}_{0,1}$	$\widehat{\alpha}_{1,1}$	$\widehat{\beta}_1$	$\widehat{\alpha}_{0,2}$	$\widehat{\alpha}_{1,2}$	$\widehat{\beta}_2$	$\widehat{\alpha}_{0,3}$	$\widehat{\alpha}_{1,3}$	$\widehat{\beta}_3$	$\widehat{\alpha}_{0,4}$	$\widehat{\alpha}_{1,4}$	$\widehat{\beta}_4$
500	*YW*	3.7485 (2.2379)	0.2020 (0.0606)	0.3018 (0.4865)	1.0659 (2.9153)	0.1000 (0.0481)	0.1400 (0.3900)	1.7496 (2.7513)	0.2637 (0.1718)	0.2418 (0.5040)	2.9849 (1.1587)	0.1506 (0.0548)	0.1042 (0.2844)
	CLS	3.6449 (1.6691)	0.2001 (0.0588)	0.3189 (0.3744)	1.0465 (3.1801)	0.1005 (0.0460)	0.1307 (0.3783)	2.0562 (1.1121)	0.2491 (0.1023)	0.1852 (0.2070)	3.0544 (.8965)	0.1516 (0.0568)	0.0825 (0.2043)
	CML	3.7738 (1.4683)	0.1978 (0.0486)	0.3092 (0.3195)	1.0310 (0.4960)	0.1061 (0.0332)	0.1580 (0.0673)	1.8995 (0.8898)	0.2476 (0.0742)	0.2277 (0.1819)	2.8213 (0.7674)	0.1466 (0.0479)	0.1484 (0.1854)
700	*YW*	3.8179 (1.8345)	0.1989 (0.0507)	0.2925 (0.4071)	0.9987 (2.4671)	0.1004 (0.0411)	0.1525 (0.3204)	1.8295 (1.1860)	0.2559 (0.0893)	0.2287 (0.2304)	3.0259 (0.7957)	0.1504 (0.0492)	0.0934 (0.1886)
	CLS	3.7126 (1.4393)	0.2048 (0.0494)	0.2937 (0.3167)	0.8501 (2.8455)	0.1007 (0.0412)	0.1514 (0.3310)	1.9777 (0.8758)	0.2447 (0.0833)	0.2057 (0.1639)	3.0106 (0.7578)	0.1475 (0.0455)	0.0987 (0.1714)
	CML	3.8195 (1.2224)	0.1989 (0.0413)	0.2909 (0.2654)	1.0266 (0.4011)	0.1060 (0.0275)	0.1526 (0.0556)	1.9385 (0.7481)	0.2527 (0.0633)	0.2117 (0.1515)	2.8864 (0.5936)	0.1491 (0.0399)	0.1295 (0.1407)
1000	*YW*	3.8497 (1.4982)	0.2015 (0.0437)	0.2804 (0.3287)	1.0680 (2.0111)	0.1000 (0.0344)	0.1412 (0.2554)	1.9431 (0.9391)	0.2502 (0.0709)	0.2104 (0.1804)	3.0093 (0.7372)	0.1506 (0.0401)	0.0968 (0.1728)
	CLS	3.8565 (1.2504)	0.2021 (0.0435)	0.2704 (0.2745)	0.9723 (2.2739)	0.0990 (0.0339)	0.1410 (0.2704)	1.9803 (0.7746)	0.2491 (0.0718)	0.1973 (0.1424)	3.0226 (0.6239)	0.1497 (0.0417)	0.0939 (0.1432)
