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SUR QUELQUES PROBLEMES INVERSES : RESULTATS D’IDENTIFIABILITE DE STABILITE ET DE RECONSTRUCTION

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Les problèmes inverses suscitent de nos jours l'attention de nombreux scientifiques et ingénieurs et ont un impact grandissant dans différent domaines par leurs applications concrètes qui commencent à apparaitre. Parmi ces applications, on peut citer la détection des défauts géométriques de structures (cavité, fissure, inhomogénéite, surface de décollement) en vue de l'intégrité de ces structures à long terme, la détermination des points de source (application à la pollution des eaux de surface, et en sismologie).

Un problème inverse suppose que le problème direct est bien posé, en d'autres termes, étant donné l'état d'un système, nous avons une description mathématique classique de cet état incluant l'existence , l'unicité de la solution correspondant au problème mathématique ainsi que la stabilité et la continuité par rapport aux données. Mais si l'une des fonctions ou paramétres décrivant l'état est à déterminer en fonction d'une donnée supplementaire (mesure au bord ou à l'interieur du domaine) , on aboutit alors un problème inverse. De tels types de problèmes existent pour des équations elliptiques, paraboliques ou hyperboliques. La théorie mathématique sous jacente est ‘ l'identifiabilité ” (unicité de la solution du problème inverse), les conditions de ‘ stabilité” (continuité de l'opérateur qui à une mesure associe l'inconnue (géométrie du domaine ou le terme source)).

Dans ce travail nous proposons l'étude de deux types de problèmes inverses; problème inverse d'identification de fissures et problème inverse d'identification du terme source. Les techniques utilisées sont, l'analyse harmonique et dérivation de domaine pour le premier type de problème, les résultats de contrôlabilité et les propriétés des opérateurs intégraux de Volterra pour le second. Dans ce contexte, notre travail est composé de quatre chapitres dont le contenu se résume comme suit.

Dans le premier chapitre [57], nous étudions un problème inverse d'identification de fissure débouchante dans un domaine Ω occupé par une structure plane hétérogène. Le problème considéré rentre dans le cadre modèle du Laplacien (qui modélise les phénomènes de conduction stationnaire thermique ou électrique). Nous prouvons un résultat d'identifiabilité par une mesure unique au bord correspondant à un choix particulier de flux. Le résultat utilisé est le théorème d'Holmgren. Dans le cas des fissures droites et grâce aux techniques de dérivation de domaine nous établissons un résultat de stabilité en distinguant, la stabilité par rapport à une variation de l'angle, et celle par rapport à une variation de longueur. Enfin un résultat d'approximation pour l'équation de la chaleur est obtenu.
Les chapitres deux, trois et quatre traitent l'étude de problèmes inverses d'identification de termes sources.Les problèmes considerés correspondent à l'équation des ondes pour les deux premiers et au système de Petrovsky pour le dernier. Notons que les résultats utilisés pour ces trois chapitres sont des résultats de contrôlabilité et les propriétés des opérateurs intégraux de Volterra.
Le deuxième chapitre [58], traite de l'identifiabilité, la stabilité et la reconstruction de points de sources dans un arbre hétérogène (multistructure unidimensionelle). L'identifiabilité est obtenue grâce à l'inégalité d'observabilité démontrée par la méthode des multiplicateurs dans ce type de domaine, les propriétés de l'opérateur intégral de Volterra permettent alors de conclure. La stabilité utilise la construction explicite de fonctions tests appropriées. En se basant sur un résultat de contrôlabilité exacte et la décomposition spectrale, on obtient un schéma de reconstruction de ces points de sources
Dans le troisième chapitre [59]on étudie l'identifiabilité, la stabilité et la reconstruction de points de source par observation interieure d'un domaine borné de $\mathbb{R}^n, n \geq 1$. Pour cela la définition d'ensemble stratégique est introduite et des exemples sont présentés. On montre que ces points de sources sont determinés de manière unique par une mesure sur ce type d'ensemble. Par des techniques similaires aux chapitre deux, nous donnons un résultat de stabilité pour des termes sources particuliers ainsi qu'un schéma de reconstruction de ces points.
Dans le quatrième chapitre [60] on considére le problème correspondant au système de Petrovsky en dimension une avec différentes conditions au bord. On obtient des résultats analogues aux deux chapitres précédents, en utilisant la même démarche.

Chapter 1

Identifiability and stability results of one emerging crack in heteregeneous media by one boundary measurements

1.1 Introduction

A practical and theoretical very interesting inverse problem concerns the problem of determination of cracks by overdetermined boundary measurements and recently has been the subject of great interest [31, 3, 4, 8, 9]. More precisely, the problem consists in finding the shape and the location of cracks inside a body by applying heat fluxes (or current fluxes) on the boundary and measuring the induced temperatures (or potentials) on the boundary (or a part of it). The three main steps are usually the identifiability (unique solvability of the problem), the stability (small perturbations of the data give rise to small perturbations of the cracks) and finally the reconstruction (build appropriate processes in order to find a good approximation of the unknowns). This last step having a meaning once the two first ones are established.

The identifiability problem has started with the pioneering work of Friedman-Vogelius [31] for a single buried crack, where they showed that the crack is uniquely determined by two measurements when the conductivity coefficient a is smooth. This type of results were extended to the case of multiple buried cracks by Bryan-Vogelius [20] and by Alessandrini and others for nonsmooth conductivity coefficient [4] and finally by Andrieux-Ben Abda-Jaoua [9] for one emerging crack with the help of one measurement when the conductivity coefficient is constant.

The stability property were first considered in [31] where they gave a Lipschitz bound for the line containing the linear crack for a constant conductivity coefficient; for the same problem, a global Lipschitz continuity result were established in [3] under some conditions on the set of admissible cracks (see also [2] for weaker results). At the end, a local Lipschitz stability result for one emerging linear crack with one measurement and a constant was proved in [9].

For numerical reconstruction algorithm, we may cite the works [64, 47, 21, 8, 10, 11].

The aim of this chapter is twofold: first we prove the identifiability result for piecewise constant conductivity coefficients (model of composite materials and usually called interface problems [46, 62]) for one emerging crack with the help of one measurement on a piece of the boundary and special heat flux. We secondly establish local Lipschitz continuity for one linear crack, extending the ideas of [9]. Finally we show that we can have an approximation of any emerging crack for the non stationary heat equation for an appropriate flux which actually gives an approximation of the flux of the stationary problem.

1.2 Identifiability

Let Ω be a bounded connected open set of the plane with a Lipschitz boundary Γ. We assume that the conductivity coefficient a is piecewise constant in Ω, more precisely

$$a = a_1\chi_{\Omega_1} + a_2\chi_{\Omega_2},$$

where Ω_1 and Ω_2 are two connected open sets with Lipschitz boundaries such that

$$\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}, \bar{\Omega}_1 \cap \bar{\Omega}_2 = I,$$

where I is a C^1 curve usually called the interface between Ω_1 and Ω_2 and a_1, a_2 are positive real numbers such that $a_1 \neq a_2$. The above asumption was chosen for simplicity and could be easily extended to multicomponents $\Omega_j, j = 1, \cdots, J$, with $J \in \mathbb{N}^*$.

We suppose that Ω contains exactly one perfectly conducting crack σ emerging at a known point S of the boundary Γ. Without loss of generality we can suppose that $S \in \partial\Omega_1$. In the whole paper, a crack is supposed to be a C^1 non self-intersecting curve and Ω_σ will mean the domain $\Omega \setminus \sigma$. The boundary Γ will be parametrized by the arclength s with origin at S.

To describe the boundary current flux ϕ that we shall use, let us fix three points P, Q, R on Γ such that $0 < s(R) < s(Q) < s(P)$ and such that $\Gamma \setminus PR$ contains a part of $\partial\Omega_i$ of positive measure, for $i = 1$ and 2 (see Figure 1.1). Define ϕ by

$$\phi(x) = \begin{cases} 1 & \text{on} \qquad RQ, \\ -\frac{|RQ|}{|PQ|} & \text{on} \qquad QP, \\ 0 & \text{elsewhere.} \end{cases} \tag{1.1}$$

Note that ϕ satisfies

$$\int_\Gamma \phi(s)\, ds = 0.$$

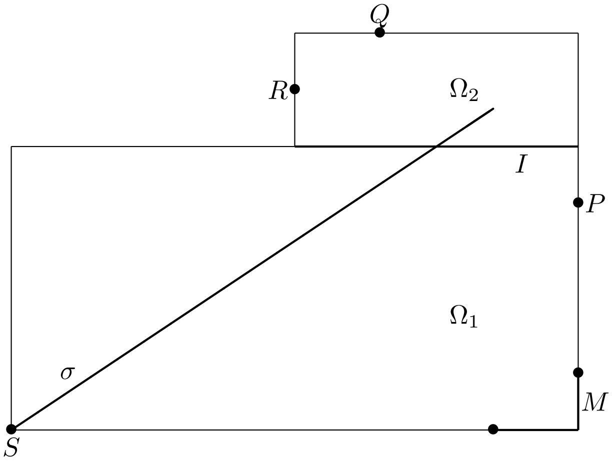

Figure 1.1: A general situation

The direct problem may be written as follows: let $u_\sigma \in H^1(\Omega_\sigma)$ be the solution of

$$\begin{cases} \text{div } (a\nabla u_\sigma) = 0 & \text{in} \quad \Omega_\sigma, \\ a\dfrac{\partial u_\sigma}{\partial n} = 0 & \text{on} \quad \sigma, \\ a\dfrac{\partial u_\sigma}{\partial n} = \phi & \text{on} \quad \Gamma, \end{cases} \tag{1.2}$$

that we normalize by requiring that

$$\int_\Gamma u_\sigma(s)\, ds = 0. \tag{1.3}$$

The rigorous formulation of problem (1.2) is the following one: set

$$V_\sigma = \{v \in H^1(\Omega_\sigma) \text{ satisfying (1.3)}\}.$$

Then $u_\sigma \in V_\sigma$ is the unique solution of

$$\int_{\Omega_\sigma} a(x)\nabla u_\sigma(x) \cdot \nabla v(x)\, dx = \int_\Gamma \phi(s)v(s)\, ds, \forall v \in V_\sigma. \tag{1.4}$$

The temperature will be measured on a part M of the boundary Γ with a positive measure not intersecting the segment (on Γ) RP and not containing S (see Figure 1.1).

The identifiability result is summarized in the next Theorem (compare with Theorem 1 of [9])

Theorem 1.2.1 *Let σ and σ' be two emerging cracks in Ω ending at the same point S of Γ and let $u_\sigma \in V_\sigma$ (resp. $u_{\sigma'} \in V_{\sigma'}$) be the solution of (1.4) with the same flux ϕ. If*

$$u_\sigma = u_{\sigma'} \text{ on } M,$$

then $\sigma = \sigma'$.

Proof: We follow the proof of Theorem 1 of [9] with the necessary adaptations. By Proposition 2.1 of [4] there exists a stream function $\omega_\sigma \in H^1(\Omega)$ (unique up to an additive constant) that satisfies

$$\nabla w_\sigma = -(a\nabla u_\sigma)^\perp = \begin{pmatrix} a\dfrac{\partial u_\sigma}{\partial x_2} \\ -a\dfrac{\partial u_\sigma}{\partial x_1} \end{pmatrix} \text{ a.e. in } \Omega_\sigma, \tag{1.5}$$

$$\text{div }(a^{-1}\nabla w_\sigma) = 0 \text{ in } \Omega_\sigma, \tag{1.6}$$

$$w_\sigma = K \text{ on } \sigma, \tag{1.7}$$

$$w_\sigma = \varphi \text{ on } \Gamma, \tag{1.8}$$

where K is a fixed constant and φ is defined by

$$\varphi(s) = \int_0^s \phi(t)\, dt + K.$$

Similarly define $w_{\sigma'} \in H^1(\Omega)$ the solution of (1.5) to (1.8) with σ' instead of σ (with the same constant K, which is always possible because w_σ and $w_{\sigma'}$ are defined up to a constant). Let us now set

$$w = w_\sigma - w_{\sigma'}.$$

Then w satisfies

$$\begin{cases} \text{div }(a^{-1}\nabla w) = 0 & \text{in} \quad \Omega \setminus (\sigma \cup \sigma'), \\ w = 0 & \text{on} \quad M, \\ a^{-1}\frac{\partial w}{\partial n} = 0 & \text{on} \quad M. \end{cases} \tag{1.9}$$

Indeed owing to (1.5) we have

$$a^{-1}\frac{\partial w}{\partial n} = a^{-1}(\nabla w)^\perp \cdot t = (\nabla u_\sigma - \nabla u_{\sigma'}) \cdot t \text{ on } M,$$

and by the assumption that $u_\sigma = u_{\sigma'}$ on M, we get

$$a^{-1}\frac{\partial w}{\partial n} = 0 \text{ on } M.$$

From (1.9) and Proposition 2.2 of [4] (Holmgren's uniqueness theorem for second order operator in divergence form), we conclude that $w = 0$ on the external connected component of $\Omega \setminus (\sigma \cup \sigma')$ and therefore $w = 0$ on its boundary which contains $\sigma \cup \sigma'$. In other words, we have

$$w_\sigma = w_{\sigma'} = K \text{ on } \sigma \cup \sigma'. \tag{1.10}$$

Suppose now that $\sigma \neq \sigma'$. Then there exists $x_0 \in \sigma' \setminus \sigma$ which is an interior point of Ω_σ. Contrary to [9], we cannot directly apply the maximum principle to w_σ because it is not harmonic on the whole of Ω_σ. But since it is harmonic in each

$\Omega_{i\sigma}$, by the maximum principle, $w_\sigma^i = w_{\sigma|\Omega_i}$ achieves its minimum on $\partial\Omega_{i\sigma}$. We then need to distinguish between the following cases:

First case: $x_0 \in \Omega_{1\sigma}$ and w_σ^1 achieves its minimum on $\partial\Omega_{1\sigma}\backslash \overset{\circ}{I}$. In that case, that minimum is equal to $\varphi(S) = K$ and since (1.10) implies that

$$w_\sigma^1(x_0) = K,$$

w_σ^1 achieves its minimum in an interior point of $\Omega_{1\sigma}$ and therefore $w_\sigma^1 = K$ in the connected part $\tilde{\Omega}_{1\sigma}$ of $\Omega_{1\sigma}$ containing x_0. Since the boundary of $\tilde{\Omega}_{1\sigma}$ contains a part I_1 (of positive measure) of $\overset{\circ}{I}$ and the fact that w_σ is solution of (1.6), we deduce that w_σ^2 satisfies

$$\begin{cases} \Delta w_\sigma^2 = 0 & \text{in} \quad \Omega_{2\sigma}, \\ w_\sigma^2 = K & \text{on} \quad I_1, \\ \frac{\partial w_\sigma^2}{\partial n} = 0 & \text{on} \quad I_1. \end{cases}$$

By Holmgren's uniqueness theorem and a cascade effect, we conclude that

$$w_\sigma = K \text{ in } \Omega_\sigma,$$

which is impossible because $\phi \neq 0$.

Second case: $x_0 \in \Omega_{1\sigma}$ and w_σ^1 achieves its minimum at an interior point x_1 of I. Then either w_σ^2 achieves its minimum on $\partial\Omega_{2\sigma}\backslash \overset{\circ}{I}$ and the arguments of the first case lead to a contradiction, or w_σ^2 achieves its minimum on $\overset{\circ}{I}$ and then by the continuity of w_σ through I, we have

$$w_\sigma^1(x_1) = \min_{\Omega_{1\sigma}} w_\sigma^1 = \min_{\Omega_{2\sigma}} w_\sigma^2 = w_\sigma^2(x_1).$$

This implies that

$$\begin{cases} \forall x \in \Omega_{1\sigma} : w_\sigma^1(x) > w_\sigma^1(x_1), \\ \forall x \in \Omega_{2\sigma} : w_\sigma^2(x) > w_\sigma^2(x_1), \end{cases} \tag{1.11}$$

otherwise by the maximum principle, w_σ^1 and w_σ^2 would be constant which is impossible. But (1.11) and Lemma 3.4 of [32] yield

$$\frac{\partial w_\sigma^1}{\partial n_1}(x_1) < 0 \text{ and } \frac{\partial w_\sigma^2}{\partial n_2}(x_1) < 0,$$

where n_1 (resp. n_2) means the exterior normal vector along $\partial\Omega_1$ ($\partial\Omega_2$). This is again a contradiction because (1.6) implies that

$$a_1^{-1}\frac{\partial w_\sigma^1}{\partial n_1}(x_1) + a_2^{-1}\frac{\partial w_\sigma^2}{\partial n_2}(x_1) = 0.$$

The two remaining cases when $x_0 \in \Omega_{2\sigma}$ are treated similarly exchanging the indices 1 and 2. ■

1.3 Stability for straight cracks

If Σ denotes the set of admissible cracks, the above identifiability result means that the application

$$\eta : \Sigma \longrightarrow L^2(M) : \sigma \longrightarrow f_\sigma = u_{\sigma|\Gamma}$$

is injective. The stability means that η^{-1} is continuous once Σ is equipped with an appropriate topology. As in [9], one can prove this continuity for straight cracks when Σ is compact (for the Hausdorff metric). We now show that if the interface I and the crack σ are straight lines this map is even locally Lipschitz, by proving first Lipschitz stability with respect to the length and secondly with respect to the angle.

Following [9], we shall use domain derivative method [50, 66, 28]. Consider a family of mappings

$$F_h = Id + h\theta,$$

where $\theta \in (W^{1,\infty}(\Omega_\sigma))^2$ is such that $F_h(\Omega_\sigma) = \Omega_{\sigma_h}$ for some admissible crack σ_h and $\theta = 0$ on Γ; the parameter h being a real number. Clearly there exists $h_0 > 0$ small enough such that for all $|h| \leq h_0$, F_h is a diffeomorphism from Ω_σ into $\Omega_{\sigma_h} := F_h(\Omega_\sigma)$.

As in Theorem 1.5 of [28] (in the elasticity setting, see also [9]), we can now calculate the first derivative in the direction of θ of the solution u_σ of problem (1.2)-(1.3) with respect to the domain.

Theorem 1.3.1 *Let $u_h = u_{\sigma_h} \circ F_h$, $u_{\sigma_h} \in V_{\sigma_h}$ being the solution of*

$$\int_{\Omega_{\sigma_h}} a(F_h^{-1}(y))\nabla u_{\sigma_h}(y) \cdot \nabla v(y)\, dy = \int_\Gamma \phi(s)v(s)\, ds, \forall v \in V_{\sigma_h}. \tag{1.12}$$

Then it admits the expansion

$$u_h = u_\sigma + hu_1 + o(h), \tag{1.13}$$

where $u_1 \in V_\sigma$ *is the unique solution of*

$$\begin{aligned}\int_{\Omega_\sigma} a\nabla u_1 \cdot \nabla v\,dx &= \int_{\Omega_\sigma} a\left[(\frac{\partial\theta}{\partial M} + (\frac{\partial\theta}{\partial M})^t)\nabla u_\sigma\right]\cdot\nabla v\,dx \\ &- \int_{\Omega_\sigma} a(\nabla u_\sigma\cdot\nabla v)\mathrm{div}\,\theta\,dx, \forall v \in V_\sigma,\end{aligned} \tag{1.14}$$

where $\frac{\partial\theta}{\partial M}$ *is the Jacobian matrix of* θ *and the mapping*

$$h \to o(h)$$

is C^1 *from* $]-h_0, h_0[$ *to* V_σ *and satisfies*

$$\lim_{h\to 0}\frac{o(h)}{h} = 0. \tag{1.15}$$

Proof: Starting from the variational problem (1.12) and performing the change of variables $y = F_h(x)$, u_h is solution of

$$a_h(u_h, v) = \int_\Gamma \phi(s)v(s)\,ds, \forall v \in V_\sigma, \tag{1.16}$$

where

$$a_h(u,v) = \int_{\Omega_\sigma} a(x)((I+h\frac{\partial\theta}{\partial M})^{-1}\nabla u)\cdot((I+h\frac{\partial\theta}{\partial M})^{-1}\nabla v)(1+h\mathrm{div}\,\theta+h^2\det\frac{\partial\theta}{\partial M})\,dx.$$

As the matrix $(I + h\frac{\partial\theta}{\partial M})^{-1}$ admits the expansion

$$(I + h\frac{\partial\theta}{\partial M})^{-1} = \sum_{k=0}^{\infty}(-1)^k h^k (\frac{\partial\theta}{\partial M})^k, \tag{1.17}$$

for h small enough, the bilinear form is holomorphic with respect to h (for h small enough). Therefore for h small enough, the operator $A_h \in \mathcal{L}(V_\sigma, V'_\sigma)$ associated with a_h (in the sense that $A_h u(v) = a_h(u,v)$, for all $u, v \in V_\sigma$) admits the expansion

$$A_h = A_0 + hA_1 + R_h,$$

where $A_0, A_1, R_h \in \mathcal{L}(V_\sigma, V'_\sigma)$ satisfy

$$\begin{aligned} &A_0 u(v) = \int_{\Omega_\sigma} a(x)\nabla u(x)\cdot\nabla v(x)\,dx, \forall u, v \in V_\sigma, \\ &\|R_h\|_{\mathcal{L}(V_\sigma, V'_\sigma)} \leq Ch^2, \end{aligned}$$

for some positive constant C. Using Neumann's series, we get for h small enough that A_h is invertible and satisfies

$$A_h^{-1} = A_0^{-1} + hA_0^{-1}A_1A_0^{-1} + R'_h, \tag{1.18}$$

where $R'_h \in \mathcal{L}(V_\sigma, V'_\sigma)$ is a remainder such that

$$\|R'_h\|_{\mathcal{L}(V_\sigma, V'_\sigma)} \leq C'h^2,$$

for some positive constant C'. As (1.16) means that

$$u_h = A_h^{-1}F$$

with $F \in V'_\sigma$ defined by

$$F(v) = \int_\Gamma \phi(s)v(s)\,ds, \forall v \in V_\sigma,$$

the identity (1.18) implies the expansion (1.13) for u_h.

It then remains to show that $u_1 = A_0^{-1}A_1A_0^{-1}F$ satisfies (1.14). For that purpose, we insert the expansion (1.13) and (1.17) in problem (1.16) to get

$$\begin{aligned} \int_{\Omega_\sigma} a(x)[\nabla u_\sigma + h\text{div}\,\theta\nabla u_\sigma \quad &+ \quad h\{\nabla u_1 - (\frac{\partial\theta}{\partial M} + (\frac{\partial\theta}{\partial M})^t)\nabla u_\sigma\} + r_h]\cdot\nabla v\,dx \\ &= \quad \int_\Gamma \phi(s)v(s)\,ds, \forall v \in V_\sigma, \end{aligned}$$

where r_h is a remainder satisfying

$$\|r_h\|_{L^2(\Omega_\sigma)} \leq Ch^2,$$

for some $C > 0$. As u_σ satisfies (1.4), the above identity becomes

$$\int_{\Omega_\sigma} a(x)\left[\text{div}\,\theta\nabla u_\sigma + \{\nabla u_1 - (\frac{\partial\theta}{\partial M}) + (\frac{\partial\theta}{\partial M})^t)\nabla u_\sigma\} + h^{-1}r_h\right]\cdot\nabla v\,dx = 0, \forall v \in V_\sigma.$$

This implies (1.14) by taking the limit on h. ■

In view of (1.13), we directly have for all $h, h' \in]-h_0, h_0[$:

$$\|f_{\sigma_h} - f_{\sigma_{h'}}\|_{L^2(M)} = \|u_h - u_{h'}\|_{L^2(M)} = \|(h-h')u_1 + o(h) - o(h')\|_{L^2(M)}. \quad (1.19)$$

Consequently, if $\|u_1\|_{L^2(M)} > 0$, then by the C^1-regularity of $o(h)$ and the property (1.15), there exists $h_1 > 0$ (depending on u_σ) such that for all $h, h' \in]-h_1, h_1[$, one has

$$|h-h'|\|u_1\|_{L^2(M)} \leq 2\|f_{\sigma_h} - f_{\sigma_{h'}}\|_{L^2(M)}. \quad (1.20)$$

Indeed (1.19) and trace theorem yield

$$\|f_{\sigma_h} - f_{\sigma_{h'}}\|_{L^2(M)} \geq |h-h'|\|u_1\|_{L^2(M)} - c\|o(h) - o(h')\|_{H^1(\Omega_\sigma)}.$$

By the finite increment theorem, there exists $\xi \in]h', h[$ (if $h' < h$) such that

$$\|f_{\sigma_h} - f_{\sigma_{h'}}\|_{L^2(M)} \geq |h-h'|(\|u_1\|_{L^2(M)} - c\|o'(\xi)\|_{H^1(\Omega_\sigma)}).$$

By the property (1.15), it is then always possible to chose h_1 small enough such that

$$\|u_1\|_{L^2(M)} - c\|o'(\xi)\|_{H^1(\Omega_\sigma)} \geq \frac{1}{2}\|u_1\|_{L^2(M)}, \forall h, h' \in]-h_1, h_1[,$$

which proves (1.20).

If we set

$$\Sigma_1 = \{\sigma_h = F_h(\sigma), \forall |h| < h_1\}, D_1 = \eta(\Sigma_1),$$

then the estimate (1.20) means that the mapping

$$\eta^{-1}{}_{|D_1} : f_\sigma \to \sigma,$$

is Lipschitz continuous. Therefore to prove the local Lipschitz continuity of η^{-1}, we first show that u_1 does not vanish identically on M for appropriate choices of θ.

1.3.1 Stability with respect to the length

Let σ be a straight crack emerging at $S \in \Gamma$ and denote by $T \in \Omega$ its second extremity (T can belong to the interface). In that case, we take θ satisfying $\theta \equiv (1, 0)$ in a fixed connected neighbourhood of T (in the Euclidean coordinates x_1, x_2 centered at S and such that the positive x_1-axis contains the crack), vanishing in a neighbourhood of Γ and such that $\theta \cdot n = 0$ on σ. We now distinguish the case T not on the interface to the case T on the interface.

