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THÈME

Géométrie et dynamique globales des variétés Lorentziennes

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Géométrie et dynamique globales des variétés Lorentziennes

Résumé en français

Etant donné un groupe de Lie G et une variété M, on considère la question suivante: sous quelles conditions G agit-il isométriquement sur M tout en préservant une métrique Lorentzienne? Cette question revient à classifier les couples (G, M) où G est un groupe de Lie et M une variété Lorentzienne sur lequel G agit isométriquement. Le cas où M est compact a été complètement étudié par S. Adams, G. Stuck [6] et d'une manière indépendante par A. Zeghib [43]. Dans le cas général, une nouvelle approche a été introduite par N. Kowalsky [26, 27]. En considérant seulement le cas de groupes de Lie simples, ses résultats portent sur le groupe et non pas sur la variété. Signalons que l'approche de Kowalsky est de nature très algébrique. Dans nos travaux on a introduit une nouvelle approche plutôt géométrique. L'intérêt de notre approche est qu'on passe du cas simple au cas plus général où le groupe est semisimple; nos réponses portent tout aussi bien sur le groupe que sur la variété.

Contents

Introduction

One of the main problems in geometry is the determination of Lie groups which can act on manifolds, by preserving a well determined geometric structure (Riemannian, pseudo-Riemannian .....), that is the Lie groups that may be embedded in the isometry group of a manifold with a geometric structure. As any Lie group may be splitted into a semi-simple part and a solvable part (Levi decomposition), the problem can be treated in two parts, firstly by determining the semi-simple groups and then the solvable groups. We solve the general problem by merging the two obtained parts.

It is well known that a Lie group that acts properly on a manifold, preserves a Riemannian metric. From this, the natural question that arises is: what about the pseudo-Riemmanian case? and more precisely in which case a Lie group admits an isometric action on a pseudo-Riemmanian manifold? This leads to a classification of couples (G, M) where G is a Lie group and M is a pseudo Riemannian manifold with an isometric action of G on M. Since any connected Lie group admits a left-invariant pseudo Riemannian metric, to make sense of this classification, we consider isometric actions with some dynamic hypotheses. From the geometric point of view, the Lorentzian case comes immediately after the Riemannian one, hence it is obvious to begin first with the study of the Lorentzian case. This was first investigated in the compact case. The problem was: how can a compact Lorentz manifold have a noncompact group of symmetry. The answer was given by S. Adams and G. Stuck [7], A. Zeghib [44]

In [6, 43], one finds a general classification up to local isomorphism of locally faithful isometric action of connected Lie groups on compact Lorentz manifolds.

Theorem 5.1.3 ([6, 43]) *Let G be a connected Lie group acting locally faithfully and isometrically on a compact connected Lorentz manifold M, then the Lie algebra* $\mathfrak{g}$ is isomorphic to $\mathfrak{q} \oplus \mathfrak{a} \oplus \mathfrak{l}$, *where* $\mathfrak{l}$ *is compact semisimple*, $\mathfrak{a}$ *abelian and*

$$\mathfrak{q} \in \{Heisenberg\ Lie\ algebras\} \cup \{rationally\ twisted\ Heisenberg\ Lie\ algebras\}$$
$$\cup \{\mathfrak{sl}(2, \mathbb{R})\} \cup \{the\ affine\ group\} \cup \{o\}.$$

In the general case, one new aspect in Kowalsky's work is to deal with group actions on Lorentz manifolds (possibly noncompact). Obviously, in this kind of work, we can compensate the noncompactness with a dynamical counterpart for ensuring some kind of recurrence. A simple and rather weak condition (which is natural in the compact case) used by Kowalsky is the nonproperness of the action.

It should be noted that the consideration of the noncompact case has a big importance at least from a physical point of view, according to which compact spacetimes have little interest (for example any compact spacetime is nonchronologic). Having a nonproper isometry group is a manifestation of the non-Riemannian character of the geometry of space-time.

Briefly, Kowalsky has proved in [26],

Theorem 5.2.1 ([26]) *If a simple Lie group with finite center acts isometrically, locally freely and nonproperly on a Lorentz manifold, then it is locally isomorphic to* $SL(2, R)$.

Theorem 5.2.2 ([26]) *If a simple Lie group with finite center acts isometrically, locally faithfully and nonproperly on a Lorentz manifold then it is locally isomorphic to either* $O(1, n)$, $n \geq 2$ *or* $O(2, n)$, $n \geq 3$.

Remark *The groups* $O(1, 1)$ *and* $O(2, 2)$ *are not simple.*

From this, a natural question arises: if a group locally isomorphic to $O(1, n)$, $O(2, n)$, or $SL(2, \mathbb{R})$ acts on a Lorentz manifold then, what can we say about this manifold? In [27], Kowalsky announced the following theorems concerning the action of $O(1, n)$, $n \geq 3$ and the action of $O(2, n)$, $n \geq 6$.

Theorem 5.2.3 ([27]) *Let* G *be locally isomorphic to* $O(1, n)$, $n \geq 3$, *and suppose that* G *acts on a manifold* M *preserving a Lorentz metric. Then all noncompact stabilizers have a Lie algebra isomorphic to either* $\mathfrak{o}(1, n)$, $\mathfrak{o}(1, n-1)$, *or* $\mathfrak{o}(n-1) \ltimes \mathbb{R}^{n-1}$.

Theorem 5.2.4 ([27]) *Let* G *be locally isomorphic to* $O(2, n)$, $n \geq 6$, *with* G *having finite center. Suppose that* G *acts nontrivially on a manifold* M *preserving a Lorentz metric. Then all noncompact stabilizers have a Lie algebra isomorphic to* $\mathfrak{o}(1, n)$.

Theorem 5.2.5 ([27]) *Let* G *and* M *be as in Theorem* 5.2.4 *above, and assume there is a point of* M *with noncompact stabilizer. Then the universal cover* $\widetilde{M}$ *is Lorentz isometric to* *a warped product $L \times_\omega \widetilde{AdS_{n+1}}$, where $\widetilde{AdS_{n+1}}$ is the simply connected $(n+1)$-dimensional Lorentz space of constant curvature -1, and L is a Riemannian manifold. Further, the induced action of the universal cover $\widetilde{G}$ on $\widetilde{M}$ is via the canonical $\widetilde{G}$-action on $\widetilde{AdS_{n+1}}$ and the trivial action on L.*

Inspired by the works of A. Zeghib, given in [40, 41, 42, 43] we give in this thesis proofs of all unpublished theorems of N. Kowalsky, with an extension to the semisimple case. Our first result is following theorem which implies theorem 5.2.1 of Kowalsky:

Theorem 5.3.1 ([8]) *Let G be a semi-simple Lie group of finite center acting isometrically and non-properly on a Lorentz manifold M. Suppose that no (local) factor of G is locally isomorphic to $SL(2,R)$. Then there is a point with a non-compact stabilizer. In particular, the restriction of the action of G to its orbit is nonproper. More precisely, the stabilizer of some point contains a non-trivial unipotent one-parameter group. (In other words, a non-proper Lorentz G-space contains a non-proper G-homogeneous orbit).*

The following theorems discuss the case where the orbit obtained in theorem 5.3.1 is of Lorentz type that is the restriction of the metric to this orbit has lorentz signature.

Theorem 5.3.5 ([8]) *Let G be a connected semi-simple Lie group acting isometrically on a Lorentz manifold (M,g) of dimension ≥ 3. Suppose that no (local) factor of G is locally isomorphic to $SL(2,R)$ and that there exists a non-proper orbit N of Lorentz type. Then, $\dim N \geq 3$ and up to a finite cover, G factors $G = G_1 \times G_2$, where:*
1) G_1 possesses an orbit N_1 which is a Lorentz space of constant (non-vanishing) curvature, and G_1 equals $Isom^0(N_1)$.
2) There is a G-invariant neighborhood U of N which is a warped product $L \times_\omega N_1$.
3) The factor N_1 corresponds to G_1-orbits, and G_2 acts along the L-factor.

In the general case we have the following result. Note that the noncompact stabilizer assumption appearing in Kowalski statement is here weakened to nonproperness.

Theorem 5.3.7 ([18]) *Let G be a semisimple group with finite center and no local $SL(2,R)$-factor, acting isometrically and nonproperly on a Lorentz manifold M. Then*
(1) G has a local factor G_1 isomorphic to $O(1,n)$ or $O(2,n)$.
(2) There exists a Lorentz manifold S, isometric, up to finite cover, to dS_n or AdS_{n+1}, depending whether G_1 is isomorphic to $O(1,n)$ or $O(2,n)$, and an open subset of M, in which each G_1-orbit is homothetic to S.
(3) Any such orbit as above has a G_1-invariant neighborhood isometric to a warped product $L \times_w S$, for L a Riemannian manifold.

We observe that Kowalsky's theorem 5.2.2 is obtained as a corollary of theorem 5.3.7

For simple group acting nonproperly with degenerate orbit we have the following theorems .

Theorem 5.3.8 ([18]) *Suppose G is a connected group with finite center, locally isomorphic to $O(1,n)$ or $O(2,n)$ for $n \geq 3$. If G acts isometrically on a Lorentz manifold and has a degenerate orbit O with noncompact stabilizer $G(x)$, then*
(1) G is locally isomorphic to $O(1,n)$;
(2) The Lie algebra $\mathfrak{g}(x)$ of $G(x)$ is isomorphic to $o(n-1) \ltimes R^{n-1}$;
(3) The orbit O is locally homothetic to the light cone in the Minkowski space.

Theorems 5.3.5 and 5.3.8 imply Kowalsky's Theorems 5.2.3 and 5.2.4. The following result implies Theorem 5.2.5 above.

Theorem 5.3.9 ([18]) *If G, a group with finite center locally isomorphic to $O(2,n)$, $n \geq 3$, acts isometrically on a Lorentz manifold M, with some noncompact stabilizer, then, up to finite covers, M is equivariantly isometric to a warped product $L \times_w AdS_{n+1}$ of a Riemannian manifold L with the anti-de Sitter space AdS_{n+1}.*

For $O(1,n)$-actions, we can also describe orbits with compact isotropy (Proposition 5.3.5). Nonproper $O(1,n)$-actions for which compact isotropy are very similar to the standard action on $\mathbb{R}^{1,n}$.

As a fusion, we can give the following "full" theorem:

Theorem 5.3.10 ([18]) *Let G be a semisimple Lie group with finite center and no local $SL(2,R)$-factor, acting nonproperly and isometrically on a Lorentz manifold M. Then, G has a simple local factor G_1 that acts nonproperly. There are two possibilities for G_1:*
(1) $G_1 = O(2,n)$. In this case, there is a Lorentz manifold S isometric, up to finite cover, to AdS_{n+1}, such that all G_1-orbits are homothetic to S. In fact, up to finite cover, M is a warped product $L \times_w AdS_{n+1}$.
(2) $G_1 = O(1,n)$. There are open sets U and V such that $M = U \sqcup \partial V \sqcup V$, where
- *For any $x \in U$, there exists S, isometric to dS_n up to finite cover, such that the G_1-orbit of x is homothetic to S. A neighborhood of x is G_1-equivariantly isometric to a warped product $L \times_w S$, for some Riemannian manifold L.*
- *Orbits of G_1 on the boundary of U are either fixed points or locally homothetic to the light cone in Minkowski space $R^{1,n.}$*
- *For any $x \in V$, there exists T, isometric to H^n up to finite cover, such that the G_1-orbit*

of x is homothetic to T. A neighborhood of x is G_1-equivariantly isometric to a warped product $L \times_w T$, for some Lorentz manifold L.

1

Linear pseudo-Riemannian geometry

1.1 Pseudo-Riemannian inner products

A pseudo-Riemannian inner product on a vector space E is a nondegenerate symmetric bilinear form g on E. A vector space E with a pseudo-Riemannian inner product is called a pseudo-Riemannian inner product space. If g is positive definite, that is $g(X, X) > 0$ for $X \neq 0$ in E, g is called an inner product.

1.1.1 Index

Let g be a symmetric bilinear form on E. The index of g is the largest integer that is the dimension of a subspace $V \subset E$ on which the restriction of g to V is negative definite.

1.1.2 Pullback

Let E be a vector space and (F, h) be a pseudo-Riemannian inner product space. Let f be a linear bijection from E to F. Then the pseudo-Riemannian inner product f^*h on E defined by $f^*h(X, Y) = h(f(X), f(Y))$ is called the *pullback* of h by f.

1.1.3 Orthogonality

Let E be a pseudo-Riemannian inner product space and let X and Y in E. Then we say that X and Y are orthogonal if $g(X, Y) = 0$. In this case we write $X \perp Y$.

In a pseudo-Riemannian inner product space we distinguish three kinds of vectors: isotropic (or null, or lightlike), spacelike, and timelike according to $X \perp X$, $g(X, X) > 0$ or $g(X, X) < 0$.

The set of isotropic vectors is called the isotropic cone, and the set of totally isotropic vectors is called the kernel of g.

A vector subspace of a pseudo-Riemannian inner product space is called lightlike if the restriction of the pseudo-Riemannian inner product on it is null.

The subsets A and B of E are orthogonal if $X \perp Y$ for all X in A and Y in B. In this situation we write $A \perp B$.

Let A be a subset of E. We define $A^{\perp} = \{X \in E, \{X\} \perp A\}$; $A^{\perp}$ is a vector subspace of E called A perp.

An orthonormal basis of E is a set $\{e_1, ..., e_n\}$ such that $g(e_i, e_j) = \delta_{ij} = 1$ if $i = j$ an 0 if $i \neq j$

Proposition 1.1.1 *Every pseudo-Riemannian inner product space has an orthonormal basis.*

Signature of a pseudo-Riemannian inner product

Let E be a pseudo-Riemannian inner product space and $e_1 ... e_n$ be an orthonormal basis of E. Let p be the number of positive signs in $\{g(e_i, e_i)\}$ and q the number of negative signs. Then (p, q) is called the signature of g. This signature is independent of the choice of the basis.

Proposition 1.1.2 *In a pseudo-Riemannian inner product space of signature (p, q) any lightlike vector subspace have dimension inferior or equal to $min(p, q)$.*

Proof. Let $\{e_1, ..., e_n\}$ be an orthonormal basis of E such that e_i is timelike for $i \leq p$ and let V be a subspace on which the restriction of the pseudo-Riemannian inner product is null. It suffices to see that the two maps

$$\begin{array}{rcl} \Phi_1 : V & \to & E \\ x = \sum_{i=1}^{p+q} \alpha_i e_i & \longmapsto & \sum_{i=1}^{p} \alpha_i e_i \end{array} \quad \text{and} \quad \begin{array}{rcl} \Phi_2 : V & \to & E \\ x = \sum_{i=1}^{p+q} \alpha_i e_i & \longmapsto & \sum_{i=p+1}^{p+q} \alpha_i e_i \end{array}$$

are injective. Indeed let $x = \sum_{i=1}^{p+q} \alpha_i e_i$ in V such that $\sum_{i=1}^{p} \alpha_i e_i = 0$. Since $\{e_1, ..., e_n\}$ is a basis of E and V is lightlike we have $\alpha_i = 0$ for $i \leq p$ and $\sum_{i=p+1}^{p+q} \alpha_i^2 = 0$ then $x = 0$ so Φ_1 is injective.

By the same argument we prove that Φ_2 is injective. ■

Proposition 1.1.3 (Witt Thoreme) *Let E be a pseudo-Riemannian inner product space and $e_1, ..., e_n$ an orthonormal basis of E. Then the index of E is the number of negative signs in the signature of g.*

Remark 1.1.1 *Let g be a symmetric bilinear form on a vector space E; this gives a quadratic form q on E defined by $q(X) = g(X, X)$, conversely a quadratic form q gives a symmetric bilinear form g such that q and g are linked by the formula*

$$g(X, Y) = \frac{1}{4}[q(X + Y) - q(X - Y)].$$

Example 1.1.1 *Let $\mathbb{R}^{1,1}$ be the space $\mathbb{R}^2$ with the pseudo-Riemannian inner product defined by the quadratic form $q(x, y) = xy$. Then*

$$\{(x, 0), x \in \mathbb{R}\}^{\perp} = \{(x, 0), x \in \mathbb{R}\}$$

and

$$\{(0, x), x \in \mathbb{R}\}^{\perp} = \{(0, x), x \in \mathbb{R}\}.$$

This example shows that in general $V + V^{\perp} \neq E$. But we have that $\dim V + \dim V^{\perp} = \dim E$ and $(V^{\perp})^{\perp} = V$.

1.1.4 Conformal equivalence

Two pseudo-Riemannian inner products on the same space are conformally equivalent if they are equal up to a multiplicative constant called conformal factor. If g is pseudo-Riemannian inner product on a vector space E, then the set of pseudo-Riemannian inner products conformally equivalent to g is called the conformal class of g.

The following theorem asserts that a pseudo-Riemannian inner product space is determined up to a conformal factor by the nullcone.

Theorem 1.1.1 *Let E be an n dimensional vector space with $n \geq 2$ and let g, h be two nondegenerate pseudo-Riemannian inner products on E. Then g and h are conformally equivalents if and only if they have the same nullcone.*

For the proof we refer to [11]

In the Lorentz case we obtain the following result as a corollary,

Corollary 1.1.1 *Two Lorentzian inner products are conformally equivalent with a positive conformal factor if and only if they have the same nullcone.*

1.1.5 Lorentz inner product

Let E be a vector space. A Lorentz inner product on E is a pseudo-Riemannian inner product of signature $(1, n)$. The importance of this particular case comes from two points. The first one is that the Lorentz situation is the framework of General Relativity, the second point is that the Lorentz case is the nearest to the Riemannian case. As corollary of Propositions 1.1.2 and 1.1.3 we have:

Corollary 1.1.2 *The index of a Lorentz inner product space is 1.*

Corollary 1.1.3 *In the Lorentz inner product space the dimension of a maximal lightlike subspace is 1.*

1.2 Semi-orthogonal groups

Let (E, g) and (F, h) be pseudo-Riemannian inner product spaces. We say that E and F are isometric if there exists a linear bijection f from E to F such that g is the pullback of h by f, in this case f is called a linear isometry of (E, g) to (F, h).

Let (E, g) be a pseudo-Riemannian inner product space, the set of linear isometry from E to itself form a group called the linear isometry group of (E, g). Since E has an orthonormal basis $\{e_1, ..., e_n\}$ such that $g(e_i, e_i) = -1$, for $1 \leq i \leq p$ and $g(e_i, e_i) = 1$ for $p+1 \leq i \leq n$, then it is isometric to the space $\mathbb{R}^n$ equipped with the pseudo-Riemannian inner product $< X, Y >= -x_1y_1 - ... - x_py_p + x_{p+1}y_{p+1} + ... + x_ny_n$. We will denote this space by $\mathbb{R}^{p,n-p}$. The isometry group of (E, g) is isomorphic to the isometry group of $\mathbb{R}^{p,n-p}$. This shows that the study of pseudo-Riemannian inner product spaces returns to the study of the space $\mathbb{R}^{p,n-p}$.

1.2.1 The group $O(p, q)$

Let $O(<>)$ be the isometry group of $\mathbb{R}^{p,n-p}$. A linear map of $\mathbb{R}^{p,n-p}$ to itself is an isometry if its matrix M satisfies

$$M^t I_{p,q} M = I_{p,q} \qquad (*)$$

where $q = n - p$ and $I_{p,q} = \begin{bmatrix} -I_p & 0 \\ 0 & +I_q \end{bmatrix}$.

The group $O(<>)$ is identified with the subgroup $O(p, q)$ of $GL_{p+q}(\mathbb{R})$ of matrices satisfying the relation $(*)$. The elements of $O(p, q)$ have determinant 1 or -1. The set of elements having determinant 1 is a subgroup denoted by $SO(p,q)$ and it contains the component of the identity which is denoted by $SO_0(p,q)$.

We can prove that $O(p,q)$ is homeomorphic to $O(p) \times O(q) \times \mathbb{R}^{p+q}$, then if p and q are > 0, $O(p,q)$ has 4 components. The following proposition gives a characterization of the elements of $O(p,q)$ by their action on the orthonormal basis.

Proposition 1.2.1 *The map f from $\mathbb{R}^{p,n-p}$ to itself is an isometry if and only if it sends the canonical orthonormal basis to orthonormal basis.*

Corollary 1.2.1 *The group $O(p,q)$ acts transitively on the set of vectors having the same length square.*

2

Pseudo-Riemannian geometry versus Riemannian geometry

2.1 Pseudo-Riemannian manifold

A pseudo Riemannian manifold is a manifold M provided with a metric tensor g that is a symmetric nondegenerate $(0,2)$ tensor field on M of constant index. A Riemannian manifold is a pseudo Riemannian manifold with index 0, and a Lorentz manifold is merely a pseudo-Riemannian manifold with index 1.

Let (M,g) be a pseudo-Riemannian manifold, then there exists a unique free torsion connection ∇ on M compatible with the metric g. This connection is called the Levi-Civita connection of (M,g). It is the analog of the Levi-Civita connection in Riemannian geometry and it satisfies the same formula which is,

$$X.g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

and

$$g(\nabla_X Y, Z) = X.g(Y,Z)+Y.g(Z,X)-Z.g(X,Y)-g(X,[Y,\ Z])+g(Y,[Z,\ X])+g(Z,[X,\ Y]).$$

Like in Riemannian geometry, we can define (by the same formula) all objects defined by the connection ∇ (geodesics, parallel translation, curvature,...). But in pseudo-Riemannian geometry the degeneration of the metric is an obstruction to the definition of some objects. To avoid this problem we define these objects in the nondegenerate case. For example in pseudo-Riemannian geometry the second fundamental form is defined for nondegenerate submanifolds and the, sectional curvature is defined for nondegenerate planes. We say that the manifold has constant sectional curvature k if the Riemannian curvature tensor R is given by

$$R_{XY}Z = k[g(Y,Z)X - g(X,Z)Y],$$

and by [37] only in this situation, the sectional curvature is continuously extended to non-degenerate planes.

By [30] if the sectional curvatures of all nondegenerate planes of a Lorentz manifold M of dimension ≥ 3 are bounded from above or from below, then M has constant sectional curvature. We find in [21] a similar result using sectional curvatures of timelike planes. More precisely if the sectional curvatures of timelike planes are bounded from above then the manifold has constant curvature.