	CML	3.8541 (1.1151)	0.2005 (0.0337)	0.2827 (0.2456)	1.0173 (0.3467)	0.1064 (0.0242)	0.1520 (0.0489)	1.9512 (0.6529)	0.2530 (0.0528)	0.2099 (0.1315)	2.9333 (0.5014)	0.1467 (0.0331)	0.1198 (0.1173)
1500	*YW*	3.9570 (1.1298)	0.2009 (0.0352)	0.2582 (0.2442)	0.9605 (1.6392)	0.1027 (0.0294)	0.1442 (0.2112)	1.9434 (0.7223)	0.2529 (0.0641)	0.2075 (0.1398)	2.9807 (0.5710)	0.1524 (0.0335)	0.0998 (0.1323)
	CLS	3.9118 (1.0529)	0.1989 (0.0339)	0.2670 (0.2256)	0.9250 (1.9299)	0.1003 (0.0268)	0.1434 (0.2228)	1.9847 (0.6306)	0.2521 (0.0613)	0.1985 (0.1161)	2.9980 (0.5275)	0.1483 (0.0320)	0.1011 (0.1224)
	CML	3.9587 (0.9243)	0.1992 (0.0285)	0.2605 (0.2043)	0.9870 (0.2857)	0.1072 (0.0198)	0.1518 (0.0404)	1.9699 (0.4880)	0.2515 (0.0423)	0.2042 (0.0974)	2.9571 (0.4120)	0.1492 (0.0267)	0.1116 (0.0961)
2000	*YW*	3.9342 (1.0090)	0.2020 (0.0301)	0.2599 (0.2132)	0.9200 (1.4388)	0.1004 (0.0255)	0.1590 (0.1837)	1.9614 (0.6670)	0.2525 (0.0524)	0.2064 (0.1248)	2.9944 (0.5065)	0.1497 (0.0308)	0.1016 (0.1188)
	CLS	3.9208 (0.9343)	0.2010 (0.0299)	0.2628 (0.2012)	0.7393 (1.6693)	0.0995 (0.0243)	0.1665 (0.1971)	1.9587 (0.5469)	0.2503 (0.0514)	0.2023 (0.1008)	3.0212 (0.4735)	0.1488 (0.0298)	0.0962 (0.1061)
	CML	3.9715 (0.8070)	0.2003 (0.0246)	0.2556 (0.1820)	1.0064 (0.2430)	0.1008 (0.0168)	0.1511 (0.0353)	1.9667 (0.4642)	0.2503 (0.0367)	0.2062 (0.0948)	2.9822 (0.3516)	0.1480 (0.0237)	0.1076 (0.0820)
3000	*YW*	3.9600 (0.8051)	0.2001 (0.0248)	0.2573 (0.1747)	0.9698 (1.1801)	0.1006 (0.0200)	0.1516 (0.1530)	1.9871 (0.5134)	0.2505 (0.0441)	0.2036 (0.0974)	2.9770 (0.4240)	0.1505 (0.0246)	0.1043 (0.0960)
	CLS	3.9715 (0.7363)	0.2009 (0.0245)	0.2531 (0.1597)	0.7279 (1.4127)	0.1004 (0.0195)	0.1647 (0.1651)	1.9748 (0.4609)	0.2486 (0.0435)	0.2016 (0.0817)	3.0028 (0.3724)	0.1507 (0.0238)	0.0972 (0.0844)
	CML	3.9997 (0.6644)	0.2004 (0.0199)	0.2492 (0.1465)	0.09950 (0.1942)	0.1002 (0.0151)	0.1512 (0.0294)	1.9903 (0.3568)	0.2515 (0.0308)	0.2017 (0.0715)	3.0022 (0.2918)	0.1494 (0.0192)	0.1017 (0.0704)

