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**Etude Arithmétique et Combinatoire des Fonctions de
Partition et Surpartition**

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By: **MOHAMMED LAMINE NADJI**

Topic

**Arithmetic and Combinatorial Study of Partition and
Overpartition Functions**

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ABSTRACT

This thesis investigates novel aspects of the arithmetic and combinatorial properties of integer partitions, with a focus on restricted partition and overpartition sets. It addresses classical partition problems while introducing new classes of partitions and overpartitions subject to intricate restriction criteria, particularly multi-restricted partitions and partitions with constraints on part multiplicities. Using combinatorial techniques and generating functions, the study advances the theoretical understanding of partitions by uncovering new structural and arithmetic properties.

A key contribution of this work is the detailed exploration of partitions where parts are simultaneously ℓ -regular and t -distinct. Here, ℓ -regularity restricts parts to values not divisible by ℓ , while t -distinctness ensures that no part appears more than $t - 1$ times in a partition. This dual restriction defines a new partition class, leading to original recurrence relations, bijections, and explicit congruence families. These results are extended to biregular partitions, biregular overpartitions, and further generalized to newly introduced partition classes, including s -modular, s -congruent, and s -duplicate partitions. These extensions provide a broader framework for understanding the impact of combinatorial constraints on integer partitions.

By establishing connections with well-known classical partition theorems, such as the Andrews-Göllnitz-Gordon theorem, the research introduces a new combinatorial foundation for these partition classes, linking them to classical partition theory through the behavior of partitions into parts simultaneously s -congruent and t -distinct. Furthermore, the thesis extends the study of arithmetic properties in partitions and overpartitions enumerated by the Andrews-Göllnitz-Gordon theorem, deriving novel congruences for these partitions, as well as new congruences of 2 and 3-regular multipartitions.

Beyond arithmetic properties, a novel combinatorial study of overpartitions into distinct parts is conducted. The thesis also provides an in-depth examination of Arndt compositions under various constraints, including restricted part sequences, palindromic structures, and colored variants.

These findings expand the theoretical framework of restricted partition and overpartition functions, reinforcing their role in combinatorial number theory. This research not only deepens our understanding of integer partitions but also suggests new avenues for future study, particularly in recurrence relations, congruences, and restricted partition classes. With potential applications in modular forms, Lie algebras, and related fields, this thesis serves as a valuable reference for ongoing and future research in partition theory.

Keywords : Partition, Overpartition, Congruence, Andrews- Göllnitz-Gordon theorem, POD, Unrestricted even part, Distinct odd part, Tripartition, Regular, Schur's partition, Distinct part, Arndt, Composition, Fibonacci number, Generating function, Rank, Crank.

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Mohammed Lamine Nadji
USTHB, February 2025

this is ten percent luck
twenty percent skill
fifteen percent concentrated power of will
five percent pleasure
fifty percent pain
*and a hundred percent reason to remember **THE NAME***

Create, Explore, Expand, Conquer.

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INTRODUCTION

The theory of partitions, a branch situated at the intersection of combinatorics and number theory, explores the myriad ways in which integers can be expressed as sums of positive integers. Since Euler's groundbreaking work on partition identities and Ramanujan's renowned congruences, this field has evolved significantly, becoming deeply intertwined with modular forms, q -series, generating functions, Lie algebra, and beyond. This interplay has provided profound insights into the arithmetic properties and combinatorial structures of the various partition functions. This thesis extends these classical foundations, delving into advanced properties of restricted partitions and overpartitions, introducing novel classes, establishing original results, and broadening existing theories to multi-restricted frameworks.

Aims and objectives

The primary objective of this thesis is to investigate the arithmetic and combinatorial properties of new classes of restricted partitions along two main axes: partitions that satisfy complex combinations of divisibility and multiplicity constraints, and those where the multiplicity of parts falls within specified residue classes modulo a given positive integer. While classical partition results predominantly address unrestricted partitions or those subject to simple constraints, this work delves into the realm of multi-restricted partitions. Specifically, it examines partitions that are simultaneously ℓ -regular (where no part is divisible by ℓ) and t -distinct (where each part appears fewer than t times). These combined conditions define a novel class of partitions, endowed with distinctive arithmetic and combinatorial properties that remain largely unexplored within the classical partition theory framework.

This thesis further aims to investigate the arithmetic and combinatorial properties of specific known classes of restricted partitions and overpartitions by developing new combinatorial identities and arithmetic properties. The results presented herein contribute to advancing the field of partition theory.

Thesis structure

The thesis is structured into six major parts, each offering a unique contribution to partition theory through the investigation of combinatorial and arithmetic properties. These parts collectively advance the field by introducing novel partition classes, establishing new identities, and deriving congruences, thereby providing deeper insights into both classical and contemporary aspects of integer partitions and compositions.

Overview of restricted partition sets and new contributions

A significant part of this work is dedicated to the development of novel partition types, including partitions into parts that are simultaneously regular and distinct, t -Schur's partitions and overpartitions, as well as s -modular, s -congruent, and s -duplicate partitions. Each of these classes introduces unique arithmetic and combinatorial properties, offering fresh insights that enhance our understanding of integer partitions.

I. Multi-restricted Sets of Partitions: This thesis defines and analyzes partitions into parts that are both ℓ -regular and t -distinct, a dual restriction that creates intricate structural properties. By employing generating functions dissections, the study derives new congruences for these partitions, including specific cases with congruences modulo 2, 3, 4, 6, and 12. These results open pathways for a deeper understanding of more complex partition behavior under multi-restrictions.

This work also expands upon the celebrated classical Schur's partitions, specifically those into parts congruent to ± 1 modulo 6, and provides a generating function for the partitions into parts that differ by at least 3 where no consecutive multiples of 3 appear through combinatorial reasoning. These new results broaden the scope of classical Schur's partitions, enhancing their impact on integer partitions and related topics.

II. Congruences and Multiplicities: Further expanding the scope, the thesis examines three novel partition classes:

- s -Modular partitions, a class consisting of partitions into parts with a number of occurrences (i.e., multiplicity) congruent to 0 or 1 modulo s .
- s -Congruent partitions, which generalize Sellers' partitions into parts not congruent to 2 modulo 4.
- s -Duplicate partitions, of which the partitions having distinct odd parts and enumerated by the function $\text{pod}(n)$ are a special case.

Each of these partition classes is rigorously defined, and their combinatorial properties are explored, with connections drawn to the classical Andrews-Göllnitz-Gordon theorem.

III. Arithmetic Properties of ℓ -Regular Multipartitions: Multipartitions extend traditional partitions by allowing an integer to be partitioned into multiple independent partitions. This part is dedicated to the arithmetic properties of ℓ -regular tripartitions, particularly through the function $T_\ell(n)$, which counts the number of tripartitions into ℓ -regular parts, by deriving congruences modulo 2, 3, 6, 9 and 12.

IV. Combinatorial Properties of Overpartitions Into Distinct Parts: Overpartitions, in which the first occurrence of each part may be overlined, are examined under distinct-part conditions in this part. The focus lies on the combinatorial properties of overpartitions into distinct parts, leading to new identities for the Rank and Crank of an overpartition. This exploration reveals deeper structural patterns within overpartitions, enriching the study of their combinatorial behavior and applications.

V. Arndt Compositions and Linear Recurrence Relations: The final part introduces Arndt compositions, which are compositions subject to specific ordering constraints, and examines their connections to linear recurrence relations such as the Padovan and Narayana sequences. By deriving generating functions for Arndt compositions under various constraints, including palindromic and colored forms, this part demonstrates how these compositions relate to

recurrence relations and combinatorial identities. These findings broaden the applications of Arndt compositions within combinatorial theory and the study of integer compositions.

Research contributions and impact

The contributions of this thesis extend partition theory by rigorously investigating new classes of restricted partitions and deriving various congruences for them. Employing generating functions and various combinatorial techniques, this work establishes original results that strengthen the theoretical framework for partitions, overpartitions, and compositions. By forging connections between these new classes of partitions, this research lays a foundation for further explorations in theory of partitions.

The findings of this thesis offer both immediate and future applications, opening new pathways for investigating arithmetic and combinatorial properties within related fields. Potential directions for further research include extending these results to additional topics, exploring their applications and examining deeper connections with classical combinatorial identities.

In summary, this work makes a contribution to partition theory, enhancing its mathematical depth and practical implications within the broader contexts of number theory and combinatorics.

PART I:**PRELIMINARIES**

CHAPTER 1

INTRODUCTION

1.1 The beginnings of the theory of partitions

Partition theory, also known as integer partitions, might well be described as the bridge between various fields in mathematics, such as combinatorics, number theory, Lie algebra, and more. These connections may even extend to theories outside of mathematics, such as statistical mechanics [38]. This field was first introduced in mathematical literature in a letter [142] from 1674 by Leibniz addressed to Bernoulli. In that letter, Leibniz posed a groundbreaking question regarding integer partitions, inquiring about the number of ways a positive integer n can be decomposed into a sum of other integers. In other words, for any positive integer n , what is the number of partitions of n ?

Definition 1.1. A partition λ of a positive integer n is a finite sequence of natural numbers which sum to n . We write $\lambda_1^{\alpha_1} + \lambda_2^{\alpha_2} + \cdots + \lambda_k^{\alpha_k} = n$, where each part λ_i occurs α_i times with $\lambda_i, \alpha_i \geq 1$ for $1 \leq i \leq k$. α_i is also referred to as the multiplicity of the part λ_i . We use $p(n)$ to denote the number of partitions of n . By convention, we let $p(0) = 1$ since the empty partition \emptyset is the only partition of 0.

Notation 1.1. Let $|\lambda|$ denote the sum of the parts and $\ell(\lambda)$ denote the total number of parts in the partition λ .

For example, the partitions of $1, \dots, 5$ are listed in Table 1.1.

n	$p(n)$	Partitions of n
0	1	\emptyset
1	1	(1)
2	2	(2), (1 ²)
3	3	(3), (2, 1), (1 ³)
4	5	(4), (3, 1), (2 ²), (2, 1 ²), (1 ⁴)
5	7	(5), (4, 1), (3, 2), (3, 1 ²), (2 ² , 1), (2, 1 ³), (1 ⁵)

TABLE 1.1: Partitions of $0, 1, \dots, 5$.

After 66 years of obscurity, the theory of partitions resurfaced. In 1740, Naudé raised a number of questions in his letter to Euler. One of these questions sparked the first in-depth research, leading to results that have formed the cornerstone of partition theory to this day.

Euler’s fascination with the question concerning the number of partitions of 50 into 7 distinct parts indeed marked a significant milestone in the history. His rigorous investigation into this problem culminated in his seminal solution in 1741, which not only provided a resolution to the specific inquiry but also laid the groundwork for subsequent advancements in the field. These elementary, but remarkable, results were presented in his fundamental treatise on analysis, *Introductio in Analysin Infinitorum* [87]. The work was divided into two parts: an analytic and a geometric section. It was designed to eliminate the necessity of using the methods of differential and integral calculus. In particular, the inception of integer partitions occurred in a chapter entirely devoted to the derivation of the basic formulas in the algebraic theory of number partitions.

For 140 years, the theory remained entirely without any practical application. In 1882, Sylvester, whose favorite field was higher algebra, lacked the patience to comprehend the theory of partitions through the study of elliptic functions. Consequently, he pioneered an entirely new method [196] based on Ferrers’ suggestion in 1853 for deriving the properties of partitions by employing a graphical approach reliant on point lattices. Sylvester’s geometrical representation introduced no new theories to the field; however, it provided an alternative means other than analysis. The latter method of addressing the problem is commonly referred to as the "Constructive Theory of Partitions".

Definition 1.2. *Ferrers diagram is a method of geometric representation of partitions using rows of dots. Each row represents the value of the part $\lambda_i \in \lambda$ for $1 \leq i \leq k$. The rows are ordered in non-increasing order, with the largest part represented in the top row and the smallest one in the bottom row. Analogously, in Young diagram, we use boxes instead of dots. In this thesis, sometimes we abuse notations when we use boxes in Ferrers diagram.*

For example, the partition of 14 into $\lambda = (5, 3^2, 2, 1)$ would be represented by the following Ferrers and Young diagrams.



FIGURE 1.1: Ferrers and Young diagrams of the partition $\lambda = (5, 3^2, 2, 1)$.

Definition 1.3. *For the partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$, define the sum $\lambda + \beta$ to be a partition $(\lambda_1 + \beta_1, \lambda_2 + \beta_2, \dots)$, where the shorter partition may be padded with zeros.*

Figure 1.2 shows the Young diagram of the sum of the partitions $\lambda = (6, 5, 4)$ and $\beta = (4, 3)$.

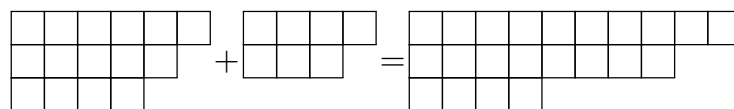


FIGURE 1.2: Young diagram of $\lambda + \beta = (10, 8, 4)$.

Definition 1.4. For the partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$, define the union $\lambda \cup \beta$ to be a partition with all the parts $\{\lambda_i, \beta_i\}$ arranged in nonincreasing order, with repetitions.

Figure 1.3 shows the definition of the operation using Young diagram of the partitions $\lambda = (4)$ and $\beta = (3, 2)$.

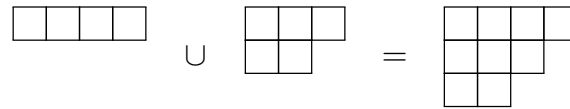


FIGURE 1.3: Young diagram of $\lambda \cup \beta = (4, 3, 2)$.

Definition 1.5. The conjugation of a Young (resp. Ferrers) diagram of the partition λ is formed by reflecting the diagram across its diagonal (transposing) to obtain the new partition λ^c .

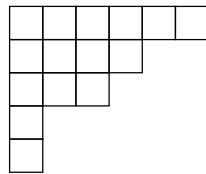


FIGURE 1.4: Young diagram of the conjugate partition of $(5, 3^2, 2, 1)$.

Definition 1.6. The k -modular Ferrers diagram, which is a modification of the Ferrers diagram, represents a partition in such a way that each part is depicted by a left-justified row of k 's with an r at the right end, where $1 \leq r \leq k$.

For example, Figure 1.5 illustrates the 3-modular Ferrers diagram of $\lambda = (11^2, 9^2, 4, 3)$ with the diagram of $(4^2, 3^2, 2, 1)$.

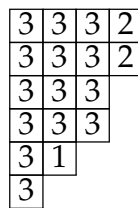


FIGURE 1.5: The 3-modular Ferrers diagram of $\lambda = (11^2, 9^2, 4, 3)$.

Definition 1.7. Given an integer partition λ , its Durfee square is the largest square with size r contained in its Young (resp. Ferrers) diagram.

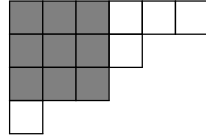


FIGURE 1.6: The partition $(6, 4, 3, 1)$ has a Durfee square of size 3.

1.2 Generating functions

In 1730, in order to solve the general linear recurrence problem, De Moivre [132] introduced a revolutionary tool called "*The Generating Functions*". Approximately two decades later, Euler [87] used these generating functions to solve Naudé's question, which remains a fundamental tool in the study of integer partitions to this day. Let the following expression:

$$(1 + x^a y)(1 + x^b y)(1 + x^c y)(1 + x^d y)(1 + x^e y) \cdots \quad (1.1)$$

When the above factors are multiplied, we obtain some coefficient of x^n . Since each number of the sequence a, b, c, d, e, \dots can appear at most once in the partition λ of n , then each x^n term represents a single partition of magnitude n into distinct parts before like terms are collected. Therefore, it becomes immediately apparent how many different ways a given positive integer n can be the sum of any desired number of positive integers from the sequence a, b, c, d, e , etc. Similarly, the exponent on y in the term $y^m x^n$ records the number of parts of the partition formed by the exponent on x . In conclusion, with replacing x by q and y by z , we arrive at the fact that the number of partitions of n into m distinct parts satisfies the following generating function:

$$\sum_{n, m \geq 0} D(n, m) z^m q^n = \prod_{i \geq 1} (1 + zq^i), \quad (1.2)$$

where $D(n, m)$ denotes the number of partitions of n into m distinct parts. Now, following Euler's analytic reasoning allows us to find a simple recurrence relation for $D(n, m)$. From the right hand side of (1.2), we obtain

$$\prod_{i \geq 1} (1 + zq^i) = (1 + zq) \prod_{i \geq 1} (1 + zq^{i+1}) = (1 + zq) \prod_{i \geq 1} (1 + (zq)q^i),$$

from which, we deduce that

$$\begin{aligned} \sum_{n, m \geq 0} D(n, m) z^m q^n &= (1 + zq) \sum_{n, m \geq 0} D(n, m) (zq)^m q^n \\ &= \sum_{m \geq 0, n \geq 0} D(n, m) z^m q^{n+m} + \sum_{m \geq 1, n \geq 0} D(n, m-1) z^m q^{n+m} \\ &= \sum_{m \geq 0, n \geq m} D(n-m, m) z^m q^n + \sum_{m \geq 1, n \geq m} D(n-m, m-1) z^m q^n, \end{aligned}$$

where the third line follows from the substitution $n \rightarrow n - m$. Then by comparing the coefficients of $z^m q^n$ on both sides of the above equation, thus for all integers $n, m \geq 1$,

$$D(n, m) = D(n - m, m) + D(n - m, m - 1), \quad (1.3)$$

where $D(n, m) = 0$ if $n \leq 0$ and $m > 0$, $m \leq 0$ and $n > 0$, or $n < m$. By convention, we define $D(0, 0) = 1$.

From a combinatorial perspective, we can derive the recurrence relation (1.3) by dividing the set of partitions of n into m distinct parts into two disjoint sets. The first set consists of partitions with parts greater than 1, while the second set comprises partitions containing a part of size 1. By subtracting one from each part in every partition of the first set yields partitions of $n - m$ into m parts. Similarly, by eliminating the part 1 from each partition in the second subset and decrementing each remaining part by one, we obtain partitions of $n - m$ into $m - 1$ distinct parts.

Utilizing the recurrence relation formula (1.3), Euler successfully computed that the number of partitions of 50 into 7 distinct parts is $D(50, 7) = 522$.

After observing the aforementioned result regarding partitions into distinct parts, Euler realized that this concept may easily be extended by removing the distinctness condition from the partitions. That is, if we let $p(n, m)$ denote the number of partitions of n into m parts, then the generating function for $p(n, m)$ is

$$\sum_{n, m \geq 0} p(n, m) z^m q^n = \prod_{i \geq 1} (1 + zq^i + z^2q^{2i} + z^3q^{3i} + z^4q^{4i} + \dots) = \prod_{i \geq 1} \frac{1}{1 - zq^i}.$$

In the same way as before, we can deduce that for all integers $n, m \geq 1$,

$$p(n, m) = p(n - 1, m - 1) + p(n - m, m), \quad (1.4)$$

where $p(n, m) = 0$ if $n \leq 0$ and $m > 0$, $m \leq 0$ and $n > 0$, or $n < m$. By convention, we define $p(0, 0) = 1$.

After studying the generating function of partitions, Euler delved into the exploration of the infinite product:

$$\prod_{i \geq 1} (1 - q^i) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \dots$$

A few years later, he substantiated his celebrated Pentagonal Numbers Theorem:

$$\prod_{i \geq 1} (1 - q^i) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = 1 + \sum_{n=1}^{\infty} (-1)^n (q^{\frac{n(3n+1)}{2}} + q^{\frac{n(3n-1)}{2}}),$$

which marked the inception of the theory of theta functions and modular forms [131]. This theorem became instrumental in devising the most efficient method to compute $p(n)$.

Theorem 1.1 (Euler). *For any nonnegative integer n , we have*

$$p(n) = p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) + p(n - 12) + p(n - 15) - \dots,$$

where by convention let $p(k) = 0$ for negative k .

Proof. We have that

$$\left(\sum_{n \geq 0} p(n) q^n \right) \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} \right) = 1.$$

Comparing the coefficients of q^n on each side, we arrive at the desired result. \square

The first partition identity discovered in the theory of partitions was by Euler [87]. Later, Legendre rephrased this identity purely in terms of partitions [25, p.24].

Theorem 1.2 (Euler). *For every positive integer n , the number of partitions of n into distinct parts $D(n)$ is equal to the number of partitions of n into odd parts $O(n)$.*

Proof. We begin with the expressions:

$$\sum_{n \geq 0} D(n)q^n = \prod_{i \geq 1} (1 + q^i), \text{ and } \sum_{n \geq 0} O(n)q^n = \prod_{i \geq 1} \frac{1}{(1 - q^{2i-1})}.$$

Using the identity $(1 - q^{2i}) = (1 + q^i)(1 - q^i)$, we can rewrite the first sum as:

$$\sum_{n \geq 0} D(n)q^n = \prod_{i \geq 1} \frac{(1 - q^{2i})}{(1 - q^i)},$$

from which, we deduce that:

$$\prod_{i \geq 1} \frac{(1 - q^{2i})}{(1 - q^i)} = \prod_{i \geq 1} \frac{(1 - q^2)(1 - q^4) \cdots}{(1 - q^1)(1 - q^2) \cdots} = \prod_{i \geq 1} \frac{1}{(1 - q^{2i-1})} = \sum_{n \geq 0} O(n)q^n.$$

Thus, we have shown that:

$$\sum_{n \geq 0} D(n)q^n = \sum_{n \geq 0} O(n)q^n.$$

\square

n	Partitions of n into distinct parts	Partitions of n into odd parts
1	(1)	(1)
2	(2)	(1 ²)
3	(3), (2, 1)	(3), (1 ³)
4	(4), (3, 1)	(3, 1), (1 ⁴)
5	(5), (4, 1), (3, 2)	(5), (3, 1 ²), (1 ⁵)

TABLE 1.2: Euler's identity for the first five integers.

After the development of generating function proofs, bijective proofs were introduced to establish a correspondence between each partition of n into distinct parts with a partition of n into odd parts. Given a partition into distinct parts, one can determine the corresponding partition into odd parts. When starting with a partition into odd parts and transforming it into a partition into distinct parts, the goal is to ensure all parts are distinct. Therefore, for every two identical parts, we merge them into a part of twice the size. This procedure is

repeated until all parts are distinct. The following example demonstrates this procedure:

$$\begin{aligned}
 9 + 7 + 7 + 7 + 5 + 3 + 3 + 3 + 3 &\mapsto 9 + (7 + 7) + 7 + 5 + (3 + 3) + (3 + 3) \\
 &\mapsto 14 + 9 + 7 + 5 + 6 + 6 \\
 &\mapsto 14 + 9 + 7 + 5 + (6 + 6) \\
 &\mapsto 14 + 12 + 9 + 7 + 5.
 \end{aligned}$$

Now we need to find the inverse transformation. Starting with a partition into distinct parts, we want to obtain a partition into odd parts. Therefore, we split every even part into two equal halves and repeat this procedure until all the parts are odd. Applying this method to the partition obtained in the previous example yields:

$$\begin{aligned}
 14 + 12 + 9 + 7 + 5 &\mapsto 9 + (7 + 7) + 7 + (6 + 6) + 5 \\
 &\mapsto 9 + 7 + 7 + 7 + 5 + (3 + 3) + (3 + 3) \\
 &\mapsto 9 + 7 + 7 + 7 + 5 + 3 + 3 + 3 + 3.
 \end{aligned}$$

Bijjective proofs exhibit remarkable diversity, employing a wide array of methods and tools that range from combinatorial arguments to geometric interpretations, showcasing the creativity and depth of mathematical reasoning. Certain partition identities are fortunate to have both generating function proofs and bijective proofs, providing multiple perspectives on the same result. However, not all partition identities enjoy this dual proof structure; some are supported solely by generating functions or bijective methods. Moreover, finding a bijective proof is not always straightforward since it can vary in difficulty from elementary to highly complex, often requiring innovative and nonintuitive approaches. This variability highlights both the richness and the challenge inherent in the study of partition identities. In 1969, Andrews [16] extended this proof to generalize Euler's identity to other sets of partitions that can also be associated through a merging/splitting procedure. This extension broadened the scope of bijective proofs by demonstrating that similar techniques could be applied to a wider variety of partition identities. Andrews' work showed that the principles underlying Euler's identity were not isolated, but rather part of a more extensive framework of partition theory, capable of encompassing various other sets of partitions through analogous transformations.

Sylvester [196] emerged as the next major contributor to the field of partition identities, building upon Euler's work in the late nineteenth century.

Theorem 1.3 (Sylvester). *For every positive integers k and n , the number of partitions of n with largest part k is equal to the number of partitions of n into exactly k parts.*



FIGURE 1.7: The conjugate of a partition of n with the largest part k is a partition of n into exactly k parts.

1.3 q -Hypergeometric functions

In 1812, Gauss presented his seminal work [97] to the Royal Society of Sciences in Göttingen, in which he analyzed the infinite series

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; z \right] = 1 + \frac{ab}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} z^3 + \dots$$

treating it as a function of a, b, c, d, z , where it is assumed that $c \neq 0, -1, -2, \dots$, to ensure that the denominators of the series remain free from zero factors. In light of Gauss' paper, his series is frequently referred to as *Gauss' series*. However, since the special case $a = 1, b = c$ yields the geometric series

$${}_2F_1 \left[\begin{matrix} 1, c \\ c \end{matrix} ; z \right] = F(1, c; c; z) = 1 + z + z^2 + z^3 + \dots,$$

Gauss' series is also known as "*the ordinary hypergeometric series*". Some celebrated functions that can be expressed using the ordinary hypergeometric series are:

$${}_2F_1 \left[\begin{matrix} -a, b \\ b \end{matrix} ; z \right] = F(-a, b; b; z) = (1+z)^a,$$

$$z {}_2F_1 \left[\begin{matrix} 1, 1 \\ 2 \end{matrix} ; -z \right] = zF(1, 1; 2; -z) = \log(1+z),$$

where $|z| < 1$. Before Gauss, Euler [87] had discovered several results for hypergeometric series, including his transformation formula

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; z \right] = (1-z)^{c-a-b} F(c-a, c-b; c; z), \quad |z| < 1.$$

Three decades after Gauss' paper, Heine [113, 114] introduced the series

$$1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)} z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})} z^2 + \dots,$$

where it is assumed that $q \neq 1$ and $c \neq 0, -1, -2, \dots$. This series converges absolutely for $|z| < 1$ when $|q| < 1$ and it tends to Gauss' series as $q \rightarrow 1$, since

$$\lim_{q \rightarrow 1} \frac{1-q^a}{1-q} = a.$$

The series presented by Heine is usually referred to as Heine's series, the basic hypergeometric series, or q -hypergeometric series.

Notation 1.2 (Pochhammer symbols).

$$(a)_n := \begin{cases} a(a+1)(a+2)\cdots(a+n-1) & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

Notation 1.3 (q -Pochhammer symbols). Let $q \in \mathbb{C}$ such that $|q| < 1$.

$$(a; q)_n := \begin{cases} \prod_{i=0}^{n-1} (1 - aq^i) & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

Moreover, we shall use the compact notations

$$(a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n \text{ for } |q| < 1, \\ (a_1, a_2, \dots, a; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a; q)_\infty.$$

Analogous to Gauss' series notation $F(a, b; c; z)$, Heine's series is denoted by $\phi(a, b; c; q, z)$. However, it is more convenient to define Heine's series as

$${}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix}; q, z \right] = \sum_{n \geq 0} \frac{(a; q)_n (b; q)_n}{(q; q)_n (c; q)_n} z^n,$$

and it is assumed that $c \neq q^m$ for $m = 0, 1, 2, \dots$

The systematic study of basic hypergeometric series began with Heine [115] in 1878, who introduced his famous transformation, relating different q -series. Key foundational contributions were made by Euler, who advanced the study of series and infinite products, and Gauss, who developed the theory of hypergeometric series and introduced the Gauss hypergeometric function. Bailey [39], who made many contributions of his own, provided a concise overview in his monograph. Ample references can be found in the works of Atkin and Swinnerton-Dyer [32], Hardy and Wright [112], MacMahon [153], and Dickson's History of the theory of numbers [79]. Among the subsequent systematic developments in the theory, those by Hahn [109, 110, 111] and Sears [189, 190, 191] are noteworthy. For extensive references and detailed expositions, see Andrews [23, 24]. Overall, the theory of q -series centers on identities.

Definition 1.8. The ordinary hypergeometric series with r numerator parameters a_1, a_2, \dots, a_r and s denominator parameters b_1, b_2, \dots, b_s is defined by

$${}_rF_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z \right] = \sum_{n \geq 0} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_s)_n} \frac{z^n}{n!}$$

A basic hypergeometric series, or q -hypergeometric series, is a q -analogue and generalization of the ordinary hypergeometric series. A series $\{u_n\}_{n \in \mathbb{Z}}$ is called hypergeometric if the ratio of successive terms

$$\frac{u_{n+1}}{u_n} = \frac{(a_1 + n)(a_2 + n) \cdots (a_n + n)}{(1 + n)(b_1 + n) \cdots (b_s + n)} z^n$$

is a rational function of n . If the ratio is a rational function of q^n , then the series is referred to as a q -hypergeometric series.

Definition 1.9. The basic hypergeometric series ${}_r\phi_s$ is defined by

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix} ; q, z \right] = \sum_{n \geq 0} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n (b_2; q)_n \cdots (b_s; q)_n} \left\{ (-1)^n q^{\binom{n}{2}} \right\}^{1+s-r} z^n,$$

where $q \neq 0$ when $r > s + 1$.

The *binomial theorem* is a fundamental result, widely used in fields such as combinatorics, that describes the algebraic expansion of powers of a binomial. In terms of hypergeometric series, the binomial theorem is as:

$${}_1F_0(a; -; z) = \sum_{n \geq 0} \frac{(a)_n}{n!} z^n = (1 - z)^{-a} \quad |z| < 1.$$

The q -analog of the binomial theorem was derived by Cauchy [65], Heine [114] and many other mathematicians. The q -binomial theorem is given by

$${}_1\phi_0(a; -; q, z) = \sum_{n \geq 0} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1 \text{ and } |q| < 1.$$

The q -hypergeometric series are crucial in partition theory, serving as generating functions that count partitions with specific properties, such as those with parts of particular types or sizes. These series provide powerful tools for deriving partition identities, proving combinatorial theorems, and exploring deeper relationships within the structure of integer partitions.

1.4 Restricted partition functions

One may impose some constraints such as requiring distinct parts, odd parts, parts that belong to certain residue classes modulo some positive integer, and so on. A search of the literature suggests that tens of restricted partition functions exist, each with different types of constraints. For instance, by limiting the number of appearances α_i of each part λ_i in a partition of n such that $\alpha_i < \ell$, we obtain an ℓ -distinct partition of n . The number of such partitions of n is denoted by $p^{(\ell)}(n)$. For example, $p^{(3)}(6) = 7$, with the partitions

$$(6), (5, 1), (4, 2), (4, 1^2), (3^2), (3, 2, 1), (2^2, 1^2).$$

This set of restricted partitions has been subject to some studies such as Fink et al. [92] and Hagis [108].

For a positive integer $\ell > 1$, a partition is called ℓ -regular if none of the parts are divisible by ℓ . The number of ℓ -regular partitions of n is denoted by $b_\ell(n)$. For example, $b_3(8) = 13$ with the partitions

$$(8), (7, 1), (5, 2, 1), (5, 1^3), (4^2), (4, 2^2), (4, 2, 1^2), (4, 1^4), (2^4), (2^3, 1^2), (2^2, 1^4), (2, 1^6), (1^8).$$

Euler's partition identity was proved using a combinatorial mapping involving Ferrers graph by Sylvester [196]. In 1883, his student Glaisher [98], generalized this bijection to prove a broader theorem, which reduces to Euler's identity in the special case when $\ell = 2$.

Theorem 1.4 (Glaisher). *For an integer $\ell > 1$, the number of partitions of n in which no part is divisible by ℓ equals the number of partitions of n in which parts appear fewer than ℓ times, such that*

$$\sum_{n \geq 0} p^{(\ell)}(n)q^n = \prod_{i \geq 1} \frac{(1 - q^{\ell i})}{(1 - q^i)} = \sum_{n \geq 0} b_\ell(n)q^n.$$

Proof. We have that

$$\sum_{n \geq 0} b_\ell(n)q^n = \prod_{\substack{i \geq 1 \\ \ell \nmid i}} \frac{1}{1 - q^i} \quad \text{and} \quad \sum_{n \geq 0} p^{(\ell)}(n)q^n = \prod_{i \geq 1} \frac{1 - q^{i\ell}}{1 - q^i}.$$

If we extend the infinite product for $p^{(\ell)}(n)$, we see that

$$\prod_{i \geq 1} \frac{1 - q^{i\ell}}{1 - q^i} = \frac{1 - q^\ell}{1 - q} \cdot \frac{1 - q^{2\ell}}{1 - q^2} \cdot \frac{1 - q^{3\ell}}{1 - q^3} \cdots,$$

where each term in the numerator cancels with the corresponding multiple of ℓ in the denominator. The resulting product after cancellation is the infinite product for $b_\ell(n)$. Thus

$$\sum_{n \geq 0} b_\ell(n)q^n = \sum_{n \geq 0} p^{(\ell)}(n)q^n.$$

From a combinatorial perspective, for a partition λ of n where each part appears fewer than ℓ times, split the parts divisible by ℓ^t into ℓ^t parts, where $t \geq 0$ is the highest power of ℓ that divides the part. This transformation results in a partition with no part divisible by ℓ . For the inverse, given any partition of n whose no part is divisible by ℓ , let the number of appearances α_i of each part λ_i in the base ℓ expansion such that $\alpha_i = \sum_r a_r^i \ell^r$ where $0 \leq a_r^i < \ell$. Then, construct the partition where the part $\lambda_i \ell^r$ appears exactly a_r^i times. \square

For example, consider $n = 6$ with the partitions (6) and (2^3) . We have:

$$\begin{aligned} (6) &\Rightarrow 3 \mid 6 \Rightarrow (2^3), \\ (2^3) &\Rightarrow 3 = 1 \cdot 3 \Rightarrow (2 \cdot 3)^1 \Rightarrow (6). \end{aligned}$$

At the beginning of the twentieth century, an extensive literature sprang from the previous discoveries of Euler, Sylvester, and Glaisher. In 1894, two identities of significant importance in partitions theory appeared in a paper by Rogers [182] for the first time but went unnoticed back then. However, after 19 years, these two identities were rediscovered by Ramanujan, who addressed them in a letter to Hardy. Hardy was unable to prove them and sent them to Littlewood and MacMahon. Although neither could prove them, MacMahon recognized the link between the product series identities and the partition identities. In 1919, a quarter

century after their first appearance, the mystery was solved through a collaboration between Rogers and Ramanujan [183]. Around the same time, particularly in 1917, and separated from the developments in British mathematics by the First World War, Schur [187] also independently proved the Rogers-Ramanujan identities.

Theorem 1.5 (First Rogers-Ramanujan identity). *For every positive integer n , the number of partitions of n such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5.*

Theorem 1.6 (Second Rogers-Ramanujan identity). *For every positive integer n , the number of partitions of n such that the difference between two consecutive parts is at least 2 and the smallest part is larger than 1 is equal to the number of partitions of n into parts congruent to 2 or 3 modulo 5.*

In terms of generating functions, the Rogers-Ramanujan identities can be written as

$$\sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty}, \text{ and } \sum_{n \geq 0} \frac{q^{n(n+1)}}{(q; q)_n} = \frac{1}{(q^2; q^5)_\infty (q^3; q^5)_\infty}.$$

Over the years, the Rogers-Ramanujan identities have acquired the status of the most celebrated identities in the field, with dozens of proofs involving different techniques being published. Notable contributions include those by Watson [201], Garsia and Milne [99], and Bressoud [46]. More generally, a partition identity of the Rogers-Ramanujan type is any partition identity stating that "for every positive integer n , the number of partitions of n with certain difference conditions equals the number of partitions of n with certain congruence conditions".

One can notice that Euler’s identity and the Rogers-Ramanujan first identity have a similar structure. Specifically, "the number of partitions of n such that the consecutive parts differ by at least k equals the number of partitions of n into parts congruent to ± 1 modulo $k + 3$ ". In fact, Euler’s identity corresponds to $k = 1$ and the first Rogers-Ramanujan identity to $k = 2$. Thus, we can wonder if this identity is also true for $k = 3$. In Table 1.3, we list the partitions of such an identity for $k = 3$ for $n = 1, \dots, 9$.

n	parts $\equiv \pm 1$ modulo 6	consecutive parts differ by ≥ 3
1	(1)	(1)
2	(1 ²)	(2)
3	(1 ³)	(3)
4	(1 ⁴)	(4)
5	(1 ⁵), (5)	(5), (4, 1)
6	(1 ⁶), (5, 1)	(6), (5, 1)
7	(1 ⁷), (5, 1 ²), (7)	(7), (6, 1), (5, 2)
8	(1 ⁸), (5, 1 ³), (7, 1)	(8), (7, 1), (6, 2)
9	(1 ⁹), (5, 1 ⁴), (7, 1 ²)	(9), (8, 1), (7, 2), (6, 3)

TABLE 1.3: An illustration of the identity for $k = 3$ and $n = 1, \dots, 9$.

Table 1.3 shows that there are three partitions of 9 into parts congruent to ± 1 modulo 6, but four partitions where consecutive parts differ by at least 3, suggesting the conjecture does

not hold. In 1946, Lehmer [143] proved that for any $k \geq 3$, there is no subset $N \subset \mathbb{N}$ such that for all n , the number of partitions of n with consecutive parts differing by at least k equals the number of partitions of n into parts belonging to N . Ten years later, Alder [31] relaxed the original conjecture to an inequality that might still hold for all k . However, two decades before Lehmer's result, Schur [188] discovered a correct modification of the failed conjecture for $k = 3$ by excluding certain subsets of those partitions with difference conditions. He showed that by excluding specific subsets of partitions with difference conditions, the number of partitions of n with consecutive parts differing by at least 3 is equinumerous to the number of partitions of n into distinct parts congruent to ± 1 modulo 3.

Theorem 1.7 (Schur). *Let $A(n)$ denote the number of partitions of n into parts congruent to ± 1 modulo 6, $B(n)$ denote the number of partitions of n into distinct parts congruent to ± 1 modulo 3, and $C(n)$ denote the number of partitions of n into parts differ by at least 3 where no consecutive multiples of 3 appear. Then for all n ,*

$$A(n) = B(n) = C(n).$$

This theorem sheds new light on why the conjecture fails for $n = 9$, as the partition $(6, 3)$ was included in the count. As expected, Schur's theorem also became a highly influential partition identity, and several different proofs provided using a variety of techniques. These techniques include bijections by Bressoud [44] and Bessenrodt [48], the method of weighted words by Alladi and Gordon [7], and recurrences by Andrews [14, 17, 20]. Among these proofs, Bressoud's bijection was compellingly clever.

Starting with a partition λ into distinct parts congruent to ± 1 modulo 3, transform it into a new partition β by merging pairs of parts that differ by at most two, beginning with the first part. For example,

$$\lambda = \begin{pmatrix} 11 \\ 10 \\ 8 \\ 5 \\ 2 \\ 1 \end{pmatrix} \rightarrow \lambda' = \begin{pmatrix} 11 \\ 10+8 \\ 5 \\ 2+1 \end{pmatrix}$$

The merged pairs always add up to multiples of 3. Consecutive multiples of 3 cannot appear in this way, since these multiples must differ by at least $3(2+i)$, where i is the number of positive integers between two multiples of 3. Next, we subtract consecutive multiples of 3 from each part of λ' , starting with subtracting 0 from the bottom part and continuing upward. The multiples will form a new column to the right.

$$\lambda' = \begin{pmatrix} 11 \\ 10+8 \\ 5 \\ 2+1 \end{pmatrix} \rightarrow \begin{pmatrix} 11-9 \\ 10+8-6 \\ 5-3 \\ 2+1-0 \end{pmatrix} \quad \begin{pmatrix} 9 \\ 6 \\ 3 \\ 0 \end{pmatrix}$$

Now, we rearrange the first column of λ' in descending order.

$$\lambda'' = \begin{pmatrix} 12 \\ 3 \\ 2 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 9 \\ 6 \\ 3 \\ 0 \end{pmatrix}$$

Finally, we add the multiples of 3 from each row in the right column to each part of λ'' .

$$\beta = \begin{pmatrix} 21 \\ 9 \\ 5 \\ 2 \end{pmatrix}$$

The last transformation always results in a partition β into consecutive parts that differ by at least 3, where no consecutive multiples of 3 appear. This transformation is invertible, even if the inverse is quite demanding. Hence, it is a bijection, which proves Schur's theorem.

Schur's modification of the original argument prompted the need for a generalization stating that "for any positive integer n , the number of partitions of n such that the parts differ by at least k and no two consecutive multiples of k appear equals the number of partitions of n into parts satisfying certain congruence conditions". Despite Alder [30] showing that such a generalization is not possible for $k \geq 4$ in 1948, during the 1960s, Göllnitz [101] and Gordon [103] independently demonstrated that such a generalization exists for $k = 2$, which corresponds to the case $a = 1$ in what is known as the Göllnitz-Gordon identities. These Rogers-Ramanujan type identities, with a story similar to that of Rogers and Ramanujan, were initially discovered by Göllnitz. They remained unknown since they were included in his unpublished 1961 honors baccalaureate thesis, until Gordon independently rediscovered them in 1965. Schur's theorem was also generalized in two different ways by Andrews [18, 19].

Theorem 1.8 (Göllnitz-Gordon identities). *Fix a to be either 1 or 3. Given an integer n , the number of partitions of n in which parts are congruent to 4 or $\pm a$ modulo 8 , is equal to the number of partitions of n in which parts are non-repeating and non-consecutive, with any two even parts differing by at least 4 , and with all parts $\geq a$.*

In terms of generating functions, these two identities can be written as

$$\sum_{n \geq 0} \frac{q^{n^2}(-q; q^2)_n}{(q^2; q^2)_n} = \frac{1}{(q; q^8)_\infty (q^4; q^8)_\infty (q^7; q^8)_\infty},$$

and

$$\sum_{n \geq 0} \frac{q^{n(n+2)}(-q; q^2)_n}{(q^2; q^2)_n} = \frac{1}{(q^3; q^8)_\infty (q^4; q^8)_\infty (q^5; q^8)_\infty}.$$

In 1961, Gordon [102] finally found the first generalization of the Rogers-Ramanujan identities.

Theorem 1.9 (Gordon). Let $A_{k,i}(n)$ denote the number of partitions of n into parts not congruent to $0, \pm i$ modulo $2k + 1$. Let $B_{k,i}(n)$ denote the number of partitions of n of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_j$, where $\lambda_j \geq \lambda_{j+1}$, and $\lambda_j - \lambda_{j+k-1} \geq 2$ and at most $i - 1$ parts are ≤ 1 . Then for $1 \leq i \leq k$, for every integer $n \geq 0$,

$$A_{k,i}(n) = B_{k,i}(n).$$

Gordon's theorem was analytically proved by Andrews [12] in 1966, and in [22], he discovered the generating function version of Gordon's theorem by using q -difference equations which has been called the Andrews-Gordon identity (see Kurşungöz [136]).

Theorem 1.10 (Andrews). For $k \geq i \geq 1$,

$$\begin{aligned} \sum_{n \geq 0} B_{k,i}(n)q^n &= \sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_i + \cdots + N_{k-1}}}{(q; q)_{N_1 - N_2} (q; q)_{N_2 - N_3} \cdots (q; q)_{N_{k-1}}} \\ &= \frac{(q^i, q^{2k+1-i}, q^{2k+1}; q^{2k+1})_\infty}{(q; q)_\infty} = \sum_{n \geq 0} A_{k,i}(n)q^n. \end{aligned}$$

In particular, Andrews [22] obtained the next formula for the number $B_{k,i}(n, m)$, where $B_{k,i}(n, m)$ denotes the number of partitions enumerated by $B_{k,i}(n)$ with exactly m parts. A combinatorial proof of the generating function for $B_{k,i}(n, m)$ was given by Kurşungöz [136] by introducing the notion of the Gordon marking of a partition.

Theorem 1.11 (Andrews). For $k \geq i \geq 1$,

$$\sum_{n, m \geq 0} B_{k,i}(n, m)z^m q^n = \sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_i + \cdots + N_{k-1}} z^{N_1 + \cdots + N_{k-1}}}{(q; q)_{N_1 - N_2} (q; q)_{N_2 - N_3} \cdots (q; q)_{N_{k-1}}}.$$

Over time, the Göllnitz-Gordon identities have acquired considerable attention. This culminated in a major combinatorial generalization by Andrews [13]. A general theorem that reduces to the Göllnitz-Gordon identities in the special cases in which $k = i = 2, k = i + 1 = 2$.

Theorem 1.12 (Andrews-Göllnitz-Gordon). Let i and k be integers with $0 < i \leq k$. Let $C_{k,i}(n)$ denote the number of partitions of n into parts not congruent to 2 modulo 4 and not congruent to $0, \pm(2i - 1)$ modulo $4k$. Let $D_{k,i}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \dots, \lambda_m)$ of n in which no odd part is repeated, $\lambda_j \geq \lambda_{j+1}$, $\lambda_j - \lambda_{j+k-1} \geq 2$ if λ_j odd, $\lambda_j - \lambda_{j+k-1} > 2$ if λ_j even, and at most $i - 1$ parts are ≤ 2 . Then

$$C_{k,i}(n) = D_{k,i}(n).$$

In 1980, Bressoud [45, Eq. (3.8)] found the generating function version of Andrews-Göllnitz-Gordon theorem.

Theorem 1.13 (Bressoud). For $k \geq i \geq 1$,

$$\begin{aligned} \sum_{n \geq 0} D_{k,i}(n)q^n &= \sum_{N_1 \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + N_2^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})}}{(q^2; q^2)_{N_1 - N_2} (q^2; q^2)_{N_2 - N_3} \cdots (q^2; q^2)_{N_{k-1}}} \\ &= \frac{(q^2; q^4)_\infty (q^{2i-1}, q^{4k-2i+1}, q^{4k}; q^{4k})_\infty}{(q; q)_\infty} = \sum_{n \geq 0} C_{k,i}(n)q^n. \end{aligned}$$

More precisely, Bressoud [45] presented the generating function of $D_{k,i}(n, m)$, the number of partitions enumerated by $D_{k,i}(n)$ with exactly m parts.

Theorem 1.14 (Bressoud). For $k \geq i \geq 1$,

$$\sum_{m, n \geq 0} D_{k,i}(n, m)z^m q^n = \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1})} z^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1 - N_2} (q^2; q^2)_{N_2 - N_3} \cdots (q^2; q^2)_{N_{k-1}}}.$$

By considering partitions where parts of fixed parity are distinct, we derive two renowned restricted partition functions: the $\text{pod}(n)$ function, which enumerates the number of partitions of n wherein the odd parts are distinct and the even parts are unrestricted, and the $\text{ped}(n)$ function, which counts the number of partitions of n into distinct even parts with unrestricted odd parts. The generating functions for these partition functions are given by

$$\sum_{n \geq 0} \text{pod}(n)q^n = \prod_{n \geq 1} \frac{(1 + q^{2n-1})}{(1 - q^{2n})}$$

and

$$\sum_{n \geq 0} \text{ped}(n)q^n = \prod_{n \geq 1} \frac{(1 + q^{2n})}{(1 - q^{2n-1})}.$$

The $\text{pod}(n)$ is widely studied in the literature, particularly in the works of Andrews [13, 21], as well as Berkovich and Garvan [40]. It has various combinatorial interpretations [197, A006950], including connections to Schröder partitions and lattice paths. From an algebraic perspective, it is noteworthy that this function equals the number of nilpotent conjugacy classes in the Lie algebras of skew-symmetric $n \times n$ matrices. This fascinating connection underscores the significance of $\text{pod}(n)$ in representation theory and invites further exploration of its broader implications. Additionally, $\text{pod}(n)$ has been discussed in several works spanning different fields, including those of Sills [192], Ferrari [89], Drake [80], and Zaletel and Mong [208].

Notation 1.4. Each set of partitions will be denoted by $\mathbb{C}(n)$ and corresponds to an enumeration function $C(n)$, where C can represent any letter or symbol.

Notation 1.5. Note that if $a(n)$ denotes the number of partitions with certain properties, then we define $a(n) := 0$ for any n that is a negative integer.

1.5 Overpartitions

The discoveries in partitions theory are driven by a significant resurgence in the study of basic hypergeometric series, leading to combinatorial interpretations of many q -series as statements of partitions and overpartitions, a generalization of partitions. The overpartitions have proven useful in several combinatorial studies of basic hypergeometric series.

An overpartition of n is a nonincreasing sequence of positive integers whose sum is n in which the first occurrence of each part may be overlined. The number of overpartition of n , denoted by $\bar{p}(n)$, satisfies the generating function:

$$\begin{aligned} \sum_{n \geq 0} \bar{p}(n)q^n &= \prod_{i \geq 1} \frac{(1 + q^i)}{(1 - q^i)} \\ &= 1 + 2q + 24q^2 + 8q^3 + 14q^4 + \dots \end{aligned}$$

For example, the 8 overpartitions of 3 are

$$(3), (\bar{3}), (2, 1), (\bar{2}, 1), (2, \bar{1}), (\bar{2}, \bar{1}), (1^3), (\bar{1}, 1^2).$$

Although not originally called "overpartitions", these structures were used by Andrews [15] in 1967 to give combinatorial interpretations of the q -binomial theorem, Heine's transformation and Lebesgue's identity. Then in 1987, they were used by Joichi and Stanton [129] in an algorithmic theory of bijective proofs of q -series identities. The term overpartitions was coined first by Corteel [66] and then studied explicitly by Corteel and Lovejoy [67] who revealed their generality by giving combinatorial interpretations for several q -series identities. Before that, they appeared in many different names, like dotted partitions [49] and superpartitions [81]. Previous to these works, Hardy and Ramanujan, who anticipated the importance of the generating function for overpartitions in their celebrated paper that gave birth to the circle method [107], noticed that the generating function for overpartitions is the reciprocal of the classical theta function:

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{i \geq 1} \frac{(1 - q^i)}{(1 + q^i)}.$$

While overpartitions appeared occasionally in the 20th century literature, it was perhaps not until the early years of the 21st century when a series of papers revealed the significance and potentials of overpartitions. These papers proved that overpartitions can be used in bijective proofs of Ramanujan's ${}_1\psi_1$ summation and the q -Gauss summation [66, 68]. In the two subsequent decades, further q -series identities found combinatorial proofs thanks to overpartitions [69, 93, 206, 210]. Moreover, Andrews' theory of q -difference equations, which encompasses the Rogers-Ramanujan-Gordon identities, was also applied to overpartitions [35, 72, 148, 151]. Beyond that, a theory of plane overpartitions was developed [52, 73] and connections were made between overpartitions and Maass forms [53, 54, 55], and much, much more, including many discoveries that have no counterpart for ordinary partitions.

Classical partition identities also found to have overpartition analogues such as those of Rogers-Ramanujan, Gordon, Capparelli and Schur's theorem. For instance, Chen et al. [71], Corteel and Mallet [74], Corteel et al. [75], Lovejoy [145, 147, 149, 150, 152], Dousse [82, 83, 84],

Dousse and Lovejoy [85] and Raghavendra and Padmavathamma [184]. Notably, Lovejoy [147] generalized Schur's theorem with the following result.

Theorem 1.15 (Lovejoy). *Let $A(n, k)$ denote the number of overpartitions of n into parts congruent to 1 or 2 modulo 3 with k nonoverlined parts. Let $B(n, k)$ denote the number of overpartitions $\lambda_1 + \lambda_2 + \cdots + \lambda_s$ of n , having k non-overlined parts and satisfying the difference conditions*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 0 + 3\chi(\overline{\lambda_{i+1}}) & \text{if } \lambda_{i+1} \equiv 1, 2 \pmod{3}, \\ 1 + 3\chi(\overline{\lambda_{i+1}}) & \text{if } \lambda_{i+1} \equiv 0 \pmod{3}, \end{cases}$$

where $\chi(\overline{\lambda_{i+1}}) = 1$ if λ_{i+1} is overlined and 0 otherwise. Then for all $k, n \geq 0$,

$$A(k, n) = B(k, n).$$

Among these analogues, several are related to the Andrews-Göllnitz-Gordon theorem. For example, Lovejoy [146] provided an overpartition analogue of this theorem for $i = k$, and He et al. [116] established an overpartition analogue for the Andrews-Göllnitz-Gordon theorem for $k \geq i \geq 1$ by introducing the Göllnitz-Gordon marking of an overpartition.

To demonstrate Lovejoy's analogue theorem [146], define an odd number $2n - 1$ to be *unattached* in the overpartition λ , if it occurs as a part, but $2n, \overline{2n}$ and $\overline{2n - 1}$ do not. Additionally, let the valuation $v_\lambda(2n)$ of an even natural number relative to an overpartition λ be the number of occurrences of $2n, \overline{2n}$ and $\overline{2n - 1}$, unless $2n - 1$ occurs unattached, in which case $v_\lambda(2n) = 1$.

Theorem 1.16 (Lovejoy). *Let $\overline{E}_k(n)$ denote the number of overpartitions of n into parts not divisible by $2k - 1$. Let $\overline{F}_k(n)$ denote the number of overpartitions λ of n such that $v_\lambda(2a) \geq k - 1$ for all a and such that*

$$v_\lambda(2a) + v_\lambda(2a + 2) \geq \begin{cases} k + 1 & \text{if } \overline{2a} \text{ and } \overline{2a - 1} \text{ both occur,} \\ k & \text{if } \overline{2a} \text{ or } \overline{2a - 1} \text{ occurs (but not both),} \\ k & \text{if } 2a - 1 \text{ occurs unattached,} \\ k - 1 & \text{otherwise.} \end{cases}$$

Then $\overline{E}_k(n) = \overline{F}_k(n)$.

For an overpartition λ , let f_{λ_i} (resp. $f_{\overline{\lambda_i}}$) denote the number of appearances of the part $\lambda_i \in \lambda$ (resp. $\overline{\lambda_i} \in \lambda$). By the definition of an overpartition, it is clear that $f_{\overline{\lambda_i}} = 1$ or 0.

Theorem 1.17 (He et al.). *For $k \geq i \geq 1$, let $O_{k,i}(n)$ denote the number of overpartitions of n of the form $(\overline{1}^{f_1}, 1^{f_1}, \overline{2}^{f_2}, 2^{f_2}, \dots)$ such that*

- $f_{\overline{1}} + f_2 \leq i - 1$.
- $f_{\overline{2\lambda_i}} + f_{2\lambda_i} + f_{\overline{2\lambda_i+1}} + f_{2\lambda_i+2} \leq k - 1$.
- If $f_{2\lambda_i+1} \geq 1$, then $f_{2\lambda_i+2} \leq k - 2$.

For $k > i \geq 1$, let $P_{k,i}(n)$ denote the number of overpartitions of n with non-overlined parts not congruent to $0, \pm(2i-1) \pmod{4k-2}$ and let $P_{k,k}(n)$ denote the number of overpartitions of n into parts not divisible by $2k-1$. Then for $k \geq i \geq 1$ and $n \geq 0$,

$$O_{k,i}(n) = P_{k,i}(n).$$

Specifically, He et al. [116] obtained the generating function version of Theorem 1.17 by using Bailey's lemma and a change in the base due to Bressoud et al. [47].

Theorem 1.18 (He et al.). For $k \geq i \geq 1$,

$$\begin{aligned} \sum_{n \geq 0} O_{k,i}(n)q^n &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} (-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} \\ &\quad \times \frac{q^{2(N_1^2 + N_2^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1})} (1 + q^{2N_i})}{(q^2; q^2)_{N_1 - N_2} \cdots (q^2; q^2)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}} \\ &= \frac{(-q; q)_\infty (q^{2i-1}, q^{4k-1-2i}, q^{4k-2}, q^{4k-2})_\infty}{(q; q)_\infty} = \sum_{n \geq 0} P_{k,i}(n)q^n. \end{aligned}$$

Additionally, He et al. [116] gave a combinatorial interpretation for $O_{k,i}(n, m)$, the number of overpartitions counted by $O_{k,i}(n)$ into exactly m parts, by introducing the Göllnitz-Gordon marking of an overpartition.

Theorem 1.19 (He et al.). For $k \geq i \geq 1$,

$$\begin{aligned} \sum_{n \geq 0} O_{k,i}(n, m)z^m q^n &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} (-q^{2-2N_1}; q^2)_{N_1-1} (-q^{1-2N_1}; q^2)_{N_1} \\ &\quad \times \frac{q^{2(N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1})} (1 + q^{2N_i})z^{N_1 + \dots + N_{k-1}}}{(q^2; q^2)_{N_1 - N_2} \cdots (q^2; q^2)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}}. \end{aligned}$$

Overpartitions are also related to the fields of Lie algebras [130], mathematical physics [81, 94, 95] and supersymmetric functions [81].

1.6 Congruences

Congruences, the study of divisibility properties of the partition functions, were first explored by Ramanujan in 1919. His investigation was inspired by a table of partition values $p(n)$ for $n \leq 200$, arranged in five columns by MacMahon. Ramanujan observed that the numbers in the last column were consistently divisible by 5. In his groundbreaking 1919 paper [179], he proved the following remarkable congruences for all $n \geq 0$:

$$\begin{aligned} p(5n + 4) &\equiv 0 \pmod{5}, \\ p(7n + 5) &\equiv 0 \pmod{7}. \end{aligned}$$

In a subsequent 1920 note [180], he also announced a proof of the congruence

$$p(11n + 6) \equiv 0 \pmod{11}.$$

Ramanujan's proofs of these congruences were primarily analytical, relying on q -series identities, and they offered little in terms of combinatorial interpretation.

For the next two decades, the combinatorial side of Ramanujan's congruences posed a significant challenge. However, in 1944, Dyson [78] made a breakthrough by conjecturing the first combinatorial interpretation of the congruences for $p(5n + 4)$ and $p(7n + 5)$ by introducing the concept of the rank of a partition.

Definition 1.10. *The rank of a partition λ equals its largest part minus its total number of parts.*

Dyson conjectured that the partitions of $5n + 4$ (resp. $7n + 5$) could be evenly divided into 5 (resp. 7) classes based on their rank modulo 5 (resp. 7). This conjecture was eventually proved ten years later by Atkin and Swinnerton-Dyer [32]. However, the rank failed to provide a similar interpretation for the modulo 11 congruence, prompting Dyson to hypothesize the existence of another statistic, which he termed the crank. The concept of the crank was later formalized by Andrews and Garvan [28] in 1988.

Definition 1.11. *For a partition λ the largest part is λ_1 , $\omega(\lambda)$ denotes the number of 1's in λ , and $\mu(\lambda)$ is the number of parts of λ larger than $\omega(\lambda)$. Then, the crank $c(\lambda)$ is given by*

$$c(\lambda) := \begin{cases} \lambda_1 & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0. \end{cases}$$

For example, the crank of the partition $\lambda = (4, 1, 1)$ is $c(\lambda) = -1$ since $\omega(\lambda) = 2$ and $\mu(\lambda) = 1$. The generating function for the number of partitions of n with crank m , denoted by $M(n, m)$, is given by

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(n, m) c^m q^n = \prod_{n=1}^{\infty} \frac{(1 - q^n)}{(1 - cq^n)(1 - c^{-1}q^n)}.$$

The ℓ -regular partitions function $b_\ell(n)$ has been studied by many authors from an arithmetic perspective. For example, Cui and Gu [64] gave some infinite families of congruences modulo 2 for some ℓ -regular partition functions, for $\ell = 2, 4, 5, 8, 13, 16$. Hirschhorn and Sellers [118] proved that $b_5(n)$ is odd if and only if $12n + 1$ is a perfect square. For further results on this topic see for example [133, 200, 203] and references therein.

The $\text{pod}(n)$ function has been studied from an arithmetic point of view by many mathematicians. It was first studied arithmetically by Hirschhorn and Sellers [119]. In their paper, they derived an infinite family of Ramanujan-type congruences for any integers $n, \alpha \geq 0$, such that

$$D_4(3^{2\alpha+3}n + (23 \times 2^{2\alpha+2} + 1)/8) \equiv 0 \pmod{3}.$$

Later on, Radu and Sellers [177] established several new congruences, such as

$$\begin{aligned}\text{pod}(135n + 8) &\equiv 0 \pmod{5}, \\ \text{pod}(135n + 107) &\equiv 0 \pmod{5}, \\ \text{pod}(567n + 260) &\equiv 0 \pmod{7}, \\ \text{pod}(675n + 647) &\equiv 0 \pmod{25}, \\ \text{pod}(3375n + 1997) &\equiv 0 \pmod{125}.\end{aligned}$$

In 2015, Wang [199], relying heavily on results from Lovejoy and Osburn [144], presented some new additional congruences for $D_4(n)$ modulo 3, 5, and 9. If p is a prime with $p \equiv 1 \pmod{3}$, and $n, \alpha \geq 0$ such that $pn \equiv 5 \pmod{8}$, then

$$\begin{aligned}\text{pod}((3p^{6\alpha+5}n + 1)/8) &\equiv 0 \pmod{3}, \\ \text{pod}(5^{2\alpha+2}n + (11 \cdot 5^{2\alpha+1} + 1)/8) &\equiv 0 \pmod{5}, \\ \text{pod}((3p^{18\alpha+17}n + 1)/8) &\equiv 0 \pmod{9}.\end{aligned}$$

The arithmetic properties of $\bar{p}(n)$ have been extensively studied in the literature. For example, Hirschhorn and Sellers [121] proved that for all $n \geq 0$,

$$\begin{aligned}\bar{p}(27n + 18) &\equiv 0 \pmod{3}, \\ \bar{p}(27n) &\equiv \bar{p}(3n) \pmod{3}.\end{aligned}$$

For further results in this direction, we refer the reader to [139, 140, 202, 207] and the references therein.

Lovejoy [145] studied the ℓ -regular overpartitions function $\bar{A}_\ell(n)$, which counts the number of overpartitions of n with no parts divisible by ℓ . Several congruences are known for the sequences $\bar{A}_3(n)$ and $\bar{A}_4(n)$. For example, Shen [186] proved the following congruences:

$$\begin{aligned}\bar{A}_3(4n + 1) &\equiv 0 \pmod{2}, & \bar{A}_3(4n + 3) &\equiv 0 \pmod{6}, \\ \bar{A}_3(9n + 3) &\equiv 0 \pmod{6}, & \bar{A}_3(9n + 6) &\equiv 0 \pmod{24}.\end{aligned}$$

Andrews [11] defined the function $\bar{C}_{k,i}(n)$ as the number of overpartitions of n in which no part is divisible by k and only parts congruent to $\pm i \pmod{k}$ may be overlined. The generating function of $\bar{C}_{k,i}(n)$ is given by

$$\sum_{n \geq 0} \bar{C}_{k,i}(n)q^n = \frac{(q^k; q^k)_\infty (-q^i; q^k)_\infty (-q^{k-i}; q^k)_\infty}{(q; q)_\infty},$$

for $k \geq 3$ and $1 \leq i \leq \lfloor k/2 \rfloor$. Andrews proved the following two congruences:

$$\begin{aligned}\bar{C}_{3,1}(9n + 3) &\equiv 0 \pmod{3}, \\ \bar{C}_{3,1}(9n + 6) &\equiv 0 \pmod{3}.\end{aligned}$$

Notice that $\bar{A}_3(n) = \bar{C}_{3,1}(n)$. For further results on this topic see for example [3, 63].

1.7 Asymptotics

Since the early development of the theory of integer partitions, mathematicians have been deeply invested in discovering an exact formula for the partition function $p(n)$. In 1918, after an analytical study of $p(n)$, Hardy and Ramanujan [107] established the following asymptotic formula as n tends to infinity:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Hardy, Ramanujan and MacMahon were amazed by how accurately this approximation formula matched the exact value of $p(200)$, which MacMahon had computed. Their insight led them to derive the following more precise formula:

$$p(n) = \frac{1}{2\sqrt{2}} \sum_{k=1}^{a\sqrt{n}} \sqrt{k} \sum_{\substack{0 \leq h \leq k \\ (h,k)=1}} \omega_{h,k} e^{\frac{-2\pi i n h}{k}} \frac{d}{dn} \left(\exp\left(\frac{\pi\sqrt{\frac{2}{3}}\left(n - \frac{1}{24}\right)}{k}\right) \right),$$

where $\omega_{h,k}$ denotes a 24th root of unity, (h, k) is the greatest common divisor of h and k , and a is an arbitrary constant, subject to the condition that n be larger than a certain threshold $n_0(a)$ which depends on a .