2.1.1 Killing vector fields

Definition 2.1.1 *A vector field X on a pseudo-Riemannian manifold (M, g) is called a Killing vector field if $L_X g = 0$ where L_X is the Lie derivative corresponding to X.*

Let ψ_t be the flow of X, since

$$L_X g(v, v) = \lim_{t \to 0} \frac{1}{t} (g(d\psi_t v, d\psi_t v) - g(v, v))$$

and

$$\psi_s \psi_t = \psi_{s+t},$$

we can prove that X is a Killing vector fields if and only if ψ_t is an isometry for all t. The set of all killing vector fields on the n dimensional pseudo-Riemannian manifold M denoted by $k(M)$, is a Lie subalgebra of $\mathfrak{X}(M)$ whose dimension is at lest $n(n+1)/2$. We end this subsection by the following proposition which characterizes the elements of $k(M)$.

Proposition 2.1.1 *Let X be a vector field on a pseudo-Riemannian manifold M, then the following assertions are equivalent.*
1- $X \in k(M)$,
2- $Xg(V, W) = g([X, V], W) + g(V, [X, W])$ for all V, W in $\mathfrak{X}(M)$,
3- $g(\nabla_V X, W) + g(\nabla_W X, V) = 0$ for all V, W in $\mathfrak{X}(M)$.

2.1.2 Isometry group

An isometry from a pseudo-Riemannian manifold (M, g) to a pseudo-Riemannian manifold (N, h) is a diffeomorphism f from M to N such that, for all x in M, $D_x f$ is a linear isometry from $T_x M$ to $T_{f(x)} N$; in this case we say that M and N are isometric and then have the same geometric properties. The set of all isometries from M to M is denoted by $Isom(M)$. By [34] $Isom(M)$ equipped with the compact-open topology is a Lie group, its natural action E on M defined by $E(f, x) = f(x)$ is smooth and a map f from N to $Isom(M)$ is smooth if and only if $Eo(f \times Id)$ is smooth.

Let $isom(M)$ be the Lie algebra of $Isom(M)$. It is anti-isomorphic to $k(M)$ by the following map: for $X \in Isom(M)$ we define $\tilde{X} \in k(M)$ by $\tilde{X}_p = \frac{d}{dt} |_{t=0} \psi_t(p)$.

2.2 Some causality theory

Definition 2.2.1 *Let $(M,\ g)$ be a pseudo-Riemannian manifold. The set of tangent vectors to M is divided into four kinds of vectors*

- *spacelike if* $g(u, u) > 0$ *or* $u = 0$,
- *null or lightlike if* $u \neq 0$ *and* $g(u, u) = 0$,
- *timelike if* $g(u, u) < 0$,
- *nonspacelike or causal if* $g(u, u) \leq 0$.

The set of null vectors in T_pM is called the *nullcone* at p or the *lightlike cone.*

A vector field X is spacelike (respectively timelike, null, or nonspacelike) if X_p is spacelike (respectively timelike, null, or nonspacelike) for all p in M

Definition 2.2.2 *Let* (M, g) *be a pseudo-Riemannian manifold and let* $\alpha(t)$ *be a curve in* M*; then we say that*

- α *is spacelike curve if* $\alpha'(t)$ *is spacelike for all* t,
- α *is timelike if* $\alpha'(t)$ *is timelike for all* t,
- α *is lightlike if* $\alpha'(t)$ *is lightlike for all* t,
- α *is nonspacelike or causal if* $\alpha'(t)$ *is nonspacelike for all* t.

2.2.1 Time orientation

In this subsection, (M, g) is a Lorentz manifold.

A timecone $C(u)$ of a timelike tangent vector u is the set of timelike vectors v such that $g(u, v) < 0$. We remark that the set of timelike vectors is the union of $C(u)$ and $C(-u)$ and then in each tangent space there are two timecones.

We denote $\mathfrak{C}(M) = \{C(u),\ u \text{ tangent to } M\}$, a time orientation of M is a section of $\mathfrak{C}(M)$. This is equivalent to the existence of a timelike vector field X on M, then a time orientation separates the nonspacelike vectors at each point into two classes called future directed and past directed. A Lorentz manifold with time orientation is called a space-time.

Let p and q be two points in a space-time M, we say that q is in the chronological future of p and we write $p \ll q$ if there is a future directed piecewise smooth timelike curve from p to q. The set of these points is called the chronological future of p and denoted by $I^+(p)$. By the same way we define the chronological past of p and denote it by $I^-(p)$. Replacing timelike curve by nonspacelike curve, we introduce $J^-(p)$ and $J^+(p)$; the causal past and future of the point p. If q is in $J^+(p)$ we write $p \leq q$.

We remark that $I^-(p)$ and $I^+(p)$ are both open but $J^-(p)$ and $J^-(p)$ are generally not open nor closed.

The collection $\{I^-(p) \cap I^+(q),\ p, q \in M\}$ is a base of topology called the Alexandrov topology.

A space-time M is said to be chronological if M contains no closed timelike curves and is called causal if it contains no closed nonspacelike curves. Since two metrics on M are conformal if and only if they have the same ligthlike cone hence chronological and causal are conformal properties.

Example 2.2.1 • *Every compact space time is not chronological since this space is covered by a finite collection* $\{I^+(p_i)\}$.
• *The quotient space* $\mathbb{R}^{1,1}/\mathbb{Z}$ *with its metric induced by the quadratic form* xy *of* $\mathbb{R}^{1,1}$ *is a chronological space but not a causal space.*

A causally convex set is a set which has connected intersection with any causal curve. A space-time is said to be strongly causal if each of its points has an arbitrary small causally convex neighborhood.

The following theorem shows that strongly causal space-time are characterized by the Alexandrov topology.

Theorem 2.2.1 ([29], [35]) *Let* M *be a space-time and let* τ *be the Alexandrov topology induced on* M*. Then the following assertions are equivalent,*
• M *is strongly causal.*
• τ *is equivalent to the given manifold topology.*
• τ *is Hausdorff.*

2.2.2 The Lorentz distance

Let p and q be two points of the space-time M. Let $\Omega_{p,q}$ be the space of all future directed causal curves from p to q. For a piecewise smooth curve $\alpha \in \Omega_{p,q}$ we define the length of α by

$$L(\alpha) = \sum_{i=0}^{i=n-1} \int_{t_i}^{t_{i+1}} (-g(\alpha'(t), \alpha'(t)))^{\frac{1}{2}}$$

where α is smooth on each interval $[t_i, t_{i+1}]$.

Definition 2.2.3 *Let* p *in* M*. We define the Lorentz distance by* $d(p,q) = 0$ *if* $q \notin J^+(p)$ *and* $d(p,q) = \sup\limits_{\alpha \in \Omega_{p,q}} L(\alpha)$ *if* $q \in J^+(p)$.

We remark that unlike the Riemannian case, (M, d) is not a metric space since d takes its values in $\mathbb{R}_+ \cup \{+\infty\}$ and that $d(p,p)$ is 0 or $+\infty$ according to $p \in I^+(p)$ or $p \notin I^+(p)$.

In Riemannian geometry there is an equivalence between isometry of the Riemannian manifold (M, g) and isometry of the metric space (M, d) where d is the distance induced by g. The Lorentzian version of this fact is given by the following theorem.

Theorem 2.2.2 ([11]) *Let (M, g) be a strongly causal space time. Let (N, h) be a space-time and let $f : M \to N$ be a map, then the following assertions are equivalent:*
- *f is a distance homothetic map that is $d_h(f(p), f(q)) = cd_g(p, q)$.*
- *f is a smooth homothetic map.*

Corollary 2.2.1 *Each map of a strongly causal space-time into itself is an isometry if and only if it preserves the Lorentz distance.*

2.2.3 Completeness

A space-time (M, g) is said to be timelike complete if all timelike geodesics are defined on the whole real numbers set. By the same way we define the null, spacelike and nonspacelike complete spaces.

For strongly causal space-time we have the following fact.

Theorem 2.2.3 *Let (M, g) be a strongly causal space-time, then the conformal class of g contains a metric h such that the space-time (M, h) is null and timelike complete.*

First, Clarke [16] proved this for null completeness, and then Beem [12] proved it for a large class of space-times which contains the class of strongly causal space-time.

2.3 Geometry of submanifolds

2.3.1 Induced connection

Let N be a submanifold of the pseudo-Riemannian manifold (M, g) and let h be the pullback of the metric g by the inclusion map $i : N \subset M$. A vector field on i is a smooth map from N to TM which assigns to x in N a vector $V_x \in T_xM$. A set $\mathfrak{X}_M(N)$ of these vector fields is an $\mathfrak{F}(N)-$module, where $\mathfrak{F}(N)$ is the ring of smooth functions from N to $\mathbb{R}$.

Suppose that h is nondegenerate and has constant index, then (N, h) is itself a pseudo-Riemannian manifold. Let ∇^N and ∇^M be the Levi-Civita connection of N and M respectively. The question which arises here is the relationship between ∇^N and ∇^M.

Remark 2.3.1 *In the Lorentz case if T_xN has the same causal character for all x in N then this causal character is attributed to N, that is if h_x is positive definite for each $x \in N$, N is said to be spacelike submanifold. If h_x is Lorentzian for each x, N is said to be timelike and if h_x is degenerate for all x, N is said to be lightlike.*

Let N be a pseudo-Riemannian submanifold of (M, g) then,

$$T_xM = T_xN \oplus (T_xN)^{\perp}.$$

Let v in T_xM, so $v = \tan v + nor(v)$ where $\tan v \in T_xN$ and $nor(v) \in (T_xN)^{\perp}$; this gives an orthogonal projections tan from T_xM to T_xN and nor from T_xM to $(T_xN)^{\perp}$. These projections are extended to a projection on the $\mathfrak{F}(N)-$module $\mathfrak{X}_M(N)$.

Let $\tilde{\nabla}$ be the projection of ∇^M by tan that is

$$\begin{array}{rcl} \tilde{\nabla} : \mathfrak{X}(N) \times \mathfrak{X}(N) & \to & \mathfrak{X}(N) \\ (X, V) & \to & \tan(\nabla^M_X V) \end{array},$$

then $\tilde{\nabla}$ is a torsion free connection such that

$$Xh(V, W) = h(\tilde{\nabla}_X V, W) + h(V, \tilde{\nabla}_X W),$$

hence $\tilde{\nabla}$ is the Levi-Civita connection of N so $\tilde{\nabla} = \nabla^N$.

2.3.2 Geodesic submanifold

Let N be a submanifold of a pseudo-Riemannian manifold M. For $v \in T_pM$ let $\overline{v}$ be its class in $\frac{T_pM}{T_pN}$. If V and W are two vector fields on N, we define

$$\Pi(V, W) = \overline{\nabla^M_V W}.$$

Then, Π is symmetric and $\mathcal{F}(N)$-bilinear.

Definition 2.3.1 *We say that N is totally geodesic if* $\Pi = 0$.

This is equivalent to the fact that N contains the geodesics tangent to itself. More precisely if γ is a geodesic such that $\gamma'(0)$ is tangent to N, then there is an $\epsilon > 0$ such that $\gamma(]-\epsilon,\ \epsilon[) \subset N$.

If N is nondegenerate then Π is called the second fundamental form tensor of N.

Example 2.3.1 *Let M and N be pseudo-Riemannian manifolds and f an homothety from M to N (for example an isometry). Let $Graph(f) = \{(x,\ f(x)),\ x \in M\ \}$ be the graph of f and let $(\alpha(t), \beta(t))$ be a geodesic in $M \times N$ tangent to $Graph(f)$ at the point $(x_0,\ f(x_0))$ then $(\alpha'(0), \beta'(0)) \in Graph(D_{x_0}f)$.This implies the existence of some $v \in T_{x_0}M$ such that $(\alpha'(0), \beta'(0)) = (v, D_{x_0}f(v))$. Let $\gamma(t)$ be a geodesic in M with initial velocity v, then $f(\gamma(t))$ is a geodesic in N with initial velocity $D_{x_0}f(v)$. The curve $(\gamma(t),\ f(\gamma(t)))$ is a geodesic in $M \times N$ such that $(\gamma'(0), (fo\gamma)'(0)) = (v, D_{x_0}f(v))$. Then $(\alpha(t),\ \beta(t)) = (\gamma(t),\ f(\gamma(t)))$ and $Graph(f)$ is a geodesic submanifold of $M \times N$.*

Definition 2.3.2 *A nondegenerate submanifold N of M is called umbilic if for any $x \in N$, II_x has the form $II_x = g.\eta_x$, where η_x is some normal vector to T_xN. Furthermore, if the map $x \in N \to \eta_x$, is parallel (along N), N is called spherical submanifold.*

In this case, the vector field (along N) $x \longrightarrow \eta_x$ is called the shape vector field. The totally geodesic submanifolds correspond to the case $\eta_x = 0$, for all $x \in N$.

2.3.3 Constant curvature and geodesic lightlike hypersurfaces

A totally geodesic lightlike hypersurface N in a pseudo Riemannian manifold M of dimension d is an $d-1$ dimensional totally geodesic submanifold of M such that the restriction of the metric of M to N is degenerate, that is $T_xN \cap (T_xN)^{\perp} \neq 0$.
The following theorem proves that in Lorentz geometry, the existence of many totally geodesic lightlike hypersurfaces through every point of a space implies the constant curvature of this space.

Theorem 2.3.1 ([42]) *Let M be a Lorentz manifold such that for each x in M there are a totally geodesic lightlike hypersurfaces $N_1, ..., N_n$ such that the collection $\{T_xN_1^{\perp}, ..., T_xN_n^{\perp}\}$ spans T_xM. Then M has constant sectional curvature.*

2.3.4 Lightlike geodesic hypersurfaces in Lorentz geometry

Let M be a Lorentz manifold, and $Isom(M)$ be its isometric group. Let m be in M and let $stab(m)$ be the isotropy subgroup of m. Let C_m the set of lightlike directions in T_mM tangent to a totally geodesic lightlike hypersurface. Then, we have the following theorem.

Theorem 2.3.2 ([42]) *If $stab(m)$ is noncompact, then $C_m \neq \phi$.*

Proof. Let f_n be a non-bounded sequence in $stab(m)$. Recall that d is the dimension of M; let g the Lorentz metric. The graph $Graph(f) \subset M \times M$ is an isotropic totally geodesic d-dimensional submanifold of $M \times M$, equipped with the metric $g \oplus (-g)$. The graphs $Graph(f_n)$ converge to E, a d-dimensional, isotropic, totally geodesic submanifold, which is no longer a graph, since f_n is divergent (see [42] or [17], 7.4). The intersection $E \cap (\{x\} \times M)$ is nontrivial, but has dimension at most 1, because it is isotropic, and M is Lorentzian. Therefore, the projection $\mathcal{F}_x$ of E is a totally geodesic lightlike hypersurface in $M \times \{x\}$. ■

3

Dynamics and semisimple Lie groups

We begin with some very general definitions concerning Lie algebras and Lie groups.

Definition 3.0.3 *Let A be a finite dimensional Lie algebra. Then:*
1) A is simple if A is nonabelian and has no proper nonzero ideal,
2) A is semisimple if A has no nonzero abelian ideal, or equivalently A has no nonzero solvable ideal., i.e. $radA = 0$ where $radA$ is the solvable radical of A.

Remark 3.0.2 *In the definition of a simple Lie algebra we can replace the non abelian condition by the condition dimension of A is ≥ 2*

Remark 3.0.3 *If A is simple then $[A,\ A] = A$.*

Example 3.0.2 • *$A/radA$ is semisimple for every finite dimensional Lie algebra A.*
• *Any 3-dimensional Lie algebra is either solvable or simple since a one dimensional Lie algebra is solvable, any 2-dimensional Lie algebra has a basis $\{X, Y\}$ such that $[X,\ Y] = Y$ then it is solvable and that the solvability of $\mathfrak{a}$ and $A/\mathfrak{a}$ implies the solvability of A.*
• $\mathfrak{sl}(2, \mathbb{k})$ *is simple for any field $\mathbb{k}$ of characteristic $\neq 2$*

Proposition 3.0.1 *Any semisimple Lie algebra A is a direct sum of a finite family $\{\mathfrak{a}_i\}$ of simple ideals (this decomposition is unique) and every ideal of A is a sum of some various $\mathfrak{a}_i$'s.*

3.1 The Killing form

Definition 3.1.1 *Let A be a Lie algebra. The symmetric bilinear map*

$$k : A \times A \longmapsto \mathbb{k},\ \ (X, Y) \longmapsto Tr(adX \circ adY)$$

where adX is the linear transformation from A to A defined by

$$adX(Y) = [X,\ Y]$$

is called the killing form of A.

Through this map we may determine if the algebra A is semisimple or solvable. More precisely, we have the following theorem.

Theorem 3.1.1 (Cartan) *Let A be a Lie algebra then,*
1) A is solvable if and only if its killing form k satisfies $k(X, Y) = 0$ for all $X \in A$ and all $Y \in [A,\ A]$.
2) A is semisimple if its killing form is nondegenerate.

3.2 Cartan decomposition

Let $\mathfrak{g}$ be a Lie algebra. A Cartan decomposition of $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ such that:
1- $[\mathfrak{l},\ \mathfrak{l}] \subset \mathfrak{l}$, $[\mathfrak{l},\ \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p},\, \mathfrak{p}] \subset \mathfrak{l}$
2- The killing form k of $\mathfrak{g}$ is negative definite on $\mathfrak{l}$ and positive definite on $\mathfrak{p}$.

3.2.1 Existence of Cartan decomposition

Let θ be an involution of $\mathfrak{g}$, that is an automorphism with square equal to the identity. We say that θ is a Cartan involution of $\mathfrak{g}$ if the symmetric bilinear form

$$k_\theta(X, Y) = -k(X, \theta(Y)),$$

where k is the killing form of $\mathfrak{g}$, is positive definite. A Cartan decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ of $\mathfrak{g}$ determines a Cartan involution by the formula $\theta = \mathbb{I}$ on $\mathfrak{l}$ and $\theta = -\mathbb{I}$ on $\mathfrak{p}$.
Conversely, let θ be a Cartan involution of $\mathfrak{g}$. Then we have $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$, where:
$\mathfrak{l} = \{X \in \mathfrak{g}, \theta(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g}, \theta(X) = -X\}$.
It is easy to show that $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}$, $[\mathfrak{l}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{l}$. Since k_θ is positive definite, k is negative definite on $\mathfrak{l}$ and positive definite on $\mathfrak{p}$.
The following proposition affirms the existence of Cartan involution in the semisimple case.

Proposition 3.2.1 *If $\mathfrak{g}$ is a real semisimple Lie algebra, then $\mathfrak{g}$ has a Cartan involution and any two Cartan involutions of $\mathfrak{g}$ are conjugate via an inner automorphism of $\mathfrak{g}$.*

3.3 Roots decomposition

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g} = \mathfrak{l} + \mathfrak{p}$ be a Cartan decomposition. Let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$. For $\lambda \in \mathfrak{a}^*$ we write

$$g_\lambda = \{X \in \mathfrak{g}/ \, [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

If g_λ is nontrivial and $\lambda \neq 0$, we call λ a root of $\mathfrak{g}$ or a root of ($\mathfrak{g}$, a) and g_λ is called a restricted root space. The set of restricted roots of $\mathfrak{g}$ is denoted by Δ.

Proposition 3.3.1 *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ with respect to the Cartan involution θ. Let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$. Then, we have:*
a) $\mathfrak{g}=g_0 \oplus \bigoplus_{\lambda\in\Delta} g_\lambda$ *(orthogonal direct sum),*
b) $[g_\lambda, \ g_\gamma] = g_{\lambda+\gamma}$ *and* $\theta(g_\lambda) = g_{-\lambda}$.

The decomposition $\mathfrak{g}=g_0 \oplus \bigoplus_{\lambda\in\Delta} g_\lambda$ is called the restricted-root space decomposition of $\mathfrak{g}$.

Example 3.3.1 ***Root space decomposition of*** $\mathfrak{sl}(n, \mathbb{R})$
Let $G = SL(n, \mathbb{R})$ and let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ its Lie algebra. Let k be the killing form of $\mathfrak{sl}(n,\mathbb{R})$ and $\theta : \mathfrak{sl}(n,\mathbb{R}) \mapsto \mathfrak{sl}(n, \mathbb{R})$ be the map defined by $\theta(X) = -X^t$. Then, θ is a Cartan involution and

$$\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{l} \oplus \mathfrak{p}$$

where $\mathfrak{l}$ is the space of skew symmetric matrices of $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{p}$ is the space of symmetric matrices of $\mathfrak{sl}(n, \mathbb{R})$. The space $\mathfrak{a}$ of diagonal matrices with trace 0 is a maximal Abelian subspace of $\mathfrak{p}$. Let $f_i \in \mathfrak{a}^*$ such that for d in $\mathfrak{a}$ $f_i(d) = d_{ii}$. Then the set of restricted roots of $\mathfrak{sl}(n, \mathbb{R})$ is the set of functions $f_{ij} = f_i - f_j$ with $i \neq j$ and $\mathfrak{g}_{f_{ij}} = \{X \in \mathfrak{sl}(n,\mathbb{R})/ [A, \ X] = f_{ij}(A).X, \forall A \in \mathfrak{a}\}$ is the space of all matrices (X_{lk}) such that $X_{lk} = 0$ for $(l, k) \neq (i, j)$. The dimension of $\mathfrak{g}_{f_{ij}}$ is 1.
For $n = 2$, we have $\dim \mathfrak{a} = 1$, the set of restricted roots is $\{f_{12}, -f_{12}\}$ and the dimension of $\mathfrak{g}_{f_{12}}$ is 1.