Table 4.2. Sample mean and root mean square error RMSE (in bracket) for Model 2.

	T.V	2	0.1	0.3	1	0.35	0.1	3	0.1	0.5	4	0.15	0.4
Size	*Est.*	$\widehat{\alpha}_{0,1}$	$\widehat{\alpha}_{1,1}$	$\widehat{\beta}_1$	$\widehat{\alpha}_{0,2}$	$\widehat{\alpha}_{1,2}$	$\widehat{\beta}_2$	$\widehat{\alpha}_{0,3}$	$\widehat{\alpha}_{1,3}$	$\widehat{\beta}_3$	$\widehat{\alpha}_{0,4}$	$\widehat{\alpha}_{1,4}$	$\widehat{\beta}_4$
500	*YW*	3.1532 (4.2515)	0.0981 (0.0523)	0.2414 (0.3215)	1.0544 (3.4590)	0.3462 (0.0908)	0.1038 (0.2229)	2.7742 (2.8848)	0.1061 (0.0658)	0.4976 (0.3033)	3.9152 (3.2748)	0.1547 (0.0616)	0.3887 (0.3334)
	CLS	3.2085 (3.5946)	0.0978 (0.0535)	0.2390 (0.3042)	1.1089 (3.2800)	0.3479 (0.0841)	0.1237 (0.2686)	3.2756 (1.8711)	0.1003 (0.0577)	0.4879 (0.2154)	4.2575 (2.5428)	0.1520 (0.0526)	0.3791 (0.2426)
	CML	2.3213 (1.9170)	0.0988 (0.0296)	0.2865 (0.1720)	1.0448 (0.2918)	0.3546 (0.0391)	0.0988 (0.0363)	2.9662 (0.8813)	0.1018 (0.0315)	0.5034 (0.1205)	4.0484 (1.3843)	0.1501 (0.0295)	0.3995 (0.1563)
700	*YW*	2.9876 (3.6054)	0.0973 (0.0481)	0.2549 (0.2969)	1.1291 (3.0994)	0.3449 (0.0791)	0.1022 (0.1988)	2.9210 (2.2453)	0.1045 (0.0570)	0.4921 (0.2357)	3.7708 (3.0535)	0.1526 (0.0589)	0.4091 (0.3404)
	CLS	2.8920 (3.3110)	0.0994 (0.0497)	0.2522 (0.2727)	1.1562 (3.0399)	0.3502 (0.0848)	0.1133 (0.2510)	3.2134 (1.8883)	0.1000 (0.0522)	0.4966 (0.2000)	4.1381 (2.2845)	0.1488 (0.0475)	0.4023 (0.2257)
	CML	2.2176 (1.6800)	0.0997 (0.0240)	0.2908 (0.1485)	1.0276 (0.2529)	0.3536 (0.0351)	0.1017 (0.0339)	3.0003 (0.7426)	0.1018 (0.0262)	0.4970 (0.1043)	3.9955 (1.1201)	0.1487 (0.0248)	0.4046 (0.1283)
1000	*YW*	2.8769 (3.2709)	0.0977 (0.0403)	0.2603 (0.2542)	1.1250 (2.5705)	0.3474 (0.0728)	0.0960 (0.1613)	2.8178 (2.0971)	0.1067 (0.0544)	0.4963 (0.2281)	3.9809 (2.4575)	0.1524 (0.0477)	0.3925 (0.2515)
	CLS	2.7353 (2.9025)	0.0993 (0.0423)	0.2615 (0.2248)	1.0820 (2.6078)	0.3462 (0.0696)	0.1285 (0.2106)	3.1489 (1.6340)	0.1009 (0.0457)	0.4991 (0.1823)	4.3408 (1.9704)	0.1473 (0.0421)	0.3906 (0.1879)
