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p-ADIC ANALYSIS
AND SEQUENCES OF NUMBERS

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Notation

The following notation will be used throughout this thesis.

1. $\mathbb{N}$: The set of all natural numbers.
2. p : Denotes a fixed prime.
3. $\mathbb{Z}$: The ring of integers.
4. $\mathbb{Z}_p$: The ring of p-adic integers.
5. $\mathbb{Q}$: The field of rational numbers.
6. $\mathbb{R}$: The field of real numbers.
7. $\mathbb{Q}_p$: The field of p-adic rational numbers.
8. $\mathbb{C}$: The field of complex numbers.
9. $\mathbb{C}_p$: The completion of the algebraic closure of $\mathbb{Q}_p$.
10. $|\cdot|$: Denote the p-adic absolute value.
11. v_p : The normalized exponential valuation of $\mathbb{C}_p$ with $|p| = p^{-v_p(p)} = p^{-1}$.
12. B_n : The n-th Bernoulli number.
13. $B_n(x_1, \cdots, x_n)$: The complete exponential Bell polynomials with variables $x_1, \ldots, x_n$.
14. $B_{<r}(a)$: The open ball of radius r and center a.
15. $B_{<r}(a)$: The closed ball of radius r and center a.
16. $S_r(a)$: The sphere of radius r and center a.
17. $e_k(X_1, \ldots, X_m)$: The k-th elementary symmetric function of variable $X_1, \ldots, X_m$.
18. $s_k(X_1, \ldots, X_m)$: The k-th power sum symmetric polynomial of variable $X_1, \ldots, X_m$.
19. $h_k(X_1, \ldots, X_m)$: The complete homogeneous symmetric polynomial of variable $X_1, \ldots, X_m$.
20. H_m : The m-th harmonic number.
21. $H_m^{(s)}$: The m-th generalized harmonic number.
22. $H(n)$: The generalized harmonic number with $m = p - 1$.
23. $H(s_1, \ldots, s_k; m)$: The multiple harmonic sum.
24. $H(\{s\}^l)$: The multiple harmonic sum with $m = p - 1$.
25. $s(n, j)$: Stirling numbers of the first kind.

26. $\mathbb{K}[[x]]$: The set of formal power series over $\mathbb{K}$.

27. $||f||$: The Gauss norm of a power series.

28. $\mathbb{C}_p\{x\} := \left\{ f = \sum_{n\geq 0} a_n x^n \in \mathbb{C}_p[[x]] : a_n \to 0 \right\}$.

29. $(x)_n = x(x-1)\cdots(x-n+1)$: The Pochhammer Symbol of falling factorial.

30. For a given field $\mathbb{K}$ with characteristic zero, k an integer and $x \in \mathbb{K}$, the binomial coefficient is defined by

$$\binom{x}{k} = \begin{cases} \frac{(x)_k}{k!} = \frac{x(x-1)\cdots(x-(k-1))}{k!} & \text{if } k \geq 0, \\ 0 & \text{if } k < 0. \end{cases}$$

31. $\mathbb{F}_{p^n}$: The finite field with cardinal p^n.

Abstract

This thesis is essentially devoted to the study of extended congruences, especially, ones involving binomial coefficients, multiple harmonic sums and generalized harmonic numbers.

The methods utilized are various. At the beginning, we shall develop a technique with the aid of p-adic analysis, this would allow us to generalize some results.

We shall present some known results on extended congruences. Next, we will introduce new families of identities and congruences, they relate certain types of products of binomial coefficients with multiples harmonic numbers and generalized harmonic numbers. We may also note that congruences presented in the thesis have coefficients in the field p-adic complex numbers, and are also valid over the ring of p-adic integers and integers.

We shall conclude this thesis with a succinct study of common congruences in the mathematical literature which contains generalized harmonic numbers, as well as well-known families of numbers and polynomials.

Key words : p-adic analysis, p-adic numbers, generalized harmonic numbers, binomial coefficient, multiple harmonic numbers, congruences, extended congruences.

Résumé

Cette thèse est consacrée essentiellement à l'étude des congruences étendues, en particulier, les congruences qui contiennent les coefficients binomiaux, les nombres harmoniques multiples et généralisés.

Les méthodes utilisées sont variées. Au début, nous developperons une technique issue de l'analyse p-adique, ce qui nous permettera de généraliser quelques résultats.

Nous présenterons quelques traveaux sur les congruences étendues. Ensuite, nous introduirons de nouvelles familles d'identités et de congruences, elles relient certains types de produits de coefficients binomiaux aux nombres harmoniques homogènes multiples et aux nombres harmoniques généralisés. Il faut bien noter que les congruences présentes dans cette thèse ont des coefficients dans les nombres complexes p-adiques, elles restent valables dans l'anneau des entiers p-adiques et les entiers relatifs.

Nous conclurons cette thèse avec une étude sommaire des congruences répandues dans la littérature mathématique qui contiennent les nombres harmoniques généralisés ainsi que d'autres familles de nombres et polynômes.

Mots-Clés : Analyse p-adique, nombres p-adiques, nombres harmoniques généralisés, coefficient binomial, normbres harmoniques multiples, congruences, congruences étendues .

Introduction

The p-adic numbers were discovered by K. Hensel around the end of the nineteenth century. In the course of one hundred years, the theory of p-adic numbers has penetrated into several areas of mathematics, including number theory, algebraic geometry, algebraic topology and analysis.

The field of p-adic numbers $\mathbb{Q}_p$ is defined to the completion of $\mathbb{Q}$ with respect to the p-adic absolute value (see [11] for more details). The ring p-adic integers in the other hand are defined to be the closed unit ball of $\mathbb{Q}_p$. The p-adic norm turns out to be ultrametric, thus many peculiar proprieties are satisfied by $\mathbb{Q}_p$. Given a finite extension $\mathbb{K}$ of $\mathbb{Q}_p$ of degree n, the p-adic absolute value is extended by the formula

$$|\alpha| = |N_{\mathbb{K}/\mathbb{Q}_p}(\alpha)|^{\frac{1}{n}} = |\prod_{i=1}^{d} \sigma_i(\alpha)|^{\frac{[\mathbb{K}:\mathbb{Q}_p[\alpha]]}{n}} = |min_\alpha(0)|^{\frac{[\mathbb{K}:\mathbb{Q}_p[\alpha]]}{n}},$$

where $\sigma_1(\alpha), \ldots, \sigma_d(\alpha)$ are all the conjugates of α and $min(x)$ is the minimal polynomial of α over $\mathbb{Q}_p$ with degree d. Equivalently the p-adic valuation is extended as follows

$$v_p(\alpha) = \frac{1}{n} v_p(N_{\mathbb{K}/\mathbb{Q}_p}(\alpha)).$$

The p-adic field $\mathbb{Q}_p$ is not algebraically closed, in fact it admits algebraic extensions of arbitrary large degrees, its algebraic closure $\mathbb{Q}_p^{\text{alg}}$ is the union of all finite extensions of $\mathbb{Q}_p$. The field $\mathbb{Q}_p^{\text{alg}}$ is not complete. We define $\mathbb{C}_p$ to be its completion with respect to the p-adic absolute value.

The field $\mathbb{C}_p$ is complete, ultrametric, non-locally compact, algebraically closed, with an infinite residual field isomorphic to $\mathbb{F}_{p^\infty} = \bigcup_{n\in\mathbb{N}} \mathbb{F}_{p^n}$ (see [31], [36] for more details), we also have that

$$|\mathbb{C}_p^\times| = \{|x| : x \in \mathbb{C}_p\} = \{p^v : v \in \mathbb{Q}\} = p^{\mathbb{Q}} \quad \text{and} \quad v_p(\mathbb{C}_p^\times) = \{v_p(x) : x \in \mathbb{C}_p\} = \mathbb{Q}.$$

The congruence relation on $\mathbb{C}_p$ is defined by

$$a \equiv b \pmod{p^v} \quad \text{if and only if} \quad |a-b| \leq |p|^v \quad \text{where } v \in \mathbb{Q}.$$

p-adic numbers and p-adic analysis have many applications in mathematics, especially in number theory.

This thesis is divided into three chapters. The organization of the present work is as follows:

In chapter 1, we announce several analogues of classical theorems in calculus such as p-adic mean value theorem, p-adic Roll's theorem, also we present a p-adic version of the maximum principal much known in complex analysis. Next, we establish the "Higher p-adic Mean Value Theorem" which generalizes a theorem proved by A. Robert [31], [32]. It states that with proper restrictions for x, t, and the prime p, and for a given restricted power series $f(x)=\sum_{n\geq 0} a_n x^n \in \mathbb{C}_p\{x\}$, we have

$$\left| f(x+t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x)}{k!} t^k \right| \leq \left| \frac{t^m}{m} \right| \left|\left| f^{(m)} \right|\right|.$$

We give some applications of the HMVT, for instance we prove a weak version of Kazandzidiz congruence.

Chapter 2 is devoted to some extended congruences for binomial coefficients and generalized harmonic numbers. We begin this chapter by giving a short description of multiple harmonic sums. Next, we present two formulas due to M. Hoffman

$$\begin{aligned} e_n =& \frac{1}{n!}\ B_n(s_1, -1!s_2, 2!s_3, -3!s_4, \dots, (-1)^{n-1}(n-1)!s_n), \\ h_n =& \frac{1}{n!} B_n(s_1, 1!s_2, 2!s_3, \dots, (n-1)!s_n). \end{aligned}$$

When applying this theorem to $x_i = \frac{1}{i^n}$ we find a formula for (star) homogeneous multiple harmonic sums in terms of generalized harmonic numbers.

Bayat's theorem which asserts that for $p \geq n+3$, we have

$$H(m) \equiv \begin{cases} 0 \quad \pmod{p^2} & \text{if} \quad m \text{ is odd}, \\ 0 \quad \pmod{p} & \text{if} \quad m \text{ is even}, \end{cases}$$

will be of great importance when it comes to reducing congruences in this thesis.

In section 2.2, we shall explore J. Rosen approach for optimizing long expressions that may occur in congruences for $\binom{kp-1}{p-1}$ with a large power of p, they involve the fewest multiple harmonic sums among those congruences holding modulo the same power of p. These coefficients are given by polynomials (called extremal polynomials) in k. For a large enough prime we can write

$$\binom{kp-1}{p-1} \equiv \sum_{j=0}^{n} b_{j,n}(k) p^j H(\{1\}^j) \pmod{p^{2n+3}}.$$

In section 2.3, we will establish two formulas that relates the product $\prod_{\omega^n=1} \binom{\omega x-1}{p-1}$ as well as its inverse to generalized and (star) homogeneous multiple harmonic sums, this would allow us to derive new identities and congruences. More precisely, we establish the following two formal identities

$$\prod_{\omega^n=1} \binom{\omega x-1}{p-1} = \sum_{k=0}^{p-1} (-1)^k H(\{n\}^k) x^{kn} \quad \text{and} \quad \frac{1}{\prod_{\omega^n=1} \binom{\omega x-1}{p-1}} = \sum_{k\geq 0} H^\star(\{n\}^k) x^{kn}.$$

Using Bayat's theorem we can prove that for $v_p(\alpha) \geq \frac{\epsilon_n - 1}{n} - 1$ and $p \geq 5n+3$

$$\prod_{\omega^n=1} \binom{\alpha\omega p-1}{p-1} \equiv 1 - (\alpha p)^n H(n) - \frac{(\alpha p)^{2n}}{2} H(2n) - \frac{(\alpha p)^{3n}}{3} H(3n) - \frac{(\alpha p)^{4n}}{4} H(4n) \pmod{p^{2nv_p(\alpha)+2n+2\epsilon_n}}.$$

and in particular, when $n \geq 2$, and $v_p(\alpha) \geq \frac{\epsilon_n}{n} - 1$ this congruence can be further reduced to

$$\prod_{\omega^n=1} \binom{\alpha\omega p-1}{p-1} \equiv 1 - (\alpha p)^n H(n) - \frac{(\alpha p)^{2n}}{2} H(2n) \pmod{p^{2nv_p(\alpha)+2n+2\epsilon_n}},$$

where, for convenience we have denoted $\epsilon_n = 1$ if n is even and $\epsilon_n = 2$ if n is odd. We may note that congruences considered in this section are congruences in $\mathbb{C}_p$ which are also valid in $\mathbb{Z}_p$ and $\mathbb{Z}$.

In section 2.3.2, by using elementary operations and Bayat's cancellation theorem we prove for $p \geq 13$,

$$H(1) \equiv -\sum_{k=2}^{8} \frac{p^{k-1}}{k} H(k) \pmod{p^9}.$$

In the remaining sections, by specifications of values n and α, we provide several applications of these two formal identities. For $n = 1$, we prove that for non-zero α (with $v_p(\alpha) \geq 0$) and $p \geq 11$, we have

$$\binom{\alpha p - 1}{p - 1} \equiv 1 - \alpha p H(1) - \frac{(\alpha p)^2}{2} H(2) - \frac{(\alpha p)^3}{3} H(3) - \frac{(\alpha p)^4}{4} H(4) + \frac{\alpha^4 p^2}{2} (\frac{1}{\alpha} H(1) + \frac{p}{2} H(2))^2 \pmod{p^{3v_p(\alpha)+7}}.$$

For $n = 2$ with $p \geq 13$ and $v_p(\alpha) \geq -\frac{1}{2}$, we show

$$\binom{\alpha p - 1}{p - 1}\binom{(\alpha+1)p - 1}{p - 1} \equiv 1 - (\alpha p)^2 H(2) + \frac{(\alpha p)^4}{2}(H(2)^2 - 2H(4)) - \frac{(\alpha p)^6}{3} H(6) \pmod{p^{6v_p(\alpha)+8}}$$

For $n = 3$, we prove For $p \geq 17$,

$$\binom{\omega_1 p - 1}{p - 1}\binom{\omega_2 p - 1}{p - 1} \equiv 1 - p^3 H(3) + \frac{p^6}{2}(H(3)^2 - H(6)) - \frac{p^9}{3} H(9) \pmod{p^{12}},$$

where ω_1, ω_2 are the third roots of unity.

We conclude Chapter 2 by giving some congruences for $\prod_{\omega^n=1} \binom{\alpha\omega p-1}{p-1}^{-1}$. We illustrate by some examples. When $n = 2, 3, 4$ and $\alpha = 1$, we find

$$\frac{1}{\binom{2p-1}{p-1}} \equiv 1 + p^2 H(2) + \frac{p^4}{2}(H(2)^2 + H(4)) + \frac{p^6}{3} H(6) \pmod{p^8},$$
$$\frac{1}{\binom{\omega_1 p-1}{p-1}\binom{\omega_2 p-1}{p-1}} \equiv 1 + p^3 H(3) + \frac{p^6}{2}(H(3)^2 + H(6)) + \frac{p^9}{3} H(9) \pmod{p^{12}},$$
$$\frac{1}{\binom{2p-1}{p-1}\binom{ip-1}{p-1}\binom{(i+1)p-1}{p-1}} \equiv 1 + p^4 H(4) + \frac{p^8}{2}(H(4)^2 + H(8)) + \frac{p^{12}}{3} H(12) \pmod{p^{14}}.$$

In Chapter 3, we begin by a brief introduction of the shuffle and stuffle product also known as harmonic product due to the following relation

$$H(\boldsymbol{s}; n) H(\boldsymbol{t}; n) = \sum_{\boldsymbol{r} \in \boldsymbol{s} \circledast \boldsymbol{t}} H(\boldsymbol{r}; n),$$

As an application of this formula we get

$$H(a)H(b) = H(b, a) + H(a, b) + H(a + b),$$
$$H(a)H(b, c) = H(a, b, c) + H(b, a, c) + H(b, c, a) + H(a + b, c) + H(b, a + c).$$

In section 3.2 we give an extension of kazandzidis congruence established by J. Zhao.

In section 3.3, we mainly prove

$$\sum_{j=1}^{p-1} \frac{H_j^{(s_1)} H_j^{(s_3)}}{j^{s_2}} = -H(s_1, s_2, s_3) + H(s_3, s_1 + s_2) + H(s_1 + s_2 + s_3) + H(s_3) H(s_1, s_2).$$

and

$$-\sum_{j=1}^{p-1} \frac{H_j^{(s_1)} H_j^{(s_3)} H_j^{(s_4)}}{j^{s_2}} = H(s_4) \sum_{j=1}^{p-1} \frac{H_j^{(s_1)} H_j^{(s_3)}}{j^{s_2}} + H(s_1, s_3, s_2, s_4) + H(s_3, s_1, s_2, s_4)$$
$$+ H(s_3, s_1 + s_2, s_4) + H(s_1 + s_3, s_2, s_4) + H(s_1, s_2 + s_3, s_4) + H(s_1 + s_2 + s_3, s_4).$$

When substituting with special values of s_i, we get various congruences. We start by determining $\sum_{j=1}^{p-1} \frac{H_j^{(s_1)} H_j^{(s_3)}}{j^{s_2}}$ modulo p, we prove that if $w = s_1 + s_2 + s_3$ is odd then

$$\sum_{j=1}^{p-1} \frac{H_j^{(s_1)} H_j^{(s_3)}}{j^{s_2}} \equiv \left[\frac{(-1)^{s_1+1}}{2w} \binom{w}{s_1} + \frac{(-1)^{s_3} + 2(-1)^{s_1+s_2}}{2w} \binom{w}{s_3} \right] B_{p-w} \pmod{p}.$$

In section 3.3.4 we prove some congruences modulo p that are not covered in previous section. For instance we prove that for $p \geq 11$, we have

$$\begin{aligned} 2\sum_{j=1}^{p-1} \frac{H_j^{(2)} H_j}{j^3} \equiv -3\sum_{j=1}^{p-1} \frac{H_j^{(3)} H_j}{j^2} \equiv -6\sum_{j=1}^{p-1} \frac{H_j^{(3)} H_j^{(2)}}{j} \equiv -3\sum_{j=1}^{p-1} \frac{(H_j)^2}{j^4} \equiv 6\sum_{j=1}^{p-1} \frac{H_j^{(4)} H_j}{j} \\ \equiv B_{p-3}^2 \pmod{p}, \end{aligned}$$

In section 3.3.5, we determine some congruences modulo p^2 since determining (MHS) of length three seems to be a much more involved problem. For example we show that for $p \geq 7$,

$$\frac{5}{4}\sum_{j=1}^{p-1} \frac{H_j^2}{j^2} \equiv -\frac{10}{7}\sum_{j=1}^{p-1} \frac{H_j^{(2)} H_j}{j} \equiv pB_{p-5} \pmod{p^2}.$$

In section 3.3.6 we determine modulo p some summations that use the sum $\sum_{j=1}^{p-1} \frac{H_j^{(s_1)} H_j^{(s_3)} H_j^{(s_4)}}{j^{s_2}}$. We prove that for $p \geq 11$,

$$\begin{aligned} -\frac{1}{13}\sum_{j=1}^{p-1} \frac{(H_j^{(2)})^2 H_j^{(3)}}{j^2} &\equiv \frac{3}{83}\sum_{j=1}^{p-1} \frac{(H_j^{(2)})^3}{j^3} \equiv B_{p-9} \pmod{p}, \\ -\frac{8}{21}\sum_{j=1}^{p-1} \frac{(H_j^{(2)})^2 H_j}{j^2} &\equiv \frac{1}{3}\sum_{j=1}^{p-1} \frac{(H_j^{(2)})^3}{j} \equiv B_{p-7} \pmod{p}. \end{aligned}$$

In the last section, using results on summation of harmonic numbers due to J. Spieß [16], [37] we give some congruences that involve some special cases of the product $\prod_{\omega^n=1} \binom{\omega\alpha p-1}{k}^{-1}$, for instance we prove that for modulo $p^{2v_p(\alpha)+4}$,

$$\sum_{k=1}^{p-1} \frac{H_k}{\binom{\alpha p-1}{k}\binom{\alpha p+k}{k}} \equiv ((1-\alpha^2)p + (\alpha p)^2)H(1).$$

This thesis is concluded by giving some open questions and remarks that we encountered during preparation of this work.

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Chapter 1

The higher p-adic mean value theorem

We assume the reader is familiar with standard notions in p-adic analysis. The interested reader may consult [1], [2],[8],[11], [19], [30], [31], [36], for proofs as well as more insight on p-adic analysis.

