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ON SOME FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES AND THEIR APPLICATIONS

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Introduction

The theory of derivatives of non-integer order goes back to the genesis of differential calculus, the philosopher and creator of modern calculus G.W.Leibniz gave some remarks in his list to L'Hospital [41], dated 30 September 1695, on the meaning and possibility of fractional derivative of order 1/2.
For three centuries the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics useful only for mathematicians. However a rigorous investigation was first carried out by Liouville in a series of papers from 1832-1837, where he defined the first outcast of an operator of fractional integration. Later investigations and further developments by among others Riemann led to the construction of the integral-based Riemann-Liouville fractional integral operator, which has been a valuable cornerstone in fractional calculus ever since. Prior to Liouville and Riemann, Euler took the first step in the study of fractional integration when he studied the simple case of fractional integrals of monomials of arbitrary real order in the heuristic fashion of the time; it has been said to have lead him to construct the gamma function for fractional powers of the factorial. Abel, Letnikov, Wayl, Hadamard, Grnwald and many other, well-known in this domain of the past and present. An early attempt by Liouville was later purified by the Swedish mathematician Holmgren, who in 1865 made important contributions to the growing study of fractional calculus. But it was Riemann who reconstructed it to fit Abel's integral equation, and thus

made it vastly more useful.

In the last few decades many authors pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various real materials, e.g. polymers. It has been shown that new fractional-order models are more adequate than previously used integer-order models. Fundamental physical consideration in favour of the use of models based on derivatives of non-integer order.

In recent years a considerable interest has been shown in the so-called fractional calculus, which allows us to consider integration and differentiation of any order, not necessarily integer. To a large extent this is due to the applications of the fractional calculus to problems in different areas of physics and engineering. The fractional calculus can be considered an old and yet novel topic, it has undergone a rapid development. One of the emerging branches of this study is the theory of fractional evolution equations, i.e. evolution equations where the integer derivative with respect to time is replaced by a derivative of fractional order. The increasing interest in this class of equations is motivated both by their application to problems from viscoelasticity, heat conduction in materials with memory, electrodynamics with memory, and also because they can be employed to approach nonlinear conservation laws.

Today there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types, but the Riemann-Liouville operator is still the most frequently used when fractional integration is performed.

Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, see [7, 47, 54].

This thesis is concerned to the study of abstract fractional differential equations. Our work is organized as follows.

In chapter 1, we state the basic notations, definitions and properties concerning functional analysis and fractional calculus which are used throughout the thesis to obtain our results.

In chapter 2, we study the linear fractional evolution equation

$$\frac{d^\alpha u(t)}{dt^\alpha} \; + \; (A - B(t))u(t) = f(t), \quad t > t_0$$

in a Banach space X, where $0 < \alpha \leq 1$, u is an X-valued function on $R^+ = [0, \infty)$ and f is a given abstract function on R^+ with values in X. We assume that $-A$ is a linear closed operator defined on a dense set S in X into X, $\{B(t) : t \in R^+\}$ is a family of linear bounded operators defined on X into X . It is assumed that $-A$ generates an analytic semigroup $Q(t)$ such that $\|Q(t)\| \leq M$ for all $t \in R^+, Q(t)h \in S, \|AQ(t)h\| \leq \frac{M}{t}\|h\|$ for every $h \in X$ and all $t \in (0, \infty)$. We prove the existence of optimal mild solutions for the considered equation, we use the Gelfand-Shilov principle to prove existence, and then the Bochner almost periodicity condition to show that solutions are weakly almost periodic. As an application, we study a fractional partial differential equation of parabolic type, see [22].

In chapter 3, we are concerned with the semilinear fractional differential equation

$$\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = f(t, u(t)), \quad t > t_0$$

in Banach space$X$,where $0 < \alpha \leq 1, t \geq 0$, we assume that $-A$ is the infinitesimal generator of an analytic c_0-semigroup $Q(t)$ satisfying the exponential stability, f is uniformly almost periodic function defined on $R^+ \times X_q$ into X satisfies the hypothesis

F: There are numbers $L \geq 0$ and $0 \leq \eta \leq 1$ such that

$$|f(t_1, u_1) - f(t_2, u_2)| \leq L(|t_1 - t_2|^\eta + |u_1 - u_2|_q)$$

for all $(t_1, u_2), (t_2, u_2)$ in $R^+ \times X_q$, where X is a real or complex Banach space with norm $|.|$, A^q is the fractional power and X_q is the Banach space $D(A^q)$ endowed with the norm $|u|_q = |A^q u|$. We use the theory of fractional calculus to establish the existence and uniqueness of almost periodic solutions of a class of semilinear fractional differential equations, then as continuation of [22], we prove under suitable conditions that their optimal mild solutions are also weakly almost periodic, see [23].

In chapter 4, We consider the nonlinear fractional integrodifferential equation

$$\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = f(t, u(t)) + \int_{t_0}^{t} a(t-s)g(s, u(s))ds,$$

$$u(t_0) \; = \; u_0,$$

in Banach space X, where $0 < \alpha \leq 1, t > t_0$, let J denote the closure of the interval $[t_0, T)$, $t_0 < T \leq \infty$ and let $-A$ be the infinitesimal generator of an analytic semigroup $Q(t), t \geq 0$, the function a is real-valued and locally integrable on $[0, \infty)$,and the nonlinear maps f and g are defined on $[0, \infty) \times X$ into X. Our basic tools are the use of Gelfand-Shilov principle and fractional powers of operators to establish the existence and uniqueness of local mild then local classical solutions of a class of nonlinear fractional integrodifferential equations, as an application, we study nonlinear integro-partial differential equation of fractional order, see [24].

Contents

Chapter 1

Preliminaries

1.1 Almost periodic functions

The theory of almost periodic functions with values in a Banach space was developed by Bohr, Bochner, Neumann and others, the existence of almost periodic solutions of abstract differential equations has been considered in several works; see for example [1, 2, 62, 63, 64, 65, 66]. From their results, we will mention.

1.1.1 Bochner's Characterization of Almost Periodicity

Let $C_b(R, X)$ denote the usual Banach space of bounded continuous functions from R into X under the supremum norm $|.|_\infty$. Given a function $f : R \longrightarrow X$ and $\omega \in R$, we define the $\omega-$ translate of f as $f_\omega(t) = f(t+\omega)$, $t \in R$. We will denote by $H(f) = \{f_\omega : \omega \in R\}$ the set of all translates of f.

Definition 1.1

A function $f \in C_b(R, X)$ is said to be almost periodic if and only if $H(f)$ is relatively compact in $C_b(R, X)$. We note that almost periodic functions can as well be characterized in terms of relatively dense sets in R of $\tau-$ almost periods.

Definition 1.2

A function $f : R \longrightarrow X$ is called almost periodic if

(i) f is continuous, and

(ii) for each $\epsilon > 0$ there exists $l(\epsilon) > 0$, such that every interval I of length $l(\epsilon)$ contains a number τ such that $|f(t+\tau) - f(t)| < \epsilon$ for all $t \in R$.

Let X be a uniformly convex Banach space equipped with a norm $|.|$ and X^* its topological dual space. It is well known, see [48, 65], that f is almost periodic if and only if for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that $(f(t+s_n))$ is uniformly convergent in $t \in R$. Using the above fact we will say:

Definition 1.3

$f : R \longrightarrow X$ is weakly almost periodic if for every sequence of numbers (s'_n) there exists a subsequence (s_n) such that every $(f(t+s_n))$ is convergent in the weak sense, uniformly in $t \in R$. In other words, for every $x^* \in X^*$, the sequence $(< x^*, f(t+s_n) >)$ is uniformly convergent in $t \in R$, where $< .,. >$ denotes duality $< X^*, X >$.

Let Y denote a Banach space and Ω an open subset of Y.

Definition 1.4

A continuous function $f : R \times \Omega \longrightarrow X$ is called uniformly almost periodic if for every $\epsilon > 0$ and every compact set $K \subset \Omega$ there exists a relatively dense set P_ϵ in R such that $|f(t+\tau, u) - f(t, u)| \leq \epsilon$ for all $t \in R$, $\tau \in P_\epsilon$ and all $u \in K$, see [2].

Let $f(t, u)$ be almost periodic in $t \in R$ uniformly for $u \in K$, then for every sequence of real numbers (s'_n) there exists a subsequence (s_n) and a function $g(t, u)$ such that $f(t+s_n, u) \longrightarrow g(t, u)$ uniformly on $R \times K$ as $n \longrightarrow \infty$, where K is a compact set in Ω, see Hamaya [36, p.188]. Also we will say: $f : R \times \Omega \longrightarrow X$ is weakly almost periodic if for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that every $(f(t+s_n,u))$ is convergent in the weak sense, uniformly on $R \times K$. In other words, for every $u^* \in X^*$, the sequence $(< u^*, f(t+s_n,u) >)$ is uniformly convergent on $R \times K$.

1.1.2 Bohr's Characterization of Almost Periodicity

Bohr's theory of almost periodic functions has been extensively studied, especially in connection with differential equations.

Almost periodic solutions of ordinary differential systems are vector valued functions defined on the set R of real numbers. But the notion of almost periodicity makes sense on any additive group other than R. Indeed, the Bohr definition for an almost periodic function is valid for vector doubly infinite sequences defined on the set Z of integers. This is important since infinite sequences are candidate solutions of difference equations. Also, the generalizations of almost periodic functions-asymptotic almost periodicity by Frechet, pseudo almost periodicity by Zhang can be defined on sequences.

Definition 1.5

A sequence $f : Z \to R^n$ is said to be almost periodic if for any $\epsilon > 0$ there exists an integer $l(\epsilon) > 0$ such that each interval of length l contains an integer τ for which

$$|f(n+\tau) - f(n)| < \epsilon, \quad n \in Z.$$

Note that in the process of discretization a periodic function such as $f(t) = sin\ t$ over R does not lead to a periodic sequence in the sense that the sequence $(sin\ nh),\ n \in Z,\ h \in (0, \infty),\ h \neq 2\pi$, is not periodic with integer period. However, such a sequence is almost periodic. For example, $(sin\ n)$ is almost periodic.

If $f : R \to R^n$ is an almost periodic function, then $(f(n))$ is an almost periodic sequence. Conversely, if x is an almost periodic sequence, then there exists an almost periodic function $f : R \to R^n$ such that $f(n) = x(n)$ for $n \in Z$. Furthermore, the limit sequence is also an almost periodic sequence. Recall that

Definition 1.6

$f : Z \times R^n \to R^n$ is said to be almost periodic in n uniformly for $x \in R^n$, or uniformly almost periodic if for any $\epsilon > 0$ and any compact set $K \subset R^n$, there exists a positive integer $l = l(\epsilon, K)$ such that any interval of length l contains an integer τ for which

$$|f(n+\tau, x) - f(n, x)| < \epsilon,\ n \in Z,\ x \in K.$$

The hull of f, denoted by $H(f)$, is defined to be the set of all $g : Z \times R^n \to R^n$ such that there exists an integer sequence (h_k) and $lim f(n+h_k, x) = g(n, x)$ uniformly on $Z \times K$, where K is any compact set in R^n.

Difference equations are more appropriate that their continuous counter parts in cases when processes evolve in stages. For example population may grow or decline through non-overlapping generations. Besides, difference equations are well known for simulating continuous models for numerical purposes. Indeed, many discrete models in population dynamics and neural dynamic systems are proposed and well studied with respect to their stability, permanence, bifurcation, chaotic behavior, oscillation, periodicity, etc, see [16].

If f is an almost periodic function, the following limit

$$a(\lambda, f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\lambda t} dt$$

for all $\lambda \in R$, exists and is called the Bohr transform of f. As is known, there is an at most countable set of reals λ such that the above limit differs from zero. This set will be denoted by $\sigma_b(f)$ and called Bohr spectrum of f. The approximation theorem says that for every almost periodic function f, there exists a sequence of trigonometric polynomials $P_n(t) = \sum_{k=1}^{N_n} a_{k,n} e^{\lambda_{k,n} t}$, where $\lambda_{k,n} \in \sigma_b(f)$ and $a_{k,n} \in X$ for all k, n, that converges uniformly in $t \in R$ to f as $n \to \infty$, see [46].

1.2 Semigroups

A family of linear operators $\{T(t) : \ t \geq 0\}$ on a Banach space X such that

(i) $T(0) = I, \ I$ is the identity operator and

(ii) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$

is called a semigroup.

Definition 1.7

Let X be a Banach space and let C be an injective operator in $L(X)$. A family $\{T(t);\ t \geq 0\}$ in $L(X)$ is called an exponentially bounded C-semigroup if the following conditions are satisfied:

(i) $T(0) = C$,

(ii) $T(t+s)C = T(t)T(s)$ for $t, s \geq 0$,

(iii) $T(.)x : [0, \infty) \to X$ is continuous for any $x \in X$,

(iv) There are $M \geq 0$ and $a \in R$ such that $\|T(t)\| \leq Me^{at}$ for $t \geq 0$.

A C-semigroup $\{S(t)\}_{t\geq 0}$ is said to be exponentially stable if there are $N, \nu > 0$ two constants such that $\|S(t)\| \leq Ne^{-\nu t}$, for all $t \geq 0$.

We define an operator A as follows:

$$D(A) = \{x \in X : \lim_{h \to 0^+} \frac{(T(h)x - Cx)}{h} \in R(C)\}$$

and for all $x \in D(A)$

$$Ax = C^{-1} \lim_{h \to 0^+} \frac{(T(h)x - Cx)}{h}.$$

This operator is called the generator of $(T(t))_{t\geq 0}$. It is known that A is closed but not necessarily densely defined.

Next we define the operator

$$Gx = \lim_{t \to 0} C^{-1} \frac{(T(t)x - Cx)}{t}$$

$$D(G) = \{x \in R(C) : \exists \lim_{t \to 0^+} C^{-1} \frac{(T(t)x - Cx)}{t}\}.$$

The complete infinitesimal generator of $(T(t))_{t\geq 0}$ is defined to be the operator $\overline{G}$ if the range $R(C)$ is dense in X.

Lemma 1.1

Let C be an injective linear operator and let $(T(t))_{t\geq 0}$ be a C-semigroup with generator A. Then, the following assertions hold true:

(i) $T(t)T(s) = T(s)T(t)$, for all $t, s \geq 0$,

(ii) If $x \in D(A)$, then $T(t)x \in D(A)$, $AT(t)x = T(t)Ax$ and for all $t \geq 0$

$$\int_0^t T(\xi)Axd\xi = T(t)x - Cx,$$

(iii) $\int_0^t T(\xi)xd\xi \in D(A)$ and $A\int_0^t T(\xi)xd\xi = T(t)x - Cx$ for every $x \in X$ and $t \geq 0$,

(iv) A is closed and satisfies $C^{-1}AC = A$,

(v) $R(C) \subset \overline{D(A)}$,

(vi) If $R(C)$ is dense in X, then $\overline{D(G)} = X$ and $G \subset A$, see [46].

1.3 Proporties in Banach Space

1.3.1 Banach Fixed Point Theorem

A fixed point of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself, that is, $Tx = x$.

Definition 1.8 (Contraction)

Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called a contraction on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$, $d(Tx, Ty) \leq \alpha d(x, y)$.

Contraction theorem 1.2

Consider a metric space $X = (X, d)$, where $X \neq \emptyset$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point, see [30].

1.3.2 Convexity and Reflexivity

Definition 1.9

Let $(X, \|.\|)$ be a real normed linear space and X^* be its dual space. The space X is said to be strictly convex if, $x = y$ whenever $x, y \in S(X)$ and $\frac{x+y}{2} \in S(X)$, where $S(X) = \{x \in X;\ \|x\| = 1\}$.

Definition 1.10

The space X is uniformly convex if, for every sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ we have, $\lim_{n\to\infty} \|x_n - y_n\| = 0$ whenever, $\lim_{n\to\infty} \|x_n + y_n\| = 2$, see [35].

Definition 1.11

Let $(X, \|.\|)$ be a real Banach space and let $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X, respectively. For any subset A of X, we denote by $conv(A)$ (resp., $\overline{conv}(A)$) the convex hull (resp., the closed convex hull) of Clarkson [17] who introduced the concept of uniform convexity, and it is known that uniform convexity implies reflexivity of Banach spaces. There are different uniform geometric properties which have been defined between the uniform convexity and the reflexivity of Banach spaces. Huff [38] introduced the nearly uniform convexity of Banach spaces. He has proved that the class of nearly uniformly convexifiable spaces is strictly between superreflexive and reflexive Banach spaces.

A Banach space X is called uniformly convex if for each $\epsilon > 0$ there is

$\delta > 0$ such that for $x, y \in S(X)$, the inequality $\|x - y\| > \epsilon$ implies that

$$\|\frac{1}{2}(x + y)\| < 1 - \delta.$$

For any $x \notin B(X)$, the drop determined by x is the set

$$D(x, B(X)) = conv(\{x\}) \cup B(X).$$

Rolewicz [57], basing on Daneš̃ drop theorem [19], introduced the notion of drop property for Banach spaces.