If $T \notin I$, then by Theorem 4.4.4.13 of [33], u_σ admits a decomposition into a regular part $u_{\sigma R} \in H^2(\Omega_\sigma)$ and a singular one

$$c_\sigma r^{1/2} \cos(\frac{\varphi}{2}),$$

where c_σ is a constant (the so-called stress intensity factor), when (r, φ) are polar coordinates centered at T with $\varphi \in [0, 2\pi]$, the half-lines $\varphi = 0$ and $\varphi = 2\pi$ containing the crack σ. In other words, we have

$$u_\sigma = u_{\sigma R} + c_\sigma r^{1/2} \cos(\frac{\varphi}{2}). \tag{1.21}$$

Theorem 1.3.2 *If $T \notin I$ and if the coefficient c_σ is different from 0, then u_1 does not vanish on the whole of M.*

Proof: We argue as in Theorem 3 of [9]: Let $\mathcal{W}_\sigma$ be the potential energy of problem (1.4):

$$\mathcal{W}_\sigma = \frac{1}{2} \int_{\Omega_\sigma} a |\nabla u_\sigma|^2 \, dx.$$

Then as in [28] and [29], we can show that

$$\frac{\partial \mathcal{W}_\sigma}{\partial \Omega}(\Omega_\sigma) \cdot \theta = \int_\Gamma \phi u_1 \, ds, \tag{1.22}$$

$$\frac{\partial \mathcal{W}_\sigma}{\partial \Omega}(\Omega_\sigma) \cdot \theta = -\frac{\pi c_\sigma^2 a(T)}{4}, \tag{1.23}$$

where $\frac{\partial \mathcal{W}_\sigma}{\partial \Omega}(\Omega_\sigma) \cdot \theta$ is the domain derivative of the energy in the direction of θ, i. e.,

$$\frac{\partial \mathcal{W}_\sigma}{\partial \Omega}(\Omega_\sigma) \cdot \theta = \lim_{h \to 0} \frac{\mathcal{W}_{\sigma_h} - \mathcal{W}_\sigma}{h},$$

where $\mathcal{W}_{\sigma_h}$ is the potential energy of problem (1.12):

$$\mathcal{W}_{\sigma_h} = \frac{1}{2} \int_{\Omega_{\sigma_h}} a(F_h^{-1}(x)) |\nabla u_{\sigma_h}|^2 \, dx.$$

For the first identity using (1.4) and (1.12) with $v = u_\sigma$ and $v = u_{\sigma_h}$ respectively and taking the difference we obtain

$$\frac{\partial \mathcal{W}_\sigma}{\partial \Omega}(\Omega_\sigma) \cdot \theta = \lim_{h \to 0} \frac{1}{2h} \int_\Gamma \phi(s) \{u_{\sigma_h}(s) - u_\sigma(s)\} \, ds.$$

Using the fact that $u_{\sigma_h} = u_h$ on Γ and the expansion (1.13), we arrive at (1.22).

For the second identity we start with (1.22) and use (1.4) with $v = u_1$ (which belongs to V_σ) to obtain

$$\frac{\partial \mathcal{W}_\sigma}{\partial \Omega}(\Omega_\sigma) \cdot \theta = \int_{\Omega_\sigma} a(x) \nabla u_\sigma(x) \cdot \nabla u_1(x)\, dx.$$

Thanks to (1.14) with $v = u_\sigma$, we get

$$\frac{\partial \mathcal{W}_\sigma}{\partial \Omega}(\Omega_\sigma) \cdot \theta = \int_{\Omega_\sigma} a\{ \left[(\frac{\partial \theta}{\partial M} + (\frac{\partial \theta}{\partial M})^t) \nabla u_\sigma \right] \cdot \nabla u_\sigma \, dx - |\nabla u_\sigma|^2 \text{div } \theta \}\, dx.$$

If for $r > 0$ small enough, we denote by $\Omega_{\sigma r} = \Omega_\sigma \setminus B(T, r)$, then the above identity implies that

$$\frac{\partial \mathcal{W}_\sigma}{\partial \Omega}(\Omega_\sigma) \cdot \theta = \lim_{r \to 0} \int_{\Omega_{\sigma r}} a\{ \left[(\frac{\partial \theta}{\partial M} + (\frac{\partial \theta}{\partial M})^t) \nabla u_\sigma \right] \cdot \nabla u_\sigma \, dx - |\nabla u_\sigma|^2 \text{div } \theta \}\, dx.$$

Applying Green's formula in $\Omega_{i\sigma} \setminus B(T, r)$, for $i = 1, 2$, we get

$$\begin{aligned} \frac{\partial \mathcal{W}_\sigma}{\partial \Omega}(\Omega_\sigma) \cdot \theta &= \lim_{r \to 0} \left\{ \int_{\Omega_{\sigma r}} a[-2 \sum_{i,j=1,2} \theta_i \partial_j (\partial_i u_\sigma \partial_j u_\sigma) + \sum_{i,j=1,2} \theta_j \partial_j (\partial_i u_\sigma)^2]\, dx \right. \\ &+ \left. \int_{\partial B(T,r)} a[2 \sum_{i,j=1,2} \theta_i n_j \partial_i u_\sigma \partial_j u_\sigma - \sum_{i,j=1,2} \theta_j n_j (\partial_i u_\sigma)^2]\, ds \right\}, \end{aligned}$$

where for shortness we have written ∂_j for $\frac{\partial}{\partial x_j}$. Note that there is no interface term since $\theta \equiv 0$ on I. By Leibniz's rule the above expression becomes

$$\begin{aligned} \frac{\partial \mathcal{W}_\sigma}{\partial \Omega}(\Omega_\sigma) \cdot \theta &= \lim_{r \to 0} \left\{ -2 \int_{\Omega_{\sigma r}} a(\theta \cdot \nabla u_\sigma) \Delta u_\sigma \, dx \right. \\ &+ \left. \int_{\partial B(T,r)} a[2(\theta \cdot \nabla u_\sigma) \frac{\partial u_\sigma}{\partial n} - \theta \cdot n |\nabla u_\sigma|^2]\, ds \right\}. \end{aligned}$$

As u_σ is solution of (1.2) the first term of the above right-hand side is zero. Therefore we have

$$\frac{\partial \mathcal{W}_\sigma}{\partial \Omega}(\Omega_\sigma) \cdot \theta = \lim_{r \to 0} \int_{\partial B(T,r)} a[2(\theta \cdot \nabla u_\sigma) \frac{\partial u_\sigma}{\partial n} - \theta \cdot n |\nabla u_\sigma|^2]\, ds.$$

Now we make use of the expansion (1.21) of u_σ and the special form of θ near T to get

$$\lim_{r\to 0}\int_{\partial B(T,r)} a[2(\theta\cdot\nabla u_\sigma)\frac{\partial u_\sigma}{\partial n} - \theta\cdot n|\nabla u_\sigma|^2]\,ds = -\frac{\pi c_\sigma^2 a(T)}{4}.$$

The two above identities show (1.23).

Assume now that $u_1 \equiv 0$ on M. Fix a connected neighbourhood D of Γ such that $\theta \equiv 0$ on D. Since u_1 is solution of (1.14), it satisfies

$$\begin{cases} \text{div}\,(a\nabla u_1) = 0 & \text{in} \quad D, \\ \frac{\partial u_1}{\partial n} = 0 & \text{on} \quad \Gamma. \end{cases}$$

By Proposition 2.2 of [4], u_1 is identically equal to 0 in D and then $u_1 \equiv 0$ on Γ. Owing to (1.22) and (1.23), we get $c_\sigma = 0$, which is a contradiction with our assumption. ■

The case $T \in I$ is now more complicated because the singular exponent is not equal to 1/2 in general [46, 45, 62]. For simplicity, we recall that we assume here that the interface is a straight line. We now distinguish the two following cases:

1) If σ and I are on the same line (see Figure 1.2), then by Theorem 4.2 of [62], u_σ admits the splitting

$$u_\sigma = u_{\sigma R} + c_\sigma r^{1/2}\psi(\varphi),$$

where $u^i_{\sigma R} \in H^2(\Omega_i)$, $i = 1, 2$, c_σ is a constant and ψ is the function given by

$$\psi(\varphi) = \begin{cases} a_2\cos(\varphi/2) & \text{if} \quad 0 \le \varphi \le \pi \\ a_1\cos(\varphi/2) & \text{if} \quad \pi \le \varphi \le 2\pi. \end{cases}$$

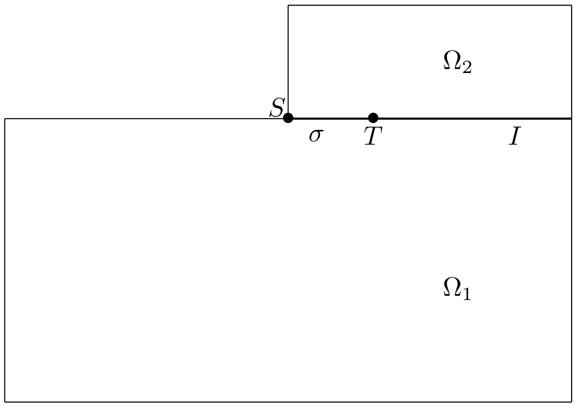

Figure 1.2: The case σ and I collinear

In that case, Theorem 1.3.2 still holds because (1.22) is always valid and (1.23) is replaced by

$$\frac{\partial \mathcal{W}_\sigma}{\partial \Omega}(\Omega_\sigma) \cdot \theta = -\gamma c_\sigma^2,$$

for some positive real number γ.

2) If σ and I are not on the same line (see Figure 1.3), then the proof of (1.23) fails since after the application of Green's formula an interface term remains which has no reason to be zero. For that reason, we have adapted the method of section 3.3.3 in [9] (stability with respect to the angle) to the above case.

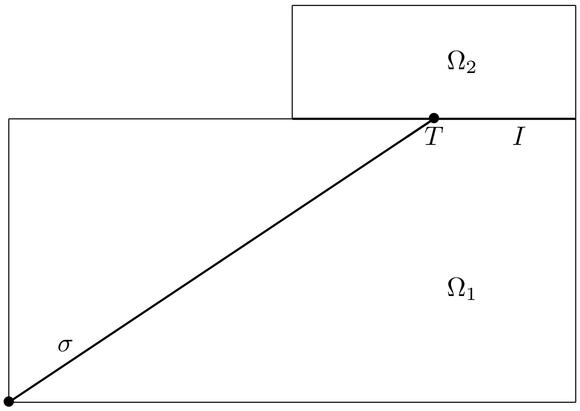

Figure 1.3: The case σ and I not collinear with $T \in I$

Theorem 1.3.3 *Assume that $T \in I$ and that σ and I are not on the same line. Let θ and $\tilde{\theta}$ be two vector fields satisfying the properties of this subsection, such that $\theta = \tilde{\theta}$ in a neighbourhood V of the crack and satisfying*

$$(\theta - \tilde{\theta}) \cdot n \neq 0 \text{ on } I. \tag{1.24}$$

Let us set u_1 (resp. $\tilde{u}_1$) the θ-derivative of u_σ (resp. $\tilde{\theta}$-derivative). Then either $u_1 \neq 0$ on M or $\tilde{u}_1 \neq 0$ on M.

Proof: Assume that $u_1 = 0$ on M and $\tilde{u}_1 = 0$ on M. Since θ is equal to zero in a neighbourhood W of the boundary Γ, and since u_1 is the solution of (1.14), we see that

$$\begin{cases} \Delta u_1^i = 0 & \text{in} \quad W \cap \Omega_i, i = 1, 2, \\ \frac{\partial u_1}{\partial n} = 0 & \text{on} \quad M, \\ u_1 = 0 & \text{on} \quad M. \end{cases}$$

Therefore by Holmgren's uniqueness theorem, we deduce that

$$u_1 = 0 \text{ in } W. \tag{1.25}$$

Similarly, $\tilde{u}_1$ is also equal to 0 in W.

Now as $\theta = \tilde{\theta}$ in V and by (1.14), we readily see that $u_1 - \tilde{u}_1$ satisfies

$$\begin{cases} \Delta(u_1^i - \tilde{u}_1^i) = 0 & \text{in} \quad V \cap \Omega_i, i = 1, 2, \\ u_1 - \tilde{u}_1 = 0 & \text{on} \quad W \cap \sigma, \\ \frac{\partial(u_1 - \tilde{u}_1)}{\partial n} = 0 & \text{on} \quad W \cap \sigma. \end{cases}$$

Again Holmgren's uniqueness theorem yields that

$$u_1 = \tilde{u}_1 \text{ in } V. \tag{1.26}$$

Let us now take as test function v in (1.14) a function in $H^1(\Omega)$ (regular in each Ω_i) satisfying

$$\begin{cases} \Delta v^i = 0 & \text{in} \quad \Omega_i, i = 1, 2, \\ a_1 \frac{\partial v^1}{\partial n_1} - a_2 \frac{\partial v^2}{\partial n_1} = 0 & \text{on} \quad I, \end{cases} \tag{1.27}$$

where n_1 means the normal vector on I directed from Ω_1 to Ω_2. Then by integration by parts in $\Omega_{i\sigma} \setminus B(T, r)$, for $r > 0$ small enough, $i = 1, 2$ and taking the limit as

$r \to 0$, we get

$$\begin{aligned}
&\int_{\Omega_\sigma} a[-u_1\Delta v + \sum_{i,j=1,2} \theta_i\partial_j(\partial_i u_\sigma \partial_j v) \qquad (1.28)\\
&+ \sum_{i,j=1,2} \theta_j\partial_i(\partial_i u_\sigma \partial_j v) - \sum_{i,j=1,2} \theta_j\partial_j(\partial_i u_\sigma \partial_i v)]\, dx\\
&+ \int_\Gamma a[u_1\frac{\partial v}{\partial n} - (\theta\cdot\nabla u_\sigma)\frac{\partial v}{\partial n} - (\theta\cdot\nabla v)\frac{\partial u_\sigma}{\partial n} + (\nabla u_\sigma\cdot\nabla v)\theta\cdot n]\, ds\\
&+ \int_{\sigma^+\cup\sigma^-} a[u_1\frac{\partial v}{\partial n} - (\theta\cdot\nabla u_\sigma)\frac{\partial v}{\partial n} - (\theta\cdot\nabla v)\frac{\partial u_\sigma}{\partial n} + (\nabla u_\sigma\cdot\nabla v)\theta\cdot n]\, ds\\
&+ \int_I [au_1\frac{\partial v}{\partial n_1} - a(\theta\cdot\nabla u_\sigma)\frac{\partial v}{\partial n_1} - a(\theta\cdot\nabla v)\frac{\partial u_\sigma}{\partial n_1} + a(\nabla u_\sigma\cdot\nabla v)\theta\cdot n_1]_I\, ds = 0,
\end{aligned}$$

where σ^+ (resp. σ^-) is the "upper part" (resp. "lower part") of σ; on σ^+ (resp. σ^-), n means the normal vector directed from σ^+ to σ^- (resp. σ^- to σ^+), and $[w]_I$ is the jump of w through I, i.e., $[w]_I = w^1 - w^2$.

By Leibniz's rule and the fact that u_σ as well as v are harmonic in each $\Omega_{i\sigma}$, we have

$$\begin{aligned}
&\int_{\Omega_\sigma} a[-u_1\Delta v + \sum_{i,j=1,2} \theta_i\partial_j(\partial_i u_\sigma \partial_j v)\\
&+ \sum_{i,j=1,2} \theta_j\partial_i(\partial_i u_\sigma \partial_j v) - \sum_{i,j=1,2} \theta_j\partial_j(\partial_i u_\sigma \partial_i v)]\, dx = 0.
\end{aligned}$$

As $\theta \equiv 0$ in a neighbourhood of Γ the boundary term on Γ is zero. For the terms on the crack, we recall that u_σ satisfies

$$\frac{\partial u_\sigma}{\partial n} = 0 \text{ on } \sigma,$$

while θ is chosen such that $\theta\cdot n = 0$ on σ, therefore we have

$$\begin{aligned}
&\int_{\sigma^+\cup\sigma^-} a[u_1\frac{\partial v}{\partial n} - (\theta\cdot\nabla u_\sigma)\frac{\partial v}{\partial n} - (\theta\cdot\nabla v)\frac{\partial u_\sigma}{\partial n} + (\nabla u_\sigma\cdot\nabla v)\theta\cdot n]\, ds\\
&= \int_{\sigma^+\cup\sigma^-} a[u_1\frac{\partial v}{\partial n} - (\theta\cdot\nabla u_\sigma)\frac{\partial v}{\partial n}]\, ds.
\end{aligned}$$

Finally on I, we remark that

$$[au_1\frac{\partial v}{\partial n}]_I = u_1^1[a\frac{\partial v}{\partial n}]_I = 0,$$

by the continuity of u_1 through I and the transmission condition satisfied by v in (1.27). For the second term, by (1.27) and the continuity of u_σ through I, we have successively

$$[a(\theta\cdot\nabla u_\sigma)\frac{\partial v}{\partial n_1}]_I = a_1\frac{\partial v^1}{\partial n_1}[\theta\cdot\nabla u_\sigma]_I = a_1\frac{\partial v^1}{\partial n_1}\theta\cdot n_1[\frac{\partial u_\sigma}{\partial n_1}]_I.$$

Using again (1.27), we arrive at

$$[a(\theta\cdot\nabla u_\sigma)\frac{\partial v}{\partial n_1}]_I = \theta\cdot n_1[a\frac{\partial v}{\partial n_1}\frac{\partial u_\sigma}{\partial n_1}]_I.$$

Similarly as v is continuous through I and u_σ satisfies $[a\frac{\partial u_\sigma}{\partial n_1}]_I = 0$, we get

$$[a(\theta\cdot\nabla v)\frac{\partial u_\sigma}{\partial n_1}]_I = \theta\cdot n_1[a\frac{\partial u_\sigma}{\partial n_1}\frac{\partial v}{\partial n_1}]_I.$$

Since we clearly have

$$[a(\nabla u_\sigma\cdot\nabla v)\theta\cdot n_1]_I = \theta\cdot n_1([a\frac{\partial u_\sigma}{\partial n_1}\frac{\partial v}{\partial n_1}]_I + [a\frac{\partial u_\sigma}{\partial t_1}\frac{\partial v}{\partial t_1}]_I),$$

where t_1 is the unit vector orthogonal to n_1, the above identities imply that

$$\begin{aligned}&\int_I [au_1\frac{\partial v}{\partial n_1} - a(\theta\cdot\nabla u_\sigma)\frac{\partial v}{\partial n_1} - a(\theta\cdot\nabla v)\frac{\partial u_\sigma}{\partial n_1} + a(\nabla u_\sigma\cdot\nabla v)\theta\cdot n_1]_I\,ds\\ &= \int_I([a\frac{\partial u_\sigma}{\partial t_1}\frac{\partial v}{\partial t_1}]_I - [a\frac{\partial u_\sigma}{\partial n_1}\frac{\partial v}{\partial n_1}]_I)\theta\cdot n_1\,ds.\end{aligned}$$

Summing up, the identity (1.28) is reduced to

$$\begin{aligned}&\int_{\sigma^+\cup\sigma^-} a[u_1\frac{\partial v}{\partial n} - (\theta\cdot\nabla u_\sigma)\frac{\partial v}{\partial n}]\,ds\\ &+\int_I([a\frac{\partial u_\sigma}{\partial t_1}\frac{\partial v}{\partial t_1}]_I - [a\frac{\partial u_\sigma}{\partial n_1}\frac{\partial v}{\partial n_1}]_I)\theta\cdot n_1\,ds = 0.\end{aligned}\tag{1.29}$$

As $\tilde{u}_1$ satisfies the same identity than u_1 with $\tilde{\theta}$ instead of θ, by difference, using (1.26) and the fact that $\theta = \tilde{\theta}$ on σ, we obtain

$$\int_I([a\frac{\partial u_\sigma}{\partial t_1}\frac{\partial v}{\partial t_1}]_I - [a\frac{\partial u_\sigma}{\partial n_1}\frac{\partial v}{\partial n_1}]_I)(\theta - \tilde{\theta})\cdot n_1\,ds = 0.\tag{1.30}$$

In this identity, take first

$$\begin{cases} v^1 = r^k \cos(k\varphi), \\ v^2 = r^k \cos(k\varphi), \end{cases} \tag{1.31}$$

for all $k \in \mathbb{N}^\star$, where (r, φ) are polar coordinates centered at T and such that $\varphi = 0$ or π on the interface I.

Then computing ∇v^i, the previous identity becomes

$$\int_I x^{k-1}(a_1 \frac{\partial u_\sigma^1}{\partial x} - a_2 \frac{\partial u_\sigma^2}{\partial x})(\theta - \tilde{\theta}) \cdot n_1 \, dx = 0, \forall k \in \mathbb{N}^\star,$$

where (x, y) are usual Euclidean coordinates centered at T and such that the x-axis contains I. This yields

$$(a_1 \frac{\partial u_\sigma^1}{\partial x} - a_2 \frac{\partial u_\sigma^2}{\partial x})(\theta - \tilde{\theta}) \cdot n_1 \equiv 0 \text{ on } I.$$

And by the assumption (1.24), we get

$$a_1 \frac{\partial u_\sigma^1}{\partial x} - a_2 \frac{\partial u_\sigma^2}{\partial x} \equiv 0 \text{ on } I_0,$$

for some nonempty open subset I_0 of I. By the continuity of u_σ through I, we conclude that

$$\frac{\partial u_\sigma^1}{\partial x} = \frac{\partial u_\sigma^2}{\partial x} = 0 \text{ on } I_0, \tag{1.32}$$

since $a_1 \neq a_2$.

Secondly in (1.30), take as test functions

$$\begin{cases} v^1 = r^k \sin(k\varphi), \\ v^2 = \frac{a_1}{a_2} r^k \sin(k\varphi), \end{cases} \tag{1.33}$$

for all $k \in \mathbb{N}^\star$. Then as before we get

$$\int_I x^{k-1} a_1 (\frac{\partial u_\sigma^1}{\partial y} - \frac{\partial u_\sigma^2}{\partial y})(\theta - \tilde{\theta}) \cdot n_1 \, dx = 0, \forall k \in \mathbb{N}^\star.$$

This implies that

$$\frac{\partial u_\sigma^1}{\partial y} = \frac{\partial u_\sigma^2}{\partial y} \text{ on } I_0.$$

Since u_σ satisfies the transmission condition

$$a_1\frac{\partial u_\sigma^1}{\partial y} - a_2\frac{\partial u_\sigma^2}{\partial y} = 0 \text{ on } I,$$

we deduce that

$$\frac{\partial u_\sigma^1}{\partial y} = \frac{\partial u_\sigma^2}{\partial y} = 0 \text{ on } I_0. \tag{1.34}$$

By (1.32), (1.34) and Holmgren's uniqueness theorem, u_σ is constant in Ω; which is impossible because ϕ is different from zero.

Remark 1.3.4 In the setting of the above theorem, the crack tip of σ is on the interface I. Consequently when you perturb the crack, you automatically perturb the interface. Therefore the idea of condition (1.24) of that Theorem is that we can choose $\tilde{\theta}$ equal to θ near the crack tip but different from θ in a small neighbourhood of σ. More precisely we can take $\tilde{\theta} = \theta + \eta$, where η is a smooth function such that $\eta \cdot n \neq 0$ on I, η has a support in $B(T, \delta_2)$, is equal to 0 in $B(T, \delta_1)$ and in a neighbourhood of σ, where $0 < \delta_1 < \delta_2$ are small in order that the support of $\tilde{\theta}$ is the same than θ. Since θ and $\tilde{\theta}$ are equal in a neighbourhood of σ, they induce the same perturbation of σ.

The same idea will be exploited in Theorem 1.3.5 for the stability with respect to the angle when σ crosses the interface I.

1.3.2 Stability with respect to the angle

As in [9], we take here θ in the following form: fix $\mathcal{V}_1$ and $\mathcal{V}_2$ two open neighbourhoods of σ such that

$$\begin{cases} \bar{\mathcal{V}}_1 \subset \mathcal{V}_2 \cup \{S\}, \\ \bar{\mathcal{V}}_2 \subset \Omega \cup \{S\}, \\ \bar{\mathcal{V}}_1 \cap \Gamma = \{S\}, \end{cases}$$

as illustrated by figure 1.4. We then require that

$$\begin{cases} \theta = \begin{pmatrix} -y \\ x \end{pmatrix} & \text{in} \quad \bar{\mathcal{V}}_1, \\ \theta \equiv 0 & \text{in} \quad \Omega \setminus \bar{\mathcal{V}}_2. \end{cases}$$

Moreover, if σ does not cross the interface I (in the sense that $\bar{\sigma} \cap \bar{I} \neq \emptyset$), we suppose that $\mathcal{V}_2 \cap I = \emptyset$. In that case, we can directly apply Theorem 4 of [9] in order to have $u_1 \neq 0$ on M. It then remains to analyze the case when σ crosses the interface I.

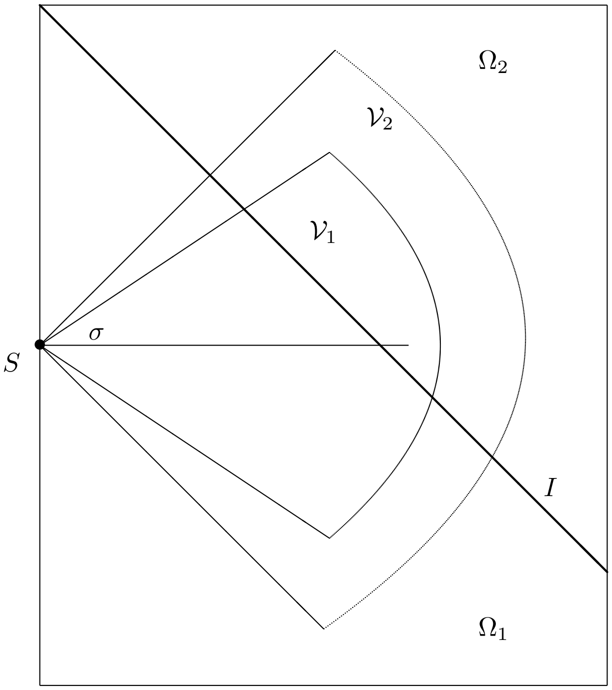

Figure 1.4: Neighbourhoods of σ

Theorem 1.3.5 *Assume that σ crosses the interface I. Let θ and $\tilde{\theta}$ be two vector fields such that $\theta = \tilde{\theta}$ in a neighbourhood V of σ and satisfying (1.24). Then $u_1 \neq 0$ on M or $\tilde{u}_1 \neq 0$ on M.*

Proof: Similar to the proof of Theorem 1.3.3. ■

1.3.3 Local Lipschitz continuity

We now show how the previous results allow to show the local Lipschitz continuity of the mapping η^{-1}. Since a neighbourhood of a fixed straight crack σ is determined by the length parameter h_l and the angle parameter h_a, we actually defined the diffeomorphism

$$F_h = Id + h_l\theta_l + h_a\theta_a,$$

where $h = (h_l, h_a)$ and $\theta_l \in (W^{1,\infty}(\Omega))^2$ and $\theta_a \in (W^{1,\infty}(\Omega))^2$ are vector fields satisfying the assumptions of subsection 3.1 and 3.2 respectively and such that $u_{1l} \neq 0$ on M and $u_{1a} \neq 0$ on M, where u_{1l} (resp. u_{1a}) means the θ_l-derivative of u_σ (resp. the θ_a-derivative of u_σ).