A few years later, in 1937, Rademacher [175] refined the method of Hardy and Ramanujan, providing an expression for $p(n)$ as a convergent series:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{\frac{1}{2}} \left[\frac{d}{dx} \frac{\sin\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{\frac{1}{2}}\right)}{\left(x - \frac{1}{24}\right)^{\frac{1}{2}}} \right]_{x=n},$$

where

$$A_k(n) = \sum_{\substack{0 \leq h \leq k \\ (h,k)=1}} \omega_{h,k} e^{\frac{-2\pi i n h}{k}}.$$

1.8 Multipartitions

According to Fayers [88], a *multipartition* of n is an r -tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$ of partitions such that $|\lambda^{(1)}| + |\lambda^{(2)}| + \dots + |\lambda^{(r)}| = n$. As with partitions, we write the unique multipartition of \emptyset as 0. The generating function for $P_k(n)$, the number of multipartitions of n , is given by

$$\sum_{n \geq 0} P_k(n) q^n = \prod \frac{1}{(1 - q^n)^k}.$$

In the literature, we can find various scattered results about multipartitions such as in the representation theory of Lie algebra [88, 56] and physics [57], where multipartitions playing an important auxiliary role.

The arithmetic properties of $P_k(n)$ have been studied by Atkin [33], Cheema et al. [61], Gupta et al [104, 105], and many others. Atkin [33] presented an extensive study on the arithmetic behaviour of $P_k(n)$, building on his success in proving Ramanujan's "11ⁿ" conjecture. For more on multipartitions, we refer the reader to [27].

1.9 Compositions

A *composition* of a positive integer n is a sequence of positive integers $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_\ell)$ such that $\sigma_1 + \sigma_2 + \dots + \sigma_\ell = n$. The integers σ_i are called *parts* of the composition, and n is referred to as the *weight* of σ and is denoted by $|\sigma|$. The number of compositions of n into m parts, denoted by $c_m(n)$, satisfies the generating function:

$$\sum_{n \geq 0} c_m(n)q^n = \frac{q^m}{(1-q)^m}.$$

For example, the 8 compositions of 4 are

$$(4), (3, 1), (1, 3), (2, 2), (2, 1, 1), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1).$$

It is well-known that the number of compositions of n is 2^{n-1} for all $n \geq 1$. The enumeration of compositions with respect to some parameters, such as the number of parts, the last part, the number of peaks, the number of fixed points, etc., is a classical problem in enumerative combinatorics. For more information on the combinatorics of compositions, we refer the reader to the book of Heubach and Mansour [159].

1.10 Combinatorial proofs

In our combinatorial proofs given in this thesis, we establish a bijective function $f : \mathbb{A}(n) \rightarrow \mathbb{B}(n)$ between two sets of partitions $\mathbb{A}(n)$ and $\mathbb{B}(n)$, creating a one-to-one correspondence where $\mathbb{A}(n)$ and $\mathbb{B}(n)$ are finite sets of partitions of fixed size. Each part λ_i of a partition $\lambda \in \mathbb{A}(n)$ is mapped to either a single part β_i or a sequence of parts $\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_m}$ in a partition $\beta \in \mathbb{B}(n)$. Thus, the bijective function f and its inverse f^{-1} operate on the individual parts of λ and β , respectively. In other words, we map a multiset of parts to another multiset of parts, all summing to n , such that for $\lambda = (\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_k^{\alpha_k})$ and $\beta = (\beta_1^{w_1}, \beta_2^{w_2}, \dots, \beta_k^{w_k})$, we have

$$\begin{aligned} \beta &= f(\lambda) = f((\lambda_1^{\alpha_1}, \dots, \lambda_k^{\alpha_k})) := (f(\lambda_1^{\alpha_1}), \dots, f(\lambda_k^{\alpha_k})) = (\cup_{i=1}^k f(\lambda_i^{\alpha_i})), \\ \lambda &= f^{-1}(\beta) = f^{-1}((\beta_1^{w_1}, \dots, \beta_k^{w_k})) := (f^{-1}(\beta_1^{w_1}), \dots, f^{-1}(\beta_k^{w_k})) = (\cup_{i=1}^k f^{-1}(\beta_i^{w_i})). \end{aligned}$$

For example, the set of partitions of 6 into distinct parts is given by

$$\mathbb{A}(6) = \{(6), (5, 1), (4, 2), (3, 2, 1)\}.$$

Meanwhile, the set of partitions of 6 into odd parts is given by

$$\mathbb{B}(6) = \{(5, 1), (3^2), (3, 1^3), (1^6)\}.$$

Therefore, the bijection between the sets $\mathbb{A}(6)$ and $\mathbb{B}(6)$ is given by

$$\begin{aligned} (3^2) &= f((6)) = (f(6)), \\ (5, 1) &= f((5, 1)) = f(f(5), f(1)), \\ (3, 1^3) &= f((3, 2, 1)) = (f(3), f(2), f(1)), \\ (1^6) &= f((4, 2)) = (f(4), f(2)). \end{aligned}$$

Here and throughout, we omit the parentheses around the parts forming the image of $f(\lambda_i^{\alpha_i})$ (resp. $f^{-1}(\beta_i^{w_i})$). The image of $f(\lambda_i^{\alpha_i})$ (resp. $f^{-1}(\beta_i^{w_i})$) could result a various types of partitions. For instance, $f(\lambda_i^{\alpha_i}) = \alpha_i^{\lambda_i}$, meaning the value of the part becomes the multiplicity, and the value of the multiplicity becomes the part. Similarly, $f(\lambda_i^{\alpha_i}) = \alpha_i \lambda_i$ denotes that the resulting image is the product of the part by its multiplicity. Other examples include $f(\lambda_i^{\alpha_i}) = (\alpha_i - 1)^{\lambda_i}, \lambda_i$ or $f(\lambda_i^{\alpha_i}) = (s\lambda_i/2)^{2(\alpha_i-1)/s}, \lambda_i$, where the obtained image is a sequence of parts. In the case when $(\alpha_i - 1) = 0$ (i.e. $\alpha_i = 1$), we exclude that part from the sequence, such that $f(\lambda_i^{\alpha_i}) = \lambda_i$.

1.11 Smoot's implementation of Radu's Ramanujan-Kolberg algorithm

The Radu-Kolberg algorithm [176, 178] is a refined method for proving congruences in partition functions, building on earlier work by Kolberg [137, 138] and significantly improved by Radu through the use of modular forms and generating functions. This approach systematically decomposes partition functions into modular expressions, allowing for efficient verification of congruences that arise. The method is particularly useful in detecting congruence patterns in restricted partition classes, such as those appearing in the Andrews-Göllnitz-Gordon theorem. Nicholas Smoot's implementation [193] of the Radu-Kolberg algorithm enhances its computational efficiency, enabling the verification of more complex partition congruences that would otherwise be intractable. By automating key steps, Smoot's work has expanded the applicability of this algorithm, making it a crucial tool in modern partition theory research.

CHAPTER 2

GENERATING FUNCTION DISSECTIONS

2.1 m -Dissection

Given a series

$$A(q) = \sum_{n \geq 0} a(n)q^n,$$

and $m > 1$ an integer, the m -dissection of $A(q)$ is the expression of $A(q)$ as a sum of m sums,

$$A(q) = A_0(q) + A_1(q) + A_2(q) + \cdots + A_{m-1}(q),$$

where for each j with $0 \leq j \leq m - 1$,

$$A_j(q) = q^j \sum_{n \geq 0} a(mn + j)q^{mn}.$$

For example, we have the following 2-dissection of $(q^2, q^2)_\infty^5 / (q, q)_\infty^2 (q^4, q^4)_\infty^2 = \sum_{n=-\infty}^{\infty} q^{n^2}$, such that

$$\begin{aligned} \frac{(q^2, q^2)_\infty^5}{(q, q)_\infty^2 (q^4, q^4)_\infty^2} &= \sum_{n=-\infty}^{\infty} q^{n^2} \\ &= 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + 2q^{36} + 2q^{49} + \cdots \\ &= (1 + 2q^4 + 2q^{16} + \cdots) + 2q(1 + q^8 + q^{24} + q^{48} + \cdots) \\ &= \frac{(q^8, q^8)_\infty^5}{(q^4, q^4)_\infty^2 (q^{16}, q^{16})_\infty^2} + 2q \frac{(q^{16}, q^{16})_\infty^2}{(q^8, q^8)_\infty}. \end{aligned}$$

2.2 Dissection formulas

In this section, we list few dissection formulas which are useful in proving our main results. Ramanujan's general theta-function $f(a, b)$ [41, p. 34, 18.1] is defined as

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

The product representation of $f(a, b)$ arises from famous Jacobi's triple product identity [41, p. 35, Entry 19]

$$f(a, b) = (-a, ab)_{\infty} (-b, ab)_{\infty} (ab, ab)_{\infty}.$$

Moreover, f_k is defined by

$$f_k := (q^k, q^k)_{\infty} = \prod_{n \geq 1} (1 - q^{nk}).$$

Some special cases of theta-function are denoted by

$$\phi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q, q^2)_{\infty} (q^2, q^2)_{\infty} = \frac{f_2^5}{f_1^2 f_4^2},$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2, q^2)_{\infty}}{(q, q^2)_{\infty}} = \frac{f_2^2}{f_1},$$

$$f(-q) := f(-q^2, -q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q; q)_{\infty} = f_1,$$

$$\psi(-q) = \frac{f_1 f_4}{f_2}, \quad \text{and} \quad \phi(-q) = \frac{f_1^2}{f_2}.$$

Jacobi's identity [42, Thm. 1.3.9] is defined as

$$f_1^3 = \sum_{n \geq 0} (-1)^n (2n+1) q^{n(n+1)/2}. \quad (2.1)$$

Lemma 2.1. For all primes p and all $k, m \geq 1$, we have

$$f_{pm}^{p^{k-1}} \equiv f_m^{p^k} \pmod{p^k}. \quad (2.2)$$

Lemma 2.1 easily follows from the binomial theorem.

Let p be any odd prime and δ be any integer relatively prime to p . The Legendre symbol $\left(\frac{\delta}{p}\right)$ is defined by

$$\left(\frac{\delta}{p}\right) = \begin{cases} 1, & \text{if } \delta \text{ is a quadratic residue of } p, \\ -1, & \text{if } \delta \text{ is a quadratic non-residue of } p. \end{cases}$$

Lemma 2.2. *The following 2-dissections hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \quad (2.3)$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8}, \quad (2.4)$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}, \quad (2.5)$$

$$\frac{1}{f_1^8} = \frac{f_4^{28}}{f_2^{28} f_8^8} + 8q \frac{f_4^{16}}{f_2^{24}} + 16q^2 \frac{f_4^4 f_8^8}{f_2^{20}}. \quad (2.6)$$

Equations (2.3), (2.4) and (2.5) are a consequence of dissection formulas of Ramanujan, collected in Berndt's book [41, p. 40, Entry 25]. Equation (2.6) follows by squaring both sides of (2.4).

Lemma 2.3. *The following 3-dissections hold*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}, \quad (2.7)$$

$$\begin{aligned} \frac{1}{f_1^3} &= \frac{f_9^3}{f_3^{12}} (f_1^6 + 9q f_1^3 f_9^3 + 27q^2 f_9^6) \\ &= \frac{f_6^2 f_9^{15}}{f_3^{14} f_{18}^6} + 3q \frac{f_6 f_9^{12}}{f_3^{13} f_{18}^3} + 9q^2 \frac{f_9^9}{f_3^{12}} + 8q^3 \frac{f_9^6 f_{18}^3}{f_3^{11} f_6} + 12q^4 \frac{f_9^3 f_{18}^6}{f_3^{10} f_6^2} + 16q^6 \frac{f_{18}^{12}}{f_3^8 f_6^4 f_9^3}. \end{aligned} \quad (2.8)$$

To prove (2.7), expand equation (14.8.5) via the definition after (14.3.1) in [117]. One may obtain (2.8) by replacing q with ωq and $\omega^2 q$ in (2.7) and multiplying the two results.

Lemma 2.4. *We have the following 3-dissections*

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}, \quad (2.9)$$

$$\frac{1}{f_1 f_2} = \frac{f_9^9}{f_3^6 f_6^2 f_{18}^3} + q \frac{f_9^6}{f_3^5 f_6^3} + 3q^2 \frac{f_9^3 f_{18}^3}{f_3^4 f_6^4} - 2q^3 \frac{f_{18}^6}{f_3^3 f_6^5} + 4q^4 \frac{f_{18}^9}{f_3^2 f_6^6 f_9^3}. \quad (2.10)$$

Lemma 2.4 was proved by Chan [60].

Lemma 2.5. *The following 3-dissections hold*

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q \frac{f_3 f_{18}^2}{f_6 f_9}, \quad (2.11)$$

$$\frac{f_2}{f_1^2} = \frac{1}{\phi(-q^4)^4} (\phi(q^4)^3 + 2q \phi(q^4)^2 \psi(q^8) + 4q^2 \phi(q^4) \psi(q^8)^2 + 8q^3 \psi(q^8)^3), \quad (2.12)$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6}. \quad (2.13)$$

Identity (2.11) is equivalent to the three-dissection of $\phi(-q)$ [117, Eq. 14.3.2]. Equations (2.12) and (2.13) were proved by Hirschhorn and Sellers [122, Eq. (3), Thm. 1].

Lemma 2.6. *The following 3-dissection holds*

$$\psi(-q) = A(q^3) - q\psi(-q^9), \quad (2.14)$$

where

$$A(q^3) = \frac{f_3 f_{12} f_{18}^5}{f_6^2 f_9^2 f_{36}^2}.$$

Replacing q by $-q$ in Corollary (ii) of [41, p. 49], and using $(-q; -q)_\infty = f_2^3 / f_1 f_4$, we obtain (2.14).

Lemma 2.7. *The following 3-dissection of $1/\psi(-q)$ holds*

$$\frac{f_2}{f_1 f_4} = \frac{f_{18}^9}{f_3^2 f_9^3 f_{12}^2 f_{36}^3} + q \frac{f_6^2 f_{18}^3}{f_3^3 f_{12}^3} + q^2 \frac{f_6^4 f_9^3 f_{36}^3}{f_3^4 f_{12}^4 f_{18}^3}. \quad (2.15)$$

Replace q by $-q$ in [120, Lem. 2] and use Jacobi's triple product identity [42, Thm. 1.3.3] to write the various sums as products.

Lemma 2.8. *The following 3-dissection holds*

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9}. \quad (2.16)$$

Equation (2.16) is Corollary (ii) of [41, p. 49]

Lemma 2.9. *The following 3-dissection holds*

$$\frac{f_2^3}{f_1^3} = \frac{f_6}{f_3} + 3q \frac{f_6^4 f_9^5}{f_3^8 f_{18}} + 6q^2 \frac{f_6^3 f_9^2 f_{18}^2}{f_3^7} + 12q^3 \frac{f_6^2 f_{18}^5}{f_3^6 f_9}. \quad (2.17)$$

Identity (2.17) was proved by Toh [198, Eq. (3.1)].

Lemma 2.10. *The following 2-dissections hold*

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q \frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \quad (2.18)$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \quad (2.19)$$

$$\frac{f_3^3}{f_1^3} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}, \quad (2.20)$$

$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q \frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}, \quad (2.21)$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \quad (2.22)$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q \frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \quad (2.23)$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q \frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}. \quad (2.24)$$

Yao and Xia [207, 204] gave a proof of Equations (2.18) and (2.19). Hirschhorn et al. [123] proved Equation (2.20). Equation (2.21) follows by changing q to $-q$ in (2.20), with $(-q, -q)_\infty = f_2^3 / f_1 f_4$. Baruah and Ojah [36, 37] proved (2.22), (2.23) and (2.24).

Lemma 2.11. *The following 3-dissections hold*

$$\frac{f_4}{f_1} = \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q \frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2 \frac{f_6 f_{18} f_{36}}{f_3^3}, \quad (2.25)$$

$$\frac{f_1}{f_4} = \frac{f_6 f_9 f_{18}}{f_{12}^3} - q \frac{f_3 f_{18}^4}{f_9^2 f_{12}^3} - q^2 \frac{f_6^2 f_9 f_{36}^3}{f_{12}^4 f_{18}^2}. \quad (2.26)$$

Hirschhorn [117, (33.2.6)] proved Equation (2.25). Equation (2.26) corresponds to Lemma 1.2.23 in [172].

Lemma 2.12. *The following 2-dissections hold*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}, \quad (2.27)$$

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q \frac{f_4^2 f_{40}}{f_8 f_{10}^2}, \quad (2.28)$$

$$f_1 f_5 = \frac{f_2 f_8 f_{20} f_{40}^4}{f_4 f_{10}^2 f_{80}} - q \frac{f_4^2 f_{40}^6}{f_8 f_{10} f_{20}^2 f_{80}^2} - 2q^5 \frac{f_2 f_8 f_{20}^3 f_{80}^2}{f_4 f_{10}^2 f_{40}^2} + 2q^6 \frac{f_4^2 f_{80}^2}{f_8 f_{10}}. \quad (2.29)$$

Equation (2.27) was proved by Hirschhorn and Sellers [118]. Replacing q by $-q$ in (2.27), we obtain (2.28). By using (2.28) and (2.4), we find that $(f_1 / f_5) f_5^2 = f_1 f_5$.

Lemma 2.13. *We have the following 2-dissections*

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q \frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}, \quad (2.30)$$

$$\frac{f_9^2}{f_1^2} = \frac{f_{12}^6 f_{18}^2}{f_2^4 f_6^2 f_{36}^2} + 2q \frac{f_4^2 f_{12}^2 f_{18}}{f_2^5} + q^2 \frac{f_4^4 f_6^2 f_{36}^2}{f_2^6 f_{12}^2}, \quad (2.31)$$

$$\frac{f_1}{f_9} = \frac{f_2 f_{12}^3}{f_4 f_6 f_{18}^2} - q \frac{f_4 f_6 f_{36}^2}{f_{12} f_{18}^3}. \quad (2.32)$$

Equations (2.30) and (2.32) were proved by Xia and Yao [205]. Equation (2.31) follows by squaring both sides of (2.30).

The following result appears in the papers of Ramanujan [181, p. 212].

Lemma 2.14. *We have the following 5-dissection*

$$f_1 = f_{25}(a - q - q^2/a), \quad (2.33)$$

where

$$a := \frac{(q^{10}, q^{15}; q^{25})_\infty}{(q^{10}, q^5; q^{20})_\infty}.$$

Lemma 2.15. *For any odd prime p , we have*

$$\psi(q) = \sum_{k=0}^{(p-3)/2} q^{k(k+1)/2} f\left(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}}\right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}). \quad (2.34)$$

Furthermore, $\frac{m^2+m}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}$ for $0 \leq m \leq (p-3)/2$.

Lemma 2.16. *For any prime $p \geq 5$, we have*

$$f_1 = \sum_{\substack{k=(1-p)/2 \\ k \neq \frac{\pm p-1}{6}}}^{(p-1)/2} (-1)^k q^{k(3k+1)/2} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}, \quad (2.35)$$

where

$$\frac{\pm p-1}{6} = \begin{cases} \frac{p-1}{6} & \text{if } p \equiv 1 \pmod{6}, \\ \frac{-p-1}{6} & \text{if } p \equiv -1 \pmod{6}. \end{cases}$$

Lemmas 2.15 and 2.16 are due to Cui and Gu [64, Thm. 2.1 & 2.2].

PART II:**MULTI-RESTRICTED SETS OF PARTITIONS**

CHAPTER 3

PARTITIONS INTO PARTS SIMULTANEOUSLY REGULAR AND DISTINCT

3.1 Introduction

A search of the literature suggests that little work has been done on partitions into parts that are simultaneously ℓ -regular and t -distinct with the strong exception of the partitions into distinct odd parts (2-regular and 2-distinct). According to the work of Rødseth [185], we know that

$$\text{RD}^{(2,2)}(125n + 99) \equiv 0 \pmod{5}$$

for all $n \geq 0$, where $\text{RD}^{(\ell,t)}(n)$ denotes the number of partitions of n into parts simultaneously ℓ -regular and t -distinct. Keith [134] presented a survey on partitions into parts simultaneously regular, distinct, and/or flat, with many valuable observations on both arithmetic and combinatorial properties. One such observation is the symmetry of the generating function for $\text{RD}^{(\ell,t)}(n)$, which is given by

$$\sum_{n \geq 0} \text{RD}^{(\ell,t)}(n)q^n = \prod_{k \geq 1} \frac{(1 - q^{tk})(1 - q^{\ell k})}{(1 - q^k)(1 - q^{\ell tk})} = \frac{f_t f_\ell}{f_1 f_{\ell t}}, \quad (3.1)$$

where it can be observed that the number of partitions of n into parts simultaneously ℓ -regular and t -distinct equals the number of partitions of n into parts simultaneously t -regular and ℓ -distinct. Additionally, the bijection between these symmetric sets has been founded by Keith [135] and O'Hara [173].

Another interesting observation concerns the connection between the number $\text{RD}^{(\ell,t)}(n)$ and the McKay-Thompson series for the Monster group. For some background, Conway and Norton [70] were the first to discover the connection between the Monster group M and the j -function, where the j -invariant is given by

$$j(\tau) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \dots,$$

and before that, McKay found that the coefficients of $j(\tau)$ are sums of the dimensions of irreducible representations of the Monster group M . The first few coefficients can be expressed

as

$$\begin{aligned} 1 &= 1, \\ 196884 &= 196883 + 1, \\ 21493760 &= 21296876 + 196883 + 1, \\ 864299970 &= 842609326 + 21296876 + 2 \cdot 196883 + 2 \cdot 1. \end{aligned}$$

Later, Borchers [58] proved this relationship, which is known as Monstrous Moonshine. The McKay-Thompson series are often of the form

$$\frac{1}{q} \prod_{k \geq 1} \frac{(1 - q^{mtk})(1 - q^{m\ell k})}{(1 - q^{mk})(1 - q^{m\ell tk})},$$

which generates coefficients that match those of $\sum_{n \geq 0} \text{RD}^{(\ell, t)}(n)q^n$ when shifted by a power of q with q replaced by q^m . Thus, for certain pairs (ℓ, t) , $\text{RD}^{(\ell, t)}(n)$ may correspond to a McKay-Thompson series. A search in the Online Encyclopedia of Integer Sequences [197] reveals several matches. For example, the OEIS sequence [A112194](#), which corresponds to the McKay-Thompson series of class 54c, also matches the coefficients $\text{RD}^{(3,3)}(n)$ for $n \geq 0$.

The OEIS sequence [A003105](#) indicates that the numbers $A(n)$ of Schur's partition **Theorem 1.7** and $\text{RD}^{(2,3)}(n)$ are equal. In this context, we will demonstrate that the left-hand side of Schur's partitions is a specific instance of the class t -Schur's partitions, which is a broader class equivalent to partitions that are simultaneously 2-regular and t -distinct. Additionally, we provide a combinatorial perspective on these partition families by establishing bijections between them and demonstrating their adherence to several recurrence relations. In the third section, we establish some Ramanujan-like congruences and infinite families of congruences for $\text{RD}^{(\ell, t)}(n)$, and in the fourth section, we introduce the t -Schur's overpartitions.

3.2 Partitions into parts simultaneously 2-regular and t -distinct

In this section, we shall explore some combinatorial properties of partitions into parts simultaneously 2-regular and t -distinct (i.e. partitions into odd parts appearing at most t times), as well as other equivalent classes of partitions. By setting $\ell = 2$ in (3.1) and for $t \geq 2$, we obtain $\text{RD}^{(2, t)}(n)$, the number of partitions of n into parts simultaneously 2-regular and t -distinct. For example, $\text{RD}^{(2,3)}(12) = 6$, where the corresponding set of partitions is

$$\mathbb{RD}^{(2,3)}(12) = \{(11, 1), (9, 3), (7, 5), (7, 3, 1^2), (5^2, 1^2), (5, 3^2, 1)\}.$$

For $t \geq 2$, let $C_d^t(n)$ denote the number of partitions of n into distinct parts not congruent to 0 modulo t . For example, $C_d^5(10) = 7$, where

$$\mathbb{C}_d^5(10) = \{(9, 1), (7, 3), (7, 2, 1), (8, 2), (4, 3, 2, 1), (6, 3, 1), (6, 4)\}.$$

For any odd $t \geq 3$, we define $S^t(n)$ as the number of partitions of n into parts congruent to i modulo $2t$, where $i \in I(t)$ and $I(t) = \{1, 3, 5, 7, \dots, (2t-1)\} \setminus \{t\}$. We refer to this class of partitions as t -Schur's partitions. For $t = 3$ we obtain the classical Schur's partitions (or 3-Schur's partitions), where $I(3) = \{1, 5\}$. For example, $S^3(13) = 7$, where the corresponding set of partitions is

$$S^3(13) = \{(13), (11, 1^2), (7, 5, 1), (7, 1^6), (5^2, 1^3), (5, 1^8), (1^{13})\}.$$

In the following theorem, we establish the equivalence between these sets of partitions, explicitly demonstrating their correspondence based on the parity of t , whether it is even or odd.

Theorem 3.1. *For all $n \geq 0$ and $t \geq 2$,*

$$\begin{aligned} \text{RD}^{(2,t)}(n) &= C_d^t(n) = S^t(n) \text{ if } t \text{ is odd, and} \\ \text{RD}^{(2,t)}(n) &= C_d^t(n) \text{ if } t \text{ is even.} \end{aligned}$$

Proof. For $t \geq 2$, we have

$$\begin{aligned} \sum_{n \geq 0} \text{RD}^{(2,t)}(n) q^n &= \prod_{k \geq 1} \frac{(1 - q^{tk})(1 - q^{2k})}{(1 - q^k)(1 - q^{2tk})} \\ &= \prod_{k \geq 1} \frac{(1 + q^k)}{(1 + q^{tk})} \\ &= \sum_{n \geq 0} C_d^t(n) q^n, \end{aligned}$$

and in the case where t is odd, we obtain

$$\begin{aligned} \sum_{n \geq 0} C_d^t(n) q^n &= \prod_{k \geq 1} \frac{(1 + q^k)}{(1 + q^{tk})} \\ &= \prod_{k \geq 1} \frac{(1 - q^{t(2k-1)})}{(1 - q^{2k-1})} \\ &= \sum_{n \geq 0} S^t(n) q^n. \end{aligned}$$

We will now provide a combinatorial proof for the three parts of the theorem.

The bijection $\text{RID}^{(2,t)}(n) \leftrightarrow S^t(n)$:

Let $\lambda = (\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_k^{\alpha_k}) \in \text{RID}^{(2,t)}(n)$. Define the map $f : \text{RID}^{(2,t)}(n) \rightarrow S^t(n)$ by $\lambda \rightarrow f(\lambda) = \cup_{i=1}^k f(\lambda_i^{\alpha_i})$. Then for $1 \leq i \leq k$, we have

$$f(\lambda_i^{\alpha_i}) = \begin{cases} \lambda_i^{\alpha_i} & \text{if } \lambda_i \not\equiv 0 \pmod{t}, \\ 1^{tr\alpha_i} & \text{if } \lambda_i \equiv 0 \pmod{t}, \end{cases}$$

where each part $\lambda_i^{\alpha_i}$ can be expressed uniquely as $\lambda_i^{\alpha_i} = (tr)^{\alpha_i}$, with $r \geq 1$ being odd.

The inverse map $f^{-1} : \mathcal{S}^t(n) \rightarrow \mathbb{RID}^{(2,t)}(n)$ is defined by

$$f^{-1}(\lambda_i^{\alpha_i}) = \begin{cases} \lambda_i^{\alpha_i} & \text{if } \lambda_i > 1, \\ \lambda_i^{\alpha_i} & \text{if } \alpha_i < t, \\ (tp_1)^{a_1}, (tp_2)^{a_2}, \dots, (tp_m)^{a_m}, 1^\delta & \text{if } \alpha_i \geq t, \end{cases}$$

such that, for $\alpha_i \geq t$, α_i can be expressed as

$$\alpha_i = (tp_1)^{a_1} + (tp_2)^{a_2} + \dots + (tp_m)^{a_m} + \delta,$$

where $a_j, \delta \in \{0, 1, 2, \dots, t-1\}$, $tp_j \neq tp_l$ for any $j, l \in \{1, 2, \dots, m\}$, and $tp_j \leq \alpha_i$ with $p_j \geq 1$ being odd.

The bijection $\mathcal{C}_d^t(n) \Leftrightarrow \mathcal{S}^t(n)$:

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathcal{C}_d^t(n)$. Define the map $g : \mathcal{C}_d^t(n) \rightarrow \mathcal{S}^t(n)$ by $\lambda \rightarrow g(\lambda) = \cup_{i=1}^k g(\lambda_i)$. Then for $1 \leq i \leq k$, we have

$$g(\lambda_i) = \begin{cases} 1^{2^k} & \text{if } \lambda_i = 2^p \text{ for } p \geq 0, \\ x^{2^k} & \text{if } \lambda_i \neq 2^p \text{ for } p \geq 0, \end{cases}$$

where each part $\lambda_i \neq 2^p$ can be uniquely represented as $\lambda_i = 2^p x$ for $p \geq 0$, with $x \equiv y \pmod{2t}$ and $y \in I(t)$.

The inverse map $g^{-1} : \mathcal{S}^t(n) \rightarrow \mathcal{C}_d^t(n)$ is given by

$$g^{-1}(\lambda_i^{\alpha_i}) = \begin{cases} \lambda_i & \text{if } \alpha_i = 1, \\ \lambda_i, 2\lambda_i, 2^2\lambda_i, \dots, 2^p\lambda_i & \text{if } \alpha_i > 1, \end{cases}$$

such that, for every $\alpha_i > 1$, consider $\alpha_i = 2^0 + 2^1 + \dots + 2^p$, where $p \geq 1$.

The bijection $\mathbb{RID}^{(2,t)}(n) \Leftrightarrow \mathcal{C}_d^t(n)$:

Let $\lambda = (\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_k^{\alpha_k}) \in \mathbb{RID}^{(2,t)}(n)$. Define the map $h : \mathbb{RID}^{(2,t)}(n) \rightarrow \mathcal{C}_d^t(n)$ by $\lambda \rightarrow h(\lambda) = \cup_{i=1}^k h(\lambda_i^{\alpha_i})$.

The first step. If $\lambda_i \not\equiv 0 \pmod{t}$, in which $\alpha_i = 2^0 + 2^1 + \dots + 2^p$ for $p \geq 0$, then for $1 \leq i \leq k$, we have

$$h(\lambda_i^{\alpha_i}) = (2^0\lambda_i, 2^1\lambda_i, \dots, 2^p\lambda_i).$$

Now, If $\lambda_i \equiv 0 \pmod{t}$, then for $1 \leq i \leq k$, we have

(I) If $\lambda_i^{\alpha_i} = (tx)^{\alpha_i}$ and $x \not\equiv 0 \pmod{t}$. Then, consider $\alpha_i t = 2^0 + 2^1 + \dots + 2^p$, where $p \geq 0$ and $x \geq 1$. Therefore, we obtain $h(\lambda_i^{\alpha_i}) = (2^0x, 2^1x, \dots, 2^px)$.

(II) If $\lambda_i^{\alpha_i} = (tx)^{\alpha_i}$ and $x \equiv 0 \pmod{t}$. Then, consider $\alpha_i \lambda_i = 2^0 + 2^1 + \dots + 2^p$, where $p \geq 0$ and $x \geq 1$. Therefore, we obtain

$$h(\lambda_i^{\alpha_i}) = (2^0, 2^1, \dots, 2^p).$$

The second step. Sum all the newly obtained parts of the form 2^p in the partition $\lambda \in \mathbf{C}_d^t(n)$, where

$$b = \sum_{\substack{\lambda_i \in \lambda \\ \lambda_i = 2^p, p \geq 0}} \lambda_i.$$

Then, rewrite b as $b = 2^m + 2^{m-1} + \dots + 2^1 + 2^0$ for $m \geq 0$, where 2^m is the largest term that can be extracted from b . Moreover, when t is even, include the parts in λ that are multiples of t into the definition of b . Finally, delete all the parts that constitute b from λ and replace them with the new output of b . For example,

$$\text{for } (t, n) = (3, 17), \text{ we have: } \begin{pmatrix} 7 \\ 3^2 \\ 1^2 \end{pmatrix} \rightarrow \begin{pmatrix} 7 \\ 4, 2 \\ 2 \end{pmatrix} \rightarrow b = 4 + 2 + 2 = 8 \rightarrow \begin{pmatrix} 8 \\ 7 \end{pmatrix},$$

$$\text{and for } (t, n) = (6, 14), \text{ we have: } \begin{pmatrix} 5 \\ 3^2 \\ 1^3 \end{pmatrix} \rightarrow \begin{pmatrix} 6 \\ 5 \\ 2, 1 \end{pmatrix} \rightarrow b = 6 + 2 + 1 = 8 + 1 \rightarrow \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix}.$$

Define the inverse map $h^{-1} : \mathbf{C}_d^t(n) \rightarrow \mathbb{RD}^{(2,t)}(n)$. For every $\lambda \in \mathbf{C}_d^t(n)$, we distinguish three types of parts $\lambda_i \in \lambda$, such that:

- (i) $\lambda_i > 1$ is odd,
- (ii) λ_i is even and $\lambda_i = 2^m x$ for $m \geq 1$, where x is odd and $x \not\equiv 0 \pmod{t}$,
- (iii) $\lambda_i = 2^m$ for $m \geq 0$.

Then for $1 \leq i \leq k$, we have

- (i) $h^{-1}(\lambda_i) = \lambda_i$,
- (ii) $h^{-1}(\lambda_i) = x^{2^m}$,

(iii) Let $\Gamma = \sum_{\lambda_i \in \lambda} \lambda_i$ where $\lambda_i = 2^m$. Then consider $\Gamma = \gamma_1^{b_1} + \gamma_2^{b_2} + \dots + \gamma_r^{b_r} + 1^a$, such that each γ_j is a multiple of t and $\gamma_j \equiv t \pmod{2t}$, where $a \in \{0, 1, 2, \dots, t-1\}$ and $0 \leq b_j \leq 2$ for $1 \leq j \leq r$. Moreover, when γ_j is even, consider $\gamma_j^{b_j} = (2^m x)^{b_j}$ where $m \geq 1$ and $x \geq 1$ is odd. Then make the transformation $\gamma_j^{b_j} \rightarrow x^{2^m b_j}$. Therefore, we have $h^{-1}(\Gamma) = (\gamma_1^{b_1}, \gamma_2^{b_2}, \dots, \gamma_r^{b_r}, 1^a)$. Finally, for any newly obtained part λ_j with $\alpha_j > t$, substitute it with the new part $\alpha_j \lambda_j$. For example,

$$\text{for } (t, n) = (3, 18), \text{ we have: } (16, 2) \rightarrow \Gamma = 18 \rightarrow \Gamma = 9^2 \rightarrow (9^2),$$

$$\text{for } (t, n) = (3, 17), \text{ we have: } (16, 1) \rightarrow \Gamma = 17 \rightarrow \Gamma = 9 + 3^2 + 1^2 \rightarrow (9, 3^2, 1^2),$$

$$\text{for } (t, n) = (3, 18), \text{ we have: } (10, 5, 3) \rightarrow (5^3, 3) \rightarrow (15, 3),$$

$$\text{and for } (t, n) = (5, 14), \text{ we have: } (8, 4, 2) \rightarrow \Gamma = 14 = 6^2 + 1^2 \rightarrow 6^2 = (3 \cdot 2)^2 + 1^2 \rightarrow (3^4, 1^2).$$

□

Example 3.1. An illustration of the bijection $\mathbb{RD}^{(2,t)}(n) \Leftrightarrow \mathbf{S}^t(n)$ for $(n, t) = (12, 3)$ in Table 3.1, and for $(n, t) = (13, 5)$ in Table 3.2.

TABLE 3.1: The bijection of **Theorem 3.1** for $n = 12$ and $t = 3$.

$\mathbb{RD}^{(2,3)}(12)$	\Leftrightarrow	$\mathbb{S}^3(12)$
(11, 1)		(11, 1)
(9, 3)		(1 ¹²)
(7, 5)		(7, 5)
(7, 3, 1 ²)		(7, 1 ⁵)
(5 ² , 1 ²)		(5 ² , 1 ²)
(5, 3 ² , 1)		(5, 1 ⁷)

TABLE 3.2: The bijection of **Theorem 3.1** for $n = 13$ and $t = 5$.

$\mathbb{RD}^{(2,5)}(13)$	\Leftrightarrow	$\mathbb{S}^5(13)$
(13)		(13)
(11, 1 ²)		(11, 1 ²)
(9, 3, 1)		(9, 3, 1)
(9, 1 ⁴)		(9, 1 ⁴)
(7, 3 ²)		(7, 3 ²)
(7, 3, 1 ³)		(7, 3, 1 ³)
(7, 5, 1)		(7, 1 ⁶)
(3 ⁴ , 1)		(3 ⁴ , 1)
(3 ³ , 1 ⁴)		(3 ³ , 1 ⁴)
(5, 3 ² , 1 ²)		(3 ² , 1 ⁷)
(5 ² , 3)		(3, 1 ¹⁰)
(5 ² , 1 ³)		(1 ¹³)

Example 3.2. An illustration of the bijection $\mathbb{RD}^{(2,t)}(n) \Leftrightarrow \mathbb{C}_d^t(n)$ for $(n, t) = (14, 6)$ in Table 3.3, $(n, t) = (18, 3)$ in Table 3.4, and $(n, t) = (15, 5)$ in Table 3.5.

TABLE 3.3: The bijection of **Theorem 3.1** for $n = 14$ and $t = 6$.

$\mathbb{RD}^{(2,6)}(14)$	\Leftrightarrow	$\mathbb{C}_d^6(14)$
(7 ²)		(14)
(5 ² , 1 ²)		(10, 4)
(5, 3 ² , 1 ³)		(8, 5, 1)
(3 ⁴ , 1 ²)		(8, 4, 2)
(3 ³ , 1 ⁵)		(8, 3, 2, 1)
(7, 3, 1 ⁴)		(7, 4, 3)
(7, 3 ² , 1)		(7, 4, 2, 1)
(5, 3 ³)		(5, 4, 3, 2)

TABLE 3.4: The bijection of **Theorem 3.1** for $n = 18$ and $t = 3$.

$\mathbb{RD}^{(2,3)}(18)$	\Leftrightarrow	$\mathbb{C}_d^3(18)$
(17, 1)		(17, 1)
(13, 5)		(13, 5)
(13, 3, 1 ²)		(13, 4, 1)
(11, 7)		(11, 7)
(11, 5, 1 ²)		(11, 5, 2)
(11, 3 ² , 1)		(11, 4, 2, 1)
(7, 5 ² , 1)		(10, 7, 1)
(7, 5, 3 ²)		(7, 5, 4, 2)
(9, 7, 1 ²)		(8, 7, 2, 1)
(15, 3)		(10, 5, 2, 1)
(5 ² , 3 ² , 1 ²)		(10, 8)
(9, 5, 3, 1)		(8, 5, 4, 1)
(9 ²)		(16, 2)
(7 ² , 3, 1)		(14, 4)

TABLE 3.5: The bijection of **Theorem 3.1** for $n = 15$ and $t = 5$.

$\mathbb{RD}^{(2,5)}(15)$	\Leftrightarrow	$\mathbb{C}_d^5(15)$
(13, 1 ²)		(13, 2)
(11, 3, 1)		(11, 3, 1)
(11, 1 ⁴)		(11, 4)
(9, 3, 1 ³)		(9, 3, 2, 1)
(9, 5, 1)		(9, 4, 2)
(7 ² , 1)		(14, 1)
(7, 3 ² , 1 ²)		(7, 6, 2)
(7, 5, 3)		(7, 4, 3, 1)
(7, 5, 1 ³)		(8, 7)
(15)		(12, 3)
(3 ⁴ , 1 ³)		(12, 2, 1)
(5, 3 ³ , 1)		(6, 4, 3, 2)
(5, 3 ² , 1 ⁴)		(8, 6, 1)
(5 ² , 3, 1 ²)		(8, 4, 3)
(5 ³)		(8, 4, 2, 1)

Recurrence relations are instrumental in efficiently computing the number of partitions for a given positive integer n through compact formulas. We introduce recurrence relations for computing $\mathbb{RD}^{(2,t)}(n, k)$, $\mathbb{C}_d^t(n, k)$, and $S^t(n, k)$, which represent the number of partitions of n into k parts for each of the aforementioned classes of partitions.

Theorem 3.2. For all $n, k \geq 1$ and $t \geq 2$,

$$\text{RD}^{(2,t)}(n, k) = \sum_{i=0}^{t-1} \text{RD}^{(2,t)}(n - 2(k - i) - i, k - i),$$

where $\text{RD}^{(2,t)}(0, 0) = 1$ and $\text{RD}^{(2,t)}(n, k) = 0$ if $n < 0, k \leq 0$ and $n > 0$, or $k > n$.

Proof. Let $\mathbb{R}\text{ID}^{(2,t)}(n, k)$ denote the set of partitions of n into k odd parts, each appearing at most $t - 1$ times. We split $\mathbb{R}\text{ID}^{(2,t)}(n, k)$ into t disjoint subsets $\mathbb{R}\text{ID}^{(2,t)}(n, k; i)$ where $0 \leq i \leq t - 1$, such that each subset contains all the partitions of $\mathbb{R}\text{ID}^{(2,t)}(n, k)$ with exactly i parts of size 1 and $k - i$ parts of sizes ≥ 3 . Then, by subtracting 2 from the $k - i$ parts and removing i parts of size 1 from each partition $\lambda \in \mathbb{R}\text{ID}^{(2,t)}(n, k; i)$ for $0 \leq i \leq t$, we obtain a partition of $n - 2(k - i) - i$ into $k - i$ parts. \square

TABLE 3.6: Some values of $\text{RD}^{(2,3)}(n, k)$ for $n \leq 15$ and $k \leq 5$.

n					
1	1				
2		1			
3	1				
4		1			
5	1		1		
6		2			
7	1		2		
8		2		1	
9	1		2		
10		3		1	
11	1		4		
12		3		3	
13	1		5		1
14		4		4	
15	1		6		2
k	1	2	3	4	5

Example 3.3. For $(t, k, n) = (3, 3, 15)$, we have

$$\begin{aligned} \text{RD}^{(2,3)}(15, 3) &= \text{RD}^{(2,3)}(10, 2) + \text{RD}^{(2,3)}(11, 1) + \text{RD}^{(2,3)}(9, 3) \\ &= 3 + 1 + 2 \\ &= 6. \end{aligned}$$

Theorem 3.3. For all $n, k \geq 1$,

$$C_d^t(n, k) = \sum_{i=1}^{2^{t-1}} C_d^t(n - t(k - \ell(\pi^i)) - |\pi^i|, k - \ell(\pi^i)),$$

where the partitions $\pi^i \in \mathcal{P}(M_{(t)})$ for $1 \leq i \leq 2^{t-1}$. Here, $\mathcal{P}(M_{(t)})$ is the power set of $M_{(t)} = \{0, 1, 2, \dots, t-1\}$. By convention, we consider 0 to be the empty set \emptyset , $C_d^t(0, 0) = 1$, and $C_d^t(n, k) = 0$ if $n < 0$, $k \leq 0$ and $n > 0$, or $k > n$.

Proof. Let $\mathbb{C}_d^t(n, k)$ denote the set of partitions of n into k distinct parts not congruent to 0 modulo t . Let $M_{(t)} = \{0, 1, 2, \dots, t-1\}$, and consider $\mathcal{P}(M_{(t)})$ as the power set of $M_{(t)}$, which contains 2^{t-1} subsets (we consider 0 to be the empty set \emptyset). Consider the partitions $\pi^i \in \mathcal{P}(M_{(t)})$ such that $0 \leq \ell(\pi^i) \leq |M_{(t)}| - 1$ (by convention, we consider the length of the partition of the empty set to be 0) and $0 \leq |\pi^i| \leq t(t-1)/2$. For every partition $\lambda \in \mathbb{C}_d^t(n, k)$, λ can be split into two disjoint partitions π^i and θ^i . Here, θ^i contains exactly $k - \ell(\pi^i)$ parts of sizes $\geq (t+1)$. For each partition λ , subtract t from each part of θ^i and remove π^i . This process yields a partition of $n - t(k - \ell(\pi^i)) - |\pi^i|$ into $k - \ell(\pi^i)$ distinct part. \square

TABLE 3.7: Some values of $C_d^3(n, k)$ for $n \leq 18$ and $k \leq 4$.

n				
1	1			
2	1			
3		1		
4	1			
5	1	1		
6		2		
7	1	1	1	
8	1	1	1	
9		3		
10	1	1	2	
11	1	1	2	
12		4	4	1
13	1	2	4	
14	1	2	4	1
15		5	2	2
16	1	2	6	1
17	1	3	6	2
18		6	3	5
k	1	2	3	4

Example 3.4. For $(t, k, n) = (3, 4, 18)$, we have

$$\begin{aligned} C_d^3(18, 4) &= C_d^3(6, 4) + C_d^3(8, 3) + C_d^3(7, 3) + C_d^3(9, 2) \\ &= 0 + 1 + 1 + 3 \\ &= 5, \end{aligned}$$

where $M_{(3)} = \{0, 1, 2\}$ and $\pi^i \in \mathcal{P}(M_{(3)}) = \{\emptyset, (1), (2), (1, 2)\}$.

Theorem 3.4. Let the sets $I(t) = \{1, 3, 5, \dots, (2t - 1)\} \setminus \{t\}$ and $I'(t) = I(t) \setminus \{i, \dots, (2t - 1)\}$, where $i \in I(t)$. Then, for any odd $t \geq 3$ and for all $n, k \geq 1$,

$$S^t(n, k) = \sum_{i \in I(t)} S^t(n, k; i) + S^t(n, k; 2t + 1).$$

Moreover, we have

$$S^t(n, k; i) = S^t(n - i, k - 1) - \sum_{r \in I'(t)} S^t(n - i, k - 1; r),$$

and

$$S^t(n, k; 2t + 1) = S^t(n - 2tk, k),$$

where $S^t(n, k; 1) = S^t(n - 1, k - 1)$. By convention, we define $S^t(0, 0) = 1$. Additionally, $S^t(n, k) = 0$ if $n < 0$, $n > 0$ and $k \leq 0$, or $k > n$.

Proof. Let the set $I(t) = \{1, 3, 5, \dots, (2t - 1)\} \setminus \{t\}$ and let $S^t(n, k)$ denote the set of the t -Sucht's partitions of n into k parts. We split $S^t(n, k)$ into subsets containing all the partitions with at least one part of size $i \in I(t)$ as the smallest part in the partition, where each subset is denoted by $S^t(n, k; i)$. The remaining partitions into parts of sizes $\geq 2t + 1$ are included in the subset $S^t(n, k; 2t + 1)$. Therefore, we obtain the following dissection of $S^t(n, k)$ into $|I(t)| + 1$ disjoint subsets

$$S^t(n, k) = S^t(n, k; 1) \cup S^t(n, k; 3) \cup \dots \cup S^t(n, k; 2t - 1) \cup S^t(n, k; 2t + 1).$$

Consider the set

$$S^t(n - i, k - 1) \setminus \cup_{r \in I'(t)} S^t(n - i, k - 1; r),$$

which contains all the partitions of $n - i$ into $k - 1$ parts of sizes $\geq i$, where

$$I'(t) = I(t) \setminus \{i, \dots, (2t - 1)\}$$

and $i \in I(t)$. To obtain partitions of $n - i$ into $k - 1$ parts of sizes $\geq i$, we exclude those with at least one part of size $r \in I'(t)$ as the smallest part from $S^t(n - i, k - 1)$. By adding one part of size i to each partition $\lambda \in S^t(n - i, k - 1) \setminus \cup_{r \in I'(t)} S^t(n - i, k - 1; r)$, we arrive at the set $S^t(n, k; i)$. Therefore, we deduce that

$$|S^t(n, k; i)| = |S^t(n - i, k - 1) \setminus \cup_{r \in I'(t)} S^t(n - i, k - 1; r)|.$$

Similarly, by adding $2t$ to each part of $\lambda \in S^t(n - 2tk, k)$, we obtain the set $S^t(n, k; 2t + 1)$. Then, we have

$$|S^t(n, k; 2t + 1)| = |S^t(n - 2tk, k)|.$$

For any partition $\lambda \in S^t(n, k; i)$, remove one part of size i to obtain a partition of $n - i$ into $k - 1$ parts of sizes $\geq i$. That is, a partition $\lambda \in S^t(n - i, k - 1) \setminus \cup_{r \in I'(t)} S^t(n - i, k - 1; r)$. For the partitions of $S^t(n, k; 2t + 1)$, subtract $2t$ from each of the k parts to obtain a partition of

$n - 2tk$ into k parts. Then, we obtain

$$S^t(n, k; i) = S^t(n - i, k - 1) - \sum_{r \in I'(t)} S^t(n - i, k - 1; r),$$

and

$$S^t(n, k; 2t + 1) = S^t(n - 2tk, k).$$

□

TABLE 3.8: Some values of $S^3(n, k)$ for $n, k \leq 15$.

n															
1	1														
2		1													
3			1												
4				1											
5	1				1										
6		1				1									
7	1		1				1								
8		1		1				1							
9			1		1				1						
10		1		1		1				1					
11	1		1		1		1				1				
12		2		1		1		1				1			
13	1		2		1		1		1				1		
14		2		2		1		1		1				1	
15			3		2		1		1		1				1
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Example 3.5. For $(t, k, n) = (3, 3, 15)$, we have

$$S^3(15, 3) = S^3(15, 3; 1) + S^3(15, 3; 5) + S^3(15, 3; 7) = 2 + 1 + 0 = 3,$$

such that

- $S^3(15, 3; 1) = S^3(14, 2) = 2,$
- $S^3(15, 3; 5) = S^3(10, 2) - S^3(10, 2; 1) = S^3(10, 2) - S^3(9, 1) = 1 - 0,$
- $S^3(15, 3; 7) = S^3(6, 3) = 0,$

where $I(3) = \{1, 5\}$.

In the following theorem, we propose a generating function for the right-hand side of **Theorem 1.7**.

Theorem 3.5. Let $C_k(n)$ denote the number of partitions of n into k parts that differ by at least 3, with no consecutive multiples of 3 appearing. Then, for $k \geq 2$, we have

$$\sum_{n \geq 0} C_k(n)q^n = \frac{q^{k(3k-1)/2}(1 - q^4 - q^6 + q^{k+3} + q^{k+4} - q^{2k+3})}{(q, q)_{k-2}(1 - q^{k-1})(1 - q^k)(1 - q^6)}.$$

Proof. Consider a partition into $k \geq 2$ consecutive parts that differ by exactly 3. Then we have the following term

$$q^{k(3k-1)/2},$$

where the exponent on q counts the weight of a partition into $k \geq 2$ distinct consecutive parts that differ by exactly 3.

The generating function $1/(q, q)_k$ generates partitions into $k \geq 1$ parts. On the other hand, $q^4/(1 - q^6)$ generates partitions into exactly two parts of the form $3i - 1$ for $i \geq 1$, in which

$$\frac{q^4}{(1 - q^6)} = \sum_{i \geq 1} q^{2(3i-1)} = q^{2 \times 2} + q^{2 \times 5} + q^{2 \times 8} + \dots$$

Now, it is evident that the generating function $q^4/((q, q)_{k-2}(1 - q^6))$ generates partitions into $k \geq 2$ parts, where all the partitions contain at least two parts of size $(3i - 1)$ for $i \geq 1$. Hence, the generating function

$$\frac{q^{k(3k-1)/2}q^4}{(q, q)_{k-2}(1 - q^6)}$$

generates all partitions into $k \geq 2$ parts which contain at least two consecutive multiples of 3. In contrast, the generating function $q^{k(3k-1)/2}/(q, q)_k$ generates both partitions into $k \geq 2$ parts that satisfy the aforementioned condition and those that do not. Therefore, the number of partitions of n into $k \geq 2$ parts that differ by at least 3 where no consecutive multiples of 3 appear satisfies the following generating function

$$\begin{aligned} \sum_{n \geq 0} C_k(n)q^n &= \frac{q^{k(3k-1)/2}}{(q, q)_k} - \frac{q^{k(3k-1)/2}q^4}{(q, q)_{k-2}(1 - q^6)} \\ &= \frac{q^{k(3k-1)/2}(1 - q^4 - q^6 + q^{k+3} + q^{k+4} - q^{2k+3})}{(q, q)_{k-2}(1 - q^{k-1})(1 - q^k)(1 - q^6)}. \end{aligned}$$

□

The following Corollary is an immediate result from **Theorem 3.5**.

Corollary 3.1. Let $C(n)$ denote the number of partitions of n into parts that differ by at least 3, where no consecutive multiples of 3 appear. Then

$$\sum_{n \geq 0} C(n)q^n = \frac{1}{1 - q} + \sum_{k \geq 2} \frac{q^{k(3k-1)/2}(1 - q^4 - q^6 + q^{k+3} + q^{k+4} - q^{2k+3})}{(q, q)_{k-2}(1 - q^{k-1})(1 - q^k)(1 - q^6)}.$$

Proof. By adding $1/(1 - q)$, which generates partitions that satisfy the condition, to the generating function of Theorem 3.5, we obtain the desired result. \square

Example 3.6. For $k = 3$, we obtain

$$\frac{q^{12}}{(q, q)_3} - q^{12} \frac{q^4}{(1 - q^6)} = \left\{ (q^{12} + q^{13} + 2q^{14} + 3q^{15} + 4q^{16} + 5q^{17} + 7q^{18} + \dots) - (q^{16} + q^{17} + q^{18} + q^{19} + \dots) \right\} = q^{12} + q^{13} + 2q^{14} + 3q^{15} + 3q^{16} + 4q^{17} + 6q^{18} \dots$$

In the following table, we consider partitions into parts that differ by at least 3, excluding those that contain two consecutive multiples of 3.

$4q^{16}$	$5q^{17}$	$7q^{18}$	-	q^{16}	q^{17}	q^{18}
		(13, 4, 1)				
(11, 4, 1)	(12, 4, 1)	(12, 5, 1)				
(10, 5, 1)	(11, 5, 1)	(11, 6, 1)				
(9, 6, 1)	(10, 6, 1)	(10, 7, 1)		(9, 6, 1)	(9, 6, 2)	(9, 6, 3)
(9, 5, 2)	(10, 5, 2)	(10, 6, 2)				
	(9, 6, 2)	(11, 5, 2)				
		(9, 6, 3)				

TABLE 3.9: An illustration of Theorem 3.5.

3.3 Arithmetic properties

In this section, we shall prove several arithmetic properties of $RD^{(\ell,t)}(n)$ for the selected pairs $(2, 5)$, $(3, 4)$, $(4, 5)$, $(4, 6)$, and $(4, 9)$ as (ℓ, t) . In this vein, we derive several Ramanujan-like congruences, infinite families of congruences, in addition to some explicit results on the generating function dissections satisfied by $RD^{(\ell,t)}(n)$ in the above mentioned cases.

Theorem 3.6. We have

$$\sum_{n \geq 0} RD^{(2,5)}(2n)q^n = \frac{f_4 f_{10}^2}{f_1 f_5 f_{20}}, \tag{3.2}$$

$$\sum_{n \geq 0} RD^{(2,5)}(2n + 1)q^n = \frac{f_2^3 f_{20}}{f_1^2 f_4 f_{10}}. \tag{3.3}$$

Proof. By setting $(\ell, t) = (2, 5)$ in (3.1), substituting (2.27), and then extracting the terms involving the powers q^{2n} and q^{2n+1} from both sides of the resulting equation, we obtain (3.2) by substituting q^2 by q and (3.3) by dividing both sides by q and then replacing q^2 by q , respectively. \square

Theorem 3.7. *We have*

$$\sum_{n \geq 0} \text{RD}^{(2,5)}(4n+1)q^n = \frac{f_4^5 f_{10}}{f_1^2 f_2 f_5 f_8^2}, \quad (3.4)$$

$$\sum_{n \geq 0} \text{RD}^{(2,5)}(4n+3)q^n = 2 \frac{f_2 f_8^2 f_{10}}{f_1^2 f_4 f_5}. \quad (3.5)$$

Proof. Substituting (2.3) into (3.3), and then collecting the even and the odd powers from both sides of the resulting equation, we arrive at

$$\sum_{n \geq 0} \text{RD}^{(2,5)}(2n+1)q^{2n} = \frac{f_8^5 f_{20}}{f_2^2 f_4 f_{10} f_{16}^2}, \quad (3.6)$$

$$\sum_{n \geq 0} \text{RD}^{(2,5)}(2n+1)q^{2n+1} = 2q \frac{f_4 f_{16}^2 f_{20}}{f_2^2 f_8 f_{10}}. \quad (3.7)$$

Thus, (3.4) can be deduced from (3.6) by substituting q^2 by q . Meanwhile, (3.5) can be deduced from (3.7) by dividing both sides by q and then substituting q^2 by q . \square

Theorem 3.8. *For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, we have*

$$\text{RD}^{(2,5)} \left(20p^{2\alpha+1}(pn+i) + \frac{5p^{2\alpha+2}+1}{6} \right) \equiv 0 \pmod{2}, \quad (3.8)$$

where i is an integer and $1 \leq i \leq p-1$.

Proof. In view of (3.4) and (2.2), with $p=2$ and $k=1$, we find that

$$\sum_{n \geq 0} \text{RD}^{(2,5)}(4n+1)q^n = \frac{f_4^5 f_{10}}{f_1^2 f_2 f_5 f_8^2} \equiv f_5 \pmod{2}.$$

That is,

$$\sum_{n \geq 0} \text{RD}^{(2,5)}(20n+1)q^n \equiv f_1 \pmod{2}. \quad (3.9)$$

Define

$$\sum_{n \geq 0} a(n)q^n = f_1. \quad (3.10)$$

Combining (3.9) and (3.10), we deduce that

$$\text{RD}^{(2,5)}(20n+1) \equiv a(n) \pmod{2}, \quad (3.11)$$

Now, we consider the congruence equation

$$\frac{3m^2+m}{2} \equiv \frac{p^2-1}{24} \pmod{p},$$

which is equivalent to

$$(6m + 1)^2 \equiv 0 \pmod{p}, \quad (3.12)$$

where $-(p-1)/2 \leq m \leq (p-1)/2$ and $p \geq 5$ is a prime. Then, the congruence relation (3.12) holds if and only if $m = (\pm p - 1)/6$. Therefore, if we substitute (2.35) into (3.10) and then extract the terms in which the powers of q are congruent to $\frac{p^2-1}{24}$ modulo p and then divide by $q^{(p^2-1)/24}$, we find that

$$\sum_{n \geq 0} a \left(pn + \frac{p^2-1}{24} \right) q^{pn} = (-1)^{\frac{\pm p-1}{6}} f_{p^2},$$

which implies that

$$\sum_{n \geq 0} a \left(p^2n + \frac{p^2-1}{24} \right) q^n = (-1)^{\frac{\pm p-1}{6}} f_1,$$

and for $n \geq 0$,

$$a \left(p^2n + pi + \frac{p^2-1}{24} \right) = 0, \quad (3.13)$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$a \left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{24} \right) = (-1)^{\alpha \frac{\pm p-1}{6}} a(n). \quad (3.14)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{24}$ in (3.14) and using (3.13), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$a \left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{24} \right) = 0.$$

Again, replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{24}$ ($1 \leq i \leq p-1$) in (3.11), we get (3.8). \square

Corollary 3.2. *The following congruence is an immediate result from (3.5). For all $n \geq 0$,*

$$\text{RD}^{(2,5)}(4n+3) \equiv 0 \pmod{2}.$$

Theorem 3.9. *We have*

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(3n)q^n = \frac{f_6^4}{f_1^2 f_{12}^2}, \quad (3.15)$$

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(3n+1)q^n = \frac{f_2^2 f_3^3 f_{12}}{f_1^3 f_4 f_6^2}, \quad (3.16)$$

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(3n+2)q^n = 2 \frac{f_2 f_6 f_{12}}{f_1^2 f_4}. \quad (3.17)$$

Proof. By setting $(\ell, t) = (3, 4)$ in (3.1), substituting (2.25), and then collecting the terms involving the powers of the form q^{3n+j} for $j = 0, 1, 2$, we obtain

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(n)q^{3n} = \frac{f_{18}^4}{f_3^2 f_{36}^2}, \quad (3.18)$$

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(n)q^{3n+1} = q \frac{f_6^2 f_9^3 f_{36}}{f_3^3 f_{12} f_{18}^2}, \quad (3.19)$$

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(n)q^{3n+2} = 2q^2 \frac{f_6 f_{18} f_{36}}{f_3^2 f_{12}}. \quad (3.20)$$

Then, by substituting q^3 by q in (3.18) we obtain (3.15), and (3.16) can be obtained from (3.19) by dividing both sides by q and then replacing q^3 by q . In a similar way, we can obtain (3.17) from (3.20). \square

Theorem 3.10. *We have*

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(6n)q^n = \frac{f_3^4 f_4^5}{f_1^5 f_6^2 f_8^2}, \quad (3.21)$$

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(6n+2)q^n = 2 \frac{C(q)\phi(q^2)^3}{\phi(-q^2)^4} + 8q \frac{C(q)\phi(q^2)\psi(q^4)^2}{\phi(-q^2)^4}, \quad (3.22)$$

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(6n+3)q^n = 2 \frac{f_2^2 f_3^4 f_8^2}{f_1^5 f_4 f_6^2}, \quad (3.23)$$

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(6n+5)q^n = 4 \frac{C(q)\phi(q^2)^2\psi(q^4)}{\phi(-q^2)^4} + 16q \frac{C(q)\psi(q^4)^3}{\phi(-q^2)^4}, \quad (3.24)$$

where

$$C(q) = \frac{f_3 f_6}{f_2}.$$

Proof. Substituting (2.3) into (3.15), we obtain

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(3n)q^n = \frac{f_6^4 f_8^5}{f_2^5 f_{12}^2 f_{16}^2} + 2q \frac{f_4^2 f_6^4 f_{16}^2}{f_2^5 f_8 f_{12}^2}. \quad (3.25)$$

Extracting the terms involving the powers of the form q^{2n} and then substituting q^2 by q , we obtain (3.21). Similarly, (3.23) can be deduced from (3.25) by extracting the powers of the form q^{2n+1} , dividing both sides by q and then replacing q^2 by q .

Employing (2.12) into (3.17), we obtain

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(3n+2)q^n = 2 \frac{C(q^2)}{\phi(-q^4)^4} (\phi(q^4)^3 + 2q\phi(q^4)^2\psi(q^8) + 4q^2\phi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3), \quad (3.26)$$

where $C(q^2) = (f_6 f_{12})/f_4$. Thus, (3.22) and (3.24) follow from (3.26) by extracting the even and the odd powers from both sides, respectively. \square

From the previous results, we obtain the following Ramanujan-like congruences for the 3-regular and 4-distinct partitions.

Corollary 3.3. For all $n \geq 0$,

$$\text{RD}^{(3,4)}(3n+2) \equiv 0 \pmod{2},$$

$$\text{RD}^{(3,4)}(6n+2) \equiv 0 \pmod{2},$$

$$\text{RD}^{(3,4)}(6n+3) \equiv 0 \pmod{2},$$

$$\text{RD}^{(3,4)}(6n+5) \equiv 0 \pmod{4}.$$

Theorem 3.11. For all $n \geq 0$, we have

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(18n+1)q^n \equiv f_1 \pmod{2}, \quad (3.27)$$

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(54n+7)q^n \equiv f_1^3 \pmod{2}, \quad (3.28)$$

$$\text{RD}^{(3,4)}(18n+13) \equiv 0 \pmod{2}, \quad (3.29)$$

$$\text{RD}^{(3,4)}(54n+j) \equiv 0 \pmod{2}, \quad j = 25, 43. \quad (3.30)$$

Proof. In view of (2.2) and (3.16), with $p = 2$ and $k = 1$, we have

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(3n+1)q^n = \frac{f_2^2 f_3^3 f_{12}}{f_1^3 f_4 f_6^2} \equiv \frac{f_3 f_6}{f_1 f_2} \pmod{2}. \quad (3.31)$$

Substituting (2.18) into (3.31), and then collecting the even powers from both sides of the resulting equation, we get

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(6n+1)q^n \equiv \frac{f_2 f_3^2 f_8 f_{12}^2}{f_1^3 f_4 f_6 f_{24}} \equiv \frac{f_4}{f_1} \pmod{2}. \quad (3.32)$$

Employing (2.25) in (3.32), and then extracting the terms of the form q^{3n+j} for $j = 0, 1, 2$ from both sides of the resulting equation, we find that

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(18n+1)q^n \equiv \frac{f_4 f_6^4}{f_1^3 f_{12}^2} \equiv f_1 \pmod{2}, \quad (3.33)$$

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(18n+7)q^n \equiv \frac{f_2^2 f_3^3 f_{12}}{f_1^4 f_6^2} \equiv f_3^3 \pmod{2}, \quad (3.34)$$

$$\sum_{n \geq 0} \text{RD}^{(3,4)}(18n+13)q^n \equiv 2 \frac{f_2 f_6 f_{12}}{f_1^3} \pmod{2}. \quad (3.35)$$

Congruences (3.27) and (3.29) follow from (3.33) and (3.35), respectively. Congruences (3.28) and (3.30) follow from (3.34) by extracting the terms of the form q^{3n+j} for $j = 0, 1, 2$ from both sides. \square

From Equations (3.27) and (3.10), we deduce that

$$\text{RD}^{(3,4)}(18n + 1) \equiv a(n) \pmod{2},$$

where $a(n)$ is defined in Equation (3.10). From **Theorem 3.8** we obtain the following result.