A Banach space X has the drop property (D) if for every closed set C disjoint with $B(X)$, there exists an element $x \in C$ such that

$$D(x, B(X)) \cap C = \{x\}.$$

A Banach space X is said to have the Kadec-Klee property (or property (H)) if every weakly convergent sequence on the unit sphere is convergent in norm. In [56], Rolewicz proved that if the Banach space X has the drop property, then X is reflexive. It extended this result by showing that X has the drop property if and only if X is reflexive and has the property (H). If X be a reflexive and smooth space, then the dual space X^* is strictly convex, see [59].

1.4 Fractional Derivatives and Integrals

1.4.1 Gamma Function

Undoubtedly, one of the basic functions of the fractional calculus is Euler's

Gamma function $\Gamma(z)$, which generalizes the factorial $n!$ and allows n to take also non-integer and even complex values. We will recall some results on the Gamma function which are important for other parts of this work.
The Gamma function $\Gamma(z)$ is defined

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt,$$

which converges in the right half of the complex plane $Re(z) > 0$. Indeed, we have

$$\begin{aligned} \Gamma(x+iy) &= \int_0^\infty e^{-t} t^{x-1+iy} dt = \int_0^\infty e^{-t} t^{x-1} e^{iy\ log(t)} dt \\ &= \int_0^\infty e^{-t} t^{x-1} [cos(y\ log(t)) + i\ sin(y\ log(t))] dt. \end{aligned}$$

The expression in the square brackets is bounded for all t; converges at infinity is provided by e^{-t}, and for the convergence at $t = 0$ we must have $x = Re(z) > 1$.
One of the basic properties of the Gamma function is that it satisfies the following functional equation

$$\Gamma(z+1) = z\Gamma(z),$$

which can be easily proved by integrating by parts

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z dt = [-e^{-t} t^z]_{t=0}^{t=\infty} + z \int_0^\infty e^{-t} t^{z-1} dt = z\Gamma(z).$$

Obviously, $\Gamma(1) = 1$, for $z = 1, 2, 3, ...$, we have

$$\Gamma(n+1) = n.\Gamma(n) = n.(n-1)! = n!.$$

The Gamma function can be represented also by the limit

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)...(z+n)}$$

where we initially suppose $Re(z) > 0$. This condition can be weakened to $z \neq 0, -1, -2, ...$ in the following manner.

If $-m < Re(z) \leq -m+1$, where m is a positive integer, then

$$\begin{aligned}
\Gamma(z) &= \frac{\Gamma(z+m)}{z(z+1)...(z+m-1)} \\
&= \frac{1}{z(z+1)...(z+m-1)} \lim_{n\to\infty} \frac{n^{z+m}n!}{(z+m)...(z+m+n)} \\
&= \frac{1}{z(z+1)...(z+m-1)} \lim_{n\to\infty} \frac{(n-m)^{z+m}(n-m)!}{(z+m)(z+m+1)...(z+n)} \\
&= \lim_{n\to\infty} \frac{n^z n!}{z(z+1)...(z+n)}.
\end{aligned}$$

Therefore, the limit representation holds for all z excluding $z \neq 0, -1, -2, ...$, see [54].

1.4.2 Grünwald-Letnikov Fractional Derivatives

Let us now return to the name of the fractional calculus. It does not mean the calculus of fractions. Neither does it mean a fraction of any calculus-differential, integral or calculus of variations. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unify and generalize the notions of integer-order differentiation and n-fold integration.

Let us consider the infinite sequence of n-fold integrals and n-fold derivatives

$$\ldots, \int_a^t d\tau_2 \int_a^{\tau_2} f(\tau_1)d\tau_1,\ \int_a^t f(\tau_1)d\tau_1,\ f(t),\ \frac{df(t)}{dt},\ \frac{d^2f(t)}{dt^2}, \ldots$$

The derivative of arbitrary real order α can be consider as an interpolation of this sequence of operators; we will use for it the notation suggested and used by Davis [21], namely ${}_aD_t^{\alpha}f(t)$. The short name for derivative of arbitrary order is fractional derivatives. The words fractional integrals mean integrales of arbitrary order and correspond to negative values of α, we will denote the fractional integral of order $\beta > 0$ by ${}_aD_t^{-\beta}f(t)$.

A fractional differential equation is an equation which contains fractional derivatives; a fractional integral equation is an integral equation containing fractional integrals. Now we describe the unification of integer-order derivatives and integrals.

Let us consider a continuous function $y = f(t)$, we have its first order derivative

$$f'(t) = \frac{df}{dt} = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h}.$$

Applying this definition twice gives the second-order derivative

$$\begin{aligned} f''(t) = \frac{d^2f}{dt^2} &= \lim_{h\to 0} \frac{f'(t) - f'(t-h)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\{\frac{f(t) - f(t-h)}{h} - \frac{f(t-h) - f(t-2h)}{h}\} \\ &= \lim_{h\to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}. \end{aligned}$$

By induction

$$f^{(n)}(t) = \frac{d^nf}{dt^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t - rh), \tag{1.1}$$

where

$$\binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r!}.$$

Let us now consider the following expression generalizing the fractions in (1.1)

$$f_h^{(p)}(t) = \frac{1}{h^p}\sum_{r=0}^{n}(-1)^r\binom{p}{r}f(t-rh), \tag{1.2}$$

where p is an arbitrary integer number; n is also integer, as above. Obviously, for $p \leq r$ we have

$$\lim_{h\to 0} f_h^{(p)}(t) = f^{(p)}(t) = \frac{d^p f}{dt^p},$$

because all the coefficients in the numerator after $\binom{p}{p}$ are equal to 0. Let us consider negative values of p. For convenience, let us denote

$$\left[{p \atop r}\right] = \frac{p(p+1)...(p+r-1)}{r!}$$

Then we have

$$\binom{-p}{r} = \frac{-p(-p-1)...(-p-r+1)}{r!} = (-1)^r\left[{p \atop r}\right]$$

and replacing p in (1.2) with $-p$, we can write

$$f_h^{(-p)}(t) = \frac{1}{h^p}\sum_{r=0}^{n}\left[{p \atop r}\right]f(t-rh),$$

where p is a positive integer number. We can take $h = \frac{t-a}{n}$, and consider the limit value, either finite or infinite, of $f_h^{(-p)}(t)$, which we will denote as

$$\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-p)}(t) =_a D_t^{-p} f(t).$$

For $p = 1$, we have

$$f_h^{(-1)}(t) = h \sum_{r=0}^{n} f(t - rh).$$

Taking into account that $t - nh = a$ and the function $f(t)$ is assumed to be continuous, we conclude that

$$\lim_{\substack{h \to 0 \\ nh = t-a}} f_h^{(-1)}(t) =_a D_t^{-1} f(t) = \int_0^{t-a} f(t - z) dz = \int_a^t f(\tau) d\tau. \qquad (1.3)$$

For $p = 2$, in this case

$$\left[\begin{smallmatrix} 2 \\ r \end{smallmatrix}\right] = \frac{2.3. \ldots .(2 + r - 1)}{r!} = r + 1$$

and we have

$$f_h^{(-2)}(t) = h \sum_{r=0}^{n} (rh) f(t - rh).$$

Denoting $t + h = y$, we can write

$$f_h^{(-2)}(t) = h \sum_{r=1}^{n+1} (rh) f(t - rh),$$

and taking $h \to 0$, we obtain

$$\lim_{\substack{h \to 0 \\ nh = t-a}} f_h^{(-2)}(t) =_a D_t^{-2} f(t) = \int_0^{t-a} z f(t - z) dz = \int_a^t (t - \tau) f(\tau) d\tau. \qquad (1.4)$$

For $p = 3$, taking into account that

$$\left[\begin{smallmatrix} 3 \\ r \end{smallmatrix}\right] = \frac{3.4. \ldots .(3 + r - 1)}{r!} = \frac{(r + 1)(r + 2)}{1.2},$$

we have

$$f_h^{(-3)}(t) = \frac{h}{1.2} \sum_{r=0}^{n} (r + 1)(r + 2) h^2 f(t - rh).$$

Denoting, as above, $t + h = y$, we write

$$f_h^{(-3)}(t) = \frac{h}{1.2} \sum_{r=1}^{n+1} r(r+1)h^2 f(y - rh),$$

which can be written as

$$f_h^{(-3)}(t) = \frac{h}{1.2} \sum_{r=1}^{n+1} (rh)^2 f(y - rh) + \frac{h^2}{1.2} \sum_{r=1}^{n+1} rhf(y - rh).$$

Sinse

$$\lim_{\substack{h\to 0\\ nh=t-a}} \frac{h^2}{1.2} \sum_{r=1}^{n+1} rhf(y - rh) = \lim_{\substack{h\to 0\\ nh=t-a}} h \int_a^t (t-\tau) f(\tau) d\tau = 0.$$

Taking now $h \to 0$, we obtain

$$_aD_t^{-3} f(t) = \frac{1}{2!} \int_0^{t-a} z^2 f(t-z) dz = \frac{1}{2!} \int_a^t (t-\tau)^2 f(\tau) d\tau. \qquad (1.5)$$

Relationships (1.3)-(1.5) suggest the following general expression

$$_aD_t^{-p} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \left[{p \atop r} \right] f(t - rh) = \frac{1}{(p-1)!} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau. \qquad (1.6)$$

Now let us show that formula (1.6) is a representation of a p-fold integral. We have

$$\frac{d}{dt}({}_aD_t^{-p} f(t)) = \frac{1}{(p-2)!} \int_a^t (t-\tau)^{p-2} f(\tau) d\tau =_a D_t^{-p+1} f(t).$$

Integrating the relationship from a to t, we obtain

$$_aD_t^{-p} f(t) = \int_a^t ({}_aD_t^{-p+1} f(t)) dt,$$

$$_aD_t^{-p+1} f(t) = \int_a^t ({}_aD_t^{-p+2} f(t)) dt, \ etc.,$$

and therefore

$$
\begin{aligned}
{}_aD_t^{-p}f(t) &= \int_a^t dt \int_a^t ({}_aD_t^{-p+2}f(t))dt \\
&= \int_a^t dt \int_a^t dt \int_a^t ({}_aD_t^{-p+3}f(t))dt \\
&= \underbrace{\int_a^t dt \int_a^t dt ... \int_a^t}_{p\ times} f(t)dt.
\end{aligned}
$$

We see that the derivative of an integer order n (1.1) and the p-fold integral (1.6) of the continuous function $f(t)$ are particular cases of the general expression

$$ {}_aD_t^p f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh), \tag{1.7} $$

which represents the derivative of order m if $p = m$ and the m-fold integral if $p = -m$. This observation naturally leads to the idea of a generalization of the notions of differentiation and integration by allowing p in (1.7) to be an arbitrary real or even complex number.

The evaluation of the derivative of integer order n of the fractional derivative of order p is given by

$$ \frac{d^n}{dt^n}({}_aD_t^p f(t)) =_a D_t^p(\frac{d^n f(t)}{dt^n}) =_a D_t^{p+n} f(t), \tag{1.8} $$

only if at the lower terminal $t = a$ of the fractional differentiation we have

$$ f^{(k)}(a) = 0, \quad k = 0, 1, ..., n-1. \tag{1.9} $$

Let us now consider the fractional derivative of order q of a fractional derivative of order p

$$ {}_aD_t^q({}_aD_t^p f(t)). $$

Two cases will be considered separately: $p < 0$ and $p > 0$. In both cases we will obtain an analogue of the well-known property of integer-order differentiation

$$\frac{d^n}{dt^n}(\frac{d^m f(t)}{dt^m}) = \frac{d^m}{dt^m}(\frac{d^n f(t)}{dt^n}) = \frac{d^{m+n} f(t)}{dt^{m+n}}.$$

Case $p < 0$:

Let us first take $q < 0$, then we have

$$\begin{aligned}
{}_aD_t^q({}_aD_t^p f(t)) &= \frac{1}{\Gamma(-q)}\int_a^t (t-\tau)^{-q-1}({}_aD_\tau^p f(\tau))d\tau \\
&= \frac{1}{\Gamma(-q)\Gamma(-p)}\int_a^t (t-\tau)^{-q-1}d\tau \int_a^\tau (\tau-\xi)^{-p-1} f(\xi)d\xi \\
&= \frac{1}{\Gamma(-q)\Gamma(-p)}\int_a^t f(\xi)d\xi \int_\xi^t (t-\tau)^{-q-1}(\tau-\xi)^{-p-1}d\tau \\
&= \frac{1}{\Gamma(-p-q)}\int_a^t (t-\xi)^{-p-q-1} f(\xi)d\xi \\
&= {}_aD_t^{p+q} f(t), \qquad (1.10)
\end{aligned}$$

where the integral

$$\begin{aligned}
\int_\xi^t (t-\tau)^{-q-1}(\tau-\xi)^{-p-1}d\tau &= (t-\xi)^{-p-q-1}\int_0^1 (1-z)^{-q-1} z^{-p-1} dz \\
&= \frac{\Gamma(-q)\Gamma(-p)}{\Gamma(-p-q)}(t-\xi)^{-p-q-1}
\end{aligned}$$

is evaluated with the help of the substitution $\tau = \xi + z(t-\xi)$ and the definition of the beta function.

Let us now suppose that $0 < n < q < n+1$. Noting that $q = (n+1) + (q-n-1)$, where $q-n-1 < 0$, using the formulas (1.8) and (1.10), we obtain

$${}_aD_t^q({}_aD_t^p f(t)) = \frac{d^{n+1}}{dt^{n+1}}\{{}_aD_t^{q-n-1}({}_aD_t^p f(t))\}$$

$$= \frac{d^{n+1}}{dt^{n+1}}\{{}_aD_t^{p+q-n-1}f(t)\}$$
$$= {}_aD_t^{p+q}f(t).$$

Combining with (1.10), we coclude that if $p<0$, then for any real q

$${}_aD_t^q({}_aD_t^pf(t)) =_a D_t^{p+q}f(t). \quad (1.11)$$

Case $p>0$:

Let us assume that $0 \le m < p < m+1$, take $q<0$, we get

$$\begin{aligned}{}_aD_t^q({}_aD_t^pf(t)) &= {}_aD_t^q({}_aD_t^{p-m-1}f^{(m+1)}(t)) =_a D_t^{p+q-m-1}f^{(m+1)}(t)\\ &= \frac{1}{\Gamma(-p-q+m+1)}\int_a^t \frac{f^{(m+1)}(\tau)d\tau}{(t-\tau)^{p+q-m}}.\end{aligned}$$

Taking into account the condition $f^k(a)=0,\ k=0,1,...,m-1$, we arrive at (1.11).

Finally, let use take $0 \le n < q < n+1$. Assuming that the late formula with its condition is satisfies and taking into account that $q-n-1<0$, we obtain

$$\begin{aligned}{}_aD_t^q({}_aD_t^pf(t)) &= \frac{d^{n+1}}{dt^{n+1}}\{{}_aD_t^{q-n-1}({}_aD_t^pf(t))\}\\ &= \frac{d^{n+1}}{dt^{n+1}}\{{}_aD_t^{p+q-n-1}f(t)\}\\ &= {}_aD_t^{p+q}f(t).\end{aligned}$$

Therefore, we can conclude that if $0 \le m < p < m+1$, then the relationship (1.11) holds also for arbitrary real q, when $f^{(k)}(a)=0, \quad k=0,1,...,m-1$.

Moreover, if $0 \le m < p < m+1$ and $0 \le n < q < n+1$ and the function

$f(t)$ satisfies $f^k(a) = 0,\ k = 0, 1, ..., r-1$, where $r = max(n, m)$, then the operators of fractional differentiation ${}_aD_t^p$ and ${}_aD_t^q$ commute:

$$ {}_aD_t^q({}_aD_t^p f(t)) = {}_aD_t^p({}_aD_t^q f(t)) = {}_aD_t^{p+q} f(t). $$

See [54].