	CML	2.1892 (1.3807)	0.0997 (0.0200)	0.2871 (0.1269)	0.9968 (0.2027)	0.3522 (0.0289)	0.0992 (0.0300)	2.9805 (0.6309)	0.1020 (0.0211)	0.4979 (0.0822)	3.9658 (0.9528)	0.1497 (0.0208)	0.4066 (0.1074)
1500	*YW*	2.6768 (2.9181)	0.0976 (0.0381)	0.2720 (0.2385)	1.0265 (2.2949)	0.3518 (0.0713)	0.0946 (0.1516)	2.8699 (1.8586)	0.1062 (0.0504)	0.4930 (0.1963)	3.8072 (2.2666)	0.1531 (0.0455)	0.4023 (0.2425)
	CLS	2.6490 (2.4867)	0.0992 (0.0405)	0.2653 (0.2051)	0.9865 (2.4291)	0.3490 (0.0660)	0.1295 (0.2015)	3.1546 (1.3637)	0.0987 (0.0403)	0.5041 (0.1525)	4.1556 (1.7724)	0.1500 (0.0371)	0.3939 (0.1744)
	CML	2.0663 (1.1626)	0.0997 (0.0173)	0.2957 (0.1057)	0.9423 (0.1851)	0.3520 (0.0267)	0.1003 (0.0269)	2.9861 (0.4948)	0.1021 (0.0175)	0.4952 (0.0663)	4.0454 (0.7482)	0.1506 (0.0169)	0.3965 (0.0846)
2000	*YW*	2.4006 (2.6566)	0.0992 (0.0374)	0.2822 (0.2108)	1.0003 (2.2004)	0.3492 (0.0638)	0.1008 (0.1278)	3.0555 (1.6407)	0.1028 (0.0424)	0.4888 (0.1781)	3.9659 (1.9366)	0.1501 (0.0393)	0.4015 (0.2142)
	CLS	2.6512 (2.4281)	0.0974 (0.0322)	0.2759 (0.1891)	1.1509 (2.0610)	0.3481 (0.0576)	0.1189 (0.1618)	3.1361 (1.3094)	0.1001 (0.0374)	0.5021 (0.1373)	4.1674 (1.5758)	0.1496 (0.0351)	0.3965 (0.1597)
	CML	2.0934 (1.0927)	0.0999 (0.0145)	0.2957 (0.0958)	0.8964 (0.1881)	0.3512 (0.0235)	0.0995 (0.0233)	3.0024 (0.4292)	0.1025 (0.0151)	0.4925 (0.0582)	3.9889 (0.6421)	0.1496 (0.0141)	0.4012 (0.0750)
3000	*YW*	2.3562 (2.3901)	0.0981 (0.0304)	0.2888 (0.1920)	1.0233 (1.8678)	0.3505 (0.0547)	0.0975 (0.1129)	2.9301 (1.4245)	0.1044 (0.0411)	0.4935 (0.1460)	4.0192 (1.9367)	0.1508 (0.0346)	0.3943 (0.1958)
	CLS	2.3837 (2.1091)	0.1002 (0.0278)	0.2782 (0.1492)	0.9378 (1.9693)	0.3468 (0.0490)	0.1391 (0.1651)	3.0970 (1.1732)	0.0991 (0.0317)	0.5075 (0.1220)	4.2652 (1.4337)	0.1490 (0.0289)	0.3885 (0.1308)
	CML	2.0753 (0.8541)	0.0996 (0.0117)	0.2971 (0.0744)	0.8212 (0.2417)	0.3503 (0.0216)	0.1001 (0.0191)	2.9943 (0.3619)	0.1022 (0.0125)	0.4962 (0.0488)	4.0081 (0.5410)	0.1499 (0.0118)	0.3999 (0.0615)