Corollary 3.4. *For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, we have*

$$\text{RD}^{(3,4)}\left(18p^{2\alpha+1}(pn + i) + \frac{3p^{2\alpha+2} + 1}{4}\right) \equiv 0 \pmod{2},$$

for all $1 \leq i \leq p - 1$.

Theorem 3.12. *We have*

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(2n)q^n = \frac{f_2 f_4 f_{10}}{f_1^2 f_{20}}, \quad (3.36)$$

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(2n + 1)q^n = \frac{f_2^4 f_5 f_{20}}{f_1^3 f_4 f_{10}^2}. \quad (3.37)$$

Proof. Setting $(\ell, t) = (4, 5)$ in (3.1), we obtain

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(n)q^n = \frac{f_5 f_4}{f_1 f_{20}}. \quad (3.38)$$

Substituting (2.27) into (3.38), we arrive at

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(n)q^n = \frac{f_4 f_8 f_{20}}{f_2^2 f_{40}} + q \frac{f_4^4 f_{10} f_{40}}{f_2^3 f_8 f_{20}^2}. \quad (3.39)$$

Thus, (3.36) and (3.37) can be deduced from (3.39) by extracting the terms of the form q^{2n+j} for $j = 0, 1$ from both sides. \square

Theorem 3.13. *We have*

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(4n)q^n = \frac{f_2 f_4^5 f_5}{f_1^4 f_8^2 f_{10}}, \quad (3.40)$$

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(4n + 2)q^n = 2 \frac{f_2^3 f_5 f_8^2}{f_1^4 f_4 f_{10}}. \quad (3.41)$$

Proof. Using (2.3) in (3.36), we obtain

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(2n)q^n = \frac{f_4 f_8^5 f_{10}}{f_2^4 f_{16}^2 f_{20}} + 2q \frac{f_4^3 f_{10} f_{16}^2}{f_2^4 f_8 f_{20}}. \quad (3.42)$$

Collecting the even powers from both sides of (3.42), and then substituting q^2 by q , we obtain (3.40). Meanwhile, by extracting the terms involving powers of the form q^{2n+1} from both sides of (3.42), dividing by q and then substituting q^2 by q , we arrive at (3.41). \square

The following congruence is an immediate result from (3.41).

Corollary 3.5. For all $n \geq 0$,

$$\text{RD}^{(4,5)}(4n + 2) \equiv 0 \pmod{2}.$$

Theorem 3.14. For all $n \geq 0$,

$$\text{RD}^{(4,5)}(12n + 9) \equiv 0 \pmod{2}, \quad (3.43)$$

$$\text{RD}^{(4,5)}(20n + i) \equiv 0 \pmod{2}, \quad i = 7, 11, 15, 19, \quad (3.44)$$

$$\text{RD}^{(4,5)}(36n + j) \equiv 0 \pmod{2}, \quad j = 17, 29. \quad (3.45)$$

Proof. In view of (2.2) and (3.37), with $p = 2$ and $k = 1$, we have

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(2n + 1)q^n = \frac{f_2^4 f_5 f_{20}}{f_1^3 f_4 f_{10}^2} \equiv f_2 \frac{f_5}{f_1} \pmod{2}.$$

Employing (2.27) in the above equation, and then extracting the even and the odd powers from both sides of the resulting equation, we find that

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(4n + 1)q^n \equiv \frac{f_4 f_{10}^2}{f_1 f_{20}} \equiv \frac{f_4}{f_1} \pmod{2}, \quad (3.46)$$

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(4n + 3)q^n \equiv \frac{f_2^3 f_5 f_{20}}{f_1^2 f_4 f_{10}} \equiv \frac{f_5 f_{20}}{f_{10}} \pmod{2}. \quad (3.47)$$

Congruence (3.44) follows from (3.47) by extracting the terms of the form q^{5n+j} for $j = 1, 2, 3, 4$ from both sides.

Substituting (2.25) into (3.46), we obtain

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(4n + 1)q^n \equiv \frac{f_{12} f_{18}^4}{f_3^3 f_{36}} + q \frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2 \frac{f_6 f_{18} f_{36}}{f_3^3} \pmod{2}. \quad (3.48)$$

Extracting the terms involving powers of the form q^{3n+1} and q^{3n+2} from (3.48), dividing both sides by q and q^2 , respectively, and then substituting q^3 by q for both, we deduce

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(12n+5)q^n \equiv \frac{f_2^2 f_3^3 f_{12}}{f_1^4 f_6^2} \equiv \frac{f_2 f_3^3}{f_1^2} \pmod{2}, \quad (3.49)$$

$$\text{RD}^{(4,5)}(12n+9) \equiv 0 \pmod{2}. \quad (3.50)$$

Substituting (2.13) into (3.49), we get

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(12n+5)q^n \equiv \frac{f_6^4 f_9^6}{f_3^5 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^4} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^3} \pmod{2}. \quad (3.51)$$

Congruence (3.45) follows from (3.51) by collecting the powers of the form q^{3n+j} for $j = 1, 2$ from both sides. \square

Theorem 3.15. *We have*

$$\sum_{n \geq 0} \text{RD}^{(4,6)}(3n)q^n = \frac{f_2 f_4 f_6^4}{f_1^3 f_8 f_{12}^2}, \quad (3.52)$$

$$\sum_{n \geq 0} \text{RD}^{(4,6)}(3n+1)q^n = \frac{f_2^3 f_3^3 f_{12}}{f_1^4 f_6^2 f_8}, \quad (3.53)$$

$$\sum_{n \geq 0} \text{RD}^{(4,6)}(3n+2)q^n = 2 \frac{f_2^2 f_6 f_{12}}{f_1^3 f_8}. \quad (3.54)$$

Proof. Setting $(\ell, t) = (4, 6)$ in (3.1) and substituting (2.25), we obtain

$$\sum_{n \geq 0} \text{RD}^{(4,5)}(n)q^n = \frac{f_6 f_{12} f_{18}^4}{f_3^3 f_{24} f_{36}^2} + q \frac{f_6^3 f_9^3 f_{36}}{f_3^4 f_{18} f_{24}} + 2q^2 \frac{f_6^2 f_{18} f_{36}}{f_3^3 f_{24}}. \quad (3.55)$$

By extracting the terms involving the powers of the form q^{3n} , q^{3n+1} , and q^{3n+2} from both sides of (3.55), then dividing both sides by q for the terms involving q^{3n+1} , by q^2 for the terms involving q^{3n+2} , and finally substituting q^3 by q , we obtain (3.52), (3.53), and (3.54), respectively. \square

Corollary 3.6. *For all $n \geq 0$, $\text{RD}^{(4,6)}(3n+2) \equiv 0 \pmod{2}$.*

Theorem 3.16. *We have*

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n)q^n = \frac{f_2 f_6^3 f_9}{f_1^2 f_3 f_{18}^2}, \quad (3.56)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n+1)q^n = \frac{f_2^3 f_3}{f_1^3 f_6}. \quad (3.57)$$

Proof. Setting $(\ell, t) = (4, 9)$ in (3.1) and employing (2.30), we obtain

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(n)q^n = \frac{f_4 f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}^2} + q \frac{f_4^3 f_6}{f_2^3 f_{12}}. \quad (3.58)$$

Thus, (3.56) can be obtained from (3.58) by extracting the powers involving q^{2n} , and then substituting q^2 by q . On the other hand, (3.57) can be obtained from (3.58) by extracting the terms involving q^{2n+1} , dividing both sides by q and then substituting q^2 by q . \square

Theorem 3.17. *We have*

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n)q^n = \frac{f_2^7 f_3^7}{f_1^9 f_6^5}, \quad (3.59)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+2)q^n = 2 \frac{f_2^6 f_3^4}{f_1^8 f_6^2}, \quad (3.60)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+4)q^n = 4 \frac{f_2^5 f_3 f_6}{f_1^7}. \quad (3.61)$$

Proof. Employing (2.13) in (3.56), we find that

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n)q^n = \frac{f_6^7 f_9^7}{f_3^9 f_{18}^5} + 2q \frac{f_6^6 f_9^4}{f_3^8 f_{18}^2} + 4q^2 \frac{f_6^5 f_9 f_{18}}{f_3^7}. \quad (3.62)$$

Then, by collecting the terms involving q^{3n} , q^{3n+1} , and q^{3n+2} , we deduce that

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n)q^{3n} = \frac{f_6^7 f_9^7}{f_3^9 f_{18}^5}, \quad (3.63)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n)q^{3n+1} = 2q \frac{f_6^6 f_9^4}{f_3^8 f_{18}^2}, \quad (3.64)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n)q^{3n+2} = 4q^2 \frac{f_6^5 f_9 f_{18}}{f_3^7}. \quad (3.65)$$

Thus, (3.59) can be obtained from (3.63) by substituting q^3 by q . Similarly, (3.60) can be derived from (3.64) by dividing both sides by q and then substituting q^3 by q . In the same manner, (3.61) can be obtained from (3.65) by dividing both sides by q^2 and then substituting q^3 by q . \square

Theorem 3.18. *We have*

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(4n+1)q^n = \frac{f_2^6 f_3^2}{f_1^6 f_6^2}, \quad (3.66)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(4n+3)q^n = 3 \frac{f_2^2 f_3^2}{f_1^4}. \quad (3.67)$$

Proof. Employing (2.22) into (3.57), we obtain

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n+1)q^n = \frac{f_4^6 f_6^2}{f_2^6 f_{12}^2} + 3q \frac{f_4^2 f_{12}^2}{f_2^4}. \quad (3.68)$$

Thus, (3.66) follows from (3.68) by extracting the even powers and then substituting q^2 with q . Similarly, (3.67) is obtained from (3.68) by extracting the odd powers, dividing both sides by q , and then substituting q^2 by q . \square

Theorem 3.19. *We have*

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(8n+1)q^n = \frac{f_2^{12} f_3^4}{f_1^{12} f_6^4} + 9q \frac{f_2^4 f_6^4}{f_1^8}, \quad (3.69)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(8n+5)q^n = 6 \frac{f_2^8 f_3^2}{f_1^{10}}. \quad (3.70)$$

Proof. By substituting (2.22) into (3.66), and then extracting the terms of the form q^{2n} and q^{2n+1} from both sides of the resulting equation, we arrive at the following expressions

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(4n+1)q^{2n} = \frac{f_4^{12} f_6^4}{f_2^{12} f_{12}^4} + 9q^2 \frac{f_4^4 f_{12}^4}{f_2^8}, \quad (3.71)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(4n+1)q^{2n+1} = 6q \frac{f_4^8 f_6^2}{f_2^{10}}. \quad (3.72)$$

Thus, (3.69) can be deduced from (3.71) by substituting q^2 by q , and (3.70) can be obtained from (3.72) by dividing both sides by q and then substituting q^2 by q . \square

Theorem 3.20. *We have*

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+1)q^n = 1 + 12q \frac{f_2 f_6^5}{f_1^5 f_3}, \quad (3.73)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+3)q^n = 3 \frac{f_2^3 f_3^5}{f_1^7 f_6}, \quad (3.74)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+5)q^n = 6 \frac{f_2^2 f_3^2 f_6^2}{f_1^6}. \quad (3.75)$$

Proof. Substituting (2.17) into (3.57), we obtain

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n+1)q^n = 1 + 3q \frac{f_6^3 f_9^5}{f_3^7 f_{18}} + 6q^2 \frac{f_6^2 f_9^2 f_{18}^2}{f_3^6} + 12q^3 \frac{f_6 f_{18}^5}{f_3^5 f_9}. \quad (3.76)$$

Extracting the terms involving powers of the form q^{3n} , q^{3n+1} , and q^{3n+2} , we get

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n+1)q^{3n} = 1 + 12q^3 \frac{f_6 f_{18}^5}{f_3^5 f_9}, \quad (3.77)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n+1)q^{3n+1} = 3q \frac{f_6^3 f_9^5}{f_3^7 f_{18}}, \quad (3.78)$$

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n+1)q^{3n+2} = 6q^2 \frac{f_6^2 f_9^2 f_{18}^2}{f_3^6}. \quad (3.79)$$

Thus, by substituting q^3 by q in (3.77), we deduce that

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+1)q^n = 1 + 12q \frac{f_2 f_6^5}{f_1^5 f_3}.$$

In the above equation, note that the additional term is due to the single partition of the number 1. On the other hand, by dividing both sides by q and substituting q^3 by q in (3.78), we obtain

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+3)q^n = 3 \frac{f_2^3 f_3^5}{f_1^7 f_6}.$$

Finally, in a similar manner, we deduce from (3.79),

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+5)q^n = 6 \frac{f_2^2 f_3^2 f_6^2}{f_1^6}.$$

□

We obtain the next Ramanujan-like congruences for the 4-regular and 9-distinct partitions.

Corollary 3.7. *For all $n \geq 0$,*

$$\text{RD}^{(4,9)}(4n+3) \equiv 0 \pmod{3},$$

$$\text{RD}^{(4,9)}(6n+2) \equiv 0 \pmod{2},$$

$$\text{RD}^{(4,9)}(6n+3) \equiv 0 \pmod{3},$$

$$\text{RD}^{(4,9)}(6n+4) \equiv 0 \pmod{4},$$

$$\text{RD}^{(4,9)}(6n+5) \equiv 0 \pmod{6},$$

$$\text{RD}^{(4,9)}(6n+7) \equiv 0 \pmod{12},$$

$$\text{RD}^{(4,9)}(8n+5) \equiv 0 \pmod{6}.$$

Hirschhorn and Sellers [124] established the following Ramanujan-like congruence for the 2-regular and 4-distinct partitions for all $n \geq 0$.

$$\text{RD}^{(2,4)}(27n+17) \equiv 0 \pmod{3}.$$

3.4 t -Schur's overpartitions

For any odd $t \geq 3$, define $\bar{S}_t(n)$ to be the number of t -Schur's overpartitions of a positive integer n (a generalization of t -Schur's partitions). The number $\bar{S}_t(n)$ satisfies the following generating function

$$\sum_{n \geq 0} \bar{S}_t(n) q^n = \prod_{k \geq 1} \frac{(1 + q^{2k-1})(1 - q^{t(2k-1)})}{(1 + q^{t(2k-1)})(1 - q^{2k-1})} = \frac{f_2^3 f_t^2 f_{4t}}{f_1^2 f_4 f_{2t}^3}. \quad (3.80)$$

For example, $\bar{S}_5(6) = 8$, where the corresponding set is

$$\bar{S}_5(6) = \{(1^6), (\bar{1}, 1^5), (3, 1^3), (\bar{3}, 1^3), (3, \bar{1}, 1^2), (\bar{3}, \bar{1}, 1^2), (3^2), (\bar{3}, 3)\}.$$

For any odd $t \geq 3$, define $I_t(2n)$ to be the number of partitions of $2n$ into parts indivisible by t , where the odd parts appear only with a multiplicity of 2, and the even parts are distinct. The number $I_t(2n)$ satisfies the following generating function

$$\sum_{n \geq 0} I_t(2n) q^n = \prod_{k \geq 1} \frac{(1 + q^{2k})(1 + q^{2(2k-1)})}{(1 + q^{2tk})(1 + q^{2(2tk-t)})}. \quad (3.81)$$

For example, $I_3(14) = 8$, in which

$$\mathbb{I}_3(14) = \{(14), (10, 4), (10, 2, 1^2), (8, 4, 2), (8, 4, 1^2), (7^2), (5^2, 4), (5^2, 2, 1^2)\}.$$

Note that the coefficients $I_3(2n)$ for $n \geq 0$ match the OEIS [197] sequence [A103260](#).

Theorem 3.21. For any odd $t \geq 3$ and $n \geq 0$,

$$\bar{S}_t(n) = I_t(2n).$$

Proof. The generating function proof is straightforward and involves elementary generating function manipulations. Our focus, however, will be on the combinatorial proof.

Let $\lambda \in \bar{S}_t(n)$, and define the map $f : \bar{S}_t(n) \rightarrow \mathbb{I}_t(2n)$ by $\lambda \rightarrow f(\lambda) = \cup_{i=1}^k f(\lambda_i^{\alpha_i})$. Then for $1 \leq i \leq k$, we have

$$f(\lambda_i^{\alpha_i}) = \begin{cases} 2\lambda_i, 2^2\lambda_i, \dots, 2^{p+1}\lambda_i & \text{if } \lambda_i \text{ is not overlined, with } \alpha_i = 2^0 + 2^1 + \dots + 2^p, \text{ for } p \geq 0, \\ \lambda_i^2 & \text{if } \lambda_i \text{ is overlined.} \end{cases}$$

The inverse map $f^{-1} : \mathbb{I}_t(2n) \rightarrow \bar{S}_t(n)$ is given by

$$f^{-1}(\lambda_i^{\alpha_i}) = \begin{cases} x^{2^{p-1}} & \text{if } \lambda_i \text{ is even,} \\ \bar{\lambda}_i & \text{if } \lambda_i \text{ is odd,} \end{cases}$$

where λ_i can be uniquely expressed as $\lambda_i = 2^p x$ for $p \geq 1$ such that $x \equiv i \pmod{2t}$ and $i \in I(t)$ (where $I(t)$ represents the set of all odd positive integers $\leq 2t - 1$ except t). \square

Example 3.7. An illustration of the bijection $\bar{S}_t(n) \Leftrightarrow \mathbb{I}_t(2n)$ for $(n, t) = (7, 7)$ in Table 3.10.

TABLE 3.10: The bijection of Theorem 3.21 for $n = 7$, and $t = 7$.

$\bar{S}_7(7)$	\Leftrightarrow	$\mathbb{I}_7(14)$
(1^7)		$(8, 4, 2)$
$(1^6, \bar{1})$		$(8, 4, 1^2)$
$(3, 1^4)$		$(8, 6)$
$(\bar{3}, 1^4)$		$(8, 3^2)$
$(3, \bar{1}, 1^3)$		$(6, 4, 2, 1^2)$
$(\bar{3}, \bar{1}, 1^3)$		$(4, 3^2, 2, 1^2)$
$(3^2, 1)$		$(12, 2)$
$(3, \bar{3}, 1)$		$(6, 3^2, 2)$
$(3^2, \bar{1})$		$(12, 1^2)$
$(3, \bar{3}, \bar{1})$		$(6, 3^2, 1^2)$
$(5, 1^2)$		$(10, 4)$
$(\bar{5}, 1^2)$		$(5^2, 4)$
$(5, \bar{1}, 1)$		$(10, 2, 1^2)$
$(\bar{5}, \bar{1}, 1)$		$(5^2, 2, 1^2)$

3.5 Congruences for 3-Schur's overpartitions

We derive several Ramanujan-like congruences and generating function dissections satisfied by $\bar{S}_t(n)$ for $t = 3$.

Theorem 3.22. *We have*

$$\sum_{n \geq 0} \bar{S}_3(2n)q^n = \frac{f_2 f_8^2 f_{12}^4}{f_1 f_3 f_4^2 f_6 f_{24}^2} + q \frac{f_4^4 f_6 f_{24}^2}{f_1 f_2 f_3 f_8^2 f_{12}^2}, \quad (3.82)$$

$$\sum_{n \geq 0} \bar{S}_3(2n + 1)q^n = 2 \frac{f_4 f_{12}}{f_1 f_3}. \quad (3.83)$$

Proof. Setting $t = 3$ in (3.80), we obtain

$$\sum_{n \geq 0} \bar{S}_3(n)q^n = \frac{f_2^3 f_3^2 f_{12}}{f_1^2 f_4 f_6^3}. \quad (3.84)$$

Invoking the following dissection formula of Yao and Xia [207, (2.22)] in (3.84),

$$\frac{f_2^3 f_3 f_{12}}{f_1 f_4 f_6^3} = \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} + q \frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}},$$

along with employing (2.18), we find that

$$\sum_{n \geq 0} \bar{S}_3(n)q^n = \left(\frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q \frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}} \right) \left(\frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} + q \frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}} \right).$$

That is,

$$\sum_{n \geq 0} \bar{S}_3(n)q^n = \frac{f_4 f_{16}^2 f_{24}^4}{f_2 f_6 f_8^2 f_{12} f_{48}^2} + 2q \frac{f_8 f_{24}}{f_2 f_6} + q^2 \frac{f_8^4 f_{12} f_{48}^2}{f_2 f_4 f_6 f_{16}^2 f_{24}^2}.$$

Extracting the terms involving the powers of the form q^{2n} and q^{2n+1} from both sides of the above equation, we get

$$\sum_{n \geq 0} \bar{S}_3(n)q^{2n} = \frac{f_4 f_{16}^2 f_{24}^4}{f_2 f_6 f_8^2 f_{12} f_{48}^2} + q^2 \frac{f_8^4 f_{12} f_{48}^2}{f_2 f_4 f_6 f_{16}^2 f_{24}^2}, \quad (3.85)$$

$$\sum_{n \geq 0} \bar{S}_3(n)q^{2n+1} = 2q \frac{f_8 f_{24}}{f_2 f_6}. \quad (3.86)$$

Thus, (3.82) follows from (3.85) by replacing q^2 by q , and (3.83) follows from (3.86) by dividing both sides by q and then substituting q^2 by q . \square

Theorem 3.23. *We have*

$$\sum_{n \geq 0} \bar{S}_3(6n+1)q^n = 2 \frac{f_4^2 f_6^4}{f_1^4 f_{12}^2}, \quad (3.87)$$

$$\sum_{n \geq 0} \bar{S}_3(6n+3)q^n = 2 \frac{f_2^2 f_3^3 f_4 f_{12}}{f_1^5 f_6^2}, \quad (3.88)$$

$$\sum_{n \geq 0} \bar{S}_3(6n+5)q^n = 4 \frac{f_2 f_4 f_6 f_{12}}{f_1^4}. \quad (3.89)$$

Proof. Substituting (2.25) into (3.83), we obtain

$$\sum_{n \geq 0} \bar{S}_3(2n+1)q^n = 2 \frac{f_{12}^2 f_{18}^4}{f_3^4 f_{36}^2} + 2q \frac{f_6^2 f_9^3 f_{12} f_{36}}{f_3^5 f_{18}^2} + 4q^2 \frac{f_6 f_{12} f_{18} f_{36}}{f_3^4}.$$

Collecting the powers of the form q^{3n} , q^{3n+1} , and q^{3n+2} , we arrive at

$$\sum_{n \geq 0} \bar{S}_3(2n+1)q^{3n} = 2 \frac{f_{12}^2 f_{18}^4}{f_3^4 f_{36}^2}, \quad (3.90)$$

$$\sum_{n \geq 0} \bar{S}_3(2n+1)q^{3n+1} = 2q \frac{f_6^2 f_9^3 f_{12} f_{36}}{f_3^5 f_{18}^2}, \quad (3.91)$$

$$\sum_{n \geq 0} \bar{S}_3(2n+1)q^{3n+2} = 4q^2 \frac{f_6 f_{12} f_{18} f_{36}}{f_3^4}. \quad (3.92)$$

Thus, (3.87) follows from (3.90), (3.88) follows from (3.91), and (3.89) is obtained from (3.92). \square

Theorem 3.24. *We have*

$$\sum_{n \geq 0} \bar{S}_3(12n + 1)q^n = 2 \frac{f_2^{16} f_3^4}{f_1^{14} f_4^4 f_6^2}, \quad (3.93)$$

$$\sum_{n \geq 0} \bar{S}_3(12n + 5)q^n = 4 \frac{f_2^{15} f_3 f_6}{f_1^{13} f_4^4}, \quad (3.94)$$

$$\sum_{n \geq 0} \bar{S}_3(12n + 7)q^n = 8 \frac{f_2^4 f_3^4 f_4^4}{f_1^{10} f_6^2}, \quad (3.95)$$

$$\sum_{n \geq 0} \bar{S}_3(12n + 11)q^n = 16 \frac{f_2^3 f_3 f_4^4 f_6}{f_1^9}. \quad (3.96)$$

Proof. Employing (2.5) in both (3.87) and (3.89), we find that

$$\sum_{n \geq 0} \bar{S}_3(6n + 1)q^n = 2 \frac{f_4^{16} f_6^4}{f_2^{14} f_8^4 f_{12}^2} + 8q \frac{f_4^4 f_6^4 f_8^4}{f_2^{10} f_{12}^2}, \quad (3.97)$$

$$\sum_{n \geq 0} \bar{S}_3(6n + 5)q^n = 4 \frac{f_4^{15} f_6 f_{12}}{f_2^{13} f_8^4} + 16q \frac{f_4^3 f_6 f_8^4 f_{12}}{f_2^9}. \quad (3.98)$$

Extracting the terms involving the powers q^{2n} and q^{2n+1} from both sides of (3.97) and (3.98), we arrive at

$$\sum_{n \geq 0} \bar{S}_3(6n + 1)q^{2n} = 2 \frac{f_4^{16} f_6^4}{f_2^{14} f_8^4 f_{12}^2}, \quad (3.99)$$

$$\sum_{n \geq 0} \bar{S}_3(6n + 5)q^{2n} = 4 \frac{f_4^{15} f_6 f_{12}}{f_2^{13} f_8^4}, \quad (3.100)$$

$$\sum_{n \geq 0} \bar{S}_3(6n + 1)q^{2n+1} = 8q \frac{f_4^4 f_6^4 f_8^4}{f_2^{10} f_{12}^2}, \quad (3.101)$$

$$\sum_{n \geq 0} \bar{S}_3(6n + 5)q^{2n+1} = 16q \frac{f_4^3 f_6 f_8^4 f_{12}}{f_2^9}. \quad (3.102)$$

Thus, by substituting q^2 by q in both (3.99) and (3.100) we deduce (3.93) and (3.94), respectively. Similarly, (3.95) and (3.96) can be obtained from (3.101) and (3.102), by dividing both sides by q and q^2 and then replacing q^2 by q , respectively. \square

From the previous results, we deduce the following congruences.

Corollary 3.8. *For all $n \geq 0$,*

$$\begin{aligned}\bar{S}_3(6n + 5) &\equiv 0 \pmod{4}, \\ \bar{S}_3(12n + 7) &\equiv 0 \pmod{8}, \\ \bar{S}_3(12n + 11) &\equiv 0 \pmod{16}.\end{aligned}$$

CHAPTER 4

ON THE ARITHMETIC PROPERTIES OF PARTITIONS INTO PARTS SIMULTANEOUSLY 4-REGULAR AND 9-DISTINCT

In this chapter, we delve into the arithmetic behavior of partitions into parts simultaneously 4-regular and 9-distinct by establishing some families of congruences for $\text{RD}^{(4,9)}(n)$ modulo 4, 6, and 12, along with additional Ramanujan-like congruences modulo 24.

4.1 Congruences for $\text{RD}^{(4,9)}(n)$ modulo 4

Theorem 4.1. *For any prime $p \equiv 3 \pmod{4}$, $\alpha \geq 0$, and $n \geq 0$, we have*

$$\text{RD}^{(4,9)}\left(12p^{2\alpha+1}(pn+i) + p^{2\alpha+2} + 1\right) \equiv 0 \pmod{4},$$

for all $1 \leq i \leq p-1$.

Proof. In view of (2.2) and (3.56), with $p = k = 2$ and using (2.13) into (3.56), we find that

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(2n)q^n = \frac{f_6^7 f_9^7}{f_3^9 f_{18}^5} + 2q \frac{f_6^6 f_9^4}{f_3^8 f_{18}^2} + 4q^2 \frac{f_6^5 f_9 f_{18}}{f_3^7} \equiv \frac{f_6^7 f_9^7}{f_3^9 f_{18}^5} + 2q f_6^2 \pmod{4}.$$

If we extract the terms of the form q^{3n+1} from both sides of the above equation, we get

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n+2)q^n \equiv 2f_2^2 \pmod{4}.$$

That is,

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(12n+2)q^n \equiv 2f_1^2 \pmod{4}. \quad (4.1)$$

Define

$$\sum_{n \geq 0} a(n)q^n = f_1^2. \quad (4.2)$$

Combining Equations (4.1) and (4.2), we deduce that

$$RD^{(4,9)}(12n + 2) \equiv 2a(n) \pmod{4}. \quad (4.3)$$

Now, we consider the following congruence equation:

$$\frac{3k^2 + k}{2} + \frac{3m^2 + m}{2} \equiv \frac{p^2 - 1}{12} \pmod{p}.$$

It is equivalent to

$$(6k + 1)^2 + (6m + 1)^2 \equiv 0 \pmod{p}, \quad (4.4)$$

where $-(p-1)/2 \leq k, m \leq (p-1)/2$ and p is a prime such that $(\frac{-1}{p}) = -1$. Since $(\frac{-1}{p}) = -1$ for $p \equiv 3 \pmod{4}$, the congruence relation (4.4) holds if and only if both $k = m = (\pm p - 1)/6$. Substituting Equation (2.35) into (4.2) and then extracting the terms in which the powers of q are congruent to $(p^2 - 1)/12$ modulo p , and then divide by $q^{\frac{p^2-1}{12}}$, we find that

$$\sum_{n \geq 0} a\left(pn + \frac{p^2 - 1}{12}\right) q^{pn} = f_p^2.$$

Therefore,

$$\sum_{n \geq 0} a\left(p^2n + \frac{p^2 - 1}{12}\right) q^n = f_1^2,$$

and for all $n \geq 0$,

$$a\left(p^2n + pi + \frac{p^2 - 1}{12}\right) = 0, \quad (4.5)$$

where $1 \leq i \leq p - 1$. By induction we can prove that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha}n + \frac{p^{2\alpha} - 1}{12}\right) = a(n). \quad (4.6)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{12}$ ($1 \leq i \leq p - 1$) in Equation (4.6) and using (4.5), we find that for all nonnegative integers n and α ,

$$a\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2} - 1}{12}\right) = 0.$$

Finally, replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{12}$ in Equation (4.3) ($1 \leq i \leq p - 1$) we obtain the desired result. \square

4.2 Congruences for $\text{RD}^{(4,9)}(n)$ modulo 6

Theorem 4.2. For all integers $\alpha \geq 0$ and $n \geq 0$,

$$\text{RD}^{(4,9)}(6 \cdot 5^{2\alpha+2}n + 6 \cdot 5^{2\alpha+1}i + 5^{2\alpha+2} + 1) \equiv 0 \pmod{6},$$

where $i = 1, 2, 3, 4$.

Proof. In view of (2.2) and (3.60), with $p = 3$ and $k = 1$, we have

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n + 2)q^n = 2 \frac{f_2^6 f_3^4}{f_1^8 f_6^2} \equiv 2f_1^4 \pmod{6}. \quad (4.7)$$

Employing (2.33) into (4.7), and then extracting the terms of the form q^{5n+4} from both sides of the resulting equation, we get

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(30n + 26)q^n \equiv 2f_5^4 \pmod{6}, \quad (4.8)$$

which implies that

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(150n + 26)q^n \equiv 2f_1^4 \pmod{6}. \quad (4.9)$$

From (4.7) and (4.9), we find that

$$\text{RD}^{(4,9)}(6n + 2) \equiv \text{RD}^{(4,9)}(150n + 26) \pmod{6}.$$

Utilizing (4.9) and mathematical induction on $\alpha \geq 0$, we obtain

$$\text{RD}^{(4,9)}(6n + 2) \equiv \text{RD}^{(4,9)}(6 \cdot 5^{2\alpha+2}n + 5^{2\alpha+2} + 1) \pmod{6}. \quad (4.10)$$

From (4.8), we find that

$$\text{RD}^{(4,9)}(150n + 30i + 26) \equiv 0 \pmod{6}, \quad i = 1, 2, 3, 4. \quad (4.11)$$

Using Equations (4.10) and (4.11), we obtain the desired result. \square

4.3 Congruences for $\text{RD}^{(4,9)}(n)$ modulo 12

Theorem 4.3. For any prime $p \equiv 5 \pmod{6}$, $\alpha \geq 0$, and $n \geq 0$, we have

$$\text{RD}^{(4,9)}(6p^{2\alpha+1}(pn + i) + 3p^{2\alpha+2} + 1) \equiv 0 \pmod{12},$$

for all $1 \leq i \leq p - 1$.

Proof. In view of (2.2) and (3.61), with $p = 3$ and $k = 1$, we find that

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(6n + 4)q^n = 4 \frac{f_2^5 f_3 f_6}{f_1^7} \equiv 4 \frac{f_2^2 f_6^2}{f_1 f_3} = 4\psi(q)\psi(q^3) \pmod{12}. \quad (4.12)$$

Define

$$\sum_{n \geq 0} b(n)q^n = \psi(q)\psi(q^3). \quad (4.13)$$

From Equations (4.12) and (4.13), we have

$$\text{RD}^{(4,9)}(6n + 4) \equiv 4b(n) \pmod{12}. \quad (4.14)$$

Now, consider the congruence equation

$$\frac{k^2 + k}{2} + 3 \cdot \frac{m^2 + m}{2} \equiv \frac{4p^2 - 4}{8} \pmod{p}.$$

which is equivalent to

$$(2k + 1)^2 + 3 \cdot (2m + 1)^2 \equiv 0 \pmod{p}, \quad (4.15)$$

where $0 \leq k, m \leq (p - 1)/2$ and p is a prime number such that $\left(\frac{-3}{p}\right) = -1$. Since $\left(\frac{-3}{p}\right) = -1$ for $p \equiv 5 \pmod{6}$, the congruence relation of Equation (4.15) holds if and only if both $k = m = (p - 1)/2$. Substitute Equation (2.34) into (4.13) and extract the terms in which the powers of q are congruent to $(p^2 - 1)/2$ modulo p , and then divide by $q^{\frac{p^2-1}{2}}$, we find that

$$\sum_{n \geq 0} b\left(pn + \frac{p^2 - 1}{2}\right) q^{pn} = \psi(q^{p^2})\psi(q^{3p^2}),$$

which implies that

$$\sum_{n \geq 0} b\left(p^2n + \frac{p^2 - 1}{2}\right) q^n = \psi(q)\psi(q^3),$$

and for $n \geq 0$,

$$b\left(p^2n + pi + \frac{p^2 - 1}{2}\right) = 0, \quad (4.16)$$

where $1 \leq i \leq p - 1$. By induction, we obtain that for all $n, \alpha \geq 0$,

$$b\left(p^{2\alpha}n + \frac{p^{2\alpha} - 1}{2}\right) = b(n). \quad (4.17)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{2}$ ($1 \leq i \leq p - 1$) in (4.17) and using (4.16), we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$b\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2} - 1}{2}\right) = 0.$$

Replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{2}$ in Equation (4.14), we obtain the desired result. \square

4.4 Congruences for $RD^{(4,9)}(n)$ modulo 24

Theorem 4.4. For all $n \geq 0$,

$$RD^{(4,9)}(24n + 23) \equiv 0 \pmod{24}, \quad (4.18)$$

$$RD^{(4,9)}(48n + 29) \equiv 0 \pmod{24}, \quad (4.19)$$

$$RD^{(4,9)}(96n + 89) \equiv 0 \pmod{24}. \quad (4.20)$$

Proof. In view of (2.2) and (3.75), with $p = k = 2$, we obtain

$$\sum_{n \geq 0} RD^{(4,9)}(6n + 5)q^n = 6 \frac{f_2^2 f_3^2 f_6^2}{f_1^6} \equiv 6 \frac{f_3^2 f_6^2}{f_1^2} \pmod{24}.$$

Substituting (2.19) into the above equation, we get

$$\sum_{n \geq 0} RD^{(4,9)}(6n + 5)q^n \equiv 6 \frac{f_4^4 f_6^3 f_{12}^2}{f_2^5 f_8 f_{24}} + 12q \frac{f_4^4 f_6^4 f_8 f_{24}}{f_2^4 f_{12}} \pmod{24}. \quad (4.21)$$

If we extract the even and the odd powers from both sides of (4.21), we find that

$$\sum_{n \geq 0} RD^{(4,9)}(12n + 5)q^n \equiv 6 \frac{f_4^4 f_6^3 f_{12}^2}{f_1^5 f_4 f_{12}} \equiv 6 \frac{f_2^2 f_3^3 f_6^2}{f_1 f_4 f_{12}} \pmod{24}, \quad (4.22)$$

and

$$\sum_{n \geq 0} RD^{(4,9)}(12n + 11)q^n \equiv 12 \frac{f_2 f_3^4 f_4 f_{12}}{f_1^4 f_6} \equiv 12 \frac{f_4 f_6 f_{12}}{f_2} \pmod{24}. \quad (4.23)$$

Employing (2.20) in (4.22), we arrive at

$$\sum_{n \geq 0} RD^{(4,9)}(12n + 5)q^n \equiv 6 \frac{f_4^2 f_6^4}{f_{12}^2} + 6q \frac{f_2^2 f_6^2 f_{12}^2}{f_4^2} \pmod{24}. \quad (4.24)$$

Extracting the odd and the even powers from both sides of (4.24), we obtain

$$\sum_{n \geq 0} RD^{(4,9)}(24n + 5)q^n \equiv 6 \frac{f_2^2 f_3^4}{f_6^2} \equiv 6 f_2^2 \pmod{24}, \quad (4.25)$$

and

$$\sum_{n \geq 0} RD^{(4,9)}(24n + 17)q^n \equiv 6 \frac{f_1^2 f_3^2 f_6^2}{f_2^2} \equiv 6 \frac{f_3^2 f_6^2}{f_1^2} \pmod{24}. \quad (4.26)$$

Congruence (4.18) follows from (4.23) by collecting the terms involving q^{2n+1} from both sides, and congruence (4.19) follows from (4.25) by equating the odd powers on both sides.

If we substitute (2.19) into (4.26), and then extracting the odd powers from both sides of the resulting equation, we arrive at

$$\sum_{n \geq 0} \text{RD}^{(4,9)}(48n + 41)q^n \equiv 12 \frac{f_4 f_6 f_{12}}{f_2} \pmod{24}. \quad (4.27)$$

Congruence (4.20) is an immediate result from (4.27). □

CHAPTER 5

ARITHMETIC PROPERTIES OF PARTITIONS INTO PARTS SIMULTANEOUSLY BIREGULAR AND DISTINCT

5.1 Introduction

Recently, Keith [135] introduced the simultaneously ℓ_1 -regular, ℓ_2 -regular, and ℓ_1 -distinct partition function, in which none of the parts of the partition are divisible by ℓ_1 and ℓ_2 , and each part appears fewer than ℓ_1 times. Let $B_{\ell_1, \ell_2}^D(n)$ denotes the number of partitions of n into parts simultaneously ℓ_1 -regular, ℓ_2 -regular, and ℓ_1 -distinct, where $1 < \ell_1 < \ell_2$. The generating function of the sequence $B_{\ell_1, \ell_2}^D(n)$ is given by

$$\sum_{n \geq 0} B_{\ell_1, \ell_2}^D(n) q^n = \frac{f_{\ell_1} f_{\ell_2}}{f_1 f_{[\ell_1, \ell_2]}} \cdot \frac{f_{\ell_1} f_{\ell_1 \cdot [\ell_1, \ell_2]}}{f_{\ell_1^2} f_{\ell_1 \ell_2}}, \quad (5.1)$$

where $[\ell_1, \ell_2]$ is the least common multiple of the integers ℓ_1 and ℓ_2 . Drema and Saikia [86] proved some congruences modulo 2 and 4 for the partition function $B_{\ell_1, \ell_2}^D(n)$ for $(\ell_1, \ell_2) \in \{(2, 5), (2, 7)\}$. Continuing along this line, Buragohain and Saikia [59] proved some infinite families of congruences for the partition function $B_{\ell_1, \ell_2}^D(n)$ for the pairs $(3, 4)$, $(4, 9)$, $(5x, 5x)$, and $(7y, 7y)$, where x and y are two positive integers.

In this chapter, we extend the study of the generating function (5.1). In particular, we investigate the arithmetic properties of the partition function $B_{\ell_1, \ell_2}^t(n)$, which counts the number of partitions of n into parts indivisible by ℓ_1 and ℓ_2 , such that each part appears fewer than t times, where $\gcd(\ell_1, \ell_2) = 1$ and $\ell_1, \ell_2, t > 1$. The sequence $B_{\ell_1, \ell_2}^t(n)$ has the following generating function:

$$\sum_{n \geq 0} B_{\ell_1, \ell_2}^t(n) q^n = \frac{f_t f_{\ell_1} f_{\ell_2} f_{\ell_1 \ell_2 t}}{f_1 f_{t\ell_1} f_{t\ell_2} f_{\ell_1 \ell_2}}. \quad (5.2)$$

For example, for the triple $(\ell_1, \ell_2, t) = (2, 3, 5)$ we have that $B_{2,3}^5(20) = 9$ with the relevant partitions set given by

$$\{(19, 1), (17, 1^3), (13, 7), (13, 5, 1^2), (11, 7, 1^2), (11, 5, 1^4), (7^2, 5, 1), (7, 5^2, 1^3), (5^4)\}.$$

We give an infinite family of congruences modulo 2 for the sequence $B_{3,4}^2(n)$ by using several dissection identities to obtain the main results. Along with obtaining some congruences

modulo 3 and 4 for the sequence $B_{3,4}^3(n)$. Finally, we derive similar arithmetical results for the sequences $B_{4,9}^3(n)$ and $B_{4,9}^9(n)$. In some cases we use the Smoot's implementation [193] of Radu's Ramanujan-Kolberg algorithm to obtain some additional congruences.

5.2 Congruences modulo 2 for $B_{3,4}^2(n)$

In this section we establish an infinite family of congruences modulo 2 for the partition function $B_{3,4}^2(n)$. By setting $(\ell_1, \ell_2, t) = (3, 4, 2)$ in (5.2), we obtain the generating function

$$\sum_{n \geq 0} B_{3,4}^2(n)q^n = \frac{f_2 f_3 f_4 f_{24}}{f_1 f_6 f_8 f_{12}}. \quad (5.3)$$

Theorem 5.1. *For any odd prime p and any non-negative integers α and n , we have*

$$B_{3,4}^2 \left(2p^{2\alpha+1}(pn + i) + \frac{p^{2\alpha+2} - 1}{4} \right) \equiv 0 \pmod{2},$$

for all $1 \leq i \leq p - 1$.

Proof. Substituting Equation (2.18) into the generating function (5.3), we obtain

$$\sum_{n \geq 0} B_{3,4}^2(n)q^n = \frac{f_4^2 f_{16} f_{24}^3}{f_2 f_8^2 f_{12}^2 f_{48}} + q \frac{f_4 f_8 f_{48}}{f_2 f_{12} f_{16}}. \quad (5.4)$$

If we extract the even terms from both sides of Equation (5.4), and then replace q^2 by q , we have

$$\sum_{n \geq 0} B_{3,4}^2(2n)q^n = \frac{f_2^2 f_8 f_{12}^3}{f_1 f_4^2 f_6^2 f_{24}}. \quad (5.5)$$

By Theorem 2.1 and Equation (5.5), with $p = 2$ and $k = 1$, we obtain the congruence

$$\sum_{n \geq 0} B_{3,4}^2(2n)q^n \equiv \frac{f_2^2}{f_1} = \psi(q) \pmod{2}. \quad (5.6)$$

If

$$\sum_{n \geq 0} a(n)q^n = \psi(q), \quad (5.7)$$

then it is clear that

$$B_{3,4}^2(2n) \equiv a(n) \pmod{2}. \quad (5.8)$$

Now, consider the congruence equation

$$\frac{k^2 + k}{2} \equiv \frac{p^2 - 1}{8} \pmod{p}.$$

This congruence is equivalent to

$$(2k + 1)^2 \equiv 0 \pmod{p}, \quad (5.9)$$

where $0 \leq k \leq (p - 1)/2$ and p is an odd prime. The congruence relation (5.9) holds if and only if $k = (p - 1)/2$. Therefore, if we substitute (2.34) into (5.7), and then extract the terms in which the powers of q are $pn + (p^2 - 1)/8$, we arrive at

$$\sum_{n \geq 0} a \left(pn + \frac{p^2 - 1}{8} \right) q^{pn + \frac{p^2 - 1}{8}} = q^{\frac{p^2 - 1}{8}} \psi(q^{p^2}). \quad (5.10)$$

Dividing $q^{\frac{p^2 - 1}{8}}$ on both sides of Equation (5.10), and then replacing q^p by q , we find that

$$\sum_{n \geq 0} a \left(pn + \frac{p^2 - 1}{8} \right) q^n = \psi(q^p),$$

which implies that

$$\sum_{n \geq 0} a \left(p^2 n + \frac{p^2 - 1}{8} \right) q^n = \psi(q). \quad (5.11)$$

Moreover, for all $n \geq 0$ we have the equality

$$a \left(p^2 n + pi + \frac{p^2 - 1}{8} \right) = 0, \quad (5.12)$$

where i is an integer and $1 \leq i \leq p - 1$. Combining (5.7) and (5.11), we obtain the equality

$$a \left(p^2 n + \frac{p^2 - 1}{8} \right) = a(n). \quad (5.13)$$

By (5.13) and mathematical induction, we deduce that for $\alpha \geq 0$,

$$a \left(p^{2\alpha} n + \frac{p^{2\alpha} - 1}{8} \right) = a(n). \quad (5.14)$$

Replacing n by $p^2 n + pi + \frac{p^2 - 1}{8}$ ($1 \leq i \leq p - 1$) in Equation (5.14) and using (5.12), we deduce that for all non-negative integers n and α

$$a \left(p^{2\alpha+2} n + p^{2\alpha+1} i + \frac{p^{2\alpha+2} - 1}{8} \right) = 0.$$

Again, replacing n by $p^{2\alpha+2} n + p^{2\alpha+1} i + \frac{p^{2\alpha+2} - 1}{8}$ in (5.8), we obtain the desired result. \square

5.3 Congruences modulo 3 for $B_{3,4}^3(n)$

In this section we establish some infinite families of congruences modulo 3 for the partition function $B_{3,4}^3(n)$. By setting $(\ell_1, \ell_2, t) = (3, 4, 3)$ in (5.2), we obtain the generating function

$$\sum_{n \geq 0} B_{3,4}^3(n)q^n = \frac{f_3^2 f_4 f_{36}}{f_1 f_9 f_{12}^2}. \quad (5.15)$$

Theorem 5.2. For all non-negative integers α and n , we have

$$\sum_{n \geq 0} B_{3,4}^3(3n+1)q^n \equiv \psi(-q)\psi(-q^3) \pmod{3}, \quad (5.16)$$

$$B_{3,4}^3\left(3^{2\alpha+2}n + \frac{3^{2\alpha+2}-1}{2} + 3^{2\alpha+1}\right) \equiv 0 \pmod{3}, \quad (5.17)$$

$$B_{3,4}^3\left(3^{2\alpha+3}n + \frac{3^{2\alpha+3}-3}{2} + 3^{2\alpha+2} + 1\right) \equiv 0 \pmod{3}. \quad (5.18)$$

Proof. Substituting Equation (2.25) into the generating function (5.15), we obtain

$$\sum_{n \geq 0} B_{3,4}^3(n)q^n = \frac{f_{18}^4}{f_3 f_9 f_{12} f_{36}} + q \frac{f_6^2 f_9^2 f_{36}^2}{f_3^2 f_{12}^2 f_{18}^2} + 2q^2 \frac{f_6 f_{18} f_{36}^2}{f_3 f_9 f_{12}^2}. \quad (5.19)$$

If we extract the terms of the form q^{3n+1} , divide by q , and replace q^3 by q , we have

$$\sum_{n \geq 0} B_{3,4}^3(3n+1)q^n = \frac{f_2^2 f_3^2 f_{12}^2}{f_1^2 f_4^2 f_6^2} = \frac{\psi(-q^3)^2}{\psi(-q)^2}. \quad (5.20)$$

Applying (2.2) on (5.20), with $p = 3$ and $k = 1$, we find that

$$\sum_{n \geq 0} B_{3,4}^3(3n+1)q^n \equiv \psi(-q)^4 \equiv \psi(-q)\psi(-q^3) \pmod{3}. \quad (5.21)$$

Congruence (5.16) follows from Equation (5.21). Substituting Equation (2.14) into (5.21), we have

$$\sum_{n \geq 0} B_{3,4}^3(3n+1)q^n \equiv A(q^3)\psi(-q^3) - q\psi(-q^3)\psi(-q^9) \pmod{3}. \quad (5.22)$$

Collecting the powers of the form q^{3n+j} , for $j = 0, 1, 2$, from both sides of Equation (5.22), we have the congruences:

$$\sum_{n \geq 0} B_{3,4}^3(9n+1)q^n \equiv A(q)\psi(-q) \pmod{3}, \quad (5.23)$$

$$\sum_{n \geq 0} B_{3,4}^3(9n+4)q^n \equiv 2\psi(-q)\psi(-q^3) \pmod{3}, \quad (5.24)$$

$$B_{3,4}^3(9n+7) \equiv 0 \pmod{3}. \quad (5.25)$$

Again, by substituting Equation (2.14) into (5.24), and then collecting the terms involving q^{3n+j} for $j = 0, 1, 2$, we arrive at

$$\sum_{n \geq 0} B_{3,4}^3(27n + 4)q^n \equiv 2A(q)\psi(-q) \pmod{3}, \quad (5.26)$$

$$\sum_{n \geq 0} B_{3,4}^3(27n + 13)q^n \equiv \psi(-q)\psi(-q^3) \pmod{3}, \quad (5.27)$$

$$B_{3,4}^3(27n + 22) \equiv 0 \pmod{3}. \quad (5.28)$$

From Equations (5.21), (5.24), and (5.27) we deduce that

$$\begin{aligned} 2B_{3,4}^3(3n + 1) &\equiv B_{3,4}^3(9n + 4) \pmod{3}, \\ B_{3,4}^3(3n + 1) &\equiv B_{3,4}^3(27n + 13) \pmod{3}. \end{aligned}$$

From the last two equations and by mathematical induction on $\alpha \geq 0$, we obtain

$$2B_{3,4}^3(3n + 1) \equiv B_{3,4}^3 \left(3^{2\alpha+2}n + \frac{3^{2\alpha+2} - 1}{2} \right) \pmod{3}, \quad (5.29)$$

$$B_{3,4}^3(3n + 1) \equiv B_{3,4}^3 \left(3^{2\alpha+3}n + 12 \sum_{i=0}^{\alpha} 3^{2i} + 1 \right) \pmod{3}. \quad (5.30)$$

Using (5.25) and (5.29), we obtain Congruence (5.17). Similarly, using (5.28) and (5.30), we obtain Congruence (5.18). \square

We can prove the Congruences (5.25) and (5.28) by using the Smoot's implementation of Radu's Ramanujan-Kolberg algorithm. This implementation, known as the RaduRK package, relies on **4ti2**, a software package for algebraic, geometric, and combinatorial problems on linear spaces. For detailed instructions on installing these packages, readers can refer to [193].

Theorem 5.3. *For any nonnegative integer n , we have*

$$\begin{aligned} B_{3,4}^3(9n + 7) &\equiv 0 \pmod{3}, \\ B_{3,4}^3(27n + 22) &\equiv 0 \pmod{3}. \end{aligned}$$

Proof. The generating function in (5.15) can be described by setting $M = 36$, $r = (-1, 0, 2, 1, 0, -1, -2, 0, 1)$ in the Smoot's implementation. Additionally, if we take $m = 9$, guess $N = 12$, and take $j = 7$, then we have

$$\begin{aligned} \text{In[1]:=} & \text{RK}[12, 36, \{-1, 0, 2, 1, 0, -1, -2, 0, 1\}, 9, 7] \\ & \prod_{\delta | M} (q^\delta; q^\delta)_\infty^{r_\delta} = \sum_{n=0}^{\infty} a(n) q^n \\ & f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn + j') q^n = \sum_{g \in AB} g \cdot p_g(t) \end{aligned}$$

	N:	12
	{M, (r _δ) _{δ M} }	{36, {-1, 0, 2, 1, 0, -1, -2, 0, 1}}
	m:	9
	P _{m,r} (j):	{7}
Out[1]=	f ₁ (q):	$\frac{(q; q)_{\infty}^5 (q^4; q^4)_{\infty}^9 (q^6; q^6)_{\infty}^{12}}{q^3 (q^2; q^2)_{\infty}^8 (q^3; q^3)_{\infty}^3 (q^{12}; q^{12})_{\infty}^{15}}$
	t:	$\frac{(q^4; q^4)_{\infty}^4 (q^6; q^6)_{\infty}^2}{q (q^2; q^2)_{\infty}^2 (q^{12}; q^{12})_{\infty}^4}$
	AB:	{1}
	{p _g (t): g∈AB}	{3 - 3t - 3t ² + 3t ³ }
	Common Factor:	3

This gives us

$$\frac{f_1^5 f_4^9 f_6^{12}}{q^3 f_2^8 f_3^3 f_{12}^{15}} \sum_{n \geq 0} B_{3,4}^3(9n + 7)q^n = 3 - 3t - 3t^2 + 3t^3,$$

where $t = f_4^4 f_6^2 / (q f_2^2 f_{12}^4)$. Moreover, 3 is the greatest common divisor of the coefficients of the polynomial in the variable t . Therefore, we obtain the first congruence. The last congruence can be derived in a similar manner. □

Theorem 5.4. For any prime $p \equiv 5 \pmod{6}$, $\alpha \geq 0$, and $n \geq 0$, we have

$$B_{3,4}^3 \left(3p^{2\alpha+1}(pn + i) + \frac{3p^{2\alpha+2} - 1}{2} \right) \equiv 0 \pmod{3},$$

for all $1 \leq i \leq p - 1$.

Proof. Define

$$\sum_{n \geq 0} b(n)q^n = \psi(-q)\psi(-q^3). \tag{5.31}$$

From Equation (5.16) we have

$$B_{3,4}^3(3n + 1) \equiv b(n) \pmod{3}. \tag{5.32}$$

Now, consider the congruence equation

$$\frac{k^2 + k}{2} + 3 \cdot \frac{m^2 + m}{2} \equiv \frac{4p^2 - 4}{8} \pmod{p},$$

which is equivalent to

$$(2k + 1)^2 + 3 \cdot (2m + 1)^2 \equiv 0 \pmod{p}, \tag{5.33}$$

where $0 \leq k, m \leq (p-1)/2$ and p is a prime number such that $\left(\frac{-3}{p}\right) = -1$. Since $\left(\frac{-3}{p}\right) = -1$ for $p \equiv 5 \pmod{6}$, the congruence relation of Equation (5.33) holds if and only if both $k = m = (p-1)/2$. Substitute Equation (2.34) into (5.31) and extract the terms in which the powers of q are congruent to $(p^2-1)/12$ modulo p , and then divide by $q^{\frac{p^2-1}{12}}$, we find that

$$\sum_{n \geq 0} b \left(pn + \frac{p^2-1}{2} \right) q^{pn} = \psi(q^{p^2})\psi(q^{3p^2}),$$

which implies that

$$\sum_{n \geq 0} b \left(p^2n + \frac{p^2-1}{2} \right) q^n = \psi(q)\psi(q^3), \quad (5.34)$$

and for $n \geq 0$,

$$b \left(p^2n + pi + \frac{p^2-1}{2} \right) = 0, \quad (5.35)$$

where $1 \leq i \leq p-1$. By induction, we obtain that for all $n, \alpha \geq 0$,

$$b \left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{2} \right) = b(n). \quad (5.36)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{2}$ ($1 \leq i \leq p-1$) in (5.36) and using (5.35), we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$b \left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{2} \right) = 0.$$

Replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{2}$ in Equation (5.32), we obtain the desired result. \square

5.4 Congruences modulo 4 for $B_{3,4}^3(n)$

In this section we establish an infinite family of congruences modulo 4 for the partition function $B_{3,4}^3(n)$.

Theorem 5.5. For any prime $p \equiv 3 \pmod{4}$, $\alpha \geq 0$, and $n \geq 0$, we have

$$B_{3,4}^3 \left(6p^{2\alpha+1}(pn+i) + \frac{p^{2\alpha+2}-1}{2} \right) \equiv 0 \pmod{4},$$

for all $1 \leq i \leq p-1$.

Proof. Extracting the terms involving q^{3n} from both sides of Equation (5.19) and replace q^3 by q , we find that

$$\sum_{n \geq 0} B_{3,4}^3(3n)q^n = \frac{f_6^4}{f_1 f_3 f_4 f_{12}}. \quad (5.37)$$

Substituting Equation (2.23) in (5.37) and extracting the terms involving q^{2n} from both sides of the resulting equation, we obtain

$$\sum_{n \geq 0} B_{3,4}^3(6n)q^n = \frac{f_4^2 f_6^4}{f_1^2 f_2^2 f_{12}^2}. \quad (5.38)$$

In view of (2.2) and (5.38) with $p = k = 2$, we see that

$$\sum_{n \geq 0} B_{3,4}^3(6n)q^n = \frac{f_4^2 f_6^4}{f_1^2 f_2^2 f_{12}^2} \equiv f_1^2 \pmod{4}. \quad (5.39)$$

Define

$$\sum_{n \geq 0} c(n)q^n = f_1^2. \quad (5.40)$$

It is clear that

$$B_{3,4}^3(6n) \equiv c(n) \pmod{4}. \quad (5.41)$$

Consider the congruence equation

$$\frac{3k^2 + k}{2} + \frac{3m^2 + m}{2} \equiv \frac{p^2 - 1}{12} \pmod{p},$$

which is equivalent to

$$(6k + 1)^2 + (6m + 1)^2 \equiv 0 \pmod{p}, \quad (5.42)$$

where $-(p-1)/2 \leq k, m \leq (p-1)/2$ and p is a prime such that $p \equiv 3 \pmod{4}$. Since $\left(\frac{-1}{p}\right) = -1$, then the congruence relation (5.42) holds if and only if both $k = m = (\pm p - 1)/6$. Substitute (2.35) into (5.40) and extract the terms in which the powers of q are congruent to $(p^2 - 1)/12$ modulo p and then divide $q^{\frac{p^2-1}{12}}$, we have

$$\sum_{n \geq 0} c\left(pn + \frac{p^2 - 1}{12}\right) q^{pn} = f_{p^2},$$

which implies that for $n \geq 0$,

$$c\left(p^2n + \frac{p^2 - 1}{12}\right) = c(n),$$

and

$$c\left(p^2n + pi + \frac{p^2 - 1}{12}\right) = 0, \quad (5.43)$$

where i is an integer and $1 \leq i \leq p - 1$. By induction we see that for $n \geq 0$ and $\alpha \geq 0$,

$$c\left(p^{2\alpha}n + \frac{p^{2\alpha} - 1}{12}\right) = c(n). \quad (5.44)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{4}$ ($1 \leq i \leq p-1$) in (5.44) and using (5.43), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$c \left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{12} \right) = 0.$$

Replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{12}$ in (5.41) ($1 \leq i \leq p-1$) we get the desired result. \square

5.5 Congruences modulo 3 for $B_{4,9}^3(n)$

In this section we establish some Ramanujan-like and infinite families of congruences modulo 3 for the partition function $B_{4,9}^3(n)$. By setting $(\ell_1, \ell_2, t) = (4, 9, 3)$ in (5.2), we obtain the generating function

$$\sum_{n \geq 0} B_{4,9}^3(n)q^n = \frac{f_3 f_4 f_9 f_{108}}{f_1 f_{12} f_{27} f_{36}}. \quad (5.45)$$

Theorem 5.6. *For any nonnegative integer n , we have*

$$B_{4,9}^3(6n+4) \equiv 0 \pmod{3}, \quad (5.46)$$

$$\sum_{n \geq 0} B_{4,9}^3(18n+1)q^n \equiv f_1^4 \pmod{3}, \quad (5.47)$$

$$\sum_{n \geq 0} B_{4,9}^3(18n+7)q^n \equiv \psi(q)\psi(q^3) \pmod{3}, \quad (5.48)$$

$$B_{4,9}^3(18n+13) \equiv 0 \pmod{3}. \quad (5.49)$$

Proof. Substituting Equation (2.25) into (5.45), and extracting the terms involving q^{3n+1} from both sides of the resulting equation, we obtain

$$\sum_{n \geq 0} B_{4,9}^3(3n+1)q^n = \frac{f_2^2 f_3^4 f_{36}}{f_1^3 f_4 f_6^2 f_9}. \quad (5.50)$$

Applying (2.2) to (5.50), with $p = 3$ and $k = 1$, we find that

$$\sum_{n \geq 0} B_{4,9}^3(3n+1)q^n \equiv \frac{f_3 f_2^2 f_{36}}{f_1^3 f_4 f_6^2} \pmod{3}. \quad (5.51)$$

Substituting (2.22) into (5.51), we have

$$\sum_{n \geq 0} B_{4,9}^3(3n+1)q^n \equiv \frac{f_4^5 f_6 f_{36}}{f_2^7 f_{12}^2} + 3q \frac{f_4 f_{12}^2 f_{36}}{f_2^5 f_6} \pmod{3}. \quad (5.52)$$

Congruence (5.46) is an immediate result from the above equation.

If we extract the even powers from both sides of (5.52) and replace q^2 by q , we obtain

$$\sum_{n \geq 0} B_{4,9}^3(6n + 1)q^n \equiv \frac{f_2^5 f_3 f_{18}}{f_1^7 f_6^2} \equiv \frac{f_2^2 f_{18}}{f_1 f_3 f_6} \pmod{3}. \quad (5.53)$$

Substituting (2.14) into (5.53), we arrive at

$$\sum_{n \geq 0} B_{4,9}^3(6n + 1)q^n \equiv \frac{f_9^2}{f_3^2} + q \frac{f_{18}^3}{f_3 f_6 f_9} \pmod{3}. \quad (5.54)$$

Congruence (5.49) follows from the above equation by extracting the powers of the form q^{3n+2} .

Extracting the terms involving q^{3n} from both sides of (5.54) and then replace q^3 by q , we have

$$\sum_{n \geq 0} B_{4,9}^3(18n + 1)q^n \equiv \frac{f_3^2}{f_1^2} \equiv f_1^4 \pmod{3}. \quad (5.55)$$

Collecting the powers of the form q^{3n+1} from both sides of Equation (5.54), divide by q , and then replace q^3 by q , we obtain

$$\sum_{n \geq 0} B_{4,9}^3(18n + 7)q^n \equiv \frac{f_6^3}{f_1 f_2 f_3} \equiv \frac{f_2^2 f_6^2}{f_1 f_3} \pmod{3}. \quad (5.56)$$

Congruence (5.48) follows from (5.56) by using the definition of $\psi(q)$. □

Theorem 5.7. For all nonnegative integers α and n , we have

$$B_{4,9}^3(18 \cdot 5^{2\alpha+2}n + 18 \cdot 5^{2\alpha+1}i + 3 \cdot 5^{2\alpha+2} - 2) \equiv 0 \pmod{3},$$

for all $1 \leq i \leq p - 1$.

Proof. Substituting Equation (2.33) into (5.47) and extracting the terms involving q^{5n+4} , we obtain

$$\sum_{n \geq 0} B_{4,9}^3(90n + 73)q^n \equiv f_5^4 \pmod{3}, \quad (5.57)$$

which implies,

$$\sum_{n \geq 0} B_{4,9}^3(450n + 73)q^n \equiv f_1^4 \pmod{3}. \quad (5.58)$$

From Equations (5.47) and (5.58), we find that

$$B_{4,9}^3(18n + 1) \equiv B_{4,9}^3(450n + 73) \pmod{3}. \quad (5.59)$$

From (5.59) and by mathematical induction on $\alpha \geq 0$, we obtain

$$B_{4,9}^3(18n + 1) \equiv B_{4,9}^3(18 \cdot 5^{2\alpha+2}n + 3 \cdot 5^{2\alpha+2} - 2) \pmod{3}. \quad (5.60)$$

From (5.57), we have

$$B_{4,9}^3(450n + 90i + 73) \equiv 0 \pmod{3}, \quad i = 1, 2, 3, 4. \quad (5.61)$$

Using (5.60) and (5.61), we obtain the desired result. \square

Theorem 5.8. For any prime $p \equiv 5 \pmod{6}$, $\alpha \geq 0$ and $n \geq 0$, we have

$$B_{4,9}^3(18p^{2\alpha+1}(pn + i) + 9p^{2\alpha+2} - 2) \equiv 0 \pmod{3},$$

for all $1 \leq i \leq p - 1$.