1.4.3 Riemann-Liouville Fractional Derivatives

The Riemann-Liouville fractional derivative is defined as

$$ {}_aD_t^p f(t) = (\frac{d}{dt})^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau) d\tau, \quad m \le p < m+1. $$

Where the function $f(t)$ must be $m+1$ times continuously differentiable. For $t \ge 0$, this formula is equivalent to

$$ \begin{aligned} {}_aD_t^p f(t) &= (\frac{d}{dt})^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau) d\tau \\ &= \sum_{k=0}^{m} \frac{f^k(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau) d\tau. \end{aligned} $$

Let us suppose that the function $f(\tau)$ is continuous and integrable in every finite interval (a, t); the function $f(t)$ may have an integrable singularity of order $r < 1$ at the point $\tau = a$:

$$ \lim_{\tau \to a} (\tau - a)^r f(t) = const\ (\neq 0). $$

The integral

$$ f^{(-1)}(t) = \int_a^t f(\tau) d\tau $$

exists and has a finite value, namely equal to 0, as $t \to a$. Indeed, performing the substitution $\tau = a + y(t-a)$ and then denoting $\epsilon = t - a$, we obtain

$$ \lim_{t \to a} f^{(-1)}(t) = \lim_{t \to a} \int_a^t f(\tau) d\tau $$

$$\begin{aligned} &= \lim_{t\to a}(t-a)\int_0^1 f(a+y(t-a))dy \\ &= \lim_{\epsilon\to 0}\epsilon^{1-r}\int_0^1 (\epsilon y)^r f(a+y\epsilon)y^{-r}dy = 0, \end{aligned}$$

because $r < 1$. Therefore, we can consider the two-fold integral

$$\begin{aligned} f^{(-2)}(t) &= \int_a^t d\tau_1 \int_a^{\tau_1} f(\tau)d\tau = \int_a^t f(\tau)d\tau \int_\tau^t d\tau_1 \\ &= \int_a^t (t-\tau)f(\tau)d\tau. \end{aligned}$$

By integration, this gives the three-fold integral of $f(\tau)$

$$\begin{aligned} f^{(-3)}(t) &= \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \int_a^{\tau_2} f(\tau_3)d\tau_3 \\ &= \int_a^t d\tau_1 \int_a^{\tau_1} (\tau_1-\tau)f(\tau)d\tau \\ &= \frac{1}{2}\int_a^t (t-\tau)^2 f(\tau)d\tau, \end{aligned}$$

and by induction in the general case we have the Cauchy formula

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)}\int_a^t (t-\tau)^{n-1}f(\tau)d\tau. \tag{1.12}$$

Let us now suppose that $n \geq 1$ is fixed and take integer $k \geq 0$. Obviously, we will obtain

$$f^{(-k-n)}(t) = \frac{1}{\Gamma(n)}D^{-k}\int_a^t (t-\tau)^{n-1}f(\tau)d\tau,$$

where the symbol D^{-k} denotes k iterated integrations.

On the other hand, for a fixed $n \geq 1$ and integer $k \geq n$ the $(k-n)$-th derivative of the function $f(t)$ can be written as

$$f^{(k-n)}(t) = \frac{1}{\Gamma(n)}D^{k}\int_a^t (t-\tau)^{n-1}f(\tau)d\tau,$$

where the symbol D^k denotes k iterated differentiations.

The notion of n-fold integration is extended to non-integer values of n, the integer n in the Cauchy formula (1.12) is replaced by a real $p > 0$:

$$ {}_aD_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau. \tag{1.13} $$

In (1.12) the integer n must satisfy the condition $n \geq 1$; the corresponding condition for p is weaker: for the existence of the integral (1.13) we must have $p > 0$, is called Riemann-Liouville fractional integral of the function $f(t)$, which denoted also $I_a^p f(t)$. From (1.13), the Riemann-Liouville fractional derivative of the function $f(t)$ is defined as

$$ {}_aD_t^p f(t) = \frac{d^n}{dt^n}({}_aD_t^{-(n-p)} f(t)) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-p-1} f(\tau) d\tau, $$

where f is a continuous function and $n - 1 \leq p < n$.

The most important property of the Riemann-Liouville fractional derivative is that for $p > 0$ and $t > a$

$$ {}_aD_t^p({}_aD_t^{-p} f(t)) = f(t). $$

If the fractional derivative ${}_aD_t^p f(t)$, $(k - 1 \leq p < k)$, of a function $f(t)$ is integrable, then

$$ {}_aD_t^{-p}({}_aD_t^p f(t)) = f(t) - \sum_{j=1}^{k} [{}_aD_t^{p-j} f(t)]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(p-j+1)}. $$

The definition of the fractional differentiation of the Riemann-Liouville type played an important role in the development of the theory of fractional derivatives and integrals and for its applications in pure mathematics, see

[54].

1.4.4 Caputo's Fractional Derivative

Caputo's definition can be written as

$$ {}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha+1-n}}, \quad n-1 < \alpha < n. $$

Under natural conditions on the function $f(t)$, for $\alpha \rightarrow n$ the Caputo derivative becomes a conventional n-th derivative of the function $f(t)$.

The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the some from as for integer-order differential equations, i.e. contain the limit values of integer-order derivatives of unknown functions at the lower terminal $t = a$.

To underline the difference in the form of the initial conditions which must accompany fractional differential equations in terms of the Riemann-Liouville and the Caputo derivative, let us recall the corresponding Laplace transform fomulas for the case $a = 0$. The formula for the Laplace transform of the Riemann-Liouville fractional derivative is

$$ \int_0^{\infty} e^{-pt} \{ {}_0 D_t^{\alpha} f(t) \} dt = p^{\alpha} F(p) - \sum_{k=0}^{n-1} p^k \, {}_0 D_t^{\alpha-k-1} f(t)|_{t=0}, \; n-1 \leq \alpha < n, $$

whereas Caputo's formula, first obtained in [15], for the Laplace transform of the Caputo derivative is

$$ \int_0^{\infty} e^{-pt} \{ {}_0^C D_t^{\alpha} f(t) \} dt = p^{\alpha} F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1} f^{(k)}(0), \; n-1 < \alpha \leq n. $$

We see that the Laplace transform of the Riemann-Liouville fractional derivative allows utilizations of initial conditions of the type $\lim_{t\to a} {}_aD_t^{\alpha-k}f(t) = b_k$, (where $b_k,\ k = 1, 2, ..., n$ are given constant), which can cause problems with their physical interpretation. On the contrary, the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivatives with known physical interpretations.

Another difference is that the Caputo derivative of a constant is 0, whereas in the cases of a finite value of the lower terminal a, the Riemann-Liouville fractional derivative of a constant C is not equal to 0, but ${}_0D_t^{\alpha}C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)}$, see [54].

1.4.5 Generalized Function Approach

This approach is based on the observation that the Cauchy formula

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)}\int_a^t (t-\tau)^{n-1}f(\tau)d\tau,$$

which allows replacement of n-fold integral of the function $f(t)$ with a single integration, can be written as a convolution of the function $f(t)$ and the power function t^{n-1}:

$$f^{(-n)}(t) = f(t) * \frac{t^{n-1}}{\Gamma(n)}, \tag{1.14}$$

where both functions, $f(t)$ and t^{n-1}, are replaced with zero for $t < a$ and $t < 0$ correspondingly; the asterisk means the convolution:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau.$$

Let us consider the function $\Phi_p(t)$ defined by [34]

$$\Phi_p(t) = \{ {}^{\frac{t^{p-1}}{\Gamma(\gamma)},\quad t>0}_{\;0,\quad t\leq 0.}$$

Using the function $\Phi_p(t)$, the formula (1.14) can be considered as a particular case of the more general convolution of the function $f(t)$ and the function $\Phi_p(t)$:

$$f^{(-p)}(t) = f(t) * \Phi_p(t). \tag{1.15}$$

To handle both positive and negative values of p in the same way, it is convenient to consider the function $\Phi_p(t)$ as a generalized function. Its properties are known [34]; for our purposes it is essential that

$$\lim_{p\to -k} \Phi_p(t) = \Phi_{-k}(t)\delta^{(k)}(t), \quad k = 0, 1, ..., \tag{1.16}$$

where $\delta(t)$ is the Dirac delta function. The Dirac delta function is often used in applied problems for the description of impulse loading (impulse forces). The convolution of the k-th derivative of the delta function and $f(t)$ is given by

$$\int_{-\infty}^{\infty} f(\tau)\delta^{(k)}(t-\tau)d\tau = f^{(k)}(t).$$

Obviously, if p is a positive integer ($p = n$), then the formula (1.15) reduces to (1.14). On the other hand, it follows from the relationship (1.16) and the properties of the delta function that for negative integer values of ($p = -n,\ n > 0$)

$$f^{(k)}(t) = f(t) * \Phi_{-k}(t) = f(t) * \delta^{(k)}(t).$$

Therefore, both integer-order integrals and derivatives of a generalized function $f(t)$ can be obtained as particular cases of the convolution (1.15), which is also meaningful for non-integer values of p. This means that the formula (1.15) provides a unification of n-fold integrals and n-th order derivatives of a generalized function and an extention of these notion to real order p and that we can define the derivative of real order p of a generalized function $f(t)$, which is equal to zero for $t < a$, as

$$_a\tilde{D}_t^p f(t) = f(t) * \Phi_p(t).$$

Another property of the function $\Phi_p(t)$, which leads to important consequences, is

$$\Phi_p(t-a) * \Phi_q(t) = \Phi_{p+q}(t-a) \tag{1.17}$$

To prove (1.17), let us first suppose that $p > 0$ and $q > 0$. Then using the substitution $\tau = a + \zeta(t-a)$ and the definition of the beta function, we obtain

$$\begin{aligned}
\Phi_p(t-a) * \Phi_q(t) &= \int_a^t \frac{(\tau-a)^{p-1}}{\Gamma(p)} \frac{(t-\tau)^{q-1}}{\Gamma(q)} d\tau \\
&= \frac{1}{\Gamma(p)\Gamma(q)} \int_0^t (\tau-a)^{p-1}(t-\tau)^{q-1} d\tau \\
&= \frac{(t-a)^{p+q-1}}{\Gamma(p)\Gamma(q)} \int_0^1 \zeta^{p-1}(t-\zeta)^{q-1} d\zeta \\
&= \frac{(t-a)^{p+q-1}}{\Gamma(p+q)},
\end{aligned}$$

and analytic continuation with respect to p and q gives (1.17). It follows that if the function $f(t)$ is zero for $t < a$, then

$$(f(t) * \Phi_p(t)) * \Phi_q(t) = f(t) * (\Phi_p(t) * \Phi_q(t)) = f(t) * \Phi_{p+q}(t),$$

from which immediately follows the composition law

$${}_a\tilde{D}_t^p({}_a\tilde{D}_t^q f(t)) =_a \tilde{D}_t^q({}_a\tilde{D}_t^p f(t)) =_a \tilde{D}_t^{p+q} f(t),$$

for all p and q. The simplicity of the composition law is a great advantage of the use of generalized functions.

From formula (1.17), we directly obtain the derivative of real order p of the generalized function

$$\Phi_{q+1}(t) = \frac{t_+^q}{\Gamma(q+1)} = \{ \begin{smallmatrix} \frac{t^q}{\Gamma(q+1)}, & t>0 \\ 0, & t\leq 0 \end{smallmatrix}$$

in the form

$${}_a\tilde{D}_t^p(\frac{(t-a)^q}{\Gamma(q+1)}) = \frac{(t-a)^{p-q}}{\Gamma(1+q-p)}, \quad t > a. \qquad (1.18)$$

In the particular case $q = 0$, we obtain the fractional derivative of the Heaviside unit-step function $H(t)$:

$${}_a\tilde{D}_t^p H(t-a) = \frac{(t-a)^{-p}}{\Gamma(1-p)}, \quad t > a,$$

and, in general, for all $b < a$

$${}_b\tilde{D}_t^p H(t-a) = \{ \begin{smallmatrix} \frac{(t-a)^{-p}}{\Gamma(1-p)}, & t>a \\ 0, & b\leq t\leq a \end{smallmatrix}.$$

Putting $q = -n-1$ ($n \geq 0$) in (1.18), we obtain the fractional derivative of order p of the n-th derivative of the Dirac delta function:

$${}_a\tilde{D}_t^p \delta^{(n)}(t-a) = \frac{(t-a)^{-n-p-1}}{\Gamma(-n-p)}, \quad t > a,$$

and, in general, for all $b < a$

$$ {}_b\tilde{D}_t^p \delta^{(n)}(t-a) = \{ \begin{smallmatrix} \frac{(t-a)^{-n-p-1}}{\Gamma(-n-p)}, & t>a \\ 0, & b\leq t\leq a. \end{smallmatrix} $$

Finally, if $q - p + 1 = -n$ $(n \geq 0)$, then from formula (1.18) it follows that

$$ {}_a\tilde{D}_t^p (\frac{(t-a)^{p-n-1}}{\Gamma(p-1)}) = \delta^{(n)}(t-a), \ t > a. $$

Using the function $\Phi_p(t)$, the Riemann-Liouville definition can be written as

$$ {}_aD_t^p f(t) = \frac{d^n}{dt^n}(f(t) * \Phi_{n-p}(t)), $$

the Caputo definition can be written as

$$ {}_a^CD_t^p f(t) = (\frac{d^n f(t)}{dt^n} * \Phi_{n-p}(t)). $$

See [54].

1.4.6 Laplace Transforms of Fractional Derivatives

Let us recall some basic facts about the Laplace transform. The function $F(s)$ of the complex variable s defined by

$$ F(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt \tag{1.19} $$

is called the Laplace transform of the function $f(t)$, which is called the original. For the existence of the integral (1.19) the function $f(t)$ must be of exponential order α, which means that there exist positive constants M and T such that $e^{-\alpha t}|f(t)| \leq M$, for all $t > T$.

In other words, the function $f(t)$ must not grow faster then a certain exponential function when $t \to \infty$. The original $f(t)$ can be restored from the

Laplace transform $F(s)$ with the help of the inverse Laplace transform

$$f(t) = L^{-1}\{F(s);t\} = \int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \quad c = Re(s) > c_0,$$

where c_0 lies in the right half plane of the absolute convergence of the Laplace integral (1.19).

The Laplace transform of the convolution

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

of the two function $f(t)$ and $g(t)$, which are equal to zero for $t < 0$, is equal to the product of the Laplace transform of those functions:

$$L\{f(t) * g(t); s\} = F(s)G(s) \tag{1.20}$$

under the assumption that both $F(s)$ and $G(s)$ exist.

Another useful property which we need is the formula for the Laplace transform of the derivative of an integer order n of the function $f(t)$:

$$L\{f^{(n)}(t);s\} = s^nF(s) - \sum_{k=0}^{n-1} s^{n-k-1}f^{(k)}(0) = s^nF(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0) \tag{1.21}$$

which can be obtained from (1.19) by integrating by parts under the assumption that the corresponding integrals exist.

We can write as a convolution of the functions $g(t) = t^{p-1}$ and $f(t)$:

$${}_0D_t^{-p}f(t) = \frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}f(\tau)d\tau = t^{p-1} * f(t).$$

The Laplace transform of the function t^{p-1} is

$$G(s) = L\{t^{p-1};s\} = \Gamma(p)s^{-p}.$$

Therefore, using (1.20), we obtain the Laplace transform of the Riemann-Liouville and the Grünwald-Letnikov fractional integral:

$$L\{{}_0D_t^{-p}f(t);s\} = s^{-p}F(s). \tag{1.22}$$

Let us now turn to the evaluation of the Laplace transform of the Riemann-Liouville fractional derivative, which for this purpose, for $n-1 \leq p < n$, we write in the form

$$ {}_0D_t^{p}f(t) = g^{(n)}(t),$$
$$g(t) ={}_0 D_t^{-(n-p)}f(t)\frac{1}{\Gamma(k-p)}\int_0^t (t-\tau)^{n-p-1}f(\tau)d\tau.$$

The formula (1.21) leads to

$$L\{{}_0D_t^{p}f(t);s\} = s^{n}G(s) - \sum_{k=0}^{n-1} s^k g^{(n-k-1)}(0).$$

The Laplace transform of the function $g(t)$ is evaluated as

$$G(s) = s^{-(n-p)}F(s).$$

Also, we have

$$g^{(n-k-1)}(t) = \frac{d^{n-k-1}}{dt^{n-k-1}}\,{}_0D_t^{-(n-p)}f(t) ={}_0 D_t^{p-k-1}f(t).$$

Thus, the Laplace transform of the Riemann-Liouville fractional derivative of order $p > 0$:

$$L\{{}_0D_t^{p}f(t);s\} = s^{p}F(s) - \sum_{k=0}^{n-1} s^k[{}_0D_t^{p-k-1}f(t)]_{t=0}, \quad n-1 \leq p < n.$$

To establish the Laplace transform formula for the Caputo fractional derivative, let us write

$${}_0^CD_t^{p}f(t) = \ {}_0D_t^{-(n-p)}g(t), \quad g(t) = f^{(n)}(t), \quad n-1 < p \leq n.$$

Using (1.22) for the Laplace transform of the Riemann-Liouville fractional integral gives

$$L\{{}_0^C D_t^p f(t); s\} = s^{-(n-p)} G(s),$$

where

$$G(s) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0).$$

Using the above formulas, we arrive at the Laplace transform formula for the Caputo fractional derivative

$$L\{{}_0^C D_t^p f(t); s\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^k(0), \quad n-1 < p \leq n.$$

See [54].

Chapter 2

Weak Almost Periodic and Optimal Mild Solutions of Fractional Evolution Equations

2.1 Introduction

The object of this chapter is to study the fractional evolution equation

$$\frac{d^\alpha u(t)}{dt^\alpha} + (A - B(t))u(t) = f(t), \quad t > t_0 \tag{2.1}$$

in a Banach space X, where $0 < \alpha \leq 1$, u is an X-valued function on $R^+ = [0, \infty)$ and f is a given abstract function on R^+ with values in X. We assume that $-A$ is a linear closed operator defined on a dense set S in X into X, $\{B(t) : t \in R^+\}$ is a family of linear bounded operators defined on X into X .