As before, we then will have

$$u_h - u_{h'} = (h_l - h'_l)u_{1l} + (h_a - h'_a)u_{1a} + o(h).$$

This implies that

$$\|u_h - u_{h'}\|_{L^2(M)} \geq \|(h_l - h'_l)u_{1l} + (h_a - h'_a)u_{1a}\|_{L^2(M)} - C_1\|h - h'\|,$$

if h, h' are small enough. Consequently, we will get

$$\|u_h - u_{h'}\|_{L^2(M)} \geq C\|h - h'\|, \tag{1.35}$$

for h, h' small enough if u_{1l} and u_{1a} are linearly independent on M. This will be proved by appropriate choices of θ_l and θ_a.

Theorem 1.3.6 *Assume that the interface is a straight line and that the set Σ of admissible cracks is made of straight cracks emerging at S. Then the mapping η^{-1} is Lipschitz continuous in a neighbourhood of σ if the coefficient of singularity c_σ of u_σ is different from 0.*

Proof: If σ crosses the interface I, then Theorems 1.3.3 and 1.3.5 (see Remark 1.3.4) guarantee the existence of θ_l, $\tilde{\theta}_l$, θ_a and $\tilde{\theta}_a$ such that the corresponding derivatives u_{1l}, $\tilde{u}_{1l}$, u_{1a} and $\tilde{u}_{1a}$ of u_σ are different from 0 on M.

Assume that u_{1l} and u_{1a} (resp. $\tilde{u}_{1l}$ and $\tilde{u}_{1a}$) are linearly dependent on M. Then there exists $\lambda \neq 0$ (resp. $\tilde{\lambda} \neq 0$) such that

$$\left\{\begin{array}{lll} u_{1a} = \lambda u_{1l} & \text{on} & M, \\ \tilde{u}_{1a} = \tilde{\lambda}\tilde{u}_{1l} & \text{on} & M. \end{array}\right. \tag{1.36}$$

We now consider the two diffeomorphism

$$\begin{aligned} F_h &= Id + h(\theta_a - \lambda\theta_l), \\ \tilde{F}_h &= Id + h(\tilde{\theta}_a - \tilde{\lambda}\tilde{\theta}_l), \end{aligned}$$

for $h \in \mathbb{R}$ small enough. Then the $(\theta_a - \lambda\theta_l)$-derivative u_1 (resp. $(\tilde{\theta}_a - \tilde{\lambda}\tilde{\theta}_l)$-derivative $\tilde{u}_1$) of u_σ is equal to

$$u_1 = u_a - \lambda u_l, \tilde{u}_1 = \tilde{u}_a - \tilde{\lambda}\tilde{u}_l.$$

By (1.36), we have

$$u_1 = \tilde{u}_1 = 0 \text{ on } M.$$

Therefore the arguments of Theorem 1.3.3 lead to a contradiction if we choose (which is always possible by choosing the support of $\theta_a - \tilde{\theta}_a$ disjoint to the support of $\theta_l - \tilde{\theta}_l$)

$$(\theta_a - \tilde{\theta}_a - \lambda\theta_l + \tilde{\lambda}\tilde{\theta}_l) \cdot n \neq 0 \text{ on } I.$$

If σ does not cross the interface I, then by Theorem 1.3.2 and Theorem 4 of [9], we can conclude that the derivatives u_{1l} and u_{1a} are linearly independent on M. ■

1.4 Application to the heat conduction

In this section, we show that we can obtain an approximation of any emerging crack for the heat conduction problem associated with the stationary problem (1.2) for an appropriate choice of the flux. Namely, we consider the heat problem: Let y_σ be the solution of

$$\begin{cases} \frac{\partial y_\sigma}{\partial t}(x,t) - \text{div } (a\nabla y_\sigma)(x,t) = 0 & \text{in} \quad \Omega_\sigma \times (0,+\infty), \\ a\frac{\partial y_\sigma}{\partial n}(x,t) = 0 & \text{on} \quad \sigma \times (0,+\infty), \\ a\frac{\partial y_\sigma}{\partial n}(x,t) = \phi(x)\sin(\frac{\pi t}{T}) & \text{on} \quad \Gamma \times (0,+\infty), \\ y_\sigma(x,0) = 0 & \text{in} \quad \Omega_\sigma, \end{cases} \tag{1.37}$$

where ϕ is the flux fixed in section 2 and T is a fixed positive real number. We now prove the following result.

Theorem 1.4.1 *Under the above hypothesis and using the notation from section 2, there exists a positive constant C such that*

$$\max_{t\in[0,T]}\max_{x\in\Gamma\setminus\{S\}}|y_\sigma(x,t)-u_\sigma(x)\sin(\frac{\pi t}{T})|\leq\frac{C}{\sqrt{T}}\|\phi\|_{H^{1/2}(\Gamma)},\tag{1.38}$$

where $u_\sigma\in V_\sigma$ is the solution of (1.2).

Proof: Let us set

$$z(x,t)=u_\sigma(x)\sin(\frac{\pi t}{T})-y_\sigma(x,t).$$

Then z is solution of

$$\begin{cases}\frac{\partial z}{\partial t}(x,t)-\text{div }(a\nabla z)(x,t)=\frac{\pi}{T}\cos(\frac{\pi t}{T})u_\sigma(x) & \text{in} \quad \Omega_\sigma\times(0,+\infty),\\ a\frac{\partial z}{\partial n}(x,t)=0 & \text{on} \quad (\sigma\cup\Gamma)\times(0,+\infty),\\ z(x,0)=0 & \text{in} \quad \Omega_\sigma.\end{cases}$$

By the spectral theorem, z is given by

$$z(x,t)=\sum_{k=1}^{\infty}\varphi_k\int_0^t e^{-\lambda_k(t-s)}(f(s),\varphi_k)\,ds,$$

where $f(t)=\frac{\pi}{T}\cos(\frac{\pi t}{T})u_\sigma$, λ_k are the eigenvalues of the selfadjoint operator A of associated eigenvectors φ_k, when A is the Friedrichs extension of the triple $(a_\sigma,V_\sigma,L^2(\Omega))$, a_σ being the bilinear form appearing in (1.4) [36]. Accordingly, since the set $\{\varphi_k\}_{k\in\mathbb{N}^\star}$ is an orthonormal basis of $L^2(\Omega)$, we obtain

$$\begin{aligned}\|Az\|^2_{L^2(\Omega)} &\leq \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k\int_0^t(f(s),\varphi_k)^2\,ds\\ &\leq \frac{1}{2}\int_0^t\|f(s)\|^2_{D(A^{1/2})}\,ds,\end{aligned}$$

by Fubini's theorem. By a direct calculation, we check that

$$\int_0^T\|f(s)\|^2_{D(A^{1/2})}\,ds=\frac{\pi^2}{2T}\|u_\sigma\|^2_{D(A^{1/2})}.$$

Therefore, we get

$$\|z\|_{C([0,T],D(A))}\leq\frac{\pi}{2\sqrt{T}}\|u_\sigma\|_{D(A^{1/2})}.\tag{1.39}$$

But Theorem 4.2 of [62] directly implies that the following continuous embedding holds:

$$D(A) \hookrightarrow H^2(\Omega_{1\sigma} \setminus V) \cap H^2(\Omega_{2\sigma} \setminus V),$$

where V is a fixed neighbourhood of the crack tip. By usual trace theorem, for any $u \in D(A)$, we then have

$$u_{|\Gamma} \in H^{3/2}(\Gamma_1 \setminus \{S\}) \cap H^{3/2}(\Gamma_2 \setminus \{S\}),$$

where $\Gamma_i = \Gamma \cap \partial\Omega_i$, $i = 1, 2$. Sobolev embedding theorem and the continuity of u_σ through the interface I yield finally

$$u_{|\Gamma} \in C(\Gamma \setminus \{S\}).$$

Note that $v \in C(\Gamma \setminus \{S\})$ means that $v(s)$ is continuous for all $s \in [0, l]$, when l is the length of Γ (in other words, v is continuous on $\Gamma \setminus \{S\}$ with finite left and right limits at S that could be different). In other words, there exists $C > 0$ such that

$$\max_{x \in \Gamma \setminus \{S\}} |u(x)| \leq C\|u\|_{D(A)}, \forall u \in D(A). \tag{1.40}$$

Using this estimate in (1.39), we obtain (1.38) because $D(A^{1/2}) = V_\sigma$ and u_σ is solution of (1.4). ■

Remark 1.4.2 The estimate (1.38) shows that the knowledge of y_σ on $M \times [0, T]$ allows to have an approximation of u_σ on M, by taking T large enough. Therefore in the setting of section 3, we can have a rather good approximation of σ.

Chapter 2

Identifiability, stability and reconstruction results of point sources by boundary measurements in heteregeneous trees

2.1 Introduction

Various models of multiple-link flexible structures, consisting of finitely many interconnected flexible elements, like strings, beams, plates, shells or combinations of them, have been described recently in [41, 25, 43, 44, 22, 16]. The problem of controllability or stabilization of such structures is an expanding field. For control results, let us quote the works of Lagnese-Leugering-Schmidt [65, 42, 43] and of Dager-Zuazua [27] for 1-d. networks; the works of Puel-Zuazua [63], Lagnese [40] and the first author [54, 55, 56] for multidimensional structures. For stabilization results, we may cite the results of Chen and coauthors [22, 23, 24], of Conrad [26] and of Lagnese-Leugering-Schmidt [42].

To our knowledge inverse problems related to the above control problems on such structures are not considered at all, contrary to the case of homogeneous structures (like one string, one beam, one membrane, one plate, etc...) for which such problems have been recently developed by Yamamoto and Bruckner [68, 18, 19, 69]. Therefore the goal of this chapter is first to consider the simplest multi-dimensional problem: namely the wave equation on one-dimensional trees with some point sources (linear combination of delta functions, see below for the details) and secondly to extend the results from [17, 18] obtained for the real interval $]0, 1[$ to this system. In the case of serially connected strings this problem may be seen as a simplification of a model of earthquakes [1].

The questions in such problems are to determine the number of point sources, the location of these points and their intensity from boundary measurements. As usual the three main steps are the uniqueness (unique solvability of the problem), the stability (small perturbations of the boundary measurements give rise to small perturbations of the sources) and finally the reconstruction (build appropriate processes in order to find a good approximation of the unknowns). We answer to these questions by adapting some results from [17, 18, 68] to our system. The main ingredients are the spectral analysis of the Laplace equation on networks (see [5, 7, 12, 15, 43, 51, 52] and the references cited there), some controllability results on such structures [65, 43] and finally appropriate properties of some integral operators [68, 18]. Since the eigenvalues and eigenvectors of the Laplace equation on networks are not explicitly known, our reconstruction process is different from the one in [18] and is more close to the one in [68].

In [19] the authors consider an interior observation for the determination of the point sources in $]0, 1[$. The extension of this kind of considerations to 1-d. trees seems to be unrealistic since the point η of observation in [19] is assumed to be an irrational algebraic number. This assumption guarantees that η is never a zero of any eigenvector of the Laplace operator on $]0, 1[$ with Dirichlet boundary conditions. For 1-d. trees such a point is difficult to determine in practice.

The determination of L^2-source functions from boundary measurements on 1-d. trees may be obtained using the method developed in [67, 68] and the arguments used below.

The chapter is organized as follows: In section 2.2 we recall some notations and definitions concerning 1-d. networks and introduce the (spatial) operator, namely a second order operator on each edge with some transmission conditions at interior nodes and Dirichlet boundary conditions at exterior nodes. We further show the well-posedness of the wave equation with point sources. Some observability estimates and hidden regularities are obtained in section 2.3 and are actually based on an identity with multiplier and its consequences namely the direct and inverse inequalities for the wave equation in a tree. Section 2.4 is devoted to the proof of the uniqueness result and is based on the previous observability estimates and some properties of an integral operator between different Sobolev spaces. The stability is deduced in section 2.5 and finally the reconstruction is detailed in section 2.6.

2.2 Preliminaries

We first recall the notion of C^ν-networks, $\nu \in \mathbb{N}$, which is simply those of [13], we refer to [5, 7, 12, 14, 15] for more details.

All graphs considered here are non empty, finite and simple. Let Γ be a connected topological graph imbedded in $\mathbb{R}^m$, $m \in \mathbb{N}^*$, with n vertices $E = \{E_i : 1 \leq i \leq n\}$ and N edges $K = \{k_j : 1 \leq j \leq N\}$. Each edge k_j is a Jordan curve in $\mathbb{R}^m$ and is assumed to be parametrized by its arc length parameter x_j, such that the parametrizations

$$\pi_j : [0, l_j] \to k_j : x_j \mapsto \pi_j(x_j)$$

is ν-times differentiable, i.e., $\pi_j \in C^\nu([0, l_j], \mathbb{R}^m)$ for all $1 \leq j \leq N$.

We now define the C^ν-network G associated with Γ as the union

$$G = \cup_{j=1}^N k_j.$$

The valency of each vertex E_i is denoted by $\gamma(E_i)$. We distinguish two types of vertices: ramified (or interior) vertices $int\ E = \{E_i \in E : \gamma(E_i) > 1\}$ and boundary (or exterior) vertices $\partial E = \{E_i \in E : \gamma(E_i) = 1\}$. For shortness, we later on denote by $I_{ext} = \{i \in \{1, \cdots, n\} : \gamma(E_i) = 1\}$ and $I_{int} = \{1, \cdots, n\} \setminus I_{ext}$. For each vertex E_i, we also denote by $N_i = \{j \in \{1, \ldots, N\} : E_i \in k_j\}$ the set of edges adjacent to E_i. Note that if $E_i \in \partial E$ then N_i is a singleton that we write $\{j_i\}$. For each vertex E_i and $j \in E_i$, we further denote by

$$\nu_j(E_i) = \begin{cases} 1 & \text{if} \quad \pi_j(l_j) = E_i, \\ \text{-1} & \text{if} \quad \pi_j(0) = E_i, \end{cases}$$

the normal vector in k_j at E_i.

For a function $u : G \to \mathbb{R}$, we set $u_j = u \circ \pi_j : [0, l_j] \to \mathbb{R}$, its “restriction” to the edge k_j. We further use the abbreviations:

$$\begin{aligned} u_j(E_i) &= u_j(\pi_j^{-1}(E_i)), \\ u_j'(E_i) &= \frac{du_j}{dx_j}(\pi_j^{-1}(E_i)), \\ u_j\text{''}(E_i) &= \frac{d^2u_j}{dx_j^2}(\pi_j^{-1}(E_i)). \end{aligned}$$

Finally, differentiations are carried out on each edge k_j with respect to the arc length parameter x_j.

Let us now fix a C^2-network G which is a tree (since for such networks the direct and inverse inequalities hold, see below and for instance [65, 43]). For each edge k_j, we also fix mechanical constants $m_j > 0$ (the mass density of the string k_j) and $d_j > 0$ (the diffusion coefficient of k_j). We now consider the following wave equation:

$$\left\{ \begin{array}{l} \partial_t^2 u_j(x_j, t) - \frac{d_j}{m_j} u_j\text{''}(x_j, t) = \lambda(t) a_j(x_j) \text{ in } Q_{jT}, \forall j = 1, \cdots, N, \\ u(\cdot, t) = 0 \text{ is continuous on } G \text{ for all } t \in]0, T[, \\ \sum_{j \in N_i} d_j \frac{\partial u_j}{\partial \nu_j}(E_i, t) = 0, \forall i \in I_{int}, \forall t \in]0, T[, \\ u_{j_i}(E_i, t) = 0, \forall i \in I_{ext}, \forall t \in]0, T[, \\ u_j(x_j, 0) = \partial_t u_j(x_j, 0) = 0 \text{ in }]0, l_j[, \forall j = 1, \cdots, N, \end{array} \right. \tag{2.1}$$

where $Q_{jT} :=]0, l_j[\times]0, T[$ and $\frac{\partial u_j}{\partial \nu_j}(E_i, t) = \nu_j(E_i) u_j'(E_i, t)$ means the exterior normal derivative of $u_j(\cdot, t)$ at E_i. Above and below $\lambda \in C^1([0, T])$ is a given function satisfying

$$\lambda(0) \neq 0. \tag{2.2}$$

For all $j = 1, \cdots, N$ the datum $a_j \in (H^1(0, l_j))'$ is assumed to be in the form

$$a_j(x_j) = \sum_{k=1}^{K_j} \alpha_{jk} \delta(x_j - \xi_{jk}), \tag{2.3}$$

for some positive integer K_j, some real numbers α_{jk} different from zero and some (different) points ξ_{jk} in $]0, l_j[$, or more precisely

$$< a_j, \phi >= \sum_{k=1}^{K_j} \alpha_{jk}\phi(\xi_{jk}), \forall \phi \in H^1(0, l_j).$$

Above and below $H^p(0, l_j)$ is the standard Sobolev space of order $p \in \mathbb{N}$ on the interval $]0, l_j[$.

Our goal is to identify the datum a in the above form (i.e. the location of the point sources ξ_{jk}, the weight α_{jk} and the number K_j) from boundary measurements, namely the value of $u'_{j_i}(E_i, t)$, for $0 < t < T$ and all external vertices E_i except one.

In order to analyse the system (2.1) we introduce the following operator A on the Hilbert space $H = \Pi_{j=1}^N L^2(0, l_j)$, endowed with the inner product

$$(u, v)_H = \sum_{j=1}^N m_j \int_0^{l_j} u_j(x) v_j(x)\, dx.$$

$$\left\{ \begin{array}{l} D(A) = \{u \in H : u_j \in H^2(0, l_j) \text{ satisfying (2.5) to (2.7) hereafter}\}, \\ \forall u \in D(A) : Au = (\frac{d_j}{m_j} u_j")_{j=1}^N. \end{array} \right. \tag{2.4}$$

$$u \text{ is continuous on } G. \tag{2.5}$$

$$\sum_{j \in N_i} d_j \frac{\partial u_j}{\partial \nu_j}(E_i) = 0, \forall i \in I_{int}, \tag{2.6}$$

$$u_{j_i}(E_i) = 0, \forall i \in I_{ext}. \tag{2.7}$$

Remark that A is a negative selfadjoint operator with a compact resolvant since A is the Friedrichs extension of the triple (H, V, a) defined by

$$V = \{u \in \Pi_{j=1}^N\ H^1(0, l_j) \text{ satisfying (2.5) and (2.7)}\},$$

which is a Hilbert space with the inner product

$$(u, v)_V = \sum_{j=1}^N \int_0^{l_j} u'_j v'_j\, dx_j,$$

and

$$a(u, v) = \sum_{j=1}^N d_j \int_0^{l_j} u'_j(x_j)\ v'_j(x_j)\, dx_j. \tag{2.8}$$

The spectrum of this operator A was studied in details in [5, 6, 7, 12, 15, 14, 43, 51, 52, 53]. For our future uses we recall the following Weyl's formula: if $\{\lambda_k\}_{k=1}^{\infty}$ denotes the set of eigenvalues of the operator $-A$ in increasing order and repeated according to their multiplicity, then

$$\lim_{k\to\infty} \frac{\lambda_k}{k^2} = \pi^2 \left(\sum_{j=1}^{N} \frac{\sqrt{m_j} l_j}{\sqrt{d_j}} \right)^{-2}. \tag{2.9}$$

Let us further prove that the eigenvectors are uniformly bounded:

Lemma 2.2.1 *For all positive integer k let ϕ_k be the eigenvector of A associated with $-\lambda_k$. Then denoting by $\phi_{k,j}$ the restriction of ϕ_k to the edge j, for k large enough, we have*

$$|\phi_{k,j}(x_j)| \leq \frac{2\sqrt{2}\sqrt{l_j}}{\sqrt{m_j}}, \forall x_j \in [0, l_j]. \tag{2.10}$$

Proof: Since $\phi_{k,j}$ satisfies

$$\frac{d_j}{m_j}\phi_{k,j}" = -\lambda_k \phi_{k,j} \text{ on }]0, l_j[,$$

there exist real numbers $c_{k,j}$ and $d_{k,j}$ such that

$$\phi_{k,j}(x_j) = c_{k,j}\cos\left(\sqrt{\frac{m_j\lambda_k}{d_j}}x_j\right) + d_{k,j}\sin\left(\sqrt{\frac{m_j\lambda_k}{d_j}}x_j\right), \forall x_j \in [0, l_j].$$

Consequently by integrating the square of this expression in $[0, l_j]$ and writing for shortness $L_j = \frac{\sqrt{m_j} l_j}{\sqrt{d_j}}$, we obtain

$$\begin{aligned}\sqrt{\frac{m_j}{d_j}}\int_0^{l_j} |\phi_{k,j}(x_j)|^2\, dx_j &= c_{k,j}^2\{\frac{L_j}{2} + \frac{\sin(2\sqrt{\lambda_k}L_j)}{4\sqrt{\lambda_k}}\} + d_{k,j}^2\{\frac{L_j}{2} - \frac{\sin(2\sqrt{\lambda_k}L_j)}{4\sqrt{\lambda_k}}\} \\ &+ c_{k,j}d_{k,j}\{\frac{1}{2\sqrt{\lambda_k}} - \frac{\cos(2\sqrt{\lambda_k}L_j)}{2\sqrt{\lambda_k}}\}.\end{aligned}$$

By (2.9) for k large enough we get

$$\begin{aligned}\frac{L_j}{2} + \frac{\sin(2\sqrt{\lambda_k}L_j)}{4\sqrt{\lambda_k}} &\geq \frac{L_j}{4}, \\ \frac{L_j}{2} - \frac{\sin(2\sqrt{\lambda_k}L_j)}{4\sqrt{\lambda_k}} &\geq \frac{L_j}{4}, \\ \left|\frac{1}{2\sqrt{\lambda_k}} - \frac{\cos(2\sqrt{\lambda_k}L_j)}{2\sqrt{\lambda_k}}\right| &\leq \frac{L_j}{4}.\end{aligned}$$

Inserting these estimates in the previous identity we arrive at

$$\begin{aligned}\sqrt{\frac{m_j}{d_j}}\int_0^{l_j} |\phi_{k,j}(x_j)|^2\,dx_j &\geq (c_{k,j}^2 + d_{k,j}^2 - |c_{k,j}d_{k,j}|)\frac{L_j}{4}\\ &\geq (c_{k,j}^2 + d_{k,j}^2)\frac{L_j}{8},\end{aligned}$$

for k large enough. We conclude by noting that

$$m_j\int_0^{l_j} |\phi_{k,j}(x_j)|^2\,dx_j \leq \|\phi_{k,j}\|_H^2 = 1.$$

■

We are now ready to prove that our wave equation (2.1) is uniquely solvable and to give regularity of its solution:

Theorem 2.2.2 *The wave equation (2.1) has a unique (weak) solution u satisfying*

$$u \in C([0,T];V)\cap C^1([0,T];H).$$

Proof: We remark that the system (2.1) is equivalently written

$$\left\{\begin{array}{l}\partial_t^2 u = Au + \lambda(t)a \text{ in }]0,T[,\\ u(0)=0, \partial_t u(0) = 0\ ,\end{array}\right. \tag{2.11}$$

where $a \in V'$ is defined by

$$< a, \phi >_{V'-V} = \sum_{j=1}^{N} m_j \sum_{k=1}^{K_j} \alpha_{jk}\phi_j(\xi_{jk}), \forall\phi\in V. \tag{2.12}$$

The solution of that system is explicitly given by (using spectral expansions)

$$u(t) = \sum_{k=1}^{\infty}\frac{1}{\sqrt{\lambda_k}}\int_0^t \sin((t-s)\sqrt{\lambda_k})\lambda(s)\,ds < a, \phi_k > \phi_k,$$

or equivalently, by integration by parts in the above integral:

$$u(t) = \sum_{k=1}^{\infty}\frac{a_k(t)}{\lambda_k}\phi_k, \tag{2.13}$$

where a_k is given by

$$a_k(t) =< a, \phi_k > (\lambda(t) - \lambda(0) \cos(t\sqrt{\lambda_k}) - \int_0^t \cos((t-s)\sqrt{\lambda_k})\lambda'(s)\, ds).$$

We now remark that Lemma 2.2.1, the form of a and the smoothness of λ allow to conclude the existence of a constant M (depending on T but not on k) such that

$$|a_k(t)| \leq M, \forall k = 1, \cdots, \infty. \tag{2.14}$$

By Parseval's identity we have

$$||u(t)||_V^2 \sim ||u(t)||_{D(A^{1/2})}^2 \sim \sum_{k=1}^{\infty} \frac{|a_k(t)|^2}{\lambda_k},$$

and consequently by the estimate (2.14) we conclude that

$$||u(t)||_V^2 \leq M^2 \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \leq C, \forall t \in [0, T],$$

for some positive constant C (depending on T) since the asymptotic behaviour of the eigenvalues guarantees the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{\lambda_k}$. This means that the series $\sum_{k=1}^{\infty} \frac{a_k(t)}{\lambda_k}\phi_k$ is convergent in $L^\infty([0,T]; V)$ and then proves that

$$u \in C([0,T]; V),$$

as limit of elements from $C([0,T]; V)$ (the truncated series).

Similarly by direct calculations we have

$$||\partial_t u(t)||_H^2 = \sum_{k=1}^{\infty} \frac{|\partial_t a_k(t)|^2}{\lambda_k^2} \leq C \sum_{k=1}^{\infty} \frac{1}{\lambda_k},$$

for some positive constant C (depending on T), and we conclude as before that $u \in C^1([0,T]; H)$. ■

2.3 Some observability estimates

In this section we first recall the (standard) direct and inverse inequalities for the wave equation in a tree, obtained in [65, 43] for general hyperbolic systems using the multiplier method or the method of characteristics and that easily follows in our case using the multiplier method. Some hidden regularity for our system (2.1) and some observability estimates for an associated one are secondly deduced.

We then consider the wave equation

$$\begin{cases} \partial_t^2\phi - A\phi = f \text{ in }]0,T[, \\ \phi(0) = \phi_0, \partial_t\phi(0) = \phi_1 \ , \end{cases} \tag{2.15}$$

where (ϕ_0, ϕ_1) belongs to $V \times H$ and $f \in L^1(]0,T[; H)$. It is well known that this system has a unique solution $\phi \in C([0,T]; V) \cap C^1([0,T]; H)$. We now state the so-called identity with multiplier which is the key identity for the direct and inverse inequalities.