We begin by presenting an ultrametric version of the maximum principle. This principle fails in the case of locally compact spaces such as $\mathbb{Q}_p$ and its finite algebraic extensions. We recall that a function $f\colon D \to \mathbb{K}$ defined on a ball B of an ultrametric field $\mathbb{K}$ is said to be analytic if there are elements $u \in B$ and $a_0, a_2, \cdots \in \mathbb{K}$ such that for all $x \in B$,

$$f(x) = \sum_{n=0}^{\infty} a_n (x-u)^n.$$

Theorem 1. *[36](p-adic maximum principle) If $\mathbb{K}$ is not locally compact, then for every analytical function defined on $\mathcal{O}_{\mathbb{K}}$. let $r \in |\mathbb{K}^{\times}|$*

- *If the valuation is dense, then*

$$\sup_{|x|\leq r} |f(x)| = \sup_{|x|<r} |f(x)| = \max_{n\geq 0} |a_n| r^n.$$

- *If the the residue field in infinite then*

$$\max_{|x|\leq r} |f(x)| = \max_{|x|=r} |f(x)| = \max_{n\geq 0} |a_n| r^n.$$

1.1 The p-adic mean value theorem

We mainly focus our attention to A. Robert theorems, which has many applications, especially in establishing congruences for special numbers. Let $\mathbb{K}$ be an ultrametric field, before proceeding further we may introduce the following sub-vector space which will be in use next sections

$$\mathbb{K}\{x\} = \left\{ f = \sum_{n\geq 0} a_n x^n \in \mathbb{K}[[x]] : a_n \to 0 \right\}.$$

For a power series $f = \sum_{n\geq 0} a_n x^n \in \mathbb{K}\{x\}$, we define its Gauss norm (or sup norm) to be

$$||f|| = \sup_{n\geq 0} |a_n|.$$

This subspace is the completion of the polynomial space for the Gauss norm (see [31] page 233 for more details), we also have

$$\mathbb{K}[x] \subset \mathbb{K}\{x\} \subset \mathbb{K}[[x]].$$

Theorem 2. *[17], [31], [32] (p-adic mean value theorem (MVT) of order one and two) Let $(\mathbb{E}, |\cdot|)$ an ultrametric Banach space over $\mathbb{K}$, and $f(x) \in \mathbb{E}[x]$ a polynomial with coefficients in $\mathbb{E}[x]$, suppose that $x, t \in \mathbb{E}$ with $|x| \leq 1$, then*

1. *If $|t| \leq |p|^{\frac{1}{p-1}}$, then $|f(x+t) - f(x)| \leq |t| \, ||f^{'}||$.*
2. *If $p \geq 3$ and $|t| \leq |p|^{\frac{1}{p-2}}$, then $\left|f(x+t) - f(x) - tf^{'}(x)\right| \leq \left|\frac{t^2}{2}\right| \left|\left|f^{''}\right|\right|$,*

   furthermore, if $p = 2$, we still have the same result as long as we restrict the values of $|t| \leq |2|^{\frac{1}{2}}$.

Proof. We shall establish a more general statement in next section. □

1.1.1 Some applications of the MVT of order one and two

A. Robert together with his students M. Zuber [49], A. Junod [17], have investigated various applications of the p-adic mean value theorem. In this section we announce without proofs many of their results.

Theorem 3. *[31], [32](p-adic Fixed point theorem) Let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_p$, $R = B_{\leq 1}(\mathbb{K})$ its closed unit ball, and $f \in \mathbb{K}[x]$ a restricted formal power series with $||f|| \leq 1$. Assume*

$$\left|\left|f^{'}\right|\right| < 1 \text{ and } \inf_{x \in R} |f(x) - x| \leq |p|^{\frac{1}{p-1}}.$$

Then f has a fixed point in R.

Theorem 4. *[31], [32]*

1. *Let $f \in \mathbb{C}_p[[x]]$ with $r_f > 1$ (hence $f \in \mathbb{C}_p\{x\}$), suppose that f has two distinct zeros $a \neq b$ in $B_{\leq 1}$ satisfying*

   $$|a - b| \leq |p|^{\frac{1}{p-1}}.$$

   Then, f' has a zero $B_{\leq 1}$.

2. *Let $f \in \mathbb{C}_p[[x]]$ with $r_f > 1$ (hence $f \in \mathbb{C}_p\{x\}$), suppose that f has two distinct zero in $a \neq b$ in $B_{\leq 1}$ satisfying*

   $$|a - b| < |p|^{\frac{1}{p-1}}.$$

   Then, f' has a zero in $B_{<1}$.

Corollary 1. *[31], [32] (p-adic Roll's theorem)*

1. *Let $f \in \mathbb{C}_p[[x]]$ with $r_f > 1$. Then for each $a, b \in \mathcal{O}_{\mathbb{C}_p}$ with $|a - b| \leq |p|^{\frac{1}{p-1}}$, there is a point $\xi \in \mathcal{O}_{\mathbb{C}_p}$ such that*

   $$f(b) - f(a) = (b - a)f'(\xi).$$

2. *If* $|a-b| < |p|^{\frac{1}{p-1}}$, *there is a point* $\eta \in \mathcal{M}_{\mathbb{C}_p}$ *such that*

$$f(b) - f(a) = (b-a)f'(\xi).$$

The following theorem is due to M. Zuber, it is an application of the MVT to congruences for a class of polynomials called "Appell family" or "Appell sequence" which generalizes the definition of Bernoulli and Euler's polynomials.

Theorem 5. *[31], [32], [17], [49] (Zuber Theorem) For an Appell family* $(A(t))_{n\in\mathbb{N}}$ *in* $\mathbb{Z}_p[t]$ *(that is* $A_n'(t) = nA_{n-1}(t)$ *and* $A_0(t) \neq 0$*) the following conditions are equivalents*

1. $A_{np}(t) \equiv A_n(t^p) \mod np\mathbb{Z}_p[t]$ *for* $n \geq 0$.
2. *There exists* $a \in \mathbb{Z}_p$ *such that* $A_{np}(a) \equiv A_n(a^p) \mod np\mathbb{Z}_p$ *for* $n \geq 0$.
3. *There exists* $a \in \mathbb{Z}_p$ *such that* $A_{np}(a) \equiv A_n(a) \mod np\mathbb{Z}_p$ *for* $n \geq 0$.

Proof. The original proof given by Zuber uses ("Spitzer identity" see [49] pages 16-17) and ("Barsky's theorem" see [49] pages 15-16) . The reader can find a more simpler approach in [17] (pages 7-8) that uses solely the MVT. □

Applications of the MVT and Zuber theorem are numerous we give a short list of some results. Let us first introduce the n-th Euler polynomials of order r by means of the generating function

$$\left(\frac{2}{e^x+1}\right)^r e^{xt} = \sum_{n\geq 0} E_n^r(t)\frac{x^n}{n!}$$

and n-th Bernoulli polynomials of order r

$$\left(\frac{x}{e^x-1}\right)^r e^{xt} = \sum_{n\geq 0} B_n^r(t)\frac{x^n}{n!}.$$

We have that $E_n^0(t) = B_n^0(t) = t^n$ and when $r = 1$ we get the usual polynomials of Euler $E_n(t)$ and Bernoulli $B_n(t)$. One should take precaution that $B_n^r(t)$ is not the r-th power of $B_n(t)$.

Theorem 6. *[17] For odd prime* p *and non-negative integer* n*, and for* $a \in \mathbb{Z}_p$*, we have*

1. $E_{np}^r(t) \equiv E_n^r(t^p) \mod np\mathbb{Z}_p[t]$.
2. *For odd prime* $E_{m+np}^r(t) \equiv E_{m+n}^r(t) \mod np\mathbb{Z}_p$.
3. *For a positive integer* $m \geq 1$ *and* $p-1$ *not dividing* $m+n$*, then the Kummer congruence can be improved as follows*
$$\frac{B_{m+np}(a)}{m+np} \equiv \frac{B_{m+n}(a)}{m+n} \mod np\mathbb{Z}_p.$$
4. *If* c *is not divisible by* $(p-1)p^s$ *(s≥ 0) and* $p-1$ *does not divide* m*, then*
$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{B_{m+kc}}{m+kc} \in p^{\min\{m-1,n(s-1)\}}.$$
5. *Let* $\sigma_p(r)$ *be the sum of digits in the* p*-adic expansion of* $r \geq 1$*. Then* $p^{\sigma_p(r)}B_n^r(a) \in \mathbb{Z}_p$.
6. *For all prime* p*, we have* $p^rB_{np}^r(t) \equiv p^rB_n^r(t^p) \mod \frac{np}{2}\mathbb{Z}_p[t]$.
7. $p^rB_{np}^r(a) \equiv p^rB_n^r(a) \mod \frac{np}{2}\mathbb{Z}_p$.

1.2 Higher p-adic mean value theorem HMVT

In this section, we generalize the p-adic mean value theorem, a result proved by A. Robert see [31], [32]. To prove this generalization we may follow the same steps as in ([31] pages 243-245 and 249-250), only we observe that a slight modification of the proof is made to find a more general statement.

Lemma 1. *Let D be the differentiation operator defined on the vector space $\mathbb{C}_p\{x\}$, then for all non-negative integers $n \in \mathbb{N}$, we have*

$$\left|\left|\frac{D^n}{n!}\right|\right| \leq 1.$$

Proof. Let $g(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{C}_p\{x\}$, then

$$D^n g(x) = g^{(n)}(x) = \sum_{i\geq n} i(i-1)\dots(i-n+1)a_i x^{i-n}.$$

By definition of the norm operator we get

$$\begin{aligned}\left|\left|\frac{D^n g}{n!}\right|\right| = \left|\left|\frac{g^{(n)}}{n!}\right|\right| &= \sup_{i\geq n}\left|\frac{i(i-1)\dots(i-n+1)}{n!}a_i\right| \\ &= \sup_{i\geq n}\left|\binom{i}{n}a_i\right| \leq \sup_{i\geq n}|a_i| \leq \sup_{i\geq 0}|a_i| = ||g||\,,\end{aligned}$$

which means that $\left|\left|\frac{D^n g}{n!}\right|\right| \leq ||g||$ and $\left|\left|\frac{D^n}{n!}\right|\right| \leq 1$.

□

Theorem 7. *(Higher mean value theorem HMVT) Let $f(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{C}_p\{x\}$, then*

$$\left|f(x+t) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x)}{k!}t^k\right| \leq \left|\frac{t^m}{m}\right|\left|\left|f^{(m)}\right|\right|$$

for all $x, t \in \mathbb{C}_p$, provided that $|x| \leq 1$ and

$$|t| \leq |p|^{\frac{1}{p-m}} \text{ if } m < p \qquad \text{and} \qquad |t| \leq |p|^{\frac{1}{p}} \text{ if } m = p.$$

Proof. We first consider the case $m < p$. Using the Taylor expansion we write

$$f(x+t) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x)}{k!}t^k = t^m \sum_{k\geq m}\frac{t^{k-m}}{k(k-1)...(k-m+1)}\frac{D^{k-m}f^{(m)}(x)}{(k-m)!}.$$

Hence, when $|x| \leq 1$, the ultrametric inequality implies

$$\left|f(x+t) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x)}{k!}t^k\right| \leq |t^m| \sup_{k\geq m}\left|\frac{t^{m-k}}{k(k-1)..(k-m+1)}\right|\left|\left|\frac{D^{k-m}f^{(m)}}{(k-m)!}\right|\right|.$$

According to Lemma 1

$$\left|\left|\frac{D^{k-m}f^{(m)}}{(k-m)!}\right|\right| \leq \left|\left|f^{(m)}\right|\right| \qquad \text{and} \qquad \left|\left|\frac{D^{k-m}f^{(m)}}{(k-m)!}\right|\right| \to 0 \quad (k\to\infty).$$

Therefore, it only remains to show that

$$\left|\frac{t^{k-m}}{k(k-1)..(k-m+1)}\right| \leq 1 \quad (\text{ for } k \geq m).$$

Notice that the prime number p divides at most one of the consecutive integers $k, k-1, \dots, k-m+1$. If $v_p(k-l) = v \geq 1$ for some $l \in \{0, \dots, m-1\}$, then $k \geq p^v + l$, a fortiori, $k \geq p^v$, we also have

$$|k-n| = 1 \text{ if } n \neq l \qquad \text{and} \qquad |k-l| = |p|^v \text{ if } n = l,$$

therefore

$$\left|\frac{t^{k-m}}{k(k-1)..(k-m+1)}\right| \leq \frac{|t|^{p^v-m}}{|p|^v} \leq |p|^e,$$

where e is the exponent

$$e = \frac{p^v - m}{p-m} - v \geq \frac{p^v-1}{p-1} - v = (1 + p + \cdots + p^{v-1}) - v \geq 0,$$

since the mapping $x \mapsto \frac{p^v-x}{p-x}$ is increasing in the interval $[0, p)$.

Next, we consider the case $m = p$. We write

$$f(x+t) - \sum_{k=0}^{p-1} \frac{f^{(k)}(x)}{k!} t^k = \frac{t^p}{p} \sum_{k \geq p} \frac{pt^{k-p}}{k(k-1)...(k-p+1)} \frac{D^{k-p} f^{(p)}(x)}{(k-p)!},$$

and as in the previous proof it only remains to show that

$$\left|\frac{pt^{k-p}}{k(k-1)...(k-p+1)}\right| \leq 1.$$

Assume $v_p(k) = v \geq 1$. Then the condition for t is $|t| \leq |p|^{\frac{1}{p}}$ which is equivalent to say that $|t|^p \leq |p|$, thus

$$\frac{|pt^{k-p}|}{|p|^v} = \frac{|t|^{k-p}}{|p|^{v-1}} \leq \frac{|t|^{p^v-p}}{|p|^{v-1}} = \frac{|t|^{p(p^{v-1}-1)}}{|p|^{v-1}} \leq |p|^{p^{v-1}-1-(v-1)} = |p|^{p^{v-1}-v}$$

and $p^{v-1} - v \geq 0$ for $v \geq 1$. The other cases when $v_p(k-l) = v \geq 1$ for some $l \in \{1, \dots, p\}$ are proved in a similar manner. □

Theorem 8. *(HMVT for ultrametric Banach spaces) Let $(\mathbb{E}, |\cdot|)$ an ultrametric Banach space over $\mathbb{K}$, and $f(x) \in \mathbb{E}[x]$ a polynomial with coefficients in $\mathbb{E}[x]$, suppose that $x, t \in \mathbb{E}$ with $|x| \leq 1$, then*

$$\left|f(x+t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x)}{k!} t^k\right| \leq \left|\frac{t^m}{m}\right| \left|\left|f^{(m)}\right|\right|,$$

provided that

$$|t| \leq |p|^{\frac{1}{p-m}} \textit{ if } m < p \qquad \textit{and} \qquad |t| \leq |p|^{\frac{1}{p}} \textit{ if } m = p.$$

Proof. Just replace in the above proof $|a_k|$ by $||a_k||$ and $|f(t)|$ by $||f(t)||$, whenever necessary. □

An equivalent reformulation of the HMVT states as follows.

Theorem 9. *(HMVT Congruence form) Let $f(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{C}_p\{x\}$, then*

$$f(x+t) \equiv \sum_{k=0}^{m-1} \frac{f^{(k)}(x)}{k!} t^k \pmod{p^{mv_p(t) - \log_p(||f^{(m)}||)}} \qquad \textit{if } m < p,$$

$$f(x+t) \equiv \sum_{k=0}^{p-1} \frac{f^{(k)}(x)}{k!} t^k \pmod{p^{pv_p(t) - \log_p(||f^{(p)}||)) - 1}} \qquad \textit{if } m = p,$$

for all $x, t \in \mathbb{C}_p$ such that $v_p(x) \geq 0$, and

$$v_p(t) \geq \frac{1}{p-m} \textit{ if } m < p \qquad \textit{and} \qquad v_p(t) \geq \frac{1}{p} \textit{ if } m = p.$$

Proof. Set $||f^{(m)}|| = |\alpha| = p^{-v(\alpha)}$ for some $\alpha \in \mathbb{C}_p$. Taking the base p logarithm we find $\log_p(||f^{(m)}||) = \log_p(|\alpha|) = -v_p(\alpha)$. When $m < p$ the equation in the previous theorem is equivalent to

$$\left| f(x+t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x)}{k!} t^k \right| \leq |t^m| \left|\left| f^{(m)} \right|\right| = p^{-v_p(t^m)} p^{-v_p(\alpha)} = p^{-v_p(t^m) + \log_p(||f^{(m)}||)}.$$

Therefore, using properties of valuations and the definition of congruence relation on $\mathbb{C}_p$ we get the desired result. □

In the following we shall give straightforward generalizations as well as a proof for Kazandzidis congruence.

1.2.1 First application

Theorem 10. *Let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_p$, R its ring of integers with maximal ideal P. For $n \in \mathbb{N}$ (or $n \in \mathbb{Z}_p$) and $m \leq \min(p, n)$, we have*

$$(1+x)^n \equiv \sum_{k=0}^{m-1} \binom{n}{k} x^k \mod pn(n-1)\cdots(n-m+1)x^{m-1}R,$$

provided that $x \in mpR$.

Proof. Consider $f(x) = (1+x)^n$, then

$$f^{(m)}(x) = n(n-1)\cdots(n-(m-1))(1+x)^{n-m} \qquad (n \geq m),$$
$$\left|\left| f^{(m)} \right|\right| = |n(n-1)\cdots(n-(m-1))| \qquad (n \geq 0).$$

For $|x| \leq |p|^{\frac{1}{p-m}}$ (or $\leq |p|^{\frac{1}{m}}$), we have

$$|(1+x)^n - \sum_{k=0}^{m-1} \binom{n}{k} x^k| \leq \left| \frac{x^m}{m} \right| \left|\left| f^{(m)} \right|\right| \leq |n(n-1)\cdots(n-(m-1))x^{m-1}| \left| \frac{x}{m} \right|.$$

As $x \in mpR$ implies that $\left|\frac{x}{m}\right| \leq |p|$, the above result yields immediately. □

1.2.2 Second application. Kazandzizis congruence

Proposition 1. *([17] pages 6-7) Let n, k be two integers with $k \geq 0$, then*

- *For all primes p, we have*
$$\binom{np}{kp} \equiv \binom{n}{k} \mod np\mathbb{Z}_p.$$
- *For odd primes p, we have*
$$\binom{np}{kp} \equiv \binom{n}{k} \mod \binom{n}{2} p^2\mathbb{Z}_p.$$

The following application generalizes the idea of A. Junod [17]. We apply the HMVT of order three and four ($p \geq 5$) to the polynomial

$$f(t) = (r(x)t + x^p + 1)^n \qquad \text{where} \qquad r(x) = \frac{(x+1)^p - (x^p+1)}{p}.$$

As $r(x) \in \mathbb{Z}[x] \subset \mathbb{Q}_p[x]$, the coefficients of $f(t)$ belong to the $\mathbb{Q}_p$-Banach space

$$\mathbb{E} = \{g(x) \in \mathbb{Q}_p[x] : \deg(g(x)) \leq np\}.$$

Substituting with $t = 0$ and $h = p$. We have

$$r(x) = \frac{(x+1)^p - (x^p+1)}{p} = \frac{\sum_{i=0}^{p} \binom{p}{i} x^i - x^p - 1}{p} = \frac{1}{p} \sum_{i=1}^{p-1} \binom{p}{i} x^i.$$

Hence, $|r(x)| = 1$ since p divide $\binom{p}{i}$ for $1 \leq i \leq p-1$ only once. Now, differentiating with respect to t, then substituting $t = 0$, we find

$$f'(0) = nr(x)(x^p+1)^{n-1}, \qquad f''(0) = n(n-1)r(x)^2(x^p+1)^{n-2},$$

$$f^{(3)}(0) = n(n-1)(n-2)r(x)^3(x^p+1)^{n-3}, \qquad f^{(4)}(0) = n(n-1)(n-2)(n-3)r(x)^4(x^p+1)^{n-4},$$

and more generally we can prove that

$$\frac{p^m}{m!} f^{(m)}(0) = p^m \binom{n}{m} r(x)^m (x^p+1)^{n-m} = \binom{n}{m} ((x+1)^p - (x^p+1))^m (x^p+1)^{n-m}. \tag{1.1}$$

We now look for the coefficients of x^{kp} $(0 \leq k \leq n)$.