It is assumed that $-A$ generates an analytic semigroup $Q(t)$ such that $\|Q(t)\| \leq M$ for all $t \in R^+, Q(t)h \in S, \|AQ(t)h\| \leq \frac{M}{t}\|h\|$ for every $h \in X$ and all $t \in (0, \infty)$.

Let X be a uniformly convex Banach space equipped with a norm $\|.\|$ and X^* its topological dual space. N'Guerekata [49] gave necessary conditions to ensure that the so-called optimal mild solutions of $u'(t) = Au(t) + f(t)$ are weakly almost periodic.

Following Gelfand and Shilov [34], we define the fractional integral of order $\alpha > 0$ as

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

also, the fractional derivative of the function f of order $0 < \alpha < 1$ as

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(s)(t-s)^{-\alpha} ds,$$

where f is an abstract continuous function on the interval $[a, b]$ and $\Gamma(\alpha)$ is the Gamma function, see [47, 54].

Definition 2.1

By a solution of (2.1), we mean a function u with values in X such that:

(1) u is continuous function on R^+ and $u(t) \in D(A)$,

(2) $\frac{d^\alpha u}{dt^\alpha}$ exists and continuous on $(0, \infty)$, $0 < \alpha < 1$, and u satisfies (2.1) on $(0, \infty)$.

We shall first obtain the Green function for the fractional abstract differential equation

$$\frac{d^\alpha u(t)}{dt^\alpha} = Au(t), \tag{I}$$

where

$$u(0) = u_0 \in S. \tag{II}$$

We apply the fractional integral operator on both sides of (I) and using (II), it is easy to get

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{Au(\theta)}{(t-\theta)^{1-\alpha}} d\theta. \tag{III}$$

Theorem If A has an analytic semigroup, then the Cauchy problem (I), (II) has the unique solution

$$u(t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) u_0 d\theta,$$

where $\zeta_\alpha(\theta)$ is a probability density function defined on $(0, \infty)$. The Laplace transform of ζ_α is given by

$$\int_0^\infty e^{-pt} \zeta_\alpha(t) dt = F_\alpha(p) = \sum_{j=0}^\infty \frac{(-p)^j}{\Gamma(1 + j\alpha)}, \quad 0 < \alpha < 1.$$

Proof Applying the Laplace transform

$$v(p) = \int_0^\infty e^{-pt} u(t) dt, \quad p > 0,$$

to (III) yields

$$v(p) = \frac{1}{p} u_0 + \frac{1}{p^\alpha} A v(p) = (p^\alpha I - A)^{-1} p^{\alpha - 1} u_0 = p^{\alpha - 1} \int_0^\infty e^{-p^\alpha \theta} Q(\theta) u_0 d\theta,$$

where I is identity operator defined on X. Since $\|Q(t)\| \leq K$, it follows that the integral in the late formula is absolutely convergent for all $p > 0$. Consider the one-sides stable probability density $\rho_\alpha(t)$, whose Laplace transform is given by

$$\int_0^\infty \rho_\alpha(t) e^{-pt} dt = e^{-p\alpha},$$

where $0 < \alpha < 1$. Using the two late formulas, we get

$$v(p) = \int_0^\infty e^{-pt} [\int_0^\infty \rho_\alpha(\theta) Q(\frac{t^\alpha}{\theta^\alpha}) u_0 d\theta] dt.$$

Now we can invert the last Laplace transform, to get

$$u(t) = \int_0^\infty \rho_\alpha(\theta) Q(\frac{t^\alpha}{\theta^\alpha}) u_0 d\theta = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) u_0 d\theta,$$

where

$$\zeta_\alpha(t) = \frac{1}{\alpha} t^{-1-1/\alpha} \rho_\alpha(t^{-1/\alpha}).$$

Thus, we can deduce that the solution u exists if even $u_0 \in X$.

Let us solve the following Cauchy problem

$$\frac{d^\alpha u(t)}{dt^\alpha} = Au(t) + f(t), \qquad (IV)$$

$$u(0) = u_0 \in S, \qquad (V)$$

where f is an abstract function defined on $[0, \infty]$ and with values in E, see [25].

Theorem If f satisfies a uniform Hölder condition, with exponent $\beta \in (0, 1]$, then the unique solution of the Cauchy problem (IV), (V) is given by

$$u(t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) u_0 d\theta + F(t),$$

where

$$F(t) = \alpha \int_0^t \int_0^\infty \theta (t - \eta)^{\alpha - 1} \zeta_\alpha(\theta) Q((t - \eta)^\alpha \theta) f(\eta) d\theta d\eta.$$

Proof If v and g are the Laplace transform of u and f, respectively, then

$$v(p) = p^{\alpha-1}(p^\alpha I - A)^{-1} u_0 + (p^\alpha I - A)^{-1} g(p).$$

Consequently, we get formally

$$u(t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) u_0 d\theta + F(t).$$

To prove that the last formula represents the unique solution of the Cauchy problem (IV), (V) it suffices to prove that $F(t) \in S$ for every $t \in [0, \infty)$. In other words, we must prove the existence of the double integral

$$H(t) = \alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \zeta_\alpha(\theta) A Q((t-\eta)^\alpha \theta) f(\eta) d\theta d\eta.$$

In fact, we notice that

$$\begin{aligned} H(t) &= \alpha \int_0^t \int_0^\infty \theta (t-\eta)^{\alpha-1} \zeta_\alpha(\theta) A Q((t-\eta)^\alpha \theta)[f(\eta) - f(t)] d\theta d\eta \\ &+ \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) f(t) d\theta - f(t). \end{aligned}$$

Since f satisfies a uniform Hölder condition with exponent $\beta \in (0, 1]$, it follows that

$$H(t) \leq K \frac{t^\beta}{\beta} + (K+1)\|f(t)\|.$$

Thus $F(t) \in S$ for each $t \in [0, \infty]$ and $H(t) = AF(t)$, see [25].

It is suitable to rewrite equation (2.1) in the form

$$u(t) = u(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} [(B(s) - A) u(s) + f(s)] ds. \tag{2.2}$$

According to [25-29], the equation (2.2) is equivalent to the integral equation

$$\begin{aligned} u(t) &= \int_0^\infty \zeta_\alpha(\theta) Q((t-t_0)^\alpha \theta) u(t_0) d\theta \\ &+ \alpha \int_{t_0}^t \int_0^\infty \theta (t-s)^{\alpha-1} \zeta_\alpha(\theta) Q((t-s)^\alpha \theta) F(s) d\theta ds, \end{aligned} \tag{2.3}$$

where $F(t) = B(t)u(t) + f(t)$ and ζ_α is a probability density function defined on $(0, \infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\theta x}\zeta_\alpha(\theta)d\theta = \sum_{j=0}^{\infty}\frac{(-x)^j}{\Gamma(1+\alpha j)}, 0 < \alpha \leq 1, x > 0,$$

A continuous solution of the integral equation (2.3) is called a mild solution of (2.1).

The theory of almost periodic functions with values in a Banach space was developed by Bohr, Bochner, von Neumann, and others, [1, 12]. See also [2, 20, 46, 49, 50, 62].

Definition 2.2

A function $f : R \longrightarrow X$ is called (Bochner) almost periodic if

(i) f is strongly continuous, and

(ii) for each $\epsilon > 0$ there exists $l(\epsilon) > 0$, such that every interval I of length $l(\epsilon)$ contains a number τ such that $\sup_{t\in R} \|f(t+\tau) - f(t)\| < \epsilon$.

2.2 Optimal Mild Solutions

As in N'Guerekata [49], let Ω_f denote the set of mild solutions $u(t)$ of (2.1) which are bounded over R^+; that is

$$\mu(u) = \sup_{t\in R^+} \|u(t)\| < \infty, \tag{2.4}$$

We assume here that $\Omega_f \neq \emptyset$, and recall that:

A bounded mild solution $\tilde{u}(t)$ of (2.1) is called optimal mild solution of (2.1) if

$$\mu(\tilde{u}) \equiv \mu^* = \inf_{u\in\Omega_f} \mu(u). \tag{2.5}$$

Theorem 2.1

Assume that $\Omega_f \neq \emptyset$ and $f : R^+ \longmapsto X$ is a nontrivial strongly continuous function, then (2.1) has a unique optimal mild solution.

Compare with [65, Theorem 1.1, p.138] and [49, Theorem 1, p.673]. Our proof is based on the following lemma.

Lemma 2.2

([40, Corollary 8.2.1]). If K is a non-empty convex and closed subset of a uniformly convex Banach space X and $v \notin K$, then there exists a unique $k_0 \in K$ such that $|v - k_0| = \inf_{k \in K} |v - k|$.

Proof of Theorem 2.1

It suffices to prove that Ω_f is a convex and closed set because the trivial solution $0 \notin \Omega_f$, then we use lemma 2.2 to deduce the uniqueness of the optimal mild solution, see [49].

For the convexity of Ω_f, we consider two distinct bounded mild solutions $u_1(t)$ and $u_2(t)$, and a real number $0 \leq \lambda \leq 1$ and let

$$u(t) = \lambda u_1(t) + (1 - \lambda)u_2(t), \quad t \in R.$$

For every $t_0 \in R$, $u(t)$ is continuous and (see[49]) has the integral representation

$$u(t) = T(t - t_0)u(t_0) + \int_{t_0}^{t} S(t - s)F(s)ds, \quad t \geq t_0, \tag{2.6}$$

where

$$T(t) = \int_0^\infty \zeta_\alpha(\theta)Q(t^\alpha\theta)d\theta \ , \ S(t) = \alpha \int_0^\infty \theta t^{\alpha-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)d\theta.$$

We have

$$u(t_0) = \lambda u_1(t_0) + (1-\lambda)u_2(t_0),$$

then $u(t)$ is a mild solution of (1.1). We note that $u(t)$ is bounded over R since

$$\mu(u) = \sup_{t\in R^+} \|u(t)\| \leq \lambda\mu(u_1) + (1-\lambda)\mu(u_2) < \infty,$$

we conclude that

$$u(t) \in \Omega_f.$$

Now we show that Ω_f is closed.

Let $u_n \in \Omega_f$ a sequence such that $\lim_{n\to\infty} u_n(t) = u(t), t \in R$. For all $t_0 \in R$ and $t \geq t_0$ we have

$$u_n(t) = T(t-t_0)u_n(t_0) + \int_{t_0}^{t} S(t-s)[B(s)u_n(s) + f(s)]ds, \tag{2.7}$$

It is clearly that $T(t-t_0)$ and $S(t-s)$ are continuous operators, then for every fixed t and t_0 with $t \geq t_0$, we have

$$\begin{aligned}
\lim_{n\to\infty} T(t-t_0)u_n(t_0) &= \lim_{n\to\infty}\int_0^\infty \zeta_\alpha(\theta)Q((t-t_0)^\alpha\theta)u_n(t_0)d\theta \\
&= \int_0^\infty \zeta_\alpha(\theta)Q((t-t_0)^\alpha\theta)d\theta \lim_{n\to\infty} u_n(t_0) \\
&= T(t-t_0)\lim_{n\to\infty} u_n(t_0) \\
&= T(t-t_0)u(t_0).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\lim_{n\to\infty}\int_{t_0}^{t} S(t-s)[B(s)u_n(s) + f(s)]ds &= \int_{t_0}^{t} S(t-s)[\lim_{n\to\infty} B(s)u_n(s) + f(s)]ds \\
&= \int_{t_0}^{t} S(t-s)F(s)ds.
\end{aligned}$$

Then we deduce that

$$u(t) = T(t-t_0)u(t_0) + \int_{t_0}^{t} S(t-s)F(s)ds,$$

for all $t_0 \in R, t \geq t_0$, which means that $u(t)$ is a mild solution of (2.1). Finally we show that $u(t)$ is bounded over R. We can write (2.6) as

$$\begin{aligned} u(t) &= T(t-t_0)u(t_0) + \int_{t_0}^{t} S(t-s)F(s)ds - u_n(t) + u_n(t) \\ &= T(t-t_0)[u(t_0) - u_n(t_0)] + \int_{t_0}^{t} S(t-s)(B(u-u_n))(s)ds + u_n(t), \end{aligned}$$

for $n = 1, 2, ...$, and every $t_0 \in R$ such that $t \geq t_0$.

Since $\int_0^\infty \zeta_\alpha(\theta)d\theta = 1$, it follows that $\|T(t)\| \leq M$,

again, since $\int_0^\infty \theta\zeta_\alpha(\theta)d\theta = 1$ (see [29, p.54]), it follows that $\|S(t)\| \leq \alpha M t^{\alpha-1}$.

Let $\|B\| \leq C$. These estimates lead to

$$\|u(t)\| \leq M\|u(t_0) - u_n(t_0)\| + \alpha MC \int_{t_0}^{t} (t-s)^{\alpha-1}\|u(s) - u_n(s)\|ds + \|u_n(t)\|.$$

Choose n large enough, for every $\epsilon_1, \epsilon_2 > 0$ we get

$$\mu(u) \leq \epsilon_1 + \epsilon_2 + \mu(u_n) < \infty.$$

Thus $u \in \Omega_f$. This completes the proof.

2.3 Weak Almost Periodic Solutions

To formulate a property of almost periodic functions, which is equivalent to Definition 2.2, we discuss the concept of normality of almost periodic functions. Namely, let $f(t)$ be almost periodic in $t \in R^+$, then for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that $f(t+s_n)$ is uniformly convergent in $t \in R^+$. see Hamaya [36, p.188]. It is well known [48, 49, 64, 65] that:

$f : R^+ \longrightarrow X$ is weakly almost periodic if for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that every $(f(t+s_n))$ is convergent in the weak sense, uniformly in $t \in R^+$. In other words, for every $u^* \in X^*$, the sequence $(< u^*, f(t+s_n) >)$ is uniformly convergent in $t \in R^+$, where $< .,. >$ denotes duality $< X^*, X >$. For each $Q(t), t \in R^+$, $Q^*(t)$ denotes the adjoint operator of $Q(t)$.

Theorem 2.3

Let $f : R^+ \longmapsto X$ be almost periodic and a nontrivial strongly continuous function, also assume that $f \in L^1(R)$ and $Q^*(t) \in L(X^*)$ for every $t \in R^+$, then the optimal mild solution of (2.1) is weakly almost periodic.

Proof

As in N'Guerekata [49], let $u(t)$ be the unique optimal mild solution of (2.1), by Theorem 2.1

$$u(t) = T(t-t_0)u(t_0) + \int_{t_0}^{t} S(t-s)F(s)ds,$$

for all $t_0 \in R, t \geq t_0$. Let (s'_n) be an arbitrary sequence of real numbers. Since f is almost periodic, we can extract a subsequence $(s_n) \subset (s'_n)$ such that $\lim_{n\to\infty} f(t+s_n) = g(t)$ uniformly in $t \in R^+$. We note that $g(t)$ is also strongly continuous. For fixed $t_0 \in R$, we can obtain a subsequence of (s_n), which again we will denote (s_n), such that

$$weak - \lim_{n\to\infty} u(t_0 + s_n) = v_0 \in X.$$

Since X is a reflexive Banach space, then the function

$$y(t) = T(t - t_0)v_0 + \int_{t_0}^{t} S(t - s)(Bu + g)(s)ds,$$

is strongly continuous. It is a mild solution of

$$\frac{d^\alpha u(t)}{dt^\alpha} + (A - B(t))u(t) = g(t), \quad t \in R^+.$$

We need the following lemmas.

Lemma 2.4

For each $t \in R^+$, we have

$$weak - \lim_{n\to\infty} u(t + s_n) = y(t).$$

Proof

We can write

$$u(t + s_n) = T(t - t_0)u(t_0 + s_n) + \int_{t_0}^{t} S(t - s)[(Bu)(s) + f(s + s_n)]ds,$$

$n = 1, 2, ...$ (see for instance [63, p.721]). Let $u^* \in X^*$, we have

$$< u^*, T(t-t_0)u(t_0+s_n) > - < u^*, T(t-t_0)v_0 >=< T^*(t-t_0)u^*, u(t_0+s_n)-v_0 >,$$

for every $n = 1, 2, ...$, we deduce that the sequence $(T(t - t_0)u(t_0 + s_n))$ converges to $T(t - t_0)v_0$ in the weak sense. Also we have

$$\begin{aligned}\int_{t_0}^{t} S(t - s)[(Bu)(s) + f(s + s_n)]ds \;-\; &\int_{t_0}^{t} S(t - s)[(Bu)(s) + g(s)]ds \\ \le\; &\| \int_{t_0}^{t} S(t - s)[f(s + s_n) - g(s)]ds\| \\ \le\; &\alpha M \int_{t_0}^{t} (t - s)^{\alpha-1}\|f(s + s_n) - g(s)\|ds.\end{aligned}$$

This leads to

$$\lim_{n\to\infty}\int_{t_0}^{t} S(t-s)[(Bu)(s)+f(s+s_n)]ds = \int_{t_0}^{t} S(t-s)[(Bu)(s)+g(s)]ds,$$

in the strong sense, then consequently in the weak sense in X.