Lemma 2.3.1 *Let $T > 0$ and let $q : G \to \mathbb{R}$ be a multiplier with the regularity $q_j \in C^1([0, l_j])$, for all $j = 1, \cdots, N$. Then for all $(\phi_0, \phi_1) \in V \times H$ and $f \in L^1(]0,T[; H)$, the solution $\phi \in C([0,T]; V) \cap C^1([0,T]; H)$ of (2.15) satisfies*

$$\begin{aligned} & \frac{1}{2}\sum_{i\in I_{ext}} \int_0^T d_{j_i} |\phi'_{j_i}(E_i,t)|^2 q_{j_i}(E_i)\nu_{j_i}(E_i)\, dt \qquad (2.16) \\ + \ & \frac{1}{2}\sum_{i\in I_{int}} \sum_{j\in N_i} \int_0^T (d_j|\phi'_j(E_i,t)|^2 + m_j|\partial_t\phi_j(E_i,t)|^2) q_j(E_i)\nu_j(E_i)\, dt \\ = \ & \frac{1}{2}\sum_{j=1}^N \int_{Q_{jT}} q'_j(m_j|\partial_t\phi_j|^2 + d_j|\phi'_j|^2)\, dx_j dt - \sum_{j=1}^N m_j \int_{Q_{jT}} f_j q_j \phi'_j \, dx_j dt \\ + \ & \sum_{j=1}^N m_j \int_0^{l_j} \partial_t\phi_j q_j \phi'_j \, dx_j \Big|_0^T . \end{aligned}$$

Proof: This identity is obtained as follows: first multiply the restriction of the first equation of (2.15) to the edge j by $m_j q_j \phi'_j$, secondly apply some integrations by part in space and time as in Lemma I.3.7 of [48] for instance and thirdly take the sum on j from 1 to N. ■

This lemma allows to obtain the direct estimate which proves the so-called hidden regularity of $\partial_n\phi$ on the external boundary.

Lemma 2.3.2 *Let $T > 0$, then there exists a positive constant c such that for all $(\phi_0, \phi_1) \in V \times H$ the solution $\phi \in C([0,T]; V) \cap C^1([0,T]; H)$ of (2.15) with $f = 0$ satisfies*

$$\sum_{i \in I_{ext}} \int_0^T |\phi'_{j_i}(E_i, t)|^2 \, dt \leq c(T+1)E_0, \tag{2.17}$$

where $E_0 = E(0)$ is the energy of the system at time $t = 0$ and we recall that

$$E(t) = \frac{1}{2}(||\partial_t \phi(t)||_H^2 + a(\phi(t), \phi(t))).$$

Proof: In the identity (2.16) we restrict ourselves to q identically equal to zero near the interior nodes and such that

$$q_{j_i}(E_i)\nu_{j_i}(E_i) = 1, \forall i \in I_{ext},$$

which is always possible. Using the boundedness of q_j and q'_j we obtain

$$\begin{aligned} \frac{1}{2} \sum_{i \in I_{ext}} \int_0^T d_{j_i} |\phi'_{j_i}(E_i, t)|^2 \, dt &\leq \frac{C_1}{2} \sum_{j=1}^N \int_{Q_{jT}} (m_j |\partial_t \phi_j|^2 + d_j |\phi'_j|^2) \, dx_j dt \qquad (2.18) \\ &+ \quad C_1 \sum_{j=1}^N m_j \int_0^{l_j} (|\partial_t \phi_j(x_j, 0) \phi'_j(x_j, 0)| + |\partial_t \phi_j(x_j, T) \phi'_j(x_j, T)|) \, dx_j, \end{aligned}$$

for some positive constant C_1 (independent of T). By the conservation of energy we have

$$\frac{1}{2} \sum_{j=1}^N \int_{Q_{jT}} (m_j |\partial_t \phi_j|^2 + d_j |\phi'_j|^2) \, dx_j dt = T E_0.$$

On the other hand by Cauchy-Schwarz's inequality, for $t \in [0, T]$ we may estimate

$$\sum_{j=1}^N m_j \int_0^{l_j} |\partial_t \phi_j(x_j, t) \phi'_j(x_j, t)| \, dx_j \leq C_2 E(t) = C_2 E(0),$$

for some positive constant C_2 (independent of T). Consequently we have

$$\sum_{j=1}^N m_j \int_0^{l_j} (|\partial_t \phi_j(x_j, 0) \phi'_j(x_j, 0)| + (|\partial_t \phi_j(x_j, T) \phi'_j(x_j, T)|) \, dx_j \leq 2C_2 E_0. \tag{2.19}$$

The above identity and the estimate (2.19) in (2.18) yield the conclusion. ■

Let us now pass to the inverse estimate.

Lemma 2.3.3 *Let $T > 0$ and fix one exterior vertex E_{i_0} of G (called the root of G). Then there exist a positive constant c' and a positive time T_0 which depend only on l_j, m_j, d_j and on the algebraic structure of G such that for all $(\phi_0, \phi_1) \in V \times H$ the solution $\phi \in C([0,T]; V) \cap C^1([0,T]; H)$ of (2.15) with $f = 0$ satisfies*

$$c'(T - T_0)E_0 \leq \sum_{i \in I_{ext} \setminus \{i_0\}} \int_0^T |\phi'_{j_i}(E_i, t)|^2 \, dt. \tag{2.20}$$

Proof: In the identity (2.16) we restrict ourselves to a multiplier q such that in the left-hand side of (2.16) the contribution of the interior nodes is nonpositive and the contribution of the node E_{i_0} is zero. So we look for q satisfying $q_{j_{i_0}}(E_{i_0}) = 0$ and for all $i \in I_{int}$:

$$\sum_{j \in N_i} m_j q_j(E_i) \nu_j(E_i) \leq 0, \tag{2.21}$$

$$\sum_{j \in N_i} d_j q_j(E_i) \nu_j(E_i) |\beta_j|^2 \leq 0, \tag{2.22}$$

for all $(\beta_j)_{j \in N_i}$ such that

$$\sum_{j \in N_i} d_j \nu_j(E_i) \beta_j = 0.$$

The first condition comes from the fact that

$$\sum_{j \in N_i} \int_0^T m_j |\partial_t \phi_j(E_i, t)|^2 q_j(E_i) \nu_j(E_i) \, dt = \int_0^T |\partial_t \phi(E_i, t)|^2 \, dt \sum_{j \in N_i} m_j q_j(E_i) \nu_j(E_i),$$

since $\phi(\cdot, t)$ is continuous at E_i. The second condition comes from the fact that

$$\sum_{j \in N_i} \int_0^T d_j |\phi'_j(E_i, t)|^2 q_j(E_i) \nu_j(E_i) \, dt = \int_0^T \sum_{j \in N_i} d_j q_j(E_i) \nu_j(E_i) |\phi'_j(E_i, t)|^2 \, dt,$$

and recalling that ϕ satisfies (2.6).

To build such a q we classify the edges of G into generations: The first generation is j_{i_0}, the second generation is the edges (different from j_{i_0}) which have a node in common with j_{i_0}, and by iteration the $(i+1)^{th}$ generation is the edges which do not belong to the i^{th} generation and have a node in common with an edge of the i^{th} generation. On each edge j of the i^{th} generation with $i \geq 1$, we now use the parametrization π_j such that $\pi_j(0)$ is a node belonging to the $(i-1)^{th}$ generation if $i \geq 2$ and $\pi_j(0) = E_{i_0}$ if $i = 1$.

With these notation we first take

$$q_{j_{i_0}}(x) = x,$$

which is nonnegative and clearly satisfies $q_{j_{i_0}}(E_{i_0}) = 0$. We now build $q \geq 0$ iteratively (from generation to generation) in order to satisfy the conditions (2.21) and (2.22). But we remark that these conditions are equivalent to the following ones: for all node E_i between an edge k of the i^{th} generation (which is unique) and the $(i+1)^{th}$ generation, we require

$$m_k q_k(E_i) \leq \sum_{j \in N_i, j \neq k} m_j q_j(E_i) \tag{2.23}$$

$$d_k q_k(E_i)|\beta_k|^2 - \sum_{j \in N_i, j \neq k} d_j q_j(E_i)|\beta_j|^2 \leq 0, \tag{2.24}$$

for all $(\beta_j)_{j \in N_i}$ such that

$$d_k \beta_k = \sum_{j \in N_i, j \neq k} d_j \beta_j.$$

This last condition (2.24) may be then equivalently written as

$$d_k q_k(E_i) \left(\sum_{j \in N_i, j \neq k} \frac{d_j}{d_k} \beta_j \right)^2 - \sum_{j \in N_i, j \neq k} d_j q_j(E_i)|\beta_j|^2 \leq 0, \tag{2.25}$$

for all $(\beta_j)_{j \in N_i \setminus \{k\}}$. To eliminate these parameters β_j, we use the estimate

$$\left(\sum_{j \in N_i, j \neq k} \frac{d_j}{d_k} \beta_j \right)^2 \leq C \sum_{j \in N_i, j \neq k} \frac{d_j^2}{d_k^2} \beta_j^2,$$

which holds for a positive constant C which normally depends on the cardinal of the set $N_i \setminus \{k\}$ and that we then estimate from above by a constant independent of that cardinal and which depends on the number of edges of G.

Therefore (2.25) holds if we have

$$Cq_k(E_i)\frac{d_j^2}{d_k} \leq d_j q_j(E_i), \forall j \in N_i \setminus \{k\},$$

or equivalently

$$Cq_k(E_i)\frac{d_j}{d_k} \leq q_j(E_i), \forall j \in N_i \setminus \{k\}. \tag{2.26}$$

In summary we are looking for q satisfying (2.23) and (2.26) at all nodes E_i between an edge k of the i^{th} generation and the $(i+1)^{th}$ generation. For that purpose we take

$$q_j(x) = \alpha(x+1), \forall j \in N_i \setminus \{k\},$$

for some $\alpha > 0$ which depends on E_i and that will be fixed below. With this choice we see that (since $q_j(E_i) = \alpha$) (2.23) and (2.26) are respectively equivalent to

$$\begin{aligned} &\frac{m_k}{\sum_{j\in N_i, j\neq k} m_j} q_k(E_i) \leq \alpha, \\ &Cq_k(E_i)\frac{d_j}{d_k} \leq \alpha. \end{aligned}$$

Since we assume that q_k is known (by inductive hypothesis) we see that the two above conditions yield a finite number of lower bounds to α and therefore such a α always exists.

By induction we have built q.

Taking such a q in the identity with multiplier we have

$$\begin{aligned} &\frac{1}{2}\sum_{j=1}^{N}\int_{Q_{jT}} q_j'(m_j|\partial_t\phi_j|^2 + d_j|\phi_j'|^2)\,dx_j dt + \sum_{j=1}^{N} m_j \int_0^{l_j} \partial_t\phi_j q_j \phi_j'\,dx_j\Big|_0^T \\ &\leq \frac{1}{2}\sum_{i\in I_{ext}\setminus\{i_0\}} \int_0^T d_{j_i}|\phi_{j_i}'(E_i,t)|^2 q_{j_i}(E_i)\nu_{j_i}(E_i)\,dt. \end{aligned} \tag{2.27}$$

We conclude by the estimate (2.19) and the fact that q_j' are uniformly bounded from below. ■

Using the direct and inverse estimates and the arguments of Theorem I.6.3 of [48], we obtain the next (weak) observability estimates:

Lemma 2.3.4 *For $a \in V'$ there exists a unique solution $v \in C([0,T];H) \cap C^1([0,T];V')$ of*

$$\begin{cases} \partial_t^2 v - Av = 0 \text{ in }]0,T[, \\ v(0) = 0, \partial_t v(0) = a \ . \end{cases} \tag{2.28}$$

Moreover for $T > T_0$ with T_0 from Lemma 2.3.3, there exist two positive constants C_1 and C_2 depending on T such that

$$C_1 \|a\|_{V'} \leq \sum_{i \in I_{ext} \setminus \{i_0\}} \|v'_{j_i}(E_i, \cdot)\|_{H^{-1}(0,T)} \leq C_2 \|a\|_{V'}, \tag{2.29}$$

where, as usual, $H^{-1}(0,T)$ is the dual space of $H_0^1(0,T)$.

Let us also give a consequence of the identity with multiplier to the solution u of problem (2.1), namely the hidden regularity of $\partial_n u$ on the external boundary:

Lemma 2.3.5 *Let $u \in C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (2.1) with datum a_j in the form (2.3) or more precisely solution of (2.11) with datum a in the form (2.12). Then for all $T > 0$ and all $i \in I_{ext}$, $u'_{j_i}(E_i, \cdot)$ belongs to $L^2(0,T)$ with the estimate*

$$\sum_{i \in I_{ext}} \|u'_{j_i}(E_i, \cdot)\|_{L^2(0,T)} \leq C(\|u\|_{C([0,T];V)} + \|u\|_{C^1([0,T];H)}), \tag{2.30}$$

for some positive constant C depending on T and a_j.

Proof: We approximate a by a sequence of $a_n \in V$ such that

$$a_n \to a \in V' \text{ as } n \to \infty. \tag{2.31}$$

Namely for n large enough, we take a_n in the form

$$a_n = \sum_{j=1}^{N} \sum_{k=1}^{K_j} \alpha_{jk} \phi_{jkn},$$

where

$$\phi_{jkn}(x) = n\phi(n(x - \xi_{jk})), \forall x \in [0, l_j],$$

with a fixed nonnegative function $\phi \in \mathcal{D}(\mathbb{R})$ with a support in $[-1,1]$ and such that $\int_{-1}^{1} \phi(x)\,dx = 1$. The above convergence property follows from the easily checked property:

$$|\int_{\mathbb{R}} n\phi(n(x-\xi_{jk}))\chi(x)\,dx - \chi(\xi_{jk})| \leq \frac{1}{\sqrt{n}}\|\chi'\|_{L^2(0,1)},$$

valid for n large enough.

We further remark that for all eigenvectors $\phi_{k'}$, we may write

$$\begin{aligned} < a_n - a, \phi_{k'} > &= (a_n, \phi_{k'})_H - < a, \phi_{k'} > \\ &= \sum_{j=1}^{N} m_j \sum_{k=1}^{K_j} \alpha_{jk} \left(\int_0^{l_j} \phi_{jkn}(x)\phi_{k'j}(x)dx - \phi_{k'j}(\xi_{jk}) \right) \\ &= \sum_{j=1}^{N} m_j \sum_{k=1}^{K_j} \alpha_{jk} \int_{-1}^{1} \phi(y) \left(\phi_{k'j}(\xi_{jk} + \frac{y}{n}) - \phi_{k'j}(\xi_{jk}) \right) dy. \end{aligned}$$

Therefore by Lemma 2.2.1 we conclude that there exists a positive constant M (independent of k') such that

$$| < a_n - a, \phi_{k'} > | \leq M, \forall k' = 1, \cdots, \infty. \tag{2.32}$$

Let u_n be the solution of (2.11) with datum a_n, which, by spectral expansions, satisfies

$$u_n \in C^2([0,T]; H) \cap C^1([0,T]; V) \cap C([0,T]; D(A)). \tag{2.33}$$

Furthermore by the property (2.31), the estimate (2.32) and Lebesgue's bounded convergence Theorem, we have (see Theorem 2.2.2)

$$u_n \to u \text{ in } C([0,T]; V) \cap C^1([0,T]; H) \text{ as } n \to \infty. \tag{2.34}$$

Due to the regularity (2.33) we can apply the identity with multiplier (2.16) from Lemma 2.3.1 to u_n to get

$$\begin{aligned} &\frac{1}{2} \sum_{i \in I_{ext}} \int_0^T d_{j_i} |u'_{nj_i}(E_i,t)|^2 q_{j_i}(E_i)\nu_{j_i}(E_i)\,dt \\ + &\frac{1}{2} \sum_{i \in I_{int}} \sum_{j \in N_i} \int_0^T (d_j|\phi'_j(E_i,t)|^2 + m_j|\partial_t u_{nj}(E_i,t)|^2) q_j(E_i)\nu_j(E_i)\,dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{j=1}^{N}\int_{Q_{jT}} q_j'(m_j|\partial_t u_{nj}|^2 + d_j|u_{nj}'|^2)\,dx_j dt - \sum_{j=1}^{N} m_j \int_{Q_{jT}} \lambda(t) a_{nj} q_j u_{nj}'\,dx_j dt \\
&+ \sum_{j=1}^{N} m_j \int_0^{l_j} \partial_t u_{nj} q_j u_{nj}'\,dx_j \Big|_0^T.
\end{aligned}
$$

Choosing q_j identically equal to zero in a neighbourhood of the points ξ_{jk} for all j, k, identically equal to zero in a neighbourhood of the interior vertices and such that

$$
q_{j_i}(E_i)\nu_{j_i}(E_i) = \frac{1}{d_{j_i}}, \forall i \in I_{ext},
$$

the above identity becomes

$$
\begin{aligned}
&\frac{1}{2}\sum_{i\in I_{ext}}\int_0^T |u_{nj_i}'(E_i,t)|^2\,dt \\
&= \frac{1}{2}\sum_{j=1}^{N}\int_{Q_{jT}} q_j'(m_j|\partial_t u_{nj}|^2 + d_j|u_{nj}'|^2)\,dx_j dt \\
&+ \sum_{j=1}^{N} m_j \int_0^{l_j} \partial_t u_{nj} q_j u_{nj}'\,dx_j \Big|_0^T,
\end{aligned}
$$

for n large enough. By Cauchy-Schwarz's inequality we arrive at

$$
\sum_{i\in I_{ext}}\int_0^T |u_{nj_i}'(E_i,t)|^2\,dt \le C(\|u_n\|^2_{C([0,T];V)} + \|u_n\|^2_{C^1([0,T];H)}),
$$

for some positive constant C depending on T and q_j (and then on a_j).

We conclude by passing to the limit in n owing to the property (2.34) and Theorem 2.2.2. ■

2.4 Uniqueness

We first recall Duhamel's principle (see for instance [68, 18]) which gives the relationship between v solution of (2.28) and u solution of (2.1).

Lemma 2.4.1 *Let $u \in C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (2.1) with datum a_j in the form (2.3) or more precisely solution of (2.11) with datum a*

in the form (2.12) and let $v \in C([0,T];H) \cap C^1([0,T];V')$ *be the unique solution of (2.28) with initial speed* a*. Then*

$$u(t) = (Kv)(t), \forall t \in]0,T[, \tag{2.35}$$

where K *is defined by*

$$(K\psi)(t) = \int_0^t \lambda(t-s)\psi(s)\,ds, \forall t \in]0,T[, \tag{2.36}$$

and is a bounded operator from $L^2(0,T)$ *into itself.*

Proof: As in Lemma 2.3.5 we approximate a by a sequence of $a_n \in V$ satisfying (2.31) and consider the solution u_n of (2.11) with datum a_n satisfying (2.33) and (2.34).

Similarly the unique solution v_n of (2.28) with initial speed a_n satisfies

$$v_n \in C^2([0,T];H) \cap C^1([0,T];V) \cap C([0,T];D(A)),$$

and by Lemma 2.3.4

$$v_n \to v \text{ in } C([0,T];H) \cap C^1([0,T];V') \text{ as } n \to \infty.$$

For u_n and v_n, we clearly have

$$u_n(t) = (Kv_n)(t), \forall t \in]0,T[, \tag{2.37}$$

since we simply check that the right-hand side of this identity satisfies the same problem than the left-hand side.

We conclude by passing to the limit in (2.37) using the continuity of K from $L^2(0,T)$ into itself and the above convergence of u_n (resp. v_n) to u (resp. v). ■

For further uses, as in [18] we need to extend the above operator K to the space $H_{-1}(0,T)$ defined as the dual space of

$${}^0H^1(0,T) = \{v \in H^1(0,T) : v(T) = 0\},$$

which is a Hilbert space with the norm

$$\|v\|_{{}^0H^1(0,T)} = \left(\int_0^T |\partial_t v(t)|^2 dt\right)^{1/2}.$$

As the identity mapping from $H^1_0(0,T)$ to ${}^0H^1(0,T)$ is clearly continuous, its adjoint operator $Id^\star$ from $H_{-1}(0,T)$ into $H^{-1}(0,T)$ is continous (but not injective). In other words, for $h \in H_{-1}(0,T)$, we have

$$< Id^\star h, \phi >_{H^{-1}(0,T)-H^1_0(0,T)} = < h, \phi >_{H_{-1}(0,T)-{}^0H^1(0,T)}, \forall \phi \in H^1_0(0,T),$$

and moreover $\|Id^\star h\|_{H^{-1}(0,T)} \leq \|h\|_{H_{-1}(0,T)}$. Note that $Id^\star h$ is simply the restriction of h to $H^1_0(0,T)$.

We can now recall the following result (mainly) proved in [18]:

Lemma 2.4.2 *If $\lambda \in C^1([0,T])$ satisfies (2.2) then the bounded operator K from $L^2(0,T)$ into itself defined by (2.36) can be extended to a bounded operator from $H_{-1}(0,T)$ onto $L^2(0,T)$ and satisfying*

$$C_1\|K\psi\|_{L^2(0,T)} \leq C_1\|\psi\|_{H_{-1}(0,T)} \leq C_2\|K\psi\|_{L^2(0,T)}, \forall \psi \in H_{-1}(0,T), \tag{2.38}$$

for some positive constants C_1, C_2.

Proof: The extension property of K and the estimates in (2.38) are proved in Lemma 2 of [18] and are based on the identity

$$(K\psi, \xi')_{L^2(0,T)} = -(\psi, F\xi)_{L^2(0,T)}, \tag{2.39}$$

valid for all $\psi \in L^2(0,T)$ and all $\xi \in {}^0H^1(0,T)$, where the operator F is defined by

$$(F\xi)(t) = \lambda(0)\xi(t) + \int_t^T \lambda'(s-t)\xi(s)\, ds, 0 < t < T,$$

and is an isomorphism from ${}^0H^1(0,T)$ onto itself.

The surjectivity property of K comes from the fact that the assumption (2.2) guarantees that the range of K as operator from $L^2(0,T)$ into itself is equal to (owing to the identity (6.3) of [68] and the change of variable $\tilde{t} = T - t$, see also section 7 of [19])

$$\{v \in H^1(0,T) : v(0) = 0\}.$$

Therefore the range of its extension is dense in $L^2(0,T)$ and then equal to $L^2(0,T)$ by its closeness, consequence of the estimate (2.38). ■

Since the estimate (2.29) from Lemma 2.3.4 is only valid for the $H^{-1}(0,T)$-norm of $v'_{j_i}(E_i,\cdot)$ we actually need to adapt the above Lemma to the space $H^{-1}(0,T)$. For that purpose we need to introduce the subspace of $L^2(0,T)$:

$$\Lambda^\perp = \{\eta \in L^2(0,T) : (\lambda,\eta)_{L^2(0,T)} = 0\},$$

which means that $\Lambda^\perp$ is the closed subspace of $L^2(0,T)$ made of functions perpendicular to λ. We denote by P the orthogonal projection (in $L^2(0,T)$) on $\Lambda^\perp$. Now we can state the

Lemma 2.4.3 *If $\lambda \in C^1([0,T])$ satisfies (2.2) then the bounded operator PK from $L^2(0,T)$ into itself can be extended to a bounded operator from $H^{-1}(0,T)$ into $L^2(0,T)$ and satisfying*

$$C_3\|PK\psi\|_{L^2(0,T)} \le \|\psi\|_{H^{-1}(0,T)} \le C_4\|PK\psi\|_{L^2(0,T)}, \forall\psi \in H^{-1}(0,T), \tag{2.40}$$

for some positive constants C_3, C_4.

Proof: Since $H^1_0(0,T)$ is a closed subspace of ${}^0H^1(0,T)$, the restriction of the operator F to $W = F^{-1}(H^1_0(0,T))$ is an isomorphism from W into $H^1_0(0,T)$. By the characterization of $H^1_0(0,T)$ we clearly have

$$W = \{\xi \in {}^0H^1(0,T) : (F\xi)(0) = 0\}.$$

Thanks to the identity

$$(F\xi)(0) = \lambda(0)\xi(0) + \int_0^T \lambda'(s)\xi(s)\,ds = -\int_0^T \lambda(s)\xi'(s)\,ds, \forall\xi \in {}^0H^1(0,T),$$

we equivalently have

$$W = \{\xi \in {}^0H^1(0,T) : \int_0^T \lambda(s)\xi'(s)\,ds = 0\}.$$

Now we introduce the differentiation operator

$$D : {}^0H^1(0,T) \to L^2(0,T) : \xi \to \xi',$$

which is an isomorphism since for any $\eta \in L^2(0,T)$, the function ξ given by

$$\xi(t) = -\int_t^T \eta(s)ds,$$

clearly belongs to ${}^0H^1(0,T)$ and satisfies $D\xi = \eta$.

Therefore W can be characterized by

$$W = \{\xi \in {}^0H^1(0,T) : D\xi \in \Lambda^\perp\}. \tag{2.41}$$

Let us now come back to our extension property: Fix $\xi \in L^2(0,T)$, then by the usual embedding of $L^2(0,T)$ into $H^{-1}(0,T)$ we may write

$$\|\psi\|_{H^{-1}(0,T)} = \sup_{\phi \in H_0^1(0,T), \phi \neq 0} \frac{|(\psi, \phi)_{L^2(0,T)}|}{\|\phi'\|_{L^2(0,T)}}.$$

As F is an isomorphism from W into $H_0^1(0,T)$ we then have

$$\|\psi\|_{H^{-1}(0,T)} \sim \sup_{\xi \in W, \xi \neq 0} \frac{|(\psi, F\xi)_{L^2(0,T)}|}{\|\xi'\|_{L^2(0,T)}}.$$

By the identity (2.39) we obtain

$$\|\psi\|_{H^{-1}(0,T)} \sim \sup_{\xi \in W, \xi \neq 0} \frac{|(K\psi, \xi')_{L^2(0,T)}|}{\|\xi'\|_{L^2(0,T)}}.$$

By the characterization (2.41) and the isomorphic property of D, we arrive at

$$\|\psi\|_{H^{-1}(0,T)} \sim \sup_{\eta \in \Lambda^\perp, \eta \neq 0} \frac{|(K\psi, \eta)_{L^2(0,T)}|}{\|\eta\|_{L^2(0,T)}}.$$

We conclude by the density of $L^2(0,T)$ into $H^{-1}(0,T)$. ■

Corollary 2.4.4 *Let $u \in C([0,T]; V) \cap C^1([0,T]; H)$ be the unique solution of (2.11) with datum a in the form (2.12) and let $v \in C([0,T]; H) \cap C^1([0,T]; V')$ be the unique solution of (2.28) with initial speed a. Then for all $T > 0$ and all $i \in I_{ext}$ we have*

$$Pu'_{j_i}(E_i, \cdot) = PKv'_{j_i}(E_i, \cdot) \text{ in } L^2(0,T). \tag{2.42}$$

Proof: As in Lemma 2.4.1 let u_n (resp. v_n) be the solution of (2.11) (resp. (2.28)) with datum a_n (resp. with initial speed a_n) with $a_n \in V$ satisfying (2.31).

For these solutions their regularity and Lemma 2.4.1 allow to write

$$u'_{nj_i}(E_i, \cdot) = Kv'_{nj_i}(E_i, \cdot) \text{ in } L^2(0,T).$$

And therefore

$$Pu'_{nj_i}(E_i,\cdot) = PKv'_{nj_i}(E_i,\cdot) \text{ in } L^2(0,T).$$

We conclude by passing to the limit in n and using Lemmas 2.4.3, 2.3.4 and 2.3.5.