Proposition 2. *(Chu-Vandermonde's convolution Identity) Let $\mathbb{K}$ be a field of characteristic zero, then for all $a, b \in \mathbb{K}$, we have*

$$\sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n}.$$

Lemma 2. *The coefficient of x^{kp} in the product $(x+1)^{mp}(x^p+1)^{n-m}$ is*

$$c_m(n,k) := \sum_{j=0}^{m} \binom{mp}{jp} \binom{n-m}{k-j} = \sum_{j=0}^{m} \binom{n-m}{k-j} \left(\binom{mp}{jp} - \binom{m}{j} \right) + \binom{n}{k}.$$

Proof. By binomial formula we have

$$(x+1)^{mp}(x^p+1)^{n-m} = \sum_{j=0}^{mp} \binom{mp}{j} x^j \sum_{i=0}^{n-m} \binom{n-m}{i} x^{ip}.$$

Identifying the coefficients $c_m(n,k)$ of x^{kp} in this product we find

$$c_m(n,k) = \sum_{j=0}^{m} \binom{mp}{jp} \binom{n-m}{k-j}.$$

The second equality is an immediate consequence of Vandermonde convolution identity. □

For instance, we have

$$c_0(n,k) = \binom{n}{k}, c_1(n,k) = \binom{n}{k},$$

$$c_2(n,k) = \binom{n-2}{k-1} \left(\binom{2p}{p} - 2 \right) + \binom{n}{k},$$

$$c_3(n,k) = \binom{n-2}{k-1} \left(\binom{3p}{p} - 3 \right) + \binom{n}{k},$$

$$c_4(n,k) = \binom{n-4}{k-2} \left(\binom{4p}{2p} - 2\binom{4p}{p} + 2 \right) + \binom{n-2}{k-1} \left(\binom{4p}{p} - 4 \right) + \binom{n}{k}.$$

In the last line we used the relation

$$\binom{n-4}{k-1}+\binom{n-4}{k-3}=\binom{n-2}{k-1}-2\binom{n-4}{k-2}.$$

Theorem 11. *We have modulo* $\binom{n}{m+1}p^{m+1}\mathbb{Z}_p$

$$\binom{np}{kp}\equiv\binom{n}{k}+\binom{n}{2}\sum_{j=0}^{2}(-1)^{2-j}\binom{2}{j}c_j(n,k)+\binom{n}{3}\sum_{j=0}^{3}(-1)^{3-j}\binom{3}{j}c_j(n,k)+\cdots+\binom{n}{m}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}c_j(n,k).$$

Proof. From (1.1) we extract the coefficient of x^{kp} from $\frac{p^l}{l!}f^{(l)}(0)$, and by previous lemma we find that it is equal to

$$\binom{n}{l}\sum_{j=0}^{l}(-1)^{l-j}\binom{l}{j}c_j(n,k).$$

One may also notice that the coefficient of x^{nk} in $\frac{p}{1!}f'(0)$ is equal to $\binom{n}{1}(c_0(n,k)-c_1(n,k))=0$. This would prove Proposition 1. ☐

Application of HMVT of order three: applying the HMVT we find

$$\left|\left|f(p)-f(0)-\frac{f'(0)}{1!}p-\frac{f''(0)}{2!}p^2\right|\right|\leq\left|\frac{p^3}{3}\right|\left|\left|f^{(3)}\right|\right|=|p|^3|n(n-1)(n-2)|,$$

that is to say

$$\left|(x+1)^{pn}-(x^p+1)^n-pnr(x)(x^p+1)^{n-1}-p^2\frac{n(n-1)}{2}r(x)^2(x^p+1)^{n-2}\right|$$
$$\leq|p|^3|n(n-1)(n-2)|.$$

Corollary 2. *We have modulo* $\binom{n}{3}p^3\mathbb{Z}_p$*, for* $n,k\in\mathbb{N}$ *and* $p\geq 5$*,*

$$\binom{np}{kp}-\binom{n}{k}\equiv\binom{n}{2}\binom{n-2}{k-1}\left(\binom{2p}{p}-2\right).$$

Note that both sides of this congruence are polynomials in n for each fixed value of k. The congruence therefore holds when n is any integer.

Corollary 3. *For* $p\geq 5$ *and* $n,k\in\mathbb{N}$*, the following assertions are equivalent*

1. $\binom{np}{kp}\equiv\binom{n}{k}\pmod{p^3}$.
2. $\binom{np}{p}\equiv n\pmod{p^3}$ *i.e* $\binom{np-1}{p-1}\equiv 1\pmod{p^3}$.
3. $\binom{2p}{p}\equiv 2\pmod{p^3}$ *i.e* $\binom{2p-1}{p-1}\equiv 1\pmod{p^3}$.

Theorem 12. *Let* n *and* k *be integers and* $p\geq 5$*. Then*

$$\binom{np}{kp}\equiv\binom{n}{k}\mod\binom{n}{2}p^3\mathbb{Z}_p. \tag{1.2}$$

In particular, we find Ljunggren's congruence

$$\binom{np}{kp}\equiv\binom{n}{k}\pmod{p^3}. \tag{1.3}$$

Proof. Replacing $n=-1$ and $k=1$ in Corollary 2 we find

$$\binom{-p}{p} = -\frac{1}{2}\binom{2p}{p} \equiv \binom{-1}{1} + \binom{-1}{2}\binom{-3}{0}\left(\binom{2p}{p} - 2\right).$$

rearranging we get

$$3\binom{2p}{p} \equiv 6 \pmod{p^3}$$

Hence, when $p \neq 3$ the second congruence holds true, and subsequently the first one yields immediately.

□

Remark 1. *For more refined results the reader can see [12], [31] pages 380-382.*

Application of HMVT of order four: as in order three, we have modulo $\binom{n}{4}p^4\mathbb{Z}_p$

$$\binom{np}{kp} - \binom{n}{k} \equiv \binom{n}{2}\binom{n-2}{k-1}\left(\binom{2p}{p} - 2\right) + \binom{n}{3}\binom{n-2}{k-1}\left(\binom{3p}{p} - 3\binom{2p}{p} + 3\right). \tag{1.4}$$

Theorem 13. *For $p \geq 5$ and $n, k \in \mathbb{N}$, the following assertions are equivalents.*

1. $\binom{np}{kp} \equiv \binom{n}{k} \pmod{p^4}$.
2. $\binom{np}{p} \equiv n \pmod{p^4}$ *i.e* $\binom{np-1}{p-1} \equiv 1 \pmod{p^4}$.
3. $\binom{2p}{p} \equiv 2 \pmod{p^4}$ *i.e* $\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}$ *(such p is called a Wolestenholme prime).*

Proof. Suppose that $\binom{2p}{p} \equiv 2 \pmod{p^4}$, then substituting with $n=-1$ and $k=1$ in the above formula, we have

$$\binom{-p}{p} = -\frac{1}{2}\binom{2p}{p} \equiv 4\binom{2p}{p} - \binom{3p}{p} - 6 \pmod{p^4},$$

and after simplification, we find

$$2\binom{3p}{p} \equiv 6 \pmod{p^4}.$$

Now, substituting with the values of $\binom{3p}{p}$ and $\binom{2p}{p}$ in (1.4), we get

$$\binom{np}{kp} - \binom{n}{k} \equiv \binom{n}{2}\binom{n-2}{k-1}(2-2) + \binom{n}{3}\binom{n-2}{k-1}(3 - 3\times 2 + 3) = 0 \pmod{p^4}.$$

Therefore, the first condition is satisfied if and only if p is a Wolstenholme prime. □

Chapter 2

Some extended congruences for binomial coefficients and generalized harmonic numbers

2.1 Multiple harmonic sums and generalized harmonic numbers

Multiple Harmonic Sums (MHS) are defined by

$$H(s_1,\dots,s_k;n) := \sum_{1\le j_1<j_2<\cdots<j_k\le n} \frac{1}{j_1^{s_1}\dots j_k^{s_k}},$$

with the conventions $H(s_1,\dots,s_k;r)=0$ for $r=0,\dots,k-1$, and, $H(\emptyset;0)=1$. They satisfy the following recurrence relation (see[42] page 2)

$$H(s_1,\dots,s_k;n) = \sum_{j=1}^{n} \frac{1}{j^{s_k}} H(s_1,\dots,s_{k-1};j-1).$$

In the case when $s_1=\cdots=s_k=s$, these sums are called the homogeneous multiple harmonic sums, and denoted

$$H(\{s\}^k;n) := H(\underbrace{s,\dots,s}_{\text{k times}};n) = \sum_{1\le j_1<j_2<\cdots<j_k\le n} \frac{1}{(j_1\dots j_k)^s}.$$

When $k=1$, we find the sequence of generalized harmonic numbers, we may denote it

$$H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s},$$

note that the superscript is omitted in the case $s=1$.

When $n=p-1$ we may simplify notations as follows

$$H(s_1,\dots,s_k) := \sum_{1\le j_1<j_2<\cdots<j_k\le p-1} \frac{1}{j_1^{s_1}\dots j_k^{s_k}} \quad \text{and} \quad H(s) := H_{p-1}^{(s)} = \sum_{k=1}^{p-1} \frac{1}{k^s}.$$

Let $X_1, X_2, \dots, X_m$ be m variables. We define the k-th elementary symmetric function

$$e_k(X_1,\dots,X_m) = \sum_{1\le j_1<j_2<\cdots<j_k\le m} X_{j_1}\cdots X_{j_k},$$

and k-th power sum symmetric polynomial

$$s_k(X_1, X_2, \ldots, X_m) = \sum_{i=1}^{m} X_i^k.$$

The elementary symmetric functions can also be characterized by

$$\prod_{i=1}^{m}(1 - X_i t) = \sum_{k=0}^{m}(-1)^k e_k t^k.$$

When fixing $m = p-1$, and $X_i = \frac{1}{i^n}$ for an arbitrary non-negative integer m, we obtain the following

$$e_k(1, \frac{1}{2^n}, \ldots, \frac{1}{(p-1)^n}) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le p-1} \frac{1}{j_1^n \ldots j_k^n} = H(\{n\}^k),$$
$$s_k(1, \frac{1}{2^n}, \ldots, \frac{1}{(p-1)^n}) = \sum_{i=1}^{p-1} \frac{1}{i^{nk}} = H(nk).$$

Another interesting symmetric function is the complete homogeneous symmetric polynomial defined by

$$h_k(X_1, X_2, \ldots, X_m) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le m} X_{i_1} X_{i_2} \cdots X_{i_k}.$$

It can also characterized by the following identity of formal power series in t

$$\sum_{k \ge 0} h_k(X_1, \ldots, X_m) t^k = \prod_{i=1}^{m} \sum_{j \ge 0} (X_i t)^j = \prod_{i=1}^{m} \frac{1}{1 - X_i t}.$$

When fixing $m = p-1$ and $X_i = \frac{1}{i^n}$ we denote

$$h_k(1, \frac{1}{2^n}, \ldots, \frac{1}{(p-1)^n}) = \sum_{1 \le j_1 \le j_2 \le \cdots \le j_k \le p-1} \frac{1}{j_1^n \ldots j_k^n} := H^\star(\{n\}^k).$$

The complete exponential Bell polynomials can be defined by means of the following exponential generating function

$$\exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) = 1 + \sum_{n \ge 1} B_n(x_1, \ldots, x_n) \frac{t^n}{n!},$$

where $\{x_1, x_2, \ldots\}$ is an arbitrary sequence. The exact expressions of them reads as

$$B_n(x_1, x_2, \ldots, x_n) = \sum_{j_1 + 2j_2 + \cdots + nj_n = n} \frac{n!}{j_1! j_2! \cdots j_n!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_n}{n!}\right)^{j_n},$$

where the summation takes over all non-negative integers $j_1, j_2, \ldots, j_n$. This explicit definition implies the following relation

$$B_n(ax_1, a^2 x_2, \ldots, a^n x_n) = a^n B_n(x_1, x_2, \ldots, x_n). \tag{2.1}$$

They also satisfy various combinatorial and algebraic identities (see for instance [5], [6], [7]).

Theorem 14. *For all non-negative integers $n \in \mathbb{N}$, we have the following formulas*

$$e_n = \frac{1}{n!} B_n(s_1, -1!s_2, 2!s_3, -3!s_4, \ldots, (-1)^{n-1}(n-1)!s_n),$$
$$h_n = \frac{1}{n!} B_n(s_1, 1!s_2, 2!s_3, \ldots, (n-1)!s_n).$$

Proof. See [13] (page 507 Proposition 1). Here we provide more details of a proof that can be found in [44] (page 1021), it uses basic properties of exponentials and logarithms.

From the identity

$$\sum_{k=0}^{n}(-1)^k e_k t^k = \prod_{i=1}^{n}(1-x_it),$$

and using well known properties of exponentials and logarithms we find

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k e_k t^k &= \prod_{i=1}^{n}(1-x_it) = \exp\left(\log(\prod_{i=1}^{n} 1-x_it)\right)\\ &= \exp\left(-\sum_{m\geq 1}\frac{(x_1t)^m}{m} - \cdots - \sum_{m\geq 1}\frac{(x_nt)^m}{m}\right)\\ &= \exp\left(\sum_{m\geq 1} -(m-1)!s_m\frac{t^m}{m!}\right)\\ &= \sum_{k=0}^{n}\frac{B_k(-s_1,-1!s_2,-2!s_3,-3!s_4,\ldots,-(k-1)!s_k)}{k!}t^k.\end{aligned}$$

Therefore, using (2.1) with $a=-1$, we deduce

$$\begin{aligned}e_k &= \frac{(-1)^k}{k!}\,B_k(-s_1,-1!s_2,-2!s_3,-3!s_4,\ldots,-(k-1)!s_k)\\ &= \frac{1}{k!}\,B_k(s_1,-1!s_2,2!s_3,-3!s_4,\ldots,(-1)^{k-1}(k-1)!s_k).\end{aligned}$$

The second one is proved in a similar manner using the identity

$$\sum_{k\geq 0} h_k(x_1,\ldots,x_n)t^k = \prod_{i=1}^{n}\frac{1}{1-x_it}.$$

□

Corollary 4. *For all non-negative integers $n\in\mathbb{N}$, the following two formulas hold true*

$$\begin{aligned}H(\{n\}^k) =& \frac{1}{k!}\,B_k(H(n),-1!H(2n),2!H(3n),\ldots,(-1)^{k-1}(k-1)!H(kn)),\\ H^\star(\{n\}^k) =& \frac{1}{k!}\,B_k(H(n),1!H(2n),2!H(3n),\ldots,(k-1)!H(kn)).\end{aligned}$$

With the aid of Maple, we can produce the following list for the homogeneous multiple harmonic numbers

$$H(\{n\}^0)=1, H(\{n\}^1)=H(n), H(\{n\}^2)=\frac{1}{2}\big(H(n)^2-H(2n)\big), \tag{2.2}$$

$$H(\{n\}^3)=\frac{1}{6}\big(H(n)^3-3H(2n)H(n)+2H(3n)\big), \tag{2.3}$$

$$H(\{n\}^4)=\frac{1}{24}\big(H(n)^4-6H(n)^2H(2n)+8H(3n)H(n)+3H(2n)^2-6H(4n)\big), \tag{2.4}$$

$$\begin{aligned}H(\{n\}^5)=\frac{1}{120}\big(&H(n)^5-10H(n)^3H(2n)+20H(n)^2H(3n)+15H(n)H(2n)^2\\ &-30H(4n)H(n)-20H(3n)H(2n)+24H(5n)\big),\end{aligned} \tag{2.5}$$

$$\begin{aligned}H(\{n\}^6)=\frac{1}{720}\big(&H(1)^6-15H(1)^4H(2)+40H(1)^3H(3)+45H(1)^2H(2)^2-90H(4)H(1)^2\\ &-120H(1)H(2)H(3)-15H(2)^3+90H(4)H(2)+144H(5)H(1)+40H(3)^2-120H(6)\big).\end{aligned} \tag{2.6}$$

For the star homogeneous multiple harmonic numbers one might replace all the minus signs by a plus sign in the above list.

Recall that Stirling numbers $s(n,j)$ of the first kind are defined by the expansion

$$(x)_m := x(x-1)(x-2)\cdots(x-(m-1)) = \sum_{j=1}^{m}(-1)^{n-j}s(m,j)x^j,$$

One can easily see that

$$(x)_{m+n} = (x)_m(x-m)_n \quad \text{and} \quad (x)_{m+n} = (x)_m(x+m)_n.$$

When substituting $n=1$ in Corollary 4, we have

$$s(m,j) = (m-1)!H(\{1\}^{j-1};m-1).$$

Before we proceed further we present two Theorems that allow cancellations modulo a power of a prime.

Theorem 15. *For any prime $p \geq 5$ and a non-negative integer m such that $p \geq m+3$, we have*

$$H(m) \equiv \begin{cases} 0 & \pmod{p^2} \quad \text{if} \quad m \text{ is odd}, \\ 0 & \pmod{p} \quad \text{if} \quad m \text{ is even}. \end{cases}$$

Proof. Originally this theorem was established by Bayat (see Theorem 3 page 559 [3]), the reader can see Lemma 2.2 page 78 [47] for more general statement and alternative proof.

By the condition $p \geq m+3$ we see that $p-1$ does not divide m, therefore, by Fermat's little theorem the map $a \mapsto a^{-m}$ is not the constant function 1 on the field $\mathbb{F}_p^\times$. Now, choose g such that $1 < g < p-1$, then $\gcd(g, p-1) = 1$, hence g is a generator of the multiplicative group $\mathbb{F}_p^\times$ and we have

$$H(m) = \sum_{k=1}^{p-1}\frac{1}{k^m} \equiv \sum_{k=1}^{p-1}(g^k)^m = \sum_{k=1}^{p-1}(g^m)^k = \frac{1-(g^m)^{p-1}}{1-g^m} \equiv 0 \pmod{p}.$$

The case of modulus p^2, we suppose m is odd, then using the binomial formula we get

$$H(m) = \frac{1}{2}\sum_{k=1}^{p-1}\frac{1}{k^m} + \frac{1}{2}\sum_{k=1}^{p-1}\frac{1}{(p-k)^m} = \frac{1}{2}\sum_{k=1}^{p-1}\frac{(p-k)^m+k^m}{k^m(p-k)^m} \equiv \frac{1}{2}\sum_{k=1}^{p-1}\frac{mpk^{m-1}}{k^m(p-k)^m} \pmod{p^2}.$$

Now, one can notice that $\frac{1}{(p-k)^m} \equiv -\frac{1}{k^m} \pmod{p}$, hence by the case modulo p

$$H(m) \equiv \frac{1}{2}\sum_{k=1}^{p-1}\frac{-mp}{k^{m+1}} = \frac{-mp}{2}H(m+1) \equiv 0 \pmod{p^2}.$$

□

Corollary 5. *Let n and k be two positive integers. Let p be a prime such that $p \geq kn+3$. Then*

$$H(\{n\}^k) \equiv H^\star(\{n\}^k) \equiv \begin{cases} 0 & \pmod{p^2} \quad \text{if} \quad kn \text{ is odd}, \\ 0 & \pmod{p} \quad \text{if} \quad kn \text{ is even}. \end{cases}$$

Proof. A more general result is found in [47]. From the previous theorem and Corollary 4 it is clear that for all positive integers k, n, we have

$$H(\{n\}^k) \equiv H^\star(\{n\}^k) \equiv 0 \pmod{p}$$

Now, suppose that nk is odd, from the Corollary 4 we find

$$H(\{n\}^k) = \text{Sum of different products of Generalized harmonic numbers} + H(nk) \equiv 0 \pmod{p^2}.$$

□

2.2 Optimized congruences for $\binom{kp-1}{p-1}$

The study of arithmetic properties of binomial coefficients has a rich history. In 1862, Wolstenholme noted that for any prime $p \geq 5$,

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

This result had been generalized by Glaisher [9], [10] as follows

$$\begin{aligned}\binom{np-1}{p-1} &\equiv 1 \pmod{p^3},\\ \binom{np-1}{p-1} &\equiv 1 - \frac{1}{3}n(n-1)B_{p-3} \pmod{p^4},\\ \binom{2p-1}{p-1} &\equiv 1 + 2pH(1) \pmod{p^4}.\end{aligned}$$

In 1995, R. J. McIntosh [23] proved that

$$\binom{2p-1}{p-1} \equiv 1 - p^2H(2) \pmod{p^5}.$$

In 2000, an equivalent congruence proved by V. Hamme states that for $p \geq 7$,

$$\binom{2p-1}{p-1} \equiv 1 + 2pH(1) \pmod{p^5}.$$

In 2010, R. Tauraso [41] sharpened this result and showed for any prime $p \geq 7$,

$$\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 + 2pH(1) + \frac{2}{3}p^3H(3) \pmod{p^6}.$$

In 2012, R. Meštrović [25] (published later in [26]), using Newton's formulas for elementary symmetric polynomials managed to show that for primes $p \geq 11$,

$$\binom{2p-1}{p-1} \equiv 1 - 2pH(1) + 4p^2H(\{1\}^2) \pmod{p^7}.$$

In 2013, J. Rosen [34] generalized and unified these results by extending congruences for the binomial coefficient $\binom{kp-1}{p-1}$ modulo p^n for arbitrary positive integers n, k. In this section, we present his linear algebra based approach which optimizes congruences for $\binom{kp-1}{p-1}$ with arbitrary length. Wolsetholme's congruence has been generalized in many directions, for a detailed survey see [28].