Lemma 2.5

$$\mu(y)=\mu(u)=\mu^*.$$

Proof

Since $u(t)$ is an optimal mild solution of (2.1), we have

$$\mu^*=\mu(u)=\sup_{t\in R}\|u(t)\|.$$

Let $u^*\in X^*$, then by lemma 2.4 we obtain

$$\lim_{n\to\infty} < u^*, u(t+s_n) > = < u^*, y(t) >,$$

for every $t\in R^+$. For each $n=1,2,...$, we have

$$\| < u^*, u(t+s_n) > \| \le \|u^*\| \; \|u(t+s_n)\| \le \|u^*\|\mu^*.$$

Therefore,

$$\| < u^*, y(t) > \| \le \|u^*\|\mu^*$$

for every $t\in R^+$, and consequently

$$\|y(t)\|\le\mu^*$$

for every $t\in R^+$, so that

$$\mu(y)\le\mu^*.$$

We suppose that $\mu(y) < \mu^*$.

Note that

$$\lim_{n\to\infty} g(t - s_n) = f(t)$$

uniformly in $t \in R^+$ because $f(t)$ is almost periodic. Since X is a reflexive Banach space, we can extract from the sequence (s_n), a subsequence which we still denote (s_n) such that $(y(t_0 - s_n))$ is weakly convergent to $z \in X$. We have

$$\lim_{n\to\infty} y(t - s_n) = T(t - t_0)z + \int_{t_0}^{t} S(t - s)F(s)ds$$

in the weak sense for every $t \in R^+$. Now we consider the function

$$Z(t) = T(t - t_0)z + \int_{t_0}^{t} S(t - s)F(s)ds.$$

It is a bounded mild solution of equation (2.1). Similarly as above, we have $\mu(Z) \le \mu(y)$; therefore, $\mu(Z) < \mu^*$, which is absurd by definition of μ^*.

Lemma 2.6

$$\mu(y) = \inf_{v\in\Omega_g} \mu(v)$$

i.e. $y(t)$ is an optimal mild solution of the equation

$$\frac{d^\alpha u(t)}{dt^\alpha} + (A - B(t))u(t) = g(t), \quad t \in R^+. \tag{2.8}$$

Proof

By lemma 2.5, $y(t)$ is bounded over R. Also $y(t)$ is a mild solution of (2.8) which implies $y(t) \in \Omega_g$. It remains to prove that $y(t)$ is optimal. Suppose it is not. Since $\Omega_g \neq \emptyset$, by Theorem 2.1, there exists a unique optimal solution $v(t)$ of (2.8). We have $\mu(v) < \mu(y)$ and

$$v(t) = T(t-t_0)v(t_0) + \int_{t_0}^{t} S(t-s)(Bu+g)(s)ds,$$

for all $t_0 \in R, t \geq t_0$. We can find a subsequence $(s_{n_k}) \subset (s_n)$ such that

$$weak - \lim_{k\to\infty} v(t-s_{n_k}) = T(t-t_0)z + \int_{t_0}^{t} S(t-s)F(s)ds \equiv V(t).$$

Noting that $V(t) \in \Omega_f$ and

$$\mu(V) \leq \mu(v) < \mu(y),$$

which is absurd. Therefore $y(t)$ is an optimal mild solution of (2.8), and in fact the only one by Theorem 2.1.

Proof of Theorem 2.3

To prove that $u(t)$ is weakly almost periodic, it suffices to show that

$$weak - \lim_{n\to\infty} u(t+s_n) = y(t)$$

uniformly in $t \in R^+$. Suppose that this does not hold; then there exists $u^* \in X^*$ such that

$$\lim_{n\to\infty} < u^*, u(t+s_n) >=< u^*, y(t) >$$

is not uniform in $t \in R$. Consequently, we can find a number $\gamma > 0$, and a sequence (t_k) with two subsequences (s'_k) and (s''_k) of (s_n) such that

$$| < u^*, u(t+s'_k) - u(t+s''_k) > | > \gamma \qquad (2.9)$$

for all $k = 1, 2, ...$

Again, let us extract two subsequences of (s'_k) and (s''_k) respectively, with the same notation, such that

$$\lim_{k\to\infty} f(t + t_k + s'_k) = g_1(t), \text{ and } \lim_{k\to\infty} f(t + t_k + s''_k) = g_2(t)$$

both uniformly in $t \in R^+$, because f is almost periodic. As we did previously, we may obtain

$$weak - \lim_{k\to\infty} f(t+t_k+s'_k) = T(t-t_0)z_1 + \int_{t_0}^{t} S(t-s)[(Bu)(s)+g_1(s)]ds \equiv y_1(t),$$

and

$$weak - \lim_{k\to\infty} f(t+t_k+s''_k) = T(t-t_0)z_2 + \int_{t_0}^{t} S(t-s)[(Bu)(s)+g_2(s)]ds \equiv y_2(t)$$

for each $t \in R^+$, where $y_1(t)$ and $y_2(t)$ are optimal mild solutions in Ω_{g_1} and Ω_{g_2}, respectively.

Since $\lim_{k\to\infty} f(t + t_k + s_k)$ exists uniformly in $t \in R^+$, and $(s'_k), (s''_k)$ are two subsequences of (s_k), we will get

$$\sup_{s\in R} \|f(s + s'_k) - f(s + s''_k)\| < \epsilon$$

if $k \geq k_0(\epsilon)$ and consequently

$$\sup_{s\in R} \|f(t + t_k + s'_k) - f(t + t_k + s''_k)\| < \epsilon$$

for $k \geq k_0(\epsilon)$, which shows that $g_1(s) = g_2(s)$ for all $s \in R^+$.

By the uniqueness of the optimal mild solution we get $y_1(t) = y_2(t), t \in R^+$.

But

$$y_1(0) = weak - \lim_{k\to\infty} u(t_k + s'_k)$$

and

$$y_2(0) = weak - \lim_{k\to\infty} u(t_k + s''_k).$$

Clearly $y_1(0) = y_2(0)$ contradicts the inequality (2.9) above. This completes the proof.

2.4 Application

Consider the partial differential equation of fractional order

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \sum_{|q|\leq 2m} a_q(x) D_x^q u(x,t) = \int_{R^n} K(x,\eta,t)u(\eta,t)d\eta + f(x,t), \quad (2.10)$$

where $t \in R^+, x \in R^n, D_x^q = D_{x_1}^{q_1}...D_{x_n}^{q_n}, D_{x_i} = \frac{\partial}{\partial x_i}$, $q = (q_1,...,q_n)$ is an n-dimensional multi-index, $|q| = q_1 + ... + q_n$.

Let $L_2(R^n)$ be the set of all square integrable functions on R^n. We denote by $C^m(R^n)$ the set of all continuous real-valued functions defined on R^n which have continuous partial derivatives of order less than or equal to m. By $C_0^m(R^n)$ we denote the set of all functions $f \in C^m(R^n)$ with compact supports. Let $H_0^m(R^n)$ be the completion of $C_0^m(R^n)$ with respect to the norm

$$\|f\|_m^2 = \sum_{|q|\leq m} \int_{R^n} |D_x^q f(x)|^2 dx.$$

It is supposed that

(i) The operator

$$A = -\sum_{|q|=2m} a_q(x) D_x^q$$

is uniformly parabolic on R^n. In other words, all the coefficients $a_q, |q| = 2m$, are continuous and bounded on R^n and

$$(-1)^m \sum_{|q|=2m} a_q(x)\xi^q \geq c|\xi|^{2m}, \quad c > 0,$$

for all $x \in R^n$ and all $\xi \neq 0$, $\xi \in R^n$, where $\xi^q = \xi_1^{q_1}...\xi_n^{q_n}$ and $|\xi|^2 = \xi_1^2 + ... + \xi_n^2$.

(ii) All the coefficients $a_q, |q| = 2m$, satisfy a uniform Hölder condition on R^n and

$$\int_{R^n} K^2(x, \eta, t)d\eta < \infty.$$

It's proved, see [25, p.438], that the operator A defined by (i) with domain of definition $S = H^{2m}(R^n)$ generates an analytic semigroup $Q(t)$ defined on $L_2(R^n)$, and that $H^{2m}(R^n)$ is dense in $X = L_2(R^n)$. Which achieves the proof of the existence of (bounded) mild solutions of the equation (2.10).

(iii) f is a nontrivial strongly continuous function defined on $R^n \times R^+$ satisfying:

For every $\epsilon > 0$ there exists $\beta > 0$ such that every interval $[a, a+\beta]$ contains at least a point τ such that

$$\int_{R^n} |f(x, t+\tau) - f(x, t)|^2 dx < \epsilon,$$

for all $t \in R^+$ and all $x \in R^n$. Applying Theorems 2.1, 2.3, stated above, we deduce that (2.10) has a unique optimal mild solution which is weakly almost periodic.

Chapter 3

Almost Periodic Solutions of Some Semilinear Fractional Differential Equations

3.1 Introduction

Many dynamical systems are represented by the following semilinear fractional differential equations:

$$\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = f(t, u(t)), \quad t > t_0 \tag{3.1}$$

in Banach space$X$,where $0 < \alpha \leq 1, t \geq 0$, we assume that $-A$ is the infinitesimal generator of an analytic c_0-semigroup $Q(t)$ satisfying the exponential stability, f is uniformly almost periodic function defined on $R \times X_q$ into X satisfies the hypothesis

F: There are numbers $L \geq 0$ and $0 \leq \eta \leq 1$ such that

$$|f(t_1, u_1) - f(t_2, u_2)| \leq L(|t_1 - t_2|^\eta + |u_1 - u_2|_q)$$

for all $(t_1, u_2), (t_2, u_2)$ in $R^+ \times X_q$, where X is a real or complex Banach space with norm $|.|$, A^q is the fractional power and X_q is the Banach space $D(A^q)$ endowed with the norm $|u|_q = |A^q u|$.

The existence of almost periodic solutions for abstract evolution equation defined on abstract Banach spaces has been studied in various works, see for instance [2, 65, 66]. By using the semigroup theory and the contraction

mapping principle, Zaidman studied in [66] the existence of almost periodic solutions for the integral equation associated to the abstract differential equation $x'(t) = Ax(t) + f(t, x)$ where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators on a Banach space. Recently, Bahaj and Sidki studied in [2] the existence of almost periodic solution of the same equation. N'Guerekata [49] gave necessary conditions to ensure that the so-called optimal mild solutions of $x'(t) = A(t) + f(t)$ are weakly almost periodic. As new in this chapter we are concerned with fractional order.
In section 2, we state the basic notations, definitions and properties which are used throughout this work to obtain our results. In section 3, we establish the existence and uniqueness of almost periodic solution over R^+ of (3.1). In section 4, we prove again the existence and uniqueness of the optimal mild solution of (3.1). In section 5, we show under necessary conditions that the optimal mild solution is also weakly almost periodic.

3.2 Preliminaries

Let X denote a real or complex Banach space endowed with the norm $|.|$ and by $\pounds(X)$ stands for the Banach algebra of bounded linear operators defined on X. For A a linear operator with domain $D(A)$, we denote by $\Re(A)$ the range of A.

Let $-A$ is the infinitesimal generator of an analytic semigroup in a Banach space and $0 \in \rho(A)$, $\rho(A)$ is the resolvent set of A. We define the fractional power A^{-q} by

$$A^{-q} = \frac{1}{\Gamma(q)} \int_0^{\infty} (t)^{q-1} Q(t) dt, q > 0.$$

For $0 < q \leq 1$, A^q is a closed linear operator whose domain $D(A^q) \supset D(A)$ is dense in X, this implies that $D(A^q)$ endowed with the graph norm

$$|u|_{D(A)} = |u| + |A^q u|, u \in D(A^q)$$

is a Banach space, clearly $A^q = (A^{-q})^{-1}$ because A^{-q} is one to one. Since $0 \in \rho(A)$, A^q is invertible, and its graph norm is equivalent to the norm $|u|_q = |A^q u|$. Thus $D(A^q)$ equipped with the norm $|.|_q$ is a Banach space denoted by X_q, for more details we refer to [32, 53].

Lemma 3.1

Let $-A$ be the infinitesimal generator of an analytic semigroup $Q(t)$. If $0 \in \rho(A)$ then

(a) $Q(t) : X \longrightarrow D(A^q)$ for every $t > 0$ and $q \geq 0$

(b) For every $u \in D(A^q)$, we have $Q(t)A^q u = A^q Q(t) u$

(c) For every $t > 0$ the operator $A^q Q(t)$ is bounded and $|A^q Q(t)|_{\pounds(X)} \leq M_q t^{-q} e^{-\delta t}$

(d) For $0 < q \leq 1$ and $u \in D(A^q)$, we have $|Q(t)u - u| \leq C_q t^q |A^q u|$, see [53, section 2.6].

Definition 3.1

A continuous function $f : R \times \Omega \longrightarrow X$ is called uniformly almost periodic if for every $\epsilon > 0$ and every compact set $K \subset \Omega$ there exists a relatively dense set P_ϵ in R such that $|f(t+\tau,u)-f(t,u)| \le \epsilon$ for all $t \in R$, $\tau \in P_\epsilon$ and all $u \in K$.

More details about this definition can be found in [36, P.188].

Lemma 3.2

Let $f : R \times \Omega \longrightarrow X$ be uniformly almost periodic and $u : R \longrightarrow \Omega$ be an almost periodic function such that $\overline{\Re(u)} \subset \Omega$, then the function $t \longrightarrow f(t,u(t))$ also is almost periodic. The proof in [62, Theorem I.2.7].

3.3 Almost Periodic Solutions

By a classical solution of (3.1) on $[0,T)$, we mean a function u with values in X such that:

1) u is continuous function on $[0,T)$ and $u(t) \in D(A)$,

2) $\frac{d^\alpha u}{dt^\alpha}$ exists and continuous on $(0,T)$, $0<\alpha<1$, and u satisfies (3.1) on $(0,T)$.

It is suitable to rewrite equation (3.1) in the form

$$u(t) = u(t_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}[-Au(s)+f(s,u(s))]ds. \qquad (3.2)$$

According to [22, 25-29], the equation (3.2) is equivalent to the integral equation

$$\begin{aligned} u(t) &= \int_0^\infty \zeta_\alpha(\theta)Q((t-t_0)^\alpha\theta)u(t_0)d\theta \\ &+ \alpha\int_{t_0}^{t}\int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,u(s))d\theta ds, \qquad (3.3) \end{aligned}$$

where ζ_α is a probability density function defined on $(0, \infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\theta x}\zeta_\alpha(\theta)d\theta = \sum_{j=0}^{\infty} \frac{(-x)^j}{\Gamma(1+\alpha j)}, 0 < \alpha \leq 1, x > 0, \tag{3.4}$$

also,we have

$$\int_0^\infty \theta^\eta \zeta_\alpha(\theta)d\theta \leq 1, 0 \leq \eta \leq 1,$$

for the proof of existence and uniqueness of solution of (3.3) we refer to [25, theorem 3.1 p.435], also as different method see [26, section 2, p. 824-827].

By a mild solution of (3.1), we mean a continuous solution of the integral equation (3.3).

When $-A$ generates a semigroup with negative exponent, we deduce that if $u(.)$ is a bounded mild solution of (3.1) on R^+, then we take the limit as $t_0 \longrightarrow -\infty$ on the right-hand side of (3.3) and using (3.4), we obtain

$$u(t) = \alpha \int_{-\infty}^{t} \int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)f(s, u(s))d\theta ds. \tag{3.5}$$

Conversely, if $u(.)$ is a bounded continuous function and (3.5) is verified, then $u(.)$ is a mild solution of (3.1).

Theorem 3.3

Let $-A$ be the infinitesimal generator of an analytic semigroup $\{Q(t)\}_{t\geq 0}$ satisfying $|Q(t)|_{\pounds(X)} \leq Me^{\beta t}$, for all $t > 0$ and $\beta < 0$. If $f : R^+ \times X \longrightarrow X$ is uniformly almost periodic and f satisfies the assumption **(F)**, then (3.1) has a unique almost periodic (classical) solution over R for L sufficiently small enough.

We shall need the following Lemma.

Lemma 3.4

If $f : R^+ \longrightarrow X$ is almost periodic and locally Hölder continuous, then (3.1) has a unique almost periodic classical solution over R^+ given by

$$u(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)f(s)d\theta ds.$$

For the proof we can use the same technique which appear in Zaidman [64]. From Pazy [53], clearly that if $f : R^+ \longrightarrow X$ is Hölder continuous and if A generates an analytic semigroup, then the mild solution of (3.1) in fact is a classical solution.

