N° d'ordre : 315 / 2025 - C / MT

REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique
Université des Sciences et de la Technologie Houari Boumediène

Faculté de Mathématiques

THÈSE DE DOCTORAT

Présentée pour l'obtention du **grade** de **DOCTEUR**

En : MATHÉMATIQUES

Spécialité : Arithmétique Codage et Combinatoire

Par:

Safia SEFFAH

Thème:

Some Diophantine equations involving generalized sequences

Soutenue publiquement, le 15/06/2025, devant le jury composé de :

M. B. BENSEBA	Professeur à l'USTHB	Président
M. A. TOGBÉ	Professeur à l'université Purdue Northwest, USA	Directeur de thèse
M. M. O. HERNANE	Professeur à l'USTHB	Co-Directeur de thèse
M. O. KIHEL	Professeur à l'université Brock, Canada	Examinateur
M. R. BOUMAHDI	Maître de conférences /A à l'ENSM, Sidi- Abdellah	Examinateur
M. T. GARICI	Maître de conférences /A à l'USTHB	Examinateur
M. S. E. RIHANE	Maître de conférences /A à l'ENSM, Sidi- Abdellah	Invité

Contents

Dedication

To my dear parents, for your constant support, endless encouragement, and, above all, for believing in me. Your trust in my abilities, even during the most difficult moments, has been my greatest source of strength. Thank you for your eternal support.

To my beloved sister Amal Fatima Zahra, thank you for always being there, for checking on me, encouraging me, and supporting me in every way. Your presence has been a true comfort throughout this journey.

To my brothers, thank you for your love and support. To my eldest brother Hamza Ahmed Soufyane, who is like a second father to me, your care and protection mean the world to me. And to my youngest brother Youcef, who always believed in me and motivated me to keep going, thank you for your unwavering faith.

To my family uncles, aunts, cousins and especially to my grandmother, thank you for your continuous encouragement, your prayers, and your love.

This thesis is also dedicated to the memory of my dear aunt Faiza SAFFAH, who was like a mother to me. Her pride in my accomplishments, her support, and her prayers have accompanied me every step of the way. May her soul rest in peace.

To my best friends Hadjer LEMGHITI, Khadidja MOUSSAOUI, Nada AZZOUZ, Houda AZZOUZ, Ahlem BOUCHERIKHA, and Meriem ASSAS, thank you for being more than just friends, you are like sisters to me. I am truly blessed to have you in my life. Your presence, your love, and your support during every step of this doctoral journey have meant everything to me. In both the joyful and the difficult moments, you stood by me, uplifted me, and never let me fall. You are a treasure in my life, and I will forever be grateful to you.

To my dear friends Zahia, Samah, Faiza, Selma, Asma, Zahra, Amina, Salima with whom I shared unforgettable memories during my undergraduate and master's studies, thank you for all the laughter, the kindness, and the support you've given me along the way. I will always be grateful to you for the warmth and friendship you brought into my life during those years.

To my classmates from the Master 2 ACC, class of 2017-2019, thank you for being such a wonderful part of my academic journey. We truly were like a family, and I carry with me many fond memories of our time together. I will always be grateful for your warmth, friendship, and the joyful moments that made those years so special.

I would also like to express my heartfelt gratitude to Zahra AMEUR, your wisdom, your guidance, and your sincere advice throughout my thesis journey have deeply shaped and encouraged me. You are a true role model, and I feel lucky to have followed your example.

To my friend Dyhia FEDELA and Rachid MAMMERI who supported me throughout my undergraduate and master's years, offering support, help, and sincere encouragement. I will always remain deeply thankful to you.

Finally, I would like to dedicate a special thought to my two wonderful friends Chèfiath ADEGBINDIN and Mariama Ndao FAYE I met during the CIMPA schools in Benin and

Senegal. We shared unforgettable experiences during those events, and from that point on, our friendship continued to grow. Despite the distance, we have stayed in touch, supporting and encouraging each other through life's journey. Thank you both for your kindness, your presence, and the beautiful memories we've created together.

Acknowledgements

First and foremost, I am deeply grateful to Allah for giving me the strength, patience, and determination to live this journey and complete it successfully. Without His guidance and mercy, none of this would have been possible.