■

We are now ready to formulate the uniqueness result:

Theorem 2.4.5 *Let* u^1 *(resp.* u^2*) in* $C([0,T];V) \cap C^1([0,T];H)$ *be the unique solution of (2.11) with datum* a^1 *(resp.* a^2*) in the form*

$$< a^l, \phi >_{V'-V} = \sum_{j=1}^{N} m_j \sum_{k=1}^{K_j^l} \alpha_{jk}^l \phi_j(\xi_{jk}^l), \forall \phi \in V, l = 1, 2,$$

for some positive integers K_j^l*, real numbers* α_{jk}^l *and points* $\xi_{jk}^l \in]0, l_j[$*. Fix one exterior vertex* E_{i_0} *of* G *and* $T > T_0$ *with* $T_0 > 0$ *from Lemma 2.3.3. If for all* $i \in I_{ext} \setminus \{i_0\}$

$$(u^1)'_{j_i}(E_i,t) = (u^2)'_{j_i}(E_i,t), \forall t \in (0,T),$$

as elements of $L^2(0,T)$*, then* $a^1 = a^2$*, or equivalently* $K_j^1 = K_j^2$*,* $\alpha_{jk}^1 = \alpha_{jk}^2$ *and* $\xi_{jk}^1 = \xi_{jk}^2$.

Proof: We remark that $u = u^1 - u^2$ satisfies (2.11) with datum $a = a^1 - a^2$ which is still in the form (2.12). By the assumption we further have

$$u'_{j_i}(E_i,\cdot) = 0 \text{ in } L^2(0,T), \forall i \in I_{ext} \setminus \{i_0\}.$$

This implies that

$$Pu'_{j_i}(E_i,\cdot) = 0 \text{ in } L^2(0,T), \forall i \in I_{ext} \setminus \{i_0\}.$$

Therefore by Corollary 2.4.4 and Lemma 2.4.3 we get

$$v'_{j_i}(E_i,\cdot) = 0 \text{ in } H^{-1}(0,T), \forall i \in I_{ext} \setminus \{i_0\},$$

where v is the unique solution of (2.28) with initial speed a. The application of Lemma 2.3.4 allows to conclude that $a = 0$. ■

2.5 Stability

For fixed positive integers $K_j, j = 1, \cdots, N$, we denote by

$$\Sigma = \{A = \left((\alpha_{jk}, \xi_{jk})_{k=1}^{K_j}\right)_{j=1}^{N} : \alpha_{jk} \in \mathbb{R} \setminus \{0\}, \xi_{jk} \in]0, l_j[\}.$$

The above uniqueness result implies that the mapping

$$\eta : \Sigma \to (L^2(0,T))^M : A := \left((\alpha_{jk}, \xi_{jk})_{k=1}^{K_j}\right)_{j=1}^{N} \to u' := (u'_{j_i}(E_i, t))_{i \in I_{ext} \setminus \{i_0\}},$$

where u is the unique solution of (2.11) with datum a in the form (2.12), is injective. The stability means that the inverse mapping $\eta^{-1} : u' \to A$ is continuous once Σ is equipped with the natural distance

$$d(A^1, A^2) = \sum_{j=1}^{N} \sum_{k=1}^{K_j} (|\alpha_{jk}^1 - \alpha_{jk}^2| + |\xi_{jk}^1 - \xi_{jk}^2|),$$

when

$$A^l := \left(\left(\alpha_{jk}^l, \xi_{jk}^l\right)_{k=1}^{K_j}\right)_{j=1}^{N}, l = 1, 2.$$

We actually will show a slightly weaker result than the continuity of this mapping by only showing that the inverse of the restriction of η to the ball $B(A, \epsilon)$ is locally Lipschitz for some $\epsilon > 0$ small enough depending on A. Namely we take

$$\epsilon \leq \frac{1}{2} \min_{j,k \neq k'} |\xi_{jk} - \xi_{jk'}|, \tag{2.43}$$

$$\epsilon \leq \frac{1}{2} \min_{j,k} |\xi_{jk}|, \tag{2.44}$$

$$\epsilon \leq \frac{1}{2} \min_{j,k} |\xi_{jk} - l_j|, \tag{2.45}$$

$$\epsilon \leq \frac{1}{2} \min_{j,k} |\alpha_{jk}|. \tag{2.46}$$

Under these assumptions we can prove the following conditional stability result:

Theorem 2.5.1 *Fix one exterior vertex* E_{i_0} *of* G *and* $T > T_0$ *with* $T_0 > 0$ *from Lemma 2.3.3. Suppose that* $A^2 = \left(\left(\alpha_{jk}^2, \xi_{jk}^2\right)_{k=1}^{K_j}\right)_{j=1}^{N}$ *is in* $\Sigma \cap B(A, \epsilon)$ *with* $\epsilon > 0$

satisfying the above constraints. Then there exists a constant C *depending on* T, $\min_{j,k\neq k'}|\xi_{jk}-\xi_{jk'}|$ *and* $\min_{j,k}|\alpha_{jk}|$ *such that*

$$\sum_{j=1}^{N}\sum_{k=1}^{K_j}(|\alpha_{jk}-\alpha^2_{jk}|+|\xi_{jk}-\xi^2_{jk}|)\leq C(1+\sqrt{\epsilon})\sum_{i\in I_{ext}\setminus\{i_0\}}\|u'_{j_i}(E_i,t)-(u^2)'_{j_i}(E_i,t)\|_{L^2(0,T)}.$$

Proof: The proof of Theorem 2.4.5 clearly shows that

$$\|a-a^2\|_{V'}\leq C\sum_{i\in I_{ext}\setminus\{i_0\}}\|u'_{j_i}(E_i,t)-(u^2)'_{j_i}(E_i,t)\|_{L^2(0,T)}. \tag{2.47}$$

Therefore it remains to estimate from below the norm of $a-a^2$ in V'. For that purpose we recall that

$$\|a-a^2\|_{V'}=\sup_{\phi\in V,\phi\neq 0}\frac{|<a-a^2,\phi>|}{\|\phi\|_V},$$

and use appropriate test functions ϕ. First we take

$$\begin{aligned}\phi^{(jk)}_j(x_j) &= \phi_1(\frac{x_j-\xi_{jk}}{\epsilon}) \text{ on }]0,l_j[,\\ \phi^{(jk)}_{j'} &= 0 \text{ if } j'\neq j,\end{aligned}$$

where ϕ_1 is a fixed function defined by

$$\phi_1(\hat{x})=\begin{cases}-2(3/2+\hat{x}) \text{ if } -3/2<\hat{x}\leq -1,\\ \hat{x} \text{ if } -1<\hat{x}\leq 0,\\ -\hat{x} \text{ if } 0\leq\hat{x}<1,\\ 2(3/2-\hat{x}) \text{ if } 1\leq\hat{x}<3/2,\\ 0 \text{ else.}\end{cases}$$

With this choice we have

$$\begin{aligned}<a-a^2,\phi^{(jk)}> &= \alpha_{jk}\phi^{(jk)}(\xi_{jk})-\alpha^2_{jk}\phi^{(jk)}(\xi^2_{jk})\\ &= \alpha^2_{jk}(\phi^{(jk)}(\xi_{jk})-\phi^{(jk)}(\xi^2_{jk})),\end{aligned}$$

since $\phi^{(jk)}(\xi_{jk})=0$. By the finite increment theorem and the fact that $|\xi_{jk}-\xi^2_{jk}|<\epsilon$, we then obtain

$$|<a-a^2,\phi^{(jk)}>|=\frac{|\alpha^2_{jk}|}{\epsilon}|\xi_{jk}-\xi^2_{jk}|.$$

This estimate yields

$$|\alpha^2_{jk}||\xi_{jk}-\xi^2_{jk}| \leq \epsilon| < a-a^2, \phi^{(jk)} > | \leq \epsilon \|a-a^2\|_{V'}\|\phi^{(jk)}\|_V,$$

and leads to

$$|\alpha^2_{jk}||\xi_{jk}-\xi^2_{jk}| \leq C\sqrt{\epsilon}\|a-a^2\|_{V'}, \tag{2.48}$$

for some positive constant C since one readily checks that $\|\phi^{(jk)}\|_V = \frac{C}{\sqrt{\epsilon}}$.

From the fourth assumption on ϵ, we have

$$|\alpha^2_{jk}| \geq m/2,$$

where $m = \min_{j,k} |\alpha_{jk}|$. These two estimates finally give

$$|\xi_{jk}-\xi^2_{jk}| \leq \frac{2C\sqrt{\epsilon}}{m}\|a-a^2\|_{V'}. \tag{2.49}$$

Now we take

$$\begin{aligned} \phi^{(jk)}_j(x_j) \quad &= \quad \phi_2(\frac{x_j-\xi_{jk}}{\delta}) \text{ on }]0, l_j[, \\ \phi^{(jk)}_{j'} \quad &= 0 \quad \text{if } j' \neq j, \end{aligned}$$

where $\delta = \frac{1}{2}\min_{j,k\neq k'} |\xi_{jk}-\xi_{jk'}|$ and ϕ_2 in the form

$$\phi_2(\hat{x}) = \begin{cases} \hat{x}+1 \text{ if } -1 < \hat{x} \leq 0, \\ 1-\hat{x} \text{ if } 0 \leq \hat{x} < 1, \\ 0 \text{ else.} \end{cases}$$

With this choice we have

$$\begin{aligned} < a-a^2, \phi^{(jk)} > \quad &= \quad \alpha_{jk}\phi^{(jk)}(\xi_{jk}) - \alpha^2_{jk}\phi^{(jk)}(\xi^2_{jk}) \\ &= \quad (\alpha_{jk}-\alpha^2_{jk})\phi^{(jk)}(\xi_{jk}) + \alpha^2_{jk}(\phi^{(jk)}(\xi_{jk}) - \phi^{(jk)}(\xi^2_{jk})), \\ &= \quad (\alpha_{jk}-\alpha^2_{jk}) + \alpha^2_{jk}(\phi^{(jk)}(\xi_{jk}) - \phi^{(jk)}(\xi^2_{jk})). \end{aligned}$$

Therefore by the finite increment theorem we obtain as before

$$|\alpha_{jk}-\alpha^2_{jk}| \leq | < a-a^2, \phi^{(jk)} > | + \frac{1}{\delta}|\alpha^2_{jk}||\xi_{jk}-\xi^2_{jk}|,$$

and by the estimate (2.48) we get

$$|\alpha_{jk} - \alpha^2_{jk}| \leq | < a - a^2, \phi^{(jk)} > | + \frac{C\sqrt{\epsilon}}{\delta} \|a - a^2\|_{V'}.$$

Since $\|\phi^{(jk)}\|_V = \frac{C_1}{\sqrt{\delta}}$ for some $C_1 > 0$, we have obtained

$$|\alpha_{jk} - \alpha^2_{jk}| \leq (\frac{C_1}{\sqrt{\delta}} + \frac{C\sqrt{\epsilon}}{\delta}) \|a - a^2\|_{V'}. \tag{2.50}$$

The estimates (2.47), (2.49) and (2.50) lead to the conclusion. ■

In the above theorem if like in [18] we are only interested in the stability of the locations of the point sources, i.e. if we assume that $\alpha^2_{jk} = \alpha_{jk}$, then we can obtain a more accurate estimate under less assumptions on ϵ, namely we have the

Theorem 2.5.2 *Fix one exterior vertex E_{i_0} of G and $T > T_0$ with $T_0 > 0$ from Lemma 2.3.3. Suppose that $A^2 = \left(\left(\alpha_{jk}, \xi^2_{jk} \right)_{k=1}^{K_j} \right)_{j=1}^{N}$ is in $\Sigma \cap B(A, \epsilon)$ with $\epsilon > 0$ satisfying (2.43) to (2.45). Then there exists a constant C depending on T, $\min_{j,k\neq k'} |\xi_{jk} - \xi_{jk'}|$ and $\min_{j,k} |\alpha_{jk}|$ such that*

$$\sum_{j=1}^{N} \sum_{k=1}^{K_j} |\xi_{jk} - \xi^2_{jk}| \leq C\sqrt{\epsilon} \sum_{i \in I_{ext} \setminus \{i_0\}} \|u'_{j_i}(E_i, t) - (u^2)'_{j_i}(E_i, t)\|_{L^2(0,T)}.$$

Proof: It suffices to take

$$\begin{aligned} \phi^{(jk)}_j(x_j) \quad &= \quad \phi_2(\frac{x_j - \xi_{jk}}{\epsilon}) \text{ on }]0, l_j[, \\ \phi^{(jk)}_{j'} \quad &= 0 \quad \text{if } j' \neq j, \end{aligned}$$

with the same ϕ_2 as before and use the above arguments. ■

Remark 2.5.3 The constraints (2.44) and (2.45) could be suppressed but this requires choices of more tricky test functions $\phi^{(jk)}$ and will modify the constants in the estimates of Theorems 2.5.1 and 2.5.2. We do not treat these cases for the sake of simplicity. ■

2.6 Reconstruction

For the reconstruction of the point sources from boundary measurements we follow the point of view of [68] which consists in using the following exact controllability result:

Lemma 2.6.1 *Fix one exterior vertex* E_{i_0} *of* G *and* $T > T_0$ *with* $T_0 > 0$ *from Lemma 2.3.3. Then for every* $\phi \in V$*, there exist unique controls* $v_{j_i} \in H_0^1(0,T)$*,* $i \in I_{ext} \setminus \{i_0\}$*, such that the (weak) solution* $\psi \in C([0,T];H) \cap C^1([0,T];V')$ *of*

$$\begin{cases} \partial_t^2 \psi_j(x_j,t) - \frac{d_j}{m_j}\psi_j"(x_j,t) = 0 \text{ in } Q_{jT}, \forall j = 1, \cdots, N, \\ \psi(\cdot,t) = 0 \text{ is continuous on } G \text{ for all } t \in]0,T[, \\ \sum_{j\in N_i} d_j \frac{\partial \psi_j}{\partial \nu_j}(E_i,t) = 0, \forall i \in I_{int}, \forall t \in]0,T[, \\ \psi_{j_i}(E_i,t) = v_{j_i}, \forall i \in I_{ext} \setminus \{i_0\}, \forall t \in]0,T[, \\ \psi_{j_{i_0}}(E_{i_0},t) = 0, \forall t \in]0,T[, \\ \psi_j(x_j,0) = \phi_j(x_j), \partial_t \psi_j(x_j,0) = 0 \text{ in }]0,l_j[, \forall j = 1, \cdots, N, \end{cases} \tag{2.51}$$

satisfies

$$\psi(\cdot,T) = \partial_t \psi(\cdot,T) = 0. \tag{2.52}$$

Proof: This is a direct consequence of Lemmas 2.3.2, 2.3.3 and 2.3.4 and of the Hilbert Uniqueness Method of Lions [48, Th. I.6.4] (see for instance [65, 43, 27] for such applications to networks). Note that ψ is only a weak solution of the system (2.51) with the final conditions (2.52) in the sense that ψ is the unique solution of (using the transposition method)

$$\begin{aligned} \sum_{j=1}^N m_j \int_{Q_{jT}} \psi_j f_j \, dxdt &= -< \partial_t \varphi(0), \phi >_{V'-V} \\ &- \sum_{i\in I_{ext}\setminus\{i_0\}} d_{j_i}\nu_{j_i}(E_i) < \varphi'_{j_i}(E_i,\cdot), v_{j_i} >_{H^{-1}(0,T)-H_0^1(0,T)}, \end{aligned} \tag{2.53}$$

for all $f \in L^1(0,T;H), \varphi_0 \in H, \varphi_1 \in V'$, where $\varphi \in C([0,T];H) \cap C^1([0,T];V')$ is the unique solution of (whose existence follows from Lemma 2.3.4)

$$\begin{cases} \partial_t^2 \varphi = A\varphi + f \text{ in }]0,T[, \\ \varphi(T) = \varphi_0, \partial_t \varphi(T) = \varphi_1 \ . \end{cases}$$

■

In view of Lemma 2.6.1 we can define a bounded linear operator $\Pi : V \to H_0^1(0,T)^M$, by

$$\phi \to (v_{j_i})_{i\in I_{ext}\setminus\{i_0\}},$$

where M is the cardinal of $I_{ext}\setminus\{i_0\}$ and v_{j_i} are the controls from the above Theorem driving the system (2.51) to rest at time T.

We further use the adjoint $K^\star_{L^2}$ of the operator K as (bounded) operator from $L^2(0,T)$ into itself and which is given by (see section 6 of [68])

$$(K^\star_{L^2}\eta)(t) = \int_t^T \lambda(s-t)\eta(s)\,ds, 0 < t < T,$$

for all $\eta \in L^2(0,T)$. By the assumption (2.2) we even have (see section 6 of [68])

$$R(K^\star_{L^2}) = {}^0H^1(0,T).$$

Consequently for all $\psi \in {}^0H^1(0,T)$ there exists a unique $\eta \in L^2(0,T)$ solution of (since $\ker K^\star_{L^2} = R(K)^\perp = \{0\}$)

$$K^\star_{L^2}\eta = \psi,$$

equivalently, η is solution of the Volterra equation of the first kind

$$\int_t^T \lambda(s-t)\eta(s)\,ds = \psi(t), 0 < t < T.$$

We then define the mapping Φ from ${}^0H^1(0,T)$ to $L^2(0,T)$ by

$$\psi \to \eta := \Phi\psi,$$

when η is solution of the above integral equation. This means that

$$K^\star_{L^2}\Phi = Id \text{ on } {}^0H^1(0,T). \tag{2.54}$$

Now we can formulate our reconstruction result:

Theorem 2.6.2 *Fix one exterior vertex E_{i_0} of G and $T > T_0$ with $T_0 > 0$ from Lemma 2.3.3. For all $k = 1, \cdots, \infty$ we define*

$$\theta_k = \Phi\Pi\phi_k.$$

Let $u \in C([0,T];V) \cap C^1([0,T];H)$ *be the unique solution of (2.11) with datum* a *in the form (2.12). Then for all* $k = 1, \cdots, \infty$ *we have*

$$< a, \phi_k >= - \sum_{i \in I_{ext} \setminus \{i_0\}} d_{j_i} \nu_{j_i}(E_i) (u'_{j_i}(E_i, \cdot), (\theta_k)_{j_i})_{L^2(0,T)}, \tag{2.55}$$

and then a *may be reconstructed by*

$$a = \sum_{k=1}^{\infty} < a, \phi_k > \phi_k = - \sum_{k=1}^{\infty} \left(\sum_{i \in I_{ext} \setminus \{i_0\}} d_{j_i} \nu_{j_i}(E_i) (u'_{j_i}(E_i, \cdot), (\theta_k)_{j_i})_{L^2(0,T)} \right) \phi_k.$$

Proof: Applying the identity (2.53) with $\varphi = v$, where v is the unique solution of (2.28) with initial speed a we have:

$$< a, \phi_k >= - \sum_{i \in I_{ext} \setminus \{i_0\}} d_{j_i} \nu_{j_i}(E_i) < v'_{j_i}(E_i, \cdot), (\Pi \phi_k)_{j_i} >_{H^{-1}(0,T) - H^1_0(0,T)} . \tag{2.56}$$

To conclude we need to show that

$$< v'_{j_i}(E_i, \cdot), (\Pi \phi_k)_{j_i} >_{H^{-1}(0,T) - H^1_0(0,T)} = (u'_{j_i}(E_i, \cdot), (\theta_k)_{j_i})_{L^2(0,T)}. \tag{2.57}$$

Let us first prove that there exists $h_{j_i} \in H_{-1}(0,T)$ such that

$$u'_{j_i}(E_i, \cdot) = K h_{j_i}, \tag{2.58}$$

and satisfying

$$< v'_{j_i}(E_i, \cdot), \chi >_{H^{-1}(0,T) - H^1_0(0,T)} =< h_{j_i}, \chi >_{H_{-1}(0,T) - {}^0H^1(0,T)}, \forall \chi \in H^1_0(0,T). \tag{2.59}$$

Indeed the identity (2.58) clearly follows from Lemmas 2.3.5 and 2.4.2; moreover using an approximation sequence of a_n as usual, the corresponding u_n and v_n satisfy

$$v'_{nj_i}(E_i, \cdot) \to h_{j_i} \text{ in } H_{-1}(0,T), \text{ as } n \to \infty,$$

due to Lemmas 2.3.5 and 2.4.2, while by Lemma 2.3.4 we have

$$v'_{nj_i}(E_i, \cdot) \to v'_{j_i}(E_i, \cdot) \text{ in } H^{-1}(0,T), \text{ as } n \to \infty.$$

The identity (2.59) then follows from the two above convergence properties and the continuity of the mapping $Id^\star$ from $H_{-1}(0,T)$ into $H^{-1}(0,T)$.

Now by the definition of θ_k and (2.54) we may write

$$K^{\star}_{L^2}\theta_k = K^{\star}_{L^2}\Phi\Pi\phi_k = \Pi\phi_k.$$

Therefore using (2.59) and the above identity, the left-hand side of (2.57) may be transformed as follows

$$\begin{aligned} < v'_{j_i}(E_i,\cdot), (\Pi\phi_k)_{j_i} >_{H^{-1}(0,T)-H^1_0(0,T)} &= \ < h_{j_i}, (\Pi\phi_k)_{j_i} >_{H_{-1}(0,T)-{}^0H^1(0,T)} \\ &= \ < h_{j_i}, K^{\star}_{L^2}(\theta_k)_{j_i} >_{H_{-1}(0,T)-{}^0H^1(0,T)}, \end{aligned}$$

and from the embeddings ${}^0H^1(0,T) \hookrightarrow L^2(0,T) \hookrightarrow H_{-1}(0,T)$, we get

$$< h_{j_i}, K^{\star}_{L^2}(\theta_k)_{j_i} >_{H_{-1}(0,T)-{}^0H^1(0,T)} = (Kh_{j_i}, (\theta_k)_{j_i})_{L^2(0,T)}.$$

This proves (2.57) since the above right-hand side coincides with the right-hand side of (2.57) due to (2.58). ■

Remark 2.6.3 In [18] Bruckner and Yamamoto use another method for the reconstruction of point sources in the real interval $]0,1[$. This method consists in solving a finite system of nonlinear equations but it relies on the fact that the series

$$\sum_{k=1}^{\infty} \frac{\phi_k(x)\phi_k(\xi)}{\lambda_k}, \text{ for } x,\xi \in]0,1[$$

is differentiable in $x \in [0,\xi]$, this differentiability property being proved by computing explicitly the above series by Fourier analysis, which is possible since $\lambda_k = k^2\pi^2$ and $\phi_k(x) = \sqrt{2}\sin(k\pi x)$. In our case the calculation of the above series as well as its differentiability cannot be easily obtained since the eigenvalues and eigenvectors are not explicitly known and since λ_k behaves like k^2 (the series of the derivatives being not absolutely convergent). For the real interval $]0,1[$, our method gives an alternative way to reconstruct the source a without any restriction on a, i.e., for any values of the parameters in (2.12). The only drawback is that a is obtained as a Fourier series which may converge slowly. ■

Chapter 3

Identifiability, stability and reconstruction results of sources by interior measurements

3.1 Introduction

Inverse problems of distributed parameter systems is in our days an expanding field. Here we are mainly concerned with the determination of some sources using some observations. As usual in such problems the three main steps are the uniqueness (unique solvability of the problem), the stability (small perturbations of the measurements give rise to small perturbations of the sources) and finally the reconstruction (build appropriate processes in order to find a good approximation of the unknowns).

The resolution of such problems using control results of distributed systems (like the wave equation, Petrowsky systems, etc...) have been recently developed, in particular by Yamamoto and coautors [68, 18, 19, 69]. The main idea is to use some observability estimates and controllabillity results, using for instance the so-called multiplier method and the Hilbert Uniqueness Method [48], to deduce the uniqueness and the reconstruction process. For the wave equation this method successfully leads to the reconstruction of point sources in 1-dimensional domains by boundary observations in [18, 19, 38, 58]. In higher dimensional domains the same technique leads to the reconstruction of smoother unknown sources using boundary observations [67, 68]. In [19] the authors consider interior pointwise observations for the determination of the point sources in $]0,1[$. For the the standard Petrovsky system (vibrations of beams or plates), pointwise and line observations are treated in a similar spirit in [69].

We consider the inverse problem of determining wave sources in bounded domains.We show that the interior observation on a part of the domain determines uniquely the sources. We further establish conditional stabilities for some particular unknown sources. we finally give a reconstructing scheme. We answer to this question by adapting some results from [17, 18, 68, 58]. The main ingredients are first the existence of some observability estimates obtained in practice by some interior controllability results [48, 49] and second appropriate properties of some integral operators [68, 18]. Since the eigenvalues and eigenvectors of the Laplace equation are not explicitly known, our reconstruction process is different from the one in [18] and is more close to the one in [68].

The chapter is organized as follows: In section 3.2 we recall the wave equation with some special sources and present the inverse problem we have in mind. In section 3.3 we introduce the notion of strategic subset, which means that some observability estimates hold for this domain, we further present some examples of strategic subsets. Section 3.4 is devoted to the proof of the uniqueness result and is based on the previous observability estimates and some properties of an integral operator between different Sobolev spaces. The conditional stability for some particular unknown sources, i. e. linear combination of Dirac functions or approximations of them (see below for the specific definitions), is deduced in section 3.5. Finally the reconstruction is detailed in section 3.6 .

3.2 Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with a Lipschitz boundary Γ. On this domain we consider the following wave equation:

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) = \lambda(t)a(x) \text{ in } Q_T, \\ u(x,t) = 0 \text{ on } \Sigma_T, \\ u(x,0) = \partial_t u(x,0) = 0 \text{ in } \Omega, \end{cases} \tag{3.1}$$

where $Q_T := \Omega\times]0,T[$, $\Sigma_T := \Gamma\times]0,T[$. Above and below $\lambda \in C^1([0,T])$ is a given function satisfying

$$\lambda(0) \neq 0. \tag{3.2}$$

The unknow source a is assumed to be in $L^2(\Omega)$ (or in $H^{-1}(\Omega)$ if $n = 1$) so that our wave equation (3.1) has a unique (weak) solution u satisfying

$$u \in C([0,T];V) \cap C^1([0,T];H),$$

where for shorthness we write $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$.

Our goal is to identify the unknown source a from interior measurements, namely the value of $\partial_t u(x,t)$, for $0 < t < T$ and all x in a fixed subdomain ω of Ω.

3.3 Some observability estimates

As usual the identifiability results for the system (3.1) is based on some observability estimates for the unique solution $v \in C([0,T];H) \cap C^1([0,T];V')$ of

$$\begin{cases} \partial_t^2 v - \Delta v = 0 \text{ in }]0,T[, \\ v(0) = 0, \partial_t v(0) = a \ . \end{cases} \tag{3.3}$$

Accordingly we make the following definition:

Definition 3.3.1 *A subdomain ω of Ω is said to be strategic if there $T > 0$ large enough and two positive constants C_1 and C_2 depending on T such that*

$$C_1\|a\|_{V'} \leq (\int_0^T \int_\omega |v(x,t)|^2 \, dxdt)^{1/2} \leq C_2\|a\|_{V'}, \tag{3.4}$$

when v is the unique solution of (3.3).