Table 4.3. Sample mean and root mean square error RMSE (in bracket) for Model 3.

T.V	3.5	0.1	0.15	2		1	0.45	0.1	1	
Size	$\widehat{\alpha}_{0,1}$	$\widehat{\alpha}_{1,1}$	$\widehat{\beta}_1$	$\widehat{r}_1$	*median*	$\widehat{\alpha}_{0,2}$	$\widehat{\alpha}_{1,2}$	$\widehat{\beta}_2$	$\widehat{r}_2$	*median*
500	3.3854 (0.9416)	0.0945 (0.0430)	0.1671 (0.1549)	2.0989 (0.3409)	2	0.8321 (0.4026)	0.4455 (0.1055)	0.1135 (0.0502)	1.0384 (0.1708)	1
700	3.4609 (0.7829)	0.0999 (0.0357)	0.1518 (0.1290)	2.0539 (0.2695)	2	0.8965 (0.3601)	0.4536 (0.0955)	0.1113 (0.0429)	1.0254 (0.1366)	1
1000	3.4921 (0.6866)	0.0965 (0.0299)	0.1523 (0.1110)	2.0548 (0.2308)	2	0.9020 (0.3107)	0.4523 (0.0763)	0.1102 (0.0351)	1.0186 (0.1096)	1
1500	3.4718 (0.5527)	0.0982 (0.0248)	0.1510 (0.0879)	2.0334 (0.1786)	2	0.9085 (0.2799)	0.4485 (0.0614)	0.1103 (0.0296)	1.0122 (0.0897)	1
2000	3.4643 (0.4809)	0.0992 (0.0221)	0.1537 (0.0784)	2.0265 (0.1507)	2	0.9510 (0.2602)	0.4506 (0.0545)	0.1106 (0.0264)	1.0100 (0.0791)	1
3000	3.4934 (0.3952)	0.0993 (0.0175)	0.1520 (0.0625)	2.0117 (0.1202)	2	0.9974 (0.2418)	0.4992 (0.0456)	0.1103 (0.0225)	1.0053 (0.0654)	1
T.V	2.5	0.6	0.25	2		2	0.25	0.1	3	
Size	$\widehat{\alpha}_{0,3}$	$\widehat{\alpha}_{1,3}$	$\widehat{\beta}_3$	$\widehat{r}_3$	*median*	$\widehat{\alpha}_{0,4}$	$\widehat{\alpha}_{1,4}$	$\widehat{\beta}_4$	$\widehat{r}_4$	*median*
500	2.3470 (0.9039)	0.5827 (0.1436)	0.2728 (0.1632)	2.0982 (0.3288)	2	1.9092 (0.6265)	0.2424 (0.0533)	0.1095 (0.0921)	3.1484 (0.5231)	3
700	2.3958 (0.7320)	0.5937 (0.1188)	0.2654 (0.1319)	2.0649 (0.2722)	2	1.9590 (0.5286)	0.2436 (0.0463)	0.1040 (0.0726)	3.1144 (0.4212)	3
1000	2.4655 (0.6067)	0.5963 (0.0996)	0.2561 (0.1044)	2.0353 (0.2060)	2	1.9516 (0.4478)	0.2462 (0.0383)	0.1040 (0.0665)	3.0789 (0.3423)	3
1500	2.5038 (0.4938)	0.5955 (0.0824)	0.2505 (0.0814)	2.0243 (0.1760)	2	1.9992 (0.3739)	0.2475 (0.0291)	0.1027 (0.0521)	3.0324 (0.2531)	3
2000	2.4844 (0.4219)	0.5965 (0.0688)	0.2522 (0.0712)	2.0220 (0.1522)	2	1.9769 (0.3188)	0.2477 (0.0262)	0.1008 (0.0446)	3.0401 (0.2180)	3
3000	2.4818 (0.3370)	0.5982 (0.0576)	0.2510 (0.0578)	2.0160 (0.1166)	2	1.9855 (0.2555)	0.2493 (0.0221)	0.1017 (0.0365)	3.0209 (0.1845)	3

4.7.2 Real data study

In this paragraph, we consider the data set of size 140 observations, recorded for the number of infections by Campylobacteriosis in Quebec-Canada, collected every 28 days, starting from January 1990 to October 2000. This time series was presented by Ferland *et al.* (2006), who proposed the $INGARCH$ model. The visualization of the considered time series is shown in Figure 4.3, while Table 4.4 summarizes some descriptive statistics.

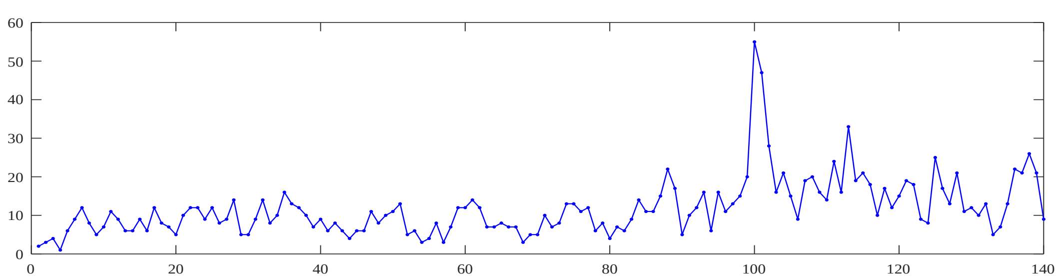

Figure 4.3. Trajectory of the Campylobacteriosis time series.