I would like to express my deepest gratitude to my supervisors, who both played essential and complementary roles in this journey. My gratitude to them is beyond words.

To Professor Mohand Ouamar HERNANE, thank you for being the first to guide me at the very beginning of this research. Your support, your thoughtful guidance, and the working sessions we had early on were fundamental to shaping the direction of my thesis. Despite the personal challenges you faced, you never stopped supporting me. You remained present and committed, especially during the final stages of my thesis, helping facilitate the submission of my defense.

To Professor Alain Togbé, thank you for your constant presence and engagement, even from a distance. Through regular Zoom meetings, your follow-ups, and your genuine concern for the progress of my research, you made me feel truly supported throughout the entire process. You were always available, even during holidays, to provide advice, direction, and motivation. Your dedication and care were invaluable.

I am also sincerely thankful to both of my supervisors for encouraging me to take part in unforgettable academic experiences that marked my journey. Thanks to their recommendation and support, I had the opportunity to attend two CIMPA schools held in Benin and Senegal, where I met inspiring researchers, took part in high-level courses in mathematics, and had the chance to improve my English in a truly international environment.

I would also like to sincerely thank Professor Alain Togbé for everything he did for me during my stay at his university "Purdue University Northwest". His warm welcome, constant support, and generous availability made my research internship not only academically enriching but also personally memorable. I am truly grateful for the time he dedicated to guiding me, for his valuable advice, and for the trust he placed in me throughout that period. These moments were not only important for my academic growth, but also precious on a personal level, and I will always carry them with me.

I would like to extend my sincere thanks to the members of my thesis committee for kindly accepting to be part of my jury.

To Professors Boualem BENSEBA, Tarek GARICI and Rachid BOUMAHDI, thank you not only for kindly agreeing to be part of my thesis committee, but also for having been my teachers during my master's studies. It is thanks to the strong academic foundation you provided, and the doctoral entrance exam you organized, that I was able to pursue this path. I am truly grateful for the knowledge, the discipline, and the encouragement that shaped my early research journey.

To Professor Omar KIHEL, thank you sincerely for accepting to be part of the jury and for making the effort to travel from abroad to attend my defense. Your presence is an honor, and I deeply appreciate your time and commitment.

I would also like to sincerely thank Doctor Salah Eddine RIHANE for his valuable collaboration throughout the preparation of the papers related to this thesis. His experience, insight, and efficiency greatly enriched our work, and I truly appreciated the clarity and responsiveness he brought to our joint research efforts. Working with him was both inspiring and highly motivating.

I would also like to express my sincere gratitude to Professor Tarek GARICI for being an inspiring supervisor during my Master's 2 studies. It was under his guidance that I was first introduced to the field of Diophantine equations and continued fractions. His clear explanations, patience, and dedication made these topics accessible and sparked my interest in them. I am truly grateful for all the knowledge and insight I gained from him, and for the solid foundation he provided in these areas of mathematics.

Abstract

This thesis focuses on the study of Diophantine equations involving generalized sequences. The work begins with problems concerning Fermat and Mersenne numbers expressed as products of two k-Fibonacci numbers. Subsequently, we address an analogous problem involving k-Pell numbers.

In another direction, we investigate repdigits that can be written as products of Fibonacci and Lucas numbers or as products of two k-Fibonacci numbers. Furthermore, we determine all k-Pell numbers that can be expressed as almost repdigits.

The primary tools employed in this thesis include Baker's theory of linear forms in logarithms of algebraic numbers and the Baker-Davenport reduction method [7], specifically we use the version developed by de Dujella and Pethő [26].

Notation

$(F_n)_{n\geqslant 0}$	The Fibonacci sequence.
$(L_n)_{n\geqslant 0}$	The Lucas sequence.
$(P_n)_{n\geqslant 0}$	The Pell sequence.
$(F_n^{(k)})_{n\geqslant 2-k}$	The k-generalized Fibonacci sequence.
$(L_n^{(k)})_{n\geqslant 2-k}$	The k-generalized Lucas sequence.
$(P_n^{(k)})_{n\geqslant 2-k}$	The k-generalized Pell sequence.
$\mathbb{N}$	The set of natural numbers.
$\mathbb{Z}$	The set of integers.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{K}$	Algebraic number field.
η	Algebraic number.
$h(\eta)$	Absolute logarithmic height of an algebraic number η.
log	Natural logarithm.
$\\| \cdot \\|$	The distance from the nearest integer.
$N_{\mathbb{K}/\mathbb{Q}}(\alpha)$	The norm of α in a number field $\mathbb{K}$ relative to $\mathbb{Q}$.
$\nu_2(x)$	Exponent of 2 in the factorization of x.