We define the set $AP(X) = \{\varphi : R \longrightarrow X, \varphi \text{ is almost periodic}\}$ with the usual supremum norm over R^+ which denoted by $|.|_\infty$. We define on the set $AP(X)$ a mapping

$$T\varphi(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)A^qQ((t-s)^\alpha\theta)f(s, A^{-q}\varphi(s))d\theta ds. \quad (3.6)$$

We show that T is well defined. Let $\varphi \in AP(X)$, using a standard properties of the almost-periodicity, we have $N = \sup_{t\in R^+} |f(t, A^{-q}\varphi(t))| < \infty$. By Lemma 3.1.c, we have

$$|T\varphi(t)| \leq \alpha N M_q \int_{-\infty}^{t} \int_{0}^{\infty} \theta^{1-q}\zeta_\alpha(\theta)(t-s)^{-\alpha q+\alpha-1}e^{-\delta\theta(t-s)^\alpha}d\theta ds.$$

Set $\eta = t - s$, we obtain

$$|T\varphi(t)| \leq \alpha N M_q \int_{0}^{\infty} \int_{0}^{\infty} \theta^{1-q}\zeta_\alpha(\theta)(\eta)^{-\alpha q+\alpha-1}e^{-\delta\theta(\eta)^\alpha}d\theta d\eta.$$

By using the properties of the probability density function ζ_α, and the definition of the gamma function we conclude that $T\varphi$ exists.

Lemma 3.5

The operator T is well defined, and maps $AP(X)$ into itself.

Proof

It follows from Lemma 3.2 that for $\varphi \in AP(X)$, $t \longrightarrow f(t, A^{-q}\varphi(t))$ is almost periodic. Hence, for each $\epsilon > 0$ there exists a set P_ϵ relatively dense in R such that

$$|f(t+\tau, A^{-q}\varphi(t+\tau)) - f(t, A^{-q}\varphi(t))| \leq \epsilon,$$

for all $t \in R^+$ and $\tau \in P_\epsilon$. Therefore, the map T defined by (3.6) satisfies

$$\begin{aligned}
& |T\varphi(t+\tau) - T\varphi(t)| \\
= & |\alpha \int_{-\infty}^{t+\tau} \int_0^\infty \theta(t+\tau-s)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t+\tau-s)^\alpha\theta)f(s, A^{-q}\varphi(s))d\theta ds \\
- & \alpha \int_{-\infty}^{t} \int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t-s)^\alpha\theta)f(s, A^{-q}\varphi(s))d\theta ds| \\
= & |\alpha \int_{-\infty}^{t} \int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t-s)^\alpha\theta)f(s+\tau, A^{-q}\varphi(s+\tau))d\theta ds \\
- & \alpha \int_{-\infty}^{t} \int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t-s)^\alpha\theta)f(s, A^{-q}\varphi(s))d\theta ds| \\
\leq & \epsilon\alpha M_q \int_{-\infty}^{t} \int_0^\infty \theta^{1-q}\zeta_\alpha(\theta)(t-s)^{-\alpha q+\alpha-1}e^{-\delta\theta(t-s)^\alpha} d\theta ds.
\end{aligned}$$

Thus the function $T\varphi$ is almost periodic and $T : AP(X) \longrightarrow AP(X)$.

Proof of Theorem 3.3

Consider the mapping from the Banach space $AP(X)$ into itself defined by

$$T\varphi(t) = \alpha \int_{-\infty}^{t} \int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)A^q Q((t-s)^\alpha\theta)f(s, A^{-q}\varphi(s))d\theta ds.$$

Let $\varphi_1, \varphi_2 \in AP(X)$, by using Lemma 3.1.c and assumption **(F)** we get

$$|T\varphi_1 - T\varphi_2| \leq \alpha L M_q |\varphi_1 - \varphi_2|_\infty \int_{-\infty}^{t} \int_0^\infty \theta^{1-q}\zeta_\alpha(\theta)(t-s)^{-\alpha q+\alpha-1}e^{-\delta\theta(t-s)^\alpha} d\theta ds,$$

again use the substitution $\eta = t - s$, we obtain

$$|T\varphi_1 - T\varphi_2|_\infty \le \alpha L M_q |\varphi_1 - \varphi_2|_\infty \int_0^\infty \int_0^\infty \theta^{1-q} \zeta_\alpha(\theta)(\eta)^{-\alpha q + \alpha - 1} e^{-\delta\theta(\eta)^\alpha} d\theta d\eta,$$

It is known from above that the double integral in the right-hand side of the inequality exists, then we choose L sufficiently small, thus T is a strict contraction. By the contraction mapping theorem there exists $\varphi \in AP(X)$ such that

$$\varphi(t) = \alpha \int_{-\infty}^t \int_0^\infty \theta(t-s)^{\alpha-1} \zeta_\alpha(\theta) A^q Q((t-s)^\alpha\theta) f(s, A^{-q}\varphi(s)) d\theta ds. \quad (3.7)$$

Since A^q is closed, applying A^{-q} on both sides of (3.7), we get

$$A^{-q}\varphi(t) = \alpha \int_{-\infty}^t \int_0^\infty \theta(t-s)^{\alpha-1} \zeta_\alpha(\theta) Q((t-s)^\alpha\theta) f(s, A^{-q}\varphi(s)) d\theta ds. \quad (3.8)$$

We show that the solution φ of (3.8) is Hölder continuous on R^+. By Lemma 3.1.d, for every β satisfying $0 < \beta < 1 - q$ and for every $h > 0$, we have

$$|(Q(h) - I)A^q Q(t-s)| \le C_\beta h^\beta |A^{q+\beta} Q(t-s)|. \quad (3.9)$$

Also for $h \ge 0$, we can write

$$\begin{aligned} |Q((t + h - s)^\alpha\theta)| &= |Q((t+h-s)^\alpha\theta - (t-s)^\alpha\theta^* - h^\alpha\theta^*)Q(h^\alpha\theta^*)Q((t-s)^\alpha\theta^*)| \\ &\le M^*|Q(h^\alpha\theta^*)Q((t-s)^\alpha\theta^*)|, \quad (3.10) \end{aligned}$$

where $\theta^* = \frac{\theta}{2}$ (to ensure that Q is defined) and M^* is a constant. Using (3.9), (3.10) and Lemma 3.1.c, we get

$$|\varphi(t + h) - \varphi(t)|$$

$$
\begin{aligned}
&\leq \; |\alpha M^* \int_{-\infty}^{t}\int_0^\infty \theta[(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}]\zeta_\alpha(\theta) \\
&\qquad\qquad (Q(h^\alpha\theta^*) - I)A^q Q((t-s)^\alpha\theta^*)f(s, A^{-q}\varphi(s))d\theta ds| \\
&+ \; |\alpha \int_t^{t+h}\int_0^\infty \theta(t+h-s)^{\alpha-1}\zeta_\alpha(\theta) \\
&\qquad\qquad A^q Q((t+h-s)^\alpha\theta)f(s, A^{-q}\varphi(s))d\theta ds| \\
&\leq \; \alpha M^* C_\beta h^{\alpha\beta} N M_{q+\beta}\int_{-\infty}^{t}\int_0^\infty |\theta[(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}]\zeta_\alpha(\theta) \\
&\qquad\qquad \theta^{*^\beta}(t-s)^{-\alpha(q+\beta)}\theta^{*^{-(q+\beta)}} exp[-\delta(t-s)^\alpha\theta^*]|d\theta ds \\
&+ \; \alpha N M_q \int_t^{t+h}\int_0^\infty |\theta(t+h-s)^{\alpha-1}\zeta_\alpha(\theta) \\
&\qquad\qquad (t+h-s)^{-\alpha q}\theta^{-q} exp(-\delta(t+h-s)^\alpha\theta)|d\theta ds.
\end{aligned}
$$

We can estimate each term of the inequality separately to get

$$
|\varphi(t+h) - \varphi(t)| \leq C h^\beta,
$$

which means that φ is Hölder continuous on R^+. From assumption **(F)** we have

$$
|f(t, A^{-q}\varphi(t)) - f(s, A^{-q}\varphi(s))| \leq L(|t-s|^\eta + |\varphi(t) - \varphi(s)|).
$$

Therefore $t \longrightarrow f(t, A^{-q}\varphi(t))$ is Hölder continuous on R^+. Let φ be the solution of (3.7) and consider the equation

$$
\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = f(t, A^{-q}\varphi(t)). \tag{3.11}
$$

By Lemma 3.4, (3.11) has a unique solution given by

$$
\psi(t) = \alpha\int_{-\infty}^{t}\int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)f(s, A^{-q}\varphi(s))d\theta ds. \tag{3.12}
$$

Moreover, $\psi(t) \in D(A) \subset D(A^q)$ for all $t \in R^+$. Applying A^q on both sides of (3.12), we get

$$A^q\psi(t) = \alpha \int_{-\infty}^{t} \int_0^{\infty} \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)A^qQ((t-s)^\alpha\theta)f(s, A^{-q}\varphi(s))d\theta ds = \varphi(t). \tag{3.13}$$

Clearly $\psi(t) = A^{-q}\varphi(t)$ is a solution of (3.1), the uniqueness of ψ follows from the uniqueness of the solution of (3.7) and (3.11). This completes the proof of the theorem.

In next sections, let us consider X be a uniformly convex Banach space equipped with a norm $|.|$ and X^* its topological dual space and Ω an open subset of X.

3.4 Optimal Mild Solutions

As in N'Guerekata [48, 49], we consider in X the equation (3.1) with the following assumptions:

F1: $A : D(A) \subset X \longmapsto X$ is a linear operator generates a c_0-semigroup of bounded linear operators $Q(t), t > 0$ satisfying $\sup_{t\in R^+} |Q(t)| < \infty$,

F2: $f : R^+ \times \Omega \longmapsto X$ is a nontrivial strongly continuous function and is convex in u.

Let us denote by Ω_f the set of all mild solutions $u(t)$ of (3.1) which are bounded over R^+, that is

$$\mu(u) = \sup_{t\in R^+} |u(t)| < \infty. \tag{3.14}$$

We assume here that $\Omega_f \neq \emptyset$, and we recall the following:

A bounded mild solution $\tilde{u}(t)$ of (3.1) is called an optimal mild solution of (3.1) if

$$\mu(\tilde{u}) \equiv \mu^* = \inf_{u \in \Omega_f} \mu(u). \tag{3.15}$$

Theorem 3.6

Assume that $\Omega_f \neq \emptyset$ and the assumptions **(F1-F2)** are hold, then (3.1) has a unique optimal mild solution. (Compare with [65, theorem 1.1, p.138] and [49, theorem 1, p. 673]) Our proof is based on the following lemma.

Lemma 3.7

If K is a non-empty convex and closed subset of a uniformly convex Banach space X and $v \notin K$, then there exists a unique $k_0 \in K$ such that $|v - k_0| = \inf_{k \in K} |v - k|$, see [40, Corollary 8.2.1].

Proof of Theorem 3.6

It suffices to prove that Ω_f is a convex and closed set because the trivial solution $0 \notin \Omega_f$, then we use Lemma 3.7 to deduce the uniqueness of the optimal mild solution.

For the convexity of Ω_f, we consider two distinct bounded mild solutions $u_1(t)$ and $u_2(t)$, and a real number $0 \leq \lambda \leq 1$ and let

$$u(t) = \lambda u_1(t) + (1 - \lambda) u_2(t), \quad t \in R^+.$$

For every $t_0 \in R$, $u(t)$is continuous and (see [28]) has the integral representation

$$u(t) = T(t - t_0)u(t_0) + \int_{t_0}^{t} S(t - s) f(s, u(s)) ds, t \geq t_0, \tag{3.16}$$

where

$$T(t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) d\theta, \quad S(t) = \alpha \int_0^\infty \theta t^{\alpha-1} \zeta_\alpha(\theta) Q(t^\alpha \theta) d\theta.$$

We have

$$u(t_0) = \lambda u_1(t_0) + (1-\lambda) u_2(t_0)$$

and $f(t, u)$ is convex in u, then $u(t)$ is a mild solution of (3.1).

We note that $u(t)$ is bounded over R^+ since

$$\mu(u) = \sup_{t \in R^+} |u(t)| \leq \lambda \mu(u_1) + (1-\lambda)\mu(u_2) < \infty,$$

we conclude that $u(t) \in \Omega_f$.

Now we show that Ω_f is closed, let a sequence $u_n \in \Omega_f$ such that

$$\lim_{n \to \infty} u_n(t) = u(t), \quad t \in R^+.$$

For all $t_0 \in R$ and $t \geq t_0$, we have

$$u_n(t) = T(t-t_0) u_n(t_0) + \int_{t_0}^t S(t-s) f(s, u_n(s)) ds, \tag{3.17}$$

It's clearly that $T(t-t_0)$ and $S(t-s)$ are continuous operators, then for every fixed t and t_0 with $t \geq t_0$, we have

$$\begin{aligned}
\lim_{n \to \infty} T(t-t_0) u_n(t_0) &= \lim_{n \to \infty} \int_0^\infty \zeta_\alpha(\theta) Q((t-t_0)^\alpha \theta) u_n(t_0) d\theta \\
&= \int_0^\infty \zeta_\alpha(\theta) Q((t-t_0)^\alpha \theta) d\theta \lim_{n \to \infty} u_n(t_0) \\
&= T(t-t_0) \lim_{n \to \infty} u_n(t_0) \\
&= T(t-t_0) u(t_0).
\end{aligned}$$

Similarly we have

$$\begin{aligned}\lim_{n\to\infty}\int_{t_0}^{t} S(t-s)f(s,u_n(s))ds &= \int_{t_0}^{t} S(t-s)\lim_{n\to\infty} f(s,u_n(s))ds\\ &= \int_{t_0}^{t} S(t-s)f(s,u(s))ds.\end{aligned}$$

Then we deduce that

$$u(t) = T(t-t_0)u(t_0) + \int_{t_0}^{t} S(t-s)f(s,u(s))ds,$$

for all $t_0 \in R, t \geq t_0$, which means that $u(t)$ is a mild solution of (3.1). Finally we show that $u(t)$ is bounded over R^+. We can write (3.16) as

$$\begin{aligned}u(t) &= T(t-t_0)u(t_0) + \int_{t_0}^{t} S(t-s)f(s,u(s))ds - u_n(t) + u_n(t)\\ &= T(t-t_0)[u(t_0) - u_n(t_0)] + \int_{t_0}^{t} S(t-s)[f(s,u(s)) - f(s,u_n(s))]ds\\ &+ u_n(t),\end{aligned}$$

for every $n = 1, 2, ...,$ and every $t_0 \in R$ such that $t \geq t_0$.

Let $M = \sup_{t\in R^+} |Q(t^\alpha \theta)| < \infty$, since $\int_0^\infty \zeta_\alpha(\theta)d\theta = 1$, then $|T(t)| \leq M$, again (see [29, p.54]) since $\int_0^\infty \theta\zeta_\alpha(\theta)d\theta = 1$, then $|S(t)| \leq M\alpha|t|^{\alpha-1}$, by assumption (F), we have

$$|f(s,u(s)) - f(s,u_n(s))| \leq L|u(s) - u_n(s)|_q.$$

These estimates lead to

$$|u(t)| \leq M|u(t_0) - u_n(t_0)| + \alpha ML\int_{t_0}^{t} |t-s|^{\alpha-1}|u(s) - u_n(s)|_q ds + |u_n(t)|.$$

Choose n large enough, for every $\epsilon > 0$ we get

$$|u(t)| \leq M\epsilon + \alpha ML\epsilon\int_{t_0}^{t} |t-s|^{\alpha-1}ds + \mu(u_n),$$

then we have $\mu(u) \leq \epsilon_1 + \epsilon_2 + \mu(u_n) < \infty$. Thus $u \in \Omega_f$. This completes the proof of the theorem.

3.5 Weak almost Periodic Solutions

In order to formulate a property of almost periodic functions, which is equivalent to Definition 3.1, we discuss the concept of normality of almost periodic functions. Namely, let $f(t,u)$ be almost periodic in $t \in R^+$ uniformly for $u \in K$, then for every sequence of real numbers (s'_n) there exists a subsequence (s_n) and a function $g(t,u)$ such that

$$f(t+s_n,u) \longrightarrow g(t,u)$$

uniformly on $R \times K$ as $n \longrightarrow \infty$, where K is a compact set in Ω, see Y. Hamaya [36, p.188]. It is well known [48, 65] that:

$f : R \times \Omega \longrightarrow X$ is weakly almost periodic if for every sequence of real numbers (s'_n) there exists a subsequence (s_n) such that every $(f(t+s_n,u))$ is convergent in the weak sense, uniformly on $R \times K$. In other words, for every $u^* \in X^*$, the sequence $(< u^*, f(t+s_n,u) >)$ is uniformly convergent on $R \times K$, where $< .,. >$ denotes duality $< X^*, X >$. For each $Q(t), t \in R^+$, $Q^*(t)$ denotes the adjoint operator of $Q(t)$.

Theorem 3.8

Let $f(t,u)$ be almost periodic and assume that **(F1-F2)** are hold, also assume that $f \in L^1(R^+ \times \Omega, X)$ and $Q^*(t) \in L(X^*)$ for every $t \in R^+$, then the optimal mild solution of (3.1) is weakly almost periodic.