Table 4.4. Descriptive statistics for the Campylobacteriosis time series.

Sample size	Minimum	Maximum	Median	Mean	Variance	Skewness	Kurtosis
140	1	55	10	10.6929	55.5237	2.4981	13.2290

A look at Table 4.4, the Campylobacteriosis time series is over dispersed, therefore, our proposed model appears to be appropriate to fit the underlying time series.

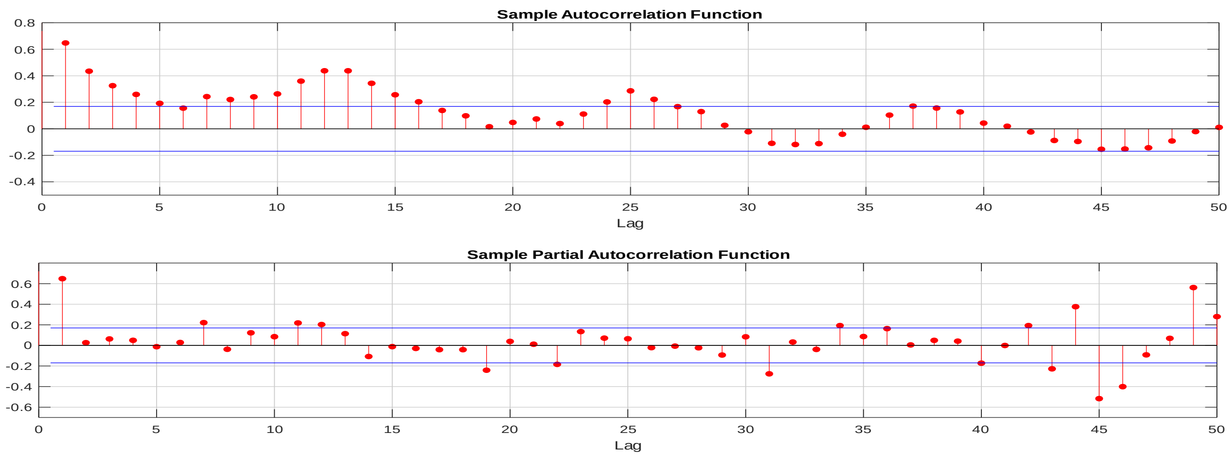

Figure 4.4. ACF and PACF of the Campylobacteriosis time series.

Furthermore, by analyzing the empirical autocorrelation function (ACF) and the empirical partial autocorrelation function ($PACF$) in Figure 4.4, the time series presents periodicity of season $S = 13$, due to the nature of data which is collected every 28 days, that makes 13 seasons per year. Table 4.5 gives the estimation results, where the estimated parameters $(r_s,\ s = 1, ..., 13)$ of each period, by maximizing the conditional Log-likelihood function :

Table 4.5. Estimation results using the $PNBINGARCH_{13}(1,1)$.

s	1	2	3	4	5	6	7	8	9	10	11	12	13
$\alpha_{0,s}$	.0152	.0358	.0125	.0120	.0297	.1596	.0604	.0144	.0340	.0834	.1015	.0565	.0115
$\alpha_{1,s}$	.0026	.0625	.0005	.0203	.0030	.0156	.0083	.0011	.2046	.0157	.0007	.0061	.0001
β_s	.7968	.0057	.4222	.8257	.3808	.3345	.9874	.5591	.3908	.0128	.4990	.5833	.6227
r_s	47	13	27	24	41	29	21	42	6	40	46	34	42

We can see that the $PNBINGARCH_{13}(1,1)$ model fits adequately the data series, which is confirmed by the empirical residual autocorrelation function. Indeed, Figure 4.5 shows that the residuals of our model do not indicate any statistical significant autocorrelation. So, the adequacy of the model is not statistically rejected. Moreover, one can easily observe that the periodic behavior of the residual autocorrelation for the fitted model $PNBINGARCH_{13}(1,1)$

has been completely disappeared thus the periodic feature has been taken into account by this model.