Proof. Define

$$\sum_{n \geq 0} e(n)q^n = \psi(q)\psi(q^3). \quad (5.62)$$

Combining Equations (5.48) and (5.62), we find that

$$B_{4,9}^3(18n + 7) \equiv e(n) \pmod{3}. \quad (5.63)$$

By using the proof of Theorem 5.4, and replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{2}$ in (5.63), we obtain the desired result. \square

5.6 Congruences modulo 4 for $B_{4,9}^3(n)$

In this section we establish some Ramanujan-like congruences modulo 4 for the partition function $B_{4,9}^3(n)$.

Theorem 5.9. For any nonnegative integer n , we have

$$B_{4,9}^3(18n + 10) \equiv 0 \pmod{4}, \quad (5.64)$$

$$B_{4,9}^3(18n + 16) \equiv 0 \pmod{4}. \quad (5.65)$$

Proof. From (2.2) and Equation (5.50), with $p = k = 2$, we see that

$$\sum_{n \geq 0} B_{4,9}^3(3n + 1)q^n = \frac{f_2^2 f_3^4 f_{36}}{f_1^3 f_4 f_6^2 f_9} \equiv \frac{f_1 f_{36}}{f_4 f_9} \pmod{4}. \quad (5.66)$$

Substituting (2.26) into (5.66), we have

$$\sum_{n \geq 0} B_{4,9}^3(3n + 1)q^n \equiv \frac{f_6 f_{18} f_{36}}{f_{12}^3} - q \frac{f_3 f_{18}^4 f_{36}}{f_9^3 f_{12}^3} - q^2 \frac{f_6^2 f_{36}^4}{f_{12}^4 f_{18}^2} \pmod{4}. \quad (5.67)$$

If we extract the terms of the form q^{3n} from both sides of (5.67) and then replace q^3 by q , we obtain

$$\sum_{n \geq 0} B_{4,9}^3(9n + 1)q^n \equiv \frac{f_2 f_6 f_{12}}{f_4^3} \pmod{4}. \quad (5.68)$$

Congruence (5.64) follows from the above equation by extracting the odd powers. If we collect the powers of the form q^{3n+2} from both sides of (5.67), divide by q^2 and then replace q^3 by q , we have

$$\sum_{n \geq 0} B_{4,9}^3(9n + 7)q^n \equiv 3 \frac{f_2^2 f_{12}^4}{f_4^4 f_6^2} \pmod{4}. \tag{5.69}$$

Congruence (5.65) follows from the above equation by extracting the odd powers. □

We can refine the previous congruences by using the Smoot's implementation of Radu's Ramanujan-Kolberg algorithm.

Theorem 5.10. *For any nonnegative integer n , we have*

$$\begin{aligned} B_{4,9}^3(18n + 10) &\equiv 0 \pmod{12}, \\ B_{4,9}^3(18n + 16) &\equiv 0 \pmod{36}. \end{aligned}$$

Proof. The generating function in (5.6) can be described by setting $M = 108$, $r = (-1, 0, 1, 1, 0, 1, -1, 0, -1, -1, 0, 1)$ in the Smoot's implementation. Additionally, if we take $m = 18$, guess $N = 12$, and take $j = 10$, then we have

In[2]:=	RK[12, 108, {-1, 0, 1, 1, 0, 1, -1, 0, -1, -1, 0, 1}, 18, 10]
	$\prod_{\delta M} (q^\delta; q^\delta)_\infty^{r_\delta} = \sum_{n=0}^{\infty} a(n) q^n$
	$f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn + j') q^n = \sum_{g \in AB} g \cdot p_g(t)$
	N: 12
	{M, (r _δ) _{δ M} }: {108, {-1, 0, 1, 1, 0, 1, -1, 0, -1, -1, 0, 1}}
	m: 18
	P _{m,r} (j): {10}
Out[2]=	f ₁ (q): $\frac{(q; q)_\infty^8 (q^4; q^4)_\infty^8 (q^6; q^6)_\infty^{12}}{q^7 (q^2; q^2)_\infty^4 (q^{12}; q^{12})_\infty^{24}}$
	t: $\frac{(q^4; q^4)_\infty^4 (q^6; q^6)_\infty^2}{q (q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty^4}$
	AB: {1}
	{p _g (t) : g ∈ AB} {108 t - 72 t ² - 396 t ³ - 48 t ⁴ + 276 t ⁵ + 120 t ⁶ + 12 t ⁷ }
	Common Factor: 12

This gives us

$$\frac{f_1^8 f_4^8 f_6^{12}}{q^7 f_2^4 f_{12}^{24}} \sum_{n \geq 0} B_{4,9}^3(18n + 10)q^n = 108t - 72t^2 - 396t^3 - 48t^4 + 276t^5 + 120t^6 + 12t^7,$$

where $t = f_4^4 f_6^2 / (q f_2^2 f_{12}^4)$. Moreover, 12 is the greatest common divisor of the coefficients of the polynomial in the variable t . Therefore, we obtain the first congruence. The last congruence can be derived in a similar manner. \square

5.7 Congruences modulo 3 for $B_{4,9}^9(n)$

In this section we establish some infinite families of congruences modulo 3 along with several Ramanujan-like congruences for the partition function $B_{4,9}^9(n)$. By setting $(\ell_1, \ell_2, t) = (4, 9, 9)$ in (5.2), we obtain the generating function

$$\sum_{n \geq 0} B_{4,9}^9(n) q^n = \frac{f_4 f_9^2 f_{324}}{f_1 f_{36}^2 f_{81}}. \quad (5.70)$$

Theorem 5.11. *For any nonnegative integer n , we have*

$$B_{4,9}^9(9n + 7) \equiv 0 \pmod{3}, \quad (5.71)$$

$$B_{4,9}^9(18n + 10) \equiv 0 \pmod{3}, \quad (5.72)$$

$$B_{4,9}^9(54n + 37) \equiv 0 \pmod{3}, \quad (5.73)$$

$$\sum_{n \geq 0} B_{4,9}^9(162n + 19) \equiv f_1^4 \pmod{3}, \quad (5.74)$$

$$\sum_{n \geq 0} B_{4,9}^9(162n + 73) \equiv 4\psi(q)\psi(q^3) \pmod{3}, \quad (5.75)$$

$$B_{4,9}^9(162n + 127) \equiv 0 \pmod{3}. \quad (5.76)$$

Proof. Substituting (2.25) in the generating function (5.70) and extracting the powers of the form q^{3n+1} from both sides of the resulting equation, we obtain

$$\sum_{n \geq 0} B_{4,9}^9(3n + 1) q^n = \frac{f_2^2 f_3^5 f_{108}}{f_1^4 f_6^2 f_{12} f_{27}}. \quad (5.77)$$

In view of (2.2) and Equation (5.77), with $p = 3$ and $k = 1$, we find that

$$\sum_{n \geq 0} B_{4,9}^9(3n + 1) q^n \equiv \frac{f_2^2 f_3^4 f_{108}}{f_1 f_6^2 f_{12} f_{27}} \pmod{3}. \quad (5.78)$$

Substituting (2.16) into (5.78), we have

$$\sum_{n \geq 0} B_{4,9}^9(3n + 1) q^n \equiv \frac{f_3^3 f_9^2 f_{108}}{f_6 f_{12} f_{18} f_{27}} + q \frac{f_3^4 f_{18}^2 f_{108}}{f_6^2 f_9 f_{12} f_{27}} \pmod{3}. \quad (5.79)$$

Congruence (5.71) follows from the above equation by collecting the terms of the form q^{3n+2} from both sides.

Collecting the terms containing q^{3n} from (5.79) and then replacing q^3 by q , we obtain

$$\sum_{n \geq 0} B_{4,9}^9(9n+1)q^n \equiv \frac{f_1^3 f_3^2 f_{36}}{f_2 f_4 f_6 f_9} \pmod{3}. \quad (5.80)$$

Collecting the terms containing q^{3n+1} from (5.79), dividing by q and replacing q^3 by q , we obtain

$$\sum_{n \geq 0} B_{4,9}^9(9n+4)q^n \equiv \frac{f_1^4 f_6^2 f_{36}}{f_2^2 f_3 f_4 f_9} \pmod{3}. \quad (5.81)$$

From (5.80), we see that

$$\sum_{n \geq 0} B_{4,9}^9(9n+1)q^n \equiv \frac{f_{36}}{f_2 f_4 f_6} \pmod{3}. \quad (5.82)$$

Congruence (5.72) follows from the above equation by collecting the odd powers. If we extract the even powers from (5.82) along with invoking (2.10), we find that

$$\sum_{n \geq 0} B_{4,9}^9(18n+1)q^n \equiv \frac{f_9^9}{f_3^7 f_6^2 f_{18}^2} + q \frac{f_9^6 f_{18}}{f_3^6 f_6^3} + 3q^2 \frac{f_9^3 f_{18}^4}{f_3^5 f_6^4} - 2q^3 \frac{f_{18}^7}{f_3^4 f_6^5} + 4q^4 \frac{f_{18}^{10}}{f_3^3 f_6^6 f_9^3} \pmod{3}. \quad (5.83)$$

Congruence (5.73) follows from (5.83) by extracting the terms of the form q^{3n+2} . If we collect the powers of the form q^{3n+1} from both sides of the equation (5.83), we find that

$$\sum_{n \geq 0} B_{4,9}^9(54n+19)q^n \equiv \frac{f_3^6 f_6}{f_1^6 f_2^3} + 4q \frac{f_6^{10}}{f_1^3 f_2^6 f_3^3} \equiv f_3^4 + 4q \frac{f_6^8}{f_3^4} \pmod{3}. \quad (5.84)$$

Congruences (5.74) and (5.76) follow from (5.84) by extracting the terms involving q^{3n+j} for $j = 0, 2$. Equating the powers of the form q^{3n+1} on both sides of (5.84), we get

$$\sum_{n \geq 0} B_{4,9}^9(162n+73)q^n \equiv 4 \frac{f_2^8}{f_1^4} \equiv 4 \frac{f_2^2 f_6^2}{f_1 f_3} \pmod{3}. \quad (5.85)$$

Congruence (5.75) follows from the above equation by using the definition of $\psi(q)$. □

Theorem 5.12. For all $\alpha \geq 0$ and $n \geq 0$, we have

$$B_{4,9}^9(162 \cdot 5^{2\alpha+2}n + 162 \cdot 5^{2\alpha+1}i + 27 \cdot 5^{2\alpha+2} - 8) \equiv 0 \pmod{3}. \quad (5.86)$$

for all $1 \leq i \leq p-1$.

Proof. Employing (2.33) in (5.74) and extracting the terms of the form q^{5n+4} , we arrive at

$$\sum_{n \geq 0} B_{4,9}^9(810n+667)q^n \equiv f_5^4 \pmod{3}, \quad (5.87)$$

which implies,

$$\sum_{n \geq 0} B_{4,9}^9(4050n + 667)q^n \equiv f_1^4 \pmod{3}. \quad (5.88)$$

From (5.74) and (5.88), we find that

$$B_{4,9}^9(162n + 19) \equiv B_{4,9}^9(4050n + 667) \pmod{3}. \quad (5.89)$$

Using (5.89) and by mathematical induction on $\alpha \geq 0$, we obtain

$$B_{4,9}^9(162n + 19) \equiv B_{4,9}^9(162 \cdot 5^{2\alpha+2}n + 27 \cdot 5^{2\alpha+2} - 8) \pmod{3}. \quad (5.90)$$

From (5.87), we have

$$B_{4,9}^9(4050n + 810i + 667) \equiv 0 \pmod{3}, \quad i = 1, 2, 3, 4. \quad (5.91)$$

Now it is clear the desired result. \square

Similarly, we have the following result.

Theorem 5.13. *For any prime $p \equiv 5 \pmod{6}$, $\alpha \geq 0$ and $n \geq 0$, we have*

$$B_{4,9}^9(162p^{2\alpha+1}(pn + i) + 81p^{2\alpha+2} - 8) \equiv 0 \pmod{3},$$

where i is an integer and $1 \leq i \leq p - 1$.

5.8 Congruences modulo 4 for $B_{4,9}^9(n)$

In this section we establish some congruences modulo 4 for the partition function $B_{4,9}^9(n)$.

Theorem 5.14. *For any nonnegative integer n , we have*

$$B_{4,9}^9(9n + 4) \equiv 0 \pmod{4}, \quad (5.92)$$

$$B_{4,9}^9(9n + 7) \equiv 0 \pmod{4}, \quad (5.93)$$

$$B_{4,9}^9(54n + 28) \equiv 0 \pmod{4}, \quad (5.94)$$

$$B_{4,9}^9(54n + 46) \equiv 0 \pmod{4}, \quad (5.95)$$

$$B_{4,9}^9(216n + 10) \equiv 3f_1^2 \pmod{4}, \quad (5.96)$$

$$B_{4,9}^9(216n + 118) \equiv 0 \pmod{4}. \quad (5.97)$$

Proof. In view of (2.2) and (5.77) with $p = k = 2$, we see that

$$\sum_{n \geq 0} B_{4,9}^9(3n + 1)q^n = \frac{f_2^2 f_3^5 f_{108}}{f_1^4 f_6^2 f_{12} f_{27}} \equiv \frac{f_3 f_{108}}{f_{12} f_{27}} \pmod{4}. \quad (5.98)$$

Congruences (5.92) and (5.93) follow from (5.98) by collecting the powers of the form q^{3n+j} for $j = 1, 2$.

Extracting the terms of the form q^{3n} from both sides of (5.98) along with substituting (2.26), we arrive at

$$\sum_{n \geq 0} B_{4,9}^9(9n + 1)q^n \equiv \frac{f_6 f_{18} f_{36}}{f_{12}^3} - q \frac{f_3 f_{18}^4 f_{36}}{f_9^3 f_{12}^3} - q^2 \frac{f_6^2 f_{36}^4}{f_{12}^4 f_{18}^2} \pmod{4}. \quad (5.99)$$

Collecting the terms of the form q^{3n+j} for $j = 0, 2$ from both sides of (5.99), we have

$$\sum_{n \geq 0} B_{4,9}^9(27n + 1)q^n \equiv \frac{f_2 f_6 f_{12}}{f_4^3} \pmod{4}, \quad (5.100)$$

$$\sum_{n \geq 0} B_{4,9}^9(27n + 19)q^n \equiv 3 \frac{f_2^2 f_{12}^4}{f_4^4 f_6^2} \pmod{4}. \quad (5.101)$$

Congruences (5.94) and (5.95) follow from equations (5.100) and (5.101) by extracting the odd powers.

By (2.2) and Equation (5.77), with $p = k = 2$, we find that

$$\sum_{n \geq 0} B_{4,9}^9(9n + 1)q^n \equiv \frac{f_1 f_{36}}{f_9 f_4} \pmod{4}. \quad (5.102)$$

Employing (2.32) into (5.102) and extracting the odd powers from both sides of the resulting equation, we arrive at

$$\sum_{n \geq 0} B_{4,9}^9(18n + 10)q^n \equiv 3 \frac{f_3 f_{18}^3}{f_6 f_9^3} \pmod{4}. \quad (5.103)$$

Collecting the terms containing q^{3n} from (5.103), we have

$$\sum_{n \geq 0} B_{4,9}^9(54n + 10)q^n \equiv 3 \frac{f_1 f_6^3}{f_3^3 f_2} \pmod{4}. \quad (5.104)$$

Substituting (2.21) into (5.104) and extracting the even powers, we find that

$$\sum_{n \geq 0} B_{4,9}^9(108n + 10)q^n \equiv 3 \frac{f_2^2 f_6^2}{f_3^4} \equiv 3f_2^2 \pmod{4}. \quad (5.105)$$

Congruences (5.96) and (5.97) are true from the above equation. \square

As in the proof of **Theorem 5.5**, by using (5.96), we have the following result.

Theorem 5.15. For any prime $p \equiv 3 \pmod{4}$, $\alpha \geq 0$ and $n \geq 0$, we have

$$B_{4,9}^9 \left(216p^{2\alpha+1}(pn + i) + 18p^{2\alpha+2} - 8 \right) \equiv 0 \pmod{4}, \quad (5.106)$$

for all $1 \leq i \leq p - 1$.

Corollary 5.1. For all $n \geq 0$, we have $B_{4,9}^9(9n + 7) \equiv 0 \pmod{12}$.

CHAPTER 6

ARITHMETIC PROPERTIES OF BIREGULAR OVERPARTITIONS

6.1 Introduction

Inspired by the various framework on overpartitions, ℓ -regular overpartitions, Andrews' singular overpartitions [11], and particularly the concept of t -Schur's overpartitions (which also satisfy the definition of $(2, t)$ -biregular overpartitions for any odd $t \geq 3$), in this chapter we investigate biregular overpartitions as a generalization. For a pair of coprime integers $\ell_1, \ell_2 > 1$ ($\gcd(\ell_1, \ell_2) = 1$), an (ℓ_1, ℓ_2) -biregular overpartition of n is a partition in which none of the parts are divisible by ℓ_1 and ℓ_2 . Let $\bar{B}_{\ell_1, \ell_2}(n)$ denote the number of (ℓ_1, ℓ_2) -biregular overpartitions of n . The generating function for the sequence $\bar{B}_{\ell_1, \ell_2}(n)$ is given by

$$\begin{aligned} \sum_{n \geq 0} \bar{B}_{\ell_1, \ell_2}(n)q^n &= \prod_{n \geq 1} \frac{(1+q^n)(1+q^{\ell_1 \ell_2 n})(1-q^{\ell_1 n})(1-q^{\ell_2 n})}{(1+q^{\ell_1 n})(1+q^{\ell_2 n})(1-q^n)(1-q^{\ell_1 \ell_2 n})} \\ &= \frac{f_2 f_{\ell_1}^2 f_{\ell_2}^2 f_{2\ell_1 \ell_2}}{f_1^2 f_{2\ell_1} f_{2\ell_2} f_{\ell_1 \ell_2}^2}. \end{aligned} \quad (6.1)$$

In Section 6.2, we prove several congruences for the sequence $\bar{B}_{4,3}(n)$. We use several dissection identities to obtain the main results. Additionally, we use the Smoot's implementation [193] of Radu's Ramanujan-Kolberg algorithm to obtain some additional congruences. Finally, in Sections 6.3, 6.4, and 6.5 we obtain similar results for the pairs $(4, 9)$, $(8, 3)$, and $(8, 9)$.

6.2 Congruences for $\bar{B}_{4,3}(n)$

In this section, we establish several congruences for the sequence $\bar{B}_{4,3}(n)$. By setting $(\ell_1, \ell_2) = (4, 3)$ in (6.1), we obtain the generating function

$$\sum_{n \geq 0} \bar{B}_{4,3}(n)q^n = \frac{f_2 f_3^2 f_4^2 f_{24}}{f_1^2 f_6 f_8 f_{12}^2}. \quad (6.2)$$

Theorem 6.1. For all integers $n \geq 0$, we have

$$\bar{B}_{4,3}(4n + 2) \equiv 0 \pmod{4}, \quad (6.3)$$

$$\bar{B}_{4,3}(8n + 4) \equiv 0 \pmod{8}, \quad (6.4)$$

$$\bar{B}_{4,3}(8n + 6) \equiv 0 \pmod{16}, \quad (6.5)$$

$$\bar{B}_{4,3}(16n + 8) \equiv 0 \pmod{32}. \quad (6.6)$$

Proof. First, substituting Equation (2.20) into (6.2), we obtain the generating function

$$\sum_{n \geq 0} \bar{B}_{4,3}(n)q^n = \frac{f_4^6}{f_2^4 f_8^2} + 2q \frac{f_4^3 f_6 f_{24}^2}{f_2^3 f_3^3}. \quad (6.7)$$

Extracting the terms involving q^{2n} from both sides of (6.7) and then replacing q^2 by q , we obtain

$$\sum_{n \geq 0} \bar{B}_{4,3}(2n)q^n = \frac{f_2^6}{f_1^4 f_4^2}. \quad (6.8)$$

Similarly, by extracting the terms of the form q^{2n+1} from both sides of Equation (6.7), dividing by q , and then replacing q^2 by q , we have

$$\sum_{n \geq 0} \bar{B}_{4,3}(2n + 1)q^n = 2 \frac{f_2^3 f_3 f_{12}^2}{f_1^3 f_6^3}. \quad (6.9)$$

Using the 2-dissection of Equation (2.5) into (6.8), we find that

$$\sum_{n \geq 0} \bar{B}_{4,3}(2n)q^n = \frac{f_4^{12}}{f_2^8 f_8^4} + 4q \frac{f_8^4}{f_2^4}. \quad (6.10)$$

Equating the even and odd powers on both sides of Equation (6.10), we have

$$\sum_{n \geq 0} \bar{B}_{4,3}(4n)q^n = \frac{f_2^{12}}{f_1^8 f_4^4}, \quad (6.11)$$

$$\sum_{n \geq 0} \bar{B}_{4,3}(4n + 2)q^n = 4 \frac{f_4^4}{f_1^4}. \quad (6.12)$$

Note that Equation (6.12) implies Congruence (6.3). Substituting Equation (2.17) into (6.9), we obtain the following expression:

$$\sum_{n \geq 0} \bar{B}_{4,3}(2n + 1)q^n = 2 \frac{f_{12}^2}{f_6^2} + 6q \frac{f_6 f_9^5 f_{12}^2}{f_7^7 f_{18}} + 12q^2 \frac{f_9^2 f_{12}^2 f_{18}^2}{f_3^6} + 24q^3 \frac{f_{12}^2 f_{18}^5}{f_3^5 f_6 f_9}. \quad (6.13)$$

Substituting Equation (2.6) into (6.11), we obtain that

$$\sum_{n \geq 0} \bar{B}_{4,3}(4n)q^n = \frac{f_4^{24}}{f_2^{16} f_8^8} + 8q \frac{f_4^{12}}{f_2^{12}} + 16q^2 \frac{f_8^8}{f_2^8}. \quad (6.14)$$

After extracting the terms containing q^{2n+j} , for $j = 0, 1$, from both sides of the above equation and simplification, we have

$$\sum_{n \geq 0} \bar{B}_{4,3}(8n)q^n = \frac{f_2^{24}}{f_1^{16} f_4^8} + 16q \frac{f_4^8}{f_1^8}, \quad (6.15)$$

$$\sum_{n \geq 0} \bar{B}_{4,3}(8n+4)q^n = 8 \frac{f_2^{12}}{f_1^{12}}. \quad (6.16)$$

Substituting (2.5) into (6.12), bringing out the term containing q^{2n+1} from both sides of the resulting equation, we find that

$$\sum_{n \geq 0} \bar{B}_{4,3}(8n+6)q^n = 16 \frac{f_2^6 f_4^4}{f_1^{10}}. \quad (6.17)$$

Note that Equations (6.16) and (6.17) imply (6.4) and (6.5), respectively. Finally, substituting (2.5) into (6.15), we arrive at

$$\sum_{n \geq 0} \bar{B}_{4,3}(8n)q^n = \frac{f_4^{48}}{f_2^{32} f_8^{16}} + 32q \frac{f_4^{36}}{f_2^{28} f_8^8} + 224q^2 \frac{f_4^{24}}{f_2^{24}} + 512q^3 \frac{f_4^{12} f_8^8}{f_2^{20}} + 256q^4 \frac{f_8^{16}}{f_2^{16}}. \quad (6.18)$$

Extracting the terms in (6.18) that involves q^{2n} and then replacing q^2 by q , we find that

$$\sum_{n \geq 0} \bar{B}_{4,3}(16n)q^n = \frac{f_2^{48}}{f_1^{32} f_4^{16}} + 224q \frac{f_2^{24}}{f_1^{24}} + 256q^2 \frac{f_4^{16}}{f_1^{16}}. \quad (6.19)$$

Also, if we extract the terms involving q^{2n+1} from both sides of Equation (6.18), divide by q and replace q^2 by q , we obtain

$$\sum_{n \geq 0} \bar{B}_{4,3}(16n+8)q^n = 32 \frac{f_2^{36}}{f_1^{28} f_4^8} + 512q \frac{f_2^{12} f_4^8}{f_1^{20}}. \quad (6.20)$$

Congruence (6.6) follows from (6.20). \square

If we extract the terms of the form q^{3n+j} for $j = 1, 2$ from Equation (6.13) we obtain the following corollary.

Corollary 6.1. For all integers $n \geq 0$, we have

$$\begin{aligned} \bar{B}_{4,3}(6n + 3) &\equiv 0 \pmod{6}, \\ \bar{B}_{4,3}(6n + 5) &\equiv 0 \pmod{12}. \end{aligned}$$

We can expand upon the congruences provided in **Theorem 6.1** by using the Smoot’s implementation of Radu’s Ramanujan-Kolberg algorithm. This implementation, known as the RaduRK package, relies on **4ti2**, a software package for algebraic, geometric, and combinatorial problems on linear spaces. For detailed instructions on installing these packages, readers can refer to [193]. In this section, we use the RaduRK software to prove **Theorem 6.2**.

Theorem 6.2. For all integers $n \geq 0$, we have

$$\begin{aligned} \bar{B}_{4,3}(32n + 16) &\equiv 0 \pmod{2^8}, \\ \bar{B}_{4,3}(64n + 32) &\equiv 0 \pmod{2^{11}}, \\ \bar{B}_{4,3}(128n + 64) &\equiv 0 \pmod{2^{14}}, \\ \bar{B}_{4,3}(256n + 128) &\equiv 0 \pmod{2^{17}}, \\ \bar{B}_{4,3}(512n + 256) &\equiv 0 \pmod{2^{20}}, \\ \bar{B}_{4,3}(1024n + 512) &\equiv 0 \pmod{2^{23}}. \end{aligned}$$

Proof. The generating function in (6.8) can be described by setting $M = 4, r = (-4, 6, -2)$ in the Smoot’s implementation. Additionally, if we take $m = 16$, guess $N = 4$, and take $j = 8$, then we have

$$\begin{aligned} \text{In[3]} := & \text{RK}[4, 4, \{-4, 6, -2\}, 16, 8] \\ & \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \sum_{n=0}^\infty a(n) q^n \\ & f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^\infty a(mn + j') q^n = \sum_{g \in AB} g \cdot p_g(t) \end{aligned}$$

N:	4
{M, (r _δ) _{δ M} }:	{4, {-4, 6, -2}}
m:	16
P _{m,r} (j):	{8}
f ₁ (q):	$\frac{(q; q)_m^{28} (q^2; q^2)_m^{12}}{q^4 (q^4; q^4)_m^{48}}$
t:	$\frac{(q; q)_m^8}{q (q^4; q^4)_m^8}$
AB:	{1}
{p _g (t) : g ∈ AB}	{536 870 912 + 83 886 080 t + 4 325 376 t ² + 77 824 t ³ + 256 t ⁴ }
Common Factor:	256

Out[3]=

This gives us

$$\frac{f_1^{28} f_2^{12}}{q^4 f_4^{40}} \sum_{n \geq 0} \overline{B}_{4,3}(16(2n) + 8)q^n = 536870912 + 83886080t + 4325376t^2 + 77824t^3 + 256t^4,$$

where $t = f_1^8 / (q f_4^8)$. Moreover, $256 = 2^8$ is the greatest common divisor of the coefficients of the polynomial in the variable t . Therefore, we obtain the first congruence. The remaining congruences can be derived in a similar manner. \square

From the results obtained in **Theorem 6.2**, we propose the following conjecture:

Conjecture 6.1. *For all integers $n, m \geq 0$, we have*

$$\overline{B}_{4,3}(2^{m+5}n + 2^{m+4}) \equiv 0 \pmod{2^{3m+8}}.$$

For a prime p and $n \in \mathbb{N}$, the p -adic valuation of n is the highest power of p that divides n . It is denoted by $v_p(n)$. It is clear that, $v_p(n) \geq b$ if and only if $n \equiv 0 \pmod{p^b}$. We will use a valuation tree to represent the results of the previous theorem (cf. [160]). The valuation tree begins with a root, placed at the top, that represents the set of natural numbers. Since the 2-adic valuation is not constant, this vertex is split onto two new vertices (one per residue class modulo 2), which form the next level. To the first vertex one associates the indices $\{n \in \mathbb{N} : n \equiv 0 \pmod{2}\}$ and to the second one $\{n \in \mathbb{N} : n \equiv 1 \pmod{2}\}$. From Equation (6.9), we know that $v_2(\overline{B}_{4,3}(2n + 1)) \geq 1$, so this vertex is marked by $\boxed{\geq 1}$. We do not have a lower bound for the first vertex, so its indices are split according to its residues modulo 4, into $\{n \equiv 2 \pmod{4}\}$ and $\{n \equiv 0 \pmod{4}\}$. From (6.3), we know that $v_2(\overline{B}_{4,3}(4n + 2)) \geq 2$, so this vertex is marked by $\boxed{\geq 2}$. In Figure 6.1, we show the steps of the tree associated with the sequence $\overline{B}_{4,3}(n)$.

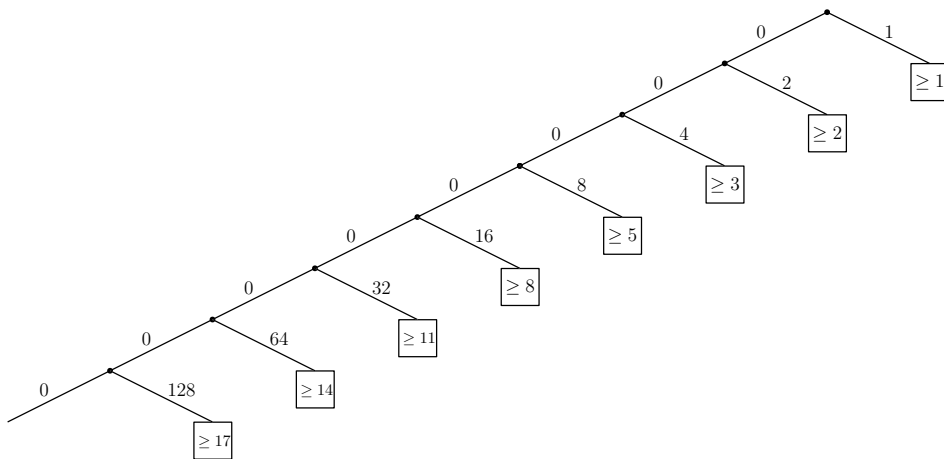


FIGURE 6.1: The valuation tree of the sequence $\overline{B}_{4,3}(n)$.

Open Problem 6.1. *Find an expression for the 2-adic valuation of the sequence $\overline{B}_{4,3}(2^\ell n + 2^{\ell+1})$ for all $\ell \geq 1$.*

Theorem 6.3. For all integers $n \geq 0$, we have

$$\sum_{n \geq 0} \bar{B}_{4,3}(12n+1)q^n \equiv 2f_1^2 \pmod{4}, \quad (6.21)$$

$$\bar{B}_{4,3}(12n+7) \equiv 0 \pmod{12}, \quad (6.22)$$

$$\bar{B}_{4,3}(24n+j) \equiv 0 \pmod{24}, \quad j = 12, 20. \quad (6.23)$$

Moreover,

$$\bar{B}_{4,3}(12n+1) \equiv 0 \pmod{4}. \quad (6.24)$$

if and only if $12n+1$ is not a square.

Proof. First, we extract the terms of the form q^{3n} from Equation (6.13):

$$\sum_{n \geq 0} \bar{B}_{4,3}(6n+1)q^n = 2\frac{f_4^2}{f_2^2} + 24q\frac{f_4^2 f_6^5}{f_1^5 f_2 f_3}. \quad (6.25)$$

Applying (2.2), with $p = 3$ and $k = 1$, in Equation (6.25), we obtain

$$\sum_{n \geq 0} \bar{B}_{4,3}(6n+1)q^n \equiv 2\frac{f_4^2}{f_2^2} \pmod{3}. \quad (6.26)$$

By extracting the odd powers from both sides of the above equation, we find that

$$\bar{B}_{4,3}(12n+7) \equiv 0 \pmod{3}. \quad (6.27)$$

Analogously, from (2.2) (with $p = k = 2$) and Equation (6.25) we have

$$\sum_{n \geq 0} \bar{B}_{4,3}(6n+1)q^n \equiv 2\frac{f_4^2}{f_2^2} \pmod{4}. \quad (6.28)$$

Collecting the terms containing q^{2n} from both sides of (6.28) and using (2.2), we obtain (6.21). Additionally, equating the odd powers on both sides of Equation (6.28), we obtain the congruence

$$\bar{B}_{4,3}(12n+7) \equiv 0 \pmod{4}. \quad (6.29)$$

Congruence (6.22) follows from (6.27) and (6.29). Finally, using (2.2), with $p = 3$ and $k = 1$, in Equation (6.16), we obtain

$$\sum_{n \geq 0} \bar{B}_{4,3}(8n+4)q^n \equiv 8\frac{f_2^{12}}{f_1^{12}} \equiv 8\frac{f_6^4}{f_3^4} \pmod{24}. \quad (6.30)$$

Congruence (6.23) follows from (6.30) by extracting the powers involving q^{3n+j} for $j = 1, 2$ from both sides.

Finally, define

$$\sum_{n \geq 0} a(n)q^n = f_1^2. \quad (6.31)$$

Combining (6.21) and (6.31), we deduce that

$$\bar{B}_{4,3}(12n + 1) \equiv 2a(n) \pmod{4}.$$

Therefore, $\bar{B}_{4,3}(12n + 1) \equiv 0 \pmod{4}$ if and only if $12n + 1$ is a square (since $f_1^2 \equiv f_2 \pmod{2}$). \square

6.3 Congruences for $\bar{B}_{4,9}(n)$

In this section, we derive some congruences for the counting sequence $\bar{B}_{4,9}(n)$. By setting $(\ell_1, \ell_2) = (4, 9)$ in the generating function (6.1), we obtain the expression

$$\sum_{n \geq 0} \bar{B}_{4,9}(n)q^n = \frac{f_2 f_4^2 f_9^2 f_{72}}{f_1^2 f_8 f_{18} f_{36}^2}. \quad (6.32)$$

Substituting Equation (2.31) in the generating function (6.32), and extracting the terms of the form q^{2n+i} , for $i = 0, 1$, we obtain the following generating functions:

$$\sum_{n \geq 0} \bar{B}_{4,9}(2n)q^n = \frac{f_2^2 f_6^6 f_9 f_{36}}{f_1^3 f_3^2 f_4 f_{18}^4} + q \frac{f_2^6 f_3^2 f_{36}}{f_1^5 f_4 f_6^2 f_9}, \quad (6.33)$$

$$\sum_{n \geq 0} \bar{B}_{4,9}(2n + 1)q^n = 2 \frac{f_2^4 f_6^2 f_{36}}{f_1^4 f_4 f_{18}^2}. \quad (6.34)$$

Theorem 6.4. *For all integers $n \geq 2$, we have*

$$\bar{B}_{4,9}(4n) \equiv 0 \pmod{3}. \quad (6.35)$$

Proof. From Lemma 2.1, with $p = 3$ and $k = 1$, and Equation (6.33), we have

$$\sum_{n \geq 0} \bar{B}_{4,9}(2n)q^n \equiv \frac{f_2^2 f_{36}}{f_4 f_{18}^2} + q \frac{f_1 f_{36}}{f_3^3 f_4} \pmod{3}. \quad (6.36)$$

Substituting (2.21) into (6.36), we have

$$\sum_{n \geq 0} \bar{B}_{4,9}(2n)q^n \equiv \frac{f_2^2 f_{36}}{f_4 f_{18}^2} + q \frac{f_2 f_4 f_{12}^2 f_{36}}{f_6^7} - q^2 \frac{f_2^3 f_{12}^6 f_{36}}{f_4^3 f_6^9} \pmod{3}. \quad (6.37)$$

If we extract the powers of the form q^{2n+j} for $j = 0, 1$ from both sides of (6.37), we find that

$$\sum_{n \geq 0} \bar{B}_{4,9}(4n)q^n \equiv \frac{f_1^2 f_{18}}{f_2 f_9^2} - q \frac{f_1^3 f_6^6 f_{18}}{f_2^3 f_3^9} \equiv \frac{f_1^2 f_{18}}{f_2 f_9^2} - q \frac{f_6^5 f_{18}}{f_3^8} \pmod{3}, \quad (6.38)$$

$$\sum_{n \geq 0} \bar{B}_{4,9}(4n+2)q^n \equiv \frac{f_1 f_2 f_6^2 f_{18}}{f_3^7} \pmod{3}. \quad (6.39)$$

Substituting Equation (2.11) into (6.38), we arrive at

$$\sum_{n \geq 0} \bar{B}_{4,9}(4n)q^n \equiv 1 - 2q \frac{f_3 f_{18}^3}{f_6 f_9^3} - q \frac{f_6^5 f_{18}}{f_3^8} \equiv 1 - 3q \frac{f_6^5 f_{18}}{f_3^8} \pmod{3}. \quad (6.40)$$

Note that we used the following congruence

$$\frac{f_6^5 f_{18}}{f_3^8} \equiv \frac{f_3 f_6^6 f_{18}}{f_6 f_9^3} \equiv \frac{f_3 f_{18}^3}{f_6 f_9^3} \pmod{3}.$$

Congruence (6.35) follows from (6.40). \square

Theorem 6.5. For all integers $\alpha \geq 0$ and $n \geq 0$,

$$\bar{B}_{4,9}(12 \cdot 5^{2\alpha+2}n + 12 \cdot 5^{2\alpha+1}i + 2 \cdot 5^{2\alpha+2}) \equiv 0 \pmod{3},$$

where $i = 1, 2, 3, 4$.

Proof. From Equation (6.39), we have

$$\sum_{n \geq 0} \bar{B}_{4,9}(4n+2)q^n \equiv \frac{f_2 f_6^2 f_{18}}{f_1^2 f_3^6} \pmod{3}. \quad (6.41)$$

Substituting Equation (2.11) into (6.41), extracting the powers of the form q^{3n} , and then replacing q^3 by q , we find that

$$\sum_{n \geq 0} \bar{B}_{4,9}(12n+2)q^n \equiv \frac{f_2 f_3^6}{f_1^{14} f_6^2} \equiv f_1^4 \pmod{3}. \quad (6.42)$$

Employing (2.33) in (6.42), and extracting the terms involving q^{5n+4} , we have

$$\sum_{n \geq 0} \bar{B}_{4,9}(60n+50)q^n \equiv f_5^4 \pmod{3}, \quad (6.43)$$

which implies that

$$\sum_{n \geq 0} \bar{B}_{4,9}(300n+50)q^n \equiv f_1^4 \pmod{3}. \quad (6.44)$$

From Equations (6.42) and (6.44), we find that

$$\bar{B}_{4,9}(12n + 2) \equiv \bar{B}_{4,9}(300n + 50) \pmod{3}. \quad (6.45)$$

Utilizing (6.44) and mathematical induction on α , we obtain

$$\bar{B}_{4,9}(12n + 2) \equiv \bar{B}_{4,9}(12 \cdot 5^{2\alpha+2}n + 2 \cdot 5^{2\alpha+2}) \pmod{3}. \quad (6.46)$$

From (6.43), we have

$$\bar{B}_{4,9}(300n + 60i + 50) \equiv 0 \pmod{3}, \quad i = 1, 2, 3, 4. \quad (6.47)$$

Using Equations (6.46) and (6.47), we obtain the desired result. \square

Theorem 6.6. *For all integers $n \geq 0$, we have*

$$\bar{B}_{4,9}(4n + 3) \equiv 0 \pmod{8}, \quad (6.48)$$

$$\sum_{n \geq 0} \bar{B}_{4,9}(12n + 1)q^n \equiv 2f_1^2 \pmod{8}, \quad (6.49)$$

$$\bar{B}_{4,9}(12n + 9) \equiv 0 \pmod{8}. \quad (6.50)$$

Proof. Using (2.2) into (6.34), with $p = k = 2$, we obtain

$$\sum_{n \geq 0} \bar{B}_{4,9}(2n + 1)q^n \equiv 2 \frac{f_2^2 f_6^2 f_{36}}{f_4 f_{18}^2} \pmod{8}. \quad (6.51)$$

Congruence (6.48) follows from the above equation by equating the odd powers on both sides. If we extract the even powers from both sides of (6.51), we obtain

$$\sum_{n \geq 0} \bar{B}_{4,9}(4n + 1)q^n \equiv 2 \frac{f_1^2 f_3^2 f_{18}}{f_2 f_9^2} \pmod{8}. \quad (6.52)$$

Substituting (2.16) into (6.52), we find that

$$\sum_{n \geq 0} \bar{B}_{4,9}(4n + 1)q^n \equiv 2f_3^2 - 4q \frac{f_3^3 f_{18}^3}{f_6 f_9^3} \pmod{8}. \quad (6.53)$$

Congruence (6.49) follows by extracting the terms involving q^{3n} from both sides of (6.53), then replace q^3 by q . Similarly, congruence (6.50) follows from (6.53). \square

From Equations (6.49) and (6.31), we deduce that

$$\bar{B}_{4,9}(12n + 1) \equiv 2a(n) \pmod{8},$$

where $a(n)$ is defined in Equation (6.31). From **Theorem 6.3** we obtain the following result.

Corollary 6.2. For any prime $p \equiv 3 \pmod{4}$, $\alpha \geq 0$, and $n \geq 0$, we have

$$\bar{B}_{4,9} \left(12p^{2\alpha+1}(pn+i) + p^{2\alpha+2} \right) \equiv 0 \pmod{8}, \quad (6.54)$$

for all $1 \leq i \leq p-1$.

6.4 Congruences for $\bar{B}_{8,3}(n)$

In this section, we establish some congruences for the enumeration function $\bar{B}_{8,3}(n)$. By setting $(\ell_1, \ell_2) = (8, 3)$ in (6.1), we obtain the generating function

$$\sum_{n \geq 0} \bar{B}_{8,3}(n)q^n = \frac{f_2 f_3^2 f_8^2 f_{48}}{f_1^2 f_6 f_{16} f_{24}^2}. \quad (6.55)$$

Substituting (2.19) into (6.55), we obtain the generating function:

$$\sum_{n \geq 0} \bar{B}_{8,3}(n)q^n = \frac{f_4^4 f_8 f_{12}^2 f_{48}}{f_2^4 f_{16} f_{24}^3} + 2q \frac{f_4 f_6 f_8^3 f_{48}}{f_2^3 f_{12} f_{16} f_{24}}.$$

Therefore, we get

$$\sum_{n \geq 0} \bar{B}_{8,3}(2n)q^n = \frac{f_2^4 f_4 f_6^2 f_{24}}{f_1^4 f_8 f_{12}^3} \quad \text{and} \quad \sum_{n \geq 0} \bar{B}_{8,3}(2n+1)q^n = 2 \frac{f_2 f_3 f_4^3 f_{24}}{f_1^3 f_6 f_8 f_{12}}.$$

Employing (2.5) in the previous left equation and (2.22) in the previous right equation, we obtain

$$\sum_{n \geq 0} \bar{B}_{8,3}(2n)q^n = \frac{f_4^{15} f_4 f_6^2 f_{24}}{f_2^{10} f_8^5 f_{12}^3} + 4q \frac{f_4^3 f_6^2 f_8^3 f_{24}}{f_2^6 f_{12}^3}, \quad (6.56)$$

$$\sum_{n \geq 0} \bar{B}_{8,3}(2n+1)q^n = 2 \frac{f_4^9 f_6^2 f_{24}}{f_2^8 f_8 f_{12}^3} + 6q \frac{f_4^5 f_{12} f_{24}}{f_2^6 f_8}. \quad (6.57)$$

By extracting the powers of the form q^{2n+j} for $j = 0, 1$ in (6.56) and (6.57), we have

$$\sum_{n \geq 0} \bar{B}_{8,3}(4n)q^n = \frac{f_2^{15} f_3^2 f_{12}}{f_1^{10} f_4^5 f_6^3}, \quad \sum_{n \geq 0} \bar{B}_{8,3}(4n+1)q^n = 2 \frac{f_2^9 f_3^2 f_{12}}{f_1^8 f_4 f_6^3},$$

$$\sum_{n \geq 0} \bar{B}_{8,3}(4n+2)q^n = 4 \frac{f_2^3 f_3^2 f_4^3 f_{12}}{f_1^6 f_6^3}, \quad \text{and} \quad \sum_{n \geq 0} \bar{B}_{8,3}(4n+3)q^n = 6 \frac{f_2^5 f_6 f_{12}}{f_1^6 f_4}.$$

Corollary 6.3. For all integers $n \geq 0$, we have

$$\begin{aligned}\overline{B}_{8,3}(4n+2) &\equiv 0 \pmod{4} \\ \overline{B}_{8,3}(4n+3) &\equiv 0 \pmod{6}.\end{aligned}$$

Theorem 6.7. For all integers $n \geq 0$, we have

$$\overline{B}_{8,3}(24n+i) \equiv 0 \pmod{12}, \quad i = 6, 10, 14, 18, 22. \quad (6.58)$$

$$\overline{B}_{8,3}(48n+j) \equiv 0 \pmod{3}, \quad j = 24, 40. \quad (6.59)$$

Proof. From the previous generating functions we have

$$\sum_{n \geq 0} \overline{B}_{8,3}(4n)q^n \equiv \frac{f_2^6}{f_1^4 f_4^2} \pmod{3}. \quad (6.60)$$

Substituting (2.5) into (6.60), we have

$$\sum_{n \geq 0} \overline{B}_{8,3}(4n)q^n \equiv \frac{f_4^{12}}{f_2^8 f_8^4} + 4q \frac{f_8^4}{f_2^4} \pmod{3}. \quad (6.61)$$

Extracting the even powers from both sides of Equation (6.61) and then using (2.6), we obtain

$$\sum_{n \geq 0} \overline{B}_{8,3}(8n)q^n \equiv \frac{f_2^{12}}{f_1^8 f_4^4} \equiv \frac{f_4^{24}}{f_2^{16} f_8^8} + 8q \frac{f_4^{12}}{f_2^{12}} + 16q^2 \frac{f_8^8}{f_2^8} \pmod{3}. \quad (6.62)$$

Again, if we collect the odd powers from both sides of Equation (6.62), we arrive at

$$\sum_{n \geq 0} \overline{B}_{8,3}(16n+8)q^n \equiv 8 \frac{f_2^{12}}{f_1^{12}} \equiv 8 \frac{f_6^4}{f_3^4} \pmod{3}. \quad (6.63)$$

Congruence (6.59) follows from Equation (6.63) by extracting the powers of the form q^{3n+j} for $j = 1, 2$. We also have

$$\sum_{n \geq 0} \overline{B}_{8,3}(4n+2)q^n \equiv 4 \frac{f_{12}^2}{f_6^2} \pmod{12}. \quad (6.64)$$

Congruence (6.58) follows from (6.64) by extracting the powers of the form q^{6n+j} for $j = 1, 2, 3, 4, 5$. \square

Theorem 6.8. For all integers $n \geq 0$, we have

$$\bar{B}_{8,3}(24n + 20) \equiv 0 \pmod{4}, \quad (6.65)$$

$$\sum_{n \geq 0} \bar{B}_{8,3}(48n + 4)q^n \equiv 2f_1^2 \pmod{4}, \quad (6.66)$$

$$\sum_{n \geq 0} \bar{B}_{8,3}(48n + 12)q^n \equiv 2\psi(q)^2 \pmod{4}, \quad (6.67)$$

$$\bar{B}_{8,3}(48n + 28) \equiv 0 \pmod{4}, \quad (6.68)$$

$$\sum_{n \geq 0} \bar{B}_{8,3}(144n + 36)q^n \equiv 2\psi(q)^2 \pmod{4}, \quad (6.69)$$

$$\bar{B}_{8,3}(144n + j) \equiv 0 \pmod{4}, \quad j = 84, 132. \quad (6.70)$$

Proof. We know that

$$\sum_{n \geq 0} \bar{B}_{8,3}(4n)q^n \equiv \frac{f_3^2 f_4 f_{12}}{f_1^2 f_2 f_6^3} \pmod{4}. \quad (6.71)$$

Using (2.19) in (6.71), we find that

$$\sum_{n \geq 0} \bar{B}_{8,3}(4n)q^n \equiv \frac{f_4^5 f_{12}^3}{f_2^6 f_6^2 f_8 f_{24}} + 2q \frac{f_4^2 f_8 f_{24}}{f_2^5 f_6} \pmod{4}. \quad (6.72)$$

If we extract the odd powers from both sides of Equation (6.72), we obtain

$$\sum_{n \geq 0} \bar{B}_{8,3}(8n + 4)q^n \equiv 2 \frac{f_2^2 f_4 f_{12}}{f_1^5 f_3} \equiv 2 \frac{f_4 f_{12}}{f_1 f_3} \pmod{4}. \quad (6.73)$$

Substituting (2.25) into (6.73), we obtain

$$\sum_{n \geq 0} \bar{B}_{8,3}(8n + 4)q^n \equiv 2 \frac{f_{12}^2 f_{18}^4}{f_3^4 f_{36}^2} + 2q \frac{f_6^2 f_9^3 f_{12} f_{36}}{f_3^5 f_{18}^2} + 4q^2 \frac{f_6 f_{12} f_{18} f_{36}}{f_3^4} \pmod{4}. \quad (6.74)$$

Collecting the powers of the form q^{3n+j} for $j = 0, 1, 2$ from both sides of Equation (6.74), we find the following two congruences and (6.65).

$$\sum_{n \geq 0} \bar{B}_{8,3}(24n + 4)q^n \equiv 2 \frac{f_4^2 f_6^4}{f_1^4 f_{12}^2} \equiv 2f_2^2 \pmod{4}, \quad (6.75)$$

$$\sum_{n \geq 0} \bar{B}_{8,3}(24n + 12)q^n \equiv 2 \frac{f_2^2 f_3^3 f_4 f_{12}}{f_1^5 f_6^2} \equiv 2 \frac{f_3^3 f_4 f_{12}}{f_1 f_6^2} \pmod{4}. \quad (6.76)$$

Congruences (6.66) and (6.68) follow from Equation (6.75) by equating the even and odd powers on both sides.

Substituting (2.20) into (6.76), we arrive at

$$\sum_{n \geq 0} \bar{B}_{8,3}(24n + 12)q^n \equiv 2 \frac{f_4^4}{f_2^2} + 2q \frac{f_{12}^4}{f_6^2} \pmod{4}. \quad (6.77)$$

Extracting the terms involving q^{2n+j} for $j = 0, 1$ from both sides of (6.77), we have

$$\sum_{n \geq 0} \bar{B}_{8,3}(48n + 12)q^n \equiv 2 \frac{f_2^4}{f_1^2} \pmod{4}, \quad (6.78)$$

$$\sum_{n \geq 0} \bar{B}_{8,3}(48n + 36)q^n \equiv 2 \frac{f_6^4}{f_3^2} \pmod{4}. \quad (6.79)$$

Congruence (6.67) follows from (6.78) by using the definition of $\psi(q)$. Congruences (6.69) and (6.70) follow from (6.79) by extracting the terms of the form q^{3n+j} for $j = 0, 1, 2$ along with using the definition of $\psi(q)$. \square

From Equations (6.66) and (6.31), we find that $\bar{B}_{8,3}(48n + 4) \equiv 2a(n) \pmod{4}$. From **Theorem 6.3** we obtain the following result.

Corollary 6.4. *For any prime $p \equiv 3 \pmod{4}$, $\alpha \geq 0$, and $n \geq 0$, we have*

$$\bar{B}_{8,3}\left(48p^{2\alpha+1}(pn + i) + 4p^{2\alpha+2}\right) \equiv 0 \pmod{4}, \quad (6.80)$$

where i is an integer and $1 \leq i \leq p - 1$.

Theorem 6.9. *For any prime $p \equiv 3 \pmod{4}$, $\alpha \geq 0$, and $n \geq 0$, we have*

$$\begin{aligned} \bar{B}_{8,3}\left(48p^{2\alpha+1}(pn + i) + 12p^{2\alpha+2}\right) &\equiv 0 \pmod{4}, \\ \bar{B}_{8,3}\left(144p^{2\alpha+1}(pn + i) + 36p^{2\alpha+2}\right) &\equiv 0 \pmod{4}, \end{aligned}$$

for all $1 \leq i \leq p - 1$.

Proof. Define

$$\sum_{n \geq 0} d(n)q^n = \psi(q)^2. \quad (6.81)$$

From Equations (6.67) and (6.69), we find that

$$\bar{B}_{8,3}(48n + 12) \equiv \bar{B}_{8,3}(144n + 36) \equiv d(n) \pmod{4}. \quad (6.82)$$

Now, consider the congruence equation

$$\frac{k^2 + k}{2} + \frac{m^2 + m}{2} \equiv \frac{p^2 - 1}{4} \pmod{p}, \quad (6.83)$$

which is equivalent to

$$(2k + 1)^2 + (2m + 1)^2 \equiv 0 \pmod{p},$$

where $0 \leq k, m \leq (p-1)/2$ and p is a prime such that $\left(\frac{-1}{p}\right) = -1$. Since $\left(\frac{-1}{p}\right) = -1$ for $p \equiv 3 \pmod{4}$, the congruence relation (6.83) holds if and only if both $k = m = (p-1)/2$. Substituting Equation (2.34) into (6.81), and then extracting the terms in which the powers of q are congruent to $\frac{p^2-1}{4}$ modulo p , and then divide by $q^{\frac{p^2-1}{4}}$, we find that

$$\sum_{n \geq 0} d\left(pn + \frac{p^2-1}{4}\right) q^{pn} = \psi(q^{p^2})^2,$$

which implies that

$$\sum_{n \geq 0} d\left(p^2n + \frac{p^2-1}{4}\right) q^n = \psi(q)^2.$$

Moreover, for all $n \geq 0$,

$$d\left(p^2n + pi + \frac{p^2-1}{4}\right) = 0, \quad (6.84)$$

where i is an integer and $1 \leq i \leq p-1$. By induction we see that for $n \geq 0$ and $\alpha \geq 0$,

$$d\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{4}\right) = d(n). \quad (6.85)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{4}$ ($1 \leq i \leq p-1$) in (6.85) and using (6.84), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$d\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{4}\right) = 0.$$

Again, replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{4}$ into Equation (6.82) ($1 \leq i \leq p-1$), we obtain the desired results. \square

6.5 Congruences for $\overline{B}_{8,9}(n)$

In this section, we establish some congruences for the enumeration function $\overline{B}_{8,9}(n)$. By setting $(\ell_1, \ell_2) = (8, 9)$ in (6.1), we obtain the generating function

$$\sum_{n \geq 0} \overline{B}_{8,9}(n) q^n = \frac{f_2 f_8^2 f_9^2 f_{144}}{f_1^2 f_{16} f_{18} f_{72}^2}. \quad (6.86)$$

We demonstrate the following theorem through the Radu's Ramanujan-Kolberg algorithm.

Theorem 6.10. For all integers $n \geq 0$, we have

$$\overline{B}_{8,9}(24n + 16) \equiv 0 \pmod{6}, \tag{6.87}$$

$$\overline{B}_{8,9}(72n + 24) \equiv 0 \pmod{3}, \tag{6.88}$$

$$\overline{B}_{8,9}(72n + 48) \equiv 0 \pmod{3}, \tag{6.89}$$

$$\overline{B}_{8,9}(72n + 60) \equiv 0 \pmod{3}. \tag{6.90}$$

Proof. The generating function in (6.86) can be described by setting $M = 144$ and $r = (-2, 1, 0, 0, 0, 2, 2, 0, -1, -1, 0, 0, 0, -2, 1)$ in the Smoot's implementation. Additionally, if we take $m = 24$, guess $N = 12$, and take $j = 16$, then we have

$$\text{In[4]:= RK}[12, 144, \{-2, 1, 0, 0, 0, 2, 2, 0, -1, -1, 0, 0, 0, -2, 1\}, 24, 16]$$

$$\prod_{\delta \mid M} (q^\delta; q^\delta)_{\infty}^{r_\delta} = \sum_{n=0}^{\infty} a(n) q^n$$

$$f_1(q) \cdot \prod_{j' \in P_{m,r}(j)} \sum_{n=0}^{\infty} a(mn + j') q^n = \sum_{g \in AB} g \cdot p_g(t)$$

N:	12
{M, (r _δ) _{δ M} }:	{144, {-2, 1, 0, 0, 0, 2, 2, 0, -1, -1, 0, 0, 0, -2, 1}}
m:	24
P _{m,r} (j):	{16}
f ₁ (q):	$\frac{(q; q)_{41} (q^4; q^4)_{16} (q^6; q^6)_{31}}{q^{15} (q^2; q^2)_{29} (q^3; q^3)_{11} (q^{12}; q^{12})_{48}}$
Out[4]= t:	$\frac{(q^4; q^4)_4 (q^6; q^6)_2}{q (q^2; q^2)_2 (q^{12}; q^{12})_4}$
AB:	{1}
{p _g (t): g∈AB}	{-3 188 646 - 143 489 070 t + 655 798 194 t ² + 1 693 643 418 t ³ - 4 642 550 478 t ⁴ - 4 735 401 750 t ⁵ + 9 270 426 186 t ⁶ + 6 433 490 610 t ⁷ - 6 899 482 962 t ⁸ - 4 699 203 498 t ⁹ + 1 291 505 526 t ¹⁰ + 1 425 158 766 t ¹¹ + 326 914 374 t ¹² + 25 799 790 t ¹³ + 577 806 t ¹⁴ + 1734 t ¹⁵ }
Common Factor:	6

Notice that 6 is a common factor. Therefore, we obtain the first congruence. The remaining congruences can be derived in a similar manner. □

Theorem 6.11. For all integers $\alpha \geq 0$ and $n \geq 0$,

$$\overline{B}_{8,9}(24 \cdot 5^{2\alpha+2}n + 24 \cdot 5^{2\alpha+1}i + 4 \cdot 5^{2\alpha+2}) \equiv 0 \pmod{3},$$

where $i = 1, 2, 3, 4$.

Proof. Employing (2.30) into (6.86) and extracting the even and the odd powers from both sides of the resulting equation, we find respectively

$$\sum_{n \geq 0} \bar{B}_{8,9}(2n)q^n = \frac{f_4^2 f_6^6 f_9 f_{72}}{f_1^3 f_3^2 f_8 f_{18}^2 f_{36}^2} + q \frac{f_2^4 f_3^2 f_4^2 f_{18}^2 f_{72}}{f_1^5 f_6^2 f_8 f_9 f_{36}^2}, \quad (6.91)$$

$$\sum_{n \geq 0} \bar{B}_{8,9}(2n+1)q^n = 2 \frac{f_2^2 f_4^2 f_6^2 f_{72}}{f_1^4 f_8 f_{36}^2}. \quad (6.92)$$

In view of Congruence (2.2) and (6.91) with $p = 3$ and $k = 1$, we find that

$$\sum_{n \geq 0} \bar{B}_{8,9}(2n)q^n \equiv \frac{f_4^2 f_{72}}{f_8 f_{36}^2} + q \frac{f_1 f_2^4 f_4^2 f_{18}^2 f_{72}}{f_6^2 f_8 f_9 f_{36}^2} \pmod{3}. \quad (6.93)$$

Substituting (2.32) into the above equation and collecting the even powers from both sides of the resulting equation, we have

$$\sum_{n \geq 0} \bar{B}_{8,9}(4n)q^n \equiv \frac{f_2^2 f_{36}}{f_4 f_{18}^2} + 2q \frac{f_1^4 f_2^3 f_{36}}{f_3 f_4 f_6 f_9} \equiv \frac{f_2^2 f_{36}}{f_4 f_{18}^2} + 2q \frac{f_1 f_{36}}{f_4 f_9} \pmod{3}. \quad (6.94)$$

Again, using Equation (2.32) in (6.94) and collecting the odd powers from both sides of the resulting equation, we arrive at

$$\sum_{n \geq 0} \bar{B}_{8,9}(8n+4)q^n \equiv 2 \frac{f_1 f_6^3 f_{18}}{f_2^2 f_3 f_9^2} \equiv 2 \frac{f_1 f_2 f_6^2 f_{18}}{f_3 f_9^2} \pmod{3}. \quad (6.95)$$

Substituting (2.9) into (6.95), and extracting the terms involving q^{3n} from both sides of the resulting equation, we have

$$\sum_{n \geq 0} \bar{B}_{8,9}(24n+4)q^n \equiv 2 \frac{f_2^3 f_3^2}{f_1^2 f_6} \equiv 2f_1^4 \pmod{3}. \quad (6.96)$$

Employing (2.33) in (6.96), and extracting the terms involving q^{5n+4} , we obtain

$$\sum_{n \geq 0} \bar{B}_{8,9}(120n+100)q^n \equiv 2f_5^4 \pmod{3}, \quad (6.97)$$

which implies that,

$$\sum_{n \geq 0} \bar{B}_{8,9}(600n+100)q^n \equiv 2f_1^4 \pmod{3}. \quad (6.98)$$

From Equations (6.96) and (6.98), we find that

$$\bar{B}_{8,9}(24n+4) \equiv \bar{B}_{8,9}(600n+100) \pmod{3}. \quad (6.99)$$

Utilizing Equation (6.98) and mathematical induction on α , we obtain

$$\overline{B}_{8,9}(24n + 4) \equiv \overline{B}_{8,9}(24 \cdot 5^{2\alpha+2}n + 4 \cdot 5^{2\alpha+2}) \pmod{3}. \quad (6.100)$$

From (6.97), we have

$$\overline{B}_{8,9}(600n + 120i + 100) \equiv 0 \pmod{3}, \quad i = 1, 2, 3, 4. \quad (6.101)$$

Using Equations (6.100) and (6.101), we obtain the desired result. \square

Theorem 6.12. *For all integers $n \geq 0$, we have*

$$\overline{B}_{8,9}(4n + 3) \equiv 0 \pmod{8}, \quad (6.102)$$

$$\sum_{n \geq 0} \overline{B}_{8,9}(12n + 1)q^n \equiv 2f_1^2 \pmod{8}, \quad (6.103)$$

$$\overline{B}_{8,9}(12n + 5) \equiv 0 \pmod{8}. \quad (6.104)$$

Proof. Using (2.2) in (6.92), with $p = k = 2$, we see that

$$\sum_{n \geq 0} \overline{B}_{8,9}(2n + 1)q^n \equiv 2 \frac{f_4^2 f_6^2 f_{72}}{f_8 f_{36}^2} \pmod{8}. \quad (6.105)$$

Congruence (6.102) follows from (6.105) by extracting the odd powers from both sides. If we collect the even powers from both sides of the above equation, we arrive at

$$\sum_{n \geq 0} \overline{B}_{8,9}(4n + 1)q^n \equiv 2 \frac{f_2^2 f_3^2 f_{36}}{f_4 f_{18}^2} \pmod{8}. \quad (6.106)$$

Substituting q by q^2 in (2.11), and then use it in the above equation, we obtain

$$\sum_{n \geq 0} \overline{B}_{8,9}(4n + 1)q^n \equiv 2f_3^2 - 4q^2 \frac{f_3^2 f_6 f_{36}^3}{f_{12} f_{18}^3} \pmod{8}. \quad (6.107)$$

Congruences (6.103) and (6.104) follow from the above equation by extracting the terms of the form q^{3n+j} for $j = 0, 1$ from both sides. \square

From Equations (6.103) and (6.31), we have $\overline{B}_{8,9}(12n + 1) \equiv 2a(n) \pmod{8}$. From **Theorem 6.3** we obtain the following result.

Corollary 6.5. *For any prime $p \equiv 3 \pmod{4}$, $\alpha \geq 0$, and $n \geq 0$, we have*

$$\overline{B}_{8,9} \left(12p^{2\alpha+1}(pn + i) + p^{2\alpha+2} \right) \equiv 0 \pmod{8},$$

for all $1 \leq i \leq p - 1$.

PART III:

CONGRUENCES AND MULTIPLICITIES

CHAPTER 7

s -MODULAR, s -CONGRUENT AND s -DUPLICATE PARTITIONS

7.1 Introduction

Throughout the history of integer partitions, most studies have focused on partitions with restrictions on the parts. Examples include partitions into odd (or even) parts [141, 156], ℓ -regular partitions [64, 77], partitions into parts that belong to certain residue classes modulo some positive integer [10, 157], and more. Nevertheless, the multiplicities (i.e. number of occurrences) of parts have not received significant attention in the literature on integer partitions, as reflected in the limited number of related papers. Many of these few papers are dedicated to the partitions where parts appear fewer than m times [92, 108].