Introduction

Different branches of mathematics have presented several problems that have been of interest to mathematicians.

Number theory is one of these branches that is used to study positive integers and their properties such as divisibility, prime factorization, or solvability of equations. Here, we focus on solving some equations. It's about " Diophantine equations".

The study of Diophantine equations dates back to Greek antiquity. They are introduced by the greek mathematician Diophantus of Alexandria who devoted to this type of equations a series of books in the history of Mathematics.

A Diophantine equation is a polynomial equation with integer coefficients whose solutions are sought in integers. For example, the most famous Diophantine equation is the linear equation of the form

$$ax + by = c, \tag{1}$$

where a, b and c are given integers, and x and y are variables.

We can also consider equations with a larger number of variables or with a higher degree. For example, we take the equation

$$x^2 + y^2 = z^2. \tag{2}$$

If (x, y, z) is a solution of (2), this triplet of integers is called a "Pythagorean triple", with reference to Pythagoras.

When we increase the power of the variables in equation (2), we obtain the famous equation

$$x^n + y^n = z^n, \tag{3}$$

known as Fermat's conjecture. He claimed that equation (3) cannot be solved when $n \geqslant 3$. Later, this equation was resolved by Andrew wiles.

Another type of Diophantine equations was studied as a Master's thesis. It was on Diophantine equations

$$ax^2 - by^4 = c, \qquad c = \pm 1, 2.$$

The theory of continued fractions was used for solving these equations.

From the above equations, we ask about the method used to solve them and if it can be applied to all type of Diophantine equations. This question was the 10-th problem of Hilbert among his 23 problems that he posed them in 1900, he was asking for a general method for solving all Diophantine equations as follows:

"Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined

by a finite number of operations whether the equation is solvable in integers".

In 1970, Matiyasevich's answered Hilbert and proved that such method does not exist. His result is the following.

Theorem 0.1 (Y. Matiyasevich)**.** *There is no algorithm which, for a given arbitrary Diophantine equation, would tell whether the equation has an integer solution or not.*

However, this didn't stop the study on Diophantine equations. Many mathematicians have found techniques and approaches for solving certain families of Diophantine equations.

In this thesis, we consider some Diophantine equations involving linear recurrent sequences such as Fibonacci, Lucas and Pell sequences. We will use the Baker's theory, mainly based on linear form in logarithms of algebraic numbers and a reduction algorithm originally introduced by Baker and Davenport. The introduced concepts have an extremely useful and interesting applications in the study of the Diophantine equations throughout our thesis.

Alan Baker and Linear forms in logarithms

We return again to the 23-th Hilbert's problems. More precisely, to the 7-th problem which motivated the development of the theory of linear forms in logarithms. This problem asked to prove the transcendence of the number α^β for any algebraic number $\alpha \neq 0, 1$ and any irrational algebraic number β. This problem was proved independently by Gelfond and Schneider in 1934. They proved that, if $\alpha_1, \alpha_2 \neq 0$ are algebraic numbers such that $\log \alpha_1, \log \alpha_2$ are linearly independent over $\mathbb{Q}$, then

$$\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0$$

for all algebraic numbers β_1, β_2.

Gelfond also noticed that a similar theorem would hold for an arbitrarily many logarithmic of algebraic numbers. Indeed, in 1966 Alan Baker generalized that result by proving that if $\alpha_1, \cdots, \alpha_n$ are algebraic numbers $\neq 0, 1$ such that $\log \alpha_1, \cdots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$, then

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots \beta_n \log \alpha_n \neq 0$$

for any algebraic numbers $\beta_0, \beta_1, \cdots, \beta_n$ that are not all zero.