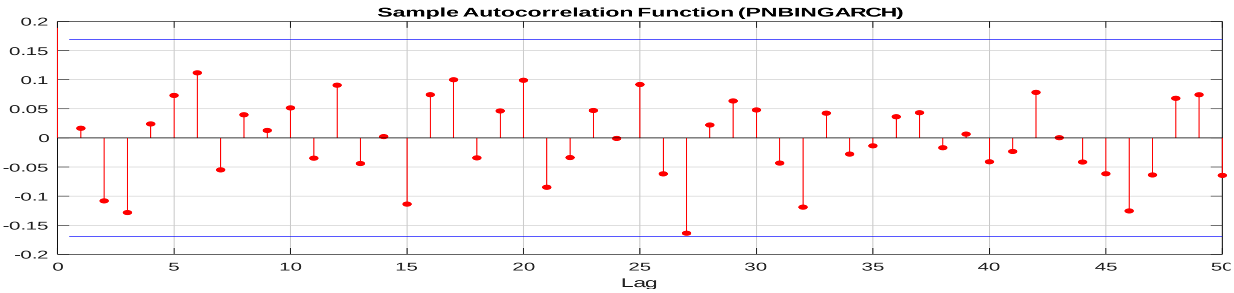

Figure 4.5. Autocorrelation of the residual time series.

The proposed model $PNBINGARCH_{13}(1,1)$ shows an improvement, in terms of residuals, comparing to the $INGARCH(1,13)$ model (Ferland *et al.* 2006), the $NBINGARCH(1,1)$ model (Zhu 2011) and also the $PINGARCH_{13}(1,1)$ model (Bentarzi and Bentarzi 2017*b*). This fact is clearly visible through Table 4.6 which illustrates the computed Sum of Squared Errors (SSE) for the four models.

Table 4.6. Computed SSE for each model.

Model	$INGARCH(1,13)$	$NBINGARCH(1,1)$	$PINGARCH_{13}(1,1)$	$PNBINGARCH_{13}(1,1)$
SSE	4373	4238	2882	2411

Furthermore, Figure 4.6 exhibits the adjusted series for the proposed model and also for the $NBINGARCH(1,1)$ model (Zhu 2011), such as the series values are shown in blue, while the red line denotes the adjusted series. The fitted values of the $PNBINGARCH_{13}(1,1)$ model seem to be suitable for the real data set values. It should be noted that the size of our time series is minimal to estimate such a number (52) of parameters, therefore the selected model can be optionally improved to a larger size.