Restricted roots of $\mathfrak{o}(p,q)$

Let $G = O(p,q)$ and let $\mathfrak{so}(p,q)$ its Lie algebra, then $M \in \mathfrak{so}(p,q)$ if M has the form

$$M = \begin{bmatrix} A & B \\ B^t & D \end{bmatrix}$$

where A is $p \times p$ skew symmetric matrix and D is $q \times q$ skew symmetric matrix; the map $\theta = -M^t$ is a Cartan involution and we find

$$\mathfrak{l} = \{M \in \mathfrak{so}(p,q), B = 0\},$$

$$\mathfrak{p} = \{M \in \mathfrak{so}(p,q), A = D = 0\}.$$

Let $\mathfrak{a}$ be the maximal Abelian subalgebra of $\mathfrak{p}$ of matrices of the form

$$M = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix}$$

where

$$B = \begin{bmatrix} 0 & 0.. & a_q \\ 0 & a_i & 0 \\ a_1 & 0.. & 0 \end{bmatrix}$$

Let $\alpha_i \in \mathfrak{a}^*$ defined by $\alpha_i(M) = a_i$, then the functions α_i, $-\alpha_i$, $\alpha_i + \alpha_j$, $\alpha_i - \alpha_j$ $i \neq j$ are restricted roots of $\mathfrak{so}(p,q)$.

3.4 KAK decomposition

For semisimple Lie groups we have the following theorem called the KAK decomposition

Theorem 3.4.1 *Let G be a semisimple Lie group, let $\mathfrak{g}$ its Lie algebra and let $\mathfrak{g} = l \oplus p$ be any Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$ and let $A = \exp \mathfrak{a}$ be the associated connected subgroup. Let K be the subgroup of G associated to l. Then $G = KAK$, that is every element of G is the product of an element of K, an element of A and an element of K moreover K is compact if G has finite centre.*

This theorem plays an important role in the dynamics of semisimple Lie groups. For the proof of this theorem see [1]

3.4.1 KAK decomposition of $GL(n, \mathbb{R})$

Let $M \in GL(n, \mathbb{R})$ $K = O(n)$. Then by the polar decomposition, there exist $k \in K$ and a positive definite symmetric matrix s such that $M = ks$ but $s = s_1^t a s_1$ for some s_1 in $O(n)$ and a diagonal matrix a with positive elements, then $M = ks_1^t a s_1$. This proves the following theorem.

Theorem 3.4.2 *$GL(n, \mathbb{R})$ =KAK, where A is the subgroup of $GL(n, \mathbb{R})$ consisting of diagonal matrices and $K = O(n)$.*

3.4.2 KAK decomposition of $O(1, n)$

The following theorem implies that the KAK decomposition is internal in $O(1, n)$.

Theorem 3.4.3 *Let M in $O(1, n)$ and let $M = kak$ its KAK decomposition in $GL(n, \mathbb{R})$ then k and a are in $O(1, n)$.*

For a detailed proof of this theorem see [15].

3.5 Group actions

Let $\mathcal{C}$ be a category whose object are sets with additional structures and whose morphisms are functions preserving the additional structures, let G be a group and let M be an object of $\mathcal{C}$. An action of G on M is an homomorphism ρ from G to the group $Hom(M)$ of all morphisms from M to M (in $\mathcal{C}$). This induces a map τ from $G \times M$ to M defined by $\tau(g, m) = \rho(g).m$. This map satisfy the two following conditions

$$\tau(g_1, \tau(g_2, m)) = \tau(g_1.g_2, m)$$

and

$$\tau(1, m) = m.$$

Generally $\tau(g, m)$ is denoted by $g.m$.
Conversely, a map τ which satisfies the two last conditions induces an action ρ given by $\rho(g)(m) = \rho(g).m$.

Definition 3.5.1 *If ρ is an action of a topological group G on a topological space M, then we say that*

- *The action ρ is free if G acts with trivial stabilizers*
- *The action ρ is faithful if it is injective*

3.5.1 From Lie group action to Lie algebra action

In our study, the category $\mathcal{C}$ is the category of Lorentz manifolds and isometries between them; the group G is a Lie group. In this case, the action ρ of G on M induces an action of the Lie algebra $\mathfrak{g}$ of G on M that is a Lie homomorphism $\overline{\rho}$ from $\mathfrak{g}$ to the Lie algebra $\mathcal{X}(M)$ of all vector fields on M, by the following construction:
For X in $\mathfrak{g}$, we associate the vector field $\overline{X}$ on M defined by

$$\overline{X}_m = \frac{d}{dt}\mid_{t=0}(\exp(tX).m)$$

We can define this vector field by differentiating the map $g \mapsto gm$ from G to M, at the element 1 of G.
Suppose G is a connected Lie group acting on a manifold M and O is an orbit. Since G is generated by any neighborhood of 1 and that the exponential map is a diffeomorphism between a neighborhood of 0 in $\mathfrak{g}$ and a neighborhood of 1 in G we have,

$$T_xO = \{\overline{X}_x, X \in \mathfrak{g}\}.$$

Definition 3.5.2 *The action ρ is said to be almost free or locally free if, $\overline{X}_m = 0$ for some point $m \in M$ implies $X = 0$; ρ is said to be almost faithful or locally faithful if the map $\overline{\rho}$ is injective*

The Gauss map

Let G be a Lie group acting isometrically on a Lorentz manifold (M, h). We recall the definition of the natural adjoint G action on $\mathcal{S}^2(\mathfrak{g})$ given by:
$(g.q)(X_1, X_2) = q(Ad_{g^{-1}}X_1, Ad_{g^{-1}}X_2)$ for q in $\mathcal{S}^2(\mathfrak{g})$, g in G, and X_1, X_2 in $\mathfrak{g}$. For each $X \in \mathfrak{g}$, let $\overline{X}$ the vector on M defined above. The map $\mathfrak{G} : M \to \mathcal{S}^2(\mathfrak{g})$ given by $x \to \mathfrak{G}_x$, where $\mathfrak{G}_x(X, Y) = h_x(\overline{X_x}, \overline{Y_x})$ is called the Gauss map. This map is equivariant, that is, $g.\mathfrak{G}_x = \mathfrak{G}_{g.x}$ for all g in G and all x in M. Indeed, for $g \in G$ and $X \in \mathfrak{g}$, we have:

$$\overline{Ad_gX_{gx}} = \frac{d}{dt}(exptAdgX.gx)|_{t=0} = \frac{d}{dt}(g.exptX.g^{-1}.gx)|_{t=0} = dg_x(\overline{X_x}).$$

Hence, for X, $Y \in G$, $x \in M$ and $g \in G$, we get (using the fact that G acts on M by isometries)

$$g.\mathfrak{G}_x(X, Y) = h_x(\overline{Ad_{g^{-1}}X_x}, \overline{Ad_{g^{-1}}Y_x}) = h_x(dg_{gx}^{-1}\overline{X_{gx}}, dg_{gx}^{-1}\overline{Y_{gx}}) = h_{gx}(\overline{X}gx, \overline{Y}gx) = \mathfrak{G}_{gx}(X, Y).$$

Observe that if G acts non-properly on M, then it does so on $\mathfrak{G}(M)$.

Classification of semi simple Lie groups actions on 1-dimensional manifold

Lie group actions on 1-dimentional manifolds have been classified by Sophus Lie [24, 31, 32] the following theorem is a weakened statement of his main theorem.

Theorem 3.5.1 ([24, 31, 32]) *If a semi simple Lie group acts non trivially on a 1-dimentional manifold then it has $\mathfrak{sl}(2,\mathbb{R})$ as a local factor.*

3.6 Proper and nonproper actions

The action τ of a topological group G on a topological space M is called proper if the set $\{g \in G, gK \cap K \neq \emptyset\}$ is compact in G for any compact K in M. This is equivalent to the assertion that for any sequences (x_n) of M and (g_n) of G, if (x_n) and $(g_n x_n)$ converge in M then some subsequence of (g_n) converges in G. Intuitively, this shows that any compact K in M moves away from his initial neighborhood by the action of any non compact set of G. If the action τ is not proper we say that it is a nonproper action. Then τ is nonproper if there is a compact subset K of M and a nonprecompact set H of G such that K cannot be moved disjoint from itself by the action of H. This is equivalent to the existence of two sequences, (x_n) of M and (g_n) of G, such that x_n converges in M and $g_n x_n$ converges in M but the closure of the set $\{g_n, n \in \mathbb{N}\}$ is not a compact set of G.

Example 3.6.1 *1) The action of the isometric group of a Riemannian manifold M on the manifold M is a proper action. In particular the stabilizer of any point is compact.*
2) The action of the group $\left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbb{R} \right\} \subset SL(2,\mathbb{R})$ on $\mathbb{R}^2$, is nonproper since it fixes the origin. Furthermore, it is nonproper on $(\mathbb{R}^2)^*$ indeed the sequence $X_n = (e^{-n}, 1)$ converge to $(0,1)$ in $(\mathbb{R}^2)^*$ and the sequence $\begin{pmatrix} e^n & 0 \\ 0 & e^{-n} \end{pmatrix}$ is unbounded in $SL(2,\mathbb{R})$ but $\begin{pmatrix} e^n & 0 \\ 0 & e^{-n} \end{pmatrix} X_n = (1, e^{-n})$ converge to $(1,0)$ in $(\mathbb{R}^2)^*$, thus the action is nonproper. We can prove that the sets $\mathbb{R}^2 \setminus \{(x,0), x \in \mathbb{R}\}$ and $\mathbb{R}^2 \setminus \{(0,\ x), x \in \mathbb{R}\}$ are maximal proper open sets of this action that is, maximal open sets of $\mathbb{R}^2$ where this action is proper.
3) If M is compact, only compact groups act properly on M.

For semisimple Lie group actions we have the following lemma due to Kowalsky.

Lemma 3.6.1 *Let G be a semisimple Lie group of finite center acting on a manifold M. Let $G = KAK$ be the KAK decomposition of G, then, G acts nonproperly iff A acts nonproperly.*

For the proof it is enough to see that K is compact.

3.6.1 Nonproper actions and locally equicontinuous actions

If M is a metric space (this is the case for a manifold, since it can be endowed with a Riemannian metric), a family F of mappings is said to be equicontinuous at a point $x \in M$ if for each $\epsilon > 0$ there exists $\eta > 0$ such that for $y \in M$ and $f \in F$ we have $d(x, y) < \eta \Rightarrow d(f(x), f(y)) < \epsilon$. The family is said to be equicontinuous on a subset $U \subseteq M$ if it is equicontinuous at every point in U. We say that the action of G is locally equicontinuous, if for any sequences (x_n) of M, and (g_n) of G, whenever (x_n) and $(g_n x_n)$ converge in M, then a subsequence of (g_n) is equicontinuous on a neighborhood of the limit x of (x_n). Therefore, a subsequence of (g_n) is converging in the group of homeomorphisms of that neighborhood, but the limit is not necessarily a restriction of an element of G. Obviously, a nonlocally equicontinuous action is nonproper. The converse is not true. The standard example of a nonproper but locally equicontinuous action is the usual linear action of $\mathbb{R}$ on the torus with dense orbits. In general, any non-closed Lie subgroup of the isometry group of a compact Riemannian manifold is locally equicontinuous but nonproper. Another example is the action of the universal cover $\tilde{G}$ of a Lie group G on G (via the canonical projection). It is always locally equicontinuous, but it is proper only if G has a finite fundamental group (because it is not faithful). Observe nevertheless:

Proposition 3.6.1 *Let G be a Lie group acting by preserving a pseudo-Riemannian structure on a manifold M. If G is the full isometry group, or if G is semisimple with finite center, then its action is nonproper iff it is non-equicontinuous.*

For the proof, recall the well-known fact that a C^0-limit of pseudo-Riemannian (smooth) isometries is a smooth isometry, and that the Lie group topology coincides with the C^0 topology. This is equivalent to saying that the isometry group is closed in the group of homeomorphisms. (Actually, this fact is true for all rigid geometric structures.). For G, a simple Lie group with finite center, recall that its image under a homomorphism into any Lie group is closed, and that G is a finite cover of it. An analogous argument applies to the semi-simple case with finite center.

Therefore, in statements (essentially within proofs) below that involve semisimple Lie groups, we will not worry about distinction between compactness and pre-compactness. A G-homogeneous space G/H is non-proper if the left G-action on it is nonproper. This is equivalent, for general G, to non-precompactness of H and to non-compactness of H when G is semi-simple.

3.6.2 Properness of abelian actions

The following is a criterion for the nonproperness of linear actions of abelian Lie groups.

Lemma 3.6.2 *Let $\{\lambda_1, \cdots, \lambda_n\}$ be a generating system in $\mathbb{R}^d$. Let $\mathbb{R}^d$ act faithfully on $\mathbb{R}^n$ by diagonal matrices as follows. For every $t \in \mathbb{R}^d$ let $M(t)$ be the matrix $M(t) = diag(e^{<\lambda_i, t>})_{1 \leq i \leq n}$, where $< ., . >$ is the usual inner product in $\mathbb{R}^d$. It defines a faithfull action. Assume V is an invariant (topological) subspace of $\mathbb{R}^d$ on which the action is non-proper. Then there exists a non-zero vector $t_0 \in \mathbb{R}^d$ and $(x, y) \in V^2$ such that $x_i = 0$ if $\lambda_i(t_0) > 0$, and $y_i = 0$ if $\lambda_i(t_0) < 0$.*

Proof. Since the action on V is non-proper, there exists a sequence (t_p) with $t_p \to +\infty$ in $\mathbb{R}^d$ and a sequence (x_p) in V such that $x_p \to x$ in V and $y_p = t_p.x_p \to y$ in V . Consider the sequence $\frac{t_p}{\|t_p\|}$; up to taking a subsequence, we may assume it has a limit t_0. Since the action is faithful, and $t_0 \neq 0$, there exists $i_0 \in \{1, \cdots, d\}$ such that $\lambda_{i_0}(t_0) \neq 0$. Notice that $\lambda_i(t_p) \to +\infty$ if $\lambda_i(t_0) > 0$ and $\lambda_i(t_p) \to -\infty$ if $\lambda_i(t_0) < 0$. Hence $x_i = 0$ if $\lambda_i(t_0) > 0$ and $y_i = 0$ if $\lambda_i(t_0) < 0$. ■

4

Homogeneous spaces and warped products

Let G be a Lie group. A G-homogeneous space M is a manifold M endowed with a transitive G-action that is an action of G on M which has one orbit.

Example 4.0.2 • $\mathbb{R}^n$ *is homogeneous under the action of the group* $O(n) \ltimes \mathbb{R}^n$.
• $\mathbb{S}^n$ *equipped with the action of* $O(n+1)$ *is a homogeneous space.*
• *If we consider* $\mathbb{S}^n$ *as a quotient of the light cone of the Lorentz space* $\mathbb{R}^{1,n+1}$, *then we can find a transitive action of* $O(1, n+1)$ *on* $\mathbb{S}^n$, *and so* $\mathbb{S}^n$ *is a homogeneous space under this action.*

Let M be a G-homogeneous space and let x be some point of M. Denote by $G(x)$ the subgroup of G of elements which stabilize x. It is called the stabilizer of x or the isotropy subgroup of x. $G(x)$ is closed in G and the set $\{G(x), x \in M\}$ is a conjugate class of subgroups of G. The map $G \to M, g \to g.x$ is a surjection and induces a diffeomorphism between M and $G/G(x)$, then we identify M with $G/G(x)$ via this diffeomorphism.

Remark 4.0.1 *By the previous paragraph, we can define a homogeneous space as a quotient of a Lie group by a closed subgoup.*

4.1 Pseudo-Riemannian homogeneous space

Let G be a Lie group acting transitively on a manifold M. Let x in M and $G(x)$ be the isotropy subgroup of x. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}(x)$ be the Lie subalgebra of

$\mathfrak{g}$ corresponding to $G(x)$. Let g_0 in G. The derivative of the map of G in G which sends g on $g_0 g g_0^{-1}$ at the identity is denoted by Ad_{g_0}, the map $Ad : g \to Ad_g$ is called the adjoint representation of G. In this section, we consider the following question: is there any pseudo-Riemannian metric on M preserved by G?
The isotropic representation map play an important role in the comprehension of this question, it is defined by:

$$\Phi : G(x) \to End(T_x M), g \to D_x g$$

where $End(T_x M)$ is the space of endomorphisms of $T_x M$ and $D_x g$ is the derivative of g at x. Note that $\mathfrak{g}/\mathfrak{g}(x)$ can be identified with $T_x M$ by the map

$$X \longrightarrow \overline{X}(x) = \frac{\partial}{\partial t} |_{t=0} \; e^{tX} x$$

For $g \in G(x)$, differentiating the relation $g e^{tX} x = (g e^{tX} g^{-1}) x$ gives

$$D_x g(\overline{X}(x)) = \overline{Adg(X)}(x)$$

This gives the following proposition.

Proposition 4.1.1 *$\Phi(g)$ is conjugate to Ad_g via the derivative of the projection map $\pi : G \to G/G(x)$ at the identity that is $D_1 \pi o Ad_g = \Phi(g) o D_1 \pi$.*

The importance of the isotropic representation map is given by the following proposition.

Proposition 4.1.2 *There is a bijective correspondence between:*
- *G-invariant pseudo-Riemannian metrics of signature (p, q) on M*
- *$\Phi(G(x))$-invariant pseudo-Riemannian inner products of signature (p, q) on $T_x M$*

As a corollary of this proposition we have,

Corollary 4.1.1 *There is a bijective correspondence between:*
- *Left-invariant pseudo-Riemannian metrics of signature (p, q) on G,*
- *Pseudo-Riemannian inner products of signature (p, q) on $\mathfrak{g}$*

A pseudo-Riemannian manifold M is homogeneous if M is homogeneous under the natural action of $Isom(M)$.

A homogeneous pseudo-Riemannian manifold is called a homogeneous space. Since a homogeneous space is completely determined by its isometry group and some point of it, then any geometrical property at one point holds at every point.

4.1.1 Homogeneous spaces and Killing vector fields

Let M be a homogeneous space. Since $IsomM$ is transitive on M, then for any p in M, the map

$$\begin{array}{rcl} IsomM & \rightarrow & M \\ f & \mapsto & f(p) \end{array}$$

is a submersion. This leads to the following result:

Proposition 4.1.3 *For a homogeneous space M the map $V \rightarrow V_p$ is a surjection from the space $k(M)$ of killing vector fields on M to the tangent space of M at p.*

Proposition 4.1.3 means that every tangent vector to a homogeneous space can be extended to a Killing vectors field on M.

4.1.2 Completeness

A homogeneous Riemannian space is complete but this does not hold in general pseudo-Riemannian geometry, however, a homogeneous pseudo-Riemannian space is not necessarily complete which is proved by the following example.

Example 4.1.1 *Let $\mathbb{R}^{1,1}$ be the Lorentz space obtained by providing $\mathbb{R}^2$ by the quadratic form $Q(x_1, x_2) = 2x_1x_2$. Let (α, β) be a point of $\mathbb{R}^2$ with $\alpha > 0$. The map*

$$(x, y) \longrightarrow (\frac{x}{\alpha}, \alpha y - \alpha\beta)$$

which is an isometry of $\mathbb{R}^2_1$ maps the Lorentz subspace

$$H = \{(x, y) \in \mathbb{R}^2, x > 0\}$$

to itself, and the point (α, β) to the point $(1, 0)$.This implies that the space H is a homogeneous space but the curve $\gamma(t) = (t, 0)$ is a geodesic having $\mathbb{R}^*_+$ as a largest domain of definition, which proves that H is not complete.

For the relationship between homogeneity and completeness in general pseudo-Riemannian geometry, we have the following result of Marsden.

Proposition 4.1.4 *A compact homogeneous pseudo-Riemannian space is complete.*

For the proof see [33]

4.2 Constant curvature homogeneous spaces

Let $\mathbb{R}^{p,n-p}$ $(p < n)$ be the space $\mathbb{R}^n$ endowed with the quadratic form

$$< X, X >= -x_1^2 - .... - x_p^2 + x_{p+1}^2 + ... + x_n^2.$$

Let $r > 0$, we define $\mathbb{S}_p^n(r)$ and $\mathbb{H}_p^n(r)$ by:

$$\mathbb{S}_p^n(r) = \{X \in \mathbb{R}^{p,n+1-p}, < X, X >= r\},$$

and

$$\mathbb{H}_p^n(r) = \{X \in \mathbb{R}^{p+1,n-p}, < X, X >= -r\}$$

The space $\mathbb{S}_p^n(r)$ is a submanifold of $\mathbb{R}^{p,n+1-p}$called the pseudosphere of radius r, and the space $\mathbb{H}_p^n(r)$ is a submanifold of $\mathbb{R}^{p+1,n-p}$ called the pseudohyperbolic space of radius r.
We notice that the space $\mathbb{S}_p^n(r)$ endowed with the restriction of the metric of $\mathbb{R}^{p,n+1-p}$ is a pseudo-Riemannian manifold of index p with sectional curvature $\frac{1}{r}$ and the restriction of the metric of $\mathbb{R}^{p+1,n-p}$ to $\mathbb{H}_p^n(r)$ gives a pseudo-Riemannian space of index p with sectional curvature $-\frac{1}{r}$.
The following theorem shows that these spaces are, up to local isomorphism, the only ones complete pseudo-Riemannian manifolds of constant non vanishing curvature.

Theorem 4.2.1 *Let M be a n-dimensional pseudo-Riemannian manifold of constant curvature r; then:*
- *If $r = 0$, every point of M has a neighborhood isometric to some neighborhood of 0 in $\mathbb{R}^{p,n-p}$.*
- *If $r \in \mathbb{R}_+^*$, then every point of M has a neighborhood isometric to some neighborhood of some point in $\mathbb{S}_p^n(r)$.*
- *If $r \in \mathbb{R}_-^*$, then every point of M has a neighborhood isometric to some neighborhood of some point in $\mathbb{H}_p^n(r)$.*

Then we distinguish three models of constant curvature spaces which are the space $\mathbb{R}^{p,n-p}$, the pseudosphere $\mathbb{S}_p^n$ and the pseudohyperbolic space $\mathbb{H}_p^n$.
The next subsections are devoted to give some information about these spaces.

4.2.1 The space $\mathbb{R}^{p,n-p}$

The natural connection $\nabla_X V = \sum X(Y^i)$ is compatible with the metric $<,>$ of $\mathbb{R}^{p,n-p}$ so it is the Levi-Civita connection of this space, the curvature vanishes in this case, therefore, we say that $\mathbb{R}^{p,n-p}$ is flat.

An isometry of $\mathbb{R}^{p,n-p}$ is the composition of a translation and an element of $O(p,n-p)$, then the isometry group of $\mathbb{R}^{p,n-p}$ is $O(p,n-p) \ltimes \mathbb{R}^n$.