Let us give some examples of strategic subdomains ω: The one-dimensional case is quite easy and use a spectral decomposition (see [39] for a similar point of view).

Lemma 3.3.2 *Let Ω be the real interval $(0,1)$, then any open subinterval ω of Ω is strategic, namely for $T>2$, the estimates (3.4) hold.*

Proof: Fix x_0 and $\epsilon>0$ such that $\omega=(x_0-\epsilon,x_0+\epsilon)$.

By the spectral theorem, v is given by

$$v(x,t)=\sum_{k=1}^{\infty} b_k \frac{\sin(k\pi t)}{k\pi}\sin(k\pi x),$$

when the initial datum is given by

$$a=\sum_{k=1}^{\infty} b_k \sin(k\pi\cdot).$$

By Ingham's inequalities [35], for $T>2$ we then have

$$\int_0^T\int_\omega |v(x,t)|^2\,dxdt \sim \sum_{k=1}^{\infty}\frac{|b_k|^2}{k^2\pi^2}\int_{x_0-\epsilon}^{x_0+\epsilon}|\sin(k\pi x)|^2\,dx.$$

By explicit calculations we have

$$\int_{x_0-\epsilon}^{x_0+\epsilon}|\sin(k\pi x)|^2\,dx=\epsilon-\frac{\sin(2k\pi\epsilon)\cos(2k\pi x_0}{4k\pi}.$$

Therefore there exists k_0 such that for $k\geq k_0$ we have

$$\epsilon/2\leq\int_{x_0-\epsilon}^{x_0+\epsilon}|\sin(k\pi x)|^2\,dx\leq 2\epsilon.$$

And the requested estimates follow. ■

In higher dimension we can still evoke the spectral decomposition for some special domains (see [39]), like a sphere or a truncated cone and a subdomain ω near the boundary. But these examples are covered by the method introduced by E. Zuazua based on the so-called HUM method of J.-L. Lions [48] and using the multiplier method:

Lemma 3.3.3 *[Zuazua, Theorem VII.2.5 of [48]] Assume that Ω is a bounded domain of $\mathbb{R}^n$, $n\geq 2$, which has a $C^{1,1}$ boundary, or is convex, or is a polygonal domain* *of the plane with a Lipschitz boundary or is a polyhedral domain of the space with a Lipschitz boundary. Fix* $x_0 \in \mathbb{R}^n$*. For a subset* S *of* $\mathbb{R}^n$ *and* $\epsilon > 0$*, set*

$$\begin{aligned} \mathcal{N}_\epsilon\left[S\right] &:= \cup_{x\in S}\{y \in \mathbb{R}^n : |y-x| < \epsilon\}, \\ \Gamma(x_0) &:= \{x \in \partial\Omega : (x-x_0)\cdot n(x) > 0\}, \end{aligned}$$

where $n(x)$ *is the unit exterior normal vector of* $\partial\Omega$ *at* x*. Then a subdomain* ω *satisfying*

$$\Omega \cap \mathcal{N}_\epsilon\left[\Gamma(x_0)\right] \subset \omega$$

for some $\epsilon > 0$*, is strategic.*

Proof: See sections 2.3 and 2.4 of [48] and in particular Theorem VII.2.5 of [48].

■

Lemma 3.3.4 *Fix* G *a (relatively) open subset of the unit sphere* S_{n-1} *of* $\mathbb{R}^n$*, with* $n = 2$ *or 3 with a smooth boundary. Let* $\Omega = \{x = re^{i\omega} \in \mathbb{R}^n : r < 1, \omega \in G\}$ *be the associated truncated cone in* $\mathbb{R}^n$*. Then for all* $\epsilon \in (0,1)$ *the subdomain* $\omega = \{x \in \Omega : 1-\epsilon < |x| < 1\}$ *is strategic.*

Proof: Let us set $C = \{x \in \partial\Omega : |x| = 1\}$. Using the technique of Theorem 4.2 of [34] with the multiplier $m(x) = x$, there exists $T_0 > 0$ such that for all $T > T_0$, the solution v of

$$\left\{\begin{array}{l} \partial_t^2 v - \Delta v = 0 \text{ in }]0,T[, \\ v(0) = v_0, \partial_t v(0) = v_1\ . \end{array}\right. \tag{3.5}$$

satisfies

$$C_3(\|v_0\|_V^2 + \|v_1\|_H^2) \le \int_0^T \int_C |\frac{\partial v}{\partial n}(x,t)|^2\, ds(x)dt \le C_4(\|v_0\|_V^2 + \|v_1\|_H^2), \tag{3.6}$$

for some positive constants C_3, C_4 depending on T. Consequently the arguments of Theorem VII.2.5 of [48] leads to the estimates (3.4). ■

In the above lemma in dimension 2, the case $G =]0,2\pi[$ is allowed, this is an example of a domain Ω with a crack for which the results below hold.

Let us finally mention that the piecewise multiplier method of Liu [49] allows to show that some “internal” subdomains are strategic:

Lemma 3.3.5 *Assume that* Ω *is either convex or has a* $C^{1,1}$ *boundary and that there exists open sets* $\Omega_j \subset \Omega$ *with a Lipschitz boundary* $\partial\Omega_j$*, and points* $x_0^j \in \mathbb{R}^n$*,* $j = 1, \cdots, J$ *such that* $\Omega_i \cap \Omega_j = \emptyset$ *if* $i \neq j$*. Set*

$$\Gamma_j := \{x \in \partial\Omega_j : (x - x_0^j) \cdot n_j(x) > 0\},$$

where $n_j(x)$ *is the unit exterior normal vector of* $\partial\Omega_j$ *at* x*. Then a subdomain* ω *satisfying*

$$\Omega \cap \mathcal{N}_\epsilon \left[\left(\cup_{j=1}^J \Gamma_j \right) \cup \left(\Omega \setminus \cup_{j=1}^J \Omega_j \right) \right] \subset \omega$$

for some $\epsilon > 0$*, is strategic.*

Proof: By Theorems 4.2 and 2.3 of [49], there exist $T > 0$ and $\delta > 0$ such that any solution v of (3.5) satisfies

$$\int_0^T \int_\omega (|\nabla v(x,t)|^2 + |\frac{\partial v}{\partial t}(x,t)|^2)\, dxdt \geq \delta(\|v_0\|_V^2 + \|v_1\|_H^2). \tag{3.7}$$

Since the energy of our system is constant, the inverse estimate

$$\int_0^T \int_\omega (|\nabla v(x,t)|^2 + |\frac{\partial v}{\partial t}(x,t)|^2)\, dxdt \leq T(\|v_0\|_V^2 + \|v_1\|_H^2)$$

clearly holds. These two estimates and the weaker norm arguments of section 2.4 in [48] allow to obtain the estimates (3.4). ■

Note that the case $J = 1$ corresponds to the case of Lemma 3.3.3. As an example (see Remark 4.3 of [48]) we may take the rectangle $\Omega =]0, l_1[\times]0, l_2[$ and $\omega =]x_1 - \epsilon, x_1 + \epsilon[\times]x_2 - \epsilon, x_2 + \epsilon[$, for any $0 < x_i < l_i$ and $\epsilon < \min_{i=1,2}\{x_i, l_i - x_i\}$.

3.4 Uniqueness

We first recall Duhamel's principle (see for instance [68, 18]) which gives the relationship between v solution of (3.3) and u solution of (3.1).

Lemma 3.4.1 *Let* $u \in C([0,T]; V) \cap C^1([0,T]; H)$ *be the unique solution of (3.1) with unknown source* $a \in L^2(\Omega)$ *(or in* V' *if* $n = 1$*) and let* $v \in C([0,T]; H) \cap C^1([0,T]; V')$ *be the unique solution of (3.3) with initial spead* a*. Then*

$$u(t) = (Kv)(t), \forall t \in]0, T[, \tag{3.8}$$

where K is defined by

$$(K\psi)(t) = \int_0^t \lambda(t-s)\psi(s)\, ds, \forall t \in]0,T[, \tag{3.9}$$

and is a bounded operator from $L^2(0,T;H)$ into itself.

We can now recall the following result proved in [68]:

Lemma 3.4.2 *If $\lambda \in C^1([0,T])$ satisfies (3.2) then the bounded operator K from $L^2(0,T;L^2(\omega))$ into itself defined by (3.9) is an isomorphism from $L^2(0,T;L^2(\omega))$ into ${}_0H^1(0,T;L^2(\omega))$, where*

$${}_0H^1(0,T;L^2(\omega)) = \{v \in H^1(0,T;L^2(\omega)) | v(t=0,\cdot) = 0\},$$

equipped with the norm

$$\left(\int_0^T \int_\omega |\partial_t v|^2\, dxdt\right)^{1/2}.$$

Let us now give a consequence to the solution u of problem (3.1):

Lemma 3.4.3 *Let $u \in C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (3.1) with unknown source $a \in L^2(\Omega)$ (or in V' if $n=1$) and let ω be a strategic subset of Ω. Then for $T>0$ large enough it holds*

$$C_1\|a\|_{V'} \le (\int_0^T \int_\omega |\partial_t u|^2\, dxdt)^{1/2} \le C_2\|a\|_{V'}, \tag{3.10}$$

for some positive constants C_1, C_2 depending on T.

Proof: By Lemmas 3.4.1 and 3.4.2 we clearly have

$$\int_0^T \int_\omega |\partial_t u|^2\, dxdt \sim \int_0^T \int_\omega |v|^2\, dxdt.$$

We then conclude from the estimates (3.4). ∎

We are now ready to formulate the uniqueness result:

Theorem 3.4.4 *Let u^1 (resp. u^2) in $C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (3.1) with unknown source a^1 (resp. a^2) in $L^2(\Omega)$ (or in V' if $n=1$). Fix a strategic subset ω of Ω and $T>0$ large enough from Definition 3.3.1. If*

$$\partial_t u^1 = \partial_t u^2 \text{ on } \omega \times (0,T),$$

then $a^1 = a^2$.

Proof: We remark that $u = u^1 - u^2$ satisfies (3.1) with $a = a^1 - a^2$. By the assumption we further have

$$\partial_t u = 0 \text{ on } \omega \times (0, T).$$

Therefore by Lemma 3.4.3 we get $a = 0$ in $H^{-1}(\Omega)$ and then if $n > 1$, in $L^2(\Omega)$ since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$. ■

3.5 Stability

Usually conditional stability results are only obtained for specific unknown sources [17, 18, 68, 58]. We here restrict ourselves to three kinds of unknown sources: The first case concerns the one-dimensional situation $n = 1$. In that case we assume that the unknown sources a^l, $l = 1, 2$ are linear combinations of Dirac functions, namely of the form:

$$< a^l, \phi >= \sum_{k=1}^{K} \alpha_k^l \phi(\xi_k^l), \forall \phi \in H^1(0, L),$$

for some positive integer K, some real numbers α_k^l different from zero and some different points ξ_k^l in $\Omega = (0, L)$. This case was considered in [18, 58] for boundary observations, but the same idea yields a stability result in the case of internal observations. Namely we have the

Theorem 3.5.1 *In the setting of the first case, fix a strategic subset ω of Ω and $T > 0$ large enough from Definition 3.3.1. Suppose that*

$$|\alpha_k^1 - \alpha_k^2| + |\xi_k^1 - \xi_k^2| \le \epsilon, \forall k = 1, \cdots, K,$$

with $\epsilon > 0$ satisfying the constraints

$$\epsilon \le \frac{1}{2} \min_{k \neq k'} |\xi_k^1 - \xi_{k'}^1|, \tag{3.11}$$

$$\epsilon \le \frac{1}{2} \min_{k} |\xi_k^1|, \tag{3.12}$$

$$\epsilon \le \frac{1}{2} \min_{k} |\xi_{jk}^1 - L|, \tag{3.13}$$

$$\epsilon \le \frac{1}{2} \min_{k} |\alpha_k^1|. \tag{3.14}$$

Then there exists a constant C depending on T, $\min_{k\neq k'}|\xi_k^1-\xi_{k'}^1|$ and $\min_k|\alpha_k^1|$ such that

$$\sum_{k=1}^{K}(|\alpha_k^1-\alpha_k^2|+|\xi_k^1-\xi_k^2|)\leq C(1+\sqrt{\epsilon})\|\partial_t u^1-\partial_t u^2\|_{L^2(0,T;L^2(\omega))}.$$

Proof: By Lemma 3.4.3 we clearly have

$$\|a^1-a^2\|_{V'}\leq C\|\partial_t u^1-\partial_t u^2\|_{L^2(0,T;L^2(\omega))}. \tag{3.15}$$

Therefore it remains to estimate from below the norm of a^1-a^2 in V'. For that purpose we recall that

$$\|a^1-a^2\|_{V'}=\sup_{\phi\in V,\phi\neq 0}\frac{|<a^1-a^2,\phi>|}{\|\phi\|_V},$$

and use appropriate test functions ϕ.

In this situation we use the test functions ϕ used in Theorem 5.1 of [58] to conclude that

$$\sum_{k=1}^{K}(|\alpha_k^1-\alpha_k^2|+|\xi_k^1-\xi_k^2|)\leq C(1+\sqrt{\epsilon})\|a^1-a^2\|_{V'},$$

for some positive constant C depending on T, $\min_{k\neq k'}|\xi_k^1-\xi_{k'}^1|$ and $\min_k|\alpha_k^1|$.

The two above estimates lead to the conclusion. ■

In the above setting for the determination of the locations of point sources only (i.e. $\alpha_k^l=1$, for all k,l), using the results from [38], the above results may be sharpened as follows:

Theorem 3.5.2 *In the setting of the first case assume that $\alpha_k^l=1$, for all $k=1,\cdots,K,l=1,2$. Fix a strategic subset ω of Ω and $T>0$ large enough from Definition 3.3.1. Suppose that*

$$|\xi_k^1-\xi_k^2|\leq\epsilon,\forall k=1,\cdots,K,$$

with $\epsilon\in(0,1/2)$ satisfying the constraint (3.11). Then there exists a constant C depending on T and K such that

$$\sum_{k=1}^{K}|\xi_k^1-\xi_k^2|\leq C\|\partial_t u^1-\partial_t u^2\|_{L^2(0,T;L^2(\omega))}.$$

Proof: From the assumptions on ξ_k^l and ϵ and Proposition 2 of [38], there exists a constant C depending on K such that

$$\sum_{k=1}^{K} |\xi_k^1 - \xi_k^2| \leq C\|a^1 - a^2\|_{V'}.$$

The conclusion then follows from the estimate (3.15). ■

The second case concerns the multi-dimensional case $n \geq 1$ (arbitrary) and takes for a^l, the $(L^2(\Omega))$ function

$$a^l(x) = \sum_{k=1}^{K} \alpha_k \delta^{-1} \varphi(\frac{x - \xi_k^l}{\delta}), l = 1, 2,$$

with a fixed positive integer K, fixed real numbers $\alpha_k \in \mathbb{R}$, and different points $\xi_k^l \in \Omega$, and finally $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with a support in $B(0, 1)$ and $\delta > 0$ small enough such that supp $\varphi(\frac{\cdot - \xi_k^l}{\delta}) = B(\xi_k^l, \delta) \subset \Omega$ and satifying

$$\begin{aligned} &B(\xi_k^1, 4\delta) \subset \Omega, \forall k = 1, \cdots, K, \\ &\delta < \frac{1}{5} \min_{k \neq k'} |\xi_k^1 - \xi_{k'}^1|. \end{aligned}$$

This last condition implies, in particular, that the functions $\varphi(\frac{\cdot - \xi_k^1}{\delta})$ have disjoint supports.

This choice is motivated by the fact that $\delta^{-1}\varphi(\frac{x - \xi_k^l}{\delta})$ tends to the Dirac function at ξ_k^l as δ goes to zero so the above choice is an approximation of the first case.

Under these assumptions we can prove the following conditional stability result:

Theorem 3.5.3 *In the setting of the second case, fix a strategic subset* ω *of* Ω *and* $T > 0$ *large enough from Definition 3.3.1. Suppose that*

$$|\xi_k^1 - \xi_k^2| \leq \epsilon, \forall k = 1, \cdots, K,$$

with $\epsilon > 0$ *satisfying the constraints*

$$\epsilon \leq \min_{k \neq k'} |\xi_k^1 - \xi_{k'}^1| - 5\delta, \tag{3.16}$$

$$\epsilon \leq \delta. \tag{3.17}$$

Then there exists a constant C depending on T and δ such that

$$\sum_{k=1}^{K} |\xi_k^1 - \xi_k^2| \le C \|\partial_t u^1 - \partial_t u^2\|_{L^2(0,T;L^2(\omega))}.$$

Proof: As in Theorem 3.5.1 it suffices to estimate from below the norm of $a^1 - a^2$ in V'.

If $n = 1$, we take

$$\phi_k(x) = \phi_1(\frac{x - \xi_k^1}{\delta}) \text{ in } \Omega,$$

where ϕ_1 is a fixed function defined by

$$\phi_1(\hat{x}) = \begin{cases} -4 - \hat{x} \text{ if } -4 < \hat{x} \le -2, \\ \hat{x} \text{ if } -2 < \hat{x} \le 2, \\ -\hat{x} + 4 \text{ if } 2 < \hat{x} \le 4, \\ 0 \text{ else.} \end{cases}$$

For that choice, by the above conditions for any $k = 1, \cdots, K$, we have

$$< a^1 - a^2, \phi_k >= \alpha_k \delta^{-1} \int_{\mathbb{R}} \left(\varphi(\frac{x - \xi_k^1}{\delta}) - \varphi(\frac{x - \xi_k^2}{\delta}) \right) \phi_1(\frac{x - \xi_k^1}{\delta})\, dx,$$

and by changes of variables we get

$$< a^1 - a^2, \phi_k >= \alpha_k \int_{-1}^{1} \varphi(y) \left(\phi_1(y) - \phi_1(\frac{\xi_k^2 - \xi_k^1 + \delta y}{\delta}) \right) dy.$$

By the finite increment theorem and the fact that $|\xi_k^1 - \xi_k^2| < \delta$, we then obtain

$$< a^1 - a^2, \phi_k >= \frac{\alpha_k}{\delta} (\xi_k^1 - \xi_k^2) \int_{-1}^{1} \varphi(y)\, dy.$$

The conclusion follows from the fact that $\|\phi_k\|_V = \frac{C_1}{\sqrt{\delta}}$ for some $C_1 > 0$.

If $n = 2$ we first take

$$\phi_k(x_1, x_2) = \phi_1(\frac{x_1 - \xi_{k1}^1}{\delta}) \phi_2(\frac{x_2 - \xi_{k2}^1}{\delta}) \text{ in } \Omega,$$

where ϕ_1 was defined above and ϕ_2 is defined by

$$\phi_2(\hat{x}) = \begin{cases} 2 + \frac{\hat{x}}{2} \text{ if } -4 < \hat{x} \leq -2, \\ 1 \text{ if } -2 < \hat{x} \leq 2, \\ 2 - \frac{\hat{x}}{2} \text{ if } 2 < \hat{x} \leq 4, \\ 0 \text{ else.} \end{cases}$$

The same arguments as before then yield

$$< a^1 - a^2, \phi_k >= \frac{\alpha_k}{\delta}(\xi^1_{k1} - \xi^2_{k1}) \int_{\mathbb{R}^2} \varphi(y_1, y_2)\, dy_1 dy_2.$$

Since $\|\phi_k\|_V = C_2$ for some $C_2 > 0$ independent of δ we conclude that

$$|\xi^1_{k1} - \xi^2_{k1}| \leq C\delta \|a^1 - a^2\|_{V'},$$

for some $C > 0$ independent of δ.

Exchanging the rule of x_1 and x_2 we may conclude

$$|\xi^1_{k2} - \xi^2_{k2}| \leq C\delta \|a^1 - a^2\|_{V'},$$

for some $C > 0$ independent of δ.

The proof is similar for $n \geq 3$. ■

The third case we want to treat is the case when $n = 1$ and the function a^l, $l = 1, 2$ is given by

$$a^l(x) = \frac{1}{|x - \xi^l|^\beta},$$

for a point $\xi^l \in \Omega$ and some $0 < \beta < \frac{1}{2}$.

Under these assumptions we can prove the following conditional stability result:

Theorem 3.5.4 *Fix a strategic subset ω of $\Omega = (0, L)$ and $T > 0$ large enough from Definition 3.3.1. There exists a positive constant κ depending on β such that if*

$$|\xi^1 - \xi^2| \leq \kappa(L - \xi^1)^{2-\beta}, \tag{3.18}$$

then there exists a constant C depending on T, β and $L - \xi^1$ such that

$$|\xi^1 - \xi^2| \leq C\|\partial_t u^1 - \partial_t u^2\|_{L^2(0,T;L^2(\omega))}.$$

Proof: As before we need to estimate from below the norm of $a^1 - a^2$ in V'.

We fix $\delta = \frac{L-\xi}{2}$ and take as test function ϕ_ξ:

$$\phi_\xi(x) = \phi(\frac{x-\xi}{\delta}),$$

where ϕ is a fixed function belonging to $C^2(\mathbb{R})$ with a support in $[0,1]$ and such that $\phi(\hat{x}) > 0$ for any $\hat{x} \in (0,1)$. By changes of variable we have

$$< a^1 - a^2, \phi_\xi >= \int_{\mathbb{R}} \frac{1}{|y|^\beta} \left(\phi(\frac{y}{\delta}) - \phi(\frac{\xi^2 - \xi^1 + y}{\delta}) \right) dy.$$

Using Taylor's expansion we then get

$$< a^1 - a^2, \phi_\xi >= c_1 \frac{\xi^2 - \xi^1}{\delta} + c_2 \frac{(\xi^2 - \xi^1)^2}{\delta^2},$$

where we have set

$$\begin{aligned} c_1 &= \int_{\mathbb{R}} \frac{1}{|y|^\beta} \phi'(\frac{y}{\delta})\, dy, \\ c_2 &= \int_{\mathbb{R}} \frac{1}{|y|^\beta} \phi"(\theta(y))\, dy, \end{aligned}$$

where $\theta(y)$ is a point between $\frac{y}{\delta}$ and $\frac{\xi^2 - \xi^1 + y}{\delta}$. Remark that by integration by parts we have

$$c_1 = \beta \delta^{1-\beta} \int_0^1 \phi(y) y^{-\beta-1}\, dy = c_1' \delta^{1-\beta}$$

which is positive due to our assumptions on ϕ. This allows to write

$$| < a^1 - a^2, \phi_\xi > | \geq \frac{|\xi^2 - \xi^1|}{\delta^2} (c_1' \delta^{2-\beta} - |c_2| |\xi^2 - \xi^1|).$$

Therefore taking

$$\kappa = \frac{c_1'}{|c_2| 2^{3-\beta}},$$

the assumption (3.18) and the above estimate yield

$$| < a^1 - a^2, \phi_\xi > | \geq \frac{c_1'}{2\delta^\beta} |\xi^2 - \xi^1|.$$

The conclusion follows from the standard identity $\|\phi_\xi\|_V = \frac{C_1}{\sqrt{\delta}}$ for some $C_1 > 0$.

■

3.6 Reconstruction

For the reconstruction of the sources from interior measurements we follow the point of view of [68] which consists in using the following exact controllability result:

Lemma 3.6.1 *Fix a strategic subset ω of Ω and $T > 0$ large enough from Definition 3.3.1. Then for every $\phi \in V$, there exists a unique control $v \in L^2(0,T;L^2(\omega))$ such that the (weak) solution $\psi \in C([0,T];H) \cap C^1([0,T];V')$ of*

$$\left\{\begin{array}{l} \partial_t^2\psi - \Delta\psi = \chi_{\omega\times(0,T)}v \ \textit{in } Q_T, \\ \psi = 0 \quad \textit{on} \ \ \Sigma_T, \\ \psi(x,0) = \phi(x), \partial_t\psi(x,0) = 0 \ \textit{in } \Omega, \end{array}\right. \tag{3.19}$$

satisfies

$$\psi(\cdot,T) = \partial_t\psi(\cdot,T) = 0. \tag{3.20}$$

Proof: This is a direct consequence of the estimates (3.4) and of the Hilbert Uniqueness Method of Lions [48, Th. VII.2.5]. Note that ψ is only a weak solution of the system (3.19) with the final conditions (3.20) in the sense that ψ is the unique solution of (using the transposition method)

$$\int_{Q_T} \psi f \, dxdt = - < \phi, \partial_t\varphi(0) >_{V-V'} + \int_0^T \int_\omega v\varphi \, dxdt \tag{3.21}$$

for all $f \in L^1(0,T;H), \varphi_0 \in H, \varphi_1 \in V'$, where $\varphi \in C([0,T];H) \cap C^1([0,T];V')$ is the unique solution of

$$\left\{\begin{array}{l} \partial_t^2\varphi = \Delta\varphi + f \ \text{in }]0,T[, \\ \varphi(T) = \varphi_0, \partial_t\varphi(T) = \varphi_1 \ . \end{array}\right.$$

■

In view of Lemma 3.6.1 we can define a bounded linear operator $\Pi : V \to L^2(0,T;L^2(\omega))$, by

$$\phi \to v,$$

where v is the control from the above Theorem driving the system (3.19) to rest at time T.