Proof

Let us consider $u(t)$ is the unique optimal mild solution of (3.1), by theorem 3.6

$$u(t) = T(t - t_0)u(t_0) + \int_{t_0}^{t} S(t - s)f(s, u(s))ds,$$

for all $t_0 \in R, t \geq t_0$. Let (s'_n) be an arbitrary sequence of real numbers. Since f is almost periodic, we can extract a subsequence $(s_n) \subset (s'_n)$ such that

$$\lim_{n\to\infty} f(t + s_n, u) = g(t, u)$$

uniformly on $R \times K$. We note that $g(t, u)$ is also strongly continuous. For fixed $t_0 \in R$, we can obtain a subsequence of (s_n), which again we will denote (s_n), such that

$$weak - \lim_{n\to\infty} u(t_0 + s_n) = v_0 \in X.$$

Since X is a reflexive Banach space, then the function

$$y(t) = T(t - t_0)v_0 + \int_{t_0}^{t} S(t - s)g(s, u(s))ds,$$

is strongly continuous. It is a mild solution of

$$\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = g(t, u(t)), t \in R^+.$$

We need the following lemmas.

Lemma 3.9

For each $t \in R^+$, we have

$$weak - \lim_{n\to\infty} u(t + s_n) = y(t).$$

Proof

We can write

$$u(t+s_n) = T(t-t_0)u(t_0+s_n) + \int_{t_0}^{t} S(t-s)f(s+s_n, u(s))ds,$$

$n = 1, 2, ...,$ (see for instance [63, p.721]) let $u^* \in X^*$, we have

$$< u^*, T(t-t_0)u(t_0+s_n) > - < u^*, T(t-t_0)v_0 > = < T^*(t-t_0)u^*, u(t_0+s_n) - v_0 >,$$

for every $n = 1, 2, ...,$ we deduce that the sequence $(T(t-t_0)u(t_0+s_n))$ converges to $T(t-t_0)v_0$ in the weak sense.

Also we have

$$\begin{aligned}
&\int_{t_0}^{t} S(t-s)f(s+s_n, u(s))ds - \int_{t_0}^{t} S(t-s)g(s, u(s))ds \\
&\qquad \leq \; |\int_{t_0}^{t} S(t-s)[f(s+s_n, u(s)) - g(s, u(s))]ds| \\
&\qquad \leq \; \int_{t_0}^{t} |S(t-s)||f(s+s_n, u(s)) - g(s, u(s))|ds \\
&\qquad \leq \; M\alpha \int_{t_0}^{t} |t-s|^{\alpha-1}|f(s+s_n, u(s)) - g(s, u(s))|ds.
\end{aligned}$$

This leads to

$$\lim_{n\to\infty} \int_{t_0}^{t} S(t-s)f(s+s_n, u(s))ds = \int_{t_0}^{t} S(t-s)g(s, u(s))ds,$$

in the strong sense, then consequently in the weak sense in X.

Lemma 3.10

$$\mu(y) = \mu(u) = \mu^*.$$

Proof

Since $u(t)$ is an optimal mild solution of (3.1), we have

$$\mu^* = \mu(u) = \sup_{t \in R^+} |u(t)|.$$

Let $u^* \in X^*$, then by lemma 3.9 we obtain

$$\lim_{n \to \infty} < u^*, u(t + s_n) >=< u^*, y(t) >,$$

for every $t \in R^+$. For each $n = 1, 2, ...$, we have

$$| < u^*, u(t + s_n) > | \leq |u^*| \; |u(t + s_n)| \leq |u^*|\mu^*.$$

Therefore, $| < u^*, y(t) > | \leq |u^*|\mu^*$ for every $t \in R^+$, and consequently $|y(t)| \leq \mu^*$ for every $t \in R^+$, so that $\mu(y) \leq \mu^*$.
We suppose that $\mu(y) < \mu^*$. Note that

$$\lim_{n \to \infty} g(t - s_n, u) = f(t, u)$$

uniformly on $R^+ \times K$ because $f(t, u)$ is almost periodic. Since X is a reflexive Banach space, we can extract from the sequence (s_n), a subsequence which we still denote (s_n) such that $(y(t_0 - s_n))$ is weakly convergent to $z \in X$. We have

$$\lim_{n \to \infty} y(t - s_n) = T(t - t_0)z + \int_{t_0}^{t} S(t - s)f(s, u(s))ds$$

in the weak sense for every $t \in R^+$. Now we consider the function

$$z(t) = T(t - t_0)z + \int_{t_0}^{t} S(t - s)f(s, u(s))ds.$$

It is a bounded mild solution of equation (3.1). Similarly as above, we have $\mu(z) \leq \mu(y)$, therefore $\mu(z) < \mu^*$, which is absurd by definition of μ^*.

Lemma 3.11

$$\mu(y) = \inf_{v \in \Omega_g} \mu(v)$$

i.e. $y(t)$ is an optimal mild solution of the equation

$$\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = g(t, u(t)), t \in R^+. \tag{3.18}$$

Proof

By lemma 3.10, $y(t)$ is bounded over R. Also $y(t)$ is a mild solution of (3.18) which means $y(t) \in \Omega_g$. It remains to prove that $y(t)$ is optimal. Suppose it is not. Since $\Omega_g \neq \emptyset$, by theorem 3.6, there exists a unique optimal solution $v(t)$ of (3.18). We have $\mu(v) < \mu(y)$ and

$$v(t) = T(t - t_0)v(t_0) + \int_{t_0}^{t} S(t - s)g(s, u(s))ds,$$

for all $t_0 \in R, t \geq t_0$. We can find a subsequence $(s_{n_k}) \subset (s_n)$ such that

$$weak - \lim_{k \to \infty} v(t - s_{n_k}) = T(t - t_0)z + \int_{t_0}^{t} S(t - s)f(s, u(s))ds \equiv V(t).$$

Noting that $V(t) \in \Omega_f$ and

$$\mu(V) \leq \mu(v) < \mu(y)$$

which is absurd. Therefore $y(t)$ is an optimal mild solution of (3.18), and in fact the only one by theorem 3.6.

Proof of Theorem 3.8

To prove that $u(t)$ is weakly almost periodic, it suffices to show that

$$weak - \lim_{n\to\infty} u(t+s_n) = y(t)$$

uniformly in $t \in R^+$. Suppose that this does not hold true; then there exists $u^* \in X^*$ such that

$$\lim_{n\to\infty} < u^*, u(t+s_n) >=< u^*, y(t) >$$

is not uniform in $t \in R^+$. Consequently, we can find a number $\gamma > 0$, and a sequence (t_k) with two subsequences (s'_k) and (s''_k) of (s_n) such that

$$| < u^*, u(t+s'_k) - u(t+s''_k) > | > \gamma \tag{3.19}$$

for all $k = 1, 2, ...$

Again, let us extract two subsequences of (s'_k) and (s''_k) respectively, with the same notation, such that

$$\lim_{k\to\infty} f(t+t_k+s'_k, u) = g_1(t,u), \text{ and } \lim_{k\to\infty} f(t+t_k+s''_k, u) = g_2(t,u)$$

both uniformly on $R^+ \times K$, because f is almost periodic. As we did previously, we may obtain

$$weak - \lim_{k\to\infty} f(t+t_k+s'_k, u) = T(t-t_0)z_1 + \int_{t_0}^{t} S(t-s)g_1(s,u(s))ds \equiv y_1(t),$$

and

$$weak - \lim_{k\to\infty} f(t+t_k+s''_k, u) = T(t-t_0)z_2 + \int_{t_0}^{t} S(t-s)g_2(s,u(s))ds \equiv y_2(t)$$

for each $t \in R^+$, where $y_1(t)$ and $y_2(t)$ are optimal mild solutions in Ω_{g_1} and Ω_{g_2}, respectively.

Since $\lim_{k\to\infty} f(t+t_k+s_k,u)$ exists uniformly on $R\times K$, and $(s'_k),(s''_k)$ are two subsequences of (s_k), we will get

$$\sup_{s\in R}|f(s+s'_k,u)-f(s+s''_k,u)|<\epsilon$$

if $k\geq k_0(\epsilon)$ and consequently

$$\sup_{s\in R}|f(t+t_k+s'_k,u)-f(t+t_k+s''_k,u)|<\epsilon$$

for $k\geq k_0(\epsilon)$, which shows that $g_1(s,u(s))=g_2(s,u(s))$ for all $s\in R^+$. By the uniqueness of the optimal mild solution we get $y_1(t)=y_2(t), t\in R$. But $y_1(0)=weak-\lim_{k\to\infty} u(t_k+s'_k)$ and $y_2(0)=weak-\lim_{k\to\infty} u(t_k+s''_k)$. Clearly $y_1(0)=y_2(0)$ contradicts the inequality (3.19) above. This completes the proof of theorem.

Chapter 4

On Some Fractional Integro-Differential Equations With Analytic Semigroups

4.1 Introduction

Some physical phenomena involving certain type of memory effects are represented by

$$\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = f(t, u(t)) + \int_{t_0}^{t} a(t-s)g(s, u(s))ds, \tag{4.1}$$

$$u(t_0) = u_0, \tag{4.2}$$

in Banach space X, where $0 < \alpha \leq 1, t > t_0$, let J denote the closure of the interval $[t_0, T)$, $t_0 < T \leq \infty$ and let $-A$ be the infinitesimal generator of an analytic semigroup $Q(t), t \geq 0$, the function a is real-valued and locally integrable on $[0, \infty)$,and the nonlinear maps f and g are defined on $[0, \infty) \times X$ into X.

This type of research has been considered in Bahuguna [3], when the equation (4.1) is given with conventional (classical) derivatives, also as several works; see for example [4, 6, 8, 9, 10, 18, 37, 51, 61] and reference listed therein. There is also an extensive literature for the same question in many mathematical models when the differential equations contains fractional derivatives, which provide an excellent instrument for the description of memory and hereditary properties of various materials and processes.

This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials.
As new in this chapter, we are concerned with fractional order. Our work is organized as follows, section 2 is devoted to review of some essential results on fractional calculus. In section 3, we establish the existence of a unique local mild solution of (4.1), (4.2). In section 4, we study the regularity of the mild solution of the considered problem and show under additional condition of Hölder continuity on a that this mild solution is in fact the classical solution. In section 5, As an example, nonlinear integro-partial differential equation of fractional order is also provided to illustrate the abstract results.

4.2 Preliminaries

Following Gelfand and Shilov [34], we define the fractional integral of order $\alpha > 0$ as

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

also, the fractional derivative of the function f of order $0 < \alpha < 1$ as

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(s)(t-s)^{-\alpha} ds,$$

where f is an abstract continuous function on the interval $[a, b]$ and $\Gamma(\alpha)$ is the Gamma function, see [22].

Definition 4.1

By a classical solution of (4.1), (4.2) on J, we mean a function u with values in X such that:

1) u is continuous function on $[0,T)$ and $u(t) \in D(A)$,

2) $\frac{d^\alpha u}{dt^\alpha}$ exists and continuous on $(0,T)$, $0 < \alpha < 1$, and u satisfies (4.1) on $(0,T)$ and the initial condition (4.2).

By a local classical solution of (4.1), (4.2) on J, we mean that there exist a T_0, $0 < T_0 < T$, and a function u defined from $J_0 = [t_0, T_0]$ into X such that u is a classical solution of (4.1), (4.2). It is suitable to rewrite the considered problem (4.1), (4.2) in the form

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1}[f(s,u(s)) - Au(s) + K(u)(s)]ds, \qquad (4.3)$$

where

$$K(u)(t) = \int_{t_0}^{t} a(t-s)g(s,u(s))ds.$$

According to [22, 25-29], the equation (4.3) is equivalent to the integral equation

$$\begin{aligned} u(t) &= \int_0^\infty \zeta_\alpha(\theta) Q((t-t_0)^\alpha \theta) u_0 \, d\theta \\ &+ \alpha \int_{t_0}^{t} \int_0^\infty \theta (t-s)^{\alpha-1} \zeta_\alpha(\theta) Q((t-s)^\alpha \theta) F(s) d\theta ds, \end{aligned} \qquad (4.4)$$

where $F(t) = f(t,u(t)) + K(u)(t)$ and ζ_α is a probability density function defined on $(0,\infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\theta x} \zeta_\alpha(\theta) d\theta = \sum_{j=0}^{\infty} \frac{(-x)^j}{\Gamma(1+\alpha j)}, 0 < \alpha \le 1, x > 0,$$

Definition 4.2

By a mild solution of (4.1), (4.2) on J, we mean a continuous function u defined from J into X satisfying the integral equation (4.4).

By a local mild solution of (4.1), (4.2) on J, we mean that there exist a T_0, $0 < T_0 < T$, and a function u defined from $J_0 = [t_0, T_0]$ into X such that u is a mild solution of (4.1), (4.2).

Let $-A$ is the infinitesimal generator of an analytic semigroup in a Banach space, then $-(A+qI)$ is invertible and generates a bounded analytic semigroup for $q > 0$ large enough. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence for convenience, we suppose that $\|Q(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$, where $\rho(-A)$ is the resolvent set of $-A$. We define the fractional power A^{-q} by

$$A^{-q} = \frac{1}{\Gamma(q)} \int_0^{\infty} (t)^{q-1} Q(t) dt, q > 0.$$

For $0 < q \leq 1$, A^q is a closed linear operator whose domain $D(A^q) \supset D(A)$ is dense in X, this implies that $D(A^q)$ endowed with the graph norm

$$\|u\|_{D(A)} = \|u\| + \|A^q u\|, u \in D(A^q)$$

is a Banach space, clearly $A^q = (A^{-q})^{-1}$ because A^{-q} is one to one. Since $0 \in \rho(A)$, A^q is invertible, and its graph norm is equivalent to the norm $\|u\|_q = \|A^q u\|$. Thus $D(A^q)$ equipped with the norm $\|.\|_q$ is a Banach space denoted by X_q, for more details we refer to [2, 3]. To state and prove the main results of this chapter, we shall require the following assumption on the map f and g:

(F): Let U be an open subset of $[0,\infty)\times X_q$, for every $(t,x)\in U$ there exist a neighborhood $V\subset U$ of (t,x) and constants $L>0, 0<\mu<1$ such that

$$\|f(s_1,u)-f(s_2,v)\|\leq L[|s_1-s_2|^{\mu}+\|u-v\|_q]$$

for all (s_1,u) and (s_2,v) in V.

4.3 Existence of Local Mild Solutions

We suppose that the analytic semigroup generated by $-A$ is bounded and that $-A$ is invertible. Furthermore, we assume that $0<T<\infty$ to establish local existence, for some different cases see [31, 43, 44, 60], we have the following Lemma:

Lemma 4.1

Let $-A$ be the infinitesimal generator of an analytic semigroup $Q(t)$.

If $0\in\rho(A)$ then

(a) $Q(t): X \longrightarrow D(A^q)$ for every $t>0$ and $q\geq 0$

(b) For every $u\in D(A^q)$, we have $Q(t)A^q u = A^q Q(t)u$

(c) For every $t>0$ the operator $A^qQ(t)$ is bounded and $\|A^qQ(t)\|\leq M_q t^{-q}$.

For more details, see [53, section 2.6].

Theorem 4.2

Suppose that the operator $-A$ generates the analytic semigroup $Q(t)$ with $\|Q(t)\|\leq M, t\geq 0$ and that $0\in\rho(-A)$. If the maps f and g satisfy **(F)** and the real valued map a integrable on J, then (4.1), (4.2) has a unique local mild solution for every $u_0 \in X_q$.