The geodesics of the space $\mathbb{R}^{p,n-p}$ are its straight lines.

4.2.2 The space $\mathbb{S}_p^n(1)$

The space $\mathbb{S}_p^n(1)$ is diffeomorphic to $\mathbb{R}^p \times \mathbb{S}^{n-p}$, then it is connected. It is known that the isometry group of $\mathbb{S}_p^n(1)$ is $O(p,n+1-p)$ which acts transitively on $\mathbb{S}_p^n(1)$ and the stabilizer of the point $(0,0,...,1)$ is isomorphic to $O(p,n-p)$, then the space $\mathbb{S}_p^n(1)$ is viewed as a homogenous space

$$\mathbb{S}_p^n(1) \cong O(p,n+1-p)/O(p,n-p)$$

with $O(p,n+1-p)$ acting by left translations.

The geodesics of $\mathbb{S}_p^n(1)$ are obtained by the intersections of the 2-dimentional vector spaces of $\mathbb{R}^{n+1}$ with $\mathbb{S}_p^n(1)$.

4.2.3 The space $\mathbb{H}_p^n(1)$

The space $\mathbb{H}_p^n(1)$ diffeomorphic to $\mathbb{R}^{n-p} \times \mathbb{S}^p$ then it is connected if $p \geq 1$. The component of $(1,0,...,0)$ in the Riemannian space $\mathbb{H}_0^n(1)$ is called the n-dimensional hyperbolic space. It is denoted by $\mathbb{H}^n$.
As in the $\mathbb{S}_p^n(1)$ case, the isometry group of $\mathbb{H}_p^n(1)$ is $O(p+1,n-p)$ and the stabilizer of $(1,0,..,0)$ is isomorphic to $O(p,n-p)$, this gives another form of $\mathbb{H}_p^n(1)$ as a homogeneous space

$$\mathbb{H}_p^n(1) \cong O(p+1,n-p)/O(p,n-p)$$

with $O(p+1,n-p)$ acting by left translations. The geodesics of $\mathbb{H}_p^n(1)$ are given by intersections of the 2-dimentional vector spaces of $\mathbb{R}^{n+1}$ with $\mathbb{H}_p^n(1)$.

4.2.4 The Lorentz case

The Lorentz situation corresponds to the case when $p = 1$ and $n \geq 2$, so we distinguish three models of these spaces.

The flat case

The flat space $\mathbb{R}^{1,n-1}$ is called the n-dimensional Minkowski space. It has a null curvature and its homogeneous version is given by

$$\mathbb{R}^{1,n-1} \cong O(1,n-1) \ltimes \mathbb{R}^n / O(1,n-1)$$

The geodesics of $\mathbb{R}^{1,n-1}$ are its straight lines.

The positive curvature case

The space $\mathbb{S}^n_1(1)$ is a connected space called the n-dimensional de Sitter space. It is denoted by dS_n, its isometry group is $O(1,n)$ and the stabilizer of each of its points is conjugated to $O(1,n-1)$, then we have

$$dS_n \cong O(1,n)/O(1,n-1).$$

The 2-dimensional case The group $SO^0(1,2)$ act transitively on the 2-dimensional hyperbolic space $\mathbb{H}^2$, moreover it act transitively on the set $\{\mathbb{H}^2 \cap P, P \text{ 2-plan}\}$ of geodesics of $\mathbb{H}^2$. The stabilizer of $\mathbb{H}^2 \cap \langle (1,0,0),(0,1,0) \rangle$ is isomorphic to $SO^0(1,1)$, this show that the 2-dimensional de Sitter space is viewed as the space of geodesics of the 2-dimensional hyperbolic space.

The negative curvature case

The space $\mathbb{H}^n_1(1)$ is the analogous of the hyperbolic space in the Riemannian geometry, it is called the n-dimensional anti-de Sitter space, we denote it by AdS_n. The isometry group of AdS_n is $O(2,n-1)$ which acts transitively on AdS_n and the stabilizer of any point of AdS_n is conjugated to $O(1,n-1)$, then,

$$AdS_n \cong O(2,n-1)/O(1,n-1)$$

is the homogenous version of AdS_n.
Let identify $\mathbb{S}^n$ with the quotient space of $(\mathbb{R}^{n+1})^*$ by the equivalence relation $\sim$, which is defined by:
$x \sim y$ if y is a product of x by a positive number. The image of AdS_n under the projection map from $(\mathbb{R}^{n+1})^*$ to $(\mathbb{R}^{n+1})^*/\sim$ endowed with the image of the metric of AdS_n, is called the Klein model of the Anti-de Sitter space.

The 3-dimensional case In the three dimensional case there is a simple model of the Anti-de Sitter space called the $SL(2,\mathbb{R})$ model. Indeed, let $\mathbb{R}^4$ be endowed with the Lorentz metric

$$< X, X >= -x_1x_4 + x_3x_2.$$

The 3−dimensional Anti-de Sitter space is the region of $\mathbb{R}^4$ where $<,>= -1$, endowed with the restriction of $<,>$, then it is identified with the group $SL(2,\mathbb{R})$ endowed with the restriction of "$-\det$" where "$\det$" is the determinant.

4.3 Warped products

In the category of pseudo-Riemannian manifolds consider the following generalization of direct product called warped product. So let (M, g) and (N, h) be two pseudo-Riemannian manifolds and $w : M \to \mathbb{R}_+^*$ be a function called warping function. The warped product $M \times_w N$, is the topological product space $M \times N$, endowed with the metric $g \oplus wh$.

The factor N is called the normal factor of the warped product. When N is a pseudo-Riemannian manifold of constant curvature some authors use the terminology generalized Robertson-Walker space. This terminology is based on the space called classical Robertson-Walker space which is the space $I \times_w N$ where I is an interval of $\mathbb{R}$ endowed with the metric $-dt^2$ and N is a Riemannian manifold.

Example 4.3.1 • ***Einstein static universe***
The Robertson-Walker space-time is any Lorentz manifold which is isometric to a Lorentz warped product $]a,\ b[\ \times_\omega N$, where $]a,\ b[$ is given the metric $-dt^2$ and N is an isotropic Riemannian manifold, that is a Riemannian manifold such that for any x in N the isotropy group Iso_x acts transitively on the unit sphere $S(T_xN)$ of the tangent space T_xN of N at x. The Robertson-Walker space-time such that $a = -\infty$, $b = +\infty$, and N equals $\mathbb{S}^{n-1}$ is called the Einstein static universe.
• ***de Sitter and Anti-de Sitter space time as warped product***
Topologically, the de Sitter space dS_n is the product $\mathbb{R} \times \mathbb{S}^{n-1}$and the anti-de Sitter space $AdS_n = \mathbb{H}_1^n(1)$ is the product $\mathbb{S}^1 \times \mathbb{R}^{n-1}$.
By [22], the space $dS_4 = \mathbb{S}_1^4(1)$ is the warped product $\mathbb{R} \times_\omega S^3$, where the warping function ω is defined by $\omega : \mathbb{R} \mapsto \mathbb{R}_+$, $\omega(t) = \cosh^2 t$, and S^3 is given the usual Riemannian metric; the space $\widetilde{AdS}_4 = \widetilde{\mathbb{H}_1^4}(1)$ may be given coordinates (t, r, α, β) for which the metric is given by: $-\cosh^2(r)(dt)^2 + dr^2 + \sinh^2(r)(d\alpha^2 + \sin^2(\alpha)d\beta^2)$ then the space $\widetilde{AdS}_4$ is the warped product

$\mathbb{H}^3 \times_\omega \mathbb{R}$ *where* $\mathbb{H}^3$ *is the three dimensional hyperbolic space and the warping function* ω *is defined by* $\omega : \mathbb{H}^3 \mapsto \mathbb{R}_+$, $\omega(r) = \cosh^2(r)$.

For time oriented warped product we have the following theorem:

Theorem 4.3.1 *A warped product* $M \times_w N$, *is time oriented iff either* M *is time oriented or 1-dimensional manifold with negative definite metric.*

4.4 Local warped product structure

Obviously a warped product $M \times_w N$ gives a pair $(\mathcal{M}, \mathcal{N})$ of transversal foliations (the leaves of $\mathcal{M}$ are $M \times \{y\}$ $y \in N$ and the leaves of $\mathcal{N}$ are $\{x\} \times N$) with $\mathcal{M}$ geodesic, that is the leaves of $\mathcal{M}$ are geodesic, so arises a natural question: in which case the converse holds? In this kind of questions it is convenient to start by local studies. For this, we need the following definition of local warped product structure

Definition 4.4.1 *Let* M *be a pseudo-Riemannian manifold. A local warped product structure on* M *is a pair* $(\mathcal{L}, \mathcal{N})$ *of transversal foliations, such that any point of* M *have a neighborhood isometric to a warped product pseudo-Riemannian manifold* $L \times_w N$, *by an isometry sending the foliation* $\mathcal{L}$ *(resp.* $\mathcal{N}$*) to the foliation of* $L \times N$ *determined by the factor* L *(resp.* N*).*

4.4.1 From isometric actions to local warped product

It is known that if the orbits of an isometric action of a Lie group on pseudo-Riemannian manifold have the same dimension then they determine a foliation. Generally this situation gives rise to a local warped product structure. For the explanation of this phenomenon we give the following theorems proved in [45].

Theorem 4.4.1 ([45]) *Let* G *be a Lie group acting (locally) isometrically on a pseudo-Riemannian manifold* M. *Suppose that the orbits have a constant dimension and thus determine a foliation* $\mathcal{N}$. *Suppose further that the leaves of* $\mathcal{N}$ *are non-degenerate, and that the isotropy group in* G *of any* $x \in M$, *acts absolutely irreducibly on* T_xN, *i.e. its complexified representation is irreducible. Suppose finally that the orthogonal of* $\mathcal{N}$ *is integrable, say it is tangent to a foliation* $\mathcal{L}$. *Then* $(\mathcal{L}, \mathcal{N})$ *determines a local warped product structure, with* $\mathcal{N}$ *as a normal factor.*

Theorem 4.4.2 ([45]) *Let G be a Lie group acting (locally) isometrically on a pseudo-Riemannian manifold M. Suppose that the orbits are non-degenerate having a constant dimension and so determine a foliation $\mathcal{N}$. Suppose that the isotropy group in G of any $x \in M$, acts absolutely irreducibly on T_xN, and that the metric on the orthogonal of $\mathcal{N}$ is definite (positive or negative), and in opposite, the metric on $\mathcal{N}$ is non-definite. Then, the orthogonal of $\mathcal{N}$ is integrable, and the action determines a local warped product.*

5

Dynamics on Lorentz manifolds

The compact case has been extensively investigated by A. Zeghib in [43], S. Adams and G. Stuck in [6]. However, some partial results in the general case have been treated by N. Kowalsky [26]. The originality in her work was based on the nonproperness of the actions, which can be regarded as some kind of compactness, that is, it enables one to get some classification results analogous to those of the compact case summarized in the second section of this chapter.

5.1 Dynamics on compact Lorentz manifolds

In this section we don't give the proofs of results concerning the classification of Lie groups actions on compact Lorentz manifolds since these latter are well known.

5.1.1 Constructions and examples

∘ Let $G = SL(2,\mathbb{R})$. Its killing form k determines a right-invariant Lorentz metric,
$< X, Y >_g = k(g^{-1}X,\, g^{-1}Y)$. Since G is a simple Lie group, then there exist a cocompact subgroup Γ of G. As $<,>$ is bi-invariant and the canonical surjection $\Pi : G \mapsto G/\Gamma$ is a covering map, the metric $<,>$ induces a Lorentz metric on G/Γ. In this case the identity component $Isom_0(G/\Gamma)$ is isomorphic to $\{f : G/\Gamma \mapsto G/\Gamma, \exists g \in G, \forall x \in G/\Gamma, f(x) = g.x\}$.

○ Let G_k be a subgroup of $SL_{k+2}(\mathbb{R})$ consisting of all matrices of the form:

$$\begin{bmatrix} 1 & * & * & \cdot & \cdot & * \\ 0 & 1 & 0 & \cdot & 0 & * \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & 0 & \cdot \\ \cdot & & & \cdot & \cdot & * \\ 0 & \cdot & \cdot & \cdot & 0 & 1 \end{bmatrix}$$

Let $\lambda = (\lambda_1....\lambda_k)$ in $\mathbb{R}^k$, the action of $\mathbb{R}$ on $(\mathbb{R}^2)^k$ defined by $t.(v_1...v_k) = (R_{\lambda_1.t}(v_1), ...R_{\lambda_k.t}(v_k))$ where $t \in \mathbb{R}$, $(v_1...v_k) \in (\mathbb{R}^2)^k$ and

$$R_{\lambda_i.t} = \begin{bmatrix} \cos(\lambda_i.t) & -\sin(\lambda_i.t) \\ \sin(\lambda_i.t) & \cos(\lambda_i.t) \end{bmatrix}$$

induces an action of $\mathbb{R}$ on G_k.

A twisted Heisenberg Lie algebra $\mathfrak{g}$ is a Lie algebra isomorphic to $h_\lambda = \mathbb{R} \ltimes G_k$ for some λ in $\mathbb{R}^k$. If $\lambda \in \mathbb{Q}_+^k$, $\mathfrak{g}$ is called a rationally twisted Heisenberg Lie algebra.

A twisted (rationally twisted) Heisenberg Lie group is a Lie group with Lie algebra isomorphic to a twisted (rationally twisted) Heisenberg Lie algebra. Further details can be found in [1].

Any rationally twisted Heisenberg Lie group admit a bi-invariant metric which passes to a Lorentz metric on its quotient by any cocompact lattice, and then this determine a homogeneous Lorentz space with noncompact isometry.

5.1.2 Classification of connected Lie groups actions on compact Lorentz manifolds

There are three cases, semisimple, solvable and the general case. In the semisimple case R. Zimmer proved in [47] that essentially $SL(2, \mathbb{R})$ is the unique noncompact semisimple Lie group which can act isometrically on a compact connected Lorentz manifold.

Theorem 5.1.1 ([47]) *Let G be a noncompact semisimple Lie group acting locally faithfully and isometrically on a compact connected Lorentz manifold M. Then, $\mathfrak{g}$ is isometric to a sum of $\mathfrak{sl}(2, \mathbb{R})$ and a compact semisimple Lie algebra $\mathfrak{l}$.*

In the solvable case we have the following result:

Theorem 5.1.2 ([43, 6]) *Let G be a solvable Lie group acting locally faithfully and isometrically on a compact connected Lorentz manifold M and let N be the nilradical of G. Then:*
1) If $G = N$ then $\mathfrak{g}$ is isomorphic to the sum of an Abelian Lie algebra $\mathfrak{a}$ and a Heisenberg (or 0) Lie algebra $\mathfrak{h}$.
2) If $G \neq N$ and N is Abelian, then $\mathfrak{g}$ is isomorphic to the sum of the Lie algebra of the affine group and an Abelian Lie algebra $\mathfrak{a}$.
3) If $G \neq N$ and N is not abelian then $\mathfrak{g}$ is isomorphic to the sum of an Abelian Lie algebra $\mathfrak{a}$ and a rationally twisted Heisenberg Lie algebra $\mathfrak{h}$.

The following theorem gives a general classification of connected Lie groups which can act isometrically on a Lorentz manifold.

Theorem 5.1.3 ([43, 6]) *Let G be a connected Lie group acting locally faithfully and isometrically on a compact connected Lorentz manifold M, then the Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathfrak{q} \oplus A \oplus \mathfrak{l}$, where $\mathfrak{l}$ is compact semisimple, A Abelian and*

$$\begin{aligned} &\mathfrak{q} \in \{Heisenberg\ Lie\ algebras\} \cup \{rationally\ twisted\ Heisenberg\ Lie\ algebras\} \\ &\quad \cup \{\mathfrak{sl}(2,\mathbb{R})\} \cup \{the\ affine\ group\} \cup \{0\}. \end{aligned}$$

The proof is based on the Levi decomposition of Lie algebras, however if $\mathfrak{R}$ is the solvable radical of $\mathfrak{g}$ and l is its semisimple part then we have:
- If $\mathfrak{l}$ is noncompact then $\mathfrak{R}$ is abelian.
- $[\mathfrak{l}, \mathfrak{R}] = 0$.

This implies that the study of nonproper actions of Lie groups on compact Lorentz manifolds reduces to the study of semisimple and solvable Lie groups actions. For a detailed proof of these theorems we refer to [43, 6].

5.2 Kowalsky approach

From now, we only consider the general case, where M is no more a priori compact.

By section 5 of chapter 3, when a Lie group G acts on M preserving a pseudo- Riemannian metric, one can consider a Gauss map from M to $\mathcal{S}^2(\mathfrak{g})$, the space of quadratic forms on the Lie algebra $\mathfrak{g}$ of G. When $\mathcal{S}^2(\mathfrak{g})$ is endowed with the natural adjoint G-action, the Gauss map is equivariant, and the G-space $\mathcal{S}^2(\mathfrak{g})$ reflects the dynamics on M. It is via this map, that the nonproperness condition is translated as a geometric condition on the induced metrics on orbits. This idea, due to Kowalsky, has become a basic tool in similar questions on the subject, e.g., Adams-Stuck [6, 7], and Bader- Nevo [9] (Note here that variants of the Gauss map, with other natural spaces instead of $\mathcal{S}^2(\mathfrak{g})$ were used by other authors, e.g. Gromov [20], Zimmer [46]). However, this is the starting point; further work in the proof is algebraic and Lie theoretic.

5.2.1 Kowalsky's main theorem

In [26] Kowalsky proved that;

Theorem 5.2.1 ([26]) *If a simple Lie group with finite center acts isometrically, locally freely and nonproperly on a Lorentz manifold, then it is locally isomorphic to $SL(2,R)$.*

Theorem 5.2.2 ([26]) *Let G be a simple Lie group with finite center acting isometrically and nonproperly on a connected Lorentz manifold. Then G is locally isomorphic to either $O(1,n)$, $n \geq 2$, or $O(2,n)$, $n \geq 3$.*

Technics of the proof

Let G be a simple Lie group with finite center acting nonproperly and isometrically on a Lorentz manifold M. By the KAK decomposition of G, we may see that A acts nonproperly (because K is compact). Then, there exists a sequence (v_n) in the Lie algebra $\mathfrak{a}$ of A and a sequence (x_n) in M such that e^{v_n} tends to infinity, (x_n) converges in M and $(e^{v_n}.x_n)$ converges to x. Like in the proof of Theorem 5.3.1 by the using of the Gauss map $\mathfrak{G}$, we find a subspace $\mathfrak{a}_0$ of dimension greater or equal to 2 and whose image $\overline{\mathfrak{a}_0}_x$ is lightlike, then of dimension one. This confirms the existence of a subspace $\mathfrak{a}_1$ of $\mathfrak{a}$ such that $\overline{\mathfrak{a}_1}_x = 0$, that is, $\overline{X}_x = 0$ for all X in $\mathfrak{a}_1$. We may take $x_n = x$ and $v_n \in \mathfrak{a}_1$. By the same process, we find another space $\mathfrak{a}_2$. We hope finding the largest space H such that $\overline{H}_x = 0$. By using this process, Kowalsky arrived to prove that if $\mathfrak{g}$ is isomorphic neither to $\mathfrak{o}(1,n)$ nor to $\mathfrak{o}\left(2,n\right)$, then, $\overline{\mathfrak{g}}_x = 0$. Thus, $\mathfrak{g}$ is a subalgebra of $\mathfrak{o}(1,n)$ and then is isomorphic to some $\mathfrak{o}\left(1,k\right)$. Contradiction.

Another proof of this theorem was proposed by S. Adams [4]; his method involves an analysis similar to Kowalsky's. In other directions, S. Adams investigated Lorentz-isometric actions, for simply connected Lie groups, with the stronger dynamical condition that some orbit either is not closed or has noncompact stabilizers ([2, 3]).

5.2.2 Kowalsky legacy

As mentioned in the introduction, in [27], the following theorems are stated. The manifold M and group G are assumed connected throughout.

Theorem 5.2.3 *Let G be locally isomorphic to $O(1,n)$, $n \geq 3$, and suppose that G acts on a manifold M preserving a Lorentz metric. Then all noncompact stabilizers have a Lie algebra isomorphic to either $o(1,n)$, $\mathfrak{o}(1,n-1)$, or $\mathfrak{o}(n-1) \ltimes \mathbb{R}^{n-1}$*

Theorem 5.2.4 *Let G be locally isomorphic to $O(2,n)$, $n \geq 6$, with G having finite center. Suppose that G acts nontrivially on a manifold M preserving a Lorentz metric. Then all noncompact stabilizers have a Lie algebra isomorphic to $\mathfrak{o}(1,n)$.*

Theorem 5.2.5 *Let G and M be as in Theorem 5.2.3 above, and assume there is a point of M with noncompact stabilizer. Then the universal cover $\widetilde{M}$ is Lorentz isometric to a warped product $L \times_\omega \widetilde{AdS_{n+1}}$, where $\widetilde{AdS_{n+1}}$ is the simply connected $(n+1)$-dimensional Lorentz space of constant curvature -1, and L is a Riemannian manifold. Further, the induced action of the universal cover $\widetilde{G}$ on $\widetilde{M}$ is via the canonical $\widetilde{G}$-action on $\widetilde{AdS_{n+1}}$ and the trivial action on L.*

Concerning Kowalsky's unpublished proofs, we notice a contribution by D. Witte Morris [39], in which he considers the homogeneous case. More precisely, he takes G locally isometric to the isometry group of de Sitter or anti-de Sitter space, respectively, and considers a nonproper Lorentz homogeneous space G/H that is, H is noncompact, and the G-action on G/H preserves some Lorentz metric. He proves that h is isomorphic to a standard copy of $\mathfrak{o}(1,n-1)$ in $\mathfrak{o}(1,n)$ or of $\mathfrak{o}(1,n)$ in $\mathfrak{o}(2,n)$; it follows that G/H is locally homothetic to de Sitter or anti-de Sitter space.