We further use the adjoint $K^\star$ of the operator K as (bounded) operator from ${}_0H^1(0,T;L^2(\omega))$ into $L^2(0,T;L^2(\omega))$ and which is given by (see section 5 of [68])

$$(K^\star\eta)(x,t) = \lambda(0)(\partial_t\eta)(x,t) + \int_t^T (\lambda'(s-t)(\partial_t\eta)(x,s) + \lambda(s-t)\eta(x,s))\,ds, 0 < t < T,$$

for all $\eta \in {}_0H^1(0,T;L^2(\omega))$. By the assumption (3.2) we even have (see [68])

$$R(K^\star) = L^2(0,T;L^2(\omega)).$$

Consequently for all $\psi \in L^2(0,T;L^2(\omega))$ there exists a unique $\eta \in {}_0H^1(0,T;L^2(\omega))$ solution of

$$K^\star\eta = \psi,$$

equivalently, η is solution of the Volterra equation of the second kind

$$\lambda(0)(\partial_t\eta)(x,t) + \int_t^T (\lambda'(s-t)(\partial_t\eta)(x,s) + \lambda(s-t)\eta(x,s))\,ds = \psi(x,t), x \in \omega, 0 < t < T.$$

We then define the mapping Φ from $L^2(0,T;L^2(\omega))$ to ${}_0H^1(0,T;L^2(\omega))$ by

$$\psi \to \eta := \Phi\psi,$$

when η is solution of the above integral equation. This means that

$$K^\star\Phi = Id \text{ on } L^2(0,T;L^2(\omega)). \tag{3.22}$$

Now we can formulate our reconstruction result:

Theorem 3.6.2 *Fix a strategic subset ω of Ω and $T > 0$ large enough from Definition 3.3.1. For all $k = 1, \cdots, \infty$ we define*

$$\theta_k = \Phi\Pi\phi_k,$$

where $\{\phi_k\}_{k=1}^{\infty}$ is an orthonormal basis(in $L^2(\Omega)$) of the Laplace operator with Dirichlet boundary condition. Let $u \in C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (3.1) with unknown source $a \in L^2(\Omega)$ (or in $H^{-1}(\Omega)$ if $n = 1$). Then for all $k = 1, \cdots, \infty$ we have

$$< a, \phi_k >= \int_0^T \int_\omega \partial_t u(x,t)\partial_t\theta_k(x,t)\,dxdt, \tag{3.23}$$

and then a may be reconstructed by

$$a = \sum_{k=1}^{\infty} < a, \phi_k > \phi_k = \sum_{k=1}^{\infty} \left(\int_0^T \int_\omega \partial_t u(x,t) \partial_t \theta_k(x,t)\, dxdt \right) \phi_k.$$

Proof: Applying the identity (3.21) with $\varphi = v$, where v is the unique solution of (3.3) with initial spead a we have:

$$< a, \phi_k >= \int_0^T \int_\omega v(x,t) \Pi \phi_k(x,t)\, dxdt. \tag{3.24}$$

To conclude we need to show that

$$\int_0^T \int_\omega v(x,t) \Pi \phi_k(x,t)\, dxdt = \int_0^T \int_\omega \partial_t u(x,t) \partial_t \theta_k(x,t)\, dxdt. \tag{3.25}$$

Indeed by the definition of θ_k and (3.22) we may write

$$K^\star \theta_k = K^\star \Phi \Pi \phi_k = \Pi \phi_k.$$

Therefore by the above identity, the left-hand side of (3.25) may be transformed as follows

$$\begin{aligned} \int_0^T \int_\omega v(x,t) \Pi \phi_k(x,t)\, dxdt &= \int_0^T \int_\omega v(x,t) K^\star \theta_k(x,t)\, dxdt \\ &= (Kv, \theta_k)_{{}_0H^1(0,T;L^2(\omega))} = (u, \theta_k)_{{}_0H^1(0,T;L^2(\omega))}, \end{aligned}$$

and the identity (3.25) follows from the definition of the inner product in ${}_0H^1(0,T;L^2(\omega))$.

$\blacksquare$

Chapter 4

Determination of point sources in vibrating beams by boundary measurements: Identifiability, stability and reconstruction results

4.1 Introduction

To our knowledge the determination of point sources by boundary measurements for the beam equation with different boundary conditions has been not yet considered. Therefore our goal is to answer to this question for two different problems by adapting some results from [17, 18, 68, 58]. The main ingredients are the spectral properties of the biharmonic operators, some controllability results [48, 37] and finally appropriate properties of some integral operators [68, 18]. For our first problem since the eigenvalues and eigenvectors of the operator are not explicitly known, our reconstruction process is different from the one in [18] and is more close to the one in [68]. On the contrary for our second system the eigenvalues and eigenvectors of the operator are explicitly known, and therefore our reconstruction process is similar to the one in [18]. The chapter is organized as follows: Section 4.2 is devoted to the first Petrovsky system. In subsection 4.2.1, we show the well-posedness of the problem, some observability estimates and hidden regularities of the solution.

Subsection 4.2.2 is devoted to the proof of the uniqueness result and is based on the previous observality estimates and some properties of an integral operator between different Sobolev spaces. The conditional stability is deduced in subsection 4.2.3 and finally the reconstruction is detailed in subsection 4.2.4. The same questions for the second Petrovsky system are treated in section 4.3 with the same subdivision into three subsections.

4.2 The first Petrovsky system

4.2.1 Preliminaries

We consider the initial boundary value problem for a beam equation

$$\begin{cases} \partial_t^2 u(x,t) + u^{(4)}(x,t) = \lambda(t)a(x) & \text{in } Q_T, \\ u(\cdot,0) = 0, \partial_t u(\cdot,0) = 0 & \text{in }]0,1[, \\ u(x,t) = u'(x,t) = 0, & \text{for } x = 0,1 \text{ and } \forall\ t \in]0,T[, \end{cases} \tag{4.1}$$

where $u^{(4)}(x,t) = \frac{\partial^4 u}{\partial x^4}(x,t)$, $u'(x,t) = \frac{\partial u}{\partial x}(x,t)$, and $Q_T :=]0,1[\times]0,T[$. Above and below $\lambda \in C^1([0,T])$ is a given function satisfying

$$\lambda(0) \neq 0. \tag{4.2}$$

The datum $a \in (H^1(0,1))'$ is assumed to be in the form

$$a(x) = \sum_{k=1}^{K} \alpha_k \delta(x - \xi_k) \tag{4.3}$$

for some positive integer K, some real numbers α_k different from zero and some (different) points ξ_k in $]0,1[$ (enumerated in increasing order), or more precisely

$$< a, \phi >= \sum_{k=1}^{K} \alpha_k \phi(\xi_k), \forall \phi \in H^1(0,1).$$

As usual $H^p(0,1)$ is the standard Sobolev space of order $p \in \mathbb{N} := \{0,1,2,\cdots\}$ on the interval $]0,1[$.

Our goal is to identify the datum a in the above form (i.e. the location of the point sources ξ_k, the weights α_k and the number K) from boundary measurements, namely the value of $u"(0,t)$, for $0 < t < T$.

In order to analyse the system (4.1) we introduce the following operator A on the Hilbert space $H = L^2(0,1)$, endowed with the inner product

$$(u,v)_H = \int_0^1 u(x)v(x)\,dx. \tag{4.4}$$

The domain of A is $D(A) = H^4(0,1) \cap H_0^2(0,1)$ and for any $u \in D(A)$ we take $Au = -u^{(4)}$. Remark that A is a negative selfadjoint operator with a compact resolvant since A is the Friedrichs extension of the triple (H, V, a) defined by

$$V = H_0^2(0,1)$$

which is a Hilbert space with the inner product

$$(u,v)_V = \int_0^1 u"(x)v"(x)\,dx, \tag{4.5}$$

and

$$a(u,v) = (u,v)_V. \tag{4.6}$$

The spectrum of this operator A is well known, namely if $\{\lambda_k\}_{k=1}^{\infty}$ denotes the set of eigenvalues of the operator $-A$ in increasing order and repeated according to their multiplicity, then $\lambda_k = \mu_k^2$ where μ_k is a root of $\cosh\sqrt{\mu_k}\cos\sqrt{\mu_k} - 1 = 0$. The eigenvalues have furthermore the asymptotics:

$$C_1 k^4 \leq \lambda_k \leq C_2 k^4, \ \ \forall k = 1, ..., \infty, \tag{4.7}$$

for some positive constants C_1 and C_2. For future purposes, we need to show that the eigenfunctions are uniformly bounded:

Lemma 4.2.1 *Let ϕ_k be the eigenfunction of $-A$ associated with λ_k. Then there exists a constant $M > 0$ (independent of k) such that*

$$|\phi_k(x)| \leq M, \forall k = 1, \cdots, \infty \ \textit{and} \ \forall x \in]0,1[.$$

Proof: By simple calculations, we see that the eigenfunctions are given by:

$$\phi_k(x) = C_k[\sin(\sqrt{\mu_k}x) - \sinh(\sqrt{\mu_k}x) - f_k(\cos(\sqrt{\mu_k}x) - \cosh(\sqrt{\mu_k}x))],$$

where

$$f_k = \frac{\sin\sqrt{\mu_k} - \sinh\sqrt{\mu_k}}{\cos\sqrt{\mu_k} - \cosh\sqrt{\mu_k}}$$

and some constant C_k. As $\mu_k \to \infty$ as $k \to \infty$ we readily show that there exists a positive constant C independent of k such that:

$$|f_k - 1| \leq C\exp(-\sqrt{\mu_k}). \tag{4.8}$$

This estimate allows to show that there exists a positive constant $C^\star$ independent of k such that:

$$|\sin(\sqrt{\mu_k}x) - \sinh(\sqrt{\mu_k}x) - f_k(\cos(\sqrt{\mu_k}x) - \cosh(\sqrt{\mu_k}x))| \leq C^\star, \quad \forall x \in [0,1]. \tag{4.9}$$

On the other hand, the constant C_k is chosen such that

$$\int_0^1 |\phi_k(x)|^2 dx = 1.$$

A careful analysis of this constant with respect to μ_k shows that

$$C_k \to 1 \ \ as \ \ k \to \infty,$$

which implies the requested estimate. ∎

We are now ready to prove that our beam equation (4.1) is uniquely solvable and to give regularity of its solution:

Theorem 4.2.2 *The beam equation (4.1) has a unique (weak) solution u satisfying*

$$u \in C([0,T];V) \cap C^1([0,T];H).$$

Proof: We remark that the system (4.1) is equivalently written

$$\left\{ \begin{array}{l} \partial_t^2 u = Au + \lambda(t)a \text{ in }]0,T[, \\ u(0) = 0, \partial_t u(0) = 0 \ , \end{array} \right. \tag{4.10}$$

where $a \in V'$ is defined by

$$< a, \phi >_{V'-V} = \sum_{k=1}^{K} \alpha_k \phi(\xi_k), \forall \phi \in V. \tag{4.11}$$

The solution of that system is given by (using spectral expansions)

$$u(t) = \sum_{k=1}^{\infty} \frac{1}{\mu_k^2} \int_0^t \sin(\mu_k(t-s))\lambda(s)\, ds < a, \phi_k > \phi_k,$$

or equivalently, by integration by parts in the above integral:

$$u(t) = \sum_{k=1}^{\infty} \frac{a_k(t)}{\mu_k^2} \phi_k, \tag{4.12}$$

where a_k is given by

$$a_k(t) =< a, \phi_k > (\lambda(t) - \lambda(0)\cos(\mu_k t) - \int_0^t \cos(\mu_k(t-s))\lambda'(s)\, ds).$$

We now remark that the form of a and Lemma 4.2.1 allow to conclude the existence of a constant C_1 (depending on T but not on k) such that

$$|a_k(t)| \leq C_1, \forall k = 1, \cdots, \infty. \tag{4.13}$$

By Parseval's identity we have

$$||u(t)||_V^2 \sim ||u(t)||_{D(A^{1/2})}^2 \sim \sum_{k=1}^{\infty} \frac{|a_k(t)|^2}{\mu_k^2},$$

and consequently we conclude that

$$||u(t)||_V^2 \leq C_1^2 \sum_{k=1}^{\infty} \frac{1}{\mu_k^2} \leq C_2, \forall t \in [0,T],$$

for some positive constant C_2 (depending on T) since (4.7) guarantees the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{\mu_k^2}$. This means that the series $\sum_{k=1}^{\infty} \frac{a_k(t)}{\mu_k^2}\phi_k$ is convergent in $L^\infty([0,T];V)$ and then proves that

$$u \in C([0,T];V),$$

as limit of elements from $C([0,T];V)$ (the truncated series).

Similarly by direct calculations we have

$$||\partial_t u(t)||_H^2 = \sum_{k=1}^{\infty} \frac{|\partial_t a_k(t)|^2}{\mu_k^4} \leq C \sum_{k=1}^{\infty} \frac{1}{\mu_k^2},$$

for some positive constant C (depending on T), and we conclude as before that $u \in C^1([0,T];H)$. ■

Let further consider the Petrovsky system

$$\begin{cases} \partial_t^2 \phi - A\phi = f \text{ in }]0,T[, \\ \phi(0) = \phi_0, \partial_t \phi(0) = \phi_1 \ , \end{cases} \tag{4.14}$$

where (ϕ_0, ϕ_1) belongs to $V \times H$ and $f \in L^1(]0,T[;H)$. It is well known that this system has a unique solution $\phi \in C([0,T];V) \cap C^1([0,T];H)$. Using the direct and inverse estimates of the system (4.14) (see Theorems IV.3.1 and IV.3.3 and Appendix I of [48] and Theorems 2.6 and 6.7 of [37]) and the arguments of Theorem IV.3.6 of [48], we obtain the next (weak) observability estimates:

Lemma 4.2.3 *For $a \in V'$ there exists a unique solution $v \in C([0,T];H) \cap C^1([0,T];V')$ of*

$$\begin{cases} \partial_t^2 v - Av = 0 \textit{ in }]0,T[, \\ v(0) = 0, \partial_t v(0) = a. \end{cases} \tag{4.15}$$

Moreover for any $T > 0$, there exist two positive constants C_1 and C_2 depending on T such that

$$C_1 \|a\|_{V'} \leq \|v"(0,.)\|_{H^{-1}(0,T)} \leq C_2 \|a\|_{V'}, \tag{4.16}$$

where, as usual, $H^{-1}(0,T)$ is the dual space of $H_0^1(0,T)$.

Let us also give a consequence of the identity with multiplier to the solution u of problem (4.1), namely the hidden regularity of $u"(0,.)$:

Lemma 4.2.4 *Let $u \in C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (4.1). Then for all $T > 0$, $u"$ belongs to $L^2(0,T)$ with the estimate*

$$\|u"(0,.)\|_{L^2(0,T)} \leq C(\|u\|_{C([0,T];V)} + \|u\|_{C^1([0,T];H)}), \tag{4.17}$$

for some positive constant C depending on T.

Proof: We set $f(x,t) = \lambda(t)a(x)$ and remark that $f \in L^1((0,T), H^{-1}(0,1))$. We now approximate f by a sequence of more regular data $f_n(x,t) = \lambda(t)a_n(x) \in L^1((0,T), L^2(0,1))$ such that

$$f_n \to f \text{ in } L^1((0,T), H^{-1}(0,1)) \text{ as } n \to \infty. \tag{4.18}$$

Namely for n large enough, we take a_n in the form

$$a_n = \sum_{k=1}^{K} \alpha_k \phi_{kn}, \tag{4.19}$$

where

$$\phi_{kn}(x) = n(\phi(n(x-\xi_k))), \forall x \in [0,1]$$

with a fixed nonnegative function $\phi \in \mathcal{D}(\mathbb{R})$ with support in $[-1,1]$ and such that $\int_{-1}^{1} \phi(x)dx = 1$.

Let u_n be the solution of (4.1) with datum a_n. Then one has

$$u_n \to u \text{ in } C([0,T]; V) \cap C^1([0,T]; H). \tag{4.20}$$

Now we may apply the identity (IV.3.15) of [48] to u_n with the multiplier q defined by

$$q(x) = (x-1)\eta(x), \forall x \in [0,1],$$

with

$$\eta(x) = \begin{cases} 1 \text{ if } x \in [0, \frac{\xi_1}{3}], \\ 0 \text{ if } x \in [\frac{2\xi_1}{3}, 1]. \end{cases}$$

This choice guarantees that $q \cdot \nu = 1$ at $x = 0$, $q \cdot \nu = 0$ at $x = 1$ and that there exists a positive integer $N(\xi_1)$ such that

$$f_n \cdot q \equiv 0, \forall n > N(\xi_1).$$

With this choice the identity (IV.3.15) of [48] yields for $n > N(\xi_1)$:

$$\begin{aligned} \frac{1}{2}\int_0^T |u"_n(0,t)|^2 dt &= \int_0^1 \partial_t u_n(x,t) q(x) u_n'(x,t) dx|_0^T \\ &+ \frac{1}{2}\int_0^T \int_0^1 q'(|\partial_t u_n'|^2 - |u"_n(x,t)|^2) dxdt \\ &+ 2\int_0^1 \int_0^T |u"_n(x,t)|^2) dxdt. \end{aligned} \tag{4.21}$$

It then remains to estimate the three terms of the above right-hand side: For the first one Cauchy-Schwarz's inequality gives

$$\begin{aligned}|\int_0^1 \partial_t u_n(x,t)q(x)u_n'(x,t)dx| &\leq C\|\partial_t u_n\|_{L^2(0,1)}\|u_n'\|_{L^2(0,1)}\\ &\leq C\|u_n\|_{C^1([0,T];L^2(0,1))}\|u_n\|_{C([0,T];H_0^2(0,1))}\\ &\leq C\|u_n\|_X^2,\end{aligned}$$

where for shortness we write $\|.\|_X = \|.\|_{C^1([0,T];L^2(0,1))} + \|.\|_{C([0,T];H_0^2(0,1))}$.

On the other hand we directly have

$$\begin{aligned}\int_0^1\int_0^T q'(|\partial_t u_n|^2 - |u_n'|^2)dxdt &\leq CT(\|\partial_t u_n\|^2_{C([0,T];L^2(0,1))} + \|u_n\|^2_{C([0,T];H_0^2(0,1))})\\ &\leq 2CT\|u_n\|_X^2,\end{aligned}$$

and similarly

$$\int_0^1\int_0^T |u"_n(x,t)|^2\, dxdt \leq T\|u_n\|^2_{C([0,T];H_0^2(0,1))} \leq T\|u_n\|_X^2.$$

The three above estimates in (4.21) yield

$$\int_0^T |u"_n(0,t)|^2 dt \leq C(1+T)\|u_n\|_X^2.$$

Passing to the limit in n in that estimate and using (4.20), we conclude that $u"(0,\cdot)$ belongs to $L^2(0,T)$ and obtain the estimate (4.17). ■

4.2.2 Uniqueness

We first recall Duhamel's principle (see for instance [68, 18]) which gives the relationship between v solution of (4.15) and u solution of (4.10).

Lemma 4.2.5 *Let $u \in C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (4.10) with datum a in the form (4.11) and let $v \in C([0,T];H) \cap C^1([0,T];V')$ be the unique solution of (4.15) with initial speed a. Then*

$$u(t) = (Kv)(t), \forall t \in]0,T[, \tag{4.22}$$

where K is defined by

$$(K\psi)(t) = \int_0^t \lambda(t-s)\psi(s)\, ds, \forall t \in]0,T[, \tag{4.23}$$

and is a bounded operator from $L^2(0,T)$ into itself.

We can now recall the following result proved in [58] (see also [18]).

Lemma 4.2.6 *If $\lambda \in C^1([0,T])$ satisfies (4.2) then the bounded operator K from $L^2(0,T)$ into itself defined by (4.23) can be extended to a bounded operator from $H_{-1}(0,T)$ onto $L^2(0,T)$ and satisfying*

$$C_1\|K\psi\|_{L^2(0,T)} \leq \|\psi\|_{H_{-1}(0,T)} \leq C_2\|K\psi\|_{L^2(0,T)}, \forall \psi \in H_{-1}(0,T), \tag{4.24}$$

for some positive constants C_1, C_2.

Here and below the space $H_{-1}(0,T)$ is defined as the dual space of

$$^0H^1(0,T) = \{v \in H^1(0,T) : v(T) = 0\},$$

which is a Hilbert space with the norm

$$\|v\|_{^0H^1(0,T)} = \left(\int_0^T |\partial_t v(t)|^2 dt\right)^{1/2}.$$

The above Lemma does not hold in the standard Sobolev space $H^{-1}(0,T)$ but we showed in Lemma 4.3 of [58] that a similar result holds in $H^{-1}(0,T)$ if we replace the operator K by the operator PK, where P is the orthogonal projection (in $L^2(0,T)$) on $\Lambda^\perp$ defined by

$$\Lambda^\perp = \{\eta \in L^2(0,T) : (\lambda, \eta)_{L^2(0,T)} = 0\}.$$

Namely we may state the (see Lemma 4.3 of [58] for the detailed proof):

Lemma 4.2.7 *If $\lambda \in C^1([0,T])$ satisfies (4.2) then the bounded operator PK from $L^2(0,T)$ into itself can be extended to a bounded operator from $H^{-1}(0,T)$ into $L^2(0,T)$ and satisfying*

$$C_3\|PK\psi\|_{L^2(0,T)} \leq \|\psi\|_{H^{-1}(0,T)} \leq C_4\|PK\psi\|_{L^2(0,T)}, \forall \psi \in H^{-1}(0,T), \tag{4.25}$$

for some positive constants C_3, C_4.

Corollary 4.2.8 *Let* $u \in C([0,T];V) \cap C^1([0,T];H)$ *be the unique solution of (4.10) with datum* a *in the form (4.11) and let* $v \in C([0,T];H) \cap C^1([0,T];V')$ *be the unique solution of (4.15) with initial speed* a*. Then for all* $T > 0$ *we have*

$$Pu\text{"}(0,\cdot) = PKv\text{"}(0,\cdot) \text{ in } L^2(0,T). \tag{4.26}$$

Proof: As in Lemma 4.2.4 let u_n (resp. v_n) be the solution of (4.10) (resp. (4.15)) with datum $a_n \in V$ (resp. with initial speed a_n) satisfying

$$a_n \to a \text{ in } V'. \tag{4.27}$$

For these solutions their regularity and Lemma 4.2.5 allow to write

$$u\text{"}_n(0,\cdot) = Kv\text{"}_n(0,\cdot) \text{ in } L^2(0,T).$$

And therefore

$$Pu\text{"}_n(0,\cdot) = PKv\text{"}_n(0,\cdot) \text{ in } L^2(0,T).$$

We conclude by passing to the limit in n and using Lemmas 4.2.7, 4.2.3 and 4.2.4. ■

We are now ready to formulate the uniqueness result:

Theorem 4.2.9 *Fix* $T > 0$*. Let* u^1 *(resp.* u^2*) in* $C([0,T];V) \cap C^1([0,T];H)$ *be the unique solution of (4.10) with datum* a^1 *(resp.* a^2*) in the form*

$$< a^l, \phi >_{V'-V} = \sum_{k=1}^{K^l} \alpha_k^l \phi(\xi_k^l), \forall \phi \in V, l = 1,2,$$

for some positive integers K^l*, real numbers* α_k^l *and points* $\xi_k^l \in]0,1[$*. If*

$$(u^1)\text{"}(0,t) = (u^2)\text{"}(0,t), \forall t \in (0,T),$$

as elements of $L^2(0,T)$*, then*

$$a^1 = a^2,$$

or equivalently

$$K^1 = K^2, \alpha_k^1 = \alpha_k^2, \xi_k^1 = \xi_k^2.$$

Proof: We remark that $u = u^1 - u^2$ satisfies (4.10) with datum $a = a^1 - a^2$ which is still in the form (4.11). By the assumption we further have

$$u"(0, \cdot) = 0 \text{ in } L^2(0, T).$$

This implies that

$$Pu"(0, \cdot) = 0 \text{ in } L^2(0, T).$$

Therefore by Corollary 4.2.8 and Lemma 4.2.7 we get

$$v"(0, \cdot) = 0 \text{ in } H^{-1}(0, T),$$

where v is the unique solution of (4.15) with initial speed a. The application of Lemma 4.2.3 allows to conclude that $a = 0$. ∎

4.2.3 Stability

For a fixed positive integer K, we denote by

$$\Sigma = \{A = (\alpha_k, \xi_k)_{k=1}^K : \alpha_k \in \mathbb{R} \setminus 0, \xi_k \in]0, 1[\}.$$

The above uniqueness result implies that the mapping

$$\eta : \Sigma \to L^2(0, T) : A := (\alpha_k, \xi_k)_{k=1}^K \to u"(0, \cdot),$$

where u is the unique solution of (4.10) with datum a in the form (4.11), is injective. The stability means that the inverse mapping $\eta^{-1} : u"(0, \cdot) \to A$ is continuous once Σ is equipped with the natural distance

$$d(A^1, A^2) = \sum_{k=1}^K (|\alpha_k^1 - \alpha_k^2| + |\xi_k^1 - \xi_k^2|),$$

when $A^l := \left(\alpha_k^l, \xi_k^l\right)_{k=1}^K, l = 1, 2$.

We actually will show a slightly weaker result than the continuity of this mapping by only showing that the inverse of the restriction of η to the ball $B(A, \epsilon)$ is Lipschitz continuous for some $\epsilon > 0$ small enough depending on A. Namely we take

$$\begin{aligned}\epsilon &\leq \frac{1}{4}\min_{k\neq k'}|\xi_k-\xi_{k'}|, && (4.28)\\ \epsilon &\leq \frac{1}{4}\min_{k}|\xi_k|, \epsilon \leq \frac{1}{4}\min_k|1-\xi_k| && (4.29)\\ \epsilon &\leq \frac{1}{2}\min_k|\alpha_k|. && (4.30)\end{aligned}$$

Under these assumptions we can prove the following conditional stability result:

Theorem 4.2.10 *Fix $T > 0$ and suppose that $A^2 = (\alpha_k^2, \xi_k^2)_{k=1}^K$ is in $\Sigma \cap B(A, \epsilon)$ with $\epsilon > 0$ satisfying the above constraints. Then there exists a constant C depending on T, $\min_{k\neq k'}|\xi_k-\xi_{k'}|$ and $\min_k|\alpha_k|$ such that*

$$\sum_{k=1}^{K}(|\alpha_k-\alpha_k^2|+|\xi_k-\xi_k^2|) \leq C\|u"(0,\cdot)-(u^2)"(0,\cdot)\|_{L^2(0,T)}. \qquad (4.31)$$

Proof: The proof of Theorem 4.2.9 clearly shows that

$$\|a-a^2\|_{V'} \leq C\|u"(0,\cdot)-(u^2)"(0,\cdot)\|_{L^2(0,T)}. \qquad (4.32)$$

Therefore it remains to estimate from below the norm of $a-a^2$ in V'. For that purpose we recall that

$$\|a-a^2\|_{V'} = \sup_{\phi\in V,\phi\neq 0}\frac{|<a-a^2,\phi>|}{\|\phi\|_V},$$

and use appropriate test functions ϕ. First we take

$$\phi^{(k)}(x) \quad = \quad \phi_1(\frac{x-\xi_k}{\delta}), \forall x \in]0,1[,$$

where $\delta = \frac{1}{4}\min_{k\neq k'}|\xi_k-\xi_{k'}|$ and ϕ_1 is a fixed function defined by

$$\phi_1(\hat{x}) = \begin{cases} 4(3/2+\hat{x})^2(4\hat{x}-3) \text{ if } -3/2 < \hat{x} \leq -1, \\ \hat{x} \text{ if } -1 < \hat{x} \leq 1, \\ 4(-3/2+\hat{x})^2(4\hat{x}-3) \text{ if } 1 \leq \hat{x} < 3/2, \\ 0 \text{ else.} \end{cases}$$

With this choice we have

$$\begin{aligned} < a - a^2, \phi^{(k)} > &= \alpha_k \phi^{(k)}(\xi_k) - \alpha_k^2 \phi^{(k)}(\xi_k^2) \\ &= \alpha_k^2(\phi^{(k)}(\xi_k) - \phi^{(k)}(\xi_k^2)), \end{aligned}$$

since $\phi^{(k)}(\xi_k) = 0$. By the finite increment theorem and the fact that $|\xi_k - \xi_k^2| < \epsilon$, we then obtain

$$| < a - a^2, \phi^{(k)} > | = \frac{|\alpha_k^2|}{\delta} |\xi_k - \xi_k^2|.$$

This estimate yields

$$|\alpha_k^2||\xi_k - \xi_k^2| \leq \delta| < a - a^2, \phi^{(k)} > | \leq \delta \|a - a^2\|_{V'} \|\phi^{(k)}\|_V,$$

and leads to

$$|\alpha_k^2||\xi_k - \xi_k^2| \leq \frac{C_1}{\sqrt{\delta}} \|a - a^2\|_{V'}, \tag{4.33}$$

for some positive constant C_1 since one readily checks that $\|\phi^{(k)}\|_V = \frac{C_1}{\delta^{\frac{3}{2}}}$.