Proposition 3. *[34] For all non-negative integers n, k, we have*

$$(n+1)^j H(\{1\}^j;n) + \sum_{i=j}^{n} (-1)^{n+j+1} \binom{i}{j} (n+1)^j H(\{1\}^j;n) = 0. \tag{2.7}$$

Proof. Define

$$f_n(t) = \binom{(n+1)(t+1)-1}{n} = \sum_{j=0}^{n} (n+1)^j H(\{1\}^j;n) t^j.$$

Then $f_n(t)$ satisfies the functional equation $f_n(t) = (-1)^n f_n(-1-t)$, by the binomial formula this implies the following

$$\begin{aligned}
\sum_{j=0}^{n} (n+1)^j H(\{1\}^j;n) t^j &= (-1)^n \sum_{j=0}^{n} (n+1)^j H(\{1\}^j;n)(-1-t)^j \\
&= \sum_{j=0}^{n} (-1)^{n+j} (n+1)^j H(\{1\}^j;n) \sum_{i=0}^{j} \binom{j}{i} t^i \\
&= \sum_{i=0}^{n} \left(\sum_{j=i}^{n} (-1)^{n+j} \binom{j}{i} (n+1)^j H(\{1\}^j;n) \right) t^i.
\end{aligned}$$

Hence, equating the coefficients of both sides and rearranging, we find Eq (2.7). □

Remark 2. *Multiplying both sides of* (2.7) *by indeterminates $c_0, c_1, c_2, \ldots$, then summing up we obtain the following*

$$\sum_{j=0}^{n} c_j (n+1)^j H(\{1\}^j;n) + \sum_{j=0}^{n} c_j \sum_{i=j}^{n} (-1)^{n+j+1} \binom{i}{j} (n+1)^j H(\{1\}^j;n) = 0,$$
$$\sum_{j=0}^{n} c_j (n+1)^j H(\{1\}^j;n) + \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{n+j+1} \binom{i}{j} c_j (n+1)^j H(\{1\}^j;n) = 0.$$

Therefore

$$\sum_{j=0}^{n} \left(c_j + (-1)^{n+j+1} \sum_{i=0}^{j} \binom{i}{j} c_i \right) (n+1)^j H(\{1\}^j;n) = 0. \tag{2.8}$$

Proposition 4. *[34] Let n be a non-negative integer. Suppose that $k, c_0, c_1, \ldots$ are arbitrary elements (indeterminates) of a given field $\mathbb{K}$. Define*

$$\beta_j = (k-1)^j + c_j + (-1)^{n+j+1} \sum_{i=0}^{j} \binom{i}{j} c_i. \tag{2.9}$$

Then the following equality holds true

$$\binom{k(n+1)-1}{n} = \sum_{j=0}^{n} \beta_j (n+1)^j H(\{1\}^j;n),$$

for any choice of indeterminates $k, c_0, c_1, \ldots$.

In particular, we have

$$\binom{kp-1}{p-1} = \sum_{j=0}^{p-1} \beta_j p^j H(\{1\}^j). \tag{2.10}$$

Proof. We have that

$$\binom{k(n+1)-1}{n} = f_n(k-1) = \sum_{j=0}^{n} (k-1)^j (n+1)^j H(\{1\}^j\,; n)$$
$$\binom{k(n+1)-1}{n} = \sum_{j=0}^{n} \left((k-1)^j + c_j + (-1)^{n+j+1} \sum_{i=0}^{j} \binom{i}{j} c_i \right) (n+1)^j H(\{1\}^j\,; n).$$

In the second line we used Eq (2.8). □

Theorem 16. *Let p be a fixed odd prime, and j an integer with $1 \le j \le p-3$. Then we have*

$$H(\{1\}^j) \equiv \begin{cases} -\frac{(j+1)}{2(j+2)} p^2 B_{p-j-2} & \pmod{p^3} \quad \text{for } j \text{ odd}, \\ -\frac{1}{j+1} p B_{p-j-1} & \pmod{p^2} \quad \text{for } j \text{ even}. \end{cases}$$

Moreover, we have the following special cases

1. *If $j = p-2$, then $H(\{1\}^j) \equiv \frac{p}{2} \pmod{p^2}$.*
2. *If $j = p-1$, then $H(\{1\}^j) \equiv -1 \pmod{p}$.*
3. *If $j \ge p$, then $H(\{1\}^j) = 0$.*

Proof. The first part is a special case $(n = 1)$ Theorem 1.6 page 75 [47]. For the second part we may use Wilson's Theorem, see [34] page 2040 for more details. □

Theorem 17. *[34] Let k be an integer. Assume that $c_0, c_1, \ldots$ are arbitrary rational numbers, and take $\beta_j \in \mathbb{Q}$ to be*

$$\beta_j = (k-1)^j + c_j + (-1)^{j+1} \sum_{i=0}^{j} \binom{i}{j} c_i.$$

Fix an odd prime p not dividing the denominator of any c_i, and let N be a non-negative integer. Define

$$E_N := \binom{kp-1}{p-1} - \sum_{j=0}^{N} \beta_j p^j H(\{1\}^j).$$

- *If $0 \le N \le p-4$, then*

$$E_N \equiv \begin{cases} \frac{B_{p-3-N}}{N+3} \left(\frac{N+2}{2} \beta_{N+1} + \beta_{N+2}\right) p^{N+3} \pmod{p^{N+4}} & \text{if } N \text{ is even}, \\ \frac{B_{p-3-N}}{N+2} \beta_{N+1} p^{N+2} \pmod{p^{N+3}} & \text{if } N \text{ is odd}. \end{cases} \tag{2.11}$$

- *If $N = p-3$, then $E_N \equiv (\frac{B_{N+1}}{2} - \beta_{N+2}) p^{N+2} \pmod{p^{N+3}}$.*
- *If $N = p-2$, then $E_N \equiv -\beta_{N+1} p^{N+1} \pmod{p^{N+2}}$.*
- *If $N \ge p-1$, then $E_N = 0$.*

In particular, we have

$$\binom{kp-1}{p-1} \equiv \sum_{j=0}^{N} \beta_j p^j H(\{1\}^j) \begin{cases} \pmod{p^{N+3}} & \textit{if} \quad N \leq p-4, N \textit{ even}, \\ \pmod{p^{N+2}} & \textit{if} \quad N \leq p-4, N \textit{ odd}, \\ \pmod{p^{N+2}} & \textit{if} \quad N = p-3, \\ \pmod{p^{N+1}} & \textit{if} \quad N = p-2, \\ \pmod{0} & \textit{if} \quad N \geq p-1. \end{cases} \tag{2.12}$$

Proof. We prove only Eq (2.11), it implies the first congruence in Eq (2.12) which will be in much use later. See [34] for more details. By definition we have

$$E_N = \binom{kp-1}{p-1} - \sum_{j=0}^{N} \beta_j p^j H(\{1\}^j) = \sum_{j=0}^{p-1} \beta_j p^j H(\{1\}^j) - \sum_{j=0}^{N} \beta_j p^j H(\{1\}^j).$$

Hence

$$E_N \equiv \sum_{N+1}^{N+3} \beta_j p^j H(\{1\}^j) \pmod{p^{N+4}}.$$

Suppose that $0 \leq N \leq p-4$ and N even, by the previous theorem we have

$$\begin{aligned} H(\{1\}^{N+1}) &\equiv -\frac{N+2}{N+3} p^2 B_{p-N-3} \pmod{p^3}, \\ H(\{1\}^{N+2}) &\equiv -\frac{1}{N+3} p B_{p-N-3} \pmod{p^2}, \\ H(\{1\}^{N+3}) &\equiv 0 \pmod{p}. \end{aligned}$$

Thus, we find

$$E_N \equiv \frac{B_{p-3-N}}{N+3} \left(\frac{N+2}{2} \beta_{N+1} + \beta_{N+2} \right) p^{N+3} \pmod{p^{N+4}}.$$

We get the first congruence in (2.12) when reducing modulo p^{N+3}. □

Definition 1. *[34] We call the congruence* (2.12) *the generalized Wolstenholme congruence associated with the data*

$$[k, (c_0, c_1, \dots), N],$$

and we will say that $b_0, \dots, b_N$ *are the generalized Wolstenholme coefficients associated with this data.*

Lemma 3. *[34] For every positive integers* b, n *the following* n *by* n *matrix*

$$M_{n,b} := \left(\binom{b+i}{j} \right)_{0 \leq i,j \leq n-1}$$

has determinant 1.

Proof. Define $n \times n$ matrices $L_n = \left(\binom{i}{j}\right)_{0 \leq i,j \leq n-1}$ and $U_{n,b} = \left(\binom{b}{j-i}\right)_{0 \leq i,j \leq n-1}$. Observe that L_n is a lower triangular that has ones in the diagonal, also $U_{n,b}$ is an upper triangular that has ones in the diagonal, so both matrices have determinant 1. By Vandermonde's convolution Identity one can see that $M_{n,b} = L_n U_{n,b}$, thus $det(M_{n,b}) = det(L_n) det(U_{n,b}) = 1$. □

We are now ready to understand the idea of Theorems 18 and 21. We are interested in finding coefficients $b_0, \dots, b_n$ such that

$$\begin{cases} b_0 & = 1 \\ b_1 & = (k-1) + c_1 + (c_0 + c_1) \\ & \vdots \\ b_n & = (k-1)^n + c_n + (-1)^{n+1} \sum_{i=0}^{n} \binom{n}{i} c_i \\ -(k-1)^{n+1} & = c_{n+1} + (-1)^{n+2} \sum_{i=0}^{n+1} \binom{n+1}{i} c_i \\ & \vdots \\ -(k-1)^{2n} & = c_{2n} + (-1)^{2n+1} \sum_{i=0}^{2n} \binom{2n}{i} c_i. \end{cases} \tag{2.13}$$

We will see in Lemma 4 that we can assume $c_n = c_{n+1} = \cdots = c_{2n} = 0$. Therefore, system (2.13) can be split into two systems as follows

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} (k-1)^0 \\ (k-1)^1 \\ \vdots \\ (k-1)^n \end{pmatrix} + D_n \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}, \tag{2.14}$$

$$-\begin{pmatrix} (k-1)^n \\ (k-1)^{n+1} \\ \vdots \\ (k-1)^{2n} \end{pmatrix} = M_n \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}, \tag{2.15}$$

where M_n and D_n is an are defined by

$$M_n = \left((-1)^{n+i} \binom{n+1+i}{j} \right)_{0 \le i,j \le n-1} \quad \text{and} \quad D_n = \left((-1)^{i+1} \binom{i}{j} + \delta_{i,j} \right)_{\substack{0 \le i \le n \\ 0 \le j \le n-1}}.$$

The matrix M_n is invertible with determinant $(-1)^{n+1}$ since it is the product of the diagonal matrix $\left((-1)^{i+1}\delta_{i,j}\right)_{0 \le i,j \le n-1}$ by $M_{n,n+1}$. Next, multiplying system (2.15) by $-D_n M_n^{-1}$ and substituting in system (2.14) we find the values $(b_i)_{0 \le i \le n}$.

We illustrate with a concrete example, let's say we want an optimized congruence for $\binom{kp-1}{p-1}$ modulo p^9, this implies $n = 3$, then we look for coefficients b_0, b_1, b_2, b_3 satisfying

$$\binom{kp-1}{p-1} \equiv b_0 + b_1 p H(1) + b_2 p^2 H(\{1\}^2) + b_3 p^3 H(\{1\}^3) \pmod{p^9},$$

By Lemma 4 we know that $(b_i)_{0\le i\le 3}$ satisfy

$$\begin{cases} b_0 & = 1 \\ b_1 & = (k-1) + 2c_1 + c_0 \\ b_2 & = (k-1)^2 - 2c_1 - c_0 \\ b_3 & = (k-1)^3 + 3c_2 + 3c_1 + c_0 \\ -(k-1)^4 & = -6c_2 - 4c_1 - c_0 \\ -(k-1)^5 & = 10c_2 + 5c_1 + c_0 \\ -(k-1)^6 & = -15c_2 - 6c_1 - c_0. \end{cases}$$

For given integers c_0, c_1, c_2. This system is split into

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} (k-1)^0 \\ (k-1)^1 \\ (k-1)^2 \\ (k-1)^3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & -2 & 0 \\ 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}, \tag{2.16}$$

$$-\begin{pmatrix} (k-1)^4 \\ (k-1)^5 \\ (k-1)^6 \end{pmatrix} = \begin{pmatrix} -1 & -4 & -6 \\ 1 & 5 & 10 \\ -1 & -6 & -15 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}. \tag{2.17}$$

Now, the matrix $M_3 = \begin{pmatrix} -1 & -4 & -6 \\ 1 & 5 & 10 \\ -1 & -6 & -15 \end{pmatrix}$ is invertible, its inverse is given by

$$M_3^{-1} = \begin{pmatrix} -15 & -24 & -10 \\ 5 & 9 & 4 \\ -1 & -2 & -1 \end{pmatrix}.$$

Multiplying system (2.17) by $-D_3 M_3^{-1}$ and substituting in system (2.16) we find

$$\begin{aligned} b_0 =& 1, \\ b_1 =& (k-1) + 5(k-1)^4 + 6(k-1)^5 + 2(k-1)^6, \\ b_2 =& (k-1)^2 - 5(k-1)^4 - 6(k-1)^5 - 2(k-1)^6, \\ b_3 =& (k-1)^3 + 3(k-1)^4 + 3(k-1)^5 + (k-1)^6. \end{aligned}$$

When replacing $k-1$ by $-k$ one can easily notice that this list of b_i is in fact equal to the last row in table (2.2). This is no surprise as we will see that $b_{j,n}(1-k) = b_{j,n}(k)$.

Suppose that $b_3 = b_4 = 0$, thus $2c_1 + c_0 = (k-1)^2$, this implies $b_1 = k(k-1)$, therefore we get

$$\binom{kp-1}{p-1} \equiv 1 + k(k-1)pH(1) \pmod{p^5}.$$

This suggest that we can take the data $[k, ((k-1)^2), 2]$ that produce $(b_0, b_1, b_2, b_3, b_4) = (1, k(k-1), 0, 0, 0)$.

In order to generalize this reasoning we need some definitions and lemmas.

Definition 2. *[34] Fix integers N, k, with $N \geq 0$. Define $V_{N,k} \subset \mathbb{Z}^{N+1}$ to be the set*

$$V_{N,k} := \left\{ (\beta_0, \beta_1, \ldots, \beta_N) : \text{ there exist } c_0, c_1, \cdots \in \mathbb{Z} \text{ s.t. } \beta_j = (k-1)^j + c_j + (-1)^{j+1} \sum_{i=0}^{j} \binom{j}{i} c_i \right\}.$$

In other words $V_{N,k}$ is the set of generalized Wolstenholme coefficients corresponding to integer data.

We similarly define $V^{\mathbb{Q}}_{N,k} \subset \mathbb{Q}^{N+1}$ to be

$$V^{\mathbb{Q}}_{N,k} := \left\{ (\beta_0, \beta_1, \ldots, \beta_N) : \text{ there exist } c_0, c_1, \cdots \in \mathbb{Q} \text{ s.t. } \beta_j = (k-1)^j + c_j + (-1)^{j+1} \sum_{i=0}^{j} \binom{j}{i} c_i \right\},$$

the set of generalized Wolstenholme coefficients corresponding to rational data.

These definitions suggest that we introduce the following set

$$\hat{V}_N := \left\{ (\beta_0, \beta_1, \ldots, \beta_N) : \exists\, c_0, c_1, \cdots \in \mathbb{Z} \text{ s.t. } \beta_j = c_j + (-1)^{j+1} \sum_{i=0}^{j} \binom{j}{i} c_i \right\},$$

which is in fact the linear part of the affine space $V_{N,k}$ as we have

$$V_{N,k} = (1, (k-1), \ldots, (k-1)^N) + \hat{V}_N.$$

Proposition 5. *[34] For all integers N, k, with $N \geq 0$, we have $V_{N,k} = V_{N,1-k}$.*

Proof. Taking $c_0 = c_1 = \cdots = 0$, we see that $(1, (-k), (-k)^2, \ldots, (-k)^N) \in V_{N,k}$, hence $V_{N,k} \subset V_{N,1-k}$. For the reverse inclusion we set $c_j = -(k-1)^j$, then we have

$$\begin{aligned} \beta_j =& (k-1)^j + c_j + (-1)^{j+1} \sum_{i=0}^{j} \binom{j}{i} c_i \\ =& (k-1)^j - (k-1)^j + (-1)^{j+1} \sum_{i=0}^{j} \binom{j}{i} (k-1)^j \\ =& (-k)^j. \end{aligned}$$

□

We following lemma says that the affine space of generalized Wolstenholme coefficients, for arbitrary k and $N = 2n$, has dimension n.

Lemma 4. *[34] For all non-negative integers n, $\hat{V}_{2n}$ is a free $\mathbb{Z}$-module of rank n, and the map*

$$\begin{aligned} \pi \colon \hat{V}_{2n} &\longrightarrow \mathbb{Z}^n \\ (\beta_0, \ldots, \beta_{2n}) &\longmapsto (\beta_{n+1}, \ldots, \beta_{2n}) \end{aligned}$$

is an isomorphism.

Proof. An element $(\beta_0, \beta_1, \ldots, \beta_{2n}) \in \hat{V}_{2n}$ is determined by the values of c_j for $0 \leq j \leq 2n$. We therefore have a surjective map

$$\begin{aligned} \varphi_n \colon \mathbb{Z}^{2n+1} &\longrightarrow \hat{V}_{2n} \\ (c_0, \ldots, c_{2n}) &\longmapsto (a_0, \ldots, a_{2n}), \end{aligned}$$

where $a_j = c_j + (-1)^{j+1} \sum_{i=0}^{j} \binom{j}{i} c_i$.

The matrix of φ_n with respect to the standard basis on $\mathbb{Z}^{2n+1}$ is given by

$$A_n = \left(\delta_{i,j} + (-1)^{j+1} \binom{j}{i} \right)_{\substack{0 \leq i \leq 2n \\ 0 \leq j \leq 2n}}.$$

Now, We identify column vectors of length $2n+1$ with the set of polynomials of degree $2n$ via the identification $(a_0, \ldots, a_{2n}) \leftrightarrow a_0 + a_1 t + \cdots + a_{2n} t^{2n}$. The jth column of A_n is therefore identified to $-t^j - (-1-t)^j$, indeed, for a fixed j we have

$$\begin{aligned} \delta_{i,j} + (-1)^{j+1} \binom{j}{i} \leftrightarrow & \sum_{i=0}^{2n} \delta_{i,j} t^i + \sum_{i=0}^{2n} (-1)^{j+1} \binom{j}{i} t^i \\ \leftrightarrow & t^j - (-1-t)^j. \end{aligned}$$

Hence, the column span of A_n is contained in the following set

$$\mathbb{E} := \{ f(t) \in \mathbb{Q}_{2n}[t] : \text{ and } f(t) = -f(-1-t) \},$$

where $\mathbb{Q}_{2n}[t]$ denotes the set of polynomials with degree less than or equal to $2n$. For $\alpha \in \mathbb{Q}$, we know that the family $\{1, (t-\alpha), (t-\alpha), \ldots, (t-\alpha)^{2n}\}$ constitute a basis for $\mathbb{Q}_{2n}[t]$, thus in order for this family to be a basis for $\mathbb{E}$, we should have

$$(t-\alpha)^m = -(-1-t-\alpha)^m \text{ where } 0 \leq m \leq 2n.$$

this implies that m must be an odd number and $\alpha = -\frac{1}{2}$. Therefore, $\mathsf{rank}(\hat{V}_{2n}) = \mathsf{rank}(A_n) \leq n$.

Consider the canonical inclusion

$$\begin{aligned} i \colon \mathbb{Z}^n &\longrightarrow \mathbb{Z}^{2n+1} \\ (c_0, \ldots, c_{n-1}) &\longmapsto (c_0, \ldots, c_{n-1}, 0, \ldots, 0). \end{aligned}$$

Then $\pi \circ \varphi_n \circ i(c_0, \ldots, c_{n-1}) = (y_0, \ldots, y_{n-1})$ where

$$y_j = (-1)^{n+j} \sum_{i=0}^{n-1} \binom{n+1+j}{i} c_i.$$

From Lemma 3 this map is bijective. Thus, π is surjective with $\pi \circ \varphi_n \circ i(\mathbb{Z}^n) = \pi(\hat{V}_{2n})$. Since $\mathsf{rank}(\hat{V}_{2n}) = \mathsf{rank}(A_n) \leq n$ we must have $\mathsf{rank}(\hat{V}_{2n}) = n$ and π bijective. $\square$

Theorem 18. *[34] (Optimized Binomial Congruences) Let integers k, n be given, with $n \geq 0$, and set $N = 2n$. There is a unique generalized binomial congruence whose coefficients $b_0, \ldots, b_{2n}$ satisfy $b_{n+1} = b_{n+2} = \cdots = b_{2n} = 0$. This congruence has $b_0, b_1, \ldots, b_n \in \mathbb{Z}$.*

In other words, for $N = 2n$, Theorem 17 produces a unique congruence of the form

$$\binom{kp-1}{p-1} \equiv \sum_{j=0}^{n} b_j p^j H(\{1\}^j) \pmod{p^{2n+3}}, \tag{2.18}$$

with $b_i \in \mathbb{Z}$, which holds for all odd primes $p \neq 2n+3$. This congruence holds $\pmod{p^{2n+2}}$ when $p = 2n+3$ and equality for $3 \leq p \leq 2n+1$.