5.3 A new Geometric approach

Our contribution is to give proofs of all theorems announced by Kowalsky. The approach here is different from that of Kowalsky (as well as from others', for instance Adams') we get proofs of the main results, using many geometric arguments. More important, we generalize results to the semisimple case, assuming there are no local $SL(2,\mathbb{R})$-factors. A semisimple Lie group is essentially a product of simple Lie groups, but, in general, a nonproper action of a product does not derive from a nonproper action of one factor. However, in the Lorentz setting, we conclude that it necessarily does, that is, the semisimple case reduces to the simple one. This is really an important fact, since it leads one to hope to reduce the remaining work to the case in which the group is solvable. Of course, the reason for this is the Levi decomposition of Lie groups, which says that a Lie group is essentially a semidirect product of a semisimple and a solvable group.

5.3.1 From nonproper actions to nonproper homogeneous spaces.

The following result allows one to get from (non-transitive) actions to homogeneous (i.e.,transitive) ones. This is a common philosophy for actions with strong dynamics in a geometric setting. The result here is in particular reminiscent of the so-called Zimmer Embedding Theorem (see for instance [47]).

Theorem 5.3.1 *Let G be a semisimple Lie group of finite center acting isometrically and nonproperly on a Lorentz manifold M. Suppose that no (local) factor of G is locally isomorphic to $SL(2,\mathbb{R})$. Then there is a point with noncompact stabilizer. In particular, the restriction of the action of G to its orbit is nonproper. More precisely, the stabilizer of some point contains a nontrivial unipotent one-parameter group. (In other words, a nonproper Lorentz G-space contains a nonproper G-homogeneous orbit.)*

Proof. In our case here, one shows that there exists a point $x \in M$ such that G admits an isotropic subspace of dimension ≥ 2 with respect to the symmetric bilinear form $\mathfrak{G}(x)$. Then the non-precompactness of the stabilizer $stab(x)$ follows.

Let $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$. Let $A = \exp \mathfrak{a}$ the associated Cartan group and K be the subgroup of G associated to l. Let $\Delta = \Delta(\mathfrak{a}, \mathfrak{g})$ be the root system of $(\mathfrak{a}, \mathfrak{g})$ and $\mathfrak{g} = \mathfrak{g}_0 \underset{\alpha\in\Delta}{\oplus} \mathfrak{g}_\alpha$ the root space decomposition where,

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : adH.X = \alpha(H).X, \forall H \in \mathfrak{a}\}$$
$$\mathfrak{g}_0 = \{X \in \mathfrak{g} : adH.X = 0, \forall H \in \mathfrak{a}\}$$

Then A acts on $\mathfrak{g}$ by diagonal matrices, since

$$Ad_{g^{-1}} = Ad_{\exp H} = e^{adH} = diag(e^{\alpha(H)})_{\alpha\in\Delta\cup\{0\}},$$

where $g^{-1} = exp(H)$, $H \in \mathfrak{a}$. It follows that A acts by diagonal matrices on $\mathcal{S}^2(\mathfrak{g})$, and this latter representation admits the following decomposition:

$$\mathcal{S}^2(\mathfrak{g}) = \underset{\lambda,\mu\in\Delta\cup\{0\}}{\oplus V_{\lambda+\mu}} \ ,$$

where V_λ is the set of symmetric bilinear forms q on $\mathfrak{g}$ satisfying

$$q(\exp(H).X_1, \exp(H).X_2) = e^{\lambda(H)}.q(X_1, X_2),$$

for all $H \in \mathfrak{a}$ and all $X_1, X_2 \in \mathfrak{g}$.

Keeping in mind that for $X_1 \in \mathfrak{g}_\alpha$ and $X_2 \in \mathfrak{g}_\beta$ we have:

$$q(\exp(H).X_1, \exp(H).X_2) = e^{(\alpha+\beta)(H)}.q(X_1, X_2).$$

It follows that the forms q in V_λ satisfy :

$$\alpha + \beta \neq \lambda \Rightarrow \mathfrak{g}_\alpha \perp \mathfrak{g}_\beta \text{ with respect to } q$$

As we mentioned above, G acts non-properly on M since $\mathfrak{G}$ is equivariant, G acts non-properly on $\mathfrak{G}(M)$. Let $G = KAK$ be the Cartan decomposition of G. Since G has a finite center, K is compact. So A acts also nonproperly on $\mathfrak{G}(M)$. By Lemma 3.6.2 , it follows that there exists $H \neq 0$, $q \in \mathfrak{G}(M)$ and $\lambda_0 \in \Delta \cup \{0\} + \Delta \cup \{0\}$ such that $\lambda_0(H) < 0$ and $q_\lambda = 0$ for all $\lambda \in \Delta \cup \{0\} + \Delta \cup \{0\}$ with $\lambda(H) < 0$. Let $x \in M$ such that $q = \mathfrak{G}_x$. Then $\underset{\alpha(H)<0}{\oplus} \mathfrak{g}_\alpha$ is isotropic with respect to $\mathfrak{G}_x$. Hence the image of $\underset{\alpha(H)<0}{\oplus} \mathfrak{g}_\alpha$ by the map $X \to \overline{X}_x$ is an isotropic subspace of T_xM, so its dimension is less than or equal to 1.

The dimension of $\underset{\alpha(H)<0}{\oplus} \mathfrak{g}_\alpha$ cannot be 0, since G is semisimple. If it equals 1, then the subalgebra $\underset{\alpha(H)\geq 0}{\oplus} \mathfrak{g}_\alpha$ has codimension 1 in $\mathfrak{g}$. By Theorem 3.5.1 this contradicts the nonexistence of an $SL(2,\mathbb{R})$-factor (only simple groups locally isomorphic to $SL(2,\mathbb{R})$ act on 1-manifolds).

We infer from this the existence of a non-zero element $X \in \underset{\alpha(H)<0}{\oplus} \mathfrak{g}_\alpha$ such that $X_x = 0$, which yields $exp(tX) \in stab(x), \forall t \in \mathbb{R}$. But elements of $\underset{\alpha(H)<0}{\oplus} \mathfrak{g}_\alpha$ are nilpotent, and thus generate non-compact groups. This finishes the proof of Theorem 5.3.1. ∎

As a consequence we have the following generalisation of Kowalsky's theorem 5.2.1 to the semisimple case.

Corollary 5.3.1 *If a semisimple Lie group with finite center acts isometrically, locally freely and nonproperly on a Lorentz manifold, then it has a local factor isomorphic to $SL(2, R)$.*

5.3.2 Warped product structure versus partial homogeneity

Let us go another step, by describing the situation according to the causal type of the orbit given in Theorem 5.3.1

Firstly we recall the following result of [42], which shows how homogeneous spaces are rare in Lorentz geometry (in comparison with the Riemannian case, say).

Theorem 5.3.2 *If (M, g) is a homogeneous Lorentz space of dimension ≥ 3, with irreducible isotropy group, then it has constant sectional curvature.*

Observe that the statement in [42] seems weaker than that above, since the isotropy group is assumed to satisfy the supplementary condition of non-precompactness. However, this follows from irreducibility. Indeed, in the same vein as [42], the principal result of [14] says how irreducibility is strong in the Lorentz setting.

Theorem 5.3.3 ([14]) *A connected Lie subgroup H (not assumed a priori to be closed) of $O(1, n)$, which does not preserve any one-dimensional isotropic subspace of $\mathbb{R}^{1,n}$ is up to conjugacy, a union of some components of some $O(1, p) \subset O(1, n)$ (canonical inclusion). In particular, if H acts irreducibly on $\mathbb{R}^{1,n}$, then H contains $SO^0(1, n)$.*

Our goal here is to relax homogeneity by considering (nontransitive) isometric group actions. This work is actually motivated by the study of isometric Lie group actions on non-compact Lorentz manifolds, for instance in the same vein as [4, 26, 27]. To simplify, we will always assume that all the given group actions are faithful.

We first ask whether there is an adaptation of Theorem 5.3.2 to non-transitive isometric actions. In this situation, we consider a group G acting isometrically on a Lorentz manifold (M, g). Each orbit is a homogeneous space. However, the causal type of the orbit may be timelike, spacelike, or lightlike, that is, the induced metric may be Lorentzian, Riemannian or degenerate, respectively. The following generalization of Theorem 5.3.2 relies on the existence of orbits of Lorentz type satisfying irreducibility. It says roughly that the space is partially of constant curvature.

Theorem 5.3.4 *Let G be a Lie group acting isometrically on a Lorentz manifold (M, g) of dimension ≥ 3. Suppose there exists an orbit N which is a (homogeneous) Lorentz space with irreducible non discrete isotropy (restricted to TN).*
Then N is a complete space of constant (sectional) curvature, and G contains the identity component $Isom^0(N)$ as a factor. Furthermore, a neighborhood of N is a warped product $L \times_\omega N$, where L is some Riemannian manifold and the factor N corresponds to the orbits of $Isom^0(N)$.

The following proposition will be useful in proofs, and also gives an exact geometric and algebraic description of N, N_1, G, G_1, ...of Theorem 5.3.5.

Proposition 5.3.1 *Let G be a connected Lie group acting isometrically and transitively on a Lorentz space N. Suppose that the isotropy group of G is irreducible. Then, N has constant curvature, G equals $Isom^0(N)$, and N is one of the following:*
i) The Minkowski flat space.
ii) The de Sitter space $SO(1,n+1)/SO(1,n)$(of positive constant curvature).
iii) A (cyclic) covering of the anti-de Sitter space $SO(2,n)/SO(1,n)$ (of negative constant curvature).

Proof. Let us consider the case where N is flat; the proof is the same in the other cases. Thus, N is locally isometric to the Minkowski space $\mathbb{R}^{1,n,}$ for which the identity component of the isometry group is the semi-direct product $SO^0(1,n) \ltimes \mathbb{R}^{1+n}$. From Theorem 5.3.3 the stabilizer in G must equal $SO^0(1,n)$ (we assumed G connected). It follows that G intersects the translation group $\mathbb{R}^{1+n}$ in an open subgroup, and therefore, G contains all translations, and hence equals $SO^0(1,n) \ltimes \mathbb{R}^{1+n}$, and N is exactly the Minkowski space. (Another way to conclude would be to observe that the full isometry group of the Minkowski space is generated by the stabilizers of its points; in fact, just stabilizers of two different points are sufficient). ■

Proof of Theorem 5.3.4 N is a Lorentz space then $\dim N \geq 2$ and by the irreducibility hypothesis $\dim N > 2$ since the two isotropic directions in such a subspace would be preserved individually, By Theorem 5.3.2 N has constant curvature. For the remainder of the proof we us the following lemmas

Lemma 5.3.1 *Let E (resp. F) be a Lorentz (resp. Euclidean) vector space. Denote by $O(E)$ and $O(F)$ their respective orthogonal groups. Let H be a Lie subgroup of $O(E) \times O(F)$, whose projection on $O(E)$ acts irreducibly on E. Then, H contains a subgroup $H' \subset O(E) \times \{1\}$, which contains the identity component of $O(E) \times \{1\}$. In particular: Any linear H-invariant mapping $f : E \to F$ ($f \circ h = f$, for any $h \in H$) is trivial. The same is true for any H-invariant bilinear antisymmetric mapping $E \times E \to F$.*

Proof. We infer from the irreducible case of Theorem 5.3.3 that the projection H' of H on $O(E)$ contains the identity component of $O(E)$ (isomorphic to $O^0(1,n)$, for $dimE = 1+n$), say $H' = O(E)$, to simplify notation. Since $O(E)$ is semisimple, the Lie algebra $\mathcal{H}'$ of H' is isomorphic to some subalgebra of the Lie algebra $\mathcal{H}$ of H, that is, there exists a homomorphism $\rho : \mathcal{H}' = \mathfrak{o}(E) \to \mathfrak{o}(F)$, such that the graph $\{(h, \rho(h)), h \in \mathfrak{o}(E)\}$ is contained in $\mathcal{H}$. Now, ρ must be trivial since a semisimple Lie group of noncompact type has no non-trivial homomorphism into a compact group. For the last two conclusions of the lemma, one can assume $F = \mathbb{R}$. The kernel of the linear mapping f is $O(E)$-invariant; hence, it is trivial by irreducibility. A similar argument yields triviality of invariant antisymmetric bilinear mappings. ∎

Lemma 5.3.2 *Let G be a connected Lie group acting isometrically on a Lorentz manifold (M, g). Let N be an orbit of G, which is of Lorentz type and has an irreducible stabilizer (when acting on the tangent space of N). Then G splits $G = K \times G_1$, where $G_1 = Isom^0(N)$, and K acts trivially on N (K is in fact precompact in the stabilizer of any point of N). Furthermore, in a neighborhood of N, all the orbits of G_1 are isometric, and thus determine a foliation. The orthogonal distribution to this foliation is integrable, and all structures are invariant under the G-action.*

Proof. The group G acts on N via a homomorphism $G \to Isom^0(N)$. Its image G_1 has irreducible isotropy, and therefore, by Proposition 5.3.1, N has constant curvature and $G_1 = Isom^0(N)$. We now check that, in fact, G_1 is contained in G, and hence G splits as claimed. For this, consider $x_0 \in N$, and denote by H its isotropy group. The orthogonal space L_{x_0} of $T_{x_0}N$ in $T_{x_0}M$ is spacelike (the metric on it is positive definite). We are in a position to apply Lemma 5.3.1 with $E = T_{x_0}N$ and $F = L_{x_0}$. It then follows that the identity component of the isotropy group of x_0 in G_1 is contained in H , and in particular in G. However, G_1 is generated by stabilizers of various points of N, and therefore, G_1 is contained in G. Next, we investigate the G_1-action on M near N. Let H be the isotropy group of x_0 in G_1. Its action on L_{x_0} is trivial. Let $\exp x_0$ denote the exponential mapping for the Lorentz metric and consider the (local) submanifold $\mathcal{L}_{x_0} = exp_{x_0}(L_{x_0})$. Then $\exp_{x_0}$ conjugates the infinitesimal action of H on $\mathcal{L}_{x_0}$ with its action on L_{x_0}. In particular, H acts trivially on this latter submanifold. That is, H is contained in the isotropy group of any point of $\mathcal{L}_{x_0}$. An obvious semi-continuity argument implies that isotropy groups cannot be bigger. Therefore, we have a foliation by G_1-orbits, all satisfying the same irreducibility condition for their isotropy groups. Let us denote this foliation by N and its tangent bundle by TN. Let L be the orthogonal distribution. The obstruction to integrability of L can be measured by means of a tensor $T : L \times L \to TN$, defined by $T(X, Y)$ is the orthogonal projection on TN of the bracket $[X, Y]$, wher X and Y are sections of L. Since the isotropy group acts trivially on L and irreducibly on TN, T is trivial, that is, L is integrable. ∎

Warped product, end of the proof of Theorem 5.3.4. By Lemma 5.3.1 G contains the identity component $Isom^0(N)$ as a factor. Our goal now is to prove that an open neighborhood of N in M is a warped product. So far, we have the orthogonal foliations N and L. One can say that the De Rham decomposition theorem is a criterion for a pair of such foliations to determine a (local) direct pseudo-Riemannian product. The condition is that (the tangent bundles of) $\mathcal{N}$ and $\mathcal{L}$ are parallel, or a priori more weakly, that leaves of $\mathcal{N}$and $\mathcal{L}$ are geodesics. There is a similar, but more complicated, criterion for warped products [25] . We will not use this criterion, but rather give a brief proof in our case. Our terminology here is close to that of [40], which may be consulted for a more complete exposition. Let N and L be (local) leaves of a point x_0 for the foliations $\mathcal{N}$ and $\mathcal{L}$, respectively. So, locally, M has an adapted topological product $L \times N$. The metric can be written $g_{(l,n)} = h_{(l,n)} \oplus m_{(l,n)}$.

• Let us show that $h_{(l,n)} = h$, that is, it does not depend on n. This is clear since G_1 acts isometrically: if $k \in G_1$, then it sends $L_{(l,n)}$ to $L_{k(l,n)}$, where $k(l,n)$ has the form (l,n) (orbits of G_1 correspond to N). Therefore $g = h \oplus m_{(l,n)}$ (the geometric meaning of this fact is that N is a geodesic foliation [40]).
• In order to understand the variation of $m_{(l,n)}$ as a function of (l,n), write $x_0 = (l_0, n_0)$, fix $l_1 \in L$ and consider the mapping $S : (l_0, n) \in N = N(l_0, n_0) \to (l_1, n) \in N(l_1, n_1)$. S commutes with the G_1-action on the G_1-orbits of (l_0, n_0) and (l_1, n_0). In particular it commutes with the isotropy actions at these two points. As shown previously these isotropy groups are the full orthogonal groups of the Lorentz scalar products on their tangent spaces. In particular, they preserve, up to a multiplicative constant, only one Lorentz scalar product. This means that S is a homothety at (l_0, n_0): the image metric equals the metric at (l_1, n_0) (along $N(l_1, n_0)$) up to a multiplicative factor $w(l_0, n_0)$. Now, since S commutes with the (full) action on orbits, it follows that w does not depend of n. That is, if $m = m_{(l_0,n)}$ is the metric on N, then $m_{(l,n)} = w(l)m$. In sum, $g = h \oplus w(l)m$, that is, M is a warped product.
• The fact that N has constant curvature follows from Theorem 5.3.2 since N has irreducible isotropy.
• Finally, the splitting of G is that given in Lemma 5.3.2, which is obviously compatible with the warped product structure. The proof of Theorem 5.3.4 is now complete. ■

5.3.3 Nonproperness versus irreducibility

Let us go a step further, and try to get rid of the irreducibility hypothesis. In fact, irreducibility is an algebraic condition which looks somehow inappropriate in our dynamic-geometric setting here. We want to substitute for it a more natural dynamical condition. Our theory is that nonproperness is good enough for this role. We find it worthwhile to "make some order" around the concept of nonproperness of actions. This will be useful in the sequel (statements and proofs).

Semisimple group actions with nonproper orbits

Without an a priori irreducibility hypothesis, we have the following generalization of Theorem 5.3.4 which is also the description of the situation in Theorem 5.3.1 when the orbit N is a Lorentz space, assuming the orbits are non-proper and the group G is semisimple (a kind of intrinsic irreducibility).

Theorem 5.3.5 *Let G be a connected semisimple Lie group acting isometrically on a Lorentz manifold (M, g) of dimension ≥ 3. Suppose that no (local) factor of G is locally isomorphic to $SL(2, R)$ and that there exists a non-proper orbit N of Lorentz type. Then,* $\dim N \geq 3$ *and up to a finite cover, G factors $G = G_1 \times G_2$, where:*
1) G_1 possesses an orbit N_1 which is a Lorentz space of constant (non-vanishing) curvature, and G_1 equals $Isom^0(N_1)$.
2) There is a G-invariant neighborhood U of N which is a warped product $L \times_\omega N_1$.
3) The factor N_1 corresponds to G_1-orbits, and G_2 acts along the L-factor.

We will first prove this theorem under an additional homogeneity assumption, that is, G acts transitively on M. This will be a generalization of Theorem 5.3.2 where one keeps the non-precompactness assumption and replaces the irreducibility one by the semi-simplicity of the ambient group. We will come back to the proof of Theorem 5.3.5, after proving the following transitive particular case.

Theorem 5.3.6 *Let (M, g) be a G-homogeneous Lorentz space of dimension ≥ 3, with non-precompact isotropy group, and G a connected semi-simple Lie group with no (local) factor locally isomorphic to $SL(2, \mathbb{R})$. Then, there is a group direct product $G = G_1 \times G_2$, and a metric direct product $M = L \times N$, where L is a Riemannian G_2-homogeneous manifold, N is a Lorentz space of constant non-vanishing curvature, and $G_1 = Isom^0(N)$.*

Proof. Let $x_0 \in M$, and a lightlike (i.e. isotropic) vector $u \in T_{x_0}M$, consider the orthogonal hyperplane $u^\perp$. Let C_{x_0} be the set of those u for which $u^\perp$ is tangent to a totally geodesic (lightlike) hypersurface, that is, $\exp(u^\perp)$ is a totally geodesic hypersurface (near x). By Theorem 2.3.2, C_x is non-empty. Let H be a totally geodesic lightlike hypersurface through x_0. Let $u \in C_{x_0} \cap (T_{x_0}H)^\perp$, $G(x_0)$ be the isotropy group and $G^0(x_0)$ be the identity component in $G(x_0)$. By Theorem 5.3.1 $G^0(x_0)$ contains a nontrivial unipotent one-parameter group. Thus $G^0(x_0)u$ is infinite or equal to $\{u\}$.

• In fact, to simplify notation in the following argument, one is allowed to suppose that $G(x_0)$ is connected that is $G(x_0) = G^0(x_0)$. Suppose $G(x_0)u = \{u\}$. Therefore, we have a G-invariant distribution of hyperplanes $x \to u(x)^\perp$ on M. This distribution is integrable, the leaf at x being the geodesic hypersurface $H_{u(x)} = \exp(u(x)^\perp)$. From this, we get that M possesses a codimension one G-invariant foliation. The quotient space is a 1-manifold. But a simple Lie group acting (non-trivially) on a 1-manifold must be locally isomorphic to $SL(2,\mathbb{R})$. This is impossible because of our assumption on G. It then follows that $G(x_0)u$ is infinite, and hence C_{x_0} generates a Lorentz subspace F_{x_0} of dimension ≥ 3.

• The same argument as above allows one to check that the isotropy group cannot preserve a space of dimension 1. Nor can it preserve a Lorentz subspace of dimension 2, since the two isotropic directions in such a subspace would be preserved individually. To summarize, one can select a subspace E_{x_0} of F_{x_0} which is Lorentz, generated by its infinite subset $E_{x_0} \cap C_{x_0}$, and on which the isotropy group acts irreducibly.

• M is a G-homogeneous Lorentz space then, we define a G-invariant distribution E on which the isotropy group, at each point, acts irreducibly. As in the proof of Theorem 5.3.4, one defines an integrability obstruction tensor for E, which must vanish, by Lemma 5.3.1, since it is antisymmetric and invariant under the isotropy group. Therefore E is integrable. We denote by $\mathcal{N}$ its tangent foliation.

• Let N be a leaf of $\mathcal{N}$ containing x_0, and G_3 the subgroup of G leaving it invariant. This subgroup acts transitively on N, since the foliation $\mathcal{N}$ is invariant under the transitive G-action. Moreover, G_3 contains the stabilizers (in G) of all the points of N, which, by construction, act irreducibly on the tangent space of each point of N. We can thus apply Theorem 5.3.4 to the G_3-action. In particular, N has constant curvature, and G_3 contains $G_1 = Isom^0(N)$, which induces a warped product $L \times_\omega N$ in a neighborhood of N in M.