In this chapter, we consider congruences of the multiplicities of parts and introduce the s -modular partitions. Additionally, considerations of congruences of the multiplicities lead to the definition of s -congruent and s -duplicate partitions. Throughout the remainder of the chapter, s is always considered to be an even positive integer greater than or equal to 4.

Definition 7.1. *A partition λ is s -modular if every part occurs with multiplicity congruent to 0 or 1 modulo s . We denote by $M_s(n)$ the number of s -modular partitions of n .*

Definition 7.2. *A partition λ is said to be s -congruent if the parts are not congruent to $2, 4, 6, \dots, (s-2)$ modulo s . We denote by $C_s(n)$ the number of s -congruent partitions of n .*

Definition 7.3. *A partition λ is s -duplicate if every part in the partition with multiplicity greater than one is congruent to 0 modulo $s/2$. We denote by $D_s(n)$ the number of s -duplicate partitions of n .*

Note that the s -congruent partitions were introduced by Ballantine and Welch [34] as partitions in which all even parts are congruent to 0 modulo s .

Through this study, we present several combinatorial properties for the three classes of partitions. We establish connections between them and show that they satisfy a number of recurrence relations. Additionally, we highlight the links between some of these classes of partitions and several celebrated theorems in the theory of partitions. It is worth noting that the OEIS [197] sequence [A096981](#) corresponds to the function $C_6(n)$, and unlike $D_4(n)$, it still lacks arithmetic studies, although it makes appearances in works such as Spector [194].

In the special case when $s = 4$, the s -duplicate become what is known in the literature as POD partitions [197, A006950], partitions wherein odd parts are distinct and even parts are unrestricted. The number $\text{pod}(n)$ satisfies the following Ramanujan-theta function

$$\sum_{n \geq 0} \text{pod}(n)q^n = \frac{1}{\psi(-q)} = \frac{1}{\sum_{n \geq 1} (-q)^{n(n+1)/2}}.$$

We can say very little about $D_s(n)$ from an arithmetic point of view, except in the most restricted cases when $s = 4$ [119, 177, 199].

One of the most fundamental identities in the theory of partitions and q -series is

$$\sum_{n \geq 0} \frac{q^{n(n+1)/2} (1+bq)(1+bq^2) \cdots (1+bq^n)}{(1-q)(1-q^2) \cdots (1-q^n)} = \prod_{m \geq 1} \frac{(1+bq^{2m})}{(1-q^{2m-1})} \quad (7.1)$$

due to Lebesgue [23, Chpt. 2]. The right hand-side generating function of (7.1) enumerates the number of partitions into distinct even parts and unrestricted odd parts (the function $\text{ped}(n)$), where the exponent on b keeps track the number of even parts. By setting $b = -1$ we obtain the famous Gauss identity

$$\sum_{n \geq 0} q^{n(n+1)/2} = \prod_{m \geq 1} \frac{(1-q^{2m})}{(1-q^{2m-1})}, \quad (7.2)$$

which is as significant as Euler's celebrated Pentagonal Numbers Theorem

$$(q; q)_{\infty} = 1 + \sum_{k \geq 1} (-1)^k (q^{(3k^2-k)/2} + q^{(3k^2+k)/2}). \quad (7.3)$$

Sylvester [196] also presented an important identity by analyzing partitions into distinct parts via Durfee squares (the largest square that is contained within a partition's Ferrers diagram). The identity is given by

$$\prod_{n \geq 1} (1+bq^n) = 1 + \sum_{k \geq 1} \frac{b^k q^{(3k^2-k)/2} (-bq; q)_{k-1} (1+bq^{2k})}{(q; q)_k}. \quad (7.4)$$

The case $b = -1$ in (7.4) is Euler's Pentagonal Numbers Theorem. Fine [96] defined the function

$$F(\alpha, \beta, \tau; q) = \sum_{n \geq 0} \frac{(\alpha q; q)_n}{(\beta q; q)_n} \tau^n \quad (7.5)$$

and studied curious transformations and iterations for $F(\alpha, \beta, \tau; q)$. One of the obtained results is the Rogers-Fine identity

$$F(\alpha, \beta, \tau; q) = \sum_{n \geq 0} \frac{(\alpha q; q)_n (\alpha \tau q / \beta; q)_n (1 - \alpha \tau q^{2n+1})}{(\beta q; q)_n (\tau; q)_{n+1}} \beta^n \tau^n q^{n^2}. \quad (7.6)$$

Andrews [21] provided a combinatorial proof of (7.6) by studying partitions in terms of $2n \times$

n Durfee rectangles. Another proof of (7.6) has been presented by Zeng [209] by utilizing Sylvester’s bijection connecting partitions into odd parts with partitions into distinct parts.

Alladi [6] presented a series expansion with different parameters for the product generating function of $\text{pod}(n)$ using 2-modular Ferrers diagrams and their Durfee squares. Several fundamental identities in the theory of partitions and q -series, including those of Sylvester, Lebesgue, Gauss, and Rogers-Fine, emerge as special cases of this series expansion. The expansion is given by

$$\frac{(-bq; q^2)_\infty}{(cq^2; q^2)_\infty} = 1 + \sum_{k \geq 1} \frac{c^k q^{2k^2-1} (-bq; q^2)_{k-1} (-bc^{-1}q; q^2)_{k-1} (bc^{-1} + q)(1 + bq^{4k-1})}{(cq^2; q^2)_k (q^2; q^2)_k}, \quad (7.7)$$

where the powers of b and c keep track of the number of odd and even parts respectively. For example, we demonstrate the relation between the series expansion for the product generating function of $\text{pod}(n)$ and Sylvester’s identity, as discussed by Alladi [6]. In (7.7), note that

$$c^k (-bc^{-1}q; q^2)_{k-1} (bc^{-1} + q) = (c + bq)(c + bq^3) \cdots (c + bq^{2k-3})(b + cq). \quad (7.8)$$

By setting $c = 0$ in (7.8), the righthand side of (7.8) equals

$$b^k q^{(k-1)^2}.$$

Thus, if $c = 0$ in (7.7), we obtain

$$(-bq; q^2)_\infty = 1 + \sum_{k \geq 1} \frac{b^k q^{3k^2-2k} (-bq; q^2)_{k-1} (1 + bq^{4k-1})}{(q^2; q^2)_k}. \quad (7.9)$$

Sylvester’s identity (7.4) follows from (7.9) by the substitutions $b \rightarrow bq$ and then $q^2 \rightarrow q$. Alladi [4] also provided a combinatorial interpretation in terms of partitions for Sylvester’s identity (7.4), demonstrating its equivalence to **Theorem 7.1**.

Theorem 7.1 (Alladi). *Let D denote the set of partitions into distinct parts, and D_3 the set of partitions into parts differing by at least 3. Let $v_d(\lambda)$ be the number of different parts of λ . Also, for $\lambda^* \in D_3$, $|\lambda^*| = \lambda_1^* + \lambda_2^* + \cdots + \lambda_k^*$, let $v_3(\lambda^*)$ denote the number of strict inequalities $\lambda_i^* - \lambda_{i+1}^* > 3$, for $i = 1, 2, \dots, k$, where $\lambda_{k+1}^* = -1$. Then we have*

$$\sum_{\lambda \in D, |\lambda|=n} c^{\ell(\lambda)} = \sum_{\lambda^* \in D_3, |\lambda^*|=n} c^{\ell(\lambda^*)} (1 + c)^{v_3(\lambda^*)}.$$

The series expansion (7.7) also establishes links with Göllnitz’s deep theorem. Alladi [5] presented the equivalence between the three parameter refinement of **Theorem 7.1** and the three parameter generalization and refinement of Göllnitz’s (Big) partition theorem [10]. Notably, the identity (7.7) holds an advantage in providing more flexibility in choosing specializations, which are used to derive several fundamental identities. To extend its applicability, we provide a generalization by employing the concept of k -modular Ferrers diagrams.

7.2 Combinatorial properties

In this section, we introduce the generating functions and several combinatorial properties for these three classes of partitions.

Theorem 7.2. *The generating function for $M_s(n)$, the number of s -modular partitions of n , is given by*

$$\sum_{n \geq 0} M_s(n)q^n = \frac{(-q; q)_\infty}{(q^s; q^s)_\infty}.$$

Proof. Since

$$\prod_{n \geq 1} (1 + q^{sn} + q^{2sn} + q^{3sn} + \dots) = \frac{1}{(q^s; q^s)_\infty}$$

is the generating function for the number of partitions where the multiplicity of every part is congruent to 0 modulo s , then

$$\begin{aligned} \sum_{n \geq 0} M_s(n)q^n &= \prod_{n \geq 1} (1 + q^n)(1 + q^{sn} + q^{2sn} + q^{3sn} + \dots) \\ &= \frac{(-q; q)_\infty}{(q^s; q^s)_\infty}, \end{aligned}$$

from which our result follows. □

For example, $M_4(8) = 10$ and the corresponding set of partitions is

$$\mathbb{M}_4(8) = \{(8), (7, 1), (6, 2), (5, 3), (5, 2, 1), (4, 3, 1), (3, 1^5), (4, 1^4), (2^4), (1^8)\}.$$

Theorem 7.3. *The generating function for $C_s(n)$, the number of s -congruent partitions of n , is given by*

$$\begin{aligned} \sum_{n \geq 0} C_s(n)q^n &= \frac{1}{(q; q^2)_\infty (q^s; q^s)_\infty} \\ &= \frac{(q^2; q^2)_\infty}{(q; q)_\infty (q^s; q^s)_\infty}. \end{aligned}$$

Proof. The generating function $1/(q^s; q^s)_\infty$ generates partitions into parts congruent to 0 modulo s . Similarly, the known generating function $1/(q; q^2)_\infty$ generates partitions into odd parts. By multiplying both generating functions together, we obtain the desired result. □

For example, $C_6(8) = 7$ such that

$$C_6(8) = \{(7, 1), (6, 1^2), (5, 3), (5, 1^3), (3^2, 1^2), (3, 1^5), (1^8)\}.$$

Theorem 7.4. *The generating function for $D_s(n)$, the number of s -duplicate partitions of n , satisfies the identity*

$$\sum_{n \geq 0} D_s(n)q^n = \frac{(-q; q)_\infty}{(-q^{s/2}; q^{s/2})_\infty (q^{s/2}; q^{s/2})_\infty}.$$

Proof. The generating function $(-q; q)_\infty / (-q^{s/2}; q^{s/2})_\infty$ generates partitions into distinct parts indivisible by $s/2$, and on the other hand, $1 / (q^{s/2}; q^{s/2})_\infty$ generates unrestricted partitions into parts multiples of $s/2$. \square

For example,

$$\mathbb{D}_6(6) = \{(6), (5, 1), (4, 2), (3, 2, 1), (3^2)\},$$

from which we conclude that $D_6(6) = 5$.

In the following theorem, we lay the foundations of the connection between $\mathbb{M}_s(n)$, $\mathbb{C}_s(n)$, and $\mathbb{D}_s(n)$ for $n \geq 0$.

Theorem 7.5. *For every positive integer $n \geq 0$,*

$$M_s(n) = C_s(n) = D_s(n).$$

Proof. We give both a generating function proof and a combinatorial proof of the result. By employing the facts $(-q; q)_\infty = (q; q^2)_\infty^{-1}$ and $(1 - q^{sn}) = (1 + q^{sn/2})(1 - q^{sn/2})$ (since s is even and ≥ 4) for every $n \geq 1$ in Theorem 7.2, we obtain

$$\begin{aligned} \sum_{n \geq 0} M_s(n)q^n &= \prod_{n \geq 1} \frac{(1 + q^n)}{(1 - q^{sn})} \\ &= \prod_{n \geq 1} \frac{1}{(1 - q^{2n-1})(1 - q^{sn})} = \sum_{n \geq 0} C_s(n)q^n \\ &= \prod_{n \geq 1} \frac{(1 + q^n)}{(1 + q^{sn/2})(1 - q^{sn/2})} = \sum_{n \geq 0} D_s(n)q^n. \end{aligned}$$

Now, we provide combinatorial proofs of the three parts of the theorem.

The bijection $\mathbb{M}_s(n) \Leftrightarrow \mathbb{C}_s(n)$ when $s = 2^p$ for $p \geq 2$:

Let $\lambda = (\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_k^{\alpha_k}) \in \mathbb{M}_s(n)$ and define the map $f : \mathbb{M}_s(n) \rightarrow \mathbb{C}_s(n)$ by $\lambda \rightarrow f(\lambda) = \cup_{i=1}^k f(\lambda_i^{\alpha_i})$ as follows. If $\lambda_i \equiv 2, 4, 6, \dots, (s-2) \pmod{s}$, then for $1 \leq i \leq k$, we have

$$f(\lambda_i^{\alpha_i}) = \begin{cases} \alpha_i^{\lambda_i} & \text{if } \alpha_i \equiv 0 \pmod{s}, \\ (\alpha_i - 1)^{\lambda_i}, \ell_i^{2^r} & \text{if } \alpha_i \equiv 1 \pmod{s}, \end{cases}$$

where each part λ_i can be expressed uniquely as $\lambda_i = 2^{r_i} \ell_i$ with ℓ_i odd. Note that if $\lambda \in \mathbb{M}_s(n)$, then $\lambda_i \equiv \pm 2^{r_i} \pmod{2^{r_i+2}}$. Else, if $\lambda_i \not\equiv 2, 4, 6, \dots, (s-2) \pmod{s}$, then

$$f(\lambda_i^{\alpha_i}) = \lambda_i^{\alpha_i}.$$

As an illustration, for $(n, s) = (18, 8)$ and the partitions $\lambda = (4, 3, 2, 1^9)$ and $\beta = (5, 4, 1^9)$, we have

$$f(\lambda) \rightarrow \begin{cases} f(4) : 4 = 2^2 \cdot 1 \rightarrow f(4) = (1-1)^4, 1^4 \rightarrow f(4) = 1^4 \\ f(3) = 3 \text{ since } 3 \not\equiv 0 \pmod{8} \text{ and } f(1^9) = 1^9 \text{ since } 1 \not\equiv 0 \pmod{8} \\ f(2) : 2 = 2 \cdot 1 \rightarrow f(2) = (1-1)^2, 1^2 \rightarrow f(2) = 1^2 \end{cases}$$

$$f(\beta) \rightarrow \begin{cases} f(5) = 5 \text{ since } 5 \not\equiv 0 \pmod{8} \\ f(4) : 4 = 2^2 \cdot 1 \rightarrow f(4) = (1-1)^4, 1^4 \rightarrow f(4) = 1^4 \\ f(1^9) = 1^9 \text{ since } 1 \not\equiv 0 \pmod{8}. \end{cases}$$

Let $\lambda = (\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_k^{\alpha_k}) \in \mathbb{C}_s(n)$ and define the inverse map $f^{-1} : \mathbb{C}_s(n) \rightarrow \mathbb{M}_s(n)$ by $\lambda \rightarrow f^{-1}(\lambda) = \bigcup_{i=1}^k f^{-1}(\lambda_i^{\alpha_i})$. If $\lambda_i \equiv 0 \pmod{s}$, then for $1 \leq i \leq k$, we have

$$f^{-1}(\lambda_i^{\alpha_i}) = \begin{cases} \lambda_i^{\alpha_i} & \text{if } \alpha_i \equiv 0, 1 \pmod{s}, \\ \alpha_i^{\lambda_i} & \text{if } \alpha_i \equiv 2, 4, 6, \dots, (s-2) \pmod{s}, \\ (\alpha_i - 1)^{\lambda_i}, \lambda_i & \text{if } \alpha_i \equiv 3, 5, 7, \dots, (s-1) \pmod{s}. \end{cases}$$

Else, if $\lambda_i \not\equiv 0 \pmod{s}$, define the vector W_{p-1} as

$$W_{p-1} = (2a_1, 2^2a_2, \dots, 2^{p-1}a_{p-1}),$$

where $a_i \in \{0, 1\}$ and $i \in \{1, 2, \dots, (p-1)\}$. The values of a_i are chosen in such a way that W_{p-1} attains a unique magnitude $|W_{p-1}|$ for which $m = \alpha_i - |W_{p-1}| \equiv 0, 1 \pmod{s}$, where m is the maximum possible value less than α_i . Consequently, we obtain the expansion

$$\lambda_i^{\alpha_i} = \lambda_i^m + \lambda_i^{2a_1} + \dots + \lambda_i^{2^{p-1}a_{p-1}},$$

where if $a_i = 0$ or $m = 0$, we exclude the corresponding part λ_i from the expansion. Then, the inverse map $f^{-1}(\lambda_i^{\alpha_i})$ is given by

$$f^{-1}(\lambda_i^{\alpha_i}) = \begin{cases} \lambda_i^{\alpha_i} & \text{if } \alpha_i \equiv 0, 1 \pmod{s}, \\ \lambda_i^m, 2a_1\lambda_i, 2^2a_2\lambda_i, \dots, 2^{p-1}a_{p-1}\lambda_i & \text{if } \alpha_i \not\equiv 0, 1 \pmod{s}. \end{cases}$$

As an illustration, for $(n, s) = (18, 8)$ and the partitions $\mu = (3, 1^{15})$ and $\gamma = (5, 1^{13})$, we have

$$f^{-1}(\mu) \rightarrow \begin{cases} f^{-1}(3) = 3 \text{ since } 3 \not\equiv 0 \pmod{8} \text{ and } u_1 = 1 \\ f^{-1}(1^{15}) : m = 15 - |(2 \cdot 1, 2^2 \cdot 1)| \equiv 1 \pmod{8} \rightarrow f^{-1}(1^{15}) = 1^9, 2, 4 \end{cases}$$

$$f^{-1}(\gamma) \rightarrow \begin{cases} f^{-1}(5) = 5 \text{ since } 5 \not\equiv 0 \pmod{8} \text{ and } u_1 = 1 \\ f^{-1}(1^{13}) : m = 13 - |(2 \cdot 0, 2^2 \cdot 1)| = 9 \equiv 1 \pmod{8} \rightarrow f^{-1}(1^{13}) = 1^9, 4. \end{cases}$$

The bijection $\mathbb{M}_s(n) \Leftrightarrow \mathbb{ID}_s(n)$ when $s = 2^p$ for $p \geq 2$:

Let $\lambda = (\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_k^{\alpha_k}) \in \mathbb{M}_s(n)$ and define the map $g : \mathbb{M}_s(n) \rightarrow \mathbb{D}_s(n)$ by $\lambda \rightarrow g(\lambda) = \cup_{i=1}^k g(\lambda_i^{\alpha_i})$. If λ_i is either a multiple of $s/2$, or a non-multiple of $s/2$ with $\alpha_i = 1$, then

$$g(\lambda_i^{\alpha_i}) = \lambda_i^{\alpha_i}.$$

Else, if λ_i is a non-multiple of $s/2$ with $\alpha_i > 1$, then let the unique expansion of the multiplicity

$$\alpha_i = b_0 + 2^p a_0 + 2^{p+1} a_1 + \dots + 2^{p+r} a_r$$

with $a_i \in \{0, 1\}$ and $r \geq 0$. Therefore, we obtain the transformation

$$\lambda_i^{\alpha_i} = (\lambda_i^{b_0}, \lambda_i^{2^p a_0}, \lambda_i^{2^{p+1} a_1}, \dots, \lambda_i^{2^{p+r} a_r}).$$

Then, for $0 \leq j \leq r$, let $\alpha'_j := 2^{p+j} a_j$ and define $g(\lambda_i^{\alpha'_j})$ by

$$g(\lambda_i^{\alpha'_j}) = \begin{cases} (\alpha'_j \lambda_i / 2)^2 & \text{if } \lambda_i \text{ is odd,} \\ (\alpha'_j \lambda_i / 4)^4 & \text{if } \lambda_i \text{ is even,} \end{cases}$$

and if $b_0 = 1$, then

$$g(\lambda_i^{b_0}) = \lambda_i.$$

As an illustration, for $(n, s) = (10, 4)$ and the partitions $\lambda = (3, 2, 1^5)$ and $\beta = (2, 1^8)$, we have

$$g(\lambda) \rightarrow \begin{cases} g(3) = 3 \text{ since } u_1 = 1 \\ g(2) = 2 \text{ since } 2 \equiv 0 \pmod{2} \\ g(1^5) : 5 = 1 + 2^2 \cdot 1 \rightarrow 1^5 = 1, 1^4 \rightarrow g(1^5) = 1, 2^2 \end{cases}$$

$$g(\beta) \rightarrow \begin{cases} g(2) = 2 \text{ since } 2 \equiv 0 \pmod{2} \\ g(1^8) : 8 = 0 + 2^2 \cdot 0 + 2^3 \cdot 1 \rightarrow 1^8 = 1^8 \rightarrow g(1^8) = 4^2. \end{cases}$$

Let $\lambda = (\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_k^{\alpha_k}) \in \mathbb{D}_s(n)$ and define the map $g^{-1} : \mathbb{D}_s(n) \rightarrow \mathbb{M}_s(n)$ by $\lambda \rightarrow g^{-1}(\lambda) = \cup_{i=1}^k g^{-1}(\lambda_i^{\alpha_i})$. If λ_i is either a multiple of $s/2$ with $\alpha_i \equiv 0, 1 \pmod{s}$, or a non-multiple of $s/2$ with $\alpha_i = 1$, then

$$g^{-1}(\lambda_i^{\alpha_i}) = \lambda_i^{\alpha_i}.$$

Now, if λ_i is a multiple of $s/2$ with $\alpha_i \not\equiv 0, 1 \pmod{s}$, in this case every part λ_i can be expressed uniquely as $\lambda_i = 2^{r_i} \ell_i$, where $\ell_i \geq 1$ is odd and $r_i \geq 1$. Then, for $n \in \{2, 4\}$, we obtain the expansion

$$\lambda_i^{\alpha_i} = \lambda_i^{\alpha_i - n} + \lambda_i^n$$

such that $\alpha_i - n \equiv 0, 1 \pmod{s}$. Therefore, we have

$$g^{-1}(\lambda_i^{\alpha_i}) = \lambda_i^{\alpha_i - n}, (n \ell_i / 2)^{2^{r_i+1}}.$$

Notice that if $\alpha_i - n = 0$, then the part $\lambda_i^{\alpha_i - n}$ is excluded from the resulting image. As an illustration, for $(n, s) = (10, 4)$ and the partitions $\mu = (3, 2^3, 1)$ and $\gamma = (4^2, 2)$, we have

$$g^{-1}(\mu) \rightarrow \begin{cases} g^{-1}(3) = 3 \\ g^{-1}(2^3) : 2 = 2^1 \cdot 1 \rightarrow 1 = 3 - 2 \rightarrow 2^3 = 2 + 2^2 \rightarrow g^{-1}(2^3) = 2, 1^4 \\ g^{-1}(2) = 2 \end{cases}$$

$$g^{-1}(\gamma) \rightarrow \begin{cases} g^{-1}(4^2) : 4 = 2^2 \cdot 1 \rightarrow 0 = 2 - 2 \rightarrow g^{-1}(4^2) = 1^8 \\ g^{-1}(2) = 2. \end{cases}$$

The bijection $\mathbb{M}_s(n) \leftrightarrow \mathbb{C}_s(n)$ when $s \neq 2^p$ for $p \geq 2$:

Let $\lambda = (\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_k^{\alpha_k}) \in \mathbb{M}_s(n)$ and define the map $h : \mathbb{M}_s(n) \rightarrow \mathbb{C}_s(n)$ by $\lambda \rightarrow h(\lambda) = \bigcup_{i=1}^k h(\lambda_i^{\alpha_i})$. If $\lambda_i \equiv 2, 4, \dots, (s-2) \pmod{s}$, then

$$h(\lambda_i^{\alpha_i}) = \begin{cases} \alpha_i^{\lambda_i} & \text{if } \alpha_i \equiv 0 \pmod{s}, \\ (\alpha_i - 1)^{\lambda_i}, \ell_i^{2^r} & \text{if } \alpha_i \equiv 1 \pmod{s}, \end{cases}$$

where

$$\lambda_i = 2^{r_i} \ell_i$$

for odd $\ell_i \geq 1$ and $r_i \geq 1$. If $\lambda_i \not\equiv 2, 4, \dots, (s-2) \pmod{s}$, then

$$h(\lambda_i^{\alpha_i}) = \begin{cases} (s\lambda_i/2)^{2\alpha_i/s} & \text{if } \alpha_i \equiv 0 \pmod{s}, \\ (s\lambda_i/2)^{2(\alpha_i-1)/s}, \lambda_i & \text{if } \alpha_i \equiv 1 \pmod{s}. \end{cases}$$

As an illustration, for $(n, s) = (18, 6)$ and the partition $\lambda = (10, 2, 1^6)$, we have

$$h(\lambda) \rightarrow \begin{cases} h(10) : 10 \equiv 4 \pmod{6}, u_1 = 1, \text{ and } 10 = 2 \cdot 5 \rightarrow h(10) = 5^2 \\ h(2) : 2 \equiv 2 \pmod{6}, u_2 = 1, \text{ and } 2 = 2 \cdot 1 \rightarrow h(2) = 1^2 \\ h(1^6) : 1 \not\equiv 2, 4 \pmod{6} \text{ and } u_3 = 6 \rightarrow h(1^6) = 3^2. \end{cases}$$

Let $\lambda = (\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_k^{\alpha_k}) \in \mathbb{C}_s(n)$ and define the inverse map $h^{-1} : \mathbb{C}_s(n) \rightarrow \mathbb{M}_s(n)$ by $\lambda \rightarrow h^{-1}(\lambda) = \bigcup_{i=1}^k h^{-1}(\lambda_i^{\alpha_i})$. If $\lambda_i \equiv 0 \pmod{s}$, then

$$h^{-1}(\lambda_i^{\alpha_i}) = \begin{cases} \alpha_i^{\lambda_i} & \text{if } \alpha_i \text{ is even,} \\ (\alpha_i - 1)^{\lambda_i}, \lambda_i & \text{if } \alpha_i \text{ is odd.} \end{cases}$$

Else, if $\lambda_i \not\equiv 0 \pmod{s}$, then when $\lambda_i = 1$, let the unique expansion of the multiplicity

$$\alpha_i = a_0 + 2a_1 + 2^2a_2 + \dots + 2^r a_r$$

where $r \geq 0$ and $a_i \in \{0, 1\}$, and define

$$h^{-1}(1^{\alpha_i}) = a_0, 2a_1, 2^2a_2, \dots, 2^r a_r.$$

Now, when $\lambda_i \geq 3$, and for some odd $\ell_i \geq 1$, we have

$$h^{-1}(\lambda_i^{\alpha_i}) = \begin{cases} \ell_i^{(\alpha_i-1)s/2}, \lambda_i & \text{if } \alpha_i \text{ is odd and } \lambda_i = s\ell_i/2, \text{ for all } \ell_i, \\ (\alpha_i - 1)\lambda_i, \lambda_i & \text{if } \alpha_i \text{ is odd and } \lambda_i \neq s\ell_i/2, \text{ for all } \ell_i, \\ \ell_i^{\alpha_i s/2} & \text{if } \alpha_i \text{ is even and } \lambda_i = s\ell_i/2, \text{ for all } \ell_i, \\ \alpha_i \lambda_i & \text{if } \alpha_i \text{ is even and } \lambda_i \neq s\ell_i/2, \text{ for all } \ell_i. \end{cases}$$

As an illustration, for $(n, s) = (18, 6)$ and the partition $\mu = (5^2, 3^2, 1^2)$, we have

$$h^{-1}(\mu) \rightarrow \begin{cases} h^{-1}(5^2) : u_1 = 2 \text{ is even and } 5 \neq 3\ell_1 \rightarrow h^{-1}(5^2) = 10 \\ h^{-1}(3^2) : u_2 = 2 \text{ is even and } 3 = 3 \cdot 1 \rightarrow h^{-1}(3^2) = 1^6 \\ h^{-1}(1^2) : u_3 = 2^1 \cdot 1 \rightarrow h^{-1}(1^2) = 2. \end{cases}$$

The bijection $\mathbb{M}_s(n) \Leftrightarrow \mathbb{D}_s(n)$ when $s \neq 2^p$ for $p \geq 2$:

Let $\lambda = (\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_k^{\alpha_k}) \in \mathbb{M}_s(n)$ and define the map $w : \mathbb{M}_s(n) \rightarrow \mathbb{D}_s(n)$ by $\lambda \rightarrow w(\lambda) = \bigcup_{i=1}^k w(\lambda_i^{\alpha_i})$. If λ_i is a non-multiple of $s/2$ with $\alpha_i > 1$, then

$$w(\lambda_i^{\alpha_i}) = \begin{cases} (s\lambda_i/2)^{2\alpha_i/s} & \text{if } \alpha_i \equiv 0 \pmod{s}, \\ (s\lambda_i/2)^{2(\alpha_i-1)/s}, \lambda_i & \text{if } \alpha_i \equiv 1 \pmod{s}. \end{cases}$$

Else,

$$w(\lambda_i^{\alpha_i}) = \lambda_i^{\alpha_i}.$$

As an illustration, for $(n, s) = (18, 6)$ and the partition $\lambda = (6, 2^6)$, we have

$$w(\lambda) \rightarrow \begin{cases} w(6) \rightarrow 6 \equiv 0 \pmod{3} \rightarrow w(6) = 6 \\ w(2^6) \rightarrow 2 \not\equiv 0 \pmod{3} \text{ and } u_2 = 6 \rightarrow w(2^6) = 6^2. \end{cases}$$

Let $\lambda = (\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}, \dots, \lambda_k^{\alpha_k}) \in \mathbb{D}_s(n)$ and define the inverse map $w^{-1} : \mathbb{D}_s(n) \rightarrow \mathbb{M}_s(n)$ by $\lambda \rightarrow w(\lambda) = \bigcup_{i=1}^k w^{-1}(\lambda_i^{\alpha_i})$. If $\lambda_i \equiv 0 \pmod{s/2}$ and $\alpha_i > 1$, then

$$\lambda_i = s\ell_i/2$$

where ℓ_i is a positive integer, and define

$$w^{-1}(\lambda_i^{\alpha_i}) = \begin{cases} \ell_i^{s\alpha_i/2}, & \text{if } \alpha_i \text{ is even,} \\ \ell_i^{s(\alpha_i-1)/2}, \lambda_i & \text{if } \alpha_i \text{ is odd.} \end{cases}$$

Else,

$$w^{-1}(\lambda_i^{\alpha_i}) = \lambda_i^{\alpha_i}.$$

As an illustration, for $(n, s) = (18, 6)$ and the partition $\mu = (6^3)$, we have

$$w^{-1}(\mu) \rightarrow w^{-1}(6^3) : 6 = 3 \cdot 2 \rightarrow w^{-1}(6^3) = 2^6, 6.$$

□

Example 7.1. An illustration of the bijection $\mathbb{M}_s(n) \Leftrightarrow \mathbb{C}_s(n) \Leftrightarrow \mathbb{D}_s(n)$ is given for some of the partitions when $(s, n) = (4, 10)$ in Table 7.1, and $(s, n) = (8, 18)$ in Table 7.2.

TABLE 7.1: The bijection of **Theorem 7.5** for $n = 10, s = 4$.

$\mathbb{M}_4(10)$	\Leftrightarrow	$\mathbb{D}_4(10)$	\Leftrightarrow	$\mathbb{C}_4(10)$
(10)		(10)		(5 ²)
(8, 2)		(8, 2)		(8, 1 ²)
(7, 2, 1)		(7, 2, 1)		(7, 1 ³)
(6, 4)		(6, 4)		(4, 3 ²)
(6, 3, 1)		(6, 3, 1)		(3 ³ , 1)
(5, 1 ⁵)		(5, 2 ² , 1)		(5, 1 ⁵)
(4, 3, 2, 1)		(4, 3, 2, 1)		(4, 3, 1 ³)
(6, 1 ⁴)		(6, 2 ²)		(3 ² , 1 ⁴)
(4, 2, 1 ⁴)		(4, 2 ³)		(4, 1 ⁶)
(3, 2, 1 ⁵)		(3, 2 ³ , 1)		(3, 1 ⁷)
(2, 1 ⁸)		(4 ² , 2)		(1 ¹⁰)
(2 ⁵)		(2 ⁵)		(4 ² , 1 ²)
(5, 1 ⁵)		(5, 2 ² , 1)		(5, 1 ⁵)
(9, 1)		(9, 1)		(9, 1)
(5, 4, 1)		(5, 4, 1)		(5, 4, 1)
(7, 3)		(7, 3)		(7, 3)

TABLE 7.2: The bijection of **Theorem 7.5** for $n = 18, s = 8$.

$\mathbb{M}_8(18)$	\Leftrightarrow	$\mathbb{D}_8(18)$	\Leftrightarrow	$\mathbb{C}_8(18)$
(9, 1 ⁹)		(9, 4 ² , 1)		(9, 1 ⁹)
(2 ⁹)		(4 ⁴ , 2)		(8 ² , 1 ²)
(8, 2, 1 ⁸)		(8, 4 ² , 2)		(8, 1 ¹⁰)
(7, 3, 1 ⁸)		(7, 4 ² , 3)		(7, 3, 1 ⁸)
(7, 2, 1 ⁹)		(7, 4 ² , 2, 1)		(7, 1 ¹¹)
(10, 1 ⁸)		(10, 4 ²)		(5 ² , 1 ⁸)
(5, 3, 2, 1 ⁸)		(5, 4 ² , 3, 2)		(5, 3, 1 ¹⁰)
(5, 4, 1 ⁹)		(5, 4 ³ , 1)		(5, 1 ¹³)
(6, 3, 1 ⁹)		(6, 4 ² , 3, 1)		(3 ³ , 1 ⁹)
(6, 4, 1 ⁸)		(6, 4 ³)		(3 ² , 1 ¹²)
(4, 3, 2, 1 ⁹)		(4 ³ , 3, 2, 1)		(3, 1 ¹⁵)
(2, 1 ¹⁶)		(8 ² , 2)		(1 ¹⁸)

Example 7.2. An illustration of the bijection $\mathbb{M}_s(n) \Leftrightarrow \mathbb{C}_s(n) \Leftrightarrow \mathbb{D}_s(n)$ is given for some of the partitions when $(s, n) = (6, 18)$ in Table 7.3, and $(s, n) = (10, 15)$ in Table 7.4.

TABLE 7.3: The bijection of **Theorem 7.5** for $n = 18, s = 6$.

$\mathbb{M}_6(18)$	\Leftrightarrow	$\mathbb{D}_6(18)$	\Leftrightarrow	$\mathbb{C}_6(18)$
$(12, 1^6)$		$(12, 3^2)$		$(12, 3^2)$
$(10, 2, 1^6)$		$(10, 3^2, 2)$		$(5^2, 3^2, 1^2)$
$(9, 3, 1^6)$		$(9, 3^3)$		$(9, 3^3)$
$(8, 4, 1^6)$		$(8, 4, 3^2)$		$(3^2, 1^{12})$
$(5, 4, 3, 1^6)$		$(5, 4, 3^3)$		$(5, 3^3, 1^4)$
$(11, 1^7)$		$(11, 3^2, 1)$		$(11, 3^2, 1)$
$(8, 3, 1^7)$		$(8, 3^3, 1)$		$(3^3, 1^9)$
$(5, 4, 2, 1^7)$		$(5, 4, 3^2, 2, 1)$		$(5, 3^2, 1^7)$
$(6, 1^{12})$		$(6, 3^4)$		$(6, 3^4)$
$(5, 1^{13})$		$(5, 3^4, 1)$		$(5, 3^4, 1)$
$(4, 2, 1^{12})$		$(4, 3^4, 2, 1)$		$(3^4, 1^6)$
(1^{18})		(3^6)		(3^6)
$(6, 2^6)$		(6^3)		(6^3)
$(5, 2^6, 1)$		$(6^2, 5, 1)$		$(6^2, 5, 1)$
$(4, 2^7)$		$(6^2, 4, 2)$		$(6^2, 1^6)$
$(3, 2^7, 1)$		$(6^2, 3, 2, 1)$		$(6^2, 3, 1^3)$
(3^6)		(9^2)		(9^2)
$(16, 2)$		$(16, 2)$		(1^{18})
$(14, 4)$		$(14, 4)$		$(7^2, 1^4)$

TABLE 7.4: The bijection of **Theorem 7.5** for $n = 15, s = 10$.

$\mathbb{M}_{10}(15)$	\Leftrightarrow	$\mathbb{D}_{10}(15)$	\Leftrightarrow	$\mathbb{C}_{10}(15)$
$(14, 1)$		$(14, 1)$		$(7^2, 1)$
$(12, 3)$		$(12, 3)$		(3^5)
$(12, 2, 1)$		$(12, 2, 1)$		$(3^4, 1^3)$
$(8, 6, 1)$		$(8, 6, 1)$		$(3^2, 1^9)$
$(8, 5, 2)$		$(8, 5, 2)$		$(5, 1^{10})$
$(8, 4, 2, 1)$		$(8, 4, 2, 1)$		(1^{15})
$(7, 6, 2)$		$(7, 6, 2)$		$(7, 3^2, 1^2)$
$(6, 5, 4)$		$(6, 5, 4)$		$(5, 3^2, 1^4)$
$(6, 5, 3, 1)$		$(6, 5, 3, 1)$		$(5, 3^3, 1)$
$(6, 4, 3, 2)$		$(6, 4, 3, 2)$		$(3^3, 1^6)$
$(3, 2, 1^{10})$		$(5^2, 3, 2)$		$(5^2, 3, 1^2)$
$(4, 1^{11})$		$(5^2, 4, 1)$		$(5^2, 1^5)$
$(5, 4, 3, 2, 1)$		$(5, 4, 3, 2, 1)$		$(5, 3, 1^7)$
$(7, 4, 3, 1)$		$(7, 4, 3, 1)$		$(7, 3, 1^5)$

It is well known that recurrence relations play a crucial role in computing the number of partitions of every positive integer n by using compact and elegant formulas for the number of partitions of large positive integers. Therefore, we establish recurrence relations for the

three classes. Let $M_s(n, k)$ denote the number of s -modular partitions of n into exactly k parts.

Theorem 7.6. For every positive integers $n, k \geq 1$,

$$M_s(n, k) = \sum_{\substack{\ell=0 \\ \ell \equiv 0, 1 \pmod{s}}}^k M_s(n - k, k - \ell),$$

where $M_s(0, 0) = 1$ and $M_s(n, k) = 0$ if $n < 0, k \leq 0$ and $n > 0, \text{ or } k > n$. Additionally, we assume that all negative arguments are equal to 0.

Proof. In every partition $\lambda \in \mathbb{M}_s(n, k)$, the part 1 has a multiplicity of ℓ for some $\ell \equiv 0, 1 \pmod{s}$. Removing the 1's from λ and subtracting 1 from its other parts yields a partition of $n - k$ into $k - \ell$ parts. If the partition λ consists only of parts of size 1, then we obtain a partition of 0, that is, $M_s(0, 0) = 1$. □

The following array contains some values of $M_4(n, k)$ for $n \leq 20$ and $k \leq 15$.

TABLE 7.5: Some values of $M_4(n, k)$ for $n \leq 20$ and $k \leq 15$.

n															
1	1														
2	1														
3	1	1													
4	1	1		1											
5	1	2			1										
6	1	2	1		1										
7	1	3	1		1	1									
8	1	3	2	1	1	1		1							
9	1	4	3		2	2			1						
10	1	4	4	1	2	2	1		1						
11	1	5	5	1	2	4	1		1	1					
12	1	5	7	3	2	4	2	1	1	1		1			
13	1	6	8	3	3	6	3		2	2			1		
14	1	6	10	5	3	6	5	1	2	2	1		1		
15	1	7	12	6	4	9	6	1	2	4	1		1	1	
16	1	7	14	10	4	9	9		2	4	2	1	1	1	
17	1	8	16	11	5	13	11	3	4	8	1		2	2	
18	1	8	19	15	7	12	15	6	4	6	5	1	2	2	1
19	1	9	21	18	9	16	18	7	5	10	6	1	2	4	1
20	1	9	24	24	11	16	23	13	5	10	9	4	2	4	2
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Example 7.3. For $(s, n, k) = (4, 20, 8)$, we have

$$\begin{aligned} \mathbb{M}_4(20, 8) &= \mathbb{M}_4(12, 8) + \mathbb{M}_4(12, 7) + \mathbb{M}_4(12, 4) + \mathbb{M}_4(12, 3) \\ &= 1 + 2 + 3 + 7 \\ &= 13. \end{aligned}$$

$\mathbb{M}_4(20, 8)$	$\mathbb{M}_4(12, 8)$	$\mathbb{M}_4(12, 7)$	$\mathbb{M}_4(12, 4)$	$\mathbb{M}_4(12, 3)$
$(3^4, 2^4)$	$(2^4, 1^4)$			
$(6, 3, 2^5, 1)$		$(5, 2, 1^5)$		
$(6, 3, 2^5, 1)$		$(4, 3, 1^5)$		
$(7, 4, 3, 2, 1^4)$			$(6, 3, 2, 1)$	
$(6, 5, 3, 2, 1^4)$			$(5, 4, 2, 1)$	
$(4^4, 1^4)$			(3^4)	
$(10, 3, 2, 1^5)$				$(9, 2, 1)$
$(9, 4, 2, 1^5)$				$(8, 3, 1)$
$(8, 5, 2, 1^5)$				$(7, 4, 1)$
$(8, 4, 3, 1^5)$				$(7, 3, 2)$
$(7, 6, 2, 1^5)$				$(6, 5, 1)$
$(7, 5, 3, 1^5)$				$(6, 4, 2)$
$(6, 5, 4, 1^5)$				$(5, 4, 3)$

TABLE 7.6: An illustration of the construction of the set $\mathbb{M}_4(20, 8)$.

Let $C_s(n, k)$ denote the number of s -congruent partitions of n into k parts such that $C_s(n, k) := |C_s(n, k)|$. First, we introduce some definitions that are needed for the proof of the next recurrence relation. Let $N(s)$ be the set consisting of s and all the odd positive integers less than s , and let $N'(\ell)$ be the set consisting of all the odd positive integers less than $\ell \in N(s)$. Denote by $C_s^\ell(n, k)$ the set of all the partitions $\lambda \in C_s(n, k)$ with at least one part of size $\ell \in N(s)$ as the smallest part in the partition, and by $C_s^{s+1}(n, k)$ the set of all the partitions $\lambda \in C_s(n, k)$ into parts of sizes greater than s . Therefore, we have the following dissection of $C_s(n, k)$ into $|N(s)| + 1$ disjoint subsets

$$C_s(n, k) = C_s^1(n, k) \cup C_s^3(n, k) \cup \cdots \cup C_s^s(n, k) \cup C_s^{s+1}(n, k). \quad (7.10)$$

Consider the subset

$$C_s(n - \ell, k - 1) \setminus \bigcup_{i \in N'(\ell)} C_s^i(n - \ell, k - 1)$$

which contains all the partitions of $n - \ell$ into $k - 1$ parts of sizes $\geq \ell$, where $\ell \in N(s)$. To obtain partitions of $n - \ell$ into $k - 1$ parts of sizes $\geq \ell$, we exclude those with at least one part of size $i \in N'(\ell)$ as the smallest part from $C_s(n - \ell, k - 1)$. Conversely, by adding one part of size $\ell \in N(s)$ to each partition $\lambda \in C_s(n - \ell, k - 1) \setminus \bigcup_{i \in N'(\ell)} C_s^i(n - \ell, k - 1)$, we arrive at $C_s^\ell(n, k)$. Therefore, for all $\ell \in N(s)$ we deduce that

$$|C_s^\ell(n, k)| = |C_s(n - \ell, k - 1) \setminus \bigcup_{i \in N'(\ell)} C_s^i(n - \ell, k - 1)|.$$

Similarly, by adding s to each part of $\lambda \in \mathbf{C}_s(n - sk, k)$, we get $\mathbf{C}_s^{s+1}(n, k)$. Therefore, we find that

$$|\mathbf{C}_s^{s+1}(n, k)| = |\mathbf{C}_s(n - sk, k)|.$$

Then, we deduce the following relations:

$$C_s^\ell(n, k) = C_s(n - \ell, k - 1) - \sum_{i \in N'(\ell)} C_s^i(n - \ell, k - 1),$$

and

$$C_s^{s+1}(n, k) = C_s(n - sk, k).$$

Theorem 7.7. For every positive integers $n, k \geq 1$, we have

$$C_s(n, k) = \sum_{\ell \in N(s)} C_s^\ell(n, k) + C_s^{s+1}(n, k).$$

Moreover, for $\ell \in N(s)$, we have

$$C_s^\ell(n, k) = C_s(n - \ell, k - 1) - \sum_{i \in N'(\ell)} C_s^i(n - \ell, k - 1),$$

and

$$C_s^{s+1}(n, k) = C_s(n - sk, k).$$

By convention, we define $C_s(0, 0) = 1$, $C_s(n, k) = 0$ if $n < 0$, $k \leq 0$ and $n > 0$, or $k > n$, and $C_s^1(n, k) = C_s(n - 1, k - 1)$.

Proof. Let the sets $N(s) = \{1, 3, 5, \dots, (s - 1), s\}$ and $N'(\ell) = \{1, 3, 5, \dots, m\}$ where $m < \ell \in N(s)$, and consider the dissection (7.10) of $\mathbf{C}_s(n, k)$. Then, for every partition $\lambda \in \mathbf{C}_s^\ell(n, k)$, remove one part of size ℓ to obtain a partition of $n - \ell$ into $k - 1$ parts, that is, a partition $\lambda \in \mathbf{C}_s(n - \ell, k - 1) \setminus \cup_{i \in N'(\ell)} \mathbf{C}_s^i(n - \ell, k - 1)$. If $\lambda \in \mathbf{C}_s^{s+1}(n, k)$, subtract s from each of the k parts of λ to obtain a partition of $n - sk$ into k parts. Therefore, we have

$$C_s^\ell(n, k) = C_s(n - \ell, k - 1) - \sum_{i \in N'(\ell)} C_s^i(n - \ell, k - 1),$$

and

$$C_s^{s+1}(n, k) = C_s(n - sk, k).$$

□

The following array contains some values of $C_4(n, k)$ for $n \leq 20$ and $k \leq 15$.

TABLE 7.7: Some values of $C_4(n, k)$ for $n \leq 20$ and $k \leq 15$.

n															
1	1														
2		1													
3	1		1												
4	1	1		1											
5	1	1	1		1										
6		2	1	1		1									
7	1	1	2	1	1		1								
8	1	3	1	2	1	1		1							
9	1	2	4	1	2	1	1		1						
10		3	3	4	1	2	1	1		1					
11	1	2	5	3	4	1	2	1	1		1				
12	1	4	4	6	3	4	1	2	1	1		1			
13	1	3	7	5	6	3	4	1	2	1	1		1		
14		4	6	9	5	6	3	4	1	2	1	1		1	
15	1	3	9	8	10	5	6	3	4	2	2	1	1		1
16	1	6	7	13	9	10	5	6	3	4	1	2	1	1	
17	1	4	12	11	15	9	10	5	6	3	4	1	2	1	1
18		5	10	18	13	16	9	10	5	6	3	4	1	2	1
19	1	4	14	16	22	14	16	9	10	5	6	3	4	1	2
20	1	7	12	23	21	24	14	16	9	10	5	6	3	4	1
k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Example 7.4. For $(s, n, k) = (4, 20, 4)$, we have

$$\begin{aligned}
 C_4(20, 4) &= C_4^1(20, 4) + C_4^3(20, 4) + C_4^4(20, 4) + C_4^5(20, 4) \\
 &= 14 + 6 + 2 + 1 \\
 &= 23,
 \end{aligned}$$

such that

- $C_4^1(20, 4) = C_4(19, 3) = 14,$
- $C_4^3(20, 4) = C_4(17, 3) - C_4^1(17, 3) = C_4(17, 3) - C_4(16, 2) = 12 - 6 = 6,$
- $C_4^4(20, 4) = C_4(16, 3) - C_4^1(16, 3) - C_4^3(16, 3) = C_4(16, 3) - C_4(15, 2) - (C_4(13, 2) - C_4(12, 1)) = 7 - 3 - (3 - 1) = 2,$
- $C_4^5(20, 4) = C_4(4, 4) = 1.$

For brevity, let $\mathbb{K}_{\geq \ell}(n - \ell, k - 1) = \mathbf{C}_s(n - \ell, k - 1) \setminus \cup_{i \in N'(\ell)} \mathbf{C}_s^i(n - \ell, k - 1).$

$\mathbb{C}_4(20,4)$	$\mathbb{K}_{\geq 1}(19,3)$	$\mathbb{K}_{\geq 3}(17,3)$	$\mathbb{K}_{\geq 4}(16,3)$	$\mathbb{C}_4(4,4)$
(17, 1 ³)	(17, 1 ²)			
(15, 3, 1 ²)	(15, 3, 1)			
(13, 5, 1 ²)	(13, 5, 1)			
(13, 3 ² , 1)	(13, 3 ²)			
(12, 4, 3, 1)	(12, 4, 3)			
(17, 1 ³)	(17, 1 ²)			
(11, 7, 1 ²)	(11, 7, 1)			
(11, 5, 3, 1)	(11, 5, 3)			
(11, 4 ² , 1)	(11, 4 ²)			
(9 ² , 1 ²)	(9 ² , 1)			
(9, 7, 3, 1)	(9, 7, 3)			
(9, 5 ² , 1)	(9, 5 ²)			
(8 ² , 3, 1)	(8 ² , 3)			
(8, 7, 4, 1)	(8, 7, 4)			
(7 ² , 5, 1)	(7 ² , 5)			
(11, 3 ³)		(11, 3 ²)		
(9, 5, 3 ²)		(9, 5, 3)		
(9, 4 ² , 3)		(9, 4 ²)		
(8, 5, 4, 3)		(8, 5, 4)		
(7 ² , 3 ²)		(7 ² , 3)		
(7, 5 ² , 3)		(7, 5 ²)		
(8, 4 ³)			(8, 4 ²)	
(7, 5, 4 ²)			(7, 5, 4)	
(5 ⁴)				(1 ⁴)

TABLE 7.8: An illustration of the construction of the set $\mathbb{C}_4(20,4)$.

Let $\mathbb{D}_s(n, k)$ denote the set of s -duplicate partitions of n into k parts with $D_s(n, k) := |\mathbb{D}_s(n, k)|$. By convention we define $D_s(0, 0) = 1$. Let $\mathbb{O}(s)$ denote the set of all partitions into distinct parts less than or equal to $s/2 - 1$. Let $\mathbb{A}(s) = \mathbb{O}(s) \cup \{\emptyset\}$, where by convention we define the empty set as the empty partition and $|\mathbb{A}(s)| = 2^{s/2-1}$. Now, consider the partition $\alpha^j \in \mathbb{A}(s)$, where $0 \leq |\alpha^j| \leq \frac{(s/2-1)(s/2)}{2}$ and $0 \leq \ell(\alpha^j) \leq s/2 - 1$ for $1 \leq j \leq 2^{s/2-1}$. By convention we consider the length of the partition of \emptyset to be 0 along with its value. Moreover, consider the partition γ^j such that $\lambda = \gamma^j \cup \alpha^j$ for some $\lambda \in \mathbb{D}_s(n, k)$. There are two kinds of partitions $\lambda \in \mathbb{D}_s(n, k)$. The first kind consists of partitions with at least one part of size $s/2$, while the second one consists of partitions into exactly $k - \ell(\alpha^j)$ parts of sizes greater than $s/2$ and $\ell(\alpha^j)$ parts of sizes $\lambda_i \leq s/2 - 1$. That is, the partitions of the second kind are of the form $\lambda = \gamma^j \cup \alpha^j$ for $1 \leq j \leq 2^{s/2-1}$. We denote by $\mathbb{D}_s(n, k; s/2)$ the subset of the first kind and by $\mathbb{D}_s(n, k; \alpha^j)$ the subset of the second. The dissection of $\mathbb{D}_s(n, k)$ into $2^{s/2-1} + 1$ disjoint subsets is given by

$$\mathbb{D}_s(n, k) = \mathbb{D}_s(n, k; s/2) \cup \mathbb{D}_s(n, k; \alpha^1) \cup \cdots \cup \mathbb{D}_s(n, k; \alpha^{2^{s/2-1}}). \quad (7.11)$$

Theorem 7.8. For every positive integers $n, k \geq 1$,

$$D_s(n, k) = D_s(n - s/2, k - 1) + \sum_{\alpha^j \in \mathbb{A}(s)} D_s\left(n - \frac{s(k - \ell(\alpha^j))}{2} - |\alpha^j|, k - \ell(\alpha^j)\right).$$

By convention, we define $D_s(0, 0) = 1$ and $D_s(n, k) = 0$ if $n < 0, k \leq 0$ and $n > 0, k > n$.

Proof. Consider the dissection (7.11) of $\mathbb{D}_s(n, k)$. For every $\lambda \in \mathbb{D}_s(n, k; s/2)$, remove one part of size $s/2$ to obtain a partition of $n - s/2$ into $k - 1$ parts. On the other hand, for $1 \leq j \leq 2^{s/2-1}$ and $\lambda \in \mathbb{D}_s(n, k; \alpha^j)$, subtract $s/2$ from the $k - \ell(\alpha^j)$ parts greater than $s/2$ and remove α^j to obtain a partition of $n - s(k - \ell(\alpha^j))/2 - |\alpha^j|$ into $k - \ell(\alpha^j)$ parts. \square

The following array contains some values of $D_4(n, k)$ for $n \leq 20$ and $k \leq 10$.

TABLE 7.9: Some values of $D_4(n, k)$ for $n \leq 20$ and $k \leq 10$.

n										
1	1									
2	1									
3	1	1								
4	1	2								
5	1	2	1							
6	1	2	2							
7	1	3	2	1						
8	1	4	3	2						
9	1	4	5	2	1					
10	1	4	6	3	2					
11	1	5	7	5	2	1				
12	1	6	9	7	3	2				
13	1	6	11	9	5	2	1			
14	1	6	13	11	7	3	2			
15	1	7	15	5	9	5	2	1		
16	1	8	17	20	12	7	3	2		
17	1	8	20	23	17	9	5	2	1	
18	1	8	23	27	22	12	7	3	2	
19	1	9	25	34	27	17	9	5	2	1
20	1	10	28	41	34	23	12	7	3	2
k	1	2	3	4	5	6	7	8	9	10

Example 7.5. For $s = 4$, we have the set $\mathbb{O}(4) = \{(1)\}$. Therefore, $\mathbb{A}(4) = \{\emptyset, (1)\}$ and the recurrence relation is given by

$$D_4(n, k) = D_4(n - 2, k - 1) + D_4(n - 2k, k) + D_4(n - 2(k - 1) - 1, k - 1). \tag{7.12}$$

For $s = 8$, we have the set $\mathcal{O}(8) = \{(1), (2), (3), (2, 1), (3, 1), (3, 2), (3, 2, 1)\}$. Therefore, $\mathbb{A}(8) = \{\emptyset, (1), (2), (3), (2, 1), (3, 1), (3, 2), (3, 2, 1)\}$ and the recurrence relation is given by

$$\begin{aligned} D_8(n, k) &= D_8(n - 4k, k - 1) \\ &+ D_8(n - 4k, k) \\ &+ D_8(n - 4(k - 1) - 1, k - 1) \\ &+ D_8(n - 4(k - 1) - 2, k - 1) \\ &+ D_8(n - 4(k - 1) - 3, k - 1) \\ &+ D_8(n - 4(k - 2) - 3, k - 2) \\ &+ D_8(n - 4(k - 2) - 4, k - 2) \\ &+ D_8(n - 4(k - 2) - 5, k - 2) \\ &+ D_8(n - 4(k - 3) - 6, k - 3). \end{aligned}$$

Example 7.6. For $(s, n, k) = (4, 13, 3)$, we have

$$\begin{aligned} D_4(13, 3) &= D_4(11, 2) + D_4(7, 3) + D_4(8, 2) \\ &= 5 + 2 + 4 \\ &= 11. \end{aligned}$$

$\mathbb{D}_4(13, 3)$	$\mathbb{D}_4(11, 2)$	$\mathbb{D}_4(7, 3)$	$\mathbb{D}_4(8, 2)$
(10, 2, 1)	(10, 1)		
(9, 2 ²)	(9, 2)		
(8, 3, 2)	(8, 3)		
(7, 4, 2)	(7, 4)		
(6, 5, 2)	(6, 5)		
(6, 4, 3)		(4, 2, 1)	
(5, 4 ²)		(3, 2 ²)	
(9, 3, 1)			(7, 1)
(7, 5, 1)			(5, 3)
(6 ² , 1)			(4 ²)
(8, 4, 1)			(6, 2)

TABLE 7.10: An illustration of the construction of the set of partitions $\mathbb{D}_4(13, 3)$.

Merca [158] proved that the partition function $\text{ped}(n)$, which enumerates the number of partitions of n wherein even parts are distinct and odd parts are unrestricted, satisfies

$$\text{ped}(n) = \sum_{k \geq 0} C_4(n - 2T_k),$$

where $T_k = k(k + 1)/2$ is the k th triangular number. In the spirit of Euler’s recurrence for the unrestricted partition function $p(n)$, we use a simple observation to prove the following recurrence for $C_4(n)$.

Theorem 7.9. *For every positive integer $n \geq 1$,*

$$C_4(n) = \sum_{k \geq 1} (-1)^{k(k+1)/2+1} C_4(n - k(k + 1)/2).$$

Proof. We have that

$$\psi(q) = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty} \tag{7.13}$$

$$= \sum_{k \geq 0} q^{k(k+1)/2}. \tag{7.14}$$

Replacing q by $-q$ in (7.14) and using $(-q; -q)_\infty = \frac{(q^2; q^2)_\infty^3}{(q; q)_\infty (q^4; q^4)_\infty}$, we obtain

$$\begin{aligned} \psi(-q) &= \frac{(q; q)_\infty (q^4; q^4)_\infty}{(q^2; q^2)_\infty} \\ &= \sum_{k \geq 0} (-q)^{k(k+1)/2}. \end{aligned}$$

Then

$$\left(\sum_{k \geq 0} (-q)^{k(k+1)/2} \right) \left(\sum_{n \geq 0} C_4(n) q^n \right) = 1,$$

from which our result follows. □

$C_4(0)$	1	$C_4(5)$	4	$C_4(10)$	16	$C_4(15)$	55	$C_4(20)$	161
$C_4(1)$	1	$C_4(6)$	5	$C_4(11)$	21	$C_4(16)$	70	$C_4(21)$	196
$C_4(2)$	1	$C_4(7)$	7	$C_4(12)$	28	$C_4(17)$	86	$C_4(22)$	236
$C_4(3)$	2	$C_4(8)$	10	$C_4(13)$	35	$C_4(18)$	105	$C_4(23)$	287
$C_4(4)$	3	$C_4(9)$	13	$C_4(14)$	43	$C_4(19)$	130	$C_4(24)$	350

TABLE 7.11: The first 24 values of the function $C_4(n)$.

Example 7.7. *For $n = 21$ and $n = 24$, we obtain*

$$\begin{aligned} C_4(21) &= C_4(20) + C_4(18) - C_4(15) - C_4(11) + C_4(6) + C_4(0) \\ &= 161 + 105 - 55 - 21 + 5 + 1 \\ &= 196, \end{aligned}$$

$$\begin{aligned} C_4(24) &= C_4(23) + C_4(21) - C_4(18) - C_4(14) + C_4(9) + C_4(3) \\ &= 287 + 196 - 105 - 43 + 13 + 2 \\ &= 350. \end{aligned}$$

Following the fact that $\text{pod}(n)$ is a special case of the s -duplicate partitions, we give a generalization for the series expansion (7.7) using the $s/2$ -modular Ferrers diagrams. That is, we wish to obtain an expansion for the product generating function

$$\sum_{n,r,l \geq 0} D_s(n,r,l)z^r b^l q^n = \prod_{n \geq 1} \frac{(1 + zbq^n)}{(1 + zbq^{sn/2})(1 - zq^{sn/2})},$$

where the powers of b and z encode the number of distinct parts and the total number of parts respectively.

Theorem 7.10. *We have*

$$\sum_{n,r,l \geq 0} D_s(n,r,l)z^r b^l q^n = 1 + \sum_{k \geq 1} A_k(q) \times \left\{ z^k q^{sk^2/2} (-zbq^{s(k-1)/2+1}; q)_{s/2-1} (-bq^{s(k-1)/2+1}; q)_{s/2-1} + (1 - zq^{sk/2})(1 - q^{sk/2}) \sum_{i=1}^{s/2-1} bz^k q^{sk^2/2-i} \right\},$$

where

$$A_k(q) = \frac{(-zbq, -zbq^2, \dots, -zbq^{s/2-1}; q^{s/2})_{k-1} (-bq, -bq^2, \dots, -bq^{s/2-1}; q^{s/2})_{k-1}}{(zq^{s/2}; q^{s/2})_k (q^{s/2}; q^{s/2})_k}.$$

Proof. Consider the $k \times k$ Durfee square D of the $s/2$ -modular Ferrers diagram of an s -duplicate partition λ of n , which is the largest square of boxes starting from the top-left corner of the diagram. This dissection of λ results a partition λ^r to the right of the square D and a partition λ^b below the square D . With regard to Durfee squares of $s/2$ -modular diagrams, there are $s/2$ cases to consider. The first case is when the bottom-right box of the square D has an entry $s/2$, while in the other $s/2 - 1$ cases, the bottom-right box of D has an entry i with $1 \leq i \leq s/2 - 1$. In the first case, λ^r is a partition into at most k parts and λ^b is a partition into parts $\leq sk/2$, and in the rest of the cases, λ^r is a partition into at most $k - 1$ parts and λ^b is a partition into parts $\leq s(k - 1)/2$. Note that both λ^r and λ^b are also s -duplicate partitions.

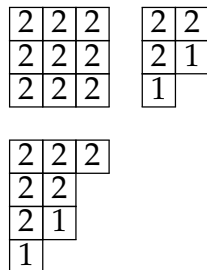


FIGURE 7.1: An illustration of the first case for $s = 4$, $k = 3$, and $\lambda = (10, 9, 7, 6, 4, 3, 1)$.

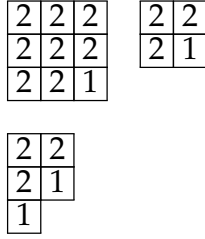


FIGURE 7.2: An illustration of the other $s/2 - 1$ cases for $s = 4$, $k = 3$, and $\lambda = (10, 9, 5, 4, 3, 1)$.