This generalization was the beginning of a new and very interesting branch in number theory known as *Baker's theory.* This earned Baker the Fields medal in 1970.

Baker-Davenport reduction

The *Baker- Davenport* reduction was introduced by Baker and Davenport in the paper [7] from 1969, in which they solved the problem which emerged in a discussion during a conference in Oberwolfach in March 1968. The problem is connected to the so-called Diophantine m-tuples, i.e. the sets of positive integers such that the product of any two of its distinct elements, increased by 1 is a perfect square.

At this conference, J. H. van Lint presented his results by attempting to extend Fermat's set, which is $\{1, 3, 8, 120\}$, to a quintuple with the same property. More precisely, if d is a positive integer such that $\{1, 3, 8, d\}$ is a Diophantine quadruple, does d have to be equal to 120? The problem was raised by Martin Gardner in his Scientific American column in March 1967.

By applying Baker's theory of linear forms in logarithms and the introduced reduction method, Baker and Davenport completely solved that problem. Their result was later generalized in several directions, which we will discuss in detail in the upcoming chapters.

In this thesis, we are interested in the solution of some Diophantine equations involving sequences. It consists of six chapters, in addition to the introduction where we give a brief overview to some Diophantine equations and the methods used to solve them.

Chapter 1 covers the prerequisites, offering essential definitions, fundamental concepts, key results, and ideas used throughout this dissertation.

In chapter 2, we study Fermat and Mersenne numbers as products of two k-Fibonacci numbers. More precisely, we solve the Diophantine equation stated in the following theorem.

Theorem 0.2 ([34]). *The Diophantine equation*

$$F_n^{(k)} F_m^{(k)} = 2^a \pm 1 \tag{4}$$

has no solutions in positive integers n, m, k and a with $3 \leqslant m \leqslant n$ and $k \geqslant 2$.

Chapter 3 addresses the same problem as Chapter 5 by considering the k-Pell sequences. In this chapter, we prove the following result.

Theorem 0.3 ([55]). *The Diophantine equations*

$$P_n^{(k)} P_m^{(k)} = 2^a \pm 1 \tag{5}$$

have no solutions in positive integers a, k, n, m with $n \geqslant m \geqslant 2$ and $k \geqslant 2$.

In chapter 4, we investigate on the solutions of the Diophantine equation

$$F_n^{(k)} L_m^{(k)} = \frac{a(10^\ell - 1)}{9}, \tag{6}$$

where repdigits are represented as products of k-Fibonacci and k-Lucas numbers. We establish the following results of equation (6).

Theorem 0.4 ([57]). *All the solutions of Diophantine equation* (6) *in positive integers n, m, ℓ, k, and a with $0 \leqslant m < n$, $k \geqslant 2$, $\ell \geqslant 2$, and $1 \leqslant a \leqslant 9$, are*

$$(a, k, l, m, n) \in \{(4, 3, 2, 1, 8), (5, 2, 2, 1, 10), (8, 2, 2, 5, 6), (8, 3, 2, 0, 8)\}.$$

Theorem 0.5 ([57]). *All the solutions of Diophantine equation* (6) *in positive integers n, m, ℓ, k, and a with $1 \leqslant n \leqslant m$, $k \geqslant 2$, $\ell \geqslant 2$, and $1 \leqslant a \leqslant 9$, are*

$$(a, k, l, m, n) \in \{(1, 2, 2, 5, 1), (1, 2, 2, 5, 2), (2, 4, 2, 5, 1), (2, 4, 2, 5, 2),$$
$$(2, 2, 2, 5, 3), (3, 2, 2, 5, 4), (4, 4, 2, 5, 3), (5, 2, 2, 5, 5), (8, 4, 2, 5, 4)\}.$$

In chapter 5, we explore Diophantine equations involving repdigits and products of two k-Fibonacci numbers. Our main result is given by the following theorem.