• There is a neighborhood O of x_0 such that the stabilizer of any point in $N \cap O$ preserves the leaves of the restriction of the foliation $\mathcal{N}$ to O (leaf by leaf) then for x near x_0 the intersection between the leaf N_x of $\mathcal{N}$ and the G_1 orbit of x is nontrivial, by irreducibility, this implies the coincidence of N_x and $G_1.x$ and then the coincidence of the foliation $\mathcal{N}$ (given by the distribution E) and that given by the factor N in the warped product. Therefore, the warped product is G -invariant, and is in particular global: $M = L \times_\omega N$.

• Now, we argue by contradiction to show that N has non-vanishing curvature, that is, N is not the Minkowski space. Indeed, N can be identified with the quotient space M/L endowed with its similarity Lorentz structure (that is, a Lorentz metric up to a (global) constant), which is preserved by the G-action. In other words, the semisimple group G acts transitively by similarity on N. However, the similarity group of the Minkowski space $\mathbb{R}^{1,n}$ is $(\mathbb{R}^+ \times O(1,n)) \ltimes \mathbb{R}^{1+n}$. Any semi-simple subgroup of it is conjugate to $O(1,n)$, and cannot act transitively.

• Since G acts transitively on $M = L \times_\omega N$ by preserving the warped product structure, all the leaves $\{l\} \times N$ are isometric, and hence have the same curvature. However, metrics at two levels l_1 and l_2 are related by a factor $w(l_1)/w(l_2)$. Curvatures are related by the inverse ratio. From constancy of the (non-vanishing) curvature, we infer that w is a constant function, that is, $M = L \times N$ is a direct product.

• Since G preserves the product structure of M and contains G_1, which is the full (identity component of the) isometry group of N, G splits as claimed. This finishes the proof of Theorem 5.3.6. ■

Corollary 5.3.2 *Let G be a semi-simple connected Lie group acting isometrically and transitively on a Lorentz space N of constant curvature.*
Suppose that the isotropy is non-compact. Then, G equals $Isom^0(N)$, and N is one of the following:
i) The de Sitter space $SO(1,n+1)/SO(1,n)$ (of positive constant curvature).
ii) A (cyclic) covering of the anti-de Sitter space $SO(2,n)/SO(1,n)$ (of negative constant curvature).

Proof. If M has constant curvature, then, it cannot have a non-trivial decomposition $M = L \times N$. Therefore, $M = N$, that is, $G = Isom^0(M) = Isom^0(N)$.
■

Proof of Theorem 5.3.5. The first point in the proof of Theorem 5.3.6 implies that the dimension of N is ≥ 3. The splitting of G is that given by Theorem 5.3.6. We apply Theorem 5.3.4 to the G_1-action in order to obtain a warped product as claimed. ■

Remark 5.3.1 *A particular case of Theorem 5.3.6 is where G is simple. It can be reformulated in this case as follows: if a simple group G acts by preserving a Lorentz metric on a quotient G/H, where H is non-compact, then this quotient has constant curvature, that is, G/H has the form $O(1,n+1)/O(1,n)$ or $O(2,n)/O(1,n)$ (up to a cyclic cover).*

Remark 5.3.2 *1) In both theorems above, the warped product is local, i.e. not the whole space is a warped product. To see this, one considers the $O(1,n)$-action on the Minkowski*

space $\mathbb{R}^{1,n.}$ If the Lorentz quadratic form is $q = -x_0^2 + x_1^2 + ... + x_n^2$ then the warped product is defined exactly on the region $q > 0$.
2) The result does not seem to be optimal, that is, it might be generalized to other groups.
3) Warped product structures on universal covers of compact Lorentz manifolds with strong dynamics are obtained, for instance, in [20, 43, 40].

5.3.4 Nonproper semisimple action: the general case

Unfortunately, the orbit obtained in Theorem 5.3.1 is a non-proper homogeneous space, but not necessarily Lorentz! The nuisance is that it can be lightlike (degenerate). In the general case we have the following theorem:

Theorem 5.3.7 *Let G be a semisimple group with finite center and no local $SL(2,\mathbb{R})$-factor, acting isometrically and nonproperly on a Lorentz manifold M. Then*
(1) G has a local factor G_1 isomorphic to $O(1,n)$ or $O(2,n)$;
(2) There exists a Lorentz manifold S, isometric, up to finite cover, to dS_n or AdS_{n+1}, depending whether G_1 is isomorphic to $O(1,n)$ or $O(2,n)$, and an open subset of M, in which each G_1-orbit is homothetic to S;
(3) Any such orbit as above has a G_1-invariant neighborhood isometric to a warped product $L \times_w S$, for L a Riemannian manifold.

Proof of Theorem 5.3.7 By Theorem 5.3.1 G has an orbit O with a noncompact stabilizer, and that if O is Lorentzian, then the situation is exactly as described in Theorem 5.3.5. If an orbit is not Lorentzian, then it is degenerate, fixed, or Riemannian. Riemannian isometries always have compact isotropy, so the existence of noncompact stabilizers implies that either O is degenerate, or some simple local factor fixes it pointwise. The proof of Theorem 5.3.7 would be finished using the following two propositions, which state that in either situation, there is a nonproper Lorentzian orbit.

Proposition 5.3.2 *Let O be a G-orbit on which some simple noncompact normal subgroup G_1 acts trivially. Then G has (near O) Lorentzian orbits with noncompact isotropy.*

Proposition 5.3.3 *Let O be a degenerate G-orbit with noncompact isotropy. Then, G has (near O) Lorentzian orbits with noncompact isotropy.*

Proof of proposition 5.3.2. Let x be a point of O. The isotropy representation $\Phi\colon G_1 \longrightarrow O(T_xM)$ is faithful and proper. No noncompact simple subgroup of $O(T_xM) \cong$

$O(1, d-1)$, can preserve an isotropic line. Therefore, Theorem 5.3.3 applies, so $\Phi(G_1)$ preserves an $(n+1)$-dimensional Lorentz subspace V_1 of T_xM for some $2 \leq n \leq d-1$; further, $\Phi(G_1)$ contains a subgroup conjugate to the standard copy of $O^0(1, n)$ in $O(1, d-1)$. Because $\Phi(G_1)$ is simple and G has no local $SL(2, \mathbb{R})$-factors, G_1 is locally isomorphic to $O(1, n)$, and $n \geq 3$. Any spacelike vector v in V_1 has Lorentzian $\Phi(G_1)$-orbit isometric to dS_n.

The stabilizer of v is locally isomorphic to $O(1, n-1)$; in particular, it is noncompact. The G_1-orbit of $\exp_x(v)$ is again of Lorentzian type, by the Gauss Lemma (see [33], 5.1). The G-orbit of $\exp_x(v)$ is Lorentzian because it contains a Lorentzian submanifold, and this orbit has noncompact isotropy, as desired. ■

Degenerate orbits: proof of proposition 5.3.3

Here we collect some algebraic facts about the structure of nonproper degenerate orbits. Suppose that x is a point of M with degenerate G-orbit. Denote this orbit by O. Recall that d is the dimension of M, and assume that G is semisimple.

Fix an isometric isomorphism of T_xM with $\mathbb{R}^{1,d-1}$, determining an isomorphism $O(T_xM) \simeq O(1, d-1)$. Let V be the image of T_xO in $\mathbb{R}^{1,d-1}$. Let $\Phi : G(x) \longrightarrow O(1, d-1)$ be the resulting isotropy representation. Because G acts properly and freely on the bundle of Lorentz frames of M, the isotropy representation is an injective, proper map. The invariant subspace V is degenerate, so $\Phi(G(x))$ leaves invariant the line $V^{\perp} \cap V$. The stabilizer of an isotropic line is conjugate in $O(1, d-1)$ to the group

$$P = (K \times \ A) \ltimes U$$

where $U \simeq \mathbb{R}^{d-2}$ is unipotent, $A = \mathbb{R}$, and $K = O(d-2)$ with the conjugation action of $K \times \ A$ on U equivalent to the standard representation of the conformal group of $\mathbb{R}^{d-2}$. Denote by $\mathfrak{p}$ the Lie algebra of P, and by $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{u}$, the subalgebras corresponding to K, A and U. Let $\varphi : \mathfrak{g}(x) \longrightarrow o(1, d-1)$ be the Lie algebra representation tangent to Φ. Note that $\mathfrak{g}/\mathfrak{g}(x)$ can be identified with T_xO by the map

$$Y \longrightarrow \overline{Y}(x) = \frac{\partial}{\partial t} |_{t=0} \, e^{tY} x$$

For $g \in G(x)$, differentiating the relation $ge^{tY}x = (ge^{tY}g^{-1})x$ gives

$$D_x g(\overline{Y}(x)) = \overline{Ad} g(Y)(x)$$

In other words, Φ restricted to V is equivalent to the representation $\overline{Ad}$ of $G(x)$ on $\mathfrak{g}/\mathfrak{g}(x)$ arising from the adjoint representation. Let $\overline{ad}$ be the representation tangent to $\overline{Ad}$.

An element Y of $\mathfrak{g}$ is called nilpotent if $ad(Y)$ is nilpotent. An element Y is $\mathbb{R}$-split if $ad(Y)$ is diagonalizable over $\mathbb{R}$.

Lemma 5.3.3 *The stabilizer subalgebra $\mathfrak{g}(x) \subset \mathfrak{g}$ has the following properties:*
(1) For all $Y \in \mathfrak{g}(x)$, the endomorphism $\overline{ad}(Y)$ has no real nonzero eigenvalues. In fact, the same is true for φ, so $\varphi(\mathfrak{g}(x))$is conjugate to a subalgebra of $\mathfrak{m} \ltimes \mathfrak{u}$.
(2) The stabilizer subalgebra $\mathfrak{g}(x)$ contains no element $\mathbb{R}$-split in $\mathfrak{g}$.
(3) There exists a subalgebra $\mathfrak{s}(x)$ in which $\mathfrak{g}(x)$ has codimension one such that $[\mathfrak{s}(x), \mathfrak{s}(x)] \subset \mathfrak{g}(x)$, and the representation of $\mathfrak{g}(x)$ on $\mathfrak{g}/\mathfrak{s}(x)$ is skew-symmetric, that is, every endomorphism is diagonalizable with purely imaginary eigenvalues.

Proof. (1) Suppose that $\overline{ad}(Y)$ has eigenvalue $\lambda > 0$. Then λ is also an eigenvalue of $\varphi(Y)$ on V . Since $\varphi(Y)$ is skew-symmetric on $V/(V^{\perp} \cap V)$, the generalized eigenspace for λ in V is one-dimensional and equals $V^{\perp} \cap V$. The trace of $\varphi(Y)$ on V is λ so the trace of $\overline{ad}(Y)$ on $\mathfrak{g}/\mathfrak{g}(x)$ is λ. Next, we will show that the trace of $ad(\varphi(Y))$ on $\varphi(\mathfrak{g}(x))$ is nonnegative. Recall that $\mathfrak{p}$ decomposes $(\mathfrak{a} \times\ \mathfrak{m}) \ltimes \mathfrak{u}$. The restriction of $ad(\mathfrak{p})$ to $\mathfrak{u}$ factors through the projection to $\mathfrak{a} \times \mathfrak{m}$, which acts as the standard conformal representation on $\mathbb{R}^{d-2}$. If $\varphi(Y)$ has eigenvalue λ on $V^{\perp} \cap V$, then it projects to $\lambda I + M$ in $\mathfrak{a} + \mathfrak{m} \simeq conf(\mathbb{R}^{d-2})$. Therefore, the characteristic roots of $ad(\varphi(Y))$ on $\mathfrak{u}$ are in $\lambda + i\mathbb{R}$. On $\mathfrak{p}/\mathfrak{u} \simeq \mathfrak{a} \oplus \mathfrak{m}$, the characteristic roots of any adjoint endomorphism are all purely imaginary. Therefore, the characteristic roots of $ad(\varphi(Y))$ on $\mathfrak{p}$ are in $(\lambda + i\mathbb{R}) \sqcup i\mathbb{R}$, implying that the trace of $ad(\varphi(Y))$ is nonnegative on $\varphi(\mathfrak{g}(x))$. Then the trace of $ad(Y)$ on $\mathfrak{g}(x)$ is also nonnegative. Finally, the trace of $ad(Y)$ on $\mathfrak{g}$ is positive, contradicting the unimodularity of $\mathfrak{g}$. If $\lambda < 0$, the same argument shows that the trace of $ad(Y)$ on $\mathfrak{g}$ is negative. Therefore, no $\overline{ad}(Y)$ has any nonzero real eigenvalues, and no $\varphi(Y)$ has any nonzero real eigenvalues on V . If $\varphi(Y)$ has a nonzero real eigenvalue on $\mathbb{R}^{1,d-1}$, then an eigenvector must be isotropic. It either lies in V or is not orthogonal to $V^{\perp} \cap V$. In either case, $\varphi(Y)$ has a nonzero real eigenvalue on V, a contradiction.

(2) If an $\mathbb{R}$-split element $H \in \mathfrak{g}(x)$, then by (1), all root vectors on which $ad(H)$ is nontrivial must project to 0 in $\mathfrak{g}/\mathfrak{g}(x)$. In this case, $\mathfrak{g}(x)$ contains a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Applying the monomorphism φ would yield a subalgebra isomorphic to $\mathfrak{sl}(2, R)$ in $\mathfrak{p}$, which is impossible.

(3) Take any Z' spanning $V^{\perp} \cap V$. Item (1) implies $\varphi(\mathfrak{g}(x))$ annihilates Z'. Take the

corresponding vector in $\mathfrak{g}/\mathfrak{g}(x)$, and let Z be any lift to $\mathfrak{g}$. Then $ad(\mathfrak{g}(x))(Z) \subset \mathfrak{g}(x)$, so $\mathfrak{s}(x) = \mathbb{R}Z + \mathfrak{g}(x)$ is the desired subalgebra. From the equivalence of $\varphi|_V$ with $\overline{ad}$, the representations $V/(V^\perp \cap V)$ and $\mathfrak{g}/\mathfrak{s}(x)$ are equivalent. The former is skew-symmetric. ■

By the proof of Theorem 5.3.1, if the G-action is nonproper, then there is $x \in M$, and an $\mathbb{R}$-split element H of $\mathfrak{g}$ such that the negative root space $\sum_{\alpha(H)<0} \mathfrak{g}_\alpha$ is isotropic at x. If G has noncompact stabilizer at some point y of M, then we may take $y = x$ above.

The subalgebra $\mathfrak{s}(x)$ of Lemma 5.3.3 is exactly the maximal subspace $\mathfrak{s} \subset \mathfrak{g}$ such that $\{\overline{Y}(x) \in T_xM : Y \in \mathfrak{s}\}$ is an isotropic subspace, In particular, $\mathfrak{s}(x)$ is unique.

Denote by $\mathfrak{a}$ and Δ the Cartan subalgebra and root system, respectively. Then we have the following proposition.

Proposition 5.3.4 *There exist J in $\mathfrak{a}$ and $S \subset \Delta$ such that*
(1) $\mathfrak{s}(x) = \mathbb{R}J + \mathfrak{g}(x)$
(2) $\alpha(J) < 0$ *for all* α *in* S
(3) $\sum_{\alpha \in S} \mathfrak{g}_\alpha \subset \mathfrak{g}(x)$
(4) $\dim \sum_{\alpha \in S} \mathfrak{g}_\alpha \geq 2$

Proof. By Theorem 5.3.1 there is H in $\mathfrak{a}$ such that the image of $\bigoplus_{\alpha(H)<0} \mathfrak{g}_\alpha$ by the map $X \to \overline{X}_x$ is an isotropic subspace of T_xM. Let

$$0 \neq X \in (\sum_{\alpha(H)<0} \mathfrak{g}_\alpha) \cap \mathfrak{g}(x).$$

There exist $J \in \mathfrak{a}$ and a nilpotent Y in $\mathfrak{g}$ such that $[J, X] = -2X$ and $[X, Y] = J$ (see [38],2.4.B). The operator $\overline{ad}(X)$ is nilpotent; on the other hand, by Lemma 5.3.3, $\overline{ad}(X)$ is skew-symmetric on $\mathfrak{g}/\mathfrak{s}(x)$, so $ad(X)(\mathfrak{g}) \subset \mathfrak{s}(x)$. Therefore, J belongs to $\mathfrak{s}(x)$.

By Lemma 5.3.3 $J \notin \mathfrak{g}(x)$. Therefore $\mathfrak{s}(x) = \mathbb{R}J + \mathfrak{g}(x)$, proving (1).

Let S be the set of α in Δ such that $\alpha(H) < 0$ and $\alpha(J) < 0$, so (2) is obviously satisfied. From the relation $[J, X] = -2X$, any α such that X has a nontrivial component in $\mathfrak{g}_\alpha$ satisfies $\alpha(J) = -2$, so any such belongs to S; in particular S is not empty.

For $\alpha \in S$, we have $\mathfrak{g}_\alpha \subset \mathfrak{s}(x)$ and

$$\mathfrak{g}_\alpha = [J, \mathfrak{g}_\alpha] \subset [\mathfrak{s}(x), \mathfrak{s}(x)] \subset \mathfrak{g}(x),$$

by Lemma 5.3.3, showing statement (3) above.

Now, replacing X by a nonzero element of some $\mathfrak{g}_\alpha$, $\alpha \in S$, we may assume that, $(-J)$ is a basic translation that is, there exists $c_\alpha < 0$ such that,

$$\alpha(J) = -2 \text{ and } \alpha(Z) = c_\alpha k(J, Z)$$

for any Z in $\mathfrak{a}$, where k denotes the Killing form. In this case, we have that for any root β, the reflection

$$\sigma_\alpha(\beta) = \beta + \beta(J)\alpha$$

is again a root (see [38],II.5.A).

Now, to show (4) it suffices to show that $\dim \mathfrak{g}_\alpha \geq 2$ or that there exists some $\gamma \neq \alpha$ also in S.

Suppose $\dim(\mathfrak{g}_\alpha) = 1$. The assumption that $\mathfrak{g}$ has no $\mathfrak{sl}(2, \mathbb{R})$-factor implies that there exists some nonzero root $\delta \neq \alpha$ such that $\delta(J) \neq 0$. We may assume $\delta(J) < 0$. If $\delta(H) < 0$, then $\delta \in S$, as desired. So suppose that $\delta(H) \geq 0$. Now let

$$\gamma = -\sigma_\alpha(\delta) = -\delta - \delta(J)\alpha$$

Then

$$\gamma(J) = -\delta(J) - \delta(J)\alpha(J) = -\delta(J)(1 - 2) = \delta(J) < 0$$

and

$$\gamma(H) = -\delta(H) - \delta(J)\alpha(H) < 0$$

so $\gamma \in S$. ■

The remainder of the subsection is devoted to the proof of proposition 5.3.3; the orbit O will be assumed degenerate with noncompact isotropy below.

The asymptotic geodesic hypersurface $\mathcal{F}_x$

Lemma 5.3.4 *For x in O, let $\mathbb{R}\eta_x$ be the null direction in T_xO. Then, the orthogonal $\eta_x^\perp$ is tangent to a lightlike geodesic hypersurface $\mathcal{F}_x$ (defined in a neighborhood of x).*

Proof. Let X be a nilpotent element of $\mathfrak{g}(x)$ given by Theorem 5.3.1, and consider the isometry $f_t = e^{tX}$, for some $t \neq 0$. The derivative D_xf fixes η_x that is $D_xf(\eta_x) = \eta_x$. We have $\{f_m\} \subset stab(x)$ then by Theorem 2.3.2 the graphs $Graph(f^m)$ converge to a d-dimensional, isotropic, totally geodesic submanifold E and the projection $\mathcal{F}_x$ of E is a totally geodesic hypersurface in $M \times \ \{x\}$. Because the derivative $D_x(f^m)$ fixes η_x for all m, the vector $(\eta_x, \eta_x) \in T_{(x,x)}E$. Because $T_{(x,x)}E$ is isotropic, its intersection with $0 \times \ T_xM$ is exactly $\mathbb{R}\eta_x$. Then $(\eta_x, 0) \in T_{(x,x)}E$, so the projection $T_x(\mathcal{F}_x) = \eta_x^\perp$, as desired. ■

Lemma 5.3.5 *The hypersurface $\mathcal{F}_x$ carries a 1-dimensional foliation $\mathcal{C}$ such that:*
(1) Any isotropic curve in $\mathcal{F}_x$ is tangent to a leaf of $\mathcal{C}$.
(2) Each leaf of $\mathcal{C}$ is an isotropic geodesic.
(3) The (local) quotient space $\mathcal{F}_x/\mathcal{C}$ inherits a Riemannian metric, infinitesimally preserved by the elements of $\mathfrak{g}$ preserving $\mathcal{F}_x$.