Proof

We fix a point (t_0, u_0) in the open subset U of $[0, \infty) \times X_q$ and choose $t_1' > t_0$ and $\epsilon > 0$ such that **(F)** holds for the functions f and g on the set

$$V = \{(t, x) \in U : t_0 \leq t \leq t_1', \|x - u_0\|_q \leq \epsilon\}. \tag{4.5}$$

Let

$$N_1 = \sup_{t_0 \leq t \leq t_1'} \|f(t, u_0)\|, \qquad N_2 = \sup_{t_0 \leq t \leq t_1'} \|g(t, u_0)\|. \tag{4.6}$$

Choose $t_1 > t_0$ such that

$$\|Q((t - t_0)^\alpha \theta) - I\| \ \|A^q u_0\| \leq \frac{\epsilon}{2},\ t_0 \leq t \leq t_1 \tag{4.7}$$

and

$$t_1 - t_0 < min\{t_1' - t_0, [\frac{\epsilon}{2} M_q^{-1} (1-q)\{(L\epsilon + N_1) + a_T (L\epsilon + N_2)\}^{-1}]^{\frac{1}{\alpha(1-q)}}\}, \tag{4.8}$$

where

$$a_T \ = \ \int_0^T |a(s)| \ ds. \tag{4.9}$$

Let $Y = C([t_0, t_1]; X)$ be endowed with the supremum norm

$$\|y\|_Y \ = \sup_{t_0 \leq t \leq t_1} \|y(t)\|.$$

Then Y is a Banach space. We define a map on Y by $\Phi y = \tilde{y}$ where $\tilde{y}$ is given by

$$\tilde{y}(t) \ = \ \int_0^\infty \zeta_\alpha(\theta) Q((t - t_0)^\alpha \theta) A^q u_0 \ d\theta$$

$$
\begin{aligned}
&+ \ \alpha \int_{t_0}^{t} \int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta) A^q Q((t-s)^\alpha\theta)[f(s, A^{-q}y(s)) \\
&\qquad\qquad + \int_{t_0}^{s} a(s-\tau) g(\tau, A^{-q}y(\tau)) d\tau] d\theta ds,
\end{aligned}
$$

Since $\int_0^\infty \zeta_\alpha(\theta)d\theta = 1$, for every $y \in Y$, $\Phi y(t_0) = A^q u_0$, and for $t_0 \le s \le t \le t_1$ we have

$$
\begin{aligned}
\Phi y(t) - \Phi y(s) &= \int_0^\infty \zeta_\alpha(\theta)[Q((t-t_0)^\alpha\theta) - Q((s-t_0)^\alpha\theta)]A^q u_0 \, d\theta \\
&+ \ \alpha \int_s^t \int_0^\infty \theta(t-\tau)^{\alpha-1}\zeta_\alpha(\theta) A^q Q((t-\tau)^\alpha\theta) \\
&\qquad [f(\tau, A^{-q}y(\tau)) + \int_{t_0}^{\tau} a(\tau-\eta) g(\eta, A^{-q}y(\eta)) d\eta] d\theta d\tau \\
&+ \ \alpha \int_{t_0}^{s} \int_0^\infty \theta(t-\tau)^{\alpha-1}\zeta_\alpha(\theta) A^q [Q((t-\tau)^\alpha\theta) - Q((s-\tau)^\alpha\theta)] \\
&\qquad [f(\tau, A^{-q}y(\tau)) + \int_{t_0}^{\tau} a(\tau-\eta) g(\eta, A^{-q}y(\eta)) d\eta] d\theta d\tau.
\end{aligned}
$$

It follows from **(F)** on the functions f and g, Lemma 4.1.c and (4.9) that $\Phi : Y \to Y$.

Let S be the nonempty closed and bounded set given by

$$
S = \{y \in Y : y(t_0) = A^q u_0, \|y(t) - A^q u_0\| \le \epsilon\}. \tag{4.10}
$$

Then for $y \in S$, we have

$$
\begin{aligned}
&\|\Phi y(t) - A^q u_0\| \le \|Q((t-t_0)^\alpha\theta) - I\| \, \|A^q u_0\| \\
&+ \ \alpha \int_{t_0}^{t} \int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta) \|A^q Q((t-s)^\alpha\theta)\| \, \|f(s, A^{-q}y(s)) - f(s, u_0)\| d\theta ds \\
&+ \ \alpha \int_{t_0}^{t} \int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta) \|A^q Q((t-s)^\alpha\theta)\| \\
&\qquad\qquad [\int_{t_0}^{s} |a(s-\tau)| \, \|g(\tau, A^{-q}y(\tau)) - g(\tau, u_0)\| d\tau] d\theta ds \\
&+ \ \alpha \int_{t_0}^{t} \int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta) \|A^q Q((t-s)^\alpha\theta)\| \, \|f(s, u_0)\| d\theta ds \\
&+ \ \alpha \int_{t_0}^{t} \int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta) \|A^q Q((t-s)^\alpha\theta)\| [\int_{t_0}^{s} |a(s-\tau)| \, \|g(\tau, u_0)\| d\tau] d\theta ds
\end{aligned}
$$

Since $\int_0^\infty \theta^{1-q}\zeta_\alpha(\theta)d\theta \leq 1$, using Lemma 4.1.c, (4.7) and (4.8) we get

$$\begin{aligned} \|\Phi y(t) - A^q u_0\| &\leq \frac{\epsilon}{2} + M_q(1-q)^{-1}\{(L\epsilon + N_1) + a_T(L\epsilon + N_2)\}(t_1 - t_0)^{\alpha(1-q)} \\ &\leq \epsilon. \end{aligned} \tag{4.11}$$

Thus $\Phi : S \to S$. Now we shall show that Φ is a strict contraction on S which will ensure the existence of a unique continuous function satisfying (4.4).

Let y and z two elements in S; then

$$\begin{aligned} \|\Phi y(t) - \Phi z(t)\| &= \|\tilde{y}(t) - \tilde{z}(t)\| \\ &\leq \alpha \int_{t_0}^{t}\int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)\|A^q Q((t-s)^\alpha\theta)\| \\ &\qquad\qquad \|f(s, A^{-q}y(s)) - f(s, A^{-q}z(s))\| d\theta ds \\ &+ \alpha \int_{t_0}^{t}\int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)\|A^q Q((t-s)^\alpha\theta)\| \\ &\quad [\int_{t_0}^{s} |a(s-\tau)| \; \|g(\tau, A^{-q}y(\tau)) - g(\tau, A^{-q}z(\tau))\| d\tau] d\theta ds \end{aligned}$$

Using assumption **(F)** on f and g, (4.9), Lemma 4.1.c and (4.8) respectively, we get

$$\begin{aligned} &\|\Phi y(t) - \Phi z(t)\| \\ &\leq \alpha L[(1 + a_T)\int_{t_0}^{t}\int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)\|A^q Q((t-s)^\alpha\theta)\| d\theta ds]\|y - z\|_Y \\ &\leq L(1 + a_T)M_q(1-q)^{-1}(t_1 - t_0)^{\alpha(1-q)}\|y - z\|_Y \\ &\leq \frac{1}{\epsilon}[(L\epsilon + N_1) + a_T(L\epsilon + N_2)]M_q(1-q)^{-1}(t_1 - t_0)^{\alpha(1-q)}\|y - z\|_Y \\ &\leq \frac{1}{2}\|y - z\|_Y. \end{aligned} \tag{4.12}$$

Thus Φ is a strict contraction map from S into S and therefore by the Banach contraction principle there exists a unique fixed point y in S such that

$$\Phi y = y = \tilde{y}. \tag{4.13}$$

Let $u = A^{-q}y$, using Lemma 4.1.b, we have

$$\begin{aligned} u(t) &= A^{-q}y(t) \\ &= \int_0^\infty \zeta_\alpha(\theta)Q((t-t_0)^\alpha\theta)u_0 \, d\theta \\ &+ \alpha\int_{t_0}^t\int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)F(s)d\theta ds, \end{aligned} \tag{4.14}$$

for every $t \in [t_0, t_1]$. Hence u is a unique local mild solution of (4.1), (4.2).

4.4 Regularity of Mild Solutions

In this section we establish the regularity of the mild solutions of (4.1), (4.2). Let J denote the closure of the interval $[t_0, T), t_0 < T \leq \infty$. In addition to the hypotheses mentioned in the earlier sections, we assume on the kernel a, that

(H): There exist constants $L_0 \geq 0$ and $0 < p \leq 1$ such that

$$|a(t_1) - a(t_2)| \leq L_0|t_1 - t_2|^p, \text{for all } t_1, t_2 \in J.$$

Theorem 4.3

Suppose that $-A$ generates the analytic semigroup Q(t) such that $\|Q(t)\| \leq M$ for all $t \geq 0$, and $0 \in \rho(-A)$. Further, suppose that the maps f and g

satisfy **(F)** and the kernel a satisfies **(H)**. Then (4.1), (4.2) has a unique local classical solution for each $u_0 \in X_q$.

Proof

From Theorem 4.2, it follows that there exist $T_0, t_0 < T_0 < T$ and a function u such that u is a unique mild solution of (4.1), (4.2) on $J_0 = [t_0, T_0)$ given by (4.14).

Let $v(t) = A^q u(t)$, then

$$v(t) = \int_0^\infty \zeta_\alpha(\theta) Q((t-t_0)^\alpha \theta) A^q u_0 \, d\theta$$
$$+ \ \alpha \int_{t_0}^t \int_0^\infty \theta (t-s)^{\alpha-1} \zeta_\alpha(\theta) A^q Q((t-s)^\alpha \theta)[\tilde{f}(s) + \int_{t_0}^s a(s-\tau)\tilde{g}(\tau) d\tau] d\theta ds, \tag{4.15}$$

where

$$\tilde{f}(t) = f(t, A^{-q} v(t)), \ \tilde{g}(t) = g(t, A^{-q} v(t)).$$

Since $u(t)$ is continuous on J_0 and the maps f and g satisfy **(F)**, it follows that $\tilde{f}$ and $\tilde{g}$ are continuous, and therefore bounded on J_0.

Let

$$M_1 = \sup_{t \in J_0} \|\tilde{f}(t)\| \text{ and } M_2 = \sup_{t \in J_0} \|\tilde{g}(t)\|. \tag{4.16}$$

By using the same method in [25, theorem 3.2], we can prove that $v(t)$ is locally Hölder continuous on J_0, then there exist a constant C such that for every $t_0' > t_0$, we have

$$\|v(t_1) - v(t_2)\| \leq C|t_1 - t_2|^p, \tag{4.17}$$

for all $t_0 < t_0' < t_1, t_2 < T_0$. Now, assumption **(F)** with (4.17) implies that there exist constants $k_1, k_2 \geq 0$ and $0 < \gamma, \eta < 1$ such that for all $t_0 < t_0' < t_1, t_2 < T_0$, we have

$$\|\tilde{f}(t_1) - \tilde{f}(t_2)\| \leq k_1|t_1 - t_2|^\gamma,$$

$$\|\tilde{g}(t_1) - \tilde{g}(t_2)\| \leq k_2|t_1 - t_2|^\eta,$$

which shows that $\tilde{f}$ and $\tilde{g}$ are locally Hölder continuous on J_0.

Let

$$h(t) = \tilde{f}(t) + \int_{t_0}^{t} a(t-\tau)\tilde{g}(\tau)d\tau$$

We shall show that $h(t)$ is locally Hölder continuous on J_0. For $t_2 \leq t_1$, we have

$$\begin{aligned}
\|h(t_1) - h(t_2)\| &= \|\tilde{f}(t_1) - \tilde{f}(t_2)\| + \int_{t_0}^{t_2} |a(t_1-\tau) - a(t_2-\tau)|\ \|\tilde{g}(\tau)\|d\tau \\
&\qquad + \int_{t_2}^{t_1} |a(t_1-\tau)|\ \|\tilde{g}(\tau)\|d\tau \\
&\leq k_1|t_1-t_2|^\gamma + M_2L_0T_0|t_1-t_2|^p + M_2a_{T_0}(2T_0)^{1-p}|t_1-t_2|^p \\
&\leq C^*|t_1-t_2|^\beta,
\end{aligned}$$

for some constants $C^* \geq 0$ and $0 < \beta < 1$. Consider the following Cauchy problem

$$\frac{d^\alpha v(t)}{dt^\alpha} + Av(t) = h(t), t > t_0, \tag{4.18}$$

$$v(t_0) = u_0. \tag{4.19}$$

By [25, theorem 3.1], (4.18), (4.19) has a unique solution v on J_0 into$X$ given by

$$v(t) = \int_0^\infty \zeta_\alpha(\theta)Q((t-t_0)^\alpha\theta)u_0 d\theta + \alpha\int_{t_0}^t\int_0^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)h(s)d\theta ds, \tag{4.20}$$

for $t > t_0$, each term on the right hand side belongs to $D(A)$, hence belongs to $D(A^q)$. Applying A^q on both sides of (4.20) and using the uniqueness of $v(t)$, we have that $A^q v(t) = u(t)$. It follows that u is the classical solution of (4.1), (4.2) on J_0, thus u is a unique local classical solution of (4.1), (4.2) on J.

4.5 Application

Consider the nonlinear integro-partial differential equation of fractional order

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \sum_{|q|\le 2m} a_q(x) D_x^q u(x,t) = F_u(x,t) + \int_{t_0}^t a(t-s)G_u(x,s)ds, \tag{4.21}$$

with the initial condition

$$u(x, t_0) = u_0(x), \tag{4.22}$$

where $t \in R^+, x \in R^n, D_x^q = D_{x_1}^{q_1}...D_{x_n}^{q_n}, D_{x_i} = \frac{\partial}{\partial x_i}$, $q = (q_1, ..., q_n)$ is an n-dimensional multi-index, $|q| = q_1 + ... + q_n$, and the operators F, G are defined as

$$F_u(x,t) = f(x, t, u(x,t), Du(x,t), ..., D^{2m-1}u(x,t)), \tag{4.23}$$

$$G_u(x,t) = g(x,t,u(x,t),Du(x,t),...,D^{2m-1}u(x,t)). \tag{4.24}$$

Let $L_2(R^n)$ be the set of all square integrable functions on R^n. We denote by $C^m(R^n)$ the set of all continuous real-valued functions defined on R^n which have continuous partial derivatives of order less than or equal to m. By $C_0^m(R^n)$ we denote the set of all functions $f \in C^m(R^n)$ with compact supports. Let $H_0^m(R^n)$ be the completion of $C_0^m(R^n)$ with respect to the norm

$$\|f\|_m^2 = \sum_{|q|\le m} \int_{R^n} |D_x^q f(x)|^2 dx.$$

It is supposed that

(i) The operator $A = -\sum_{|q|=2m} a_q(x) D_x^q$ is uniformly elliptic on R^n. In other words, all the coefficients $a_q, |q| = 2m$, are continuous and bounded on R^n and there is a positive number c such that

$$(-1)^{m+1} \sum_{|q|=2m} a_q(x)\xi^q \ge c|\xi|^{2m},$$

for all $x \in R^n$ and all $\xi \neq 0$, $\xi \in R^n$, where $\xi^q = \xi_1^{q_1}...\xi_n^{q_n}$ and $|\xi|^2 = \xi_1^2 + ... + \xi_n^2$.

(ii) All the coefficients $a_q, |q| = 2m$, satisfy a uniform Hölder condition on R^n. It is proved under these conditions, see [25, p.438], that the operator A defined by (i) with domain of definition $D(A) = H^{2m}(R^n)$ generates an analytic semigroup $Q(t)$ defined on $L_2(R^n)$, and it is well known that $H^{2m}(R^n)$ is dense in $X = L_2(R^n)$ and the initial function u_0 is an element in Hilbert space $H^{2m}(R^n)$. Which achieves the proof of the existence of mild solutions of the problem (4.21), (4.22). The operators F_u and G_u defined in

(4.23), (4.24) satisfy

(iii) There are numbers $L \geq 0$ and $0 \leq \lambda \leq 1$ such that

$$\sum_{|q|\leq 2m-1} \int_{R^n} |f(x,t,D_x^q u) - f(x,s,D_x^q v)|^2 dx \leq L(|t-s|^\lambda + \sum_{|q|\leq 2m-1} \int_{R^n} |A^q(u-v)|^2 dx).$$

for all $(t,u),(s,v)$ in $R^+ \times X_q$ and all $x \in R^n$. If the kernel a is integrable on $0 < t < T < \infty$, applying Theorems 4.2 stated above, we deduce that (4.21), (4.22) has a unique local mild solution. In addition, if the real valued map a satisfies the assumption **(H)**, again applying Theorem 4.3, we conclude that the considered problem has a unique local classical solution.

Conclusion

We have employed a new approach in the junction between "almost periodicity, optimal control and fractional calculus" that we hope will certainly attract attention from interest. We have tried in this work to show the reader the importance of fractional calculus and its potential application.
This thesis deals with the existence and uniqueness of solutions of classes of some abstract differential equations of fractional orders, $0 < \alpha \leq 1$, contain linear closed operators defined on dense sets in Banach space, these operators are assumed to be generate analytic semigroups, The solutions were obtained by using Gelfand-Shilov principle in fractional calculus and are given in terms of some probability density functions such that their Laplace transforms are indicated, see [25].
Under suitable conditions, the existence of optimal mild solutions for linear fractional evolution equations are proved, then the Bochner almost periodicity condition is used to show that these solutions are weakly almost periodic, also as application, a fractional partial differential equation of parabolic type is studied [22]. In addition, the same purpose for a class of semilinear fractional differential equations is proved under further assumptions [23]. Also, our basic tool was the fractional powers of operators to establish the existence, uniqueness and regularity of mild solutions of a class of nonlinear fractional integrodifferential equations [24].
We actually generalize a previous results by [2, 3, 49, 53, 63, 64, 65] and

reference listed therein, obtained in the context of differential equation in abstract spaces, their techniques were tested on some cases and were seen to produce satisfactory results. Examples which provide to illustrate the abstract results are given.

Since the results obtained in this research are new and this approach justifies its efficiency and presents quite promising results and provides a high degree of accuracy. The reliability of the fractional order gives this sense a wider applicability and the use of its technique presented in this work to solve some other models including the problems described in [22, 23, 24] can be an interesting investigation.

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