Theorem 0.6 ([56]). *The only solution of the Diophantine equation*

$$F_n^{(k)} F_m^{(k)} = \frac{a(10^\ell - 1)}{9} \tag{7}$$

in positive integers n, m, ℓ, k, and a with $3 \leqslant m \leqslant n$, $k \geqslant 3$, $\ell \geqslant 2$, and $1 \leqslant a \leqslant 9$, is

$$(a, k, l, m, n) = (8, 3, 2, 3, 8).$$

In chapter 6, we solve the Diophantine equation

$$P_n^{(k)} = a\left(\frac{10^{d_1}-1}{9}\right) + (b-a)10^{d_2}, \quad 0 \leqslant d_2 < d_1, \quad \text{and} \quad 0 \leqslant a, b \leqslant 9, \tag{8}$$

by proving the following result.

Theorem 0.7 ([54]). *The Diophantine equation* (2.2) *has only the following solution* $P_8^{(3)} = 545$, $P_7^{(4)} = 228$ *and* $P_7^{(5)} = 232$ *when* $P_n^{(k)}$ *has at least three digits.*

The final part of this dissertation includes a conclusion, perspectives for future work and the list of references.

Publications

This thesis is based on the results of the publications listed below. All these papers have either been published or submitted for publication.

1. M. O. Hernane, S. E. Rihane, S. Seffah and A. Togbé, *On Fermat and Mersenne numbers expressible as product of two k-Fibonacci numbers*, Boletín de la Sociedad Matemática Mexicana. 28 (2), 36 (2022). doi: 10.1007/s40590-022-00429-4

2. S. Seffah, S. E. Rihane and A. Togbé, *Almost Repdigits in k-Pell numbers*, submitted.

3. S. Seffah, S. E. Rihane and A. Togbé, *Fermat and Mersenne numbers as products of two k-Pell numbers*, submitted.

4. S. Seffah, S. E. Rihane and A. Togbé, *On equations involving repdigits and products of two k-Fibonacci*, submitted.

5. S. Seffah, S. E. Rihane and A. Togbé, *On repdigits as product of k-Fibonacci and k-Lucas numbers.* Mathematica Bohemica. (2025). https://doi.org/10.21136/MB.2025.0035-24

Chapter 1

Preliminaries

In this chapter, we present the fundamental mathematical concepts and results that are essential for understanding the problem at the heart of this thesis. Given that our research builds on advanced topics in algebraic number theory, we start by revisiting the core ideas in this field, focusing particularly on properties of binary sequences and generalized sequences. We also recall some methods used to prove our main results, which include essential tools such as linear forms in logarithms and reduction methods.

1.1 Number fields

1.1.1 Algebraic Numbers

Definition 1.1. *A complex number α is called an algebraic number if there is a polynomial $Q(x)$ with rational coefficients, different from the zero polynomial, such that $Q(\alpha) = 0$. A complex number is called transcendental if it is not algebraic.*

Theorem 1.2 (Theorem 12.1 of [25])**.** *For any algebraic number α, there is a unique polynomial*

$$P(x) = a_d x^d + \cdots + a_1 x + a_0$$

with the following properties:

(1) $P(x) \in \mathbb{Z}[x]$,

(2) $a_d > 0$ *and* $\gcd(a_0, a_1, \cdots, a_d) = 1$,

(3) $P(\alpha) = 0$,

(4) if $P_0(x) \in \mathbb{Q}[x]$ *such that* $P_0(\alpha) = 0$, *then* $P(x) \mid P_0(x)$ *in* $\mathbb{Q}[x]$,

(5) $P(x)$ *is irreducible over* $\mathbb{Q}$.

Definition 1.3. *The minimal polynomial over $\mathbb{Z}$ of an algebraic number α is the polynomial $P(x)$ described in Theorem 1.2. The minimal polynomial of α is the polynomial $g(x) = \frac{1}{a_d}P(x)$, hence, it is the irreducible monic polynomial with rational coefficients such that $g(\alpha) = 0$. The degree of an algebraic number is the degree of its minimal polynomial.*

Elements that are algebraic over $\mathbb{Q}$ and have the same minimal polynomial are called conjugates over $\mathbb{Q}$.

1.1.2 Algebraic number fields

Definition 1.4. Let α be an algebraic number. The algebraic number field $\mathbb{Q}(\alpha)$ generated by α is the smallest field which contains $\mathbb{Q}$ and α. We say that $\mathbb{Q}(\alpha)$ is a simple algebraic extension of $\mathbb{Q}$.