Proof. At any point y of a degenerate hypersurface $\mathcal{F}$, there exists a unique tangent isotropic direction C_y. These lines determine a characteristic 1- dimensional foliation $\mathcal{C}$ of $\mathcal{F}$, proving (1). Since $\mathcal{F}_x$ is totally geodesic, (2) follows. For (3), it is known (see, for instance, [41]) that $\mathcal{C}$ is transversally Riemannian if and only if $\mathcal{F}_x$ is totally geodesic. Here, transversally Riemannian means that the flow along any parameterization of $\mathcal{C}$ preserves the induced degenerate metric, or equivalently, that the degenerate metric can be projected as a Riemannian metric on the (local) quotient space $\mathcal{F}_x/\mathcal{C}$. Finally, if an isometric flow locally preserves $\mathcal{F}_x$, then it induces a local diffeomorphism of $\mathcal{F}_x/\mathcal{C}$ that is obviously an isometry by construction of the Riemannian metric. ■

Lemma 5.3.6 *The subalgebra $\mathfrak{s}(x)$ preserves the isotropic geodesic $\mathcal{C}_x$, so it preserves $\mathcal{F}_x$.*

Proof. Indeed, any Y in $\mathfrak{s}(x)$ has $\overline{Y}(x)$ isotropic, and hence the whole Y-orbit of x is isotropic. But, as stated above, isotropic curves of $\mathcal{F}_x$ are contained in leaves of $\mathcal{C}$ that is, all Y -orbits through x are contained in $\mathcal{C}_x$. The image of $\mathcal{C}_x$ by any element of the one-parameter group generated by Y is an isotropic geodesic tangent to $\mathcal{C}_x$ at some point, thus equals $\mathcal{C}_x$. ■

Lemma 5.3.7 *The action of $\sum_{\alpha\in S}\mathfrak{g}_\alpha$ on the tangent space of the (local) quotient space $\mathcal{F}_x/\mathcal{C}$ at the point corresponding to $\mathcal{C}_x$ is trivial.*

Proof. The tangent space to $\mathcal{F}_x/\mathcal{C}$ at $\mathcal{C}_x$ is identified with $\eta_x\ /\mathbb{R}\eta_x$. Note that the subspace $\mathbb{R}J + \sum_{\alpha\in S}\mathfrak{g}_\alpha$ as in Proposition 5.3.4 is in fact a subalgebra of $\mathfrak{s}(x)$. We have a representation of $\mathbb{R}J+\sum_{\alpha\in S}\mathfrak{g}_\alpha$ into the orthogonal algebra of $\eta_x\ /\mathbb{R}\eta_x$, which is endowed with a positive definite inner product. But in such an orthogonal algebra, an equality $[\rho(J),\rho(Y)] = \lambda\rho(Y)$, becomes trivial—that is $\rho(Y) = 0$ (since $\lambda \neq 0$). ■

Corollary 5.3.3 *$\sum_{\alpha\in S}\mathfrak{g}_\alpha$ acts trivially on the (local) quotient space $\mathcal{F}_x/\mathcal{C}$. That is, $\sum_{\alpha\in S}\mathfrak{g}_\alpha$ preserves individually each leaf of $\mathcal{C}$.*

Proof. The action of $\sum_{\alpha \in S} \mathfrak{g}_\alpha$ on $\mathcal{F}_x/\mathcal{C}$ is trivial, since it is a Riemannian action with a fixed point and a trivial derivative at this point. ■

Corollary 5.3.4 *Any point of $\mathcal{F}_x$ has a noncompact isotropy algebra.*

Proof. Indeed, $\sum_{\alpha \in S} \mathfrak{g}_\alpha$ has dimension ≥ 2 and has orbits of dimension 1. Therefore, stabilizers are nontrivial. They are not compact since all elements of $\sum_{\alpha \in S} \mathfrak{g}_\alpha$ are nilpotent. ■

Lemma 5.3.8 *Let Γ be the set of fixed points of $\sum_{\alpha \in S} \mathfrak{g}_\alpha$ in $\mathcal{F}_x$. Then, Γ has an empty interior (in $\mathcal{F}_x$). In particular, the orbit of any point of $\mathcal{F}_x - \Gamma$ under $\sum_{\alpha \in S} \mathfrak{g}_\alpha$, locally, coincides with its $\mathcal{C}$-leaf.*

Proof. No element of $\sum_{\alpha \in S} \mathfrak{g}_\alpha$ can fix points of an open subset of $\mathcal{F}_x$. Indeed, in general, a Lorentz transformation fixing one point and acting trivially on a tangent lightlike hyperplane at that point has a trivial derivative, and is therefore trivial. ■

Corollary 5.3.5 *No point of $\mathcal{F}_x$ has a spacelike G-orbit.*

Proof. If a point y in $\mathcal{F}_x$ has a spacelike orbit, then all orbits of points in a neighborhood of y are spacelike, as well. However, any neighborhood of y meets $\mathcal{F}_x - \Gamma$; orbits of points in here cannot be spacelike, because they contain at least one isotropic geodesic. ■

End of the proof of proposition 5.3.3

Lemma 5.3.9 *The degenerate orbit O cannot be locally contained in $\mathcal{F}_x$ that is, $\mathcal{F}_x \cap O$ does not contain an open subset of O.*

Proof. Suppose O is locally contained in $\mathcal{F}_x$. Then the group G locally preserves $\mathcal{F}_x$. From Corollary 5.3.1, the infinitesimal action of $\mathfrak{g}$ on the quotient space $\mathcal{Q} = \mathcal{F}_x/\mathcal{C}$ is not faithful. More precisely, any factor $\mathfrak{b}$ of $\mathfrak{g}$ which contains an element like J (in Proposition 5.3.4) must act trivially on $\mathcal{Q}$. However, orbits of $\mathfrak{b}$ cannot be 1-dimensional, since $\mathfrak{g}$ has no $\mathfrak{sl}(2, \mathbb{R})$-factor. Therefore, $\mathfrak{b}$ acts trivially on $\mathcal{F}_x$. As in the proof of Lemma 5.3.8 , this implies that $\mathfrak{b}$ acts trivially on M. ■

Now, thanks to Corollary 5.3.2, the proof of Proposition 5.3.3 would be finished once one proves that there is a point of $\mathcal{F}_x$ with a Lorentz orbit. It suffices to show existence of nondegenerate orbit, since by Corollary 5.3.5, points of $\mathcal{F}_x$ cannot have spacelike orbits. Assume, for a contradiction, that all G-orbits of points of $\mathcal{F}_x$ are degenerate. For any $y \in \mathcal{F}_x$, the orbit G_y locally contains the isotropic geodesic $\mathcal{C}_y$ by Lemma 5.3.8; therefore, $T_y(G_y) \subset \eta_y^\perp = T_y(\mathcal{F}_x)$. In other words, for any Killing field $X \in \mathfrak{g}$ and any $y \in \mathcal{F}_x - \Gamma$, the evaluation $X(y)$ is tangent to $\mathcal{F}_x$, so X defines a vector field in this open subset of $\mathcal{F}_x$. The flow of any $y \in \mathcal{F}_x - \Gamma$ along X for sufficiently short time is again contained in $\mathcal{F}_x$. In particular, since $x \in \mathcal{F}_x - \Gamma$, the orbit O is locally contained in $\mathcal{F}_x$, contradicting lemma 5.3.9. The proof of Proposition 5.3.3 is then finished.

As a compliment of proposition 5.3.3 we have:

Corollary 5.3.6 *There is a simple local factor G_1 of G for which the G_1-orbit of x is a point or degenerate with noncompact stabilizer. In other words, if G has a nonproper orbit that is either a point or degenerate, then a subgroup G_1 locally isomorphic to $O(1,n)$ or $O(2,n)$ has an orbit with the same properties.*

Proof. First suppose that G has a nonproper fixed or Riemannian orbit O_1. Then any local factor G_1 in the kernel of the restriction to O_1 has the desired properties.

Now suppose that G has a nonproper degenerate orbit. We have seen that some nilpotent elements stabilizing a point in the degenerate orbit O stabilize a point y with Lorentzian orbit. But, from Theorem 5.3.5, a Lorentz orbit can be nonproper only if there is a local factor G_1 acting nonproperly with $O_1 = G_1 y$ Lorentzian of constant curvature; moreover, there is a warped product $L \times_w O_1$ preserved by G, and G splits up to finite cover as $G_2 \times G_1$.

If $X \in \mathfrak{g}(y)$, the projection of X on $\mathfrak{g}_2$ generates a precompact 1-parameter group, so X cannot be nilpotent unless it belongs to $\mathfrak{g}_1$. We infer from this that G_1 acts nonproperly on the degenerate orbit $O = Gx$. Because G_1 is simple and the stabilizer $G_1(x)$ is noncompact, the orbit $G_1 x$ must be degenerate or one point. ■

5.3.5 Simple Lie group action with degenerate orbits

Theorem 5.3.5 which classifies nonproper orbits of Lorentz type and the following theorem implies Theorems 5.2.3 and 5.2.4 of Kowalsky.

Theorem 5.3.8 *Suppose G is a connected group with finite center, locally isomorphic to $O(1,n)$ or $O(2,n)$ for $n \geq 3$. If G acts isometrically on a Lorentz manifold and has a*

degenerate orbit O with noncompact stabilizer $G(x)$, then
(1) G is locally isomorphic to $O(1, n)$;
(2) The Lie algebra $\mathfrak{g}(x)$ of the $G(x)$ is isomorphic to $\mathfrak{o}(n-1) \ltimes \mathbb{R}^{n-1}$;
(3) The orbit O is locally homothetic to the light cone in the Minkowski space

Proof Now assume $\mathfrak{g} = \mathfrak{o}(1, n)$ or $\mathfrak{o}(2, n)$ for some $n \geq 3$ and that $\mathfrak{g}$ has a degenerate orbit on M with noncompact stabilizer $\mathfrak{g}(x)$. By the proof of Theorem 5.3.7, $g \cong \mathfrak{o}(1, n)$ and $\mathfrak{g}(x) \cong \mathfrak{o}(n-1) \ltimes \mathbb{R}^{n-1}$. The final point of the theorem follows from these two.

Excluding o(2, n)

Let J, $\mathfrak{a}$, and S be as in Proposition 5.3.4. Observe that there can be at most one negative root with $\mathfrak{g}_\alpha \cap \mathfrak{g}(x) \neq 0$. If X in $\mathfrak{g}_\alpha \cap \mathfrak{g}(x)$, then let $Y \in \mathfrak{g}_{-\alpha}$ be as in ([38] 2.4.B), so $[X, Y] = H_\alpha \in \mathfrak{a}$. Since $ad(X)$ is nilpotent but skew-symmetric on $\mathfrak{g}/\mathfrak{s}(x)$ (Lemma 5.5.3), the $\mathbb{R}$-split element H_α must belong to $\mathfrak{s}(x)$. Any $X' \in \mathfrak{g}_\beta$ would give rise to $H_\beta \in \mathfrak{a} \cap \mathfrak{s}(x)$. If $\alpha \neq \beta$, then for some $c \in \mathbb{R}$, the difference $H_\alpha - cH_\beta$ would be a nonzero $\mathbb{R}$-split element of $\mathfrak{g}(x)$, contradicting Lemma 5.5.3.
Suppose that $\mathfrak{g} = \mathfrak{o}(2, n)$. Let β and γ be distinct negative roots, each with $(n-2)$-dimensional root spaces. The other negative roots are $\beta - \gamma$ and $\beta + \gamma$, with one-dimensional root spaces.

First suppose $X \in \mathfrak{g}_\beta \cap \mathfrak{g}(x)$. Let L be a generator of $\mathfrak{g}_{-\beta-\gamma}$. The adjoint $(adX)^2(L)$ is a nonzero element of $\mathfrak{g}_{\beta-\gamma}$. On the other hand, Lemma 5.3.3 implies that any nilpotent element of $\mathfrak{g}(x)$ has nilpotence order 2 on $\mathfrak{g}/\mathfrak{g}(x)$. Then we would have $\mathfrak{g}_{\beta-\gamma} \cap \mathfrak{g}(x) \neq 0$, a contradiction.

Therefore, X cannot be in $\mathfrak{g}_\beta$. The same argument shows X cannot be in $\mathfrak{g}_\gamma$. Proposition 5.3.4 says that $\mathfrak{g}(x)$ contains a sum of negative root spaces with total dimension at least 2, but since $\mathfrak{g}_{\beta-\gamma}$ and $\mathfrak{g}_{\beta+\gamma}$ are 1-dimensional, and because only one can be contained in $\mathfrak{g}(x)$, we have a contradiction.

Full stabilizer

Now $\mathfrak{g}$ must be $\mathfrak{o}(1, n)$. Let α and J be as above. Let $\mathfrak{m}$ be the maximal compact subalgebra of the centralizer of J in $\mathfrak{g}$; it is isomorphic to $\mathfrak{o}(n-1)$.

Suppose $Y \in \mathfrak{s}(x) \cap \mathfrak{g}_{-\alpha}$. By the third item of Lemma 5.3.3,

$$J \in [Y, \mathfrak{g}_\alpha] \subset [\mathfrak{s}(x), \mathfrak{s}(x)] \subset \mathfrak{g}(x)$$

But this contradicts the second item of Lemma 5.3.3 Therefore, $\mathfrak{s}(x) \cap \mathfrak{g}_{-\alpha} = 0$.

On the other hand, since $\overline{ad}(X)$ is nilpotent for all $X \in \mathfrak{g}_\alpha$, Lemma 5.3.3 (3) forces

$$\mathfrak{m} \subset [\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha] \subset \overline{ad}(\mathfrak{g}_\alpha)(\mathfrak{g}) \subset \mathfrak{s}(x)$$

Since $\mathfrak{g} = \mathfrak{g}_{-\alpha} + \mathfrak{m} + \mathbb{R}J + \mathfrak{g}_\alpha$, the algebra $\mathfrak{s}(x)$ is exactly $\mathfrak{m} + \mathbb{R}J + \mathfrak{g}_\alpha$.

Suppose there were $X = cJ + M \in (\mathbb{R}J + \mathfrak{m}) \cap \mathfrak{g}(x)$ for some nonzero $c \in \mathbb{R}$.

The subspace $\mathfrak{g}_{-\alpha}$ is $ad(X)$-invariant and maps onto $\mathfrak{g}/\mathfrak{s}(x)$. But $ad(X)$ is clearly not skew-symmetric here, contradicting Lemma 5.3.3 (3). Therefore, $\mathfrak{g}(x)$ is exactly $\mathfrak{m} + \mathfrak{g}_\alpha$, which is isomorphic as a Lie algebra to $o(n-1) \ltimes \mathbb{R}^{n-1}$. This finish the proof of Theorem 5.3.8.

5.3.6 Global AdS warped product

The following result implies Theorem 5.2.5 announced by Kowalsky.

Theorem 5.3.9 *If G, a group with finite center locally isomorphic to $O(2,n)$, $n \geq 3$, acts isometrically on a Lorentz manifold M, with some noncompact stabilizer, then, up to finite cover, M is equivariantly isometric to a warped product $L \times_w AdS_{n+1}$ of a Riemannian manifold L with the anti-de Sitter space AdS_{n+1}.*

Proof. Suppose G is locally isomorphic to $O(2,n)$, $n \geq 3$, with finite center, and G acts isometrically on M. Then a neighborhood of some G-orbit is a warped product of the form $L \times_w S$, where S is isometric to AdS_{n+1}, up to finite cover. The set of orbits having a neighborhood isometric to $L \times_w S$, for some Riemannian manifold L and $w : L \longrightarrow R_+$, is open. Let us prove that this set is also closed, and thus equals the whole of M. A limit O of a sequence O_k of such orbits is a non-Riemannian orbit O of dimension $n+1$. Suppose that such a limit O has compact isotropy $G(x)$ for $x \in O$. Then $G(x)$ is contained in a maximal compact subgroup K of G. The Lie algebra $\mathfrak{k}$ is isomorphic to $\mathfrak{o}(2) \times \mathfrak{o}(n)$, which has codimension $2n$ in $\mathfrak{g}$. Since $n \geq 3$, this is impossible. Therefore, for any $x \in O$, the stabilizer $G(x)$ is noncompact. From Theorem 5.3.8, O cannot be degenerate; hence, it is Lorentzian. Then by Theorem 5.3.5, a neighborhood of this orbit is isometric to a warped product. Any orbit of M has a neighborhood isometric to $L \times_w S$, for some L and w. From this, one sees in particular that the G-action determines a foliation $\mathcal{O}$. In addition, $\mathcal{O}$ admits an orthogonal foliation $\mathcal{O}^{\perp}$. We will use the G-action to show that the pair of foliations $\mathcal{O}$ and $\mathcal{O}^{\perp}$ arise from a global warped product of the form $L \times_w AdS_{n+1}$ on a finite cover of M. Choose a point $x_0 \in M$. Let O_0 and $O_0^{\perp}$ be the leaves of x_0 in the foliations $\mathcal{O}$ and $\mathcal{O}^{\perp}$, respectively. Let H_0 be the stabilizer of x_0. Note that $O_0^{\perp}$ is a component of the fixed set $Fix(H_0)$.

The full $Fix(H_0)$ is $N(H_0)O_0^{\perp}$, where $N(H_0)$ is the normalizer of H_0 in G. It is well known that $N(H_0)/H_0$ is finite; then since $O_0^{\perp}$ is H_0-invariant, $Fix(H_0)$ has finitely-many components, each isometric to $O_0^{\perp}$.

Let i and $i^{\perp}$ denote the respective inclusions of O_0 and $Fix(H_0)$ in M. Let G acts on $Fix(H_0) \times\ O_0$ by $g(x, y) = (x, gy)$. Define a mapping :

$$\phi : Fix(H_0) \times\ O_0 \longrightarrow M, \text{ by } \phi(x, gx_0) = g(i^{\perp}(x)).$$

One sees that it is well defined, and it is in fact the G-equivariant extension of the inclusions. Next, ϕ is a covering map. Clearly ϕ is a local diffeomorphism. It is also easy to see that ϕ is surjective: the orbit of any $y \in M$ is homothetic to S. Let H_y be its stabilizer. There is some $g \in G$ conjugating H_y to H_0. Then $gy \in Fix(H_0)$, and $y = \phi(gy, g^{-1}x_0)$. Finally, ϕ is everywhere N-to-1, where $N = |N(H_0)/H_0|$, because every G-orbit in M is homothetic to S. An N-to-1 surjective local diffeomorphism is a covering map. The submanifold $Fix(H_0)$ is Riemannian. Let $L = Fix(H_0)$ with the metric pulled back by $i^{\perp}$. By G-equivariance of ϕ, all leaves $\phi(L \times \{y\})$ are isometric. Also by G-equivariance, the metrics along G-orbits $\phi(\{y\} \times\ S)$ are all homothetic, with homothety factor depending only on $y \in L$. Since S and AdS_{n+1} have a common finite cover, M is isometric, up to finite cover, to a warped product $L \times_w AdS_{n+1}$. ■

5.3.7 Full description of semisimple Lie group actions

As a fusion, we can give the following theorem:

Theorem 5.3.10 *Let G be a semisimple Lie group with finite center and no local $SL(2, \mathbb{R})$-factor, acting nonproperly and isometrically on a Lorentz manifold M. Then, G has a simple local factor G_1 that acts nonproperly. There are two possibilities for G_1:*
(1) $G_1 \cong O(2, n)$. In this case, there is a Lorentz manifold S isometric, up to finite cover, to AdS_{n+1}, such that all G_1-orbits are homothetic to S. In fact, up to finite cover, M is a warped product $L \times_w AdS_{n+1}$.

(2) $G_1 \cong O(1,n)$. There are open sets U and V such that $M = U \sqcup \partial V \sqcup V$, where:

• For any $x \in U$, there exists S, isometric to dS_n up to finite cover, such that the G_1-orbit of x is homothetic to S. A neighborhood of x is G_1-equivariantly isometric to a warped product $L \times_w S$, for some Riemannian manifold L.

• Orbits of G_1 on the boundary of U are either fixed points or locally homothetic to the light cone in Minkowski space $R^{1,n.}$

• For any $x \in V$, there exists T, isometric to $\mathbb{H}^n$ up to finite cover, such that the G_1-orbit of x is homothetic to T. A neighborhood of x is G_1-equivariantly isometric to a warped product $L \times_w T$, for some Lorentz manifold L.

Item (1) of this theorem follows from Theorems 5.3.7 and 5.3.9. We consider the case in which $G_1 \cong O(1,n)$. We will take U and V to be the points of M having G_1-orbit homothetic, up to finite cover, to dS_n and $\mathbb{H}^n$, respectively. The first point of item (2) follows from Theorem 5.3.7 and an argument just as above, to obtain the warped product on each component (see also Lemma 5.3.10). In the first subsection below, we will deduce the second point of (2) from Theorem 5.3.8. Next, we address the last point of item (2), to obtain a warped product near Riemannian G_1-orbits, as well as the global decomposition $M = U \sqcup \partial U \sqcup V$

These two subsections will complete the proof of Theorem 5.3.10.

Degenerate orbits

Let G be locally isomorphic to $O(1,n)$ for some $n \geq 3$, and acts nonproperly on M. Let U be the set of points having G-orbit homothetic, up to finite cover, to $dS_{n.}$

In order to deduce the second point of (2) from Theorem 5.3.8, it suffices to prove the following lemma:

Lemma 5.3.10 *Let $x \in \partial U$. Then the G-orbit of x is either a fixed point or degenerate with noncompact isotropy.*

Proof. The G-orbit of x cannot be Riemannian and has dimension at most n. Suppose $G(x)$ is compact, so it is contained in a maximal compact subgroup K of G. The Lie algebra $\mathfrak{k} = \mathfrak{o}(n)$ and has codimension n, so $\mathfrak{g}(x) = \mathfrak{o}(n)$; further, Gx is either Lorentzian or degenerate and n-dimensional. In either case, the isotropy representation of $\mathfrak{g}(x)$ has a 1-dimensional invariant subspace tangent to Gx and an $(n-1)$-dimensional complementary representation in $T_x(Gx)$, which is necessarily trivial. On the other hand, x is the limit of a sequence $x_i \in U$ for which the isotropy $\mathfrak{g}(x_i)$ is trivial on $T(Gx_i)^\perp$, because $\mathfrak{o}(1, n-1) \cong \mathfrak{g}(x_i)$ has no nontrivial representation in $\mathfrak{o}(d-n)$. The limit $\lim \mathfrak{g}(x_i) \subset \mathfrak{g}(x)$, and, because these subalgebras have the same dimension, they must be equal. Then continuity of the action implies that $\mathfrak{g}(x)$ is trivial on $T(G_x)^\perp$. Now, whether Gx is Lorentzian or degenerate, the isotropy $\mathfrak{g}(x)$ is trivial on all of T_xM. But isometries fixing a point and having trivial derivative at that point are trivial, so we have a contradiction.

Therefore $G(x)$ is noncompact. If the orbit Gx is Lorentzian, then the result of Theorem 5.3.5 would give that $x \in U$. Then, the G-orbit of x is fixed or degenerate. ■

Riemannian orbits

As above, we assume in this subsection that G is locally isomorphic to $O(1, n)$ for some $n \geq 3$, and that G acts nonproperly on M. By Theorem 5.3.7, the set U defined above is open and nonempty. The following proposition implies the last point of item (2) in Theorem 5.3.10, as well as the claimed decomposition of M into de Sitter orbits, fixed points and light cone orbits, and hyperbolic orbits.