The generating function for the number of partitions in the first case: Let $\lambda \in \mathbb{D}(n, k)$ whose $s/2$ -modular Ferrers diagram has a $k \times k$ Durfee square D with all boxes filled with $s/2$. The sum of the entries in the boxes of D is $sk^2/2$ and so we have the term

$$z^k q^{sk^2/2}.$$

On the other hand, the generating function for the partitions λ^b is given by

$$\frac{(-zbq, -zbq^2, \dots, -zbq^{s/2-1}; q^{s/2})_k}{(zq^{s/2}; q^{s/2})_k}. \quad (7.15)$$

In computing the generating function of λ^r , we do not need to keep track of the total number of parts. Thus the parameter z will be absent in this generating function. Instead, we are only interested in the number of distinct parts in λ^r . Then, by using the conjugation of the modular diagram, we obtain the following generating function

$$\frac{(-bq, -bq^2, \dots, -bq^{s/2-1}; q^{s/2})_k}{(q^{s/2}; q^{s/2})_k}. \quad (7.16)$$

Thus, the generating function for the partitions in the first case is

$$z^k q^{sk^2/2} \frac{(-zbq, -zbq^2, \dots, -zbq^{s/2-1}; q^{s/2})_k (-bq, -bq^2, \dots, -bq^{s/2-1}; q^{s/2})_k}{(zq^{s/2}; q^{s/2})_k (q^{s/2}; q^{s/2})_k}.$$

The generating function for the number of partitions in the other $s/2 - 1$ cases: The rest of the $s/2 - 1$ cases of partitions whose $s/2$ -modular $k \times k$ Durfee square D has a sum of entries equals to $s(k^2 - 1)/2 + i$ where $i \in \{1, 2, \dots, s/2 - 1\}$, such that i is the entry in the bottom right box of D . Again, the partitions λ^b and λ^r are the same as in the previous first case, except that the largest part of λ^b is $\leq s(k - 1)/2$ and the total number of parts of λ^r is $\leq k - 1$. Thus, the generating function for such cases would be

$$\sum_{i=1}^{s/2-1} bz^k q^{sk^2/2-i} \frac{(-zbq, \dots, -zbq^{s/2-1}; q^{s/2})_{k-1} (-bq, \dots, -bq^{s/2-1}; q^{s/2})_{k-1}}{(zq^{s/2}; q^{s/2})_{k-1} (q^{s/2}; q^{s/2})_{k-1}}.$$

For the products occurring in (7.15) and (7.16), we have

$$\begin{aligned} (-zbq, \dots, -zbq^{\frac{s}{2}-1}; q^{\frac{s}{2}})_k &= (-zbq, \dots, -zbq^{\frac{s}{2}-1}; q^{\frac{s}{2}})_{k-1} \prod_{t=1}^{s/2-1} (1 + zbq^t q^{s(k-1)/2}) \\ &= (-zbq, \dots, -zbq^{\frac{s}{2}-1}; q^{\frac{s}{2}})_{k-1} \prod_{t=0}^{s/2-2} (1 + zbq^{s(k-1)/2+1} q^t) \\ &= (-zbq, \dots, -zbq^{\frac{s}{2}-1}; q^{\frac{s}{2}})_{k-1} (-zbq^{\frac{s(k-1)}{2}+1}; q)_{\frac{s}{2}-1}, \end{aligned}$$

and

$$\begin{aligned} (-bq, \dots, -bq^{\frac{s}{2}-1}; q^{\frac{s}{2}})_k &= (-bq, \dots, -bq^{\frac{s}{2}-1}; q^{\frac{s}{2}})_{k-1} \prod_{t=1}^{s/2-1} (1 + bq^t q^{s(k-1)/2}) \\ &= (-bq, \dots, -bq^{\frac{s}{2}-1}; q^{\frac{s}{2}})_{k-1} \prod_{t=0}^{s/2-2} (1 + bq^{s(k-1)/2+1} q^t) \\ &= (-bq, \dots, -bq^{\frac{s}{2}-1}; q^{\frac{s}{2}})_{k-1} (-bq^{\frac{s(k-1)}{2}+1}; q)_{\frac{s}{2}-1}. \end{aligned}$$

The sum of the generating functions of all the cases is

$$\begin{aligned} A_k(q) \left\{ z^k q^{sk^2/2} (-zbq^{s(k-1)/2+1}; q)_{s/2-1} (-bq^{s(k-1)/2+1}; q)_{s/2-1} \right. \\ \left. + (1 - zq^{sk/2})(1 - q^{sk/2}) \sum_{i=1}^{s/2-1} bz^k q^{sk^2/2-i} \right\}, \end{aligned}$$

where

$$A_k(q) = \frac{(-zbq, -zbq^2, \dots, -zbq^{s/2-1}; q^{s/2})_{k-1} (-bq, -bq^2, \dots, -bq^{s/2-1}; q^{s/2})_{k-1}}{(zq^{s/2}; q^{s/2})_k (q^{s/2}; q^{s/2})_k}.$$

Now, the desired series expansion is obtained by summing the above expansion over k and adding one, namely,

$$\begin{aligned} \sum_{n,r,l \geq 0} D_s(n,r,l) z^r b^l q^n &= 1 + \sum_{k \geq 1} A_k(q) \times \\ &\left\{ z^k q^{sk^2/2} (-zbq^{s(k-1)/2+1}; q)_{s/2-1} (-bq^{s(k-1)/2+1}; q)_{s/2-1} \right. \\ &\left. + (1 - zq^{sk/2})(1 - q^{sk/2}) \sum_{i=1}^{s/2-1} bz^k q^{sk^2/2-i} \right\}. \end{aligned}$$

□

Example 7.8. Alladi's series expansion (7.7) can be obtained from **Theorem 7.10** by setting $s = 4$ and using the substitutions $z \rightarrow c$ and $b \rightarrow bc^{-1}$. That is,

$$\prod_{n \geq 1} \frac{(1 + zbq^{2n-1})}{(1 - zq^{2n})} = 1 + \sum_{k \geq 1} z^k q^{2k^2-1} \frac{(-zbq; q^2)_{k-1} (-bq^2; q)_{k-1}}{(-zq^2; q^2)_k (q^2; q^2)_k} \\ \times \left\{ q(1 + zbq^{2k-1})(1 + bq^{2k-1}) + b(1 - zq^{2k})(1 - q^{2k}) \right\}.$$

Then, by simplifying the term in the brackets, we get

$$\prod_{n \geq 1} \frac{(1 + zbq^{2n-1})}{(1 - zq^{2n})} = 1 + \sum_{k \geq 1} z^k q^{2k^2-1} \frac{(-zbq; q^2)_{k-1} (-bq^2; q)_{k-1}}{(-zq^2; q^2)_k (q^2; q^2)_k} (b + q)(1 + zbq^{4k-1}).$$

By using the substitutions $z \rightarrow c$ and $b \rightarrow bc^{-1}$, we obtain

$$\prod_{n \geq 1} \frac{(1 + bq^{2n-1})}{(1 - cq^{2n})} = 1 + \sum_{k \geq 1} c^k q^{2k^2-1} \frac{(-bq; q^2)_{k-1} (-bc^{-1}q^2; q)_{k-1}}{(-cq^2; q^2)_k (q^2; q^2)_k} (bc^{-1} + q)(1 + bq^{4k-1}).$$

To delve deeper into the structure of s -duplicate partitions, we focus on $D_s(n, k)$, the number of s -duplicate partitions of n into k parts. Let $D_4(n, 2)$ denote the number of 4-duplicate partitions into 2 parts of n . In the Online Encyclopedia of Integer Sequences [197], we find that the function $D_4(n, 2)$ matches the sequence [A004524](#), where it appears to have several interesting combinatorial interpretations. In the next theorem we use the sum operation on partitions.

Theorem 7.11. The generating function for the number of 4-duplicate partitions into 2 parts is given by

$$\sum_{n \geq 0} D_4(n, 2)q^n = \frac{2q^4}{(q^2; q^2)_2} + \frac{q^3}{(1 - q^2)^2}.$$

Proof. There are three types of partitions in $\mathbb{D}_4(n, 2)$: partitions into two even parts, partitions into one even part and one odd part, and partitions into two distinct odd parts. The first and the third type satisfy the same generating function with different combinatorial interpretations, which is

$$\frac{q^4}{(q^2; q^2)_2} = q^4 + q^6 + 2q^8 + 2q^{10} + \dots$$

The function $1/(q^2; q^2)_2$ generates partitions into at most two even parts. In terms of Ferrers diagram, the exponent on q^4 consists of the partitions (2, 2) and (3, 1) in the first and the third type respectively. Then we add the partitions generated by $1/(q^2; q^2)_2$ to (2, 2) and (3, 1) to obtain the desired types of partitions. Meanwhile, the second type's partitions are generated by

$$\frac{q^2}{(1 - q^2)} \cdot \frac{q}{(1 - q^2)} = q^3 + 2q^5 + 3q^7 + 4q^9 + \dots$$

By summing up these generating functions, our result follows. \square

The following result follows from **Theorem 7.8**.

Corollary 7.1. *We have that $D_4(0,2) = D_4(1,2) = D_4(2,2) = 0$, $D_4(3,2) = 1$, and $D_4(4,2) = D_4(5,2) = D_4(6,2) = 2$. For $n \geq 7$, $D_4(n,2)$ satisfies the recurrence relation*

$$D_4(n,2) = D_4(n-4,2) + 2.$$

Proof. By listing the first 6 terms of recurrence (7.12) in Example 7.5 for $k = 2$, we get

$$D_4(3,2) = D_4(1,1) + D_4(-1,2) + D_4(0,1) = 1 + 0 + 0 = 1,$$

$$D_4(4,2) = D_4(2,1) + D_4(0,2) + D_4(1,1) = 1 + 0 + 1 = 2,$$

$$D_4(5,2) = D_4(3,1) + D_4(1,2) + D_4(2,1) = 1 + 0 + 1 = 2,$$

$$D_4(6,2) = D_4(4,1) + D_4(2,2) + D_4(3,1) = 1 + 0 + 1 = 2,$$

$$D_4(7,2) = D_4(5,1) + D_4(3,2) + D_4(4,1) = 1 + 1 + 1 = 3,$$

$$D_4(8,2) = D_4(6,1) + D_4(4,2) + D_4(5,1) = 1 + 2 + 1 = 4.$$

We observe that $D_4(n,1) = 1$ for all $n \geq 1$. Thus, we conclude that $D_4(n,2) = D_4(n-4,2) + 2$ for all $n \geq 7$. \square

7.3 Partitions into parts simultaneously s -congruent and t -distinct

A t -distinct partition λ of a positive integer n is a finite sequence of positive integers such that $\lambda_1^{\alpha_1} + \lambda_2^{\alpha_2} + \cdots + \lambda_k^{\alpha_k} = n$, where $1 \leq \alpha_i < t$ and $t \geq 2$. Note that in the literature, t -distinct partitions are also defined by a gap between parts condition. We shall impose an additional restriction on the set of the s -congruent partitions to obtain a new set of partitions into parts simultaneously s -congruent and t -distinct.

Definition 7.4. *A partition into parts simultaneously s -congruent and t -distinct is a partition into parts not congruent to $2, 4, 6, \dots, (s-2)$ modulo s and appearing fewer than t times. We denote by $C_s^t(n)$ the number of partitions into parts simultaneously s -congruent and t -distinct of n .*

The generating function for $C_s^t(n)$ is given by

$$\sum_{n \geq 0} C_s^t(n) q^n = \prod_{n \geq 1} \frac{(1 - q^{t(2n-1)})(1 - q^{tsn})}{(1 - q^{2n-1})(1 - q^{sn})}. \quad (7.17)$$

For example, $C_4^4(9) = 9$ and the corresponding set is

$$C_4^4(9) = \{(9), (8, 1), (7, 1^2), (3^3), (3^2, 1^3), (5, 4), (5, 3, 1), (4, 3, 1^2), (4^2, 1)\}.$$

In further exploration of Andrews' general **Theorem 1.12**, we enlarge its scope by expanding the class of partitions enumerated by $V_{k,i}(n)$ for certain values of k and i .

Definition 7.5. Let $t \geq 3$ be a positive integer with $t \not\equiv 2, 4, 6, \dots, (s-2) \pmod{s}$. Let $E_s^t(n)$ denote the number of partitions of n into parts not congruent to $2, 4, 6, \dots, (s-2)$ modulo s and not congruent to $0, t(2r+1)$ modulo ts , where $r = 0, 1, 2, 3, \dots, s/2 - 1$.

Theorem 7.12. The generating function for $E_s^t(n)$ is given by

$$\sum_{n \geq 0} E_s^t(n)q^n = \frac{(q^2; q^2)_\infty (q^t; q^t)_\infty (q^{ts}; q^{ts})_\infty}{(q; q)_\infty (q^s; q^s)_\infty (q^{2t}; q^{2t})_\infty}.$$

Proof. The proof is straightforward and involves elementary generating function manipulations. Consider the positive integer $t \geq 3$ with $t \not\equiv 2, 4, 6, \dots, (s-2) \pmod{s}$. The generating function

$$\frac{(q^2; q^2)_\infty}{(q; q)_\infty (q^s; q^s)_\infty} \quad (7.18)$$

generates partitions into parts $\lambda_i \not\equiv 2, 4, 6, \dots, (s-2) \pmod{s}$. Now, for $r = 0, 1, 2, 3, \dots, s/2 - 1$, consider the product of generating functions

$$(q^t; q^{ts})_\infty (q^{3t}; q^{ts})_\infty (q^{5t}; q^{ts})_\infty \cdots (q^{t(s-1)}; q^{ts})_\infty. \quad (7.19)$$

For $n \geq 0$, the exponents on q in the product (7.19) are of the form

$$\begin{aligned} & t(sn+1) \\ & t(sn+3) \\ & t(sn+5) \\ & \vdots \\ & t(sn+s-1). \end{aligned}$$

Since $s \geq 4$ is even and $t \geq 3$, we obtain

$$(q^t; q^{ts})_\infty (q^{3t}; q^{ts})_\infty (q^{5t}; q^{ts})_\infty \cdots (q^{t(s-1)}; q^{ts})_\infty = \prod_{n \geq 1} (1 - q^{t(2n-1)}),$$

therefore,

$$(q^t; q^{ts})_\infty (q^{3t}; q^{ts})_\infty (q^{5t}; q^{ts})_\infty \cdots (q^{t(s-1)}; q^{ts})_\infty (q^{ts}; q^{ts})_\infty = \frac{(q^t; q^t)_\infty}{(q^{2t}; q^{2t})_\infty} (q^{ts}; q^{ts})_\infty. \quad (7.20)$$

By multiplying the generating functions (7.18) and (7.20) together, our result follows. \square

Example 7.9. For $(s, t, n) = (6, 3, 14)$, there are 13 partitions into parts not congruent to 2, 4 modulo 6 and 0, 3, 9, 15 modulo 18, such that

$$\mathbb{E}_6^3(14) = \{(13, 1), (12, 1^2), (11, 3), (7^2), (7, 6, 1), (7, 5, 1^2), (7, 1^7), (6^2, 1^2), (6, 1^8), (6, 5, 1^3), (5^2, 1^4), (5, 1^9), (1^{14})\}.$$

Theorem 7.13. Let $t \geq 3$ be a positive integer with $t \not\equiv 2, 4, 6, \dots, (s - 2)$ modulo s . Then for every positive integer $n \geq 0$,

$$C_s^t(n) = E_s^t(n).$$

Proof. By employing the fact $(q; q^2)_\infty^{-1} = (q^2; q^2)_\infty / (q; q)_\infty$ in (7.17), we obtain

$$\begin{aligned} \sum_{n \geq 0} C_s^t(n)q^n &= \frac{(q^t; q^{2t})_\infty (q^{ts}; q^{ts})_\infty}{(q; q^2)_\infty (q^s; q^s)_\infty} \\ &= \frac{(q^2; q^2)_\infty (q^t; q^t)_\infty (q^{ts}; q^{ts})_\infty}{(q; q)_\infty (q^{2t}; q^{2t})_\infty (q^s; q^s)_\infty} \\ &= \sum_{n \geq 0} E_s^t(n)q^n. \end{aligned}$$

□

The relation between the number $E_s^t(n)$ and Andrews' theorem [23, p. 114] becomes apparent by setting $s = 4$ in **Definition 7.5**. We have

$$E_4^t(n) = V_{t, (t+1)/2}(n)$$

for every odd $t \geq 3$, where $t = k = 2i - 1$. In essence, this establishes a relationship with Andrews' theorem and highlights the significance of these class of partitions in the context of the study.

Corollary 7.2. Let $t \geq 3$ be an odd integer. Then for every natural number $n \geq 0$,

$$C_4^t(n) = V_{t, (t+1)/2}(n) = W_{t, (t+1)/2}(n).$$

Note that, when $t = 3, 5$, the generating function

$$\begin{aligned} \sum_{n \geq 0} C_4^t(n)q^n &= \frac{(q^2; q^2)_\infty (q^t; q^t)_\infty (q^{4t}; q^{4t})_\infty}{(q; q)_\infty (q^4; q^4)_\infty (q^{2t}; q^{2t})_\infty} \\ &= \frac{\psi(-q^t)}{\psi(-q)} \end{aligned}$$

is the generating function for the sequences [A036018](#) and [A036026](#), respectively, in the Online Encyclopedia of Integer Sequences [197].

TABLE 7.12: An illustration of **Corollary 7.2** for $(s, t, n) = (4, 3, 12)$.

$\mathbb{C}_4^3(12)$	$\mathbb{V}_{3,2}(12)$	$\mathbb{W}_{3,2}(12)$
(12)	(1 ¹²)	(12)
(11, 1)	(11, 1)	(11, 1)
(8, 4)	(8, 4)	(8, 4)
(8, 3, 1)	(8, 1 ⁴)	(8, 3, 1)
(7, 5)	(7, 5)	(7, 5)
(7, 4, 1)	(7, 4, 1)	(7, 4, 1)
(7, 3, 1 ²)	(7, 1 ⁵)	(7, 3, 2)
(5 ² , 1 ²)	(5 ² , 1 ²)	(10, 2)
(5, 4, 3)	(5, 4, 1 ³)	(5, 4, 3)
(5, 3 ² , 1)	(5, 1 ⁷)	(6, 5, 1)
(9, 3)	(4 ³)	(9, 3)
(4, 3 ² , 1 ²)	(4, 1 ⁸)	(6, 4, 2)
(4 ² , 3, 1)	(4 ² , 1 ⁴)	(6 ²)

7.4 s -Modular, s -congruent and s -duplicate overpartitions

An overpartition of a nonnegative integer n is a nonincreasing sequence of natural numbers whose sum is n , and where the first occurrence of a number may be overlined. They were introduced by Corteel and Lovejoy [67]. The generating function of overpartition is

$$\sum_{n \geq 0} \bar{p}(n)q^n = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}}.$$

where $\bar{p}(n)$ denotes the number of overpartitions of n . Let $\bar{M}_s(n)$, $\bar{C}_s(n)$, and $\bar{D}_s(n)$ denote the number of s -modular, s -congruent, and s -duplicate overpartitions of n , respectively.

Theorem 7.14. *The generating function for $\bar{M}_s(n)$, the number of s -modular overpartitions of n , is given by*

$$\sum_{n \geq 0} \bar{M}_s(n)q^n = \frac{(-2q; q)_{\infty}(-q^s; q^s)_{\infty}}{(q^s; q^s)_{\infty}} = \frac{(-2q; q)_{\infty}(q^{2s}; q^{2s})_{\infty}}{(q^s; q^s)_{\infty}^2}.$$

Proof. The generating function for the number of overpartitions where the multiplicity of every part is congruent to 0 modulo s is given by

$$\frac{(-q^s; q^s)_{\infty}}{(q^s; q^s)_{\infty}},$$

and on the other hand, the number of overpartitions where the multiplicity of every part is congruent to 1 modulo s has generating function

$$(-2q; q)_{\infty}.$$

Then, by multiplying both generating functions together, our result follows. \square

Theorem 7.15. *The generating function for $\overline{C}_s(n)$, the number of s -congruent overpartitions of n , is given by*

$$\sum_{n \geq 0} \overline{C}_s(n) q^n = \frac{(-q; q^2)_\infty (-q^s; q^s)_\infty}{(q; q^2)_\infty (q^s; q^s)_\infty} = \frac{(q^2; q^2)_\infty^3 (q^{2s}; q^{2s})_\infty}{(q; q)_\infty^2 (q^4; q^4)_\infty (q^s; q^s)_\infty^2}.$$

Proof. The number of overpartitions into odd parts has the generating function

$$\frac{(-q; q^2)_\infty}{(q; q^2)_\infty},$$

while

$$\frac{(-q^s; q^s)_\infty}{(q^s; q^s)_\infty}$$

generates overpartitions into parts congruent to 0 modulo s . \square

Theorem 7.16. *The generating function for $\overline{D}_s(n)$, the number of s -duplicate overpartitions of n , satisfies the identity*

$$\sum_{n \geq 0} \overline{D}_s(n) q^n = \frac{(-2q; q)_\infty (-q^{s/2}; q^{s/2})_\infty}{(-2q^{s/2}; q^{s/2})_\infty (q^{s/2}; q^{s/2})_\infty} = \frac{(-2q; q)_\infty (q^s; q^s)_\infty}{(-2q^{s/2}; q^{s/2})_\infty (q^{s/2}; q^{s/2})_\infty^2}.$$

Proof. The generating function

$$\frac{(-2q; q)_\infty}{(-2q^{s/2}; q^{s/2})_\infty}$$

generates overpartitions into distinct parts indivisible by $s/2$, and

$$\frac{(-q^{s/2}; q^{s/2})_\infty}{(q^{s/2}; q^{s/2})_\infty}$$

generates overpartitions into parts congruent to 0 modulo $s/2$. \square

CHAPTER 8

ARITHMETIC PROPERTIES OF PARTITIONS ENUMERATED BY THE ANDREWS-GÖLLNITZ-GORDON THEOREM

In this chapter, our goal is to study the arithmetic properties of $C_{k,i}(n)$, the partitions enumerated by the Andrews-Göllnitz-Gordon **Theorem 1.12**, for $k \in \{3, 5, 9\}$ such that $k = 2i - 1$. Through this chapter, we will omit the parameter i from $C_{k,i}(n)$. From **Theorem 1.13** we have

$$\sum_{n \geq 0} C_k(n)q^n = \frac{(q^2; q^2)_\infty (q^k; q^k)_\infty (q^{4k}; q^{4k})_\infty}{(q; q)_\infty (q^4; q^4)_\infty (q^{2k}; q^{2k})_\infty} = \frac{\psi(-q^k)}{\psi(-q)}, \quad (8.1)$$

where $C_k(n)$ enumerates the number of partitions of n into parts not congruent to 2 modulo 4 and 0, k or $3k$ modulo $4k$. Note that, when $k = 3, 5, 9$, the generating function (8.1) matches the sequences [A036018](#), [A036026](#), and [A132975](#) (shifted by one term) in [197], respectively.

8.1 Congruences modulo 2 for $C_3(n)$

Theorem 8.1. *For any nonnegative integer n , we have*

$$\sum_{n \geq 0} C_3(6n)q^n \equiv f_1 \pmod{2}, \quad (8.2)$$

$$\sum_{n \geq 0} C_3(54n + 2)q^n \equiv f_1 \pmod{2}, \quad (8.3)$$

$$\sum_{n \geq 0} C_3(162n + 20)q^n \equiv f_1^3 \pmod{2}, \quad (8.4)$$

$$C_3(6n + 4) \equiv 0 \pmod{2}, \quad (8.5)$$

$$C_3(18n + 8) \equiv 0 \pmod{2}, \quad (8.6)$$

$$C_3(18n + 14) \equiv 0 \pmod{2}, \quad (8.7)$$

$$C_3(54n + 38) \equiv 0 \pmod{2}, \quad (8.8)$$

$$C_3(162n + 74) \equiv 0 \pmod{2}, \quad (8.9)$$

$$C_3(162n + 128) \equiv 0 \pmod{2}. \quad (8.10)$$

Proof. Setting $k = 3$ in (8.1), we obtain

$$\sum_{n \geq 0} C_3(n)q^n = \frac{f_2 f_3 f_{12}}{f_1 f_4 f_6} = \frac{\psi(-q^3)}{\psi(-q)}. \quad (8.11)$$

Substituting (2.15) into (8.11), we get

$$\sum_{n \geq 0} C_3(n)q^n = \frac{f_{18}^9}{f_3 f_6 f_9^3 f_{12} f_{36}^3} + q \frac{f_6 f_{18}^3}{f_3^2 f_{12}^2} + q^2 \frac{f_6^3 f_9^3 f_{36}^3}{f_3^3 f_{12}^3 f_{18}^3}.$$

Extracting the terms involving q^{3n+j} for $j = 0, 1, 2$ from both sides of the above equation, we deduce that

$$\sum_{n \geq 0} C_3(3n)q^n = \frac{f_6^9}{f_1 f_2 f_3^3 f_4 f_{12}^3}, \quad (8.12)$$

$$\sum_{n \geq 0} C_3(3n+1)q^n = \frac{f_2 f_6^3}{f_1^2 f_4^2}, \quad (8.13)$$

$$\sum_{n \geq 0} C_3(3n+2)q^n = \frac{f_2^3 f_3^3 f_{12}^3}{f_1^3 f_4^3 f_6^3} = \frac{\psi(-q^3)^3}{\psi(-q)^3}. \quad (8.14)$$

In view of (2.2) and (8.12), with $p = 2$ and $k = 1$, along with invoking (2.20), we find that

$$\sum_{n \geq 0} C_3(3n)q^n \equiv \frac{f_3^3}{f_1 f_2 f_4} = \frac{f_4^2 f_6^2}{f_2^3 f_{12}} + q \frac{f_{12}^3}{f_2 f_4^2} \pmod{2}. \quad (8.15)$$

Congruence (8.2) follows by equating even powers on both sides of (8.15).

Applying (2.2) on (8.13), with $p = 2$ and $k = 1$, we obtain

$$\sum_{n \geq 0} C_3(3n+1)q^n \equiv \frac{f_6^3}{f_4^2} \pmod{2}. \quad (8.16)$$

Congruence (8.5) follows from (8.16) by extracting the terms involving q^{2n+1} from both sides.

Employing (2.2) in (8.14), with $p = 2$ and $k = 1$, and using the definition of $\psi(-q)$, we find that

$$\sum_{n \geq 0} C_3(3n+2)q^n = \frac{\psi(-q^3)^2 \psi(-q^3)}{\psi(-q)^2 \psi(-q)} \equiv \frac{\psi(-q^6) \psi(-q^3)}{\psi(-q^2) \psi(-q)} \equiv \frac{f_3 f_{24}}{f_1 f_8} \pmod{2}. \quad (8.17)$$

Substituting (2.18) into (8.17), and then extracting the even powers from both sides of the resulting equation, we arrive at

$$\sum_{n \geq 0} C_3(6n+2)q^n \equiv \frac{f_2 f_3^3}{f_1^2} \pmod{2}. \quad (8.18)$$

Again, substituting (2.13) into (8.18), we get

$$\sum_{n \geq 0} C_3(6n + 2)q^n \equiv \frac{f_6^4 f_9^6}{f_3^5 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^4} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^3} \pmod{2}. \quad (8.19)$$

Congruences (8.6) and (8.7) follow from (8.19) by equating the terms containing q^{3n+1} and q^{3n+2} on both sides, respectively.

Collecting the terms involving q^{3n} from both sides of (8.19), we obtain

$$\sum_{n \geq 0} C_3(18n + 2)q^n \equiv \frac{f_2^4 f_3^6}{f_1^5 f_6^3} \equiv \frac{f_2^2}{f_1} \pmod{2}. \quad (8.20)$$

Employing (2.16) into (8.20), we get

$$\sum_{n \geq 0} C_3(18n + 2)q^n \equiv \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9} \pmod{2}. \quad (8.21)$$

If we extract the terms of the form q^{3n} from the above equation and then replace q^3 by q , we deduce that

$$\sum_{n \geq 0} C_3(54n + 2)q^n \equiv f_1 \pmod{2}. \quad (8.22)$$

Congruence (8.8) follows from (8.21) by equating the powers of the form q^{3n+2} on both sides. By extracting the terms in (8.21) that involves q^{3n+1} , dividing both sides by q and then replacing q^{3n} by q , we obtain

$$\sum_{n \geq 0} C_3(54n + 20)q^n \equiv f_3^3 \pmod{2}. \quad (8.23)$$

Congruences (8.4), (8.9), and (8.10) are immediate from Equation (8.23). \square

Theorem 8.2. For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, we have

$$C_3 \left(6p^{2\alpha+1}(pn + i) + \frac{p^{2\alpha+2} - 1}{4} \right) \equiv 0 \pmod{2}, \quad (8.24)$$

$$C_3 \left(54p^{2\alpha+1}(pn + i) + \frac{9p^{2\alpha+2} - 1}{4} \right) \equiv 0 \pmod{2}, \quad (8.25)$$

where i is an integer and $1 \leq i \leq p - 1$.

Proof. Define

$$\sum_{n \geq 0} a(n)q^n = f_1. \quad (8.26)$$

Combining (8.2), (8.3) and (8.26), we deduce that

$$C_3(6n) \equiv a(n) \pmod{2}, \quad (8.27)$$

$$C_3(54n + 2) \equiv a(n) \pmod{2}. \quad (8.28)$$

Now, we consider the congruence equation

$$\frac{3m^2 + m}{2} \equiv \frac{p^2 - 1}{24} \pmod{p},$$

which is equivalent to

$$(6m + 1)^2 \equiv 0 \pmod{p}, \quad (8.29)$$

where $-(p-1)/2 \leq m \leq (p-1)/2$ and $p \geq 5$ is a prime. Then, the congruence relation (8.29) holds if and only if $m = (\pm p - 1)/6$. Therefore, if we substitute (2.35) into (8.26) and then extract the terms in which the powers of q are congruent to $\frac{p^2-1}{24}$ modulo p and then divide by $q^{(p^2-1)/24}$, we find that

$$\sum_{n \geq 0} a\left(pn + \frac{p^2 - 1}{24}\right) q^{pn} = (-1)^{\frac{\pm p - 1}{6}} f_{p^2},$$

which implies that,

$$\sum_{n \geq 0} a\left(p^2n + \frac{p^2 - 1}{24}\right) q^n = (-1)^{\frac{\pm p - 1}{6}} f_1,$$

and for $n \geq 0$,

$$a\left(p^2n + pi + \frac{p^2 - 1}{24}\right) = 0, \quad (8.30)$$

where i is an integer and $1 \leq i \leq p - 1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha}n + \frac{p^{2\alpha} - 1}{24}\right) = (-1)^{\alpha \frac{\pm p - 1}{6}} a(n). \quad (8.31)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{24}$ in (8.31) and using (8.30), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2} - 1}{24}\right) = 0.$$

Again, replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{24}$ ($1 \leq i \leq p - 1$) in (8.27) and (8.28), we arrive at (8.24) and (8.25), respectively. \square

Theorem 8.3. For any odd prime p , $\alpha \geq 0$ and $n \geq 0$, we have

$$C_3 \left(162p^{2\alpha+1}(pn+i) + \frac{81p^{2\alpha+2}-1}{4} \right) \equiv 0 \pmod{2}, \quad (8.32)$$

where i is an integer and $1 \leq i \leq p-1$.

Proof. We have that

$$\sum_{n \geq 0} C_3(162n+20)q^n \equiv f_1^3 = \sum_{n \geq 0} (-1)^n (2n+1)q^{n(n+1)/2} \equiv \sum_{n \geq 0} q^{n(n+1)/2} = \psi(q) \pmod{2}. \quad (8.33)$$

Define

$$\sum_{n \geq 0} b(n)q^n = \psi(q). \quad (8.34)$$

Combining (8.4) and (8.34), we find that

$$C_3(162n+20) \equiv b(n) \pmod{2}. \quad (8.35)$$

Now, consider the congruence equation

$$\frac{k^2+k}{2} \equiv \frac{p^2-1}{8} \pmod{p},$$

which is equivalent to

$$(2k+1)^2 \equiv 0 \pmod{p}, \quad (8.36)$$

where $0 \leq k \leq (p-1)/2$ and p is an odd prime. Then, the congruence relation (8.36) holds if and only if $k = (p-1)/2$. Therefore, if we substitute (2.34) into (8.34) and then extract the terms in which the powers of q are $pn + \frac{p^2-1}{8}$, we arrive at

$$\sum_{n \geq 0} b \left(pn + \frac{p^2-1}{8} \right) q^{pn + \frac{p^2-1}{8}} = q^{pn + \frac{p^2-1}{8}} \psi(q^{p^2}).$$

Dividing $q^{\frac{p^2-1}{8}}$ on both sides of the above equation and then replacing q^p by q , we find that

$$\sum_{n \geq 0} b \left(pn + \frac{p^2-1}{8} \right) q^n = \psi(q^p),$$

which implies that ,

$$\sum_{n \geq 0} b \left(p^2n + \frac{p^2-1}{8} \right) q^n = \psi(q),$$

and for $n \geq 0$,

$$b\left(p^2n + pi + \frac{p^2 - 1}{8}\right) = 0, \quad (8.37)$$

where i is an integer and $1 \leq i \leq p - 1$. By mathematical induction, we deduce that for $\alpha \geq 0$,

$$b\left(p^{2\alpha}n + \frac{p^{2\alpha} - 1}{8}\right) = b(n). \quad (8.38)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{8}$ ($1 \leq i \leq p - 1$) in (8.38) and using (8.37), we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$b\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2} - 1}{8}\right) \equiv 0 \pmod{2}.$$

Replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{8}$ in (8.35), we obtain the desired result. \square

8.2 Congruences modulo 3 for $C_3(n)$

Theorem 8.4. *For any nonnegative integer n , we have*

$$\sum_{n \geq 0} C_3(n)q^n \equiv \sum_{n \geq 0} C_3(9n + 2)q^n \equiv \psi(-q)^2 \pmod{3}, \quad (8.39)$$

$$C_3(9n + 5) \equiv 0 \pmod{3}, \quad (8.40)$$

$$C_3(9n + 8) \equiv 0 \pmod{3}. \quad (8.41)$$

Proof. Utilizing (2.2) in (8.14), with $p = 3$ and $k = 1$, we find that

$$\sum_{n \geq 0} C_3(3n + 2)q^n \equiv \psi(-q^3)^2 \pmod{3}. \quad (8.42)$$

If we extract the terms of the form q^{3n} from both sides of (8.42) and then replace q^3 by q , we obtain

$$\sum_{n \geq 0} C_3(9n + 2)q^n \equiv \psi(-q)^2 \pmod{3}. \quad (8.43)$$

Congruences (8.40) and (8.41) are obtained from (8.42) by extracting the terms involving q^{3n+j} for $j = 1, 2$ from both sides. In view of (2.2) and (8.11), with $p = 3$ and $k = 1$, we get

$$\sum_{n \geq 0} C_3(n)q^n \equiv \psi(-q)^2 \pmod{3}. \quad (8.44)$$

\square

Theorem 8.5. For any prime $p \equiv 3 \pmod{4}$, $\alpha \geq 0$ and $n \geq 0$, we have

$$C_3\left(p^{2\alpha+1}(pn+i) + \frac{p^{2\alpha+2}-1}{4}\right) \equiv 0 \pmod{3}, \quad (8.45)$$

$$C_3\left(3p^{2\alpha+1}(pn+i) + \frac{3p^{2\alpha+2}+5}{4}\right) \equiv 0 \pmod{3}, \quad (8.46)$$

where i is an integer and $1 \leq i \leq p-1$.

Proof. Define

$$\sum_{n \geq 0} c(n)q^n = \psi(-q)^2. \quad (8.47)$$

Combining (8.39) and (8.47), we find that

$$C_3(n) \equiv C_3(3n+2) \equiv c(n) \pmod{3}. \quad (8.48)$$

Now, consider the congruence equation

$$\frac{k^2+k}{2} + \frac{m^2+m}{2} \equiv \frac{p^2-1}{4} \pmod{p},$$

which is equivalent to

$$(2k+1)^2 + (2m+1)^2 \equiv 0 \pmod{p}, \quad (8.49)$$

where $0 \leq k, m \leq (p-1)/2$ and p is a prime such that $\left(\frac{-1}{p}\right) = -1$. Since $\left(\frac{-1}{p}\right) = -1$ for $p \equiv 3 \pmod{4}$, the congruence relation (8.49) holds if and only if both $k = m = (p-1)/2$. Substituting (2.34) into (8.47) and then extracting the terms in which the powers of q are congruent to $\frac{p^2-1}{4}$ modulo p and then divide by $q^{\frac{p^2-1}{4}}$, we find that

$$\sum_{n \geq 0} c\left(pn + \frac{p^2-1}{4}\right) q^{pn} = \psi(q^{p^2})^2,$$

which implies that,

$$\sum_{n \geq 0} c\left(p^2n + \frac{p^2-1}{4}\right) q^n = \psi(q)^2,$$

and for $n \geq 0$,

$$c\left(p^2n + pi + \frac{p^2-1}{4}\right) = 0, \quad (8.50)$$

where i is an integer and $1 \leq i \leq p-1$. By induction we see that for $n \geq 0$ and $\alpha \geq 0$,

$$c\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{4}\right) = c(n). \quad (8.51)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{4}$ ($1 \leq i \leq p-1$) in (8.51) and using (8.50), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$c\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{4}\right) = 0.$$

Again, replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{4}$ in (8.48) ($1 \leq i \leq p-1$) we arrive at (8.45) and (8.46). \square

Corollary 8.1. For all $n \geq 0$,

$$C_3(18n + 8) \equiv 0 \pmod{6}, \quad (8.52)$$

$$C_3(18n + 14) \equiv 0 \pmod{6}. \quad (8.53)$$

Proof. Congruence (8.52) follows from (8.6) and (8.41), and (8.53) follows from (8.7) and (8.40). \square

8.3 Congruences modulo 2 for $C_5(n)$

Theorem 8.6. For any nonnegative integer n , we have

$$\sum_{n \geq 0} C_5(4n)q^n \equiv \psi(q)f_{40} \pmod{2}, \quad (8.54)$$

$$\sum_{n \geq 0} C_5(20n + 2)q^n \equiv \psi(q) \pmod{2}, \quad (8.55)$$

$$C_5(4n + 3) \equiv 0 \pmod{2}, \quad (8.56)$$

$$C_5(20n + 6) \equiv 0 \pmod{2}, \quad (8.57)$$

$$C_5(20n + 10) \equiv 0 \pmod{2}, \quad (8.58)$$

$$C_5(20n + 14) \equiv 0 \pmod{2}, \quad (8.59)$$

$$C_5(20n + 18) \equiv 0 \pmod{2}. \quad (8.60)$$

Proof. Setting $k = 5$ in (8.1), we have

$$\sum_{n \geq 0} C_5(n)q^n = \frac{f_2 f_5 f_{20}}{f_1 f_4 f_{10}} = \frac{\psi(-q^5)}{\psi(-q)}. \quad (8.61)$$

Substituting (2.27) into (8.61), we obtain

$$\sum_{n \geq 0} C_5(n)q^n = \frac{f_8 f_{20}^3}{f_2 f_4 f_{10} f_{40}} + q \frac{f_4^2 f_{40}}{f_2^2 f_8}.$$

Extracting the terms involving q^{2n+1} for $j = 0, 1$ from both sides of the above equation, we deduce that

$$\sum_{n \geq 0} C_5(2n)q^{2n} = \frac{f_8 f_{20}^3}{f_2 f_4 f_{10} f_{40}}, \quad (8.62)$$

and

$$\sum_{n \geq 0} C_5(2n+1)q^{2n+1} = q \frac{f_4^2 f_{40}}{f_2^2 f_8}. \quad (8.63)$$

By replacing q^2 by q in (8.62) and invoking (2.2), with $p = 2$ and $k = 1$, we get

$$\sum_{n \geq 0} C_5(2n)q^n = \frac{f_4 f_{10}^3}{f_1 f_2 f_5 f_{20}} \equiv f_1 f_5 \pmod{2}. \quad (8.64)$$

Substituting (2.29) into (8.64), we arrive at

$$\sum_{n \geq 0} C_5(2n)q^n \equiv \frac{f_2 f_8 f_{20} f_{40}^4}{f_4 f_{10}^2 f_{80}} - q \frac{f_4^2 f_{40}^6}{f_8 f_{10} f_{20}^2 f_{80}^2} \pmod{2}. \quad (8.65)$$

Equating the even powers of q on both sides of (8.65), we obtain

$$\sum_{n \geq 0} C_5(4n)q^n \equiv \frac{f_1 f_4 f_{10} f_{20}^4}{f_2 f_5^2 f_{40}} \pmod{2}. \quad (8.66)$$

Equating the odd powers of q on both sides of (8.65), we get

$$\sum_{n \geq 0} C_5(4n+2)q^n \equiv \frac{f_2^2 f_{20}^6}{f_4 f_5 f_{10}^2 f_{40}^2} \pmod{2}. \quad (8.67)$$

In view of (2.1) and (8.66), we arrive at

$$\sum_{n \geq 0} C_5(4n)q^n \equiv f_1^3 f_{40} \equiv \psi(q) f_{40} \pmod{2}, \quad (8.68)$$

and from (8.67), we obtain

$$\sum_{n \geq 0} C_5(4n+2)q^n \equiv f_5^3 \pmod{2}. \quad (8.69)$$

Congruence (8.54) is immediate from (8.68). Meanwhile, congruences (8.55), (8.57), (8.58), (8.59), and (8.60) follow from (8.69) by extracting the terms involving q^{5n+j} for $j = 0, 1, 2, 3, 4$ from both sides.

Extracting the terms involving the powers of q^{2n+1} from (8.63), dividing by q both sides and then replacing q^2 by q , we arrive at

$$\sum_{n \geq 0} C_5(2n+1)q^n = \frac{f_2^2 f_{20}}{f_1^2 f_4}. \quad (8.70)$$

Utilizing (2.2) in (8.70), with $p = 2$ and $k = 1$, we find that

$$\sum_{n \geq 0} C_5(2n+1)q^n \equiv \frac{f_{20}}{f_2} \pmod{2}. \quad (8.71)$$

Congruence (8.56) follows by equating the odd powers from both sides of the above equation. \square

Theorem 8.7. For any prime $p \equiv 5 \pmod{6}$, $\alpha \geq 0$ and $n \geq 0$, we have

$$C_5\left(4p^{2\alpha+1}(pn+i) + \frac{43p^{2\alpha+2} - 43}{6}\right) \equiv 0 \pmod{2}, \quad (8.72)$$

where i is an integer and $1 \leq i \leq p-1$.

Proof. Define

$$\sum_{n \geq 0} d(n)q^n = \psi(q)f_{40}. \quad (8.73)$$

Combining (8.54) and (8.73), we find that

$$C_5(4n) \equiv d(n) \pmod{2}. \quad (8.74)$$

Consider the congruence equation

$$\frac{k^2+k}{2} + 40 \cdot \frac{3m^2+m}{2} \equiv \frac{43p^2-43}{24} \pmod{p},$$

which is equivalent to

$$(2k+1)^2 + 40 \cdot (6m+1)^2 \equiv 0 \pmod{p}, \quad (8.75)$$

where $0 \leq k \leq (p-1)/2$, $-(p-1)/2 \leq m \leq (p-1)/2$ and p is a prime such that $p \equiv 5 \pmod{6}$. Then, the congruence relation (8.75) holds if and only if $k = (p-1)/2$ and $m = (\pm p-1)/6$. Therefore, if we substitute (2.34) and (2.35) into (8.73) and then extract the terms in which the powers of q are $pn + \frac{43p^2-43}{24}$, we arrive at

$$\sum_{n \geq 0} d\left(pn + \frac{43p^2-43}{24}\right) q^{pn + \frac{43p^2-43}{24}} = (-1)^{\frac{\pm p-1}{6}} q^{pn + \frac{43p^2-43}{24}} \psi(q^{p^2}) f_{40p^2}. \quad (8.76)$$

Dividing $q^{\frac{43p^2-43}{24}}$ on both sides of (8.76) and then replacing q^p by q , we find that

$$\sum_{n \geq 0} d\left(pn + \frac{43p^2-43}{24}\right) q^n = (-1)^{\frac{\pm p-1}{6}} \psi(q^p) f_{40p},$$

which implies that,

$$\sum_{n \geq 0} d\left(p^2n + \frac{43p^2 - 43}{24}\right)q^n = (-1)^{\frac{\pm p-1}{6}} \psi(q) f_{40}, \quad (8.77)$$

and for $n \geq 0$,

$$d\left(p^2n + pi + \frac{43p^2 - 43}{24}\right) = 0, \quad (8.78)$$

where i is an integer and $1 \leq i \leq p - 1$. Combining (8.73) and (8.77), we see that

$$d\left(p^2n + \frac{43p^2 - 43}{24}\right) = (-1)^{\frac{\pm p-1}{6}} d(n). \quad (8.79)$$

By (8.79) and mathematical induction, we deduce that for $\alpha \geq 0$ and $n \geq 0$,

$$d\left(p^{2\alpha}n + \frac{43p^{2\alpha} - 43}{24}\right) = (-1)^\alpha \frac{\pm p-1}{6} d(n). \quad (8.80)$$

Replacing n by $p^2n + pi + \frac{43p^2 - 43}{24}$ ($1 \leq i \leq p - 1$) in (8.80) and using (8.78), we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$d\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{43p^{2\alpha+2} - 43}{24}\right) = 0.$$

Replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{43p^{2\alpha+2} - 43}{24}$ in (8.74) we obtain the desired result. \square

From Equations (8.55) and (8.34), we deduce that $C_5(20n + 2) \equiv b(n) \pmod{2}$, where $b(n)$ is defined in Equation (8.34). From **Theorem 8.3** we obtain the following result.

Corollary 8.2. For any odd prime p , $\alpha \geq 0$ and $n \geq 0$, we have

$$C_5\left(20p^{2\alpha+1}(pn + i) + \frac{5p^{2\alpha+2} - 1}{2}\right) \equiv 0 \pmod{2}. \quad (8.81)$$

where i is an integer and $1 \leq i \leq p - 1$.

We introduce the following conjectures for the number $C_5(n)$.

Conjecture 8.1. For all $n \geq 0$, we have

$$C_5(841n + 29i + 420) \equiv 0 \pmod{5},$$

where i is an integer and $1 \leq i \leq 28$.

Conjecture 8.2. For all $n \geq 0$, we have

$$C_5(841n + 29i + 420) \equiv 0 \pmod{10},$$

where i is an integer such that $i \equiv 2, 3 \pmod{4}$ and $1 \leq i \leq 28$.

8.4 Congruences modulo 2 for $C_9(n)$

Theorem 8.8. *For any nonnegative integer n , we have*

$$\sum_{n \geq 0} C_9(24n)q^n \equiv f_1 \pmod{2}. \quad (8.82)$$

$$C_9(6n + 3) \equiv 0 \pmod{2}, \quad (8.83)$$

$$C_9(6n + 5) \equiv 0 \pmod{2}, \quad (8.84)$$

$$C_9(12n + 8) \equiv 0 \pmod{2}, \quad (8.85)$$

$$C_9(24n + 13) \equiv 0 \pmod{2}, \quad (8.86)$$

$$C_9(24n + 19) \equiv 0 \pmod{2}, \quad (8.87)$$

$$C_9(24n + 12) \equiv 0 \pmod{2}, \quad (8.88)$$

$$C_9(24n + 18) \equiv 0 \pmod{2}. \quad (8.89)$$

Proof. Setting $k = 9$ in (8.1), we have

$$\sum_{n \geq 0} C_9(n)q^n = \frac{f_2 f_9 f_{36}}{f_1 f_4 f_{18}} = \frac{\psi(-q^9)}{\psi(-q)}. \quad (8.90)$$

In view of (2.2) and (8.90), with $p = 2$ and $k = 1$, we find that

$$\sum_{n \geq 0} C_9(n)q^n \equiv \frac{f_9 f_{18}}{f_1 f_2} \pmod{2}. \quad (8.91)$$

Substituting (2.30) into (8.91), we obtain

$$\sum_{n \geq 0} C_9(n)q^n \equiv \frac{f_{12}^3}{f_2^3 f_6} + q \frac{f_6 f_{18} f_{36}}{f_{12}} \pmod{2}. \quad (8.92)$$

Equating the even and the odd powers on both sides of (8.92), we get

$$\sum_{n \geq 0} C_9(2n)q^n \equiv \frac{f_6^3}{f_1^3 f_3} \equiv \frac{f_3 f_6^2}{f_1 f_2} \pmod{2}, \quad (8.93)$$

and

$$\sum_{n \geq 0} C_9(2n + 1)q^n \equiv \frac{f_3 f_9 f_{18}}{f_6} \equiv \frac{f_9^3}{f_3} \pmod{2}. \quad (8.94)$$

Employing (2.10) in (8.93), we obtain

$$\sum_{n \geq 0} C_9(2n)q^n \equiv \frac{f_9^9}{f_3^5 f_{18}^3} + q \frac{f_6^6}{f_3^4 f_6} + q^2 \frac{f_9^3 f_{18}^3}{f_3^3 f_6^2} \pmod{2}. \quad (8.95)$$

If we extract the terms of the form q^{3n+j} for $j = 0, 1$ from both sides of (8.95), we find that

$$\sum_{n \geq 0} C_9(6n)q^n \equiv \frac{f_3^9}{f_1^5 f_6^3} \equiv \frac{f_3^3}{f_1 f_2^2} \pmod{2}, \quad (8.96)$$

and

$$\sum_{n \geq 0} C_9(6n+2)q^n \equiv \frac{f_3^6}{f_1^4 f_2} \equiv \frac{f_6^3}{f_2^3} \pmod{2}. \quad (8.97)$$

Congruence (8.85) easily follows from (8.97) by equating the odd powers on both sides.

Substituting (2.20) into (8.96), we obtain

$$\sum_{n \geq 0} C_9(6n)q^n \equiv f_4 + q \frac{f_{12}^3}{f_8} \pmod{2}. \quad (8.98)$$

Congruences (8.82), (8.88), and (8.89) follow by equating the powers of the form q^{4n+j} for $j = 0, 2, 3$ on both sides of Equation (8.98).

Extracting the terms involving q^{3n} from both sides of equation (8.94) and then and replacing q^3 by q , we find that

$$\sum_{n \geq 0} C_9(6n+1)q^n \equiv \frac{f_3^3}{f_1} \pmod{2}. \quad (8.99)$$

Congruences (8.83) and (8.84) are true from equation (8.94).

Substituting (2.20) into (8.99), we find that

$$\sum_{n \geq 0} C_9(6n+1)q^n \equiv f_4^2 + q \frac{f_{12}^3}{f_4} \pmod{2}. \quad (8.100)$$

Congruences (8.86) and (8.87) follow by equating the powers of the form q^{4n+2} and q^{4n+3} on the both sides of (8.100), respectively. \square

From Equations (8.82) and (8.26), we deduce that

$$C_9(24n) \equiv a(n) \pmod{2},$$

where $a(n)$ is defined in Equation (8.26). From **Theorem 8.2** we obtain the following result.

Corollary 8.3. *For any prime $p \geq 5$, we have*

$$C_9(24p^{2\alpha+1}(pn+i) + p^{2\alpha+2} - 1) \equiv 0 \pmod{2}, \quad (8.101)$$

where i is an integer and $1 \leq i \leq p-1$.

8.5 Congruences modulo 3 for $C_9(n)$

Theorem 8.9. For all $\alpha \geq 0$ and $n \geq 0$, we have

$$C_9(6n + 4) \equiv 0 \pmod{3}, \quad (8.102)$$

$$C_9(18n + 14) \equiv 0 \pmod{3}, \quad (8.103)$$

$$C_9(18n + 2) \equiv C_9(3^\alpha \cdot 54n + 9(3^\alpha - 1) + 8) \pmod{3}, \quad (8.104)$$

$$C_9(18n + 8) \equiv C_9(3^\alpha \cdot 54n + 27(3^\alpha - 1) + 26) \pmod{3}, \quad (8.105)$$

$$C_9(3^\alpha \cdot 54n + 54(3^\alpha - 1) + 44) \equiv 0 \pmod{3}. \quad (8.106)$$

Proof. In view of (2.2) and (8.90), with $p = 3$ and $k = 1$, along with using (2.20), we find that

$$\sum_{n \geq 0} C_9(n)q^n \equiv \frac{f_3^3 f_2 f_{36}}{f_1 f_4 f_{48}} = \frac{f_4^2 f_6^2 f_{36}}{f_2 f_{12} f_{18}} + q \frac{f_2 f_{12}^3 f_{36}}{f_4^2 f_{18}} \pmod{3}. \quad (8.107)$$

If we extract the even and the odd powers from both sides of Equation (8.107), we arrive at

$$\sum_{n \geq 0} C_9(2n)q^n \equiv \frac{f_2^2 f_3^2 f_{18}}{f_1 f_6 f_9} \pmod{3}, \quad (8.108)$$

and

$$\sum_{n \geq 0} C_9(2n + 1)q^n \equiv \frac{f_1 f_6^3 f_{18}}{f_2^2 f_9} \pmod{3}. \quad (8.109)$$

Employing (2.16) in (8.108), we obtain

$$\sum_{n \geq 0} C_9(2n)q^n \equiv f_3 f_9 + q \frac{f_3^2 f_{18}^3}{f_6 f_9^2} \pmod{3}. \quad (8.110)$$

Congruence (8.102) follows from (8.110) by equating the powers of the form q^{3n+2} on both sides.

Extracting the terms involving the powers of the form q^{3n+1} from both sides of (8.110) along with employing (2.11), we get

$$\sum_{n \geq 0} C_9(6n + 2)q^n \equiv \frac{f_9^2}{f_3^2} - 2q \frac{f_6^2 f_{18}^2}{f_3 f_9} \pmod{3}. \quad (8.111)$$

Congruence (8.103) follows by extracting the terms involving q^{3n+2} from both sides of (8.111). Extracting the powers of the form q^{3n} from both sides of (8.111), and then replacing q^3 by q , we obtain

$$\sum_{n \geq 0} C_9(18n + 2)q^n \equiv \frac{f_3^2}{f_1^2} \pmod{3}. \quad (8.112)$$

Equating the powers of the form q^{3n+1} on both sides of (8.111), dividing q and then replacing q^3 by q , we get

$$\sum_{n \geq 0} C_9(18n + 8)q^n \equiv \frac{f_2^2 f_6^2}{f_1 f_3} \pmod{3}. \quad (8.113)$$

Again, substituting (2.16) into the above equation, we find that

$$\sum_{n \geq 0} C_9(18n + 8)q^n \equiv \frac{f_6^3 f_9^2}{f_3^2 f_{18}} + q \frac{f_6^2 f_{18}^2}{f_3 f_9} \pmod{3}. \quad (8.114)$$

If we extract the terms of the form q^{3n} , and then replace q^{3n} by q , we obtain

$$\sum_{n \geq 0} C_9(54n + 8)q^n \equiv \frac{f_3^2}{f_1^2} \pmod{3}. \quad (8.115)$$

If we extract the terms of the form q^{3n+1} , divide by q , and then replace q^3 by q from both sides of equation (8.114), we deduce that

$$\sum_{n \geq 0} C_9(54n + 26)q^n \equiv \frac{f_2^2 f_6^2}{f_1 f_3} \pmod{3}. \quad (8.116)$$

By equating the powers of the form q^{3n+2} on both sides of the equation (8.114), we arrive at

$$C_9(54n + 44) \equiv 0 \pmod{3}. \quad (8.117)$$

Combining (8.112) and (8.115), we deduce that

$$C_9(18n + 2) \equiv C_9(54n + 8) \pmod{3}. \quad (8.118)$$

Combining (8.113) and (8.116), we obtain that

$$C_9(18n + 8) \equiv C_9(54n + 26) \pmod{3}. \quad (8.119)$$

Congruence (8.104) follows from (8.116), (8.118) and by mathematical induction on α , where

$$C_9(18n + 2) \equiv C_9 \left(3^\alpha \cdot 54n + 2 \sum_{i=1}^{\alpha} 3^{i+1} + 8 \right) \pmod{3}.$$

Congruence (8.105) is immediate from (8.116) and (8.119), where

$$C_9(18n + 8) \equiv C_9 \left(3^\alpha \cdot 54n + 2 \sum_{i=1}^{\alpha} 3^{i+2} + 26 \right) \pmod{3}.$$

Meanwhile, congruence (8.106) is immediate from (8.116) and (8.117) such that

$$C_9\left(3^\alpha \cdot 54n + 4 \sum_{i=1}^{\alpha} 3^{i+2} + 44\right) \equiv 0 \pmod{3}.$$

□

Corollary 8.4. For all $n \geq 0$,

$$C_9(36n + 32) \equiv 0 \pmod{6}. \tag{8.120}$$

Proof. Congruence (8.120) is an immediate result from (8.85) and (8.103). □

CHAPTER 9

ARITHMETIC PROPERTIES OF OVERPARTITION ANALOGUE OF THE ANDREWS-GÖLLNITZ-GORDON THEOREM

In this chapter, we study the arithmetic properties of the overpartition analogue of the partitions enumerated by the number $C_{k,i}(n)$, which generates the partitions of the Andrews-Göllnitz-Gordon **Theorem 1.12**, for $k = 3$ such that $k = 2i - 1$. Let $\bar{C}_{k,i}(n)$ denote the number of overpartitions into parts not congruent to 2 modulo 4 and not congruent to $0, k$ or $3k$ modulo $4k$. Through the remainder of this chapter, we will omit the parameter i from $\bar{C}_{k,i}(n)$. The number $\bar{C}_k(n)$ satisfies the generating function

$$\begin{aligned} \sum_{n \geq 0} \bar{C}_k(n)q^n &= \prod_{n \geq 1} \frac{(1+q^n)(1+q^{4n})(1+q^{2kn})(1-q^{2n})(1-q^{kn})(1-q^{4kn})}{(1+q^{2n})(1+q^{kn})(1+q^{4kn})(1-q^n)(1-q^{4n})(1-q^{2kn})} \\ &= \frac{\psi(-q^2)\psi(-q^k)^2}{\psi(-q)^2\psi(-q^{2k})} \\ &= \frac{f_2^3 f_8 f_k^2 f_{4k}^3}{f_1^2 f_4^3 f_{2k}^3 f_{8k}}. \end{aligned} \tag{9.1}$$

9.1 Generating function dissections for $\bar{C}_3(n)$

In this section, we derive several generating function dissections from the generating function of $\bar{C}_3(n)$. Setting $k = 3$ in (9.1), we obtain

$$\sum_{n \geq 0} \bar{C}_3(n)q^n = \frac{f_2^3 f_3^2 f_8 f_{12}^3}{f_1^2 f_4^3 f_6^3 f_{24}}. \tag{9.2}$$

Theorem 9.1. *We have*

$$\sum_{n \geq 0} \bar{C}_3(6n)q^n = \frac{f_2^9 f_3^6}{f_1^{10} f_4^2 f_6^3}, \quad (9.3)$$

$$\sum_{n \geq 0} \bar{C}_3(4n + 1)q^n = 2 \frac{f_2 f_4^2 f_6^5}{f_1^4 f_3^2 f_{12}^2}, \quad (9.4)$$

$$\sum_{n \geq 0} \bar{C}_3(4n + 3)q^n = 2 \frac{f_2^7 f_{12}^2}{f_1^6 f_4^2 f_6}, \quad (9.5)$$

$$\sum_{n \geq 0} \bar{C}_3(6n + 2)q^n = 2 \frac{f_2^8 f_3^3}{f_1^9 f_4^2}, \quad (9.6)$$

$$\sum_{n \geq 0} \bar{C}_3(6n + 4)q^n = 4 \frac{f_2^7 f_6^3}{f_1^8 f_4^2}. \quad (9.7)$$

Proof. Substituting (2.19) into (9.2), we obtain

$$\sum_{n \geq 0} \bar{C}_3(n)q^n = \frac{f_4 f_{12}^5}{f_2^2 f_6^2 f_{24}^2} + 2q \frac{f_8^2 f_{12}^2}{f_2 f_4^2 f_6}. \quad (9.8)$$

Extracting the even and the odd powers from both sides of the above equation, we get

$$\sum_{n \geq 0} \bar{C}_3(2n)q^n = \frac{f_2 f_6^5}{f_1^2 f_3^2 f_{12}^2}, \quad (9.9)$$

and

$$\sum_{n \geq 0} \bar{C}_3(2n + 1)q^n = 2 \frac{f_4^2 f_6^2}{f_1 f_2^2 f_3}. \quad (9.10)$$

Employing (2.13) into (9.9), we arrive at

$$\sum_{n \geq 0} \bar{C}_3(2n)q^n = \frac{f_6^9 f_9^6}{f_3^{10} f_{12}^2 f_{18}^3} + 2q \frac{f_6^8 f_9^3}{f_3^9 f_{12}^2} + 4q^2 \frac{f_6^7 f_{18}^3}{f_3^8 f_{12}^2}. \quad (9.11)$$

The generating function dissection (9.3) follows from (9.11) by equating the terms of the form q^{3n} from both sides and then replacing q^3 by q , and (9.6) follows from (9.11) by extracting the powers involving q^{3n+1} from both sides, dividing by q and then replacing q^3 by q . Similarly, we obtain the generating function dissection (9.7).

On the other hand, by using (2.23) in (9.10), we find that

$$\sum_{n \geq 0} \bar{C}_3(2n + 1)q^n = 2 \frac{f_4 f_8^2 f_{12}^5}{f_2^4 f_6^2 f_{24}^2} + 2q \frac{f_4^7 f_{24}^2}{f_2^6 f_8^2 f_{12}}. \quad (9.12)$$

The generating function dissections (9.4) and (9.5) follow from (9.12) by equating the odd and the even powers on both sides. \square

9.2 Congruences modulo 3 for $\overline{C}_3(n)$

In this section, we establish some Ramanujan-like congruences and an infinite family of congruences modulo 3 for $\overline{C}_3(n)$.

Theorem 9.2. *For any nonnegative integer n , we have*

$$\sum_{n \geq 0} \overline{C}_3(72n + 6)q^n \equiv f_1^2 \pmod{3}, \quad (9.13)$$

$$\overline{C}_3(72n + j) \equiv 0 \pmod{3}, \quad j = 18, 30, 42, 54, 66, \quad (9.14)$$

$$\overline{C}_3(108n + i) \equiv 0 \pmod{3}, \quad i = 36, 72, \quad (9.15)$$

$$\overline{C}_3(144n + 84) \equiv 0 \pmod{3}, \quad (9.16)$$

$$\overline{C}_3(144n + 132) \equiv 0 \pmod{3}. \quad (9.17)$$

Proof. Applying (2.2) on (9.3), with $p = 3$ and $k = 1$, and then substituting (2.20) into the resulting equation, we find that

$$\sum_{n \geq 0} \overline{C}_3(6n)q^n \equiv \frac{f_3^3}{f_1 f_4^2} = \frac{f_4 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4^3} \pmod{3}. \quad (9.18)$$

Equating the even and the odd powers on both sides of Equation (9.18), we get

$$\sum_{n \geq 0} \overline{C}_3(12n)q^n \equiv \frac{f_2 f_3^2}{f_1^2 f_6} \pmod{3}, \quad (9.19)$$

and

$$\sum_{n \geq 0} \overline{C}_3(12n + 6)q^n \equiv \frac{f_6^3}{f_2^3} \equiv f_6^2 \pmod{3}. \quad (9.20)$$

Congruences (9.13) and (9.14) follow from (9.20) by extracting the powers of the form q^{6n+j} for $j = 0, 1, 2, 3, 4, 5$ from both sides.

Substituting (2.13) into (9.19), we obtain

$$\sum_{n \geq 0} \overline{C}_3(12n)q^n \equiv \frac{f_6^3 f_9^6}{f_3^6 f_{18}^3} + 2q \frac{f_6^2 f_9^3}{f_3^5} + 4q^2 \frac{f_6 f_{18}^3}{f_3^4} \pmod{3}. \quad (9.21)$$

Collecting the terms of the form q^{3n} from both sides of Equation (9.21), and replacing q^3 by q , we arrive at

$$\sum_{n \geq 0} \overline{C}_3(36n)q^n \equiv \frac{f_2^3 f_3^6}{f_1^6 f_6^3} \equiv \frac{f_3^4}{f_6^2} \pmod{3}. \quad (9.22)$$

Congruence (9.15) follows from (9.22) by extracting the powers of the form q^{3n+j} for $j = 1, 2$ from both sides.

Equating the terms involving q^{3n+1} on both sides of Equation (9.21), dividing by q and then replacing q^3 by q , we get

$$\sum_{n \geq 0} \overline{C}_3(36n + 12)q^n \equiv 2 \frac{f_2^2 f_3^3}{f_1^5} \equiv 2f_1 f_2^2 f_3 \pmod{3}. \quad (9.23)$$

Substituting (2.24) into (9.23), we obtain

$$\sum_{n \geq 0} \overline{C}_3(36n + 12)q^n \equiv 2 \frac{f_2^3 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - 2q \frac{f_2 f_4^2 f_6 f_{24}^2}{f_8^2 f_{12}^2} \pmod{3}. \quad (9.24)$$

Extracting the even powers from both sides of (9.24), we get

$$\sum_{n \geq 0} \overline{C}_3(72n + 12)q^n \equiv 2 \frac{f_1^3 f_4^2 f_6^4}{f_2^2 f_3 f_{12}^2} \equiv 2 \frac{f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod{3}. \quad (9.25)$$

Congruence (9.16) follows from Equation (9.25) by equating the odd powers on both sides.

If we extract the terms involving q^{3n+2} from both sides of (9.21), divide by q^2 , replace q^3 by q , and then substitute (2.23) into the resulting equation, we find that

$$\sum_{n \geq 0} \overline{C}_3(36n + 24)q^n \equiv 4 \frac{f_2 f_6^3}{f_1^4} \equiv 4 \frac{f_2 f_6^3}{f_1 f_3} \equiv 4 \frac{f_8^2 f_{12}^5}{f_2 f_4 f_6 f_{24}^2} + 4q \frac{f_4^5 f_6 f_{24}^2}{f_2^3 f_8^2 f_{12}} \pmod{3}. \quad (9.26)$$

Collecting the odd powers from both sides of (9.26), we get

$$\sum_{n \geq 0} \overline{C}_3(72n + 60)q^n \equiv 4 \frac{f_2^5 f_3 f_{12}^2}{f_1^3 f_4^2 f_6} \equiv 4 \frac{f_2^2 f_{12}^2}{f_4^2} \pmod{3}. \quad (9.27)$$

Congruence (9.17) follows from (9.27) by extracting the odd powers from both sides. \square

Theorem 9.3. For any prime $p \equiv 3 \pmod{4}$, $\alpha \geq 0$, and $n \geq 0$, we have

$$\overline{C}_3 \left(72p^{2\alpha+1}(pn + i) + 6p^{2\alpha+2} \right) \equiv 0 \pmod{3}, \quad (9.28)$$

for all $1 \leq i \leq p - 1$.