Proof. This is equivalent to show that there exists a unique element $(b_0, \dots, b_n, 0, \dots, 0) \in V_{2n,k}$ with $b_0, \dots, b_n \in \mathbb{Z}$. Since $V_{2n,k} = (1, (k-1), \dots, (k-1)^{2n}) + \hat{V}_{2n}$ it suffices to find a unique

$$\underline{a} = (a_0, \dots, a_n, -(k-1)^{n+1}, \dots, -(k-1)^{2n})$$

in $\hat{V}_{2n,k}$. This is guaranteed by previous lemma which asserts there exists a unique $\underline{a} \in \hat{V}_{2n,k}$ such that

$$\pi(\underline{a}) = -((k-1)^{n+1}, \dots, (k-1)^{2n}).$$

□

Definition 3. *[34] For integers j, n, k, with $j, n \geq 0$, we let $b_{j,n}(k)$ denote the coefficients arising from Theorem 18. We call these extremal coefficients. We also denote by $b_{j,n}(t)$ the polynomial giving these coefficients, and call these extremal polynomials. By convention, we take $b_{j,n}(t) = 0$ for $n+1 \leq j \leq 2n$, and we say that $b_{j,n}(t)$ is not defined for $j \geq 2n+1$.*

Theorem 19. *[34] (uniqueness of optimized congruences) Let $n \geq 0$ be a fixed integer. The extremal polynomials $b_{j,n}(t)$ ($0 \leq j \leq n$) have integer coefficients, and for every prime $p \geq 2n+5$ and every integer $k \geq 1$ we have*

$$\binom{kp-1}{p-1} \equiv \sum_{j=0}^{n} b_{j,n}(k) p^j H(\{1\}^j) \pmod{p^{2n+3}} \tag{2.19}$$

This congruence holds $\mod p^{2n+2}$ for $p = 2n+3$, and is inequality for $3 \leq p \leq 2n+1$

Conjecture 1. *[34] (Linear Bernoulli Nondegeneracy Conjecture) For all odd integers $k \geq 3$, there exist infinitely many primes p for which p does not divide the numerator of the Bernoulli number B_{p-k}.*

Theorem 20. *[34] (Uniqueness of Optimized Congruences) Assume the truth of the Linear Bernoulli Nondegeneracy Conjecture. Let n, k be integers with $n \geq 0$, and suppose $b_0, \dots, b_n \in \mathbb{Q}$ are such that the congruence*

$$\binom{kp-1}{p-1} \equiv \sum_{j=0}^{n} b_j p^j H(\{1\}^j) \pmod{p^{2n+3}}$$

holds for all sufficiently large primes p. Then we have $b_j = b_{j,n}(k)$.

Theorem 21. *[34] The extremal polynomials $b_{j,n}(t)$ given in Theorem 18 can be computed explicitly as follows.*

Let M_n be the $n \times n$ invertible matrix

$$M_n = \left((-1)^{n+i} \binom{n+1+i}{j} \right)_{0 \leq i,j \leq n-1}.$$

And D_n the $(n+1) \times n$ matrix

$$D_n = \left((-1)^{i+1} \binom{i}{j} + \delta_{i,j} \right)_{\substack{0 \leq i \leq n \\ 0 \leq j \leq n-1}}.$$

Then these polynomials satisfy the following matrix equation

$$\begin{pmatrix} b_{0,n}(t) \\ b_{1,n}(t) \\ \vdots \\ b_{n,n}(t) \end{pmatrix} = \begin{pmatrix} (t-1)^0 \\ (t-1)^1 \\ \vdots \\ (t-1)^n \end{pmatrix} - D_n M_n^{-1} \begin{pmatrix} (t-1)^{n+1} \\ (t-1)^{n+2} \\ \vdots \\ (t-1)^{2n} \end{pmatrix}.$$

Corollary 6. *[34] The extremal polynomials $b_{j,n}(t)$ can also be computed by*

$$\begin{pmatrix} b_{0,n}(t) \\ b_{1,n}(t) \\ \vdots \\ b_{n,n}(t) \end{pmatrix} = \begin{pmatrix} (-t)^0 \\ (-t)^1 \\ \vdots \\ (-t)^n \end{pmatrix} - D_n M_n^{-1} \begin{pmatrix} (-t)^{n+1} \\ (-t)^{n+2} \\ \vdots \\ (-t)^{2n} \end{pmatrix}.$$

Proof. Immediate by Proposition 5 as $V_{2n,k} = V_{2n,1-k}$ implies that $b_{j,n}(1-k) = b_{j,n}(k)$. □

Here is a table for the exteremal polynomials with small values n, j

n/j	0	1	2	3
0	1			
1	1	$t^2 - t$		
2	1	$-t^4 + 2t^3 - t$	$t^4 - 2t^3 + t^2$	
3	1	$2t^6 - 6t^5 + 5t^4 - t$	$-2t^6 + 6t^5 - 5t^4 + t^2$	$t^6 - 3t^5 + 3t^4 - t^3$

Proposition 6. *[34] Let $0 \leq j \leq n$ be integers. The extremal polynomial $b_{j,n}(t) \in \mathbb{Z}[t]$ is the unique polynomial of degree at most $2n$ satisfying*

1. $b_{j,n}(t) \equiv (t-1)^j \mod (t-1)^{n+1}$.
2. $b_{j,n}(t) \equiv (-1)^j t^j \mod t^{n+1}$.

Proof. For $n+1 \leq j \leq 2n$, we have that $b_{n,k}(t) = 0$, hence this case is trivial. When $0 \leq j \leq n$, Theorem 21 shows that $\deg(b_{j,n}(t)) \leq 2n$ and that $b_{j,n}(t)$ is equal to $(t-1)^j$ plus a $\mathbb{Z}$-linear combination of $(t-1)^{n+1}, \ldots, (t-1)^{2n}$. Similarly, Corollary 6 implies that $b_{j,n}(t)$ is equal to $(-t)^j$ plus a $\mathbb{Z}$-linear combination of $(-t)^{n+1}, \ldots, (-t)^{2n}$.

For uniqueness, since $\gcd((t-1)^{n+1}, t^{n+1}) = 1$, then the Chinese remainder theorem implies that there exists a unique solution satisfying these two conditions. □

Proposition 7. *[34] The extremal polynomials $b_{j,n}(t)$ satisfy the following properties*

1. *For all non-negative integers n, $b_{0,n}(t) = 1$ and $b_{n,n}(t) = t^n(t-1)^n$.*
2. *For all non-negative integers $j \leq 2n$, $t^j(t-1)^j$ divides $b_{j,n}(t)$.*

Proof. By the previous proposition we have that both $(t-1)^n$ and t^n divide $b_{n,n}(t)$ and as their greatest common divisor is equal to one we see that the term $(t-1)^n t^n$ (which has degree $2n$) divides $b_{n,n}(t)$. Then there exists a constant c such that $b_{n,n}(t) = c(t-1)^n t^n$, this constant must be equal to 1 in order to satisfy the two conditions of the previous theorem. For the second statement we may follow a similar reasoning. □

Proposition 8. *[34] Let m be a non-negative integer. Suppose $f(t) = \sum_{j=0}^{m} a_j t^j \in \mathbb{Q}[t]$ satisfying $f(t) = f(-t-1)$. Then for all non-negative integers n with $2n \geq m$, we have*

$$\sum_{j=0}^{m} a_j b_{j,n}(t) = f(t-1).$$

2.3 Some contribution to congruences with binomial coefficients

2.3.1 Congruences for binomial coefficients with roots of unity and generalized harmonic numbers

We establish the following formula valid in every extension $\mathbb{K}$ that contains the n-th roots of unity. This formula is especially true for algebraically closed fields such as $\mathbb{C}$ and $\mathbb{C}_p$ for all odd primes p.

Theorem 22. *Let $\mathbb{K}$ be any extension of $\mathbb{Q}$ that contains the n-th roots of unity, then for all $x \in \mathbb{K}$ we have*

$$\prod_{\omega^n=1} \binom{\omega x - 1}{p-1} = \sum_{k=0}^{p-1} (-1)^k H(\{n\}^k) x^{kn}. \tag{2.20}$$

Proof. We prove that both the left hand side and the right hand side in the above equation are equal to the polynomial

$$F_n(x) = \frac{(x^n - 1)(x^n - 2^n)\dots(x^n - (p-1)^n)}{(p-1)!^n}.$$

Firstly, we show that the left hand side is equal to $F_n(x)$. Indeed, if we denote the roots of unity by $\{1, \omega_2, \dots \omega_n\}$, then the polynomial $F_n(x)$ can be written as follows

$$\begin{aligned} F_n(x) &= \prod_{j=1}^{p-1} \frac{x-j}{(p-1)!} \prod_{j=1}^{p-1} \frac{x - j\omega_2}{(p-1)!} \cdots \prod_{j=1}^{p-1} \frac{x - j\omega_n}{(p-1)!} \\ &= (1^{p-1}\omega_2^{p-1} \dots \omega_n^{p-1}) \prod_{j=1}^{p-1} \frac{x-j}{(p-1)!} \prod_{j=1}^{p-1} \frac{\omega_2^{-1}x - j}{(p-1)!} \cdots \prod_{j=1}^{p-1} \frac{\omega_n^{-1}x - j}{(p-1)!}. \end{aligned}$$

Since the product of roots $1 \cdot \omega_2 \cdots \omega_n = (-1)^{n-1}$, and p is an odd prime, then $(1 \cdot \omega_2 \cdots \omega_n)^{p-1} = 1$, and we get

$$\begin{aligned} F_n(x) &= \prod_{j=1}^{p-1} \frac{x-j}{(p-1)!} \prod_{j=1}^{p-1} \frac{\omega_2^{-1}x - j}{(p-1)!} \cdots \prod_{j=1}^{p-1} \frac{\omega_n^{-1}x - j}{(p-1)!} \\ &= \binom{x-1}{p-1} \binom{\omega_2^{-1}x - 1}{p-1} \cdots \binom{\omega_n^{-1}x - 1}{p-1} \\ &= \prod_{\omega^n=1} \binom{\omega^{-1}x - 1}{p-1} \\ &= \prod_{\omega^n=1} \binom{\omega x - 1}{p-1}. \end{aligned}$$

For the second equality one observes that

$$\begin{aligned} F_n(x) &= \frac{(x^n - 1)(x^n - 2^n)\dots(x^n - (p-1)^n)}{(p-1)!^n} \\ &= (\frac{x^n - 1}{1^n})(\frac{x^n - 2^n}{2^n})\dots(\frac{x^n - (p-1)^n}{(p-1)^n}) \\ &= (x^n - 1)(\frac{x^n}{2^n} - 1)\dots(\frac{x^n}{(p-1)^n} - 1) \\ &= \sum_{k=0}^{p-1} (-1)^k e_k(1, \frac{1}{2^n}, \dots, \frac{1}{(p-1)^n}) x^{nk} \\ &= \sum_{k=0}^{p-1} (-1)^k H(\{n\}^k) x^{kn}. \end{aligned}$$

□

Corollary 7. *We have the following identity*

$$\frac{1}{\prod_{\omega^n=1}\binom{\omega x-1}{p-1}} = \sum_{k\geq 0} H^\star(\{n\}^k)x^{kn}. \tag{2.21}$$

Proof. From the formal power series identity we have

$$\sum_{k\geq 0} h_k(1,\ldots,\frac{1}{(p-1)^n})x^{nk} = \prod_{i=1}^{p-1}\sum_{j\geq 0}(\frac{x^n}{i^n})^j = \prod_{i=1}^{p-1}\frac{1}{1-\frac{x^n}{i^n}},$$

and from the previous proof

$$\prod_{i=1}^{p-1}\frac{1}{1-\frac{x^n}{i^n}} = \prod_{i=1}^{p-1}\frac{i^n}{i^n-x^n} = \frac{1}{F_n(x)} = \frac{1}{\prod_{\omega^n=1}\binom{\omega x-1}{p-1}}.$$

□

Combining Theorem 22 with Corollaries 4, 7 one gets the following.

Corollary 8. *With the same conditions of the previous theorem, we have*

$$\begin{aligned}\prod_{\omega^n=1}\binom{\omega x-1}{p-1} &= \sum_{k=0}^{p-1}(-1)^k H(\{n\}^k)x^{kn}\\ &= \sum_{k=0}^{p-1}\frac{(-1)^k}{k!}\, B_k(H(n),-1!H(2n),\ldots,(-1)^{k-1}(k-1)!H(kn))x^{kn},\end{aligned} \tag{2.22}$$

$$\frac{1}{\prod_{\omega^n=1}\binom{\omega x-1}{p-1}} = \sum_{k\geq 0} H_k^{\star(n)}x^{kn} = \sum_{k\geq 0}\frac{1}{k!}\, B_k(H(n),1!H(2n),\ldots,(k-1)!H(kn))x^{kn}. \tag{2.23}$$

2.3.1.1 *Congruences for* $\prod_{\omega^n=1}\binom{\alpha\omega p-1}{p-1}$

The next result gives congruences for the product $\prod_{\omega^n=1}\binom{\omega x-1}{p-1}$. For convenience we may denote

$$\epsilon_n = 1 \text{ if } n \text{ is even} \quad \text{and} \quad \epsilon_n = 2 \text{ if } n \text{ is odd}.$$

Corollary 9. *We have the following congruences*

1. *For* $v_p(\alpha) \geq \frac{\epsilon_n}{n} - 1$ *and* $p \geq n+3$,

$$\prod_{\omega^n=1}\binom{\alpha\omega p-1}{p-1} \equiv 1 \pmod{p^{nv_p(\alpha)+n+\epsilon_n}}, \tag{2.24}$$

2. *For* $v_p(\alpha) \geq \frac{1}{n} - 1$ *and* $p \geq 2n+3$,

$$\prod_{\omega^n=1}\binom{\alpha\omega p-1}{p-1} \equiv 1-(\alpha p)^n H(n) \pmod{p^{2nv_p(\alpha)+2n+1}}. \tag{2.25}$$

3. *For* $v_p(\alpha) \geq \frac{1}{n} - 1$ *and* $p \geq 4n+3$,

$$\prod_{\omega^n=1}\binom{\alpha\omega p-1}{p-1} \equiv 1-(\alpha p)^n H(n) - \frac{(\alpha p)^{2n}}{2}H(2n) \pmod{p^{2nv_p(\alpha)+2n+\epsilon_n+1}}. \tag{2.26}$$

4. *For $v_p(\alpha) \geq \frac{\epsilon_n - 1}{n} - 1$ and $p \geq 5n+3$,*

$$\prod_{\omega^n=1} \binom{\alpha\omega p - 1}{p-1} \equiv 1 - (\alpha p)^n H(n) - \frac{(\alpha p)^{2n}}{2} H(2n) - \frac{(\alpha p)^{3n}}{3} H(3n) - \frac{(\alpha p)^{4n}}{4} H(4n) \pmod{p^{2nv_p(\alpha)+2n+2\epsilon_n}}. \tag{2.27}$$

In particular, when $n \geq 2$, and $v_p(\alpha) \geq \frac{\epsilon_n}{n} - 1$ the last congruence can be further reduced to

$$\prod_{\omega^n=1} \binom{\alpha\omega p - 1}{p-1} \equiv 1 - (\alpha p)^n H(n) - \frac{(\alpha p)^{2n}}{2} H(2n) \pmod{p^{2nv_p(\alpha)+2n+2\epsilon_n}}. \tag{2.28}$$

Proof. We prove only (2.27). From Corollary 8 we have

$$\prod_{\omega^n=1} \binom{\alpha\omega p - 1}{p-1} \equiv \sum_{k=0}^{5} (-1)^k H(\{n\}^k)(\alpha p)^{kn} \pmod{p^{6nv_p(\alpha)+6n}}.$$

In view of equations (2.2), (2.3), (2.4), (2.5), Theorem 15 and Corollary 5, we have modulo $p^{2nv_p(\alpha)+2n+2\epsilon_n}$, where $n \geq 1$, $v_p(\alpha) \geq \frac{\epsilon_n - 1}{n} - 1$, and $p \geq 5n+3$,

$$\prod_{\omega^n=1} \binom{\alpha\omega p - 1}{p-1} \equiv 1 - (\alpha p)^n H(n) - \frac{(\alpha p)^{2n}}{2} H(2n) - \frac{(\alpha p)^{3n}}{3} H(3n) - \frac{(\alpha p)^{4n}}{4} H(4n).$$

Moreover, when $n \geq 2$, as one may observe that $4nv_p(\alpha) + 4n + 1 \geq 2nv_p(\alpha) + 2n + 2\epsilon_n$ and $3nv_p(\alpha) + 3n + \epsilon_n \geq 2nv_p(\alpha) + 2n + 2\epsilon_n$, therefore, the terms $\frac{(\alpha p)^{3n}}{3} H(3n)$ and $\frac{(\alpha p)^{4n}}{4} H(4n)$ can be omitted from (2.27). □

2.3.2 Congruences with 1-th root of unity

2.3.2.1 *Extended congruences for* $H(1)$

Proposition 9. *([18] Theorem 3.1 page 9) The following p-adic series equality holds true*

$$H(m) = (-1)^m \sum_{l \geq 0} \binom{m+l-1}{l} p^l H(m+l). \tag{2.29}$$

In particular,

$$H(1) = -\frac{1}{2} \sum_{l \geq 1} p^l H(l+1) \quad \text{and} \quad H(2) = \sum_{l \geq 0} (l+1) p^l H(l+2).$$

Proof. We have

$$\begin{aligned} H(m) &= \sum_{k=1}^{p-1} \frac{1}{k^m} = \sum_{k=1}^{p-1} \frac{1}{(p-k)^m} = (-1)^m \sum_{k=1}^{p-1} \frac{1}{k^m (1 - \frac{p}{k})^m} \\ &= (-1)^m \sum_{k=1}^{p-1} \sum_{l \geq 0} \binom{m+l-1}{l} \frac{p^l}{k^{m+l}} \\ &= (-1)^m \sum_{l \geq 0} p^l \sum_{k=1}^{p-1} \binom{m+l-1}{l} \frac{1}{k^{m+l}} \\ &= (-1)^m \sum_{l \geq 0} \binom{m+l-1}{l} p^l H(m+l). \end{aligned}$$

□

Proposition 10. *1. For $p \geq 5$, we have*

$$H(1) \equiv -\frac{p}{2}H(2) \pmod{p^3}. \tag{2.30}$$

2. For $p \geq 11$, we have

$$H(1) \equiv -\frac{p}{2}H(2) \pmod{p^4}, \tag{2.31}$$

$$H(1) \equiv -\frac{p}{2}H(2) - \frac{p^2}{3}H(3) - \frac{p^3}{4}H(4) \pmod{p^6}. \tag{2.32}$$

Proof. We prove (2.31) and (2.32) only.

From Theorem 22 we obtain

$$\binom{p-1}{p-1} = 1 = 1 - pH(1) + \frac{p^2}{2}(H(1)^2 - H(2)) + \sum_{k=3}^{p-1}(-1)^k H(\{1\}^k)p^k.$$

Reducing modulo p^4 and using Theorems 15 or 5 for primes $p \geq 11$ gives

$$pH(1) \equiv -\frac{p^2}{2}H(2) \pmod{p^5}.$$

Dividing by p congruence (2.31) yields immediately. One can also notice that we could have used the previous proposition to show Eq(2.31).

For the second one, we have modulo p^7

$$\begin{aligned} pH(1) &\equiv \frac{p^2}{2}(H(1)^2 - H(2)) - \frac{p^3}{3!}(-3H(1)H(2) + 2H(3)) \\ &+ \frac{p^4}{4!}(3H(2)^2 - 6H(4)) \pmod{p^7}. \end{aligned}$$

Dividing by p and rearranging we find

$$\begin{aligned} H(1) &\equiv -\frac{p}{2}H(2) - \frac{p^2}{3}H(3) - \frac{p^3}{4}H(4) \\ &+ \frac{p}{2}(H(1)^2 + pH(1)H(2) + \frac{p^2}{4}H(2)^2) \pmod{p^6}. \end{aligned}$$

The nonlinear terms can be eliminated by means of the first congruence in next lemma. □

Lemma 5. *For $p \geq 11$, we have*

$$H(1)^2 + pH(1)H(2) + \frac{p^2}{4}H(2)^2 \equiv 0 \pmod{p^8}, \tag{2.33}$$

$$H(1)^3 + \frac{3p}{2}H(1)^2H(2) + \frac{3p^2}{4}H(1)H(2)^2 + \frac{p^3}{8}H(2)^3 \equiv 0 \pmod{p^{12}}, \tag{2.34}$$

$$\begin{aligned} H(1)^2 + pH(1)H(2) + \frac{2p^2}{3}H(1)H(3) + \frac{p^3}{2}H(1)H(4) + \frac{p^2}{4}H(2)^2 + \frac{p^3}{3}H(2)H(3) \\ + \frac{p^4}{4}H(2)H(4) + \frac{p^4}{9}H(3)^2 + \frac{p^5}{6}H(3)H(4) + \frac{p^6}{16}H(4)^2 \equiv 0 \pmod{p^{12}}. \end{aligned} \tag{2.35}$$

Proof. From the previous Proposition we have

$$H(1) + \frac{p}{2}H(2) \equiv 0 \pmod{p^4},$$

hence, squaring and cubing this congruence we obtain (2.33) and (2.34). For the last one we may use a similar argument and square (2.32). □

Theorem 23. *For $p \geq 13$, we have*

$$H(1) \equiv -\frac{p}{2}H(2) - \frac{p^2}{3}H(3) - \frac{p^3}{4}H(4) - \frac{p^4}{5}H(5) - \frac{p^5}{6}H(6) \pmod{p^8}, \tag{2.36}$$

$$H(1) \equiv -\sum_{k=2}^{8} \frac{p^{k-1}}{k}H(k) \pmod{p^9}. \tag{2.37}$$

Proof. Congruences are proved in quite similar argument, however the expressions involved in the proof are somewhat cumbersome. We prove only the second one.