Definition 1.5. Let $\mathbb{K} = \mathbb{Q}(\alpha)$. The degree of $\mathbb{K}$ is defined as the degree n of the minimal polynomial of α, i.e. the degree of α.

1.1.3 Absolute logarithm height of an algebraic number

For a given non-zero algebraic number η of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}\left(X-\eta^{(j)}\right)$, we denote by

$$h(\eta)=\frac{1}{d}\left(\log|a|+\sum_{j=1}^{d}\log\max\left(1,|\eta^{(j)}|\right)\right)$$

the usual absolute logarithmic height of η. In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. Below, we outline some well-known properties of the height function (Property 3.3 in [58]), which will be used as needed throughout this work.

$$h(\eta\pm\gamma)\leqslant h(\eta)+h(\gamma)+\log 2, \tag{1.1}$$

$$h(\eta\gamma^{\pm 1})\leqslant h(\eta)+h(\gamma), \tag{1.2}$$

$$h(\eta^{s})=|s|\,h(\eta)\quad (s\in\mathbb{Z}), \tag{1.3}$$

where η, γ are algebraic numbers.

1.2 Binary sequences

Definition 1.6. Let $k \geqslant 1$ be an integer. A sequence $(u_n)_{n\geqslant 0}$ of complex numbers is called linearly recurrent of order k if the recurrence

$$u_{n+k}=a_1u_{n+k-1}+a_2u_{n+k-2}+\cdots+a_ku_n \tag{1.4}$$

holds for all $n \geqslant 0$ with some fixed coefficients $a_1, \cdots, a_k \in \mathbb{C}$.

Assume that $a_k \neq 0$ (since if $a_k = 0$, the sequence $(u_n)_{n\geqslant 0}$ satisfies a linear recurrence of order smaller than k). If $a_1, \ldots, a_k \in \mathbb{Z}$ and $u_0, \ldots, u_{k-1} \in \mathbb{Z}$, then by induction on n, it follows that u_n is an integer for all $n \geqslant 0$.

Definition 1.7. Let $(u_n)_{n\geqslant 0}$ be a linear recurrent sequence of order k. The characteristic polynomial of the sequence $(u_n)_{n\geqslant 0}$ is given by

$$f(X)=X^k-a_1X^{k-1}-\cdots-a_k,$$

where $a_1, a_2, \ldots, a_k$ are the coefficients of the recurrence relation defining u_n.

Considering $\alpha_1, \ldots, \alpha_s$ are the distinct roots of $f(x)$ with multiplicities $\sigma_1, \ldots, \sigma_s$, respectively, we assume that

$$f(X)=\prod_{i=1}^{s}(X-\alpha_i)^{\sigma_i}.$$

From the theory of linear recurrence sequences, it follows that for all i there exist uniquely determined polynomials $h_i \in \mathbb{Q}(u_0, \dots, a_1, \dots, a_k, \alpha_1, \dots, \alpha_k)[x]$ of degree less than $\sigma_i (i = 1, \dots, s)$ such that

$$u_n = \sum_{i=1}^{s} h_i(n)\alpha_i^n, \quad \text{for } n \geqslant 0.$$

This dissertation focuses solely on integer recurrent sequences, which are sequences defined by recurrence relations whose coefficients and initial values are integers, i.e.,$s = k$ (all the roots of $f(x)$ are distinct). Consequently, for all $i = 1, \dots, k$ and $n \in \mathbb{Z}$, $h_i(n)$ is an algebraic number. The following result thus follows.

Proposition 1.8 (Proposition 2.0.2 in [39])**.** Suppose that $f(X) \in \mathbb{Z}[X]$ has distinct roots. Then there exist constants $c_1, \dots, c_k \in \mathbb{K} = \mathbb{Q}(\alpha_1, \dots, \alpha_k)$ such that the formula

$$u_n = \sum_{i=1}^{k} c_i \alpha_i^n \quad \text{holds for all} \quad n \geqslant 0.$$