Proof. Define

$$\sum_{n \geq 0} a(n)q^n = f_1^2. \quad (9.29)$$

From (9.13) and (9.29), we deduce that

$$\overline{C}_3(72n + 6) \equiv a(n) \pmod{3}. \quad (9.30)$$

Consider the congruence equation

$$\frac{3k^2 + k}{2} + \frac{3m^2 + m}{2} \equiv \frac{p^2 - 1}{12} \pmod{p},$$

which is equivalent to

$$(6k + 1)^2 + (6m + 1)^2 \equiv 0 \pmod{p}, \quad (9.31)$$

where $-(p-1)/2 \leq k, m \leq (p-1)/2$ and p is a prime such that $p \equiv 3 \pmod{4}$. Since $\left(\frac{-1}{p}\right) = -1$, then the congruence relation (9.31) holds if and only if both $k = m = (\pm p - 1)/6$. Substitute (2.35) into (9.29) and extract the terms in which the powers of q are congruent to $(p^2 - 1)/12$ modulo p and then divide $q^{\frac{p^2-1}{12}}$, we have

$$\sum_{n \geq 0} a\left(pn + \frac{p^2 - 1}{12}\right) q^{pn} = f_{p^2}^2,$$

which implies that for $n \geq 0$,

$$a\left(p^2n + \frac{p^2 - 1}{12}\right) = a(n),$$

and

$$a\left(p^2n + pi + \frac{p^2 - 1}{12}\right) = 0, \quad (9.32)$$

where i is an integer and $1 \leq i \leq p-1$. By induction we see that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha}n + \frac{p^{2\alpha} - 1}{12}\right) = a(n). \quad (9.33)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{12}$ ($1 \leq i \leq p-1$) in (9.33) and using (9.32), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2} - 1}{12}\right) = 0.$$

Replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{12}$ in (9.30) ($1 \leq i \leq p-1$), we arrive at (9.28). \square

9.3 Congruences modulo 6 and 18 for $\overline{C}_3(n)$

In this section, we derive a Ramanujan-like congruence modulo 18 in addition to an infinite family of congruences modulo 6 for $\overline{C}_3(n)$.

Theorem 9.4. For any prime $p \equiv 3 \pmod{4}$, $\alpha \geq 0$, and $n \geq 0$, we have

$$\overline{C}_3\left(24p^{2\alpha+1}(pn + i) + 2p^{2\alpha+2}\right) \equiv 0 \pmod{6}, \quad (9.34)$$

for all $1 \leq i \leq p - 1$.

Proof. In view of (2.2) and (9.6), with $p = 3$ and $k = 1$, we find that

$$\sum_{n \geq 0} \overline{C}_3(6n + 2)q^n \equiv 2 \frac{f_2^8 f_3}{f_1^3 f_4^2} \pmod{6}. \quad (9.35)$$

Employing (2.22) in (9.35) and then collecting the even powers from both sides of the resulting equation, we arrive at

$$\sum_{n \geq 0} \overline{C}_3(12n + 2)q^n \equiv 2 \frac{f_2^4 f_3^3}{f_1 f_6^2} \equiv 2 \frac{f_2 f_3^3}{f_1 f_6} \pmod{6}. \quad (9.36)$$

Using (2.20) in (9.36), we get

$$\sum_{n \geq 0} \overline{C}_3(12n + 2)q^n \equiv 2 \frac{f_4^3 f_6}{f_2 f_{12}} + 2q \frac{f_2 f_{12}^3}{f_4 f_6} \pmod{6}. \quad (9.37)$$

If we collect the terms of the form q^{2n} from both sides of (9.37) and then replace q^2 by q , we arrive at

$$\sum_{n \geq 0} \overline{C}_3(24n + 2)q^n \equiv 2 \frac{f_2^3 f_3}{f_1 f_6} \equiv 2f_1^2 \pmod{6}. \quad (9.38)$$

Combining equations (9.29) and (9.38), we find that

$$\overline{C}_3(24n + 2) \equiv a(n) \pmod{6}. \quad (9.39)$$

By using the proof of Theorem 9.3, and replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{12}$ in (9.39), we obtain the desired result. \square

Theorem 9.5. *For any nonnegative integer n , we have*

$$\overline{C}_3(12n + 8) \equiv 0 \pmod{18}. \quad (9.40)$$

Proof. Invoking (2.2) in (9.6), with $p = 3$ and $k = 2$, we find that

$$\sum_{n \geq 0} \overline{C}_3(6n + 2)q^n \equiv 2 \frac{f_2^8}{f_4^2} \pmod{18}. \quad (9.41)$$

Congruence (9.40) follows from equation (9.41) by extracting the odd powers from both sides. \square

9.4 Congruences modulo Powers of 2 for $\overline{C}_3(n)$

In this section, we establish some congruences modulo powers of 2 for $\overline{C}_3(n)$.

Theorem 9.6. For any nonnegative integers $n \geq 0$ and $\alpha \geq 1$, we have

$$\bar{C}_3(24n + 12) \equiv 0 \pmod{4}, \quad (9.42)$$

$$\bar{C}_3(24n) \equiv \bar{C}_3(8 \cdot 3^{\alpha+1}n) \pmod{4}, \quad (9.43)$$

$$\bar{C}_3(72n + 24) \equiv \bar{C}_3(8 \cdot 3^{\alpha+2}n + 8 \cdot 3^{\alpha+1}) \pmod{4}, \quad (9.44)$$

$$\bar{C}_3(8 \cdot 3^\alpha(3n + 2)) \equiv 0 \pmod{4}. \quad (9.45)$$

Proof. In view of (2.2) and (9.3), with $p = k = 2$, we obtain

$$\sum_{n \geq 0} \bar{C}_3(6n)q^n \equiv \frac{f_2 f_3^2}{f_1^2 f_6} \pmod{4}. \quad (9.46)$$

Employing (2.19) into (9.46), we get

$$\sum_{n \geq 0} \bar{C}_3(6n)q^n \equiv \frac{f_4^4 f_{12}^2}{f_2^4 f_8 f_{24}} + 2q \frac{f_4 f_6 f_8 f_{24}}{f_2^3 f_{12}} \pmod{4}. \quad (9.47)$$

Equating the even and the odd powers on both sides of equation (9.47), we find that

$$\sum_{n \geq 0} \bar{C}_3(12n)q^n \equiv \frac{f_2^2 f_6^2}{f_1^4 f_4 f_{12}} \equiv \frac{f_2^2 f_6^2}{f_4 f_{12}} \pmod{4}, \quad (9.48)$$

$$\sum_{n \geq 0} \bar{C}_3(12n + 6)q^n \equiv 2 \frac{f_2 f_3 f_4 f_{12}}{f_1^3 f_6} \pmod{4}.$$

Congruence (9.42) follows from (9.48) by extracting the odd powers from both sides. Collecting the terms involving q^{2n} from both sides of (9.48) and then replacing q^2 by q , we arrive at

$$\sum_{n \geq 0} \bar{C}_3(24n)q^n \equiv \frac{f_1^2 f_3^2}{f_2 f_6} \pmod{4}. \quad (9.49)$$

Substituting (2.11) into (9.49), we get

$$\sum_{n \geq 0} \bar{C}_3(24n)q^n \equiv \frac{f_3^2 f_9^2}{f_6 f_{18}} - 2q \frac{f_3^3 f_{18}^2}{f_6^2 f_9} \pmod{4}. \quad (9.50)$$

Collecting the terms of the form q^{3n+j} for $j = 0, 1, 2$ from both sides of (9.50), we find that

$$\sum_{n \geq 0} \bar{C}_3(72n)q^n \equiv \frac{f_1^2 f_3^2}{f_2 f_6} \pmod{4}, \quad (9.51)$$

$$\sum_{n \geq 0} \bar{C}_3(72n + 24)q^n \equiv 2 \frac{f_1^3 f_6^2}{f_2^2 f_3} \pmod{4}, \quad (9.52)$$

$$\bar{C}_3(72n + 48) \equiv 0 \pmod{4}. \quad (9.53)$$

From (9.49) and (9.51), we deduce that for all $n \geq 0$ and $\alpha \geq 1$,

$$\overline{C}_3(24n) \equiv \overline{C}_3(8 \cdot 3^{\alpha+1}n) \pmod{4}. \quad (9.54)$$

From (9.51) and (9.52), we deduce that for all $n \geq 0$ and $\alpha \geq 1$,

$$\overline{C}_3(72n + 24) \equiv \overline{C}_3(8 \cdot 3^{\alpha+2}n + 8 \cdot 3^{\alpha+1}) \pmod{4}. \quad (9.55)$$

From (9.54) and (9.53), we deduce that for all $n \geq 0$ and $\alpha \geq 1$,

$$\overline{C}_3(8 \cdot 3^{\alpha+1}n + 16 \cdot 3^\alpha) \equiv 0 \pmod{4}. \quad (9.56)$$

□

Theorem 9.7. For any nonnegative integers $n \geq 0$ and $\alpha \geq 1$, we have

$$\overline{C}_3(12n + 11) \equiv 0 \pmod{8}, \quad (9.57)$$

$$\overline{C}_3(24n + 5) \equiv 0 \pmod{8}, \quad (9.58)$$

$$\overline{C}_3(24n + 17) \equiv 0 \pmod{8}, \quad (9.59)$$

$$\overline{C}_3(36n + 15) \equiv 0 \pmod{8}, \quad (9.60)$$

$$\overline{C}_3(72n + 33) \equiv 0 \pmod{8}, \quad (9.61)$$

$$\overline{C}_3(108n + 99) \equiv 0 \pmod{8}, \quad (9.62)$$

$$\overline{C}_3(216n + 153) \equiv 0 \pmod{8}. \quad (9.63)$$

Proof. Invoking (2.2) in (9.4), with $p = k = 2$, we obtain

$$\sum_{n \geq 0} \overline{C}_3(4n + 1)q^n \equiv 2 \frac{f_4^2 f_6}{f_2 f_3^2} \pmod{8}. \quad (9.64)$$

Substituting (2.3) into (9.64), with the transformation of q to q^3 , we get

$$\sum_{n \geq 0} \overline{C}_3(4n + 1)q^n \equiv 2 \frac{f_4^2 f_{24}^5}{f_2 f_6^4 f_{48}^2} + 4q^3 \frac{f_4^2 f_{12}^2 f_{48}^2}{f_2 f_6^4 f_{24}} \pmod{8}. \quad (9.65)$$

Equating the even and the odd powers on both sides of the Equation (9.65), we obtain

$$\sum_{n \geq 0} \overline{C}_3(8n + 1)q^n \equiv 2 \frac{f_2^2 f_{12}^5}{f_1 f_3^4 f_{24}^2} \equiv 2 \frac{f_2^2 f_{12}}{f_1 f_6^2} \pmod{8}, \quad (9.66)$$

$$\sum_{n \geq 0} \overline{C}_3(8n + 5)q^n \equiv 4q \frac{f_2^2 f_6^2 f_{24}^2}{f_1 f_3^4 f_{12}} \equiv 4q \frac{f_2^2 f_{24}^2}{f_1 f_{12}} \pmod{8}. \quad (9.67)$$

Substituting (2.16) into both (9.66) and (9.67), we obtain

$$\sum_{n \geq 0} \bar{C}_3(8n+1)q^n \equiv 2 \frac{f_9^2 f_{12}}{f_3 f_6 f_{18}} + 2q \frac{f_{12} f_{18}^2}{f_6^2 f_9} \pmod{8}, \quad (9.68)$$

$$\sum_{n \geq 0} \bar{C}_3(8n+5)q^n \equiv 4q \frac{f_6 f_9^2 f_{24}^2}{f_3 f_{12} f_{18}} + 4q^2 \frac{f_{18}^2 f_{24}^2}{f_9 f_{12}} \pmod{8}. \quad (9.69)$$

Congruence (9.59) follows from (9.68) by extracting the terms of the form q^{3n+2} from both sides. Meanwhile, Congruence (9.58) follows from (9.69) by collecting the terms involving q^{3n} from both sides.

If we extract the terms of the form q^{3n+1} from both sides of (9.68), divide by q and then replace q^3 by q , we find that

$$\sum_{n \geq 0} \bar{C}_3(24n+9)q^n \equiv 2 \frac{f_4 f_6^2}{f_2^2 f_3} \pmod{8}. \quad (9.70)$$

By substituting (2.13) into (9.70), with the transformation of q to q^2 , we arrive at

$$\sum_{n \geq 0} \bar{C}_3(24n+9)q^n \equiv 2 \frac{f_{12}^4 f_{18}^6}{f_3 f_6^6 f_{36}^3} + 4q^2 \frac{f_{12}^3 f_{18}^3}{f_3 f_6^5} + 8q^4 \frac{f_{18}^2 f_{36}^3}{f_3 f_6^4} \pmod{8}. \quad (9.71)$$

Congruence (9.17) follows by extracting the terms involving q^{3n+1} from both sides of Equation (9.71). By extracting the terms of the form q^{3n} from both sides of (9.71), we get

$$\sum_{n \geq 0} \bar{C}_3(72n+9)q^n \equiv 2 \frac{f_4^4 f_6^6}{f_1 f_2^6 f_{12}^3} \equiv 2 \frac{f_2^2 f_6^2}{f_1 f_{12}} \pmod{8}. \quad (9.72)$$

Congruence (9.63) follows by invoking (2.16) into (9.72), and then collecting the terms of the form q^{3n+2} from both sides of the resulting equation.

In view of (2.2) and (9.5), with $k = p = 2$, we get

$$\sum_{n \geq 0} \bar{C}_3(4n+3)q^n \equiv 2 \frac{f_2 f_{12}^2}{f_1^2 f_6} \pmod{8}. \quad (9.73)$$

Substituting (2.13) into (9.73), we arrive at

$$\sum_{n \geq 0} \bar{C}_3(4n+3)q^n \equiv 2 \frac{f_6^3 f_9 f_{12}^2}{f_3^8 f_{18}^3} + 4q \frac{f_6^2 f_9^3 f_{12}^2}{f_3^7} + 8q^2 \frac{f_6 f_{12}^2 f_{18}^3}{f_3^6} \pmod{8}. \quad (9.74)$$

Congruence (9.57) follows from the above equation by extracting the powers of the form q^{3n+2} from both sides. If we collect the terms involving q^{3n} from both sides of (9.74), we get

$$\sum_{n \geq 0} \overline{C}_3(12n + 3)q^n \equiv 2 \frac{f_2^3 f_3^6 f_4^2}{f_1^8 f_6^3} \equiv 2 \frac{f_2^3 f_3^2}{f_6} \pmod{8}. \quad (9.75)$$

Substituting (2.7) into (9.75), with the transformation of q by q^2 , we obtain

$$\sum_{n \geq 0} \overline{C}_3(12n + 3)q^n \equiv 2 \frac{f_3^2 f_{12} f_{18}^6}{f_6^2 f_{36}^3} - 6q^2 \frac{f_3^2 f_{18}^3}{f_6} + 8q^6 \frac{f_3^2 f_6 f_{36}^6}{f_{12}^2 f_{18}^3} \pmod{8}. \quad (9.76)$$

Congruence (9.60) follows from (9.76) by extracting the terms of the form q^{3n+1} from both sides. If we collect the terms of the form q^{3n+2} from both sides of (9.76), divide q^2 and then replace q^3 by q , we arrive at

$$\sum_{n \geq 0} \overline{C}_3(36n + 27)q^n \equiv 2 \frac{f_1^2 f_6^3}{f_2} \pmod{8}. \quad (9.77)$$

By replacing (2.13) into (9.77), we obtain congruence (9.62) by equating the powers of the form q^{3n+2} on both sides of the resulting equation. \square

Theorem 9.8. For any nonnegative integers $n \geq 0$ and $\alpha \geq 1$, we have

$$\overline{C}_3(6n + 2) \equiv \overline{C}_3(6 \cdot 4^\alpha n + 2 \cdot 4^\alpha) \pmod{8}, \quad (9.78)$$

$$\overline{C}_3(6 \cdot 4^\alpha n + 5 \cdot 4^\alpha) \equiv 0 \pmod{8}. \quad (9.79)$$

Proof. In view of (2.2) and (9.6), with $p = k = 2$, we have

$$\sum_{n \geq 0} \overline{C}_3(6n + 2)q^n = 2 \frac{f_2^8 f_3^3}{f_1^9 f_4^2} \equiv 2 \frac{f_3^3}{f_1} \pmod{8}. \quad (9.80)$$

Employing (2.20) into (9.80), we get

$$\sum_{n \geq 0} \overline{C}_3(6n + 2)q^n \equiv 2 \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + 2q \frac{f_{12}^3}{f_4} \pmod{8}. \quad (9.81)$$

By equating the odd powers on both sides of the equation (9.81), we obtain

$$\sum_{n \geq 0} \overline{C}_3(12n + 8)q^n \equiv 2 \frac{f_6^3}{f_2} \pmod{8}. \quad (9.82)$$

If we collect the terms of the form q^{2n+j} for $j = 0, 1$ from both sides of (9.82), we find that

$$\sum_{n \geq 0} \bar{C}_3(24n + 8)q^n \equiv 2 \frac{f_3^3}{f_1} \pmod{8}, \quad (9.83)$$

$$\bar{C}_3(24n + 20) \equiv 0 \pmod{8}. \quad (9.84)$$

From (9.80) and (9.83), we deduce that for all $n \geq 0$ and $\alpha \geq 1$,

$$\bar{C}_3(6n + 2) \equiv \bar{C}_3(6 \cdot 4^\alpha n + 2 \cdot 4^\alpha) \pmod{8}. \quad (9.85)$$

From (9.85) and (9.84), we deduce that

$$\bar{C}_3(6 \cdot 4^\alpha n + 5 \cdot 4^\alpha) \equiv 0 \pmod{8}. \quad (9.86)$$

□

Theorem 9.9. For any nonnegative integers $n \geq 0$ and $\alpha \geq 1$, we have

$$\bar{C}_3(12n + 4) \equiv \bar{C}_3(12 \cdot 4^\alpha n + 4^{\alpha+1}) \pmod{16}, \quad (9.87)$$

$$\bar{C}_3(12 \cdot 4^\alpha n + 10 \cdot 4^\alpha) \equiv 0 \pmod{16}, \quad (9.88)$$

$$\bar{C}_3(12n + 10) \equiv 0 \pmod{32}. \quad (9.89)$$

Proof. In view of (2.2) and (9.7), with $p = k = 2$, we find that

$$\sum_{n \geq 0} \bar{C}_3(6n + 4)q^n \equiv 4 \frac{f_6^3}{f_2} \pmod{16}. \quad (9.90)$$

By collecting the even powers from both sides of (9.90), we get

$$\sum_{n \geq 0} \bar{C}_3(12n + 4)q^n \equiv 4 \frac{f_3^3}{f_1} \pmod{16}. \quad (9.91)$$

If we substitute (2.20) into (9.91) and then extract the terms of the form q^{2n} from both sides of the resulting equation, we arrive at

$$\sum_{n \geq 0} \bar{C}_3(24n + 16)q^n \equiv 4 \frac{f_6^3}{f_2} \pmod{16}. \quad (9.92)$$

Again, equating the even and the odd powers on both sides of (9.92), we find that

$$\sum_{n \geq 0} \bar{C}_3(48n + 16)q^n \equiv 4 \frac{f_3^3}{f_1} \pmod{16}, \quad (9.93)$$

$$\bar{C}_3(48n + 40) \equiv 0 \pmod{16}. \quad (9.94)$$

From (9.91) and (9.93), we deduce that for $n \geq 0$ and $\alpha \geq 1$,

$$\overline{C}_3(12n + 4) \equiv \overline{C}_3(12 \cdot 4^\alpha n + 4^{\alpha+1}) \pmod{16}. \quad (9.95)$$

From (9.95) and (9.94), we deduce that for $n \geq 0$ and $\alpha \geq 1$,

$$\overline{C}_3(12 \cdot 4^\alpha n + 10 \cdot 4^\alpha) \equiv 0 \pmod{16}. \quad (9.96)$$

Invoking (2.2) in (9.7), with $p = 2$ and $k = 3$, we find that

$$\sum_{n \geq 0} \overline{C}_3(6n + 4)q^n \equiv 4 \frac{f_2^3 f_6^3}{f_4^2} \pmod{32}. \quad (9.97)$$

If we collect the terms involving q^{2n+1} from both sides of the above equation, we arrive at (9.89). \square

CHAPTER 10

ARITHMETIC PROPERTIES OF PARTITIONS INTO DISTINCT ℓ -REGULAR ODD PARTS AND UNRESTRICTED EVEN PARTS

In this chapter, we are interested in the arithmetic behaviour of the function $\text{pod}_\ell(n)$, which enumerates the number of partitions of n into distinct ℓ -regular odd parts and unrestricted even parts for $\ell = 3, 9$. The number $\text{pod}_\ell(n)$ satisfies the generating function

$$\sum_{n \geq 0} \text{pod}_\ell(n) q^n = \prod_{i \geq 1} \frac{(1 + q^{2i-1})}{(1 - q^{2i})(1 + q^{\ell(2i-1)})} = \frac{f_2 f_\ell f_{4\ell}}{f_1 f_4 f_{2\ell}^2}. \quad (10.1)$$

In this vein, we establish several congruences modulo 3, 4 and 12 for $\text{pod}_\ell(n)$ for $\ell = 3, 9$.

10.1 Congruences for $\text{pod}_3(n)$

In this section, we establish several Ramanujan-like congruences modulo 12 and 4 and an infinite family of congruences modulo 4 for $\text{pod}_3(n)$. Setting $\ell = 3$ in (10.1), we obtain

$$\sum_{n \geq 0} \text{pod}_3(n) q^n = \frac{f_2 f_3 f_{12}}{f_1 f_4 f_6^2}. \quad (10.2)$$

Theorem 10.1. *For any nonnegative integer n , we have*

$$\text{pod}_3(24n + 19) \equiv 0 \pmod{12}. \quad (10.3)$$

Proof. Substituting (2.15) into (10.2), we get

$$\sum_{n \geq 0} \text{pod}_3(n) q^n = \frac{f_{18}^9}{f_3 f_6^2 f_9^3 f_{12} f_{36}^3} + q \frac{f_{18}^3}{f_3^2 f_{12}^2} + q^2 \frac{f_6^2 f_9^3 f_{36}^3}{f_3^3 f_{12}^3 f_{18}^3}. \quad (10.4)$$

If we collect the terms of the form q^{3n+1} from both sides of (10.4), divide by q and then replace q^3 by q , we obtain

$$\sum_{n \geq 0} \text{pod}_3(3n+1)q^n = \frac{f_6^3}{f_1^2 f_4^2}. \quad (10.5)$$

Employing (2.3) in (10.5) and then extracting the even powers from both sides of the resulting equation, we find that

$$\sum_{n \geq 0} \text{pod}_3(6n+1)q^n = \frac{f_3^3 f_4^5}{f_1^5 f_2^2 f_8^2}. \quad (10.6)$$

In view of (2.2) and (10.6), with $p = 3$ and $k = 1$, we get

$$\sum_{n \geq 0} \text{pod}_3(6n+1)q^n \equiv \frac{f_3^2 f_4^5}{f_1^2 f_2^2 f_8^2} \pmod{3}. \quad (10.7)$$

By using (2.19) in (10.7) and then extracting the terms involving q^{2n+1} from both sides of the resulting equation, we obtain

$$\sum_{n \geq 0} \text{pod}_3(12n+7)q^n \equiv 2 \frac{f_2^6 f_3^2 f_{12}}{f_1^6 f_4 f_6} \equiv 2 \frac{f_6 f_{12}}{f_4} \pmod{3}. \quad (10.8)$$

By collecting the odd powers from both sides of Equation (10.8), we get

$$\text{pod}_3(24n+19) \equiv 0 \pmod{3}. \quad (10.9)$$

On the other hand, in view of (2.2) and (10.6), with $p = k = 2$, we obtain

$$\sum_{n \geq 0} \text{pod}_3(6n+1)q^n \equiv \frac{f_3^3}{f_1 f_4} \pmod{4}. \quad (10.10)$$

Employing (2.20) into (10.10) and then extracting the terms of the form q^{2n+1} from both sides of the resulting equation, we find that

$$\sum_{n \geq 0} \text{pod}_3(12n+7)q^n \equiv \frac{f_6^3}{f_2^2} \pmod{4}.$$

Again, if we extract the odd powers from both sides of the above equation, we obtain

$$\text{pod}_3(24n+19) \equiv 0 \pmod{4}. \quad (10.11)$$

Congruence (10.3) follows from (10.9) and (10.11). \square

Theorem 10.2. For any nonnegative integer n , we have

$$\text{pod}_3(12n + 6) \equiv 0 \pmod{4}, \quad (10.12)$$

$$\text{pod}_3(12n + 9) \equiv 0 \pmod{4}, \quad (10.13)$$

$$\text{pod}_3(48n + 36) \equiv 0 \pmod{4}, \quad (10.14)$$

$$\text{pod}_3(96n + 60) \equiv 0 \pmod{4}, \quad (10.15)$$

$$\sum_{n \geq 0} \text{pod}_3(288n + 12)q^n \equiv 2f_1 \pmod{4}, \quad (10.16)$$

$$\text{pod}_3(288n + 204) \equiv 0 \pmod{4}, \quad (10.17)$$

$$\text{pod}_3(864n + j) \equiv 0 \pmod{4}, \quad j = 396, 684. \quad (10.18)$$

Proof. Extracting the terms of the form q^{3n} from both sides of (10.4), dividing by q and then replacing q^3 by q , we obtain

$$\sum_{n \geq 0} \text{pod}_3(3n)q^n = \frac{f_6^9}{f_1 f_2^2 f_3^3 f_4 f_{12}^3}. \quad (10.19)$$

Invoking (2.2) in (10.19), with $p = k = 2$, we get

$$\sum_{n \geq 0} \text{pod}_3(3n)q^n \equiv \frac{f_3 f_6^3}{f_1 f_2^2 f_4 f_{12}} \pmod{4}. \quad (10.20)$$

Substituting (2.18) into (10.20) and then extracting the even and the odd powers from both sides of the resulting equation, we arrive at

$$\sum_{n \geq 0} \text{pod}_3(6n)q^n \equiv \frac{f_3^4 f_8 f_{12}^2}{f_1^4 f_4 f_6^2 f_{24}} \equiv \frac{f_8 f_{12}^2}{f_2^2 f_4 f_{24}} \pmod{4}, \quad (10.21)$$

$$\sum_{n \geq 0} \text{pod}_3(6n + 3)q^n \equiv \frac{f_3^4 f_4^2 f_{24}}{f_1^4 f_2 f_6 f_8 f_{12}} \equiv \frac{f_2 f_6 f_{24}}{f_8 f_{12}} \pmod{4}. \quad (10.22)$$

Congruences (10.12) and (10.13) follow from (10.21) and (10.22) by extracting the odd powers from both sides, respectively. Extracting the even powers from both sides of (10.21), we get

$$\sum_{n \geq 0} \text{pod}_3(12n)q^n \equiv \frac{f_4 f_6^2}{f_1^2 f_2 f_{12}} \pmod{4}. \quad (10.23)$$

Employing (2.3) into (10.23) and then extracting the terms involving the odd powers from both sides of the resulting equation, we obtain

$$\sum_{n \geq 0} \text{pod}_3(24n + 12)q^n \equiv 2 \frac{f_2^3 f_3^2 f_8^2}{f_1^6 f_4 f_6} \equiv 2 \frac{f_2 f_3^2 f_8^2}{f_1^2 f_4 f_6} \pmod{4}. \quad (10.24)$$

Substituting (2.19) into (10.24), we arrive at

$$\sum_{n \geq 0} \text{pod}_3(24n + 12)q^n \equiv 2 \frac{f_4^3 f_8 f_{12}^2}{f_2^4 f_{24}} + 4q \frac{f_6 f_8^3 f_{24}}{f_2^3 f_{12}} \pmod{4}. \quad (10.25)$$

Congruence (10.14) is an immediate result from (10.25). Equating the even powers on both sides of (10.25), we obtain

$$\sum_{n \geq 0} \text{pod}_3(48n + 12)q^n \equiv 2 \frac{f_2^3 f_4 f_6^2}{f_1^4 f_{12}} \equiv 2 \frac{f_2 f_4 f_6^2}{f_{12}} \pmod{4}. \quad (10.26)$$

Congruence (10.15) follows from the above equation. Equating the even terms from both sides of (10.26), we find that

$$\sum_{n \geq 0} \text{pod}_3(96n + 12)q^n \equiv 2 \frac{f_1 f_2 f_3^2}{f_6} \pmod{4}. \quad (10.27)$$

Employing (2.9) into (10.27), we get

$$\sum_{n \geq 0} \text{pod}_3(96n + 12)q^n \equiv 2 \frac{f_3 f_9^4}{f_{18}^2} - 2q \frac{f_3^2 f_9 f_{18}}{f_6} - 4q^2 \frac{f_3^3 f_{18}^4}{f_6^2 f_9^2} \pmod{4}. \quad (10.28)$$

If we extract the terms involving q^{3n} from both sides of the above equation, we obtain

$$\sum_{n \geq 0} \text{pod}_3(288n + 12)q^n \equiv 2 \frac{f_1 f_3^4}{f_6^2} \pmod{4}. \quad (10.29)$$

Congruence (10.16) is true from (10.29). Congruence (10.17) follows from (10.28) by collecting the powers of the form q^{3n+2} from both sides. Collecting the terms of the form q^{3n+1} from both sides of (10.28) and then replacing (2.11) into the resulting equation, we arrive at

$$\sum_{n \geq 0} \text{pod}_3(288n + 108)q^n \equiv 2 \frac{f_3 f_6 f_9^2}{f_{18}} - 4q \frac{f_3^2 f_{18}^2}{f_9} \pmod{4}.$$

Congruence (10.18) follows from the above equation by collecting the terms involving q^{3n+j} for $j = 1, 2$ from both sides. \square

Theorem 10.3. For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, we have

$$\text{pod}_3\left(288p^{2\alpha+1}(pn + i) + 12p^{2\alpha+2}\right) \equiv 0 \pmod{4}, \quad (10.30)$$

where i is an integer and $1 \leq i \leq p - 1$.

Proof. Define

$$\sum_{n \geq 0} a(n)q^n = f_1. \quad (10.31)$$

Combining (10.16) and (10.31), we deduce that

$$\text{pod}_3(288n + 12) \equiv 2a(n) \pmod{4}, \quad (10.32)$$

Now, we consider the congruence equation

$$\frac{3m^2 + m}{2} \equiv \frac{p^2 - 1}{24} \pmod{p},$$

which is equivalent to

$$(6m + 1)^2 \equiv 0 \pmod{p}, \quad (10.33)$$

where $-(p-1)/2 \leq m \leq (p-1)/2$ and $p \geq 5$ is a prime. Then, the congruence relation (10.33) holds if and only if $m = (\pm p - 1)/6$. Therefore, if we substitute (2.35) into (10.31) and then extract the terms in which the powers of q are congruent to $\frac{p^2-1}{24}$ modulo p and then divide by $q^{(p^2-1)/24}$, we find that

$$\sum_{n \geq 0} a\left(pn + \frac{p^2 - 1}{24}\right) q^{pn} = (-1)^{\frac{\pm p-1}{6}} f_{p^2},$$

which implies that

$$\sum_{n \geq 0} a\left(p^2n + \frac{p^2 - 1}{24}\right) q^n = (-1)^{\frac{\pm p-1}{6}} f_1,$$

and for $n \geq 0$,

$$a\left(p^2n + pi + \frac{p^2 - 1}{24}\right) = 0, \quad (10.34)$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha}n + \frac{p^{2\alpha} - 1}{24}\right) = (-1)^{\alpha \frac{\pm p-1}{6}} a(n). \quad (10.35)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{24}$ in (10.35) and using (10.34), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2} - 1}{24}\right) = 0.$$

Again, replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{24}$ ($1 \leq i \leq p-1$) in (10.32), we get (10.30). \square

10.2 Congruences for $\text{pod}_9(n)$

In this section, we derive some Ramanujan-like congruences modulo 3, 4 and 12 and an infinite family of congruences modulo 3 for $\text{pod}_9(n)$. Setting $\ell = 9$ in (10.1), we obtain

$$\sum_{n \geq 0} \text{pod}_9(n)q^n = \frac{f_2 f_9 f_{36}}{f_1 f_4 f_{18}^2}. \quad (10.36)$$

Theorem 10.4. *For any nonnegative integer n , we have*

$$\sum_{n \geq 0} \text{pod}_9(6n)q^n \equiv f_1 \pmod{3}, \quad (10.37)$$

$$\text{pod}_9(6n + 4) \equiv 0 \pmod{3}, \quad (10.38)$$

$$\text{pod}_9(18n + 14) \equiv 0 \pmod{3}, \quad (10.39)$$

$$\sum_{n \geq 0} \text{pod}_9(54n + 2)q^n \equiv f_1 \pmod{3}, \quad (10.40)$$

$$\text{pod}_9(54n + 20) \equiv 0 \pmod{3}, \quad (10.41)$$

$$\text{pod}_9(54n + 38) \equiv 0 \pmod{3}, \quad (10.42)$$

Proof. Substituting (2.30) into (10.36), we obtain

$$\sum_{n \geq 0} \text{pod}_9(n)q^n = \frac{f_{12}^3}{f_2 f_4 f_6 f_{18}} + q \frac{f_4 f_6 f_{36}^2}{f_2^2 f_{12} f_{18}^2}. \quad (10.43)$$

Equating the even powers on both sides of (10.43), and then invoking (2.2) into the resulting equation, with $p = 3$ and $k = 1$, we find that

$$\sum_{n \geq 0} \text{pod}_9(2n)q^n = \frac{f_6^3}{f_1 f_2 f_3 f_9} \equiv \frac{f_2^2 f_6^2}{f_1 f_3 f_9} \pmod{3}. \quad (10.44)$$

Employing (2.16) in (10.44), we get

$$\sum_{n \geq 0} \text{pod}_9(2n)q^n \equiv \frac{f_6^3 f_9}{f_3^2 f_{18}} + q \frac{f_6^2 f_{18}^2}{f_3 f_9^2} \pmod{3}. \quad (10.45)$$

Collecting the terms of the form q^{3n+j} for $j = 0, 1$ from both sides of the above equation, we arrive at

$$\sum_{n \geq 0} \text{pod}_9(6n)q^n \equiv \frac{f_2^3 f_3}{f_1 f_6} \pmod{3}, \quad (10.46)$$

$$\sum_{n \geq 0} \text{pod}_9(6n + 2)q^n \equiv \frac{f_2^2 f_6^2}{f_1 f_3^2} \pmod{3}, \quad (10.47)$$

Congruence (10.38) is an immediate result from (10.45) and congruence (10.37) follows from (10.46).

Again, using (2.16) in (10.47), we obtain

$$\sum_{n \geq 0} \text{pod}_9(6n + 2)q^n \equiv \frac{f_6^3 f_9^2}{f_3^3 f_{18}} + q \frac{f_6^2 f_{18}^2}{f_3^2 f_9} \pmod{3}. \quad (10.48)$$

Congruence (10.39) follows from the above equation by equating the powers of the form q^{3n+2} from both sides. By collecting the terms of the form q^{3n} from both sides of (10.48), we find that

$$\sum_{n \geq 0} \text{pod}_9(18n + 2)q^n \equiv \frac{f_2^3 f_3^2}{f_1^3 f_6} \equiv f_3 \pmod{3}.$$

Congruences (10.40), (10.41), and (10.42) follow from the above equation by extracting the terms of the form q^{3n+j} for $j = 0, 1, 2$, respectively. \square

From Equations (10.37), (10.40), and (10.31), we deduce that

$$\text{pod}_9(6n) \equiv \text{pod}_9(54n + 2) \equiv a(n) \pmod{3}.$$

where $a(n)$ is defined in Equation (10.31). By following the proof of **Theorem 10.3** we obtain the following result.

Corollary 10.1. *For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, we have*

$$\begin{aligned} \text{pod}_9 \left(6p^{2\alpha+1}(pn + i) + \frac{p^{2\alpha+2} - 1}{4} \right) &\equiv 0 \pmod{3}, \\ \text{pod}_9 \left(54p^{2\alpha+1}(pn + i) + \frac{27p^{2\alpha+2} - 3}{12} \right) &\equiv 0 \pmod{3}. \end{aligned}$$

for all $1 \leq i \leq p - 1$.

Theorem 10.5. *For any nonnegative integer n , we have*

$$\text{pod}_9(6n + 5) \equiv 0 \pmod{4}, \quad (10.49)$$

$$\text{pod}_9(54n + 47) \equiv 0 \pmod{12}. \quad (10.50)$$

Proof. Extracting the odd powers from both sides of (10.43), dividing by q and then replacing q^2 by q , we arrive at

$$\sum_{n \geq 0} \text{pod}_9(2n + 1)q^n = \frac{f_2 f_3 f_{18}^2}{f_1^2 f_6 f_9^2}. \quad (10.51)$$

Employing (2.13) in (10.51) and then extracting the terms of the form q^{3n+2} from both sides of the resulting equation, we find that

$$\sum_{n \geq 0} \text{pod}_9(6n + 5)q^n = 4 \frac{f_2 f_6^5}{f_1^5 f_3^2}. \quad (10.52)$$

Congruence (10.49) is an immediate result from (10.52). In view of (2.2) and (10.52), with $p = 3$ and $k = 1$, we arrive at

$$\sum_{n \geq 0} \text{pod}_9(6n + 5)q^n \equiv 4 \frac{f_2 f_6^5}{f_1^2 f_3} \pmod{12}. \quad (10.53)$$

Again, by substituting (2.13) into (10.53) and then extracting the terms involving q^{3n+1} from both sides of the resulting equation, we obtain

$$\sum_{n \geq 0} \text{pod}_9(18n + 11)q^n \equiv 8 \frac{f_2^8 f_3^3}{f_1^{10}} \equiv 8 \frac{f_2^2 f_6^2}{f_1} \pmod{12}. \quad (10.54)$$

By substituting (2.16) into (10.54) and then extracting the terms involving q^{3n+2} from both sides of the resulting equation, we obtain congruence (10.50). \square

PART IV:**ARITHMETIC PROPERTIES OF ℓ -REGULAR
MULTIPARTITIONS**

CHAPTER 11

CONGRUENCES FOR ℓ -REGULAR TRIPARTITIONS FOR $\ell \in \{2, 3\}$

11.1 Introduction

An ℓ -regular tripartition of n is a triplet of ℓ -regular partitions (λ, μ, β) such that the sum of all the parts of λ , μ , and β is equal to n . For example, let $\lambda = (1, 3)$, $\mu = (5, 7)$, and $\beta = (9, 7, 1)$. Then (λ, μ, β) is a 2-regular tripartition of 33.

Let $T_\ell(n)$ denote the number of ℓ -regular tripartitions of n , where the enumerating function of $T_\ell(n)$ is given by

$$\sum_{n \geq 0} T_\ell(n) q^n = \frac{f_\ell^3}{f_1^3}. \quad (11.1)$$

Adiga and Dasappa [2] studied the arithmetic behaviour of the 3-regular tripartitions and established the following infinite families of congruences for $\alpha \geq 1$ and $n \geq 0$,

$$T_3 \left(3^{2\alpha} n + \frac{11 \cdot 3^{2\alpha-1} - 1}{4} \right) \equiv 0 \pmod{3^{2\alpha+3}},$$

and

$$T_3 \left(3^{2\alpha} n + \frac{7 \cdot 3^{2\alpha-1} - 1}{4} \right) \equiv 0 \pmod{3^{2\alpha+2}}.$$

In section 11.2, we establish several families of congruences modulo 2 and 12 for the partition function $T_2(n)$, and in section 11.3, we derive a family of congruences modulo 3 beside few Ramanujan-type congruences for the partition function $T_3(n)$. Note that the sequences $T_2(n)$ and $T_3(n)$ are known in the OEIS [197] as [A022568](#) and [A285927](#), respectively.

11.2 Congruences for $T_2(n)$

In this section, we establish several congruences for the sequence $T_2(n)$.

Lemma 11.1. *We have*

$$\sum_{n \geq 0} T_2(3n)q^n = \frac{f_2}{f_1} + 12q \frac{f_2^2 f_6^5}{f_1^6 f_3}, \quad (11.2)$$

$$\sum_{n \geq 0} T_2(3n+1)q^n = 3 \frac{f_2^4 f_3^5}{f_1^8 f_6}, \quad (11.3)$$

$$\sum_{n \geq 0} T_2(3n+2)q^n = 6 \frac{f_2^3 f_3^2 f_6^2}{f_1^7}. \quad (11.4)$$

Proof. Collecting the powers of the form q^{3n+j} for $j = 0, 1, 2$ from both sides of (2.17), we obtain the desired results. \square

Corollary 11.1. *For all $n \geq 0$,*

$$T_2(3n+1) \equiv 0 \pmod{3},$$

$$T_2(3n+2) \equiv 0 \pmod{6}.$$

Theorem 11.1. *For any prime $p \geq 5$, $\alpha \geq 0$, and $n \geq 0$, we have*

$$T_2 \left(3p^{2\alpha+1}(pn+i) + \frac{p^{2\alpha+2}-1}{8} \right) \equiv 0 \pmod{2}, \quad (11.5)$$

where i is an integer and $1 \leq i \leq p-1$.

Proof. In view of (2.2) and (11.2), with $p = 2$ and $k = 1$, we have

$$\sum_{n \geq 0} T_2(3n)q^n \equiv f_1 \pmod{2}. \quad (11.6)$$

Define

$$\sum_{n \geq 0} a(n)q^n = f_1. \quad (11.7)$$

Combining (11.6) and (11.7), we find that

$$T_2(3n) \equiv a(n) \pmod{2}. \quad (11.8)$$

Now, we consider the congruence equation

$$\frac{3m^2 + m}{2} \equiv \frac{p^2 - 1}{24} \pmod{p},$$

which is equivalent to

$$(6m+1)^2 \equiv 0 \pmod{p}, \quad (11.9)$$

where $-(p-1)/2 \leq m \leq (p-1)/2$ and $p \geq 5$ is a prime. Then, the congruence relation (11.9) holds if and only if $m = (\pm p - 1)/6$. Therefore, if we substitute (2.35) into (11.7) and

then extract the terms in which the powers of q are congruent to $\frac{p^2-1}{24}$ modulo p and then divide by $q^{(p^2-1)/24}$, we find that

$$\sum_{n \geq 0} a\left(pn + \frac{p^2-1}{24}\right) q^{pn} = (-1)^{\frac{\pm p-1}{6}} f_{p^2},$$

which implies that,

$$\sum_{n \geq 0} a\left(p^2n + \frac{p^2-1}{24}\right) q^n = (-1)^{\frac{\pm p-1}{6}} f_1, \quad (11.10)$$

and for $n \geq 0$,

$$a\left(p^2n + pi + \frac{p^2-1}{24}\right) = 0, \quad (11.11)$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{24}\right) = (-1)^{\alpha \frac{\pm p-1}{6}} a(n). \quad (11.12)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{24}$ in (11.12) and using (11.11), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{24}\right) = 0.$$

Again, replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{24}$ ($1 \leq i \leq p-1$) in (11.8), we arrive at (11.5). \square

Theorem 11.2. For any odd prime p , $\alpha \geq 0$, and $n \geq 0$, we have

$$T_2\left(9p^{2\alpha+1}(pn + i) + \frac{9p^{2\alpha+2}-1}{8}\right) \equiv 0 \pmod{2}, \quad (11.13)$$

where i is an integer and $1 \leq i \leq p-1$.

Proof. In view of (2.2) and (11.3), with $p = 2$ and $k = 1$, we see that

$$\sum_{n \geq 0} T_2(3n + 1)q^n \equiv f_3^3 \pmod{2}. \quad (11.14)$$

If we extract the terms involving q^{3n} from both sides of (11.14), and then replace q^3 by q , we get

$$\sum_{n \geq 0} T_2(9n + 1)q^n \equiv f_1^3 = \sum_{n \geq 0} (-1)^n (2n + 1)q^{n(n+1)/2} \equiv \sum_{n \geq 0} q^{n(n+1)/2} = \psi(q) \pmod{2}. \quad (11.15)$$

Define

$$\sum_{n \geq 0} b(n)q^n = \psi(q). \quad (11.16)$$

Combining (11.15) and (11.16), we obtain

$$T_2(9n + 1) \equiv b(n) \pmod{2}. \quad (11.17)$$

Now, we consider the congruence equation

$$\frac{k^2 + k}{2} \equiv \frac{p^2 - 1}{8} \pmod{p},$$

which is equivalent to

$$(2k + 1)^2 \equiv 0 \pmod{p}, \quad (11.18)$$

where $0 \leq k \leq (p - 1)/2$ and p is an odd prime. The congruence relation (11.18) holds if and only if $k = (p - 1)/2$. Therefore, if we substitute (2.34) into (11.16) and then extract the terms in which the powers of q are $pn + \frac{p^2 - 1}{8}$, we arrive at

$$\sum_{n \geq 0} b\left(pn + \frac{p^2 - 1}{8}\right) q^{pn + \frac{p^2 - 1}{8}} = q^{pn + \frac{p^2 - 1}{8}} \psi(q^{p^2}).$$

Dividing $q^{\frac{p^2 - 1}{8}}$ on both sides of the above equation and then replacing q^p by q , we find that

$$\sum_{n \geq 0} b\left(pn + \frac{p^2 - 1}{8}\right) q^n = \psi(q^p). \quad (11.19)$$

Again, by extracting the terms containing q^{pn} from both sides of (11.19), and then replacing q^p by q , we obtain

$$\sum_{n \geq 0} b\left(p^2n + \frac{p^2 - 1}{8}\right) q^n = \psi(q), \quad (11.20)$$

which implies that for $n \geq 0$,

$$b\left(p^2n + pi + \frac{p^2 - 1}{8}\right) = 0, \quad (11.21)$$

where i is an integer and $1 \leq i \leq p - 1$. By induction, we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$b\left(p^{2\alpha}n + \frac{p^{2\alpha} - 1}{8}\right) = b(n). \quad (11.22)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{8}$ in (11.22) and using (11.21), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$b\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{8}\right) = 0.$$

Again, replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{8}$ ($1 \leq i \leq p-1$) in (11.17), we obtain (11.13). \square

Theorem 11.3. For all $\alpha \geq 0$ and $n \geq 0$, we have

$$T_2\left(3^{4\alpha+4}n + \sum_{i=0}^{2\alpha+1} 3^{2i} + 3^{4\alpha+3}\right) \equiv 0 \pmod{12}, \quad (11.23)$$

$$T_2\left(3^{4\alpha+4}n + \sum_{i=0}^{2\alpha+1} 3^{2i} + 2 \cdot 3^{4\alpha+3}\right) \equiv 0 \pmod{12}, \quad (11.24)$$

$$T_2\left(3^{4\alpha+2}n + \sum_{i=0}^{2\alpha} 3^{2i} + 3^{4\alpha+1}\right) \equiv 0 \pmod{12}, \quad (11.25)$$

$$T_2\left(3^{4\alpha+2}n + \sum_{i=0}^{2\alpha} 3^{2i} + 2 \cdot 3^{4\alpha+1}\right) \equiv 0 \pmod{12}. \quad (11.26)$$

Proof. In view of (2.2) and (11.3), with $p = k = 2$, we see that

$$\sum_{n \geq 0} T_2(3n+1)q^n \equiv 3f_3f_6 \pmod{12}. \quad (11.27)$$

Collecting the terms of the form q^{3n+j} for $j = 0, 1, 2$ from both sides of equation (11.27), we get

$$\sum_{n \geq 0} T_2(9n+1)q^n \equiv 3f_1f_2 \pmod{12}, \quad (11.28)$$

$$T_2(9n+4) \equiv 0 \pmod{12}, \quad (11.29)$$

$$T_2(9n+7) \equiv 0 \pmod{12}. \quad (11.30)$$

Substituting (2.9) into (11.28), we obtain

$$\sum_{n \geq 0} T_2(9n+1)q^n \equiv 3 \frac{f_6f_9^4}{f_3f_{18}^2} - 3qf_9f_{18} - 6q^2 \frac{f_3f_{18}^4}{f_6f_9^2} \pmod{12}. \quad (11.31)$$

If we extract the terms involving q^{3n+1} from both sides of the above equation, divide by q and then replace q^3 by q , we get

$$\sum_{n \geq 0} T_2(27n+10)q^n \equiv 9f_3f_6 \pmod{12}. \quad (11.32)$$

Collecting the terms containing q^{3n+j} for $j = 0, 1, 2$ from both sides of (11.32), we get

$$\sum_{n \geq 0} T_2(81n + 10)q^n \equiv 9f_1f_2 \pmod{12}, \quad (11.33)$$

$$T_2(81n + 37) \equiv 0 \pmod{12}, \quad (11.34)$$

$$T_2(81n + 64) \equiv 0 \pmod{12}. \quad (11.35)$$

Again, by substituting (2.9) into (11.33) and extracting the powers of the form q^{3n+1} from both sides of the resulting equation, we find that

$$\sum_{n \geq 0} T_2(243n + 91)q^n \equiv 3f_3f_6 \pmod{12}. \quad (11.36)$$

From (11.27), (11.32) and (11.36), we deduce that

$$3T_2(3n + 1) \equiv T_2(27n + 10) \pmod{12}, \quad (11.37)$$

$$T_2(3n + 1) \equiv T_2(243n + 91) \pmod{12}, \quad (11.38)$$

Utilizing both (11.37) and (11.38) and by mathematical induction on $\alpha \geq 0$, we arrive at

$$3T_2(3n + 1) \equiv T_2\left(3^{4\alpha+3}n + \sum_{i=0}^{2\alpha+1} 3^{2i}\right) \pmod{12}, \quad (11.39)$$

$$T_2(3n + 1) \equiv T_2\left(3^{4\alpha+1}n + \sum_{i=0}^{2\alpha} 3^{2i}\right) \pmod{12}. \quad (11.40)$$

Using (11.39), (11.34), and (11.35), we obtain (11.23) and (11.24), respectively. Similarly, using (11.40), (11.29), and (11.30), we obtain (11.25) and (11.26), respectively. \square

11.3 Congruences for $T_3(n)$

In this section, we derive some congruences for the counting sequence $T_3(n)$.

Theorem 11.4. *We have*

$$\sum_{n \geq 0} T_3(3n)q^n = \frac{f_2^2 f_3^{15}}{f_1^{11} f_6^6} + 8q \frac{f_3^6 f_6^3}{f_1^8 f_2} + 16q^2 \frac{f_6^{12}}{f_1^5 f_2^4 f_3^3}, \quad (11.41)$$

$$\sum_{n \geq 0} T_3(3n + 1)q^n = 3 \frac{f_2 f_3^{12}}{f_1^{10} f_6^3} + 12q \frac{f_3^3 f_6^6}{f_1^7 f_2^2}, \quad (11.42)$$

$$\sum_{n \geq 0} T_3(3n + 2)q^n = 9 \frac{f_3^9}{f_1^9}. \quad (11.43)$$

Proof. Setting $\ell = 3$ in (11.1), we have

$$\sum_{n \geq 0} T_3(n)q^n = \frac{f_3^3}{f_1^3}. \quad (11.44)$$

Substituting (2.8) into (11.44), and then extracting the terms of the form q^{3n+j} for $j = 0, 1, 2$ from both sides of the resulting equation, we obtain (11.41), (11.42), and (11.43). \square

Corollary 11.2. For all $n \geq 0$,

$$T_3(3n + 1) \equiv 0 \pmod{3}, \quad (11.45)$$

$$T_3(3n + 2) \equiv 0 \pmod{9}. \quad (11.46)$$

Theorem 11.5. For any prime $p \equiv 3 \pmod{4}$, $\alpha \geq 0$, and $n \geq 0$, we have

$$T_3\left(3p^{2\alpha+1}(pn + i) + \frac{p^{2\alpha+2} - 1}{4}\right) \equiv 0 \pmod{3}, \quad (11.47)$$

where i is an integer and $1 \leq i \leq p - 1$.

Proof. In view of (2.2) and (11.44), with $p = 3$ and $k = 1$, we see that

$$\sum_{n \geq 0} T_3(n)q^n \equiv f_3^2 \pmod{3}. \quad (11.48)$$

If we extract the terms containing q^{3n} from both sides of the above equation, and then replace q^3 by q , we get

$$\sum_{n \geq 0} T_3(3n)q^n \equiv f_1^2 \pmod{3}. \quad (11.49)$$

Define

$$\sum_{n \geq 0} c(n)q^n = f_1^2. \quad (11.50)$$

Combining (11.49) and (11.50), we deduce that

$$T_3(3n) \equiv c(n) \pmod{3}. \quad (11.51)$$

Now, we consider the congruence equation

$$\frac{3k^2 + k}{2} + \frac{3m^2 + m}{2} \equiv \frac{p^2 - 1}{12} \pmod{p},$$

which is equivalent to

$$(6k + 1)^2 + (6m + 1)^2 \equiv 0 \pmod{p}, \quad (11.52)$$

where $-(p-1)/2 \leq k, m \leq (p-1)/2$ and p is a prime such that $\left(\frac{-1}{p}\right) = -1$. Since $\left(\frac{-1}{p}\right) = -1$ for $p \equiv 3 \pmod{4}$, then the congruence relation (11.52) holds if and only if both $k = m =$

$(\pm p - 1)/6$. Substituting (2.35) into (11.50) and then extracting the terms in which the powers of q are congruent to $\frac{p^2-1}{12}$ modulo p and then divide by $q^{\frac{p^2-1}{12}}$, we find that

$$\sum_{n \geq 0} c\left(pn + \frac{p^2-1}{12}\right) q^{pn} = f_{p^2}^2,$$

which implies that,

$$\sum_{n \geq 0} c\left(p^2n + \frac{p^2-1}{12}\right) q^n = f_1^2, \quad (11.53)$$

and for $n \geq 0$,

$$c\left(p^2n + pi + \frac{p^2-1}{12}\right) = 0, \quad (11.54)$$

where i is an integer and $1 \leq i \leq p-1$. By induction we see that for $n \geq 0$ and $\alpha \geq 0$,

$$c\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{12}\right) = c(n). \quad (11.55)$$

Replacing n by $p^2n + pi + \frac{p^2-1}{12}$ ($1 \leq i \leq p-1$) in (11.55) and using (11.54), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$c\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{12}\right) = 0.$$

Again, replacing n by $p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{p^{2\alpha+2}-1}{12}$ in (4.22) ($1 \leq i \leq p-1$), we arrive at (11.47). \square

Corollary 11.3. For all $n \geq 0$, we have

$$T_3(6n + 4) \equiv 0 \pmod{6}, \quad (11.56)$$

Proof. Using (2.2) in (11.42), with $p = 2$ and $k = 1$, we find that

$$\sum_{n \geq 0} T_3(3n + 1)q^n \equiv \frac{f_6^3}{f_2^4} \pmod{2}. \quad (11.57)$$

If we extract the odd terms from both sides of the above equation, we obtain

$$T_3(6n + 4) \equiv 0 \pmod{2}. \quad (11.58)$$

Congruence (11.56) follows from (11.45) and (11.58). \square

PART V:

**COMBINATORIAL PROPERTIES OF
OVERPARTITIONS INTO DISTINCT PARTS**

CHAPTER 12

OVERPARTITIONS INTO DISTINCT PARTS

12.1 Introduction

For $k \geq 2$, a k -sequence in a partition λ is a string of k consecutive integers appearing in λ . For example, the partition $\lambda = (10, 9, 7, 6, 5, 3, 1)$ contains the 3-sequence $(7, 6, 5)$ and the three 2-sequences $(10, 9)$, $(7, 6)$ and $(6, 5)$. The basic case of this class ($k = 2$) was first studied by MacMahon [154, Chpt. IV]. Let $p_2(n)$ denote the number of partitions of a positive integer n without 2-sequences. According to MacMahon [154, Chap. IV], the generating function of $p_2(n)$ is given by

$$\sum_{n \geq 0} p_2(n)q^n = 1 + \sum_{n \geq 1} \frac{q^n (q^6, q^6)_{n-1}}{(1 - q^n)(q^2, q^2)_{n-1}(q^3, q^3)_{n-1}}.$$

The higher cases of k were studied by many, such as Holroyd, Liggett and Romik [125], Andrews [26], and Bringmann and Mahlburg [50]. The generalization of this partitions, the overpartitions into distinct parts without short sequences, was studied by Bringmann, Mahlburg and Nataraj [51] and Choi, Kim and Lovejoy [76], where they presented many new combinatorial identities related to the number of such overpartitions.

In this chapter, we consider the overpartitions into distinct parts, where we prove some new identities related to the rank and crank.

12.2 Overpartitions into distinct parts

The number of overpartitions into distinct parts of a positive integer n , denoted by $\bar{p}_d(n)$, satisfies the following generating function

$$\sum_{n \geq 0} \bar{p}_d(n)q^n = \prod_{n=1}^{\infty} (1 + 2q^n) = 1 + 2q + 2q^2 + 6q^3 + 6q^4 + 10q^5 + \dots$$

Note that by distinct parts, we mean that all parts must be distinct, regardless of whether they are overlined or not. For instance, $(\bar{2}, 2, 1)$ is not included. The coefficients of this generating function matches the sequence [A032302](#) in the Online Encyclopedia of Integer Sequences [197].

Example 12.1. The 10 overpartitions into distinct parts of 5 are

$$\{(5), (\bar{5}), (4, 1), (\bar{4}, 1), (4, \bar{1}), (\bar{4}, \bar{1}), (3, 2), (\bar{3}, 2), (3, \bar{2}), (\bar{3}, \bar{2})\}.$$

Theorem 12.1. For all $i \geq 0$, we have

$$\begin{aligned}\bar{p}_d(4i+1) &\equiv 2 \pmod{8}, \\ \bar{p}_d(4i+2) &\equiv 2 \pmod{8}, \\ \bar{p}_d(4i+3) &\equiv 6 \pmod{8}, \\ \bar{p}_d(4i+4) &\equiv 6 \pmod{8}.\end{aligned}$$

Proof. Let $\bar{A}(n)$ denote the set of overpartitions of n into distinct parts. We split the set $\bar{A}(n)$ into two disjoint subsets $\bar{A}_1(n)$ and $\bar{A}_2(n)$, where $\bar{A}_1(n)$ contains two overpartitions of size n and exactly $4 \times \lfloor \frac{n-1}{2} \rfloor$ overpartitions into two distinct parts ($\lfloor \frac{n-1}{2} \rfloor = \frac{n-1}{2}$ if n is odd and equals $\frac{n}{2} - 1$ if n is even). On the other hand, the subset $\bar{A}_2(n)$ contains all the overpartitions $\lambda \in \bar{A}(n)$ with $\ell(\lambda) \geq 3$. Note that any partition into at least three distinct parts generates $8k$ overpartitions into at least three distinct parts, where $k \geq 1$. Then,

$$\bar{p}_d(n) = |\bar{A}(n)| = |\bar{A}_1(n)| + |\bar{A}_2(n)| = 2 + 4 \times \lfloor \frac{n-1}{2} \rfloor + 8k, \text{ for } k \geq 0.$$

That is,

$$\bar{p}_d(n) = \begin{cases} 2 + 4\left(\frac{n-1}{2}\right) + 8k & \text{if } n \text{ is odd,} \\ 2 + 4\left(\frac{n}{2} - 1\right) + 8k & \text{if } n \text{ is even.} \end{cases}$$

Now, we deduce that

$$2 + 4\left(\frac{n-1}{2}\right) \equiv \begin{cases} 2 \pmod{8}, \\ 6 \pmod{8}, \end{cases} \text{ for any odd } n \geq 1.$$

$$2 + 4\left(\frac{n}{2} - 1\right) \equiv \begin{cases} 2 \pmod{8}, \\ 6 \pmod{8}, \end{cases} \text{ for any even } n \geq 2.$$

Which leads to

$$\begin{cases} 2 + 4\left(\frac{n-1}{2}\right) = 8i + 2, \\ 2 + 4\left(\frac{n}{2} - 1\right) = 8i + 2, \\ 2 + 4\left(\frac{n-1}{2}\right) = 8i + 6, \\ 2 + 4\left(\frac{n}{2} - 1\right) = 8i + 6, \end{cases}, i \in \mathbb{N}.$$

Thus

$$\begin{cases} n = 4i + 1, \\ n = 4i + 2, \\ n = 4i + 3, \\ n = 4i + 4, \end{cases}, i \in \mathbb{N}.$$

From which, our result follows. □

The rank of an overpartition λ , according to Corteel and Lovejoy [67], is defined as one less than the largest part, minus the number of overlined parts that are smaller than the largest part.

Let $\bar{p}_d(n, m)$ denote the number of overpartitions of n into distinct parts with rank m .

Theorem 12.2. *We have*

$$\sum_{n, m \geq 0} \bar{p}_d(n, m) b^m q^n = 1 + \sum_{k \geq 1} \frac{2(1+b)^{k-1} b^{k-1} q^{k(k+1)/2}}{(bq; q)_k}.$$

Proof. The function $q^{k(k+1)/2} / (bq; q)_k$ generates the partitions of n into k distinct parts (following Sylvester triangle), where the exponent on b counts the largest part minus the number of parts of each partition (i.e. $\lambda_1 - \ell(\lambda)$). The exponent on b^{k-1} equals the number of parts of each partition minus one (i.e. $\ell(\lambda) - 1$). Hence, the exponent on b in $b^{k-1} q^{k(k+1)/2} / (bq; q)_k$ keeps track the number of the largest part minus one in each partition. On the other hand, $(1+b)$ refers to the possible sorts of each part, whether it is overlined or not. Now, since $(1+b)^{k-1}$ covers all the $k-1$ parts smaller than λ_1 , then this leaves the two possible sorts of λ_1 . Hence the exponent on b in $(1+b)^{k-1}$ keeps track the number of overlined parts in each overpartitions of n into distinct parts with $k-1$ parts. Then, $2(1+b)^{k-1} b^{k-1} q^{k(k+1)/2} / (bq; q)_k$ generates overpartitions of n into k distinct parts where the exponent on b records the largest part minus one minus the number of overlined parts less than the largest part. By summing over k , we obtain the desired generating function. □

Example 12.2. *For $k = 2$, we obtain*

$$\frac{2(1+b)bq^3}{(1-bq)(1-bq^2)} = 2q^3 + 2bq^3 + 2bq^4 + 2b^2q^4 + 2bq^5 + 4b^2q^5 + 2b^3q^5 + \dots,$$

where $4b^2q^5$ indicates that there are 4 overpartitions of 5 into 2 distinct parts with a rank equals to 2. Indeed we have the overpartitions $(4, \bar{1})$, $(\bar{4}, \bar{1})$, $(\bar{3}, 2)$ and $(3, 2)$.

Let $\bar{p}_d(n, t, r)$ denote the number of overpartitions of n into t distinct parts, r of which are overlined.

Theorem 12.3. *We have*

$$\sum_{n, t, r \geq 0} \bar{p}_d(n, t, r) z^t a^r q^n = \sum_{k \geq 0} \frac{(1+a)^k z^k q^{k(k+1)/2}}{(q; q)_k}.$$

Proof. The function $z^k q^{k(k+1)/2} / (q; q)_k$ generates the partitions of n into k distinct parts, where the exponent on z keeps track the number of parts and $(1 + a)^k = \sum_{i=0}^k \binom{k}{i} a^i$ generates all the possible sorts of the overpartition into k distinct parts, where the exponent on a records the number of overlined parts and the coefficients indicate the number of these overpartitions into k distinct parts with i overlined parts. \square

Example 12.3. For $k = 3$, we obtain

$$\frac{(1 + a)^3 z^3 q^6}{(1 - q)(1 - q^2)(1 - q^3)} = z^3 q^6 + 3z^3 a q^6 + 3z^3 a^2 q^6 + 3z^3 a^3 q^6 + \dots,$$

where $3z^3 a q^6$ indicates that there are 3 overpartitions of 6 into 3 distinct parts with one overlined part, in which the overpartitions are $(\bar{3}, 2, 1)$, $(3, \bar{2}, 1)$ and $(3, 2, \bar{1})$.

Recall that a Durfee square of size s is the largest square contained within a partition's Young diagram. Let λ be an overpartition into distinct parts of n with a Durfee square of size s . Then we distinguish two cases:

- **The first case:** the s^{th} part of λ equals s , which means that $\lambda = S + \lambda' + \lambda''$, where λ' is an overpartition into distinct parts less than s (below the Durfee square), λ'' is also an overpartition into exactly $s - 1$ distinct parts (to the right of the Durfee square) and $S = (s, s, \dots, s)$ with $\ell(S) = s$.