Using Maple (we prefer not to give the very long full expressions of $H(\{1\}^7)$,$H(\{1\}^8)$, $H(\{1\}^9)$, $H(\{1\}^{10})$) and reducing modulo p^{10} we obtain

$$\begin{aligned} pH(1) \equiv& \frac{p^2}{2}(H(1)^2 - H(2)) - \frac{p^3}{3!}(H(1)^3 - 3H(1)H(2) + 2H(3)) \\ &+ \frac{p^4}{4!}(-6H(1)^2H(2) + 8H(3)H(1) + 3H(2)^2 - 6H(4)) \\ &- \frac{p^5}{5!}(15H(1)H(2)^2 - 30H(4)H(1) - 20H(3)H(2) + 24H(5)) \\ &+ \frac{p^6}{6!}(-15H(2)^3 + 90H(4)H(2) - 120H(6)) - \frac{p^7}{7}H(7) - \frac{p^8}{8}H(8). \end{aligned}$$

Dividing by p and rearranging terms so we can simplify by Lemma 5 we find

$$\begin{aligned} H(1) \equiv& -\sum_{k=2}^{8} \frac{p^{k-1}}{k}H(k) \\ &+ \frac{p}{2}\big(H(1)^2 + pH(1)H(2) + \frac{2p^2}{3}H(1)H(3) + \frac{p^3}{2}H(1)H(4) + \frac{p^2}{4}H(2)^2 \\ &+ \frac{p^3}{3}H(2)H(3) + \frac{p^4}{4}H(2)H(4) + \frac{p^4}{9}H(3)^2 + \frac{p^5}{6}H(3)H(4) + \frac{p^6}{16}H(4)^2\big) \\ &- \frac{p^2}{3!}(H(1)^3 + \frac{3p}{2}H(1)^2H(2) + \frac{3p^2}{4}H(1)H(2)^2 + \frac{p^3}{8}H(2)^3). \end{aligned}$$

Hence, the non-linear terms are eliminated modulo p^9. □

Remark 3. *The interested reader may type the following simple code in maple*

$$X := seq((-1)^{i-1} factorial(i-1) * H(i), i = 1..10) :$$

$$Y := seq(\frac{(-1)^i * p^i}{factorial(i)} * CompleteBellB(i, X), i = 1..10)$$

The following Theorem had been discussed by several authors (for instance [26], [34], [41]), here we refine some of these results as well as giving a new type of congruences.

Theorem 24. *For $v_p(\alpha) \geq 0$, we have the following congruences*

1. *For $p \geq 7$, we have*

$$\binom{\alpha p - 1}{p - 1} \equiv 1 \pmod{p^{v_p(\alpha)+3}}, \tag{2.38}$$

$$\binom{\alpha p - 1}{p - 1} \equiv 1 - \alpha p H(1) \pmod{p^{2v_p(\alpha)+3}}. \tag{2.39}$$

2. *For $p \geq 11$, we have*

$$\binom{\alpha p-1}{p-1} \equiv 1+\alpha(\alpha-1)pH(1) \pmod{p^{2v_p(\alpha)+5}}, \tag{2.40}$$

$$\binom{\alpha p-1}{p-1} \equiv 1-\frac{1}{2}\alpha(\alpha-1)p^2H(2) \pmod{p^{2v_p(\alpha)+5}}. \tag{2.41}$$

3. *For non-zero α and $p \geq 11$, we have*

$$\begin{aligned}\binom{\alpha p-1}{p-1} \equiv\ & 1-\alpha pH(1)-\frac{(\alpha p)^2}{2}H(2)-\frac{(\alpha p)^3}{3}H(3)-\frac{(\alpha p)^4}{4}H(4)\\ &+\frac{\alpha^4p^2}{2}(\frac{1}{\alpha}H(1)+\frac{p}{2}H(2))^2 \pmod{p^{3v_p(\alpha)+7}}.\end{aligned} \tag{2.42}$$

Proof. The proof of the first two congruences is omitted.

For the first congruence in part two of the Theorem, we have form Corollary 8 that

$$\binom{\alpha p-1}{p-1} \equiv 1-\alpha pH(1)+(\alpha p)^2H(\{1\}^2)-(\alpha p)^3H(\{1\}^3)+(\alpha p)^4H(\{1\}^4) \pmod{p^{5v_p(\alpha)+5}}.$$

Hence, canceling terms modulo $p^{2v_p(\alpha)+5}$ using Theorem 5, we have

$$\binom{\alpha p-1}{p-1} \equiv 1-\alpha pH(1)-\frac{(\alpha p)^2}{2}H(2) \pmod{p^{2v_p(\alpha)+5}}.$$

Now, multiplying both sides of the congruence (2.31) by $\alpha^2 p$ gives

$$\alpha^2pH(1) \equiv -\frac{\alpha^2p^2}{2}H(2) \pmod{p^{2v_p(\alpha)+5}}.$$

Therefore, the first two congruences follows immediately.

For the last congruence, we have form Corollary 8 that

$$\binom{\alpha p-1}{p-1} \equiv \sum_{k=0}^{6}(-1)^kH(\{1\}^k)(\alpha p)^k \pmod{p^{7v_p(\alpha)+7}}.$$

Canceling modulo $p^{3v_p(\alpha)+7}$ we get

$$\begin{aligned}\binom{\alpha p-1}{p-1} \equiv\ & 1-\alpha pH(1)+\frac{(\alpha p)^2}{2}(H(1)^2-H(2))-\frac{(\alpha p)^3}{3!}(-H(2)H(1)+2H(3))\\ &+\frac{(\alpha p)^4}{4!}(3H(2)^2-6H(4)) \pmod{p^{3v_p(\alpha)+7}}.\end{aligned}$$

For $\alpha \neq 0$, we can rearrange as follows

$$\begin{aligned}\binom{\alpha p-1}{p-1} \equiv\ & 1-\alpha pH(1)-\frac{(\alpha p)^2}{2}H(2)-\frac{(\alpha p)^3}{3}H(3)-\frac{(\alpha p)^4}{4}H(4)\\ &+\frac{\alpha^4p^2}{2}(\frac{1}{\alpha^2}H(1)^2+\frac{p}{\alpha}H(1)H(2)+\frac{p^2}{4}H(2)^2) \pmod{p^{3v_p(\alpha)+7}},\end{aligned}$$

and the last congruence yields immediately. This concludes the proof. □

2.3.3 Congruences with second roots of unity

Proposition 11. *Let $\mathbb{K}$ be a field of characteristic 0 and p an odd prime. Then for all $x \in \mathbb{K}$*

$$\binom{x-1}{p-1}\binom{x+p-1}{p-1} = \sum_{k=0}^{p-1}(-1)^kH(\{2\}^k)x^{2k}. \tag{2.43}$$

Proof. This is immediate by Theorem 22, and by noticing that

$$\binom{-x-1}{p-1} = \binom{x+p-1}{p-1}.$$

□

In particular, when setting $x := \alpha p$ we find

$$\binom{\alpha p-1}{p-1}\binom{(\alpha+1)p-1}{p-1} = \sum_{k=0}^{p-1}(-1)^k H(\{2\}^k)(\alpha p)^{2k}.$$

By Theorem 15 and the above proposition, we deduce the following results.

Proposition 12. *We have the following congruences*

1. *For $v_p(\alpha) \geq -\frac{1}{2}$ and $p \geq 5$,*

$$\binom{\alpha p-1}{p-1}\binom{(\alpha+1)p-1}{p-1} \equiv 1 \pmod{p^{2v_p(\alpha)+3}}. \tag{2.44}$$

2. *For $v_p(\alpha) \geq -\frac{1}{2}$ and $p \geq 7$,*

$$\binom{\alpha p-1}{p-1}\binom{(\alpha+1)p-1}{p-1} \equiv 1-(\alpha p)^2 H(2) \pmod{p^{4v_p(\alpha)+5}}. \tag{2.45}$$

3. *For $v_p(\alpha) \geq -\frac{1}{2}$ and $p \geq 11$,*

$$\binom{\alpha p-1}{p-1}\binom{(\alpha+1)p-1}{p-1} \equiv 1-(\alpha p)^2 H(2) - \frac{(\alpha p)^4}{2}H(4) \pmod{p^{4v_p(\alpha)+6}}. \tag{2.46}$$

4. *For $v_p(\alpha) \geq -\frac{3}{4}$ and $p \geq 11$,*

$$\binom{\alpha p-1}{p-1}\binom{(\alpha+1)p-1}{p-1} \equiv 1-(\alpha p)^2 H(2) + \frac{(\alpha p)^4}{2}(H(2)^2-2H(4)) \pmod{p^{6v_p(\alpha)+7}}. \tag{2.47}$$

5. *For $v_p(\alpha) \geq -\frac{1}{2}$ and $p \geq 13$,*

$$\binom{\alpha p-1}{p-1}\binom{(\alpha+1)p-1}{p-1} \equiv 1-(\alpha p)^2 H(2) + \frac{(\alpha p)^4}{2}(H(2)^2-2H(4)) - \frac{(\alpha p)^6}{3}H(6) \pmod{p^{6v_p(\alpha)+8}}. \tag{2.48}$$

Proposition 13. 1. *For $p \geq 7$, we have*

$$\binom{2p-1}{p-1} \equiv 1-p^2H(2) \pmod{p^5}, \tag{2.49}$$

$$\binom{2p-1}{p-1} \equiv 1-p^2H(2) - \frac{p^4}{2}H(4) \pmod{p^6}. \tag{2.50}$$

2. *For $p \geq 11$, we have*

$$\binom{2p-1}{p-1} \equiv 1-p^2H(2) + \frac{p^4}{2}(H(2)^2-H(4)) \pmod{p^7}, \tag{2.51}$$

$$\binom{2p-1}{p-1} \equiv 1-p^2H(2) + \frac{p^4}{2}(H(2)^2-H(4)) - \frac{p^6}{3}H(6) \pmod{p^8}. \tag{2.52}$$

2.3.4 Congruences with third roots of unity

Proposition 14. *Let $\mathbb{K}$ be any extension of $\mathbb{Q}$ that contains the third roots of unity, and p an odd prime, then for all $x \in \mathbb{K}$, we have*

$$\binom{x-1}{p-1}\binom{\omega_1 x-1}{p-1}\binom{\omega_2 x-1}{p-1} = \sum_{k=0}^{p-1}(-1)^k H(\{3\}^k)x^{3k}, \tag{2.53}$$

where $\omega_1 = \frac{-1+i\sqrt{3}}{2}$, $\omega_2 = \frac{-1-i\sqrt{3}}{2}$, and $i = \sqrt{-1}$.

When substituting x by p, $-p$, $2p$, respectively we obtain the following.

Corollary 10. *Each of the following formulas holds true*

$$\binom{\omega_1 p-1}{p-1}\binom{\omega_2 p-1}{p-1} = \sum_{k=0}^{p-1}(-1)^k H(\{3\}^k)p^{3k}, \tag{2.54}$$

$$\binom{2p-1}{p-1}\binom{-\omega_1 p-1}{p-1}\binom{-\omega_2 p-1}{p-1} = \sum_{k=0}^{p-1} H(\{3\}^k)p^{3k}, \tag{2.55}$$

$$\binom{2p-1}{p-1}\binom{2\omega_1 p-1}{p-1}\binom{2\omega_2 p-1}{p-1} = \sum_{k=0}^{p-1}(-1)^k 2^{3k} H(\{3\}^k)p^{3k}. \tag{2.56}$$

With the help of (2.54) and Theorem 15 we obtain the following.

Corollary 11. 1. *For $p \geq 11$, we have*

$$\binom{\omega_1 p-1}{p-1}\binom{\omega_2 p-1}{p-1} \equiv 1 - p^3 H(3) \pmod{p^7}. \tag{2.57}$$

2. *For $p \geq 13$, we have*

$$\binom{\omega_1 p-1}{p-1}\binom{\omega_2 p-1}{p-1} \equiv 1 - p^3 H(3) - \frac{p^6}{2}H(6) \pmod{p^{10}}. \tag{2.58}$$

3. *For $p \geq 17$, we have*

$$\binom{\omega_1 p-1}{p-1}\binom{\omega_2 p-1}{p-1} \equiv 1 - p^3 H(3) + \frac{p^6}{2}(H(3)^2 - H(6)) - \frac{p^9}{3}H(9) \pmod{p^{12}}. \tag{2.59}$$

2.3.5 Congruences for $\prod_{\omega^n=1}\binom{\alpha\omega p-1}{p-1}^{-1}$

We conclude this section by remarking that in order to get congruences for $\prod_{\omega^n=1}\binom{\alpha\omega p-1}{p-1}^{-1}$ one may just replace $H(\{n\}^k)$ by $H^\star(\{n\}^k)$ or the minus signs by plus ones whenever necessary as in these illustrative examples.

Corollary 12. *We have the following congruences*

1. *For $v_p(\alpha) \geq \frac{\epsilon_n}{n} - 1$ and $p \geq n+3$*

$$\frac{1}{\prod_{\omega^n=1}\binom{\alpha\omega p-1}{p-1}} \equiv 1 \pmod{p^{nv_p(\alpha)+n+\epsilon_n}}, \tag{2.60}$$

2. *For* $v_p(\alpha) \geq \frac{1}{n} - 1$ *and* $p \geq 2n+3$

$$\frac{1}{\prod_{\omega^n=1}\binom{\alpha\omega p-1}{p-1}} \equiv 1 + (\alpha p)^n H(n) \pmod{p^{2nv_p(\alpha)+2n+1}}. \tag{2.61}$$

3. *For* $v_p(\alpha) \geq \frac{1}{n} - 1$ *and* $p \geq 4n+3$

$$\frac{1}{\prod_{\omega^n=1}\binom{\alpha\omega p-1}{p-1}} \equiv 1 + (\alpha p)^n H(n) + \frac{(\alpha p)^{2n}}{2} H(2n) \pmod{p^{2nv_p(\alpha)+2n+\epsilon_n+1}}. \tag{2.62}$$

4. *For* $v_p(\alpha) \geq \frac{\epsilon_n - 1}{n} - 1$ *and* $p \geq 5n+3$

$$\frac{1}{\prod_{\omega^n=1}\binom{\alpha\omega p-1}{p-1}} \equiv 1 + (\alpha p)^n H(n) + \frac{(\alpha p)^{2n}}{2} H(2n) + \frac{(\alpha p)^{3n}}{3} H(3n) + \frac{(\alpha p)^{4n}}{4} H(4n) \pmod{p^{2nv_p(\alpha)+2n+2\epsilon_n}}. \tag{2.63}$$

In particular, when $n \geq 2$*, and* $v_p(\alpha) \geq \frac{\epsilon_n}{n} - 1$ *the last congruence can be further reduced to*

$$\frac{1}{\prod_{\omega^n=1}\binom{\alpha\omega p-1}{p-1}} \equiv 1 + (\alpha p)^n H(n) + \frac{(\alpha p)^{2n}}{2} H(2n) \pmod{p^{2nv_p(\alpha)+2n+2\epsilon_n}}. \tag{2.64}$$

Proof. Similar to the proof of Corollary 9. □

Corollary 13. *Let* $\{1, \omega_1, \ldots, \omega_n\}$ *be the set of* n*-th roots of unity, then each of the following formulas holds true*

$$\frac{1}{\binom{\omega_1 p-1}{p-1}\binom{\omega_2 p-1}{p-1}\cdots\binom{\omega_n p-1}{p-1}} = \sum_{k\geq 0} H^\star(\{n\}^k) p^{nk}, \tag{2.65}$$

$$\frac{1}{\binom{2p-1}{p-1}\binom{-\omega_1 p-1}{p-1}\cdots\binom{-\omega_n p-1}{p-1}} = \sum_{k\geq 0} (-1)^{nk} H^\star(\{n\}^k) p^{nk}, \tag{2.66}$$

$$\frac{1}{\binom{2p-1}{p-1}\binom{2\omega_1 p-1}{p-1}\cdots\binom{2\omega_n p-1}{p-1}} = \sum_{k\geq 0} 2^{nk} H^\star(\{n\}^k) p^{nk}. \tag{2.67}$$

Corollary 14. *The following congruences hold true.*

1. *For* $p \geq 11$*, we have*

$$\frac{1}{\binom{\alpha p-1}{p-1}\binom{(\alpha+1)p-1}{p-1}} \equiv 1 + (\alpha p)^2 H(2) + \frac{(\alpha p)^4}{2} H(4) \pmod{p^{4v_p(\alpha)+6}}. \tag{2.68}$$

2. *For* $p \geq 13$*, we have*

$$\frac{1}{\binom{\alpha p-1}{p-1}\binom{(\alpha+1)p-1}{p-1}} \equiv 1 + (\alpha p)^2 H(2) + \frac{(\alpha p)^4}{2}(H(2)^2 + 2H(4)) + \frac{(\alpha p)^6}{3} H(6) \pmod{p^{6v_p(\alpha)+8}}. \tag{2.69}$$

3. *For* $p \geq 11$*, we have*

$$\frac{1}{\binom{\omega_1 p-1}{p-1}\binom{\omega_2 p-1}{p-1}} \equiv 1 + p^3 H(3) + \frac{p^6}{2} H(6) \pmod{p^9}, \tag{2.70}$$

$$\frac{1}{\binom{\omega_1 p-1}{p-1}\binom{\omega_2 p-1}{p-1}} \equiv 1 + p^3 H(3) + \frac{p^6}{2}(H(3)^2 + H(6)) + \frac{p^9}{3} H(9) \pmod{p^{12}} \tag{2.71}$$

4. *For* $p \geq 11$*, we have*

$$\frac{1}{\binom{2p-1}{p-1}\binom{ip-1}{p-1}\binom{(i+1)p-1}{p-1}} \equiv 1 + p^4 H(4) + \frac{p^8}{2} H(8) \pmod{p^{10}}, \tag{2.72}$$

$$\frac{1}{\binom{2p-1}{p-1}\binom{ip-1}{p-1}\binom{(i+1)p-1}{p-1}} \equiv 1 + p^4 H(4) + \frac{p^8}{2}(H(4)^2 + H(8)) + \frac{p^{12}}{3} H(12) \pmod{p^{14}}. \tag{2.73}$$

Corollary 15. 1. *For* $p \geq 7$*, we have*

$$\frac{1}{\binom{2p-1}{p-1}} \equiv 1 + p^2 H(2) \pmod{p^5}, \tag{2.74}$$

$$\frac{1}{\binom{2p-1}{p-1}} \equiv 1 + p^2 H(2) + \frac{p^4}{2} H(4) \pmod{p^6}. \tag{2.75}$$

2. *For* $p \geq 11$*, we have*

$$\frac{1}{\binom{2p-1}{p-1}} \equiv 1 + p^2 H(2) + \frac{p^4}{2}(H(2)^2 + H(4)) \pmod{p^7}, \tag{2.76}$$

$$\frac{1}{\binom{2p-1}{p-1}} \equiv 1 + p^2 H(2) + \frac{p^4}{2}(H(2)^2 + H(4)) + \frac{p^6}{3} H(6) \pmod{p^8}. \tag{2.77}$$

Chapter 3

Congruences with multiple harmonic sums

Investigating arithmetical properties of well-known numbers and functions (Bernoulli numbers and polynomials, Binomial coefficients, Stirling numbers, etc.), as well as, infinite summations (for instance $\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{17\pi^4}{360}$ and their finite counterpart), has been and is still a rich subject in mathematics. Numerous mathematicians utilized various combinatorial and algebraic methods to prove a variety of interesting congruences. One could consult [28] for the subject of congruences, especially, ones including binomial coefficients and harmonic numbers.

3.1 Multiple harmonic sums and quasi-shuffle product

In this section we introduce the concepts of shuffle product and quasi-shuffle product,

The shuffle product may be defined inductively as follows: given two words $\boldsymbol{s} = (s_1, \dots, s_l) = (s_1, \boldsymbol{s'})$ and $\boldsymbol{t} = (t_1, \dots, t_k) = (t_1, \boldsymbol{t'})$, we define a new multiplication that obeys to these two rules

$$\boldsymbol{s} * \emptyset = \emptyset * \boldsymbol{s} = \boldsymbol{s}, \quad (s_1, \boldsymbol{s'}) * (t_1, \boldsymbol{t'}) = s_1(\boldsymbol{s'} * \boldsymbol{t}) + t_1(\boldsymbol{s} * \boldsymbol{t'}). \tag{3.1}$$

It is worth mentioning that the sum in the above definition is formal, it is just the concatenation of words.