Proposition 5.3.5 *Let U and V be the sets of points of M having orbits homothetic up to finite cover to the de Sitter space dS_n and the hyperbolic space $\mathbb{H}^n$, respectively. Then*
(1) V is open and G-invariant isometric to a warped product $L \times_w \mathbb{H}^n$, for some Lorentz manifold L.
(2) $\partial U = \partial V$ consists of all fixed points and orbits locally homothetic to the Minkowski light cone.
(3) $M = U \sqcup \partial U \sqcup V$.

Notation: There exists a neighborhood of the 0-section of TM on which the exponential map is defined and injective on each fiber. We fix one such neighborhood and denote it by Ω below. We first collect some lemmas for the proof of proposition 5.3.5.

Maximal subalgebras of $\mathfrak{o}$(1, n) We begin with two facts about certain natural subalgebras of $\mathfrak{o}(1, n)$. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is maximal if it is not contained nontrivially in another subalgebra: if $\mathfrak{h} \subset \mathfrak{h}' \subset \mathfrak{g}$, then $\mathfrak{h} = \mathfrak{h}'$ or $\mathfrak{h}' = \mathfrak{g}$.

Consider the infinitesimal action of $\mathfrak{g} = \mathfrak{o}(1, n)$ on the projectivization $\mathbb{P}(\mathbb{R}^{1,n})$ of the standard representation with the standard inner product of type $(1, n)$. For $v \in \mathbb{R}^{1,n}$, denote by $\overline{v}$ the image in $\mathbb{P}(\mathbb{R}^{1,n})$.

• If v is spacelike, then $\mathfrak{g}(v) = \mathfrak{g}(\overline{v})$ is conjugate to $\mathfrak{g}(e_n)$, which is isomorphic to $\mathfrak{o}(1, n-1)$, with an obvious embedding in $\mathfrak{o}(1, n)$.

• For v timelike, $\mathfrak{g}(v) = \mathfrak{g}(\overline{v})$ is conjugate to $\mathfrak{g}(e_1)$, which is isomorphic to $\mathfrak{o}(n)$, with an obvious embedding in $o(1, n)$.

• Finally, for v nonzero and isotropic, $\mathfrak{g}(\overline{v})$ is conjugate to $\mathfrak{g}(\overline{e_1 + e_2})$, which is isomorphic to the parabolic subalgebra $\mathfrak{p}$ as the proof of Theorem 5.3.7. This subalgebra is isomorphic to $\mathfrak{sim}(n-1)$, the algebra of infinitesimal affine similarities of $\mathbb{R}^{n-1}$. The annihilator of a nonzero isotropic v is a codimension-1 ideal of $\mathfrak{g}(\overline{v})$, isomorphic to $\mathfrak{euc}(n-1)$, the algebra of infinitesimal affine isometries of $\mathbb{R}^{n-1}$.

Lemma 5.3.11 *(1) $\mathfrak{sim}(n-1)$ is a maximal subalgebra of $\mathfrak{o}(1, n)$.*
(2) $\mathfrak{euc}(n-1)$ is contained in exactly one maximal subalgebra, $\mathfrak{sim}(n-1)$.
(3) Let $\mathfrak{h}$ be a compact proper subalgebra of $\mathfrak{o}(1, n)$ containing two different conjugates $\mathfrak{k}$ and $\mathfrak{k}'$, each isomorphic to $\mathfrak{o}(n-1)$. Then $\mathfrak{h}$ is conjugate to $\mathfrak{o}(n)$.

Proof. (1) The subalgebra $\mathfrak{sim}(n-1)$ acts infinitesimally conformally on the projectivization of the light cone, which is conformally equivalent to $\mathbb{S}^{n-1}$. It is the infinitesimal stabilizer of one point and acts transitively on the complement of this point. Then for any $X \notin \mathfrak{sim}(n-1)$, the algebra generated by X and $\mathfrak{sim}(n-1)$ is transitive on $\mathbb{S}^{n-1}$ and contains the full stabilizer subalgebra of each point. It follows that any subalgebra properly containing $\mathfrak{sim}(n-1)$ is $\mathfrak{o}(1, n) \cong \mathfrak{conf}(\mathbb{S}^{n-1})$, so $\mathfrak{sim}(n-1)$ is a maximal subalgebra.
(2) Let $\mathfrak{h}$ be a maximal subalgebra of $\mathfrak{o}(1, n)$ containing $\mathfrak{euc}(n-1)$, and suppose that $\mathfrak{h}$ does not preserve any isotropic line in $\mathbb{R}^{1,n}$. Because the corresponding connected group $H \subset O(1, n)$ is not compact, Theorem 5.3.3 implies that $\mathfrak{h}$ preserves some Lorentz subspace of $\mathbb{R}^{1,n}$ and contains the full infinitesimal linear isometry algebra of this subspace. Because $\mathfrak{euc}(n-1)$ preserves no proper Lorentz subspace, $\mathfrak{h}$ must be equal to $\mathfrak{o}(1, n)$.
(3) Let H be the connected subgroup of $O(1, n)$ with Lie algebra $\mathfrak{h}$. If H is compact, so it is conjugate to a subgroup of $O(n)$ containing two copies of $SO(n-1)$. By an argument similar to that for $\mathfrak{sim}(n-1)$, the subalgebra $\mathfrak{o}(n-1)$ is maximal in $\mathfrak{o}(n)$, so $\mathfrak{h} = \mathfrak{o}(n)$. ■

Fixed point sets Recall that, for H a subgroup of G, the fixed set is denoted $Fix(H)$. Recall also that the exponential map is defined on Ω and injective on each fiber in it. Each component of $Fix(H)$ is a totally geodesic submanifold of M. Let D be the points of M having degenerate G-orbit with noncompact isotropy. By Theorem 5.3.8, for any $x \in D$, $\mathfrak{g}(x) = \mathfrak{o}(n-1) \ltimes \mathbb{R}^{n-1} \cong \mathfrak{euc}(n-1)$.

Lemma 5.3.12 *Let $x \in D$, and O its G-orbit. Near x, the fixed set $Fix(G(x))$ coincides with $\exp_x(T_xO^{\perp} \cap \Omega)$. It has dimension $d-n$, and intersects O along the isotropic geodesic in O through x.*

Proof. By Proposition 5.3.3, we can approximate the degenerate orbit O by Lorentzian orbits O_i with noncompact isotropy. By Theorem 5.3.5, each O_i is homothetic, up to finite cover, to dS_n. Take $x_i \in O_i$ with $x_i \to x \in O$. The limit $\lim g(x_i)$ is contained in $g(x) \cong \mathfrak{euc}(n-1)$; since $\dim(g(x_i)) = \dim(g(x))$ for all i, the limit equals $g(x)$. As in the proof of Lemma 5.3.10, triviality of $g(x_i)$ on $(T_{x_i}O_i)^{\perp}$ for all i implies that, in the limit, the isotropy $g(x)$ is trivial on $(T_xO)^{\perp}$. It is easy to see from the form of the isotropy representation of $g(x) \cong \mathfrak{euc}(n-1)$ that the maximal trivial subrepresentation in T_xM is exactly $(T_xO)^{\perp}$. Then the lemma follows with the exponential map and dimension counting. ■

Lemma 5.3.13 *Let K be a maximal compact connected subgroup of $G(x)$.Then $Fix(K)$ has dimension $d-n+1$ and is of Lorentzian type. It meets O along the isotropic geodesic in O through x.*

Proof. The subgroup K is conjugate to $SO(n-1) \subset Euc(n-1)$. The isotropy representation of K at x fixes any nonzero normal vector $\eta_x \in (T_xO)^{\perp} \cap T_xO$ and acts irreducibly on an $(n-1)$-dimensional complementary spacelike subspace $L_x \subset T_xO$. Then $Fix(K) \cap O = N_x$, where N_x is the isotropic geodesic in O through x. Now, $L_x(T_xO)^{\perp}$ has codimension one in T_xM. There is a K-invariant 1-dimensional complementary representation, which is necessarily trivial.
We have therefore a $(d-n+1)$-dimensional Lorentzian subspace L'_x containing $(T_xO)^{\perp}$, complementary to L_x, on which K acts trivially. This subspace is exactly the maximal trivial subrepresentation of K on T_xM. ■

Lemma 5.3.14 *Let $X_x = \sqcup\{T_x(Fix(K))\backslash\{0\} : K \subset G(x)$ maximal compact connected$\}$. Then X_x is open in T_xM. More precisely, $X_x = ((T_xO)^{\perp} \times P^c) \cup ((T_xO)^{\perp}\backslash\{0\})$ where P is a hyperplane in a subspace F complementary to $(TxO)^{\perp}$.*

Proof. Fix one maximal connected compact $K_0 \subset G(x)$, locally isometric to $SO(n-1)$. Let F_0 be a K_0-invariant degenerate complement to $(T_xO)^{\perp}$, and let v_0 be a nonzero isotropic vector in F_0. Then $(T_xO)^{\perp} + \mathbb{R}v_0 = T_x(Fix(K_0))$. Now any other maximal compact subgroup of $G(x)$ equals gK_0g^{-1} for some g in $G^0(x)$. The fixed subspace $T_x(Fix(gK_0g^{-1})) =$

$(T_xO)^\perp + \mathbb{R}gv_0$. As g ranges over $G^0(x)$, the projection of gv_0 to F_0 ranges over all vectors v with $< v, \eta_x > = < v_0, \eta_x >$ where η_x is a generator of $T_xO^\perp \cap T_xO$. Then the projection of $\mathbb{R}G^0(x)v_0$ to F_0 ranges over all v with $< v, \eta_x > \neq 0$. Now the lemma is proved, with $F = F_0$ and $P = F_0 \cap \eta_x^\perp$. ■

Hypersurface of degenerate orbits Let O be as above, a degenerate orbit with isotropy $\mathfrak{euc}(n-1)$. We will next show that any neighborhood of a point of O meets U or V . Then we will show that O lies in a hypersurface of degenerate orbits with the same isotropy, and, finally, that this hypersurface locally separates U from V .

Let X_x be as in Lemma 5.3.14 above. Consider $X(O) = \cup\{X_x \cap \Omega : x \in O\}$. This is an open set in the restriction $TM|O$.

Lemma 5.3.15 *The image $exp(X(O)) \subset U \cup D \cup V$, that is, for any $y \in exp(X(O))$, the stabilizer $g(y) \cong \mathfrak{o}(n), \mathfrak{euc}(n-1)$, or $o(1, n-1)$.*

Proof. Denote by Φ the restriction of the exponential map $\Phi = exp : X(O) \to M$. Reducing Ω if necessary, we may assume $\Phi(X(O))$ contains no fixed points. Each X_x is open in T_xM, so Φ has maximal rank, and is in particular an open map. Its image is an open set containing O in its closure. Any point y in this image is a regular value; the inverse image $S = \Phi^{-1}(y) \subset X(O)$ has dimension $\dim(X(O)) - \dim(M) = n$. Because exp is injective on each fiber of Ω, and hence of $X(O)$, the inverse image S projects injectively under $\pi : TM \to M$ onto some open set W in O. For any $z \in W$, there exists a maximal compact $K_z \subset G(z)$ such that $y \in Fix(K_z)$. Because $n \geq 3$, there exist z, $z' \in W$ which do not lie on a common isotropic geodesic of O, so that K_z and $K_{z'}$ are contained in $G(y)$ and are distinct. By Lemma 5.3.14, either $g(y) \cong \mathfrak{o}(n)$, or it is noncompact. In the latter case, by Theorems 5.3.5 and 5.3.8, $g(y) \cong \mathfrak{o}(1, n-1)$ or $\mathfrak{euc}(n-1)$. ■

Let $x \in O$, and let R be a neighborhood of x in $Fix(G(x))$. Let $\widehat{O}$ be the union of all the G-orbits of all points of R

Lemma 5.3.16 *For R sufficiently small, $\widehat{O}$ is a degenerate hypersurface in which every orbit is degenerate with isotropy $\mathfrak{euc}(n-1)$; further, $\widehat{O} \cup U \cup V$ is a neighborhood of $\widehat{O}$.*

Proof. From Lemma 5.3.12, R equals $exp((T_xO)^\perp \cap \Omega)$ near x and has dimension $d-n$. Since $g(x) \cong \mathfrak{euc}(n-1)$ lies in a unique maximal subalgebra of $o(1, n)$ by
Lemma 5.3.11, the orbit of any $z \in R$, is either a fixed point or degenerate. Shrink R so that the G-orbit of any $z \in R$ is degenerate; in this case, $g(z) \cong euc(n-1)$. The tangent space

$T_z\widehat{O} = T_z(Gz) + T_z(Fix(G(x)) = T_z(Gz) + (T_z(Gz))^{\perp}$, because z is also as in Lemma 5.3.12; hence, O is a degenerate hypersurface. Now consider $X(\widehat{O}) = \cup\{X(Gz) : z \in R\} \subset T\widehat{O}$, and let Φ be the restriction of exp to $X(\widehat{O})$. From Lemma 5.3.15, the image of the restriction of Φ_z to $X(Gz)$ is open and contained in $U \cup D \cup V$, and its closure is a neighborhood of Gz. If $y \in D$, then $\mathfrak{g}(y)$ is conjugate to $euc(n-1)$. But y is close to a point of $\widehat{O}$, with stabilizer also conjugate to $\mathfrak{euc}(n-1)$. Then y has the same stabilizer as a nearby point of $\widehat{O}$. The set Ω can be chosen sufficiently small such that any such y is contained in $\widehat{O}$. Then the union

$$\underset{z\in R}{\cup}\Phi_z(X(Gz))$$

is contained in $U \cup \widehat{O} \cup V$, and its closure is a neighborhood of $\widehat{O}$. The subset of points of this closure with $\dim(\mathfrak{g}(y)) \leq \dim euc(n-1)$ still forms a neighborhood of $\widehat{O}$, and is contained in $U \cup \widehat{O} \cup V$. $\blacksquare$

Lemma 5.3.17 *Let Φ be the restriction of the exponential map to $X(\widehat{O})$, in a sufficiently small neighborhood of the 0-section. One component of $\Phi(X(\widehat{O}))\backslash\widehat{O}$ lies in U, and the other is in V .*

Proof. From the last lemma, any point $x \in \widehat{O}$ has a neighborhood in which $\dim(\mathfrak{g}(x)) = \dim(\mathfrak{g}(y))$ for all y in the neighborhood; let N be this common dimension. In this neighborhood, the map $y \mapsto \mathfrak{g}(y) \in Gr^N(\mathfrak{g})$ is continuous. There are smaller neighborhoods $Z \subset W$ of x such that W is a normal neighborhood of each of its points, and, whenever z, $z' \in Z$ are such that $\mathfrak{g}(z) = Ad(g)(\mathfrak{g}(z'))$, then $gz' \in W$. (Note: We take g to be the element of $Ad^{-1}(Ad(g))$ closest to the identity.)

Suppose for a contradiction that points on opposite sides of $\widehat{O}$ have the same orbit type. Let Z_1, Z_2 be the components of $Z\backslash\widehat{O}$, and W_1, W_2 the components of $W\backslash\widehat{O}$. Take $z \in Z_1$ and $z' \in Z_2$. By assumption, $\mathfrak{g}(z)$ and $\mathfrak{g}(z')$ are conjugate in G. Let g be a conjugating element with minimal distance to the identity. Then z, $gz' \in W$ are connected by a unique geodesic, which necessarily passes through $\widehat{O}$. The common stabilizer $\mathfrak{g}(z) = \mathfrak{g}(gz')$ fixes the geodesic pointwise, so it fixes a point of $\widehat{O}$. Whether $\mathfrak{g}(z)$ is isomorphic to $o(n)$ or to $o(1, n-1)$, neither embeds in the stabilizer of a point of $\widehat{O}$. $\blacksquare$

Proof of Proposition 5.3.5 Suppose that $U \neq M$, so $\partial U \neq \phi$. By Lemma *5.3.17, V* is nonempty, furthermore the interior $int(V) \neq \phi$. The following lemma will be the key for the rest of the proof.

Lemma 5.3.18 *For $x \in int(V)$with orbit O, the isotropy representation of $\mathfrak{g}(x)$ is trivial on $(T_xO)^{\perp}$.*

Proof. The stabilizer $\mathfrak{g}(x) = \mathfrak{o}(n)$ acts on the orthogonal space $(T_xO)^\perp$ via a homomorphism $\mathfrak{o}(n) \longrightarrow \mathfrak{o}(1, d-n-1)$. There is a neighborhood W of $(x, 0)$ in $(TO)^\perp$ on which the exponential map is a diffeomorphism onto its image. For any y in $\exp_x(W \cap T_xM)$, the stabilizer $\mathfrak{g}(y) \subset \mathfrak{g}(x)$. If all orbits near O have the same dimension, then $\mathfrak{g}(x) = \mathfrak{g}(y)$ for all $y \in \exp(W \cap T_xM)$. Then the orthogonal representation of $\mathfrak{g}(x)$ is trivial. ■

Now we can show that $\partial(int(V)) = \partial U$. Suppose $x_n \in int(V)$ and $x_n \longrightarrow x$. Denote by O the orbit of x. As usual, we have $\lim \mathfrak{g}(x_n) \subset \mathfrak{g}(x)$, and $\dim(O) \leq n$. The orbit of x is either Riemannian, degenerate, or fixed.

First suppose O is Riemannian. Then $\mathfrak{g}(x)$ is compact, so it is conjugate to a subalgebra of $\mathfrak{o}(n)$. The dimension restriction forces $\mathfrak{g}(x)$ to be conjugate to $\mathfrak{o}(n)$, so $x \in V$. As in the proof of Lemma 5.3.10, the isotropy $\mathfrak{g}(x)$ is trivial on $(T_xO)^\perp$, so orbits near x have isotropy containing $\mathfrak{o}(n)$. The orbit O has a neighborhood consisting of Riemannian G-orbits. Because $\mathfrak{o}(n)$ is a maximal subalgebra, orbits near x all have isotropy $\mathfrak{g}(x)$. Then $x \in int(V)$, a contradiction.

If O is degenerate, the same argument as in Lemma 5.3.10 implies that $\mathfrak{g}(x)$ is noncompact. Then Proposition 5.3.3 implies that $x \in \partial U$. If x is a fixed point, then Proposition 5.3.2 implies that $x \in \partial U$. Lemma 5.3.10 gives item (2).

Now $U \cup \partial U \cup int(V)$ is closed. It is open by Lemma 5.3.16, so it equals M. Therefore, $int(V) = V$, and $M = U \sqcup \partial U \sqcup V$, proving item (3).

It remains to prove item (1), the warped product structure in V. The key properties are irreducibility of the isotropy along orbits and triviality of the isotropy orthogonal to orbits.

Fix $x \in V = int(V)$ with G-orbit O, and let $\mathcal{L}_x = exp_x((T_xO)^\perp \cap \Omega)$. Since $G(x)$ is irreducible on T_xO and trivial on $(TxO)^\perp$, the leaf $\mathcal{L}_x$ is exactly the set of common fixed points of $G(x)$ near x. Let $L = Fix(G(x)) \cap V$.

Define $\phi : L \times O \to V$ by $\phi(y, gx) = gy$. This map is a local diffeomorphism because, for any $y \in Fix(G(x)) \cap V$, the fixed set L coincides near y with $exp((T_y(G(y)))^\perp \cap \Omega)$. If $\phi(y, gx) = \phi(y', g'x)$, then $g^{-1}g' \in NG(G(x))$, and $g^{-1}g'y' = y$. The stabilizer $G(x)$ is a maximal compact subgroup of the semisimple connected group G, so $N_G(G(x)) = G(x)$ (see [19], 1.13.14 (4)). It follows that ϕ is injective. Any $y \in V$ has $G(y)$ compact and $\dim G(y) = \dim G(x)$; because $G(x)$ is connected, there exists $g \in G$ such that $gG(y)g^{-1} = G(x)$. Then $\phi(gy, g^{-1}x) = y$, so ϕ is surjective. In conclusion, ϕ is a G-equivariant diffeomorphism $L \times O \to V$.

The orbit O is assumed to be homothetic, up to finite cover, to $\mathbb{H}^n$. Because $\mathbb{H}^n$ is simply connected, it has no finite covers. Because any finite subgroup of $Isom(\mathbb{H}^n)$ has a common fixed point, there are no smooth finite quotients of $\mathbb{H}^n$. Therefore, the orbit O, and every other orbit of V , is globally homothetic to $\mathbb{H}^n$. The metric on V pulls back by ϕ to $L \times O$.

The transverse fibers of the product are orthogonal by construction. All L-fibers are isometric because they are related by the action of G. Also by equivariance, O-fibers in $L \times O$ correspond to G-orbits in V , so each one is homothetic to $\mathbb{H}^n$. In conclusion, there exists $w : L \to \mathbb{R}^+$ such that V is isometric to $L \times_w \mathbb{H}^n$ this finish the proof of proposition 5.3.5 and then the proof of theorem 5.3.10 is finished.

6

Further results and questions

The case of groups with infinite center

Almost work in this domain consider groups with finite center and then it is interesting to investigate the infinite center case.

The Hermite-Lorentz case

It seems very interesting to try to generalize our work to the complex case. In this situation we need to find a space analogous to AdS_n and dS_n spaces of the real case, but here the holomorphic sectional curvature plays the role of the sectional curvature. We need to find the spaces candidates to play the role of the de Sitter and anti-de Sitter spaces

The lightlike metrics

This case consists to study the dynamics of a manifold endowed with a degenerate metric. In this direction we recall the work of E. Bekkara, A. Zeghib and C. Frances given in [13] where they extended Theorem 5.3.7 to the lightlike case:

Theorem 6.0.11 *Let G be a non-compact semi-simple Lie group with finite center acting locally faithfully, isometrically and non-properly on a light-like manifold (M, h). Assume that G has no factor locally isomorphic to $SL(2, \mathbb{R})$. Then, up to a finite cover, $G = H \times H'$, where H is locally isomorphic to $O(1, n)$ and G has an orbit which is homothetic, up to a finite cover, to a metric product $Co^n \times N$, where N is a Riemannian H'-homogeneous manifold and Co^n is the light cone of the Minkowski space. The action of $H \times H'$ on $Co^n \times N$ is the product action.*

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