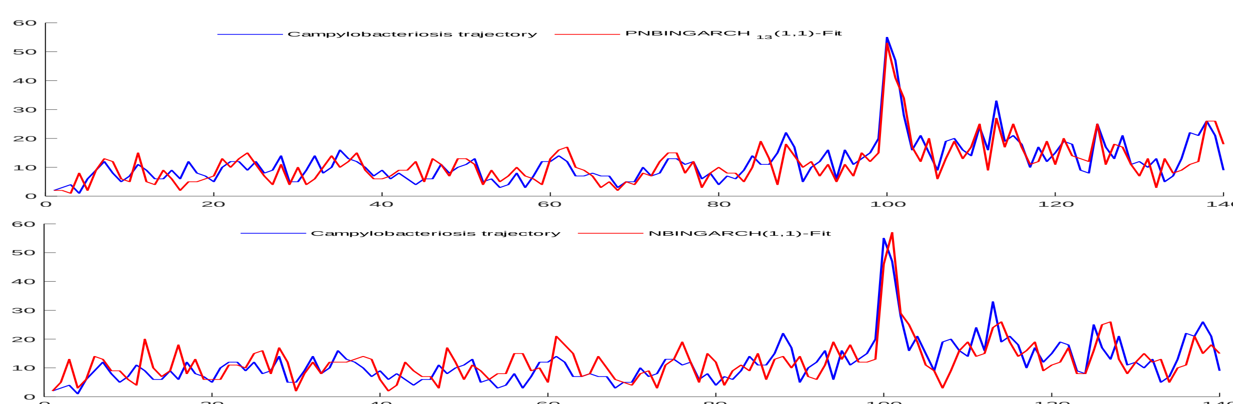

Figure 4.6. An adjusted trajectory proposed to the Campylobacteriosis time series.

Chapter 5

Conclusion and perspectives

The scope of this thesis was to present three integer-valued time series models with periodic coefficients that play a key role in the description of many stylized facts encountered in several count fields. In the second chapter, we have studied the probabilistic and statistical properties of the periodic self exciting threshold integer valued autoregressive $PSETINAR(2;1)$ model of order one with two regimes, based on the binomial thinning operator, driven by a periodic sequence of independent Poisson distributed random variables. In addition, in order to determine the unknown threshold parameters, we have adapted the Nested Sub-Sample Search ($NeSS$) algorithm, which is very useful in practice to estimate the underlying parameters of the concerned model. Moreover, we have successfully applied the $PSETINAR_7(2;1)$ model to fit the number of births in Quebec-Canada, for the year 1986. In the third chapter, we have established also the probabilistic and statistical properties of the periodic negative binomial self exciting threshold integer valued autoregressive $PNBSETINAR(2;1)$ model of order one with two regimes, based on the negative binomial thinning operator, which makes the proposed model more flexible to describe, more than the piece-wise phenomenon and the overdispersion of discrete-valued time series, the periodicity exhibited by the autocovariance structure. Furthermore, we have explored the $PNBSETINAR_7(2;1)$ model on real dataset, the number of daytime road accidents in the Schiphol area, and a comparison with existing models, shows the adequacy of the proposed model to capture the periodicity hidden in the autocovariance structure. The fourth chapter was devoted to the presentation and study the probabilistic and statistical properties of the periodic negative binomial $INGARCH\,(1,1)$ model which makes it possible to deal with overdispersion and potential extreme observations on the one hand, and the periodicity feature on the other hand, simultaneously. Last but not least, we applied the model to a real dataset, namely, Number of cases of campylobacterosis infections from January 1990 to the end of October 2000, and a comparison with existing works, considering the same dataset, showed an improvement of the proposed model. The research that was undertaken in this thesis highlighted a number of topics on which further research could be helpful. Based on the work presented in this thesis, it would be fruitful to give some perspectives and possible generalizations for future research.

i. In Chapter 2 , the high moments of the model were not calculated explicitly in terms of parameters, which would give a better impression to find applicable techniques in this case. Also, the proposed class of model can be extended to a more general one, $PSETINARMA\,(k; p_1, ..., p_k, q_1, ..., q_k)$ with an arbitrary orders of autoregressive and moving average as well as the number of thresholds.

ii. In Chapter 3 , we can hold the same views as we did in Chapter 2 . Besides, one can also define another class of models with periodic coefficients for the negative binomial integer-valued autoregressive in an arbitrary order p, or more generally, the periodic negative binomial integer-valued autoregressive moving average $INARMA(p, q)$.

iii. In Chapter 4 , it will be wise of us to try and find, in our future consideration, more positive ways to achieve more results from our work. Indeed, by taking into consideration its probabilistic and statistical features, the suggested model may be extended into the periodic negative binomial $INGARCH(p, q)$ model with arbitrary orders. Furthermore, we may suggest the periodic negative binomial $INARCH(p)$ model as a particular case of this class.

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