Given two ordered sets $(r_1, \dots, r_t)$ and $(r_{t+1}, \dots, r_n)$ we can define the shuffle product explicitly as

$$\mathsf{Shfl}((r_1, \dots, r_t), (r_{t+1}, \dots, r_n)) = \bigcup_{\sigma \in \boldsymbol{P}} (r_{\sigma(1)}, \dots, r_{\sigma(n)})$$

where $\boldsymbol{P}$ is the subset of permutations such that both $(r_1, \dots, r_t)$ and $(r_{t+1}, \dots, r_n)$ appear in their original order, more precisely

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(t) \quad \text{and} \quad \sigma^{-1}(t+1) < \cdots < \sigma^{-1}(n),$$

for example, we have

$$\begin{aligned}\mathsf{Shfl}(r_1,r_2) &= \{(r_1,r_2),(r_2,r_1)\}\,,\\ \mathsf{Shfl}(r_1,(r_2,r_3)) &= \{(r_1,r_2,r_3),(r_2,r_1,r_3),(r_2,r_3,r_1)\}\,,\\ \mathsf{Shfl}((r_1,r_2),(r_3,r_4)) &= \{(r_1,r_2,r_3,r_4),(r_1,r_3,r_2,r_4),(r_1,r_3,r_4,r_2),\\ &\qquad (r_3,r_1,r_2,r_4),(r_3,r_4,r_1,r_2),(r_3,r_1,r_4,r_2)\}.\end{aligned}$$

To see some sort of equivalence between the two definitions one may observe for instance

$$\begin{aligned}r_1 * r_2 &= r_1r_2 + r_2r_1,\\ r_1 * r_2r_3 &= r_1r_2r_3 + r_2r_1r_3 + r_2r_3r_1,\\ r_1r_2 * r_3r_4 &= r_1r_2r_3r_4 + r_1r_3r_2r_4 + r_1r_3r_4r_2 + r_3r_1r_2r_4 + r_3r_4r_1r_2 + r_3r_1r_4r_2.\end{aligned}$$

The quasi-shuffle product (or stuffle product) on the other hand is defined recursively by the following rules

$$\boldsymbol{s} \circledast \emptyset = \emptyset \circledast \boldsymbol{s} = \boldsymbol{s}, \qquad (s_1,\boldsymbol{s'}) \circledast (t_1,\boldsymbol{t'}) = s_1(\boldsymbol{s'} \circledast \boldsymbol{t}) + t_1(\boldsymbol{s} \circledast \boldsymbol{t'}) + (s_1+t_1)(\boldsymbol{s'} \circledast \boldsymbol{t'}).$$

The quasi-shuffle product is called sometimes the harmonic shuffle product (see [48]) due to the following interesting relation

$$H(\boldsymbol{s};n)H(\boldsymbol{t};n) = \sum_{\boldsymbol{r}\in \boldsymbol{s}\circledast\boldsymbol{t}} H(\boldsymbol{r};n).$$

Proposition 15. *The quasi-shuffle product of a (a word of length 1) and $\boldsymbol{s} = (s_1,\dots,s_l)$ is given by the following rule*

$$a \circledast \boldsymbol{s} = a * \boldsymbol{s} + \sum_{i=1}^{l}(s_1,\dots,s_{i-1},s_i+a,s_{i+1},\dots,s_l).$$

Therefore, we have

$$\begin{aligned}H(m;n)H(s_1,\dots,s_l;n) = &\sum_{\boldsymbol{s}\in\mathsf{Shfl}(m,(s_1,\dots,s_l))} H(\boldsymbol{s};n)\\ &+\sum_{i=1}^{l} H(s_1,\dots,s_{i-1},s_i+m,s_{i+1},\dots,s_l;n)\end{aligned} \tag{3.2}$$

In particular,

$$H(a)H(b) = H(b,a) + H(a,b) + H(a+b), \tag{3.3}$$

$$H(a)H(b,c) = H(a,b,c) + H(b,a,c) + H(b,c,a) + H(a+b,c) + H(b,a+c). \tag{3.4}$$

Proof. We prove by induction on the length l. Suppose that the relation is true for words of length $l-1$, then using (3.1), we obtain

$$\begin{aligned} a \circledast \boldsymbol{s} =&(a, \boldsymbol{s}) + s_1(a \circledast (s_2, \dots, s_l)) + (a + s_1, s_2, \dots, s_l) \\ =&(a, \boldsymbol{s}) + s_1\left(a * \boldsymbol{s'} + \sum_{i=2}^{l}(s_2, \dots, s_{i-1}, s_i + a, s_{i+1}, \dots, s_l)\right) + (a + s_1, s_2, \dots, s_l) \\ =&(a, \boldsymbol{s}) + s_1(a * \boldsymbol{s'}) + \sum_{i=1}^{l}(s_1, \dots, s_{i-1}, s_i + a, s_{i+1}, \dots, s_l) \\ =&a * \boldsymbol{s} + \sum_{i=1}^{l}(s_1, \dots, s_{i-1}, s_i + a, s_{i+1}, \dots, s_l). \end{aligned}$$

□

3.2 A refinement of Kazandzidis congruence

Theorem 25. *[46] Let n and r be non-negative integers and $p \geq 7$ be a prime. Then*

$$\frac{\binom{np}{rp}}{\binom{n}{r}} \equiv 1 + w_p nr(n-r)p^3 \pmod{p^5}.$$

Moreover,

$$\frac{\binom{np}{rp}}{\binom{n}{r}} \equiv 1 \pmod{p^4}$$

for all n, r if and only if p divides the numerator of B_{p-3}, and w_p is the unique non-negative integer strictly less than p^2 such that $w_p \equiv \frac{H(1)}{p^2} \pmod{p^2}$.

Proof. See [46] for the original proof. Here we give a more detailed one. We have that

$$\binom{np}{rp} = \frac{\prod_{j=n-r+1}^{n}(jp)_p}{\prod_{l=1}^{r}(lp)_p},$$

therefore with the help of elementary symmetric polynomials we have

$$\frac{\binom{np}{rp}}{\binom{n}{r}} \equiv \frac{\prod_{j=n-r+1}^{n}\left(1 - jpH(1) + j^2p^2H(\{1\}^2)\right)}{\prod_{l=1}^{r}\left(1 - lpH(1) + l^2p^2H(\{1\}^2)\right)} \pmod{p^5}. \tag{3.5}$$

Now, using the stuffle product of Eq (3.3) one can see

$$pH(\{1\}^2) = p\frac{H(1)^2 - H(2)}{2} \equiv -p\frac{H(2)}{2} \equiv H(1) \pmod{p^4}.$$

Substituting in (3.5), then using properties of symmetric polynomials for elements $(-j^2+j)_{n-r+1 \leq j \leq n}$, we find

$$\prod_{j=n-r+1}^{n}\left(1 - (-j^2+j)pH(1)\right) = 1 + \sum_{k=1}^{r}(-1)^k e_k p^k H(1)^k,$$

also since $|pH(1)| < 1$, we have

$$\frac{1}{\prod_{l=1}^{r}(1 - (-l^2+l)pH(1))} = \sum_{k \geq 0} h_k p^k H(1)^k.$$

Therefore,

$$\frac{\binom{np}{rp}}{\binom{n}{r}} \equiv \left(1+\sum_{j=n-r+1}^{n}(j^2-j)pH(1)\right)\left(1-\sum_{l=1}^{r}(l^2-l)pH(1)\right) \pmod{p^5}$$
$$\equiv 1-\sum_{l=1}^{r}(l^2-l)pH(1)+\sum_{j=n-r+1}^{n}(j^2-j)pH(1) \pmod{p^5}$$
$$\equiv 1+w_p nr(n-r)p^3 \pmod{p^5}.$$

□

3.3 Some congruences with multiple harmonic sums

In this section, we determine the sums $\sum_{j=1}^{p-1}\frac{H_j^{(s_1)}H_j^{(s_3)}}{j^{s_2}}$ and $\sum_{j=1}^{p-1}\frac{H_j^{(s_1)}H_j^{(s_3)}H_j^{(s_4)}}{j^{s_2}}$ modulo p and modulo p^2 in certain cases. This is done by using multiple harmonic sums of length three and four, as well as, many other results. In addition, We recover three congruences conjectured by Z.-W Sun and solved later in [40] and [27].

Z. W. Sun, in many of his papers, has conjectured a large collection of congruences and identities concerning sums with generalized harmonic numbers and binomial coefficients. For example in [39], he proposed the following problems (namely, Conjecture 1.1 and Conjecture 1.2).

Conjecture 2. *1. For primes $p \geq 5$, we have*

$$\sum_{j=1}^{p-1}\frac{H_j}{j2^j} \equiv \frac{7}{24}pB_{p-3} \pmod{p^2}, \tag{3.6}$$

$$\sum_{j=1}^{p-1}\frac{H_j^2}{j^2} \equiv \frac{4}{5}pB_{p-5} \pmod{p^2}. \tag{3.7}$$

2. Let n be a positive integer. Suppose that $p-1$ does not divide $6n$, then

$$\sum_{j=1}^{p-1}\frac{(H_j^{(2n)})^2}{j^{2n}} \equiv 0 \pmod{p}. \tag{3.8}$$

3. And for $p > 6n+1$, we have

$$\sum_{j=1}^{p-1}\frac{(H_j^{(s)})^2}{j^s} \equiv \left[\binom{3s+1}{s-1}+\frac{s}{2}\right]p\frac{B_{p-3s-1}}{3s+1} \pmod{p^2}. \tag{3.9}$$

Later, in [40] he proved (3.6), and Conjecture 1.2 (Eq (3.8) and (3.9)). Meštrović [27], on the other hand, established the second part of Conjecture 1.1 (Eq (3.7)) using congruences of (MHS) of length three that can be found in [47].

In this section, we may unify the proof of (3.7), (3.8), (3.9), as well as, proving many other congruences of similar kind.

Now, we give some results which will be using later.

Theorem 26. *[47] Let s and l be two positive integers. Let p be an odd prime such that $p \geq l+2$ and $p-1$ divides non of ls and $ks+1$ for $k=1,\dots,l$. Then*

$$H(\{s\}^l) \equiv \begin{cases} 0 & \pmod{p^2} \quad \textit{for} \quad ls-1 \textit{ even}, \\ 0 & \pmod{p} \quad \textit{for} \quad ls-1 \textit{ odd}. \end{cases}$$

In particular, when $p \geq ls+3$, the above is always true.

Theorem 27. *[47] Let s and k be two non-negative integers. Suppose also that $p \geq sk+3$, then*

$$H(\{s\}^k) \equiv \begin{cases} (-1)^k \frac{s(sk+1)p^2}{2(sk+2)} B_{p-sk-2} & \pmod{p^3} \quad \textit{for } ks \textit{ odd}, \\ (-1)^{k-1} \frac{sp}{sk+1} B_{p-sk-1} & \pmod{p^2} \quad \textit{for } ks \textit{ even}, \end{cases}$$

where $(B_n)_{n\in\mathbb{N}}$ is the sequence of Bernoulli numbers.

Theorem 28. *[47] Let s_1,s_2 be two positive integers and $p \geq 3$. Let $s_1 \equiv m, s_2 \equiv n \pmod{p-1}$ where $0 \leq m,n \leq p-2$. If m,n≥ 1 then*

$$H(s_1,s_2) \equiv \begin{cases} \frac{(-1)^n}{m+n}\binom{m+n}{m} B_{p-m-n} & \pmod{p} \quad \textit{for} \quad p \geq m+n, \\ 0 & \pmod{p} \quad \textit{for} \quad p < m+n. \end{cases}$$

Furthermore, when s_1+s_2 is even and $p > s_1+s_2+1$

$$\begin{aligned} H(s_1,s_2) \equiv p &\left((-1)^{s_1} s_2 \binom{s_1+s_2+1}{s_1} - (-1)^{s_1} s_1 \binom{s_1+s_2+1}{s_2} - s_1 - s_2\right) \\ &\times \frac{B_{p-s_1-s_2-1}}{2(s_1+s_2+1)} \pmod{p^2}. \end{aligned}$$

Theorem 29. *[47],[15] Suppose that $w := s_1+s_2+s_3$ is odd, then for primes $p > w$ we have*

$$H(s_1,s_2,s_3) \equiv \left((-1)^{s_1}\binom{w}{s_1} - (-1)^{s_3}\binom{w}{s_3}\right) \frac{B_{p-w}}{2w} \pmod{p}.$$

In particular, when $s_1 = s_3$, and s_2 is odd, we have

$$H(s_1,s_2,s_1) \equiv 0 \pmod{p}.$$

3.3.1 Congruences using MHS of length three and two

We start by establishing the following theorem which is motivated by a paper of Meštrović [27].

Theorem 30. *For all positive integers s_1, s_2, s_3, we have the following*

$$\sum_{j=1}^{p-1} \frac{H_j^{(s_1)} H_j^{(s_3)}}{j^{s_2}} = -H(s_1,s_2,s_3) + H(s_3,s_1+s_2) + H(s_1+s_2+s_3) + H(s_3)H(s_1,s_2). \tag{3.10}$$

Equivalently, we have

$$\begin{aligned} \sum_{j=1}^{p-1} \frac{H_j^{(s_1)} H_j^{(s_3)}}{j^{s_2}} =& H(s_1,s_3,s_2) + H(s_3,s_1,s_2) + H(s_3,s_1+s_2) \\ &+ H(s_1+s_3,s_2) + H(s_1,s_2+s_3) + H(s_1+s_2+s_3). \end{aligned} \tag{3.11}$$

Proof. For $j = 1, \dots, p-1$, we have

$$\frac{1}{(j+1)^{s_3}} + \cdots + \frac{1}{(p-1)^{s_3}} = -H_j^{(s_3)} + H(s_3).$$

We obtain the following

$$\begin{aligned}
H(s_1, s_2, s_3) = & \sum_{1\le i<j<k\le p-1} \frac{1}{i^{s_1} j^{s_2} k^{s_3}} = \sum_{j=2}^{p-1} \frac{1}{j^{s_2}} \sum_{i=1}^{j-1} \frac{1}{i^{s_1}} \sum_{k=j+1}^{p-1} \frac{1}{k^{s_3}} \\
= & \sum_{j=2}^{p-1} \frac{1}{j^{s_2}} (1 + \frac{1}{2^{s_1}} + \cdots + \frac{1}{(j-1)^{s_1}})(\frac{1}{(j+1)^{s_3}} + \cdots + \frac{1}{(p-1)^{s_3}}) \\
= & \sum_{j=2}^{p-1} \frac{1}{j^{s_2}} (1 + \frac{1}{2^{s_1}} + \cdots + \frac{1}{(j-1)^{s_1}})(-H_j^{(s_3)} + H(s_3)) \\
= & -\sum_{j=1}^{p-1} \frac{1}{j^{s_2}} (H_j^{(s_1)} - \frac{1}{j^{s_1}}) H_j^{(s_3)} + H(s_3) \sum_{j=1}^{p-1} \frac{1}{j^{s_2}} H_{j-1}^{(s_1)} \\
= & -\sum_{j=1}^{p-1} \frac{H_j^{(s_1)} H_j^{(s_3)}}{j^{s_2}} + \sum_{j=1}^{p-1} \frac{1}{j^{s_1+s_2}} (H_{j-1}^{(s_3)} + \frac{1}{j^{s_3}}) + H(s_3) \sum_{j=1}^{p-1} \frac{1}{j^{s_2}} H_{j-1}^{(s_1)} \\
= & -\sum_{j=1}^{p-1} \frac{H_j^{(s_1)} H_j^{(s_3)}}{j^{s_2}} + H(s_3, s_1+s_2) + H(s_1+s_2+s_3) + H(s_3) H(s_1, s_2).
\end{aligned}$$

Therefore

$$\sum_{j=1}^{p-1} \frac{H_j^{(s_1)} H_j^{(s_3)}}{j^{s_2}} = -H(s_1, s_2, s_3) + H(s_3, s_1+s_2) + H(s_1+s_2+s_3) + H(s_3) H(s_1, s_2).$$

Eq (3.11) follows immediately by the shuffle product relation

$$H(s_3) H(s_1, s_2) = H(s_3, s_1, s_2) + H(s_1, s_3, s_2) + H(s_1, s_2, s_3) + H(s_3+s_1, s_2) + H(s_1, s_3+s_2).$$

□

Remark 4. *One can easily observe that the previous proof does not depend on $p-1$, so for all positive integers n, we have*

$$\sum_{j=1}^{n} \frac{H_j^{(s_1)} H_j^{(s_3)}}{j^{s_2}} = -H(s_1, s_2, s_3; n) + H(s_3, s_1+s_2; n) + H(s_1+s_2+s_3; n) + H(s_3; n) H(s_1, s_2; n).$$

and

$$\begin{aligned}
\sum_{j=1}^{n} \frac{H_j^{(s_1)} H_j^{(s_3)}}{j^{s_2}} = & H(s_1, s_3, s_2; n) + H(s_3, s_1, s_2; n) + H(s_3, s_1+s_2; n) \\
& + H(s_1+s_3, s_2; n) + H(s_1, s_2+s_3; n) + H_n^{(s_1+s_2+s_3)}.
\end{aligned}$$

3.3.2 Congruences mod p

From Theorem 30 one can see that determining $\sum_{j=1}^{p-1} \frac{H_j^{(s_1)} H_j^{(s_3)}}{j^{s_2}}$ depends on determining MHS of length three and two, Using Theorems 29 and 28 we obtain the following result.

Theorem 31. *Suppose that w is odd, then*

$$\sum_{j=1}^{p-1}\frac{H_j^{(s_1)}H_j^{(s_3)}}{j^{s_2}} \equiv \left[\frac{(-1)^{s_1+1}}{2w}\binom{w}{s_1} + \frac{(-1)^{s_3}+2(-1)^{s_1+s_2}}{2w}\binom{w}{s_3}\right]B_{p-w} \pmod{p}. \tag{3.12}$$

Proof. From the first part of Theorem 30, and in light of Theorems 29 and 28, we have modulo p

$$\begin{aligned}\sum_{j=1}^{p-1}\frac{H_j^{(s_1)}H_j^{(s_3)}}{j^{s_2}} &\equiv -\left((-1)^{s_1}\binom{w}{s_1} - (-1)^{s_3}\binom{w}{s_3}\frac{B_{p-w}}{2w}\right) + \frac{(-1)^{s_1+s_2}}{w}\binom{w}{s_3}B_{p-w}\\ &\equiv \left[\frac{(-1)^{s_1+1}}{2w}\binom{w}{s_1} + \frac{(-1)^{s_3}+2(-1)^{s_1+s_2}}{2w}\binom{w}{s_3}\right]B_{p-w}.\end{aligned}$$

□

Corollary 16. *Let $s \geq 1$ and r is odd, then for $p > 2s+r$, we have*

$$\sum_{j=1}^{p-1}\frac{(H_j^{(s)})^2}{j^r} \equiv (-1)^{s+r}\binom{2s+r}{s}\frac{B_{p-2s-r}}{2s+r} \pmod{p}. \tag{3.13}$$

For all $s \geq 1$ and any prime $p \geq 3$ such that $p-1$ does not divide $3s$, in particular when $p \geq 3s+3$, we have

$$\sum_{j=1}^{p-1}\frac{(H_j^{(s)})^2}{j^s} \equiv \binom{3s}{s}\frac{B_{p-3s}}{3s} \pmod{p}. \tag{3.14}$$

Moreover, when s is even, then

$$\sum_{j=1}^{p-1}\frac{(H_j^{(s)})^2}{j^s} \equiv 0 \pmod{p}. \tag{3.15}$$

Proof. For the first part, we apply Theorems 26 and 29.

For the second part, we apply Theorems 26 and 28; hence the last part yields immediately since Bernoulli numbers vanishes for odd values. This would solve the first part of Sun conjecture see ([39] Conjecture 1.2) and ([40] Theorem 1.2). □

3.3.3 Some special cases

In this section, we shall prove some congruences modulo p that are not covered by Theorem 31.