From the third assumption on ϵ, we have

$$|\alpha_k^2| \geq m/2,$$

where $m = \min_k |\alpha_k|$. These two estimates finally give

$$|\xi_k - \xi_k^2| \leq \frac{2C_1}{m\sqrt{\delta}} \|a - a^2\|_{V'}.$$

Now we take

$$\phi_2^{(k)}(x) = \phi_2(\frac{x - \xi_k}{\delta}), \forall x \in]0, 1[,$$

when $\phi_2 \in \mathcal{D}(] - 1, 1[)$ satisfies $\phi_2(0) = 1$. With this choice we have

$$\begin{aligned} < a - a^2, \phi_2^{(k)} > &= \alpha_k \phi_2^{(k)}(\xi_k) - \alpha_k^2 \phi_2^{(k)}(\xi_k^2) \\ &= (\alpha_k - \alpha_k^2)\phi_2^{(k)}(\xi_k) + \alpha_k^2(\phi_2^{(k)}(\xi_k) - \phi_2^{(k)}(\xi_k^2)), \\ &= (\alpha_k - \alpha_k^2) + \alpha_k^2(\phi_2^{(k)}(\xi_k) - \phi_2^{(k)}(\xi_k^2)). \end{aligned}$$

Therefore by the finite increment theorem we obtain as before

$$|\alpha_k - \alpha_k^2| \leq | < a - a^2, \phi_2^{(k)} > | + \frac{S}{\delta} |\alpha_k^2||\xi_k - \xi_k^2|,$$

where $S = \max_{-1\leq\hat{x}\leq 1} |\phi_2'(\hat{x})|$ and by the estimate (4.33) we get

$$|\alpha_k - \alpha_k^2| \leq | < a - a^2, \phi_2^{(k)} > | + \frac{SC_1}{\delta^{\frac{3}{2}}} \|a - a^2\|_{V'}.$$

Since $\|\phi_2^{(k)}\|_V = \frac{C_2}{\delta^{\frac{3}{2}}}$ for some $C_2 > 0$, we have obtained

$$|\alpha_k - \alpha_k^2| \leq (\frac{C_2}{\delta^{\frac{3}{2}}} + \frac{SC_1}{\delta^{\frac{3}{2}}}) \|a - a^2\|_{V'}.$$

■

In the above theorem if like in [18] we are only interested in the stability of the locations of the point sources, i.e. if we assume that $\alpha_k^2 = \alpha_k$, then we can obtain a more accurate estimate under less assumptions on ϵ, namely we have the

Theorem 4.2.11 *Fix $T > 0$ and suppose that $A^2 = (\alpha_k, \xi_k^2)_{k=1}^K$ is in $\Sigma \cap B(A, \epsilon)$ with $\epsilon > 0$ satisfying (4.28) and (4.29). Then there exists a constant C depending on T, $\min_{k\neq k'} |\xi_k - \xi_{k'}|$ and $\min_k |\alpha_k|$ such that*

$$\sum_{k=1}^{K} |\xi_k - \xi_k^2| \leq C\|u"(0, \cdot) - (u^2)"(0, \cdot)\|_{L^2(0,T)}. \tag{4.34}$$

Proof: It suffices to take

$$\phi^{(k)}(x) \quad = \quad \phi_1(\frac{x - \xi_k}{\delta}) \text{ on }]0, 1[,$$

with the same ϕ_1 as before and use the above arguments. ■

4.2.4 Reconstruction

For the reconstruction of the point sources from boundary measurements we follow the point of view of [68] which consists in using the following exact controllability result:

Lemma 4.2.12 *Fix $T > 0$. Then for every $\phi \in V$, there exist a unique control $v \in H_0^1(0,T)$, such that the (weak) solution $\psi \in C([0,T];H) \cap C^1([0,T];V')$ of*

$$\begin{cases} \partial_t^2 \psi(x,t) + \psi^{(4)}(x,t) = 0 \text{ in } Q_T, \\ \psi(0,t) = \psi(1,t) = 0, \forall t \in]0,T[, \\ \psi'(0,t) = v, \psi'(1,t) = 0, \forall t \in]0,T[, \\ \psi(\cdot,0) = \phi, \partial_t \psi(\cdot,0) = 0 \text{ in }]0,1[, \end{cases} \tag{4.35}$$

satisfies

$$\psi(\cdot,T) = \partial_t \psi(\cdot,T) = 0. \tag{4.36}$$

Proof: This is a direct consequence of Lemma 4.2.3 and of the Hilbert Uniqueness Method of Lions [48, Th.IV.3.4], see also [37].

Note that ψ is only a weak solution of the system (4.35) with the final conditions (4.36) in the sense that ψ is the unique solution of (using the transposition method)

$$\int_{Q_T} \psi f \, dxdt = - < \partial_t \varphi(0), \phi >_{V'-V} + < \varphi"(0), v >_{H^{-1}(0,T)-H_0^1(0,T)}, \tag{4.37}$$

for all $f \in L^1(0,T;H), \varphi_0 \in H, \varphi_1 \in V'$, where $\varphi \in C([0,T];H) \cap C^1([0,T];V')$ is the unique solution of (whose existence follows from Lemma (4.2.3)

$$\begin{cases} \partial_t^2 \varphi = A\varphi + f \text{ in }]0,T[, \\ \varphi(T) = \varphi_0, \partial_t \varphi(T) = \varphi_1 \ . \end{cases}$$

■

In view of Lemma 4.2.12 we can define a bounded linear operator

$$\Pi : V \to H_0^1(0,T) : \phi \to v,$$

where v is the control from the above Lemma driving the system (4.35) to rest at time T.

We further use the adjoint $K_{L^2}^\star$ of the operator K as (bounded) operator from $L^2(0,T)$ into itself and which is given by (see section 6 of [68])

$$(K_{L^2}^\star \eta)(t) = \int_t^T \lambda(s-t)\eta(s) \, ds, 0 < t < T,$$

for all $\eta \in L^2(0,T)$. By the assumption (4.2) we even have (see section 6 of [68])

$$R(K^\star_{L^2}) = {}^0H^1(0,T).$$

Consequently for all $\psi \in {}^0H^1(0,T)$ there exists a unique $\eta \in L^2(0,T)$ solution of (since $\ker K^\star_{L^2} = R(K)^\perp = \{0\}$)

$$K^\star_{L^2}\eta = \psi,$$

equivalently, η is solution of the Volterra equation of the first kind

$$\int_t^T \lambda(s-t)\eta(s)\,ds = \psi(t), 0 < t < T.$$

We then define the mapping Φ from ${}^0H^1(0,T)$ into $L^2(0,T)$ by

$$\psi \to \eta := \Phi\psi,$$

when η is solution of the above integral equation. This means that

$$K^\star_{L^2}\Phi = Id \text{ on } {}^0H^1(0,T). \tag{4.38}$$

Now we can formulate our reconstruction result:

Theorem 4.2.13 *Fix $T > 0$. For all $k = 1, \cdots, \infty$ we define*

$$\theta_k = \Phi\Pi\phi_k.$$

Let $u \in C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (4.1) with datum a in the form (4.3). Then for all $k = 1, \cdots, \infty$ we have

$$< a, \phi_k >= (u"(0,\cdot), \theta_k)_{L^2(0,T)}, \tag{4.39}$$

and then a may be reconstructed by

$$a = \sum_{k=1}^{\infty} < a, \phi_k > \phi_k = \sum_{k=1}^{\infty} (u"(0,\cdot), \theta_k)_{L^2(0,T)}\phi_k.$$

Proof: Applying the identity (4.37) with $\varphi = v$, where v is the unique solution of (4.15) with initial speed a we have:

$$< a, \phi_k >=< v"(0,\cdot), \Pi\phi_k >_{H^{-1}(0,T)-H_0^1(0,T)} . \tag{4.40}$$

To conclude we need to show that

$$< v"(0,\cdot), \Pi\phi_k >_{H^{-1}(0,T)-H_0^1(0,T)}= (u"(0,\cdot), \theta_k)_{L^2(0,T)}. \tag{4.41}$$

Let us first prove that there exists $h \in H_{-1}(0,T)$ such that

$$u"(0,\cdot) = Kh, \tag{4.42}$$

and satisfying

$$< v"(0,\cdot), \chi >_{H^{-1}(0,T)-H_0^1(0,T)}=< h, \chi >_{H_{-1}(0,T)-{}^0H^1(0,T)}, \forall \chi \in H_0^1(0,T). \tag{4.43}$$

Indeed the identity (4.42) clearly follows from Lemmas 4.2.4 and 4.2.6; moreover using an approximation sequence of a_n as usual, the corresponding u_n and v_n satisfy

$$v"_n(0,\cdot) \to h \text{ in } H_{-1}(0,T), \text{ as } n \to \infty,$$

due to Lemmas 4.2.4 and 4.2.6, while by Lemma 4.2.3 we have

$$v"_n(0,\cdot) \to v"(0,\cdot) \text{ in } H^{-1}(0,T), \text{ as } n \to \infty.$$

The identity (4.43) then follows from the two above convergence properties and the continuity of the mapping $Id^\star$ from $H_{-1}(0,T)$ into $H^{-1}(0,T)$ (see [58]). Now by the definition of θ_k and (4.38) we may write

$$K^\star_{L^2}\theta_k = K^\star_{L^2}\Phi\Pi\phi_k = \Pi\phi_k.$$

Therefore using (4.43) and the above identity, the left-hand side of (4.41) may be transformed as follows

$$\begin{aligned} < v"(0,\cdot), \Pi\phi_k >_{H^{-1}(0,T)-H_0^1(0,T)} &= < h, \Pi\phi_k >_{H_{-1}(0,T)-{}^0H^1(0,T)} \\ &= < h, K^\star_{L^2}\theta_k >_{H_{-1}(0,T)-{}^0H^1(0,T)}, \end{aligned}$$

and from the embeddings ${}^0H^1(0,T) \hookrightarrow L^2(0,T) \hookrightarrow H_{-1}(0,T)$, we get

$$< h, K^\star_{L^2}\theta_k >_{H_{-1}(0,T)-{}^0H^1(0,T)}= (Kh, \theta_k)_{L^2(0,T)}.$$

This proves (4.41) since the above right-hand side coincides with the right-hand side of (4.41) due to (4.42). ∎

4.3 The second Petrovsky system

4.3.1 Preliminaries

We consider the initial boundary value problem for the beam equation with supported boundary conditions:

$$\left\{\begin{array}{ll} \partial_t^2 u(x,t) + u^{(4)}(x,t) = \lambda(t)a(x) & \text{in } Q_T, \\ u(\cdot,0) = 0, \partial_t u(\cdot,0) = 0 & \text{in }]0,1[, \\ u(x,t) = u"(x,t) = 0, & \text{for } x = 0,1 \text{ and } \forall t \in]0,T[, \end{array}\right. \tag{4.44}$$

where a is in the form (4.3).

As in section 4.2, our goal is to identify the datum a from boundary measurements, namely from the values of $u'(0,t)$, for $0 < t < T$.

In order to analyse the system (4.44) we define the operator A on the Hilbert space $H = L^2(0,1)$ endowed with the inner product (4.4) as follows:

$$\left\{\begin{array}{l} D(A) = \{u \in H^4(0,1) \cap H_0^1(0,1) : u"(0) = u"(1) = 0\}, \\ \forall u \in D(A) : Au = -u^{(4)}. \end{array}\right.$$

As before A is a negative selfadjoint operator with a compact resolvant since A is the Friedrichs extension of the triple (H, V, a), where $V = \{u \in H^2(0,1) \cap H_0^1(0,1) : u"(0) = u"(1) = 0\}$ equipped with the inner product (4.5) and a is given by (4.6).

Recall that the spectrum $\{\lambda_k\}_{k=1}^{\infty}$ of $-A$ is given by $\lambda_k = k^4\pi^4$ and the associated eigenfunctions are given by $\phi_k(x) = \sqrt{2}\sin(k\pi x)$ for all $k = 1, ..., \infty$. As in Theorem 4.2.2, we may prove the

Theorem 4.3.1 *The beam equation (4.44) has a unique (weak) solution u satisfying*

$$u \in C([0,T]; V) \cap C^1([0,T]; H).$$

Proof: The system (4.44) is equivalently written in the form (4.10) and then

$$u(t) = \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \int_0^t \sin(k^2\pi^2(t-s))\lambda(s)\,ds < a, \phi_k > \phi_k,$$

or equivalently, by integration by parts in the above integral:

$$u(t) = \sum_{k=1}^{\infty} \frac{a_k(t)}{\lambda_k} \phi_k, \tag{4.45}$$

where a_k is here given by

$$a_k(t) =< a, \phi_k > (\lambda(t) - \lambda(0)\cos(k^2\pi^2 t) - \int_0^t \cos(k^2\pi^2(t-s))\lambda'(s)\,ds).$$

The remainder of the proof is similar to the one of Theorem 4.2.2. ■

Using the direct and inverse estimates of Theorem 2.10 and 6.11 of [37] we obtain the next (weak) observability estimates:

Lemma 4.3.2 *For $a \in V'$ there exists a unique solution $v \in C([0,T];H) \cap C^1([0,T];V')$ of*

$$\begin{cases} \partial_t^2 v - Av = 0 \text{ in }]0,T[, \\ v(0) = 0, \partial_t v(0) = a. \end{cases} \tag{4.46}$$

Moreover for $T > 0$ there exist two positive constants C_1 and C_2 depending on T such that

$$C_1\|a\|_{H^{-1}(0,1)} \le \|v'(0,.)\|_{L^2(0,T)} \le C_2\|a\|_{H^{-1}(0,1)}. \tag{4.47}$$

4.3.2 Uniqueness

As in subsection 4.2.2, using Lemma 4.3.2 instead of Lemma 4.2.3, we obtain the following uniqueness result:

Theorem 4.3.3 *Fix $T > 0$. Let u^1 (resp. u^2) in $C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (4.44) with datum a^1 (resp. a^2) in the form*

$$< a^l, \phi >_{V'-V} = \sum_{k=1}^{K^l} \alpha_k^l \phi(\xi_k^l), \forall \phi \in V, l = 1, 2,$$

for some positive integers K^l, real numbers α_k^l and points $\xi_k^l \in]0,1[$. If

$$(u^1)'(0,t) = (u^2)'(0,t), \forall t \in (0,T),$$

as elements of $L^2(0,T)$, then

$$K^1 = K^2, \alpha_k^1 = \alpha_k^2, \xi_k^1 = \xi_k^2.$$

Proof: As before we see that $u = u^1 - u^2$ satisfies (4.44) with datum $a = a^1 - a^2$ and

$$u'(0, \cdot) = 0 \text{ in } L^2(0, T),$$

by the assumption. This implies that

$$Pu'(0, \cdot) = 0 \text{ in } L^2(0, T).$$

Therefore by Corollary 4.2.8 and Lemma 4.2.7 we get

$$v'(0, \cdot) = 0 \text{ in } H^{-1}(0, T),$$

and consequently

$$v'(0, \cdot) = 0 \text{ in } L^2(0, T)$$

where v is the unique solution of (4.46) with initial speed a. Lemma 4.3.2 finally yields $a = 0$. ■

4.3.3 Stability

Using the notation from subsection 4.2.3 and under the same assumptions we have the following conditional stability result:

Theorem 4.3.4 *Fix $T > 0$. Suppose that $A^2 = (\alpha_k^2, \xi_k^2)_{k=1}^K$ is in $\Sigma \cap B(A, \epsilon)$ with $\epsilon > 0$ satisfying (4.28), (4.29) and (4.30). Then there exists a constant C depending on T, $\min_{k \neq k'} |\xi_k - \xi_{k'}|$ and $\min_k |\alpha_k|$ such that*

$$\sum_{k=1}^K (|\alpha_k - \alpha_k^2| + |\xi_k - \xi_k^2|) \leq C(1 + \sqrt{\epsilon}) \|u'(0, t) - (u^2)'(0, t)\|_{L^2(0,T)}.$$

Proof: By Theorem 4.3.3 we have

$$\|a - a^2\|_{H^{-1}(0,1)} \leq C \|u'(0, t) - (u^2)'(0, t)\|_{L^2(0,T)}.$$

The conclusion now follows from the next estimate proved in Theorem 5.1 of [58]

$$\sum_{k=1}^K (|\alpha_k - \alpha_k^2| + |\xi_k - \xi_k^2|) \leq C(1 + \sqrt{\epsilon}) \|a - a^2\|_{H^{-1}(0,1)}.$$

■

If we assume that $\alpha_k^2 = \alpha_k$, then using Theorem 5.2 of [58] we can obtain

Theorem 4.3.5 *Fix $T > 0$ and suppose that $A^2 = (\alpha_k, \xi_k^2)_{k=1}^K$ is in $\Sigma \cap B(A, \epsilon)$ with $\epsilon > 0$ satisfying (4.28) and (4.29). Then there exists a constant C depending on T, $\min_{k \neq k'} |\xi_k - \xi_{k'}|$ and $\min_k |\alpha_k|$ such that*

$$\sum_{k=1}^{K} |\xi_k - \xi_k^2| \leq C\sqrt{\epsilon} |u'(0,t) - (u^2)'(0,t)\|_{L^2(0,T)}.$$

4.3.4 Reconstruction

For the reconstruction of point sources we could follow the arguments of subsection 4.2.4 and obtain a reconstruction result similar to Theorem 4.2.13. We here present an alternative result following the point of view of [18] based on the explicit knowledge of the eigenvalues and the eigenfunctions and some properties of Fourier series. This result seems to be more realistic in the pratical point of view than the first one but the prize to pay is that we need boundary observations on a timelength $\frac{1}{\pi}$. For the sake of simplicity, we only consider the case of two point sources, namely

$$(\alpha_1, \xi_1), (\alpha_2, \xi_2), 0 < \xi_1 < \xi_2 < 1.$$

Now we introduce the operator from $L^2(0, \frac{1}{\pi})$ to $L^2(0, \frac{1}{\pi})$ defined by

$$(Lf)(t) = \int_0^t \lambda'(t-s) f(s) ds, \;\; 0 \leq t \leq \frac{1}{\pi}. \tag{4.48}$$

By the assumption (4.2), we see that $-(\lambda(0) + L)^{-1}$ corresponds to the solution of a Volterra equation of second kind and therefore, $-(\lambda(0)+L)^{-1}$ is a bounded operator from $L^2(0, \frac{1}{\pi})$ into itself. We further assume

$$\int_0^{\frac{1}{\pi}} ((\lambda(0) + L)^{-1}\lambda)(t)\, dt \neq 0. \tag{4.49}$$

Henceforth we denote by $(\cdot, \cdot)$, the $L^2(0, \frac{1}{\pi})$-inner product, i.e., $(\phi, \psi) = \int_0^{\frac{1}{\pi}} \phi(t)\psi(t)dt$. Moreover, let us set

$$e_k(t) = \cos(k^2\pi^2 t), \;\; k \in \mathbb{N},$$

and

$$\psi_k = (\lambda(0) + L^\star)^{-1} e_k, \;\; k \in \mathbb{N},$$

where $L^\star{:}L^2(0, \frac{1}{\pi}) \rightarrow L^2(0, \frac{1}{\pi})$ is the adjoint operator of L given by

$$L^\star \psi(t) = \int_t^{\frac{1}{\pi}} \lambda'(s-t)\psi(s)\, ds, 0 \leq t \leq \frac{1}{\pi},$$

and consequently ψ_k is the solution of the Volterra equation of the second kind

$$\lambda(0)\psi_k(t) + \int_t^{\frac{1}{\pi}} \lambda'(s-t)\psi_k(s)\, ds = \cos(k^2\pi^2 t), 0 \leq t \leq \frac{1}{\pi}.$$

Remark 4.3.6 We directly see that the assumption (4.49) is equivalent to

$$(\lambda, \psi_0) \neq 0. \tag{4.50}$$

Now we can state our reconstruction result (compare with Theorem 3 of [18]):

Theorem 4.3.7 *Assume that (4.49) holds. Then for all* $k \geq 1$ *we have the following identity*

$$\alpha_1 \sin(k\pi\xi_1) + \alpha_2 \sin(k\pi\xi_2) = k^3\pi^4 \left(\frac{(u'(0,\cdot), \psi_0)}{(\lambda, \psi_0)}(\lambda, \psi_k) - (u'(0,\cdot), \psi_k)\right). \tag{4.51}$$

In particular, if we assume that $\alpha_1 = \alpha_2 = 1$, *then* ξ_1 *and* ξ_2 *are given by*

$$\xi_1 = \frac{1}{\pi} \arcsin\theta_1, \xi_2 = \frac{1}{\pi}\arcsin\theta_2,$$

θ_1 *and* θ_2 *being the roots of*

$$\theta^2 - a\theta + \frac{b + 4a^3 - 3a}{12a} = 0, \tag{4.52}$$

where

$$\begin{aligned} a &= \pi^4 \frac{(u'(0,\cdot), \psi_0)}{(\lambda, \psi_0)}(\lambda, \psi_1) - \pi^4 (u'(0,\cdot), \psi_1), \\ b &= 27\pi^4 \frac{(u'(0,\cdot), \psi_0)}{(\lambda, \psi_0)}(\lambda, \psi_3) - 27\pi^4 (u'(0,\cdot), \psi_3). \end{aligned}$$

Proof: We remark that (4.45) may be equivalently written

$$\begin{aligned} u(x,t) &= -2\sum_{k=1}^{\infty}\frac{\alpha_1\sin(k\pi\xi_1)+\alpha_2\sin(k\pi\xi_2)}{k^4\pi^4}(\lambda(0)+L)e_k(t)\sin(k\pi x) \\ &+ 2\sum_{k=1}^{\infty}\frac{\alpha_1\sin(k\pi\xi_1)+\alpha_2\sin(k\pi\xi_2)}{k^4\pi^4}\sin(k\pi x)\lambda(t). \end{aligned}$$

Setting

$$g(\xi,x)=2\sum_{k=1}^{\infty}\frac{\sin(k\pi\xi)\sin(k\pi x)}{k^4\pi^4},$$

the above identity may be written

$$\begin{aligned} -u(x,t) &= 2\sum_{k=1}^{\infty}\frac{\alpha_1\sin(k\pi\xi_1)+\alpha_2\sin(k\pi\xi_2)}{k^4\pi^4}(\lambda(0)+L)e_k(t)\sin(k\pi x) \\ &- \lambda(t)(\alpha_1 g(\xi_1,x)+\alpha_2 g(\xi_2,x)). \end{aligned}$$

Differentiating this identity with respect to x, we obtain

$$\begin{aligned} -u'(x,t) &= 2\sum_{k=1}^{\infty}\frac{\alpha_1\sin(k\pi\xi_1)+\alpha_2\sin(k\pi\xi_2)}{k^3\pi^3}(\lambda(0)+L)e_k(t)\cos(k\pi x) \\ &- \lambda(t)(\alpha_1 g'(\xi_1,x)+\alpha_2 g'(\xi_2,x)), \end{aligned}$$

so that we can substitute $x=0$ to get

$$\begin{aligned} -u'(0,t) &= \sum_{k=1}^{\infty}\frac{2}{k^3\pi^3}(\alpha_1\sin(k\pi\xi_1)+\alpha_2\sin(k\pi\xi_2))(\lambda(0)+L)e_k(t) \qquad (4.53) \\ &- \lambda(t)(\alpha_1 g'(\xi_1,0)+\alpha_2 g'(\xi_2,0)). \end{aligned}$$

On the other hand we note that

$$((\lambda(0)+L)e_k,(\lambda(0)+L^\star)^{-1}e_j)=\begin{cases} 0 \text{ if } k\neq j,\ k,j\geq 0, \\ \frac{1}{2\pi} \text{ if } k=j. \end{cases}$$

Therefore in (4.53) taking the $L^2(0,\frac{1}{\pi})$-inner product with $\psi_j = (\lambda(0)+L^\star)^{-1}e_j$, we obtain

$$\begin{aligned} -(u'(0,\cdot),\psi_0) &+ (\lambda,\psi_0)(\alpha_1 g'(\xi_1,0)+\alpha_2 g'(\xi_2,0)) = 0, && (4.54)\\ -(u'(0,\cdot),\psi_j) &+ (\lambda,\psi_j)(\alpha_1 g'(\xi_1,0)+\alpha_2 g'(\xi_2,0)) && (4.55)\\ &= \frac{\alpha_1\sin(j\pi\xi_1)+\alpha_2\sin(j\pi\xi_2)}{j^3\pi^4}, \forall j\geq 1. \end{aligned}$$

The identity (4.54) is equivalent to

$$\alpha_1 g'(\xi_1,0)+\alpha_2 g'(\xi_2,0) = \frac{(u'(0,\cdot),\psi_0)}{(\lambda,\psi_0)},$$

which we combine with (4.55) to obtain (4.51).

Now if we assume that $\alpha_1 = \alpha_2 = 1$, then (4.51) for $k = 1,3$ gives with the notation from the statement of the Theorem

$$\begin{aligned} \sin(\pi\xi_1)+\sin(\pi\xi_2) &= a,\\ \sin(3\pi\xi_1)+\sin(3\pi\xi_2) &= b. \end{aligned}$$

Using the trigonometric rule $\sin 3\rho = 3\sin\rho - 4\sin^3\rho$ and the above identities we further get

$$\sin(\pi\xi_1)\sin(\pi\xi_2) = \frac{b+4a^3-3a}{12a}.$$

Consequently the roots θ_1,θ_2 of (4.52) are equal to $\sin(\pi\xi_1)$ and $\sin(\pi\xi_2)$ respectively.

■

Remark 4.3.8 For an arbitrary $T > 0$, the above reconstruction scheme would work if we could find a dual family $(f_k)_{k\in\mathbb{N}}$ to $(e_k)_{k\in\mathbb{N}}$, in the sense that

$$\int_0^T e_k(t)f_l(t)\,dt = \delta_{kl}, \forall k,l\in\mathbb{N}.$$

In that case it would suffice to take

$$\psi_k = (\lambda(0)+L^\star)^{-1}f_k.$$

To our knowledge such a family is not explicitely known except if $T = \frac{n}{\pi}$, for a positive integer n.

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