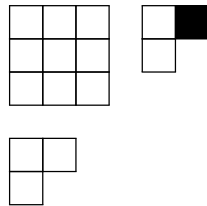


FIGURE 12.1: An illustration of the first case for $n = 15$, $\lambda = (\bar{5}, 4, 3, 2, 1)$, $s = 3$, $\lambda' = (2, 1)$ and $\lambda'' = (\bar{2}, 1)$.

- **The second case:** the s^{th} part is greater than s , which means that λ' and λ'' are overpartitions into distinct parts less or equal to s and into exactly s distinct parts, respectively.

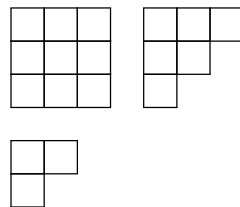


FIGURE 12.2: An illustration of the second case for $n = 18$, $\lambda = (6, 5, 4, 2, 1)$, $s = 3$, $\lambda' = (2, 1)$ and $\lambda'' = (3, 2, 1)$.

The concept of the crank for partitions is identically extended to overpartitions. By employing combinatorial reasoning on Durfee squares, we can establish the following identity.

Theorem 12.4. Let $N(n, l)$ denote the number of overpartitions into distinct parts of n with crank l . Then,

$$\sum_{n=0}^{\infty} \sum_{l=-1}^{\infty} N(n, l) c^l q^n = 2c^{-1}q + \sum_{s=1}^{\infty} 2^s c^s q^{s(3s+1)/2} (1 + 2cq^{2s+1}) W_s(q), \tag{12.1}$$

where

$$W_s(q) = \frac{c^s (-2q^2; q)_{s-1} (q; q)_s + c^{-1} q (-2cq^2; q)_{s-1} (cq; q)_s}{(cq; q)_s (q; q)_s}.$$

Proof. We split the set of overpartitions into distinct parts of n into two disjoint subsets. A subset N which contains all the overpartitions into distinct parts of n with no parts of size 1 and a subset O which contains the rest (all the overpartitions into distinct parts of n with a part of size 1). For instance, for $n = 6$, we have

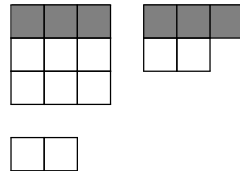
$$N = \{(6), (\bar{6}), (4, 2), (\bar{4}, 2), (4, \bar{2}), (\bar{4}, \bar{2})\}$$

and

$$O = \{(5, 1), (\bar{5}, 1), (5, \bar{1}), (\bar{5}, \bar{1}), (3, 2, 1), (\bar{3}, 2, 1), (3, \bar{2}, 1), (3, 2, \bar{1}), (\bar{3}, \bar{2}, 1), (\bar{3}, 2, \bar{1}), (3, \bar{2}, \bar{1}), (\bar{3}, \bar{2}, \bar{1})\}.$$

By following the combinatorial reasoning on Durfee squares, all the overpartitions into distinct parts are considered to be of the form $\lambda = S + \lambda' + \lambda''$.

- **The overpartitions of the subset N :** First, when the s^{th} part of λ equals s , it is sufficient to eliminate 1 from λ' since this overpartition controls the appearance of 1 as a part in λ . In this case, the crank is the largest part of λ ($c(\lambda) = \lambda_1$), which is obtained from the Durfee square and λ'' as illustrated by grey in the following Young diagram.



Then, we obtain the following generating function

$$\sum_{s=2}^{\infty} f_s[N](q) = \sum_{s=2}^{\infty} 2c^s q^{s^2} (-2q^2; q)_{s-2} \frac{(2c)^{s-1} q^{s(s-1)/2}}{(cq; q)_{s-1}},$$

where the exponent on c keeps track the largest part in the Durfee square and λ'' , with

$$(1 + 2q^2)(1 + 2q^3) \cdots (1 + 2q^{s-1}) = (-2q^2; q)_{s-2}.$$

Similarly, when the s^{th} part is greater than s in λ , we have

$$\sum_{s=1}^{\infty} g_s[N](q) = \sum_{s=1}^{\infty} c^s q^{s^2} (-2q^2; q)_{s-1} \frac{(2c)^s q^{s(s+1)/2}}{(cq; q)_s}.$$

Therefore, by adding both equations above, we obtain the generating function for the overpartitions of the subset N such that

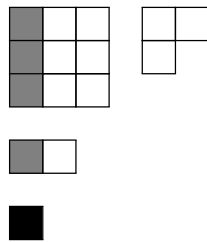
$$\begin{aligned} \sum_{s=1}^{\infty} g_s[N](q) + \sum_{s=2}^{\infty} f_s[N](q) &= \sum_{s=1}^{\infty} \left(g_s[N](q) + f_{s+1}[N](q) \right) \\ &= \sum_{s=1}^{\infty} 2^s c^{2s} q^{\frac{s(3s+1)}{2}} (1 + 2cq^{2s+1}) \frac{(-2q^2; q)_{s-1}}{(cq; q)_s}, \end{aligned}$$

where the exponent on c records the crank of the overpartitions $\lambda \in N$.

- **The overpartitions of the subset O :** First, the generating function of the overpartitions into distinct parts where every overpartition contains a part of size 1 when the s^{th} part is equal to s is given by

$$f_s[O](q) = f_s(q) - f_s[N](q) = qf_s[N](q),$$

in which $f_s(q)$ is the generating function of all the overpartitions into distinct parts where the s^{th} part is equal to s . Since the parts of size 1 can appear at most once in any overpartition, then $c(\lambda) = \mu(\lambda) - 1$. Notice that $\mu(\lambda)$ is obtained from the Durfee square and λ' (which contains no parts of size 1 as illustrated below).



Then, we have the generating function

$$\sum_{s=2}^{\infty} f_s[O](q) = 2c^{-1}q \sum_{s=2}^{\infty} 2c^s q^{s^2} (-2cq^2; q)_{s-2} \frac{2^{s-1} q^{s(s-1)/2}}{(q; q)_{s-1}},$$

where the exponent on c^{-1} equals $-\omega(\lambda)$. Similarly, when the s^{th} part is greater than s , we have

$$\sum_{s=0}^{\infty} g_s[O](q) = 2c^{-1}q \sum_{s=0}^{\infty} c^s q^{s^2} (-2cq^2; q)_{s-1} \frac{2^s q^{s(s+1)/2}}{(q; q)_s},$$

and by convention, let $(-2cq^2; q)_{-1} = 1$. Then, by adding both generating functions we arrive at

$$\begin{aligned} \sum_{s=0}^{\infty} g_s[O](q) + \sum_{s=2}^{\infty} f_s[O](q) &= 2c^{-1}q + \sum_{s=1}^{\infty} \left(g_s[O](q) + f_{s+1}[O](q) \right) \\ &= 2\frac{q}{c} + \sum_{s=1}^{\infty} 2^{s+1} c^{s-1} q^{\frac{s(3s+1)}{2}+1} (1 + 2q^{2s+1}) \frac{(-2cq^2; q)_{s-1}}{(q; q)_s}, \end{aligned}$$

where the exponent on c keeps track the crank of all $\lambda \in O$. Now, by adding both generating functions of the subsets N and O together, we attain the generating function for $N(n, l)$, the number of overpartitions into distinct parts of n with crank l . \square

When doing all the calculations with 2 replaced by 1 in (12.1), we get the following result.

Remark 12.1. Let $p_d(n, l)$ denote the number of partitions into distinct parts of n with crank l . Then,

$$\sum_{n=0}^{\infty} \sum_{l=-1}^{\infty} p_d(n, l) c^l q^n = c^{-1} q + \sum_{s=1}^{\infty} c^s q^{s(3s+1)/2} (1 + cq^{2s+1}) W'_s(q),$$

where

$$W'_s(q) = \frac{c^s (-q^2; q)_{s-1} (q; q)_s + c^{-1} q (-cq^2; q)_{s-1} (cq; q)_s}{(cq; q)_s (q; q)_s}.$$

In the following result, we propose a generating function for the number of overpartitions of n into distinct parts, with rank t and crank l , where the rank t is defined as per Dyson's definition [78].

Theorem 12.5. Let $K(n, l, t)$ denote the number of overpartitions into distinct parts of n , with rank t and crank l . Then,

$$\sum_{n=0}^{\infty} \sum_{l=-1}^{\infty} \sum_{t=-\infty}^{\infty} K(n, l, t) c^l d^t q^n = 2c^{-1} q + \sum_{s=1}^{\infty} (2cd)^s q^{s(3s+1)/2} R_s(q), \quad (12.2)$$

where

$$R_s(q) = (1 + 2q^{2s+1}) \left(\frac{c^s (-2d^{-1}q^2; q)_{s-1}}{(cdq; q)_s} + \frac{2(cd)^{-1} q (-2cd^{-1}q^2; q)_{s-1}}{(dq; q)_s} \right).$$

Proof. By using the proposed proof in Theorem 12.4, we notice that the rank of the overpartition into distinct parts $\lambda = S + \lambda' + \lambda''$ is always the largest part of λ'' minus the number of parts of λ' . Then, we have

$$\sum_{s=2}^{\infty} f_s[N](q) = \sum_{s=2}^{\infty} 2c^s q^{s^2} (-2d^{-1}q^2; q)_{s-2} \frac{(2cd)^{s-1} q^{s(s-1)/2}}{(cdq; q)_{s-1}},$$

$$\sum_{s=1}^{\infty} g_s[N](q) = \sum_{s=1}^{\infty} c^s q^{s^2} (-2d^{-1}q^2; q)_{s-1} \frac{(2cd)^s q^{s(s+1)/2}}{(cdq; q)_s}.$$

Thus we obtain the generating function of all the overpartitions with no parts of size 1 to be

$$\sum_{s=1}^{\infty} A_s[N](q) = \sum_{s=1}^{\infty} (2d)^s c^{2s} q^{s(3s+1)/2} (1 + 2cq^{2s+1}) \frac{(-2d^{-1}q^2; q)_{s-1}}{(cdq; q)_s}.$$

Now, for the second case where each partition contains a part of size 1, we have

$$\sum_{s=2}^{\infty} f_s[O](q) = 2c^{-1} q \sum_{s=2}^{\infty} 2c^s q^{s^2} (-2d^{-1}q^2; q)_{s-2} \frac{2^{s-1} d^{s-2} q^{s(s-1)/2}}{(dq; q)_{s-1}},$$

$$\sum_{s=0}^{\infty} g_s[O](q) = 2c^{-1}q + 2c^{-1}q \sum_{s=1}^{\infty} c^s q^{s^2} (-2d^{-1}q^2; q)_{s-1} \frac{2^s d^{s-1} q^{s(s+1)/2}}{(dq; q)_s}.$$

In the same manner we obtain

$$\sum_{s=1}^{\infty} B_s[O](q) = 2c^{-1}q + 2c^{-1}q \sum_{s=1}^{\infty} 2^s c^s d^{s-1} q^{s(3s+1)/2} (1 + 2cq^{2s+1}) \frac{(-2cd^{-1}q^2; q)_{s-1}}{(dq; q)_s}.$$

By adding both generating functions $A_s[N](q)$ and $B_s[O](q)$ together, we establish the required identity. \square

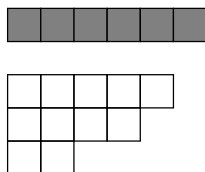
We Introduce two generating functions which generate all the overpartitions into distinct parts with crank s .

Theorem 12.6. *Let $C_s(q)$ and $G_{s-1}(q)$ denote the generating functions of overpartitions into distinct parts with crank s where all the parts are greater than 1 and with crank $s - 1$ where every overpartition contains a part of size 1, respectively. Then,*

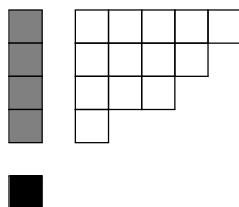
$$C_s(q) = 2q^s (-2q^2; q)_{s-2} \text{ for } s \geq 2, \tag{12.3}$$

$$G_{s-1}(q) = (2q)^{s+1} \frac{q^{s(s+1)/2}}{(q; q)_s} \text{ for } s \geq 0. \tag{12.4}$$

Proof. For the first case where all the overpartitions contain no part of size 1, we have the following Young diagram



where the grey part, constructed by $2q^s$, formes the largest part of the overpartition, which is in this case, the crank s . The rest below is an overpartition into distinct parts each less than s . From which, we obtain the generating function $(-2q^2; q)_{s-2}$ and since the smallest part can be 2, then $s \geq 2$. For the second case where all the overpartitions contain a part of size 1, we have the following Young diagram



where the grey column, formed by q^s , is $\mu(\lambda)$ and the black square is the part 1 (i.e. $2q$). To the right is an overpartition into exactly s distinct parts. Therefore, we obtain the generating function $2^s q^{s(s+1)/2} / (q; q)_s$. \square

Example 12.4. For $s = 3$, we obtain

$$C_3(q) = 2q^3(-2q^2; q)_1 = 2q^3 + 4q^5,$$

where the overpartitions are $\{(3), (\bar{3})\}$ for $n = 3$ and $\{(3, 2), (\bar{3}, 2), (3, \bar{2}), (\bar{3}, \bar{2})\}$ for $n = 5$.

$$C_2(q) = (2q)^4 \frac{q^6}{(q; q)_3} = 16q^{10} + 16q^{11} + 32q^{12} + \dots,$$

where some of these overpartitions are $\{(4, 3, 2, 1)\}$ for $n = 10$, $\{(5, 3, 2, 1)\}$ for $n = 11$ and $\{(5, 4, 2, 1), (6, 3, 2, 1)\}$ for $n = 12$.

When doing all the calculations with 2 replaced by 1 in (12.3) and (12.4), we obtain the following result.

Remark 12.2. Let $Q_s(q)$ and $K_{s-1}(q)$ denote the generating functions of partitions into distinct parts with crank s where all the parts are greater than 1 and with crank $s - 1$ where every partition contains a part of size 1, respectively. Then,

$$Q_s(q) = q^s(-q^2; q)_{s-2}, \text{ for } s \geq 2,$$

$$K_{s-1}(q) = q^{s+1} \frac{q^{s(s+1)/2}}{(q; q)_s}, \text{ for } s \geq 0.$$

PART VI:

ARNDT COMPOSITIONS AND LINEAR RECURRENCE RELATIONS

CHAPTER 13

ARNDT COMPOSITIONS WITH RESTRICTED PARTS, PALINDROMES, AND COLORED VARIANTS

13.1 Introduction

In this chapter, we study a new family of integer compositions defined from a restriction on pairwise descending components. Specifically, an *Arndt composition* of a positive integer n is defined as a composition $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_\ell)$ with a total weight of n , wherein the condition $\sigma_{2i-1} > \sigma_{2i}$ is satisfied for each positive integer i , $2i \leq \ell$. The name of this family of compositions honors Jörg Arndt, who observed that the number of these compositions is given by the well-known Fibonacci sequence F_n , see the comments of the sequence [A000045](#) in the On-Line Encyclopedia of Integer Sequences [197]. Recall that Fibonacci sequence is defined by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with the initial conditions $F_0 = 0$ and $F_1 = 1$.

The first combinatorial results on Arndt compositions come from the work of Hopkins and Tangboonduangjit [126, 127], where the authors provided the proof of the observation made by Arndt. Recently, Checa and Ramírez [62] conducted a detailed study of Arndt compositions through the use of generating functions. They also establish connections with the reduced anti-palindromic compositions introduced in 2022 by Andrews, Just, and Simay [29]. Continuing along this line, Prodinger [174] merged the concepts of Arndt compositions and Carlitz compositions, determining the associated generating function for the counting sequence of this new family of compositions.

In Section 13.2, we obtain the bivariate generating function for Arndt compositions with parts contained in a set $S \subseteq \mathbb{N}$ with respect to the length and number of parts. It is noted that under certain restrictions, the number of Arndt compositions coincides with the Padovan and Narayana's cows numbers. We provide algorithms to prove these coincidences. Additionally, we introduce a new combinatorial expression to calculate the number of Arndt compositions of length n with an even/odd number of parts.

In Section 13.3, we obtain the generating function for the number of palindromic Arndt compositions. We employ Flajolet and Prodinger's *adding-a-new slice* technique (cf.[90]) to derive the generating function with respect to the length.

Finally, in Section 13.4, we introduce an n -color version of Arndt compositions following Agarwal [8].

13.2 Arndt compositions with restricted parts

Let $\mathcal{A}(n)$ denote the set of Arndt compositions of weight n , and let $\mathcal{A} = \bigcup_{n \geq 0} \mathcal{A}(n)$. We set $\mathcal{A}(0) = \{\epsilon\}$, where ϵ denotes the empty composition (weight zero). Hopkins and Tangboonduangjit [126] proved that $|\mathcal{A}(n)| = F_n$ for all $n \geq 0$. For example, the Arndt compositions of 6 are

$$(6), (5, 1), (4, 2), (4, 1, 1), (3, 2, 1), (3, 1, 2), (2, 1, 3), (2, 1, 2, 1). \quad (13.1)$$

We let $\text{parts}(\sigma)$ denote the number of parts in a composition σ . Let $a(n)$ and $a(n, m)$ denote the number of Arndt compositions of n and the number of Arndt compositions of n with exactly m parts, respectively. It is clear that $a(n) = \sum_{m \geq 1} a(n, m)$. We introduce a bivariate generating function to count the number of Arndt compositions with respect to the weight and number of parts:

$$A(x, y) := \sum_{\sigma \in \mathcal{A}} x^{|\sigma|} y^{\text{parts}(\sigma)} = \sum_{n, m \geq 0} a(n, m) x^n y^m.$$

Checa and Ramírez [62, Thm. 1] derived the following generating function

$$\begin{aligned} A(x, y) &= \frac{1 - x - x^2 + x^3 + xy - x^3y}{1 - x - x^2 + x^3 - x^3y^2} = 1 + yx + yx^2 + (y + y^2)x^3 + (y + y^2 + y^3)x^4 \\ &\quad + (y + 2y^2 + 2y^3)x^5 + (y + 2y^2 + 4y^3 + y^4)x^6 + O(x^7). \end{aligned}$$

The bold summand in the above series means that $a(6, 1) = 1, a(6, 2) = 2, a(6, 3) = 4$, and $a(6, 4) = 1$, consistent with the list (13.1).

13.2.1 Arndt compositions with parts from a set

In Theorem 13.1, we extend this result to Arndt compositions with parts from a non-empty given set $S \subseteq \mathbb{N}$. Let $\mathcal{A}_S(n)$ denote the set of Arndt compositions of n with parts in S , and let $\mathcal{A}_S = \bigcup_{n \geq 0} \mathcal{A}_S(n)$. We let $a_S(n, m)$ denote the number of Arndt compositions of n with m parts in S and let $a_S(n) := \sum_{m \geq 1} a_S(n, m)$. Analogously, we introduce the bivariate generating function

$$A_S(x, y) = \sum_{\sigma \in \mathcal{A}_S} x^{|\sigma|} y^{\text{parts}(\sigma)} = \sum_{\substack{n \in S, \\ m \geq 0}} a_S(n, m) x^n y^m.$$

Theorem 13.1. *For $S \subseteq \mathbb{N}$, we have*

$$A_S(x, y) = \left(1 + y \sum_{i \in S} x^i \right) \sum_{m \geq 0} \left(\sum_{\substack{i, j \in S, \\ j < i}} x^{i+j} \right)^m y^{2m}. \quad (13.2)$$

Proof. Let $\sigma = (\sigma_1, \sigma_2)$ be an Arndt composition with two parts, such that $\sigma_1, \sigma_2 \in S$ and $\sigma_1 > \sigma_2 \geq \min S$. Let us suppose that $\sigma_1 = i \in S$, then the bivariate generating function for

this case is $x^i y^2 \sum_{j \in S, j < i} x^j$. Summing over $i \in S$ we have the series

$$\sum_{\substack{i, j \in S, \\ j < i}} x^{i+j} y^2.$$

Finally, the generating function follows from the fact that Arndt compositions can be regarded as the concatenation of m pairs of parts ($m \geq 0$), whose generating function is

$$\sum_{m \geq 0} \left(\sum_{\substack{i, j \in S, \\ j < i}} x^{i+j} \right)^m y^{2m},$$

along with an additional part (which may be empty). This last part has generating function $(1 + y \sum_{i \in S} x^i)$. Multiplying these two expressions yields the desired result. \square

For example, for $S = \{1, 2\}$ we obtain the generating function of Arndt compositions in $\mathcal{A}_{\{1,2\}}$ with respect to the weight and number of parts:

$$A_{\{1,2\}}(x, y) = \frac{1 + (x + x^2)y}{1 - x^3 y^2} = 1 + yx + yx^2 + \dots + y^{2i} x^{3i} + y^{2i+1} x^{3i+1} + y^{2i+1} x^{3i+2} + \dots$$

It is clear that $a_{\{1,2\}}(n) = 1$ for all $n \geq 1$, because we have only one Arndt composition of each weight, that is,

$$\mathcal{A}_{\{1,2\}} = \{\epsilon, (1), (2), (2, 1), (2, 1, 1), (2, 1, 2), (2, 1, 2, 1), (2, 1, 2, 1, 1), \dots\}.$$

Similarly, for $S = \{1, 2, 3\}$ we have

$$A_{\{1,2,3\}}(x, y) = (1 + y(x + x^2 + x^3)) \sum_{m \geq 0} (x^3 + x^4 + x^5)^m y^{2m} = \frac{1 + (x + x^2 + x^3)y}{1 - x^3 y^2 - x^4 y^2 - x^5 y^2}.$$

As a series expansion, the generating function $A_{\{1,2,3\}}(x, y)$ begins with

$$A_{\{1,2,3\}}(x, y) = 1 + yx + yx^2 + (y + y^2)x^3 + (y^2 + y^3)x^4 + (y^2 + 2y^3)x^5 + (3y^3 + y^4)x^6 + (2y^3 + 2y^4 + y^5)x^7 + (y^3 + 3y^4 + 3y^5)x^8 + O(x^9).$$

For example, Arndt compositions corresponding to the bold coefficient in the above series are

$$(3, 1, 3), (3, 2, 2), (3, 1, 2, 1), (2, 1, 3, 1), (2, 1, 2, 1, 1).$$

By letting $y = 1$, we see that the sequence $a_{\{1,2,3\}}(n)$ begins 1, 1, 1, 2, 2, 3, 4, 5, 7. This appears to match the Padovan sequence defined by the recurrence relation $P(n) = P(n - 2) + P(n - 3)$ for all $n \geq 3$, with initial values $P(0) = P(1) = P(2) = 1$ (see [197, A000931]). We establish this connection in the next theorem, giving what we believe to be a new combinatorial interpretation of the Padovan numbers.

Theorem 13.2. For all $n \geq 0$,

$$a_{\{1,2,3\}}(n) = P(n).$$

Proof. We will define a bijection to establish the equality

$$\mathcal{A}_{\{1,2,3\}}(n) = \mathcal{A}_{\{1,2,3\}}(n-2) \cup \mathcal{A}_{\{1,2,3\}}(n-3).$$

Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)$ be an Arndt composition in the set $\mathcal{A}_{\{1,2,3\}}(n-2)$, with $n \geq 2$. We now associate σ with a composition in $\mathcal{A}_{\{1,2,3\}}(n)$ as follows:

$$\sigma \rightarrow \begin{cases} (\sigma_1, \sigma_2, \dots, \sigma_k, 2), & \text{if } k \text{ is even;} \\ (\sigma_1, \sigma_2, \dots, \sigma_k, 2), & \text{if } k \text{ is odd and } \sigma_k = 3; \\ (\sigma_1, \sigma_2, \dots, \sigma_k + 1, 1), & \text{if } k \text{ is odd and } \sigma_k \in \{1, 2\}. \end{cases}$$

Similarly, given a composition $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k) \in \mathcal{A}_{\{1,2,3\}}(n-3)$, we associate σ with a composition in $\mathcal{A}_{\{1,2,3\}}(n)$ as follows:

$$\sigma \rightarrow \begin{cases} (\sigma_1, \sigma_2, \dots, \sigma_k, 3), & \text{if } k \text{ is even;} \\ (\sigma_1, \sigma_2, \dots, \sigma_k, 2, 1), & \text{if } k \text{ is odd and } \sigma_k = 3; \\ (\sigma_1, \sigma_2, \dots, \sigma_k + 1, 1, 1), & \text{if } k \text{ is odd and } \sigma_k \in \{1, 2\}. \end{cases}$$

This process is reversible, establishing a one-to-one correspondence between the sets. Therefore, $a_{\{1,2,3\}}(n)$ satisfies the same recurrence relation of the Padovan sequence with the same initial values. □

In Table 13.1, we provide an example of the algorithm described in the preceding proof.

$\mathcal{A}_{\{1,2,3\}}(10)$	$\mathcal{A}_{\{1,2,3\}}(8)$	$\mathcal{A}_{\{1,2,3\}}(7)$
(3, 2, 2, 1, 2)	(3, 2, 2, 1)	
(3, 1, 3, 1, 2)	(3, 1, 3, 1)	
(2, 1, 3, 2, 2)	(2, 1, 3, 2)	
(3, 2, 3, 2)	(3, 2, 3)	
(3, 1, 2, 1, 2, 1)	(3, 1, 2, 1, 1)	
(2, 1, 3, 1, 2, 1)	(2, 1, 3, 1, 1)	
(3, 1, 3, 2, 1)		(3, 1, 3)
(3, 2, 3, 1, 1)		(3, 2, 2)
(3, 1, 2, 1, 3)		(3, 1, 2, 1)
(2, 1, 3, 1, 3)		(2, 1, 3, 1)
(2, 1, 2, 1, 2, 1, 1)		(2, 1, 2, 1, 1)

TABLE 13.1: The $n = 10$ case of the bijection established in **Theorem 13.2**.

Open Problem 13.1. From a known identity for Padovan numbers [197, A000931] for $n \geq 5$, we have the equality

$$a_{\{1,2,3\}}(n) = \sum_{k=0}^{\lfloor (n+2)/2 \rfloor} \binom{k}{n+2-2k}, \quad n \geq 0.$$

As an open problem for the readers, we invite them to provide a combinatorial proof of this equality.

For $S = \{1, 2, 3, 4\}$ we obtain the generating function

$$A_{\{1,2,3,4\}}(x, y) = \frac{1 + x(1+x)(1+x^2)y}{1 - x^3(1+x^2)(1+x+x^2)y^2}.$$

As a series expansion, the generating function $A_{\{1,2,3,4\}}(x, y)$ begins with

$$A_{\{1,2,3,4\}}(x, y) = 1 + yx + yx^2 + (y + y^2)x^3 + (y + y^2 + y^3)x^4 + (2y^2 + 2y^3)x^5 + (y^2 + 4y^3 + y^4)x^6 + (y^2 + 5y^3 + 2y^4 + y^5)x^7 + (5y^3 + 5y^4 + 3y^5)x^8 + O(x^9).$$

For example, Arndt compositions corresponding to the bold coefficient in the above series are

$$(4, 3), \quad (4, 2, 1), \quad (4, 1, 2), \quad (2, 1, 4), \quad (3, 1, 3), \quad (3, 2, 2), \\ (3, 1, 2, 1), \quad (2, 1, 3, 1), \quad (2, 1, 2, 1, 1).$$

By letting $y = 1$, we see that the sequence $a_{\{1,2,3,4\}}(n)$ begins 1, 1, 1, 2, 3, 4, 6, 9, 13, 19. This appears to match the Narayana's cows sequence defined by the recurrence relation $N(n) = N(n-1) + N(n-3)$ for all $n \geq 3$, with the initial values $N(0) = N(1) = N(2) = 1$ (see [197, A000930]). We establish this connection in the next theorem, giving what we believe to be a new combinatorial interpretation of the Narayana's cows numbers.

Theorem 13.3. For all $n \geq 0$,

$$a_{\{1,2,3,4\}}(n) = N(n).$$

Proof. We will define a bijection to establish the equality

$$\mathcal{A}_{\{1,2,3,4\}}(n) = \mathcal{A}_{\{1,2,3,4\}}(n-1) \cup \mathcal{A}_{\{1,2,3,4\}}(n-3).$$

Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k)$ be an Arndt composition in the set $\mathcal{A}_{\{1,2,3,4\}}(n-1)$, with $n \geq 1$. We associate σ with a composition in $\mathcal{A}_{\{1,2,3,4\}}(n)$ as follows:

$$\sigma \rightarrow \begin{cases} (\sigma_1, \sigma_2, \dots, \sigma_k, 1), & \text{if } k \text{ is even;} \\ (\sigma_1, \sigma_2, \dots, \sigma_k + 1), & \text{if } k \text{ is odd and } \sigma_k = 1; \\ (\sigma_1, \sigma_2, \dots, \sigma_k, 1), & \text{if } k \text{ is odd and } \sigma_k \in \{2, 3, 4\}. \end{cases}$$

Similarly, given a composition $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_k) \in A_{\{1,2,3,4\}}(n-3)$, we associate σ with a composition in $\mathcal{A}_{\{1,2,3,4\}}(n)$ as follows:

$$\sigma \rightarrow \begin{cases} (\sigma_1, \sigma_2, \dots, \sigma_k, 3), & \text{if } k \text{ is even;} \\ (\sigma_1, \sigma_2, \dots, \sigma_k + 3), & \text{if } k \text{ is odd and } \sigma_k = 1; \\ (\sigma_1, \sigma_2, \dots, \sigma_k + 1, 2), & \text{if } k \text{ is odd and } \sigma_k \in \{2, 3\}; \\ (\sigma_1, \sigma_2, \dots, \sigma_k, 3), & \text{if } k \text{ is odd and } \sigma_k = 4. \end{cases}$$

This process is reversible, establishing a one-to-one correspondence between the sets. Therefore, $a_{\{1,2,3,4\}}(n)$ satisfies the same recurrence relation of the Narayana’s cows sequence with the same initial values. □

In Table 13.2, we provide an example of the algorithm described in the proof of Theorem 13.3.

$\mathcal{A}_{\{1,2,3,4\}}(8)$	$\mathcal{A}_{\{1,2,3,4\}}(7)$	$\mathcal{A}_{\{1,2,3,4\}}(5)$
(3, 1, 2, 1, 1)	(3, 1, 2, 1)	
(2, 1, 2, 1, 2)	(2, 1, 2, 1, 1)	
(4, 1, 2, 1)	(4, 1, 2)	
(3, 2, 2, 1)	(3, 2, 2)	
(4, 2, 2)	(4, 2, 1)	
(2, 1, 3, 1, 1)	(2, 1, 3, 1)	
(3, 1, 3, 1)	(3, 1, 3)	
(2, 1, 4, 1)	(2, 1, 4)	
(4, 3, 1)	(4, 3)	
(3, 1, 4)		(3, 1, 1)
(2, 1, 3, 2)		(2, 1, 2)
(4, 1, 3)		(4, 1)
(3, 2, 3)		(3, 2)

TABLE 13.2: The $n = 8$ case of the bijection established in Theorem 13.3.

Let ℓ be a positive integer. We let \mathcal{A}_ℓ denote the set of Arndt compositions with largest part at most ℓ . Let $A(x, y; \ell)$ denote the bivariate generating function for compositions in \mathcal{A}_ℓ respect to the number of parts and weight. From (13.2), by setting $S = \{1, 2, \dots, \ell\}$, we obtain the following corollary.

Corollary 13.1. *For $\ell \geq 2$ we have*

$$A(x, y; \ell) = \frac{(1 - x^2)(1 - x + x(1 - x^\ell)y)}{1 - x^{2+2\ell}y^2 + x^{2+\ell}(1 + x)y^2 - x(1 + x + x^2(-1 + y^2))}.$$

Let $a_\ell(n)$ be the n -th coefficient of $A(x, 1; \ell)$. We have

$$A_\ell(x) := A(x, 1; \ell) = \sum_{n \geq 0} a_\ell(n)x^n = \frac{1 - x^2}{1 - x - x^2 + x^{\ell+1}}.$$

In Table 13.3 we show the first few values of this sequence for $\ell = 3, 4, 5, 6$. We provide the first combinatorial interpretations for the sequences with $\ell = 5$ and $\ell = 6$.

Sequence n	1	2	3	4	5	6	7	8	9	10	Name
$a_3(n)$	1	1	2	2	3	4	5	7	9	12	A000931 (Padovan Seq.)
$a_4(n)$	1	1	2	3	4	6	9	13	19	28	A000930 (Narayana Seq.)
$a_5(n)$	1	1	2	3	5	7	11	17	26	40	A204631
$a_6(n)$	1	1	2	3	5	8	12	19	30	47	A225393

TABLE 13.3: Sequence $a_\ell(n)$, for $\ell = 3, 4, 5$ and $1 \leq n \leq 10$.

We let $\mathcal{A}_{\geq \ell}$ denote the set of Arndt compositions with smallest part is at least ℓ . Let $B(x, y; \ell)$ denote the bivariate generating function for compositions in $\mathcal{A}_{\geq \ell}$ respect to the weight and number of parts. From (13.2), by setting $S = \{\ell, \ell + 1, \dots\}$, we obtain the following corollary.

Corollary 13.2. *For $\ell \geq 1$ we have*

$$B(x, y; \ell) = \frac{(1 - x^2)(1 - x + x^\ell y)}{1 - x - x^2 + x^3 - x^{2\ell+1}y^2}.$$

Let $b_\ell(n)$ be the n -th coefficient of $B(x, 1; \ell)$. We have

$$B_\ell(x) := B(x, 1; \ell) = \sum_{n \geq 0} b_\ell(n)x^n = \frac{(1 - x^2)(1 - x + x^\ell)}{1 - x(1 + x - x^2 + x^{2\ell})}.$$

In Table 13.4 we show the first few values of this sequence for $\ell = 2, 3, 4, 5$. The sequences $b_\ell(n)$ are new in the OEIS.

Sequence n	2	3	4	5	6	7	8	9	10	11	12	13
$b_2(n)$	1	1	1	2	2	4	5	8	11	16	23	33
$b_3(n)$	0	1	1	1	1	2	2	3	4	6	8	11
$b_4(n)$	0	0	1	1	1	1	1	2	2	3	3	5

TABLE 13.4: Sequence $b_\ell(n)$, for $\ell = 2, 3, 4$ and $2 \leq n \leq 13$.

13.2.2 Arndt compositions with an odd or even number of parts

Another interesting example are Arndt compositions using only even (odd) parts. Let $A_E(x, y)$ (resp., $A_O(x, y)$) be the bivariate generating function of Arndt compositions using only even (odd) parts with respect to the weight and number of parts. From (13.2), by setting $E = \{2, 4, 6, \dots\}$ and $O = \{1, 3, 5, \dots\}$, we obtain the following corollary.

Corollary 13.3. *We have*

$$A_E(x, y) = \frac{(1 - x^4)(1 - x^2(1 - y))}{1 - x^2 - x^4 + x^6(1 - y^2)} \quad \text{and} \quad A_O(x, y) = \frac{(1 - x^4)(1 - x^2 + xy)}{1 - x^2 + x^6 - x^4(1 + y^2)}.$$

Let $a_E(n)$ be the n -th coefficient of $A_E(x, 1)$. We have

$$A_E(x) := A_E(x, 1) = \sum_{n \geq 0} a_E(n)x^n = \frac{1 - x^4}{1 - x^2 - x^4}.$$

Note that for all $n \geq 1$,

$$a_E(n) = \begin{cases} F_{n/2}, & \text{if } n \equiv 0 \pmod{2}; \\ 0, & \text{if } n \equiv 1 \pmod{2}. \end{cases}$$

Let $a_O(n)$ be the n -th coefficient of $A_O(x, 1)$. We have

$$\begin{aligned} A_O(x) := A_O(x, 1) &= \sum_{n \geq 0} a_O(n)x^n = \frac{1 + x - x^2 - x^4 - x^5 + x^6}{1 - x^2 - 2x^4 + x^6} \\ &= 1 + x + x^3 + x^4 + 2x^5 + x^6 + 3x^7 + 3x^8 + 6x^9 + 4x^{10} + O(x^{11}). \end{aligned}$$

While the sequence of coefficients is not in the OEIS, it appears that $a_O(2n)$ matches with the sequence [A006053](#) and $a_O(2n + 1)$ matches with [A028495](#).

Open Problem 13.2. Find bijections between the different restrictions of Arndt compositions and the objects enumerated by the sequences [A006053](#) and [A028495](#).

13.2.3 Arndt compositions in terms of q -series

In **Theorem 13.4**, we present an alternative expression for $A(x, y)$ in terms of q -series notation. This expression is derived from the generating functions of partitions of n into exactly k distinct parts. Here and throughout we will use the standard q -series notation (cf. [23]):

$$(a; q)_n = \begin{cases} \prod_{k=0}^{n-1} (1 - aq^k), & \text{if } n > 0; \\ 1, & \text{if } n = 0. \end{cases}$$

Moreover, $(a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n$, for $|q| < 1$. Let $q(n, k)$ denote the number of partitions of n into $k \geq 1$ distinct parts. The generating function of the sequence $q(n, k)$ is given by

$$\sum_{n, k \geq 0} q(n, k)x^n y^k = \frac{y^k x^{\binom{k+1}{2}}}{(x; x)_k}.$$

Theorem 13.4. The generating function for Arndt compositions with respect to the weight and the number of parts is given by

$$A(x, y) = \left(1 + \frac{xy}{1 - x}\right) \sum_{m \geq 0} \frac{y^{2m} x^{3m}}{(x; x)_2^m}.$$

Proof. Let σ be an Arndt composition. If σ has two parts, then it can be considered as a partition into two distinct parts. The generating function for this case is given by $y^2 x^3 / (x; x)_2$. The generating function follows from the fact that Arndt compositions can be regarded as the

concatenation of partitions with two distinct parts, along with an additional part (which may be empty). \square

Let $A_{2k}(x, y)$ denote the bivariate generating function for the number of Arndt compositions with exactly $2k$ parts, where $k \geq 1$. Then $A_{2k}(x, y) = y^{2k}x^{3k}/(x; x)_2^k$. Analogously, let $A_{2k+1}(x, y)$ be the bivariate generating function for the number of Arndt compositions with exactly $2k + 1$ parts, where $k \geq 0$. Then $A_{2k+1}(x, y) = y^{2k+1}x^{3k+1}/((1-x)(x; x)_2^k)$.

13.2.4 Arndt compositions with a fixed number of parts

Checa and Ramírez [62] obtained the following combinatorial expressions for the number of Arndt compositions with a fixed number of parts. For all $n, m \geq 0$ we have

$$\begin{aligned} a(n, m) &= \sum_{\ell=0}^{n-m-\lfloor \frac{m}{2} \rfloor} \binom{m+\ell-1}{\ell} \binom{n-m-\ell-1}{n-m-\lfloor \frac{m}{2} \rfloor-\ell} (-1)^{n-m-\lfloor \frac{m}{2} \rfloor-\ell} \\ &= \sum_{\ell=0}^{\lfloor (n-m-\lfloor \frac{m}{2} \rfloor)/2 \rfloor} \binom{\lfloor \frac{m}{2} \rfloor + \ell - 1}{\ell} \binom{n-2\lfloor \frac{m}{2} \rfloor - 2\ell - 1}{\lfloor \frac{m-1}{2} \rfloor}. \end{aligned}$$

We use $\lfloor x \rfloor$ to denote the integer part of x , which is the greatest integer less than or equal to x .

In the following theorem, we give an additional formula for the number of Arndt compositions into k parts.

Theorem 13.5. *The number of Arndt compositions into 2ℓ parts is given by*

$$a(n, 2\ell) = \sum_{\substack{a_1+a_2+\dots+a_\ell=n; \\ a_i \geq 3, i=1, \dots, \ell}} \prod_{i=1}^{\ell} \left\lfloor \frac{a_i - 1}{2} \right\rfloor.$$

The number of Arndt compositions into $2\ell + 1$ parts is given by

$$a(n, 2\ell + 1) = \sum_{\substack{a_1+a_2+\dots+a_\ell+a_{\ell+1}=n; \\ a_i \geq 3(1 \leq i \leq \ell), a_{\ell+1} \geq 1}} \prod_{i=1}^{\ell} \left\lfloor \frac{a_i - 1}{2} \right\rfloor.$$

Proof. Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{2\ell})$ be an Arndt composition into 2ℓ parts. This composition is composed by ℓ compositions with exactly two distinct parts. If $a_i = \sigma_{2i-1} + \sigma_{2i}$ ($i = 1, \dots, \ell$), then it is clear that $a_1 + \dots + a_\ell = n$ and $a_i \geq 3$. Since the number of compositions of n into exactly 2 parts is given by $\lfloor (n-1)/2 \rfloor$, we obtain the desired result. If the number of parts is odd, the result follows in a similar manner. \square

For example, if $\ell = 1$ we have

$$a(n, 3) = \sum_{3 \leq a+b \leq n-1} \left\lfloor \frac{a+b-1}{2} \right\rfloor = \sum_{i=3}^{n-1} \left\lfloor \frac{i-1}{2} \right\rfloor.$$

By considering the parity of n we obtain

$$a(n, 3) = \begin{cases} n^2/4 - n + 1, & \text{if } n \text{ is even;} \\ (n^2 - 4n + 3)/4, & \text{if } n \text{ is odd.} \end{cases}$$

The sequence $a(n, 3)$ corresponds to the sequence [A002620](#) in the OEIS. Similarly, we have

$$a(n, 4) = \sum_{\substack{a_1+a_2=n; \\ a_1, a_2 \geq 3}} \prod_{i=1}^2 \left\lfloor \frac{a_i - 1}{2} \right\rfloor = \sum_{i=3}^{n-3} \left\lfloor \frac{(n-i) - 1}{2} \right\rfloor \left\lfloor \frac{i - 1}{2} \right\rfloor.$$

Note that $a(n, 4)$ is the sequence [A006918](#) in the OEIS.

Open Problem 13.3. Find bijections between the different restrictions of Arndt compositions and the objects enumerated by the sequences [A002620](#) and [A006918](#).

13.3 Palindromic Arndt compositions

A *palindromic composition* is one whose sequence of parts is the same when read from left to right or from right to left. For example, the palindromic Arndt compositions of 11 are

$$(11), (5, 1, 5), (2, 1, 5, 1, 2), (4, 3, 4), (3, 1, 3, 1, 3), (2, 1, 2, 1, 2, 1, 2). \quad (13.3)$$

It is well known that the number of palindromic compositions of n is $2^{\lfloor n/2 \rfloor}$. To enumerate palindromic Arndt compositions, we must impose the original constraint $\sigma_{2i-1} > \sigma_{2i}$ with $\sigma_i = \sigma_{k-i+1}$, where k is the number of parts. Let \mathcal{PA} be the set of palindromic Arndt compositions. Note that the number of parts cannot be even since the middle summands cannot simultaneously satisfy both conditions. On the other hand, observe that the second summand must be between two adjacent greater summands since its reflection at the other end of the composition must satisfy $\sigma_3 = \sigma_{k-2} > \sigma_{k-1} = \sigma_2$. The same reasoning applies to the fourth summand, the sixth, and so on. This means that the parts preceding the summand $\sigma_{\lceil k/2 \rceil}$ must form a composition with the constraint

$$\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \sigma_5 > \dots$$

These compositions are known as *descending wiggly sums* or *down/up compositions*, which are alternating compositions beginning with a descent. They are enumerated by the OEIS sequence [A025049](#), and we denote the set of such compositions by \mathcal{W} .

A composition σ in \mathcal{PA} with k parts can be decomposed as the concatenation of a composition μ in \mathcal{W} , its reflection, and a part in the middle. Depending on the parity of the length of μ , the middle part must be greater or less than the summand immediately preceding $\sigma_{\lceil k/2 \rceil}$. Therefore, we will focus first on enumerating the compositions of \mathcal{W} .

13.3.1 Alternating compositions

Alternating compositions have been extensively studied by Bender and Canfield [43], but since we must distinguish them by the parity of their length, we will employ the procedure from Prodinger [174]; both authors utilize the *adding-a-new slice* technique (see Flajolet and Prodinger [90] for more details about this method).

Let us first examine the even case. Let $a_k(x, z)$ be the generating function of those compositions in \mathcal{W} with $2k$ summands; the variable x tracks the weight of the composition and z the size of the last summand. Likewise, we will track the number of parts with y . We define $F(x, y, z) := \sum_{k \geq 0} a_k(x, z)y^{2k}$.

Theorem 13.6. *We have*

$$F(x, y, z) = \alpha(x, y, z) + \frac{\alpha(x, y, x)}{1 - \beta(x, y, x)}\beta(x, y, z). \tag{13.4}$$

where

$$\alpha(x, y, z) = \sum_{n \geq 0} \frac{(-1)^n x^{n^2+n} y^{2n} z^n}{(xz; x)_{2n}} \quad \text{and} \quad \beta(x, y, z) = \sum_{n \geq 1} \frac{(-1)^{n+1} x^{n^2+n} y^{2n} z^n}{(1-x)(xz; x)_{2n-1}}. \tag{13.5}$$

Proof. To apply the *adding-a-new slice* technique, we need to employ the substitution

$$z^j \longrightarrow \sum_{i > j} \sum_{k < i} x^{i+k} z^k = \frac{x^{j+2} z}{(1-x)(1-xz)} - \frac{x^{2j+2} z^{j+1}}{(1-xz)(1-x^2 z)}.$$

This leads us to, for $k \geq 0$,

$$a_{k+1}(x, z) = \frac{x^2 z}{(1-x)(1-xz)} a_k(x, x) - \frac{x^2 z}{(1-xz)(1-x^2 z)} a_k(x, x^2 z),$$

and $a_0(x, z) = 1$. When multiplying $a_k(x, z)$ by y^{2k} and summing over all $k \geq 0$, we obtain

$$\begin{aligned} F(x, y, z) &= a_0(x, z) + \sum_{k \geq 0} a_{k+1}(x, z)y^{2k+2} \\ &= 1 + \frac{x^2 y^2 z}{(1-x)(1-xz)} F(x, y, x) - \frac{x^2 y^2 z}{(1-xz)(1-x^2 z)} F(x, y, x^2 z). \end{aligned}$$

Let $|x|, |y|, |z| < 1$. By iterating this equation infinitely many times, we have

$$F(x, y, z) = \alpha(x, y, z) + F(x, y, x)\beta(x, y, z),$$

where $\alpha(x, y, z)$ and $\beta(x, y, z)$ are defined as in (13.5). By substituting $z \rightarrow x$, we have $F(x, y, x) = \alpha(x, y, x) + F(x, y, x)\beta(x, y, x)$. Solving this equation we obtain the desired result. \square

Let $G(x, y, z)$ be the generating function of those compositions in \mathcal{W} with an odd number of summands.

Theorem 13.7. *We have*

$$G(x, y, z) = \frac{xyz}{1 - xz} F(x, y, xz).$$

Proof. We just need to employ the substitution

$$z^j \longrightarrow \sum_{i>j} x^i y z^i = \frac{xyz}{1 - xz} x^j z^j,$$

for the last summand of each composition enumerated by $F(x, y, z)$. From this, we obtain the result. \square

13.3.2 Generating function of the palindromic Arndt compositions

We are now in a position to give an expression for the generating function of palindromic Arndt compositions.

Theorem 13.8. *The generating function of palindromic Arndt compositions with respect to the weight, the number of parts and the size of the middle summand is given by*

$$PA(x, y, z) = \frac{xyz}{1 - xz} F(x^2, y^2, xz) + \frac{x^3 y^3 z}{(1 - x^2)(1 - xz)} F(x^2, y^2, x^2) - \frac{x^3 y^3 z}{(1 - xz)(1 - x^3 z)} F(x^2, y^2, x^3 z),$$

where $F(x, y, z)$ is given by expression (13.4).

Proof. Let $PA_1(x, y, z)$ and $PA_3(x, y, z)$ be the generating functions of compositions in \mathcal{PA} whose length modulo 4 is 1 and 3, respectively; but in these two particular cases, z will no longer track the last summand but rather the middle one.

When the length of a composition σ in \mathcal{PA} with k parts is of the form $4\ell + 1$, the summand $\sigma_{\lceil k/2 \rceil}$ must be greater than the two immediately to the right and left. Therefore, we use again the substitution

$$z^j \longrightarrow \sum_{i>j} x^i y z^i = \frac{xyz}{1 - xz} x^j z^j,$$

and evaluate $F(x, y, z)$ at $x \rightarrow x^2$ and $y \rightarrow y^2$. Therefore

$$PA_1(x, y, z) = \frac{xyz}{1 - xz} F(x^2, y^2, xz).$$

Similarly, when the length of an element in \mathcal{PA} is $4\ell + 3$, the middle summand must be smaller than the adjacent ones. Therefore, we employ the substitution

$$z^j \longrightarrow \sum_{1 \leq i < j} x^i y z^i = \frac{xyz - x^j y z^j}{1 - xz},$$

and obtain

$$PA_3(x, y, z) = \frac{xyz}{1 - xz} G(x^2, y^2, 1) - \frac{y}{1 - xz} G(x^2, y^2, xz).$$

Finally, we obtain $F(x, y, z)$ by summing and simplifying the expression $PA_1(x, y, z) + PA_3(x, y, z)$. □

As a series expansion, the generating function $PA(x, y, 1)$ begins with

$$PA(x, y, 1) = yx + yx^2 + yx^3 + yx^4 + (y + y^3)x^5 + yx^6 + (y + y^3)x^7 + (y + y^3 + y^5)x^8 + (y + y^3 + y^5)x^9 + (y + y^3 + 2y^5)x^{10} + (y + 2y^3 + 2y^5 + y^7)x^{11} + O(x^{12}).$$

The bold summand in the above series means that there are exactly 6 palindromic Arndt compositions of 11, consistent with the list (13.3).

Let $a_P(n)$ be the number of palindromic Arndt compositions of weight n , that is $a_P(n) = [x^n]PA(x, 1, 1)$. The first few values of this sequence are

$$1, 1, 1, 1, 2, 1, 2, 3, 3, 4, 6, 5, 9, 10, 13, 15, 22, 23, 34.$$

To estimate asymptotically this sequence, we proceed as Flajolet and Prodinger [90], where basically they extend Theorem 4.1 of Sedgewick and Flajolet [91] to non-rational functions. Let $f(x)$ and $g(x)$ be, respectively, the numerator and denominator of $PA(x, 1, 1)$ when we substitute backward in terms of $\alpha(x, y, z)$ and $\beta(x, y, z)$. For convenience, we omit $f(x)$, but $g(x)$ is simply

$$g(x) = (1 - x^2)(1 + x)(1 + x + x^2) \left(1 - \beta(x^2, 1, x^2)\right).$$

Numerically, the dominant zeros of $g(x)$ correspond to $\rho \approx 0.7976727085788669$ and $-\rho$, both are simple. Whence the following formula.

Corollary 13.4. *The number of palindromic Arndt compositions of n is asymptotically*

$$a_P(n) \sim -\frac{f(\rho)}{\rho g'(\rho)} \rho^{-n} + \frac{f(-\rho)}{\rho g'(-\rho)} (-\rho)^{-n} \approx \left(0.436296840800465 + (-1)^{n+1} 0.0291927761747376\right) (1.2536470023922461)^n.$$

13.3.3 Palindromic Arndt compositions with a fixed number of parts

Let $a_P(n, k)$ be the number of palindromic Arndt compositions of n with k parts. It is clear that $a_P(n, 2k) = 0$ for all $n \geq 0$. From Theorem 13.8, and using computer algebra system,

specifically *Mathematica*, we can derive explicit rational generating functions for the sequence $\{a_P(n, k)\}_n$ with k fixed. The combinatorial sums obtained during the proof of the results are well within the reach of modern computer algebra, in particular we used *Mathematica*

For example, for $k = 1, 3, 5$, and $k = 9$ we have the rational generating functions

$$\begin{aligned}\sum_{n \geq 0} a_P(n, 1)x^n &= \frac{x}{1-x} = x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + O(x^{10}), \\ \sum_{n \geq 0} a_P(n, 3)x^n &= \frac{x^5}{1-x^2-x^3+x^5} \\ &= x^5 + x^7 + x^8 + x^9 + x^{10} + 2x^{11} + x^{12} + 2x^{13} + 2x^{14} + 2x^{15} + 2x^{16} + 3x^{17} \\ &\quad + 2x^{18} + 3x^{19} + 3x^{20} + 3x^{21} + O(x^{22}), \\ \sum_{n \geq 0} a_P(n, 5)x^n &= \frac{x^8}{(1-x)^3(1+x)(1+x+x^2+x^3+x^4)} \\ &= x^8 + x^9 + 2x^{10} + 2x^{11} + 3x^{12} + 4x^{13} + 5x^{14} + 6x^{15} + 7x^{16} + 8x^{17} \\ &\quad + 10x^{18} + 11x^{19} + 13x^{20} + O(x^{21}), \\ \sum_{n \geq 0} a_P(n, 9)x^n &= \frac{x^{14}(1+x+x^2(1+x+x^2)(1+x^3+x^6))}{(1-x)^5(1+x)^3(1-x+x^2)(1+x+x^2)^2(1+x+x^2+x^3+x^4)(1+x^3+x^6)} \\ &= x^{14} + x^{15} + 3x^{16} + 3x^{17} + 6x^{18} + 7x^{19} + 12x^{20} + O(x^{21}).\end{aligned}$$

For example, $a_P(16, 5) = 7$ and $a_P(16, 7) = 3$. The corresponding palindromic Arndt compositions are

$$(2, 1, 10, 1, 2), (3, 1, 8, 1, 3), (6, 1, 2, 1, 6), (4, 1, 6, 1, 4), (3, 2, 6, 2, 3), (5, 1, 4, 1, 5), (4, 2, 4, 2, 4)$$

and

$$(2, 1, 2, 1, 4, 1, 2, 1, 2), (3, 1, 2, 1, 2, 1, 2, 1, 3), (2, 1, 3, 1, 2, 1, 3, 1, 2).$$

Except for a shift, the sequences $a_P(n, k)$ for $k = 3, 5$ correspond to the sequences [A103221](#) and [A000115](#), respectively.

Open Problem 13.4. Find bijections between the palindromic Arndt compositions and the objects enumerated by the sequences [A006053](#) and [A028495](#).

13.4 Colored Arndt compositions

Agarwal [8] introduced a generalization of compositions known as *n-color compositions*, where a part of size $m \geq 1$ can come in one of m different colors. The colors of the part m are denoted

by subscripts m_1, m_2, \dots, m_m . For instance, the n -color compositions of 4 are

$$\begin{aligned} &\{4_1\}, \{4_2\}, \{4_3\}, \{4_4\}, \{3_1, 1_1\}, \{3_2, 1_1\}, \{3_3, 1_1\}, \{1_1, 3_1\}, \{1_1, 3_2\}, \{1_1, 3_3\}, \{2_1, 2_1\}, \\ &\{2_1, 2_2\}, \{2_2, 2_1\}, \{2_2, 2_2\}, \{2_1, 1_1, 1_1\}, \{2_2, 1_1, 1_1\}, \{1_1, 2_1, 1_1\}, \{1_1, 2_2, 1_1\}, \{1_1, 1_1, 2_1\}, \\ &\{1_1, 1_1, 2_2\}, \{1_1, 1_1, 1_1, 1_1\}. \end{aligned}$$

From this point, we will refer to these n -color compositions simply as colored compositions. The number of colored compositions of weight n is given by the Fibonacci number F_{2n} . Moreover, the number of colored compositions of n with exactly m parts is given by the binomial coefficient $\binom{n+m-1}{2m-1}$. For further information on n -color compositions, we refer the reader to see [1, 9, 100, 106, 128, 155].

In analogy with these compositions, we introduce the colored version of Arndt compositions. Specifically, a *colored Arndt composition* of a positive integer n is a colored composition of n where the parts of size m can be colored with one of m different colors. For instance, the colored Arndt compositions of 4 are

$$\{4_1\}, \{4_2\}, \{4_3\}, \{4_4\}, \{3_1, 1_1\}, \{3_2, 1_1\}, \{3_3, 1_1\}, \{2_1, 1_1, 1_1\}, \{2_2, 1_1, 1_1\}. \tag{13.6}$$

Let $\mathcal{C}(n)$ denote the set of colored Arndt compositions of weight n , and let $\mathcal{C} = \bigcup_{n \geq 0} \mathcal{C}(n)$. We set $\mathcal{C}(0) = \{\epsilon\}$, where ϵ denotes the empty colored composition (weight zero). Let $c(n)$ and $c(n, m)$ denote the number of colored Arndt compositions of n and the number of colored Arndt compositions of n with exactly m parts, respectively. We introduce a bivariate generating function to count the number of colored Arndt compositions with respect to the weight and number of parts:

$$C(x, y) := \sum_{\sigma \in \mathcal{C}} x^{|\sigma|} y^{\text{parts}(\sigma)} = \sum_{n, m \geq 0} c(n, m) x^n y^m.$$

Theorem 13.9. *The generating function for colored Arndt compositions with respect to the number of parts and weight is given by*

$$C(x, y) = \frac{(1-x)^2(1+x)^3(1+x^2+x(y-2))}{1-x-3x^2-x^3(2y^2-3)-x^4(y^2-3)-x^5(3+y^2)-x^6+x^7}.$$

Proof. Let $w = (u_a, v_b)$ be a colored Arndt composition with two parts. Then $1 \leq v_b < u_a$, $1 \leq a \leq u$, and $1 \leq b \leq v$. If $u_a = j \geq 2$, then the bivariate generating function for this case is

$$\begin{aligned} &\sum_{j \geq 2} jx^j y \left(x + 2x^2 + \dots + (j-1)x^{j-1} \right) y \\ &= y^2 \sum_{j \geq 2} jx^j \frac{x - jx^j + (j-1)x^{j+1}}{(1-x)^2} = \frac{x^3(2+x+x^2)y^2}{(1-x)^4(1+x)^3}. \end{aligned}$$

Colored Arndt compositions are the concatenation of pairs of colored parts. Therefore the generating function for colored Arndt compositions with an even number of parts is given

by

$$\begin{aligned} \sum_{m \geq 0} \left(\frac{x^3}{(1-x)^3(1+x)^2} \right)^m y^{2m} \\ = \frac{(1-x)^4(1+x)^3}{1-x-3x^2+(3-2y^2)x^3+(3-y^2)x^4-(3+y^2)x^5-x^6+x^7}. \end{aligned}$$

Analogously, if the number of parts is odd, then generating functions is given by

$$\begin{aligned} \sum_{m \geq 0} \left(\frac{x^3}{(1-x)^3(1+x)^2} \right)^m y^{2m} \frac{xy}{(1-x)^2} \\ = \frac{(1-x)^2x(1+x)^3y}{1-x-3x^2-x^6+x^7-x^4(-3+y^2)-x^5(3+y^2)-x^3(-3+2y^2)}. \end{aligned}$$

Note that the generating function $xy/(1-x^2) = y \sum_{j \geq 1} jx^j$ corresponds to the last part. Adding the last two equations, we obtain the desired result. \square

As a series expansion, the generating function $C(x, y)$ begins with

$$\begin{aligned} C(x, y) = 1 + yx + 2yx^2 + (3y + 2y^2)x^3 + (4y + 3y^2 + 2y^3)x^4 \\ + (5y + 10y^2 + 7y^3)x^5 + (6y + 13y^2 + 22y^3 + 4y^4)x^6 + O(x^7). \end{aligned}$$

The bold summand in the above series means that there are exactly 9 colored Arndt compositions of 4, consistent with the list (13.6).

Corollary 13.5. *The generating function of the number of colored Arndt compositions is*

$$C(x, 1) = \sum_{n \geq 0} c(n)x^n = \frac{(1-x)^2(1+x)^3(1-x+x^2)}{1-x-3x^2+x^3+2x^4-4x^5-x^6+x^7}.$$

The number of colored Arndt compositions for $1 \leq n \leq 11$:

$$1, \quad 1, \quad 2, \quad 5, \quad 9, \quad 22, \quad 45, \quad 101, \quad 217, \quad 470, \quad 1022.$$

Theorem 13.10. *The generating function for the number of colored Arndt compositions with k parts is given by*

$$C_k(x) = \frac{x^{k+\lfloor k/2 \rfloor} (x^2 + x + 2)^{\lfloor k/2 \rfloor}}{(1-x)^{2k} (1+x)^{3\lfloor k/2 \rfloor}}.$$

Proof. By definition, it is clear that for all $\ell \geq 0$

$$C_{2\ell+1}(x) = (C_2(x))^\ell C_1(x) \quad \text{and} \quad C_{2\ell}(x) = (C_2(x))^\ell.$$

On the other hand, from [Theorem 13.9](#), we already know that

$$C_1(x) = \frac{x}{(1-x)^2} \quad \text{and} \quad C_2(x) = \frac{x^3(x^2+x+2)}{(1-x)^4(1+x)^3}.$$

Therefore

$$C_{2\ell+1}(x) = \frac{x^{3\ell+1}(x^2+x+2)^\ell}{(1-x)^{4\ell+2}(1+x)^{3\ell}} \quad \text{and} \quad C_{2\ell}(x) = \frac{x^{3\ell}(x^2+x+2)^\ell}{(1-x)^{4\ell}(1+x)^{3\ell}}.$$

Proceeding inductively yields the stated formula for $C_k(x)$. □

From [Theorem 13.10](#) a closed formula for $c(n, k)$ can be obtained, however, it may not be as straightforward to manipulate. Instead, we can also provide an asymptotic approximation.

Corollary 13.6. *For a fixed positive integer k , we have*

$$c(n, k) \sim \frac{n^{2k-1}}{2^{\lfloor k/2 \rfloor} (2k-1)!}.$$

Proof. This is a direct application of the transfer theorem (see [Theorem 5.5](#) of Sedgewick and Flajolet [\[91\]](#)). □

CONCLUSIONS

CONCLUDING REMARKS AND PERSPECTIVES

This thesis advances partition theory by exploring new classes of restricted partitions and overpartitions, addressing both combinatorial and arithmetic properties under complex modular and frequency constraints. By introducing multi-restricted partitions, such as ℓ -regular and t -distinct partitions, and developing novel partition types like t -Schur's partitions and overpartitions, this work provides valuable extensions to classical partition theory. Through rigorous applications of generating functions, congruences and combinatorial techniques, the research reveals intricate congruence relations that deepen the understanding of partition structures and their properties.

The introduction of the partition classes; s -modular, s -congruent, and s -duplicate partitions; further expands partition theory's scope, providing new insights into how integer partitions can be structured and analyzed under various conditions. The thesis also extends its impact through the study of multipartitions and Arndt compositions, linking them with linear recurrence relations and enriching their combinatorial interpretations.

The findings of this thesis offer a strong foundation for future research in partition theory and combinatorial number theory. Potential directions include further examination of these restricted partition functions, applications in linear recurrence relations, and continued exploration of connections with classical combinatorial identities. In sum, this research contributes significantly to the theoretical landscape of partition theory, establishing a framework that will support ongoing developments in the field and its applications across mathematical disciplines.

Future research could address several intriguing questions:

- Are there structures in the Monster group M or its representations that are in bijection with the partitions of the set $\mathbb{R}ID^{(\ell,t)}(n)$ for some (ℓ, t) and $n \geq 0$?
- The investigation of partitions into parts that are simultaneously 2-regular and t -distinct revealed several new classes of partitions with their combinatorial properties. How promising are these results for $\ell \geq 3$, or at least in some specific cases?
- What interesting arithmetic properties do $RD^{(\ell,t)}(n)$ and $\bar{S}_t(n)$ exhibit?
- Are there any connections between t -Schur's partitions and overpartitions and other fields or topics?
- Are there any series or product expansions using the Ramanujan theta-function for $D_s(n)$ for $s \geq 6$, analogous to the expansion $\sum_{n \geq 0} \text{pod}(n) = \frac{1}{\psi(-q)}$?
- What is the impact of s -duplicate partitions on topics where $\text{pod}(n)$ appears?

- Are there any recurrence relations, similar to that of $C_4(n)$, applicable to the remaining cases?
- What interesting arithmetic properties do these classes of partitions exhibit?
- Are there any dissections for the generating function $\frac{f_2}{f_1 f_s}$ for $s \geq 6$?
- In the context of $D_4(n, 2)$, are there any interesting combinatorial interpretations for $D_4(n, k)$ for $k \geq 3$?
- The same previous question applies to $M_s(n, k)$ and $C_s(n, k)$?
- Are there any equivalent classes to $\mathbb{E}_s^t(n)$ that satisfy specific difference conditions?
- What arithmetic properties do $E_s^t(n)$ exhibit?
- Is there an overpartition analogue for the partitions enumerated by $E_s^t(n)$?
- What arithmetic properties do $\overline{C}_s(n)$ exhibit?

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