Lemma 6. *([14] Theorem 7.2, [47] page 94 and Proposition 3.8)*

For $p \geq 11$, we have

$$\frac{1}{3}H(2,3,1) \equiv -\frac{1}{2}H(3,2,1) \equiv -H(3,1,2) \equiv -\frac{1}{2}H(1,4,1) \equiv H(4,1,1) \equiv -\frac{1}{6}B_{p-3}^2 \pmod{p}. \tag{3.16}$$

For $p \geq 7$, we have

$$H(1,2,2) \equiv -\frac{3}{2}B_{p-5} \pmod{p}. \tag{3.17}$$

For $p \geq 17$, we have

$$H(5,3,4) \equiv H(4,3,5) \equiv 0 \pmod{p}. \tag{3.18}$$

Theorem 32. *For $p \geq 11$, we have*

$$2\sum_{j=1}^{p-1}\frac{H_j^{(2)}H_j}{j^3} \equiv -3\sum_{j=1}^{p-1}\frac{H_j^{(3)}H_j}{j^2} \equiv -6\sum_{j=1}^{p-1}\frac{H_j^{(3)}H_j^{(2)}}{j} \equiv -3\sum_{j=1}^{p-1}\frac{(H_j)^2}{j^4} \equiv 6\sum_{j=1}^{p-1}\frac{H_j^{(4)}H_j}{j} \equiv B_{p-3}^2 \pmod p, \tag{3.19}$$

$$\sum_{j=1}^{p-1}\frac{H_jH_j^{(2)}}{j^2} \equiv -\frac{1}{2}B_{p-5} \pmod p. \tag{3.20}$$

For $p \geq 17$, we have

$$\sum_{j=1}^{p-1}\frac{H_j^{(5)}H_j^{(4)}}{j^3} \equiv 0 \pmod p. \tag{3.21}$$

Proof. We shall prove only the last one, since the others are proved in a similar manner.

From Theorems 30, 28 and the previous Lemma we find

$$\sum_{j=1}^{p-1}\frac{H_j^{(5)}H_j^{(4)}}{j^3} \equiv -H(5,3,4)+H(4,8) \equiv 0+\frac{165}{4}B_{p-12}=0 \pmod p.$$

□

3.3.4 Congruences mod p^2

In this section, we prove some congruences modulo p^2 in certain cases only, since determining (MHS) of length three seems to be a much more involved problem.

Lemma 7. *([47] Proposition 3.7)*

For $p \geq 7$, we have

$$\frac{10}{9}H(1,2,1) \equiv \frac{5}{3}H(2,1,1) \equiv \frac{10}{11}H(1,1,2) \equiv pB_{p-5} \pmod{p^2}, \tag{3.22}$$

$$H(1,3,1) \equiv 0 \pmod{p^2}, \tag{3.23}$$

$$H(4,1) \equiv -B_{p-5} \pmod{p^2}. \tag{3.24}$$

Corollary 17. *For $p \geq 7$, we have*

$$\sum_{j=1}^{p-1}\frac{H_j^2}{j^2} \equiv H(4) \equiv \frac{4}{5}pB_{p-5} \pmod{p^2}, \tag{3.25}$$

$$\sum_{j=1}^{p-1}\frac{H_j^{(2)}H_j}{j} \equiv -\frac{7}{10}pB_{p-5} \pmod{p^2}, \tag{3.26}$$

$$\sum_{j=1}^{p-1}\frac{H_j^2}{j^3} \equiv B_{p-5} \pmod{p^2}. \tag{3.27}$$

Proof. From Theorem 28 and the previous lemma we obtain

$$H(1,2,1) \equiv \frac{9}{10}pB_{p-5} \equiv H(1,3) \pmod{p^2}.$$

Substituting in (3.10) and using Theorem 27 we find

$$\sum_{j=1}^{p-1} \frac{H_j^2}{j^2} \equiv H(4) \equiv \frac{4}{5} p B_{p-5} \pmod{p^2}.$$

This establishes the second part of Sun conjecture see ([39] Conjecture 1.1) proved by Meštrović see [27].

From Theorems 28, 27, and the previous lemma we get

$$\sum_{j=1}^{p-1} \frac{H_j^{(2)} H_j}{j} \equiv -\frac{3}{5} p B_{p-5} - \frac{9}{10} p B_{p-5} + \frac{4}{5} p B_{p-5} \equiv -\frac{7}{10} p B_{p-5} \pmod{p^2}.$$

From Equation (3.10) and the previous lemma we obtain

$$\sum_{j=1}^{p-1} \frac{H_j^2}{j^3} \equiv H(1,4) \pmod{p^2}.$$

Since

$$H(4,1) \equiv -B_{p-5} \pmod{p^2},$$

we have from Theorem 3.2 in [47]

$$H(4,1) + H(1,4) \equiv -B_{p-5} + H(1,4) \equiv -\frac{5}{6} p B_{p-6} = 0 \pmod{p^2}.$$

□

Corollary 18. *Suppose s is even, then for $p > 3s+1$, we have*

$$\sum_{j=1}^{p-1} \frac{(H_j^{(s)})^2}{j^s} \equiv \left[\binom{3s+1}{s-1} + \frac{s}{2} \right] p \frac{B_{p-3s-1}}{3s+1} \pmod{p^2}. \tag{3.28}$$

Proof. Using the identity

$$\binom{n}{k+1} = \frac{n-k}{k+1} \binom{n}{k},$$

and from the first part of Theorem 3.10, we find modulo p^2

$$\begin{aligned}
\sum_{j=1}^{p-1} \frac{(H_j^{(s)})^2}{j^s} &\equiv -H(s,s,s) + H(s,2s) + H(3s) \\
&\equiv \left[-s + \frac{2s\binom{3s+1}{s} - s\binom{3s+1}{2s} - 3s}{2} + 3s \right] p \frac{B_{p-3s-1}}{3s+1} \\
&= \left[2 + \frac{\binom{3s+1}{s} - \frac{s}{s+1}\binom{3s+1}{s} - 3}{2} \right] sp \frac{B_{p-3s-1}}{3s+1} \\
&= \left[1 + \frac{\binom{3s+1}{s}}{s+1} \right] \frac{sp}{2} \frac{B_{p-3s-1}}{3s+1} \\
&= \left[\binom{3s+1}{s-1} + \frac{s}{2} \right] p \frac{B_{p-3s-1}}{3s+1}.
\end{aligned}$$

Congruences (3.14) and (3.28) establishes Conjecture 1.2 (see [39] and [40] Theorem 1.2) with a slightly different approach. □

3.3.5 Some special congruences using MHS of length four

In this section, we prove some congruences using some results on (MHS) of length four.

Theorem 33. *The following equality holds true*

$$-\sum_{j=1}^{p-1}\frac{H_j^{(s_1)}H_j^{(s_3)}H_j^{(s_4)}}{j^{s_2}} = H(s_4)\sum_{j=1}^{p-1}\frac{H_j^{(s_1)}H_j^{(s_3)}}{j^{s_2}} + H(s_1,s_3,s_2,s_4) + H(s_3,s_1,s_2,s_4)$$
$$+ H(s_3,s_1+s_2,s_4) + H(s_1+s_3,s_2,s_4) + H(s_1,s_2+s_3,s_4) + H(s_1+s_2+s_3,s_4). \quad (3.29)$$

Proof. From Remark 4 and Eq (3.11) we find

$$\sum_{l=1}^{p-1}\frac{1}{l^{s_4}}\sum_{j=1}^{l-1}\frac{H_j^{(s_1)}H_j^{(s_3)}}{j^{s_2}} = \sum_{l=1}^{p-1}\frac{1}{l^{s_4}}\Big[H(s_1,s_3,s_2;l-1) + H(s_3,s_1,s_2;l-1)$$
$$+ H(s_3,s_1+s_2;l-1) + H(s_1+s_3,s_2;l-1)$$
$$+ H(s_1,s_2+s_3;l-1) + H(s_1+s_2+s_3;l-1)\Big]$$
$$\sum_{j=1}^{p-1}\left(\sum_{l=j+1}^{p-1}\frac{1}{l^{s_4}}\right)\frac{H_j^{(s_1)}H_j^{(s_3)}}{j^{s_2}} = H(s_1,s_3,s_2,s_4) + H(s_3,s_1,s_2,s_4) + H(s_3,s_1+s_2,s_4)$$
$$+ H(s_1+s_3,s_2,s_4) + H(s_1,s_2+s_3,s_4) + H(s_1+s_2+s_3,s_4)$$
$$\sum_{j=1}^{p-1}\left(H(s_4) - H_j^{(s_4)}\right)\frac{H_j^{(s_1)}H_j^{(s_3)}}{j^{s_2}} = H(s_1,s_3,s_2,s_4) + H(s_3,s_1,s_2,s_4) + H(s_3,s_1+s_2,s_4)$$
$$+ H(s_1+s_3,s_2,s_4) + H(s_1,s_2+s_3,s_4) + H(s_1+s_2+s_3,s_4).$$

Hence, (3.29) yields immediately. □

Corollary 19. *For $p \geq 7$ we have*

$$\sum_{j=1}^{p-1}\frac{H_j^3}{j} \equiv \frac{3}{2}pB_{p-5} \pmod{p^2}. \quad (3.30)$$

Proof. From the previous theorem, we have modulo p^2

$$-\sum_{j=1}^{p-1}\frac{H_j^3}{j} \equiv 2H(\{1\}^4) + 2H(1,2,1) + H(2,1,1) + H(3,1)$$
$$-\sum_{j=1}^{p-1}\frac{H_j^3}{j} \equiv -\frac{2}{5}pB_{p-5} - \frac{9}{5}pB_{p-5} + \frac{3}{5}pB_{p-5} + \frac{1}{10}pB_{p-5} = -\frac{3}{2}pB_{p-5}.$$

Note that another proof can be found in [27]. □

Lemma 8. *[15] Let a, b be non-negative integers and a prime $p > 2a+2b+3$. Then*

$$H(\{2\}^a,3,\{2\}^b) \equiv \frac{(-1)^{a+b}(a-b)}{(a+1)(b+1)}\binom{2a+2b+2}{2a+1}B_{p-2a-2b-3} \pmod{p}, \quad (3.31)$$

And for primes such that $p > 2a+2b+1$, we have

$$H(\{2\}^a,1,\{2\}^b) \equiv \frac{4(-1)^{a+b}(a-b)(1-4^{-a-b})}{(2a+1)(2b+1)}\binom{2a+2b}{2a}B_{p-2a-2b-1} \pmod{p}. \quad (3.32)$$

Corollary 20. *For $p \geq 11$ we have*

$$-\frac{1}{13}\sum_{j=1}^{p-1}\frac{(H_j^{(2)})^2 H_j^{(3)}}{j^2} \equiv \frac{3}{83}\sum_{j=1}^{p-1}\frac{(H_j^{(2)})^3}{j^3} \equiv B_{p-9} \pmod p, \tag{3.33}$$

$$-\frac{8}{21}\sum_{j=1}^{p-1}\frac{(H_j^{(2)})^2 H_j}{j^2} \equiv \frac{1}{3}\sum_{j=1}^{p-1}\frac{(H_j^{(2)})^3}{j} \equiv B_{p-7} \pmod p. \tag{3.34}$$

Proof. For the first congruence, we apply the previous lemma, Theorems 33, 28 and 29 and obtain

$$\begin{aligned}\sum_{j=1}^{p-1}\frac{(H_j^{(2)})^2 H_j^{(3)}}{j^2} &= -2H(\{2\}^3,3) - 2H(2,4,3) - H(4,2,3) - H(6,3)\\ &\equiv \big[+12 - \frac{40}{3} - \frac{35}{3} - 0\big]B_{p-9} = -13B_{p-9},\end{aligned}$$

also

$$\begin{aligned}\sum_{j=1}^{p-1}\frac{(H_j^{(2)})^3}{j^3} &= -2H(\{2\}^2,3,2) - 2H(2,5,2) - H(5,2,2) - H(7,2)\\ &\equiv \big[+\frac{56}{3} - 0 + 9 - 0\big]B_{p-9} = \frac{83}{3}B_{p-9}.\end{aligned}$$

The second part of the corollary is proved in a quite similar manner. □

3.4 Some summations modulo powers of p

Theorem 34. *Let $\mathbb{K}$ be any extension of $\mathbb{Q}$ that contains the n-th roots of unity, then for all $x \in \mathbb{K}$ and $m \geq 1$ we have*

$$(-1)^{mn}\prod_{\omega^n=1}\binom{\omega x-1}{m} = \sum_{k=0}^{m}(-1)^k H(\{n\}^k;m)x^{kn},$$

$$(-1)^{mn}\frac{1}{\prod_{\omega^n=1}\binom{\omega x-1}{m}} = \sum_{k\geq 0} H^\star(\{n\}^k;m)x^{kn}.$$

Proof. Similar to the proof of Theorem 22. □

Theorem 35. *Let $\mathbb{K}$ be any extension of $\mathbb{Q}$ that contains the n-th roots of unity, then for all $v_p(\alpha) \geq 0$, $m \geq 1$ and $(a_k)_{k\geq 1}$ is any sequence in that extension, then we have modulo* $\pmod{p^{4v_p(\alpha)+4}}$

$$\begin{aligned}\sum_{k=1}^{p-1}(-1)^{kn}a_k\prod_{\omega^n=1}\binom{\omega\alpha p-1}{k} \equiv \sum_{k=1}^{p-1}a_k - (\alpha p)^n\sum_{k=1}^{p-1}a_k H_k^{(n)} + (\alpha p)^{2n}\sum_{k=1}^{p-1}a_k H(\{n\}^2;k)\\ -(\alpha p)^{3n}\sum_{k=1}^{p-1}a_k H(\{n\}^3;k),\end{aligned} \tag{3.35}$$

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{(-1)^{kn}a_k}{\prod_{\omega^n=1}\binom{\omega\alpha p-1}{k}} \equiv \sum_{k=1}^{p-1}a_k + (\alpha p)^n\sum_{k=1}^{p-1}a_k H_k^{(n)} + (\alpha p)^{2n}\sum_{k=1}^{p-1}a_k H^\star(\{n\}^2;k)\\ +(\alpha p)^{3n}\sum_{k=1}^{p-1}a_k H^\star(\{n\}^3;k).\end{aligned} \tag{3.36}$$

Lemma 9. *[16], [22], [37] We have the following formulas*

$$\sum_{k=1}^{n}\frac{H_k^{(p)}}{k^q}+\sum_{k=1}^{n}\frac{H_k^{(q)}}{k^p}=H_n^{(p)}H_n^{(q)}+H_n^{(p+q)}. \tag{3.37}$$

In particular, we have

$$\sum_{k=1}^{n}\frac{H_k}{k}=\frac{H_n^2+H_n^{(2)}}{2}. \tag{3.38}$$

We also have

$$\sum_{k=1}^{n}H_k^{(r)}=(n+1)H_n^{(r)}-H_n^{(r-1)}, \tag{3.39}$$

$$\sum_{k=1}^{n}H_k^2=(n+1)H_n^2-(2n+1)H_n+2n, \tag{3.40}$$

$$\sum_{k=1}^{n}kH_k^2=\frac{n(n+1)}{2}H_n^2-\frac{n^2-n-1}{2}H_n+\frac{n(n-3)}{4}, \tag{3.41}$$

$$\sum_{k=1}^{n}H_kH_k^{(2)}=(n+1)H_nH_n^{(2)}-\frac{2n+1}{2}H_n^{(2)}+H_n-\frac{1}{2}H_n^2, \tag{3.42}$$

$$\sum_{k=1}^{n}kH_kH_k^{(2)}=\frac{n(n+1)}{2}H_nH_n^{(2)}+\frac{1+n-n^2}{4}H_n^{(2)}-\frac{2n+3}{4}H_n+\frac{1}{4}H_n^2+\frac{3}{4}n, \tag{3.43}$$

$$\sum_{k=1}^{n}H_k^3=(n+1)H_n^3-\frac{3}{2}(2n+1)H_n^2+3(2n+1)H_n+\frac{1}{2}H_n^{(2)}-6n. \tag{3.44}$$

Corollary 21. *We have the following congruences*

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{\binom{\alpha p-1}{k}}\equiv(p-1)(1+\alpha p+(\alpha p)^2-(\alpha p)^3)\pmod{p^{v_p(\alpha)+4}}, \tag{3.45}$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k kH_k}{\binom{\alpha p-1}{k}}\equiv(p-1)((\frac{2-p}{4})+\alpha p(\frac{p-4}{4})+\frac{3(\alpha p)^2}{16}(p-8)+\frac{3(\alpha p)^2}{8})\pmod{p^{v_p(\alpha)+3}}. \tag{3.46}$$

Proof. When $n=1$ and $a_k=1$ for all $k\geq 1$, using the second congruence in the previous theorem we find modulo $\pmod{p^{v_p(\alpha)+4}}$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{\binom{\alpha p-1}{k}}\equiv\sum_{k=1}^{p-1}1+\alpha p\sum_{k=1}^{p-1}H_k+(\alpha p)^2\sum_{k=1}^{p-1}H^\star(\{1\}^2;k)+(\alpha p)^3\sum_{k=1}^{p-1}H^\star(\{1\}^3;k)$$

Using (3.39), (3.40), (3.42), (3.44), we find

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{\binom{\alpha p-1}{k}}\equiv(p-1)+(\alpha p)(p-1)+\frac{(\alpha p)^2}{2}(2(p-1))+\frac{(\alpha p)^3}{6}(-6(p-1)).$$

Hence, the first congruence yields immediately.

The second one is done with exactly the same method so we may omit the proof. □

Corollary 22. *For $p\geq 5$, we have*

$$\sum_{k=1}^{p-1}\frac{H_k}{\binom{\alpha p-1}{k}\binom{\alpha p+k}{k}}\equiv((1-\alpha^2)p+(\alpha p)^2)H(1)\pmod{p^{2v_p(\alpha)+4}}. \tag{3.47}$$

Proof. By previous theorem and identities (3.39),(3.42), we have modulo $p^{4v_p(\alpha)+4}$

$$\sum_{k=1}^{p-1} \frac{H_k}{\binom{\alpha p-1}{k}\binom{\alpha p+k}{k}} \equiv pH(1) - (p-1) + (\alpha p)^2 \frac{2(p-1)+1}{2} H(2),$$

and since for $p \geq 5$

$$H(1) \equiv \frac{p}{2} H(2) \pmod{p^3},$$

then, multiplying both sides by $\alpha^2 p$ we have modulo $p^{2v_p(\alpha)+4}$

$$\sum_{k=1}^{p-1} \frac{H_k}{\binom{\alpha p-1}{k}\binom{\alpha p+k}{k}} \equiv ((1-\alpha^2)p + (\alpha p)^2) H(1).$$

□

Conclusion and Perspectives

- We generalized the p-adic mean value theorem and as an application we proved a weak version of Kazandzidis congruence, we also established other results with same nature.
- We established two new formulas that relate to roots of unity, this allows us to find many congruences and refine many old ones.
- We determined some summations $\mod p$ and $\mod p^2$.

We suggest the following issues to be investigated.

1. Find more applications of the p-adic HMVT.
2. In chapter 2 we announced the following conjecture

   Conjecture 3. *Let m be a positive integer, then for a large enough prime we have*

   $$H(1) \equiv -\sum_{k=2}^{2m} \frac{p^{k-1}}{k} H(k) \pmod{p^{2m+2}}.$$

   Can we find a simpler way to establish this conjecture ?
3. In the thesis "The Arithmetic of Multiple Harmonic Sums" by 'J. Rosen', the author investigated extended and optimized congruences for many sort of well-known numbers and functions, such as $\binom{kp-1}{p-1}$ and the p-adic L function. In Chapter 2 section 2.3 we explained his approach with the binomial coefficients.

   Can we find algebraic integers (or simply integers) $\beta_1, \ldots, \beta_n$ such that

   $$\prod_{\omega^n=1} \binom{\alpha\omega p - 1}{p-1} \equiv \sum_{j=0}^{m} \beta_j p^j H(\{n\}^j) \pmod{p^{2mn+3}},$$

   and polynomials $\beta_{j,m,n}(t)$ satisfying $\beta_{j,m,n}(\alpha) = \beta_j$?
4. What about $\prod_{\omega^n=1} \binom{\alpha\omega p-1}{p-1}^{-1}$? Can we find algebraic integers (or integers) $\gamma_1, \ldots, \gamma_n$ such that

   $$\frac{1}{\prod_{\omega^n=1} \binom{\alpha\omega p-1}{p-1}} \equiv \sum_{j=0}^{m} \gamma_j p^j H^\star(\{n\}^j) \pmod{p^{2mn+3}}.$$
5. Find extended and optimized congruences for $\prod_{\omega^n=1} \binom{\alpha\omega p-1}{mp-1}$, where m is a positive integer.

6. In [4] the author investigated binomial transforms, and gave many applications. Can we generalize his work by studying the more general transform replacing binomial coefficient by the product $\prod_{\omega^n=1}\binom{\alpha\omega p-1}{k}$ or by its inverse.

7. Study congruences that involve $\prod_{\omega^n=1}\binom{\alpha\omega p-1}{k}$ or its inverse.

8. It would be a nice riddle if we try to find summations (called Euler summations) of the form

$$\sum_{n=1}^{\infty}\frac{H_n^{(s_1)}H_n^{(s_3)}}{j^{s_2}} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{H_n^{(s_1)}H_n^{(s_3)}H_n^{(s_4)}}{n^{s_2}}$$

   for given non-negative integers s_1, s_2, s_3, s_4, we have for instance

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}=\frac{17\pi^4}{360} \text{ and } \sum_{n=1}^{\infty}\frac{H_n}{n^3}=\frac{\pi^4}{72}.$$

   In [44] the authors gave many explicit formulas.

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