If $k = 2$, the sequence is called *binary sequence.*

Definition 1.9. Let $\{G\}_{n=0}^{\infty}$ be a sequence defined by the binary recurrence relation $G_n = AG_{n-1} + BG_{n-2}$ for $n \geqslant 2$, where A, B, G_0 and G_1 are given integers. The characteristic equation of this sequence is

$$X^2 - AX - B = 0,$$

with distinct roots α and β satisfying

$$\alpha = (A + \sqrt{D})/2 \quad \text{and} \quad \beta = (A - \sqrt{D})/2,$$

where $D = A^2 + 4B$ is the discriminant of $X^2 - AX - B$. We observe that

$$A = \alpha + \beta, \quad B = -\alpha\beta, \quad \text{and} \quad \alpha - \beta = \sqrt{D}.$$

The explicit formula for the terms of the sequence $\{G\}_{n=0}^{\infty}$ is given by the following result.

Theorem 1.10 (See page 2 in [38])**.** *If $D \neq 0$, then the explicit form for G_n is given by*

$$G_n = \frac{(G_1 - \beta G_0)\alpha^n - (G_1 - \alpha G_0)\beta^n}{\alpha - \beta}, \quad n \geqslant 0.$$

If $D = 0$, then $\alpha = \beta = A/2$ and

$$G_n = n\alpha^{n-1}G_1 - (n-1)\alpha^n G_0, \quad n \geqslant 0.$$

In the upcoming section, we will introduce some specific cases of binary sequences known as *Lucas sequences*, focusing exclusively on those used throughout this thesis.

1.3 Lucas sequences

Definition 1.11. Given two integers A and B, the Lucas sequence $\{G\}_{n=0}^{\infty}$ is defined by the recurrence relation

$$G_0 = 0, \ G_1 = 1, \ G_n = AG_{n-1} + BG_{n-2}.$$

Its characteristic equation is:

$$x^2 - Ax - B = 0.$$

It has the discriminant $D = A^2 + 4B \neq 0$ and the roots

$$\alpha = \frac{A + \sqrt{D}}{2} \quad \textit{and} \quad \beta = \frac{A - \sqrt{D}}{2},$$

where, $\alpha \neq \beta$, $\alpha + \beta = A$, $\alpha\beta = B$, and $(\alpha - \beta)^2 = D$.

Thus, from the conditions of Theorem (1.10) the terms of Lucas sequence can be written in terms of α and β as follows:

$$G_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{for} \quad n \geqslant 0.$$

Definition 1.12. With the above conditions, the sequences $U = (U_n(A, B))_{n=0}^{\infty}$ and $V = (V_n(A, B))_{n=0}^{\infty}$ given by

$$U_0 = 0,\ U_1 = 1,\ U_n = AU_{n-1} - BU_{n-2} \text{ for } n \geqslant 2, \tag{1.5}$$

$$V_0 = 2,\ V_1 = A,\ V_n = AV_{n-1} - BV_{n-2} \text{ for } n \geqslant 2, \tag{1.6}$$

are called the first and second Lucas sequences with parameters (A, B).

1.3.1 The Fibonacci sequence

Definition 1.13. The Fibonacci sequence $(F_n)_{n=0}^{\infty}$ is obtained with $(A, B) = (1, -1)$. From (1.5), the sequence of Fibonacci numbers is given by

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geqslant 2. \tag{1.7}$$

Its initial terms are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

From Theorem(1.10), we define the n-th Fibonacci number F_n by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n), \quad \text{for} \quad n \geqslant 0, \quad \text{where} \quad (\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right).$$

1.3.2 The Lucas sequence

Definition 1.14. The Lucas sequence $(L_n)_{n=0}^{\infty}$ is related to the Fibonacci sequence which is also obtained with $(A, B) = (1, -1)$. Thus, from (1.6) it follows that

$$L_0 = 2, \quad L_1 = 1, \quad \text{and} \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n \geqslant 2. \tag{1.8}$$

Its first terms are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots.$$

As the Fibonacci numbers, the n-th Lucas number is given by

$$L_n = \alpha^n + \beta^n, \quad \text{for} \quad n \geqslant 0, \quad \text{where} \quad (\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right).$$

1.3.3 The Pell sequence

Definition 1.15. The Pell sequence $(P_n)_{n=0}^{\infty}$ corresponds to $(A, B) = (2, -1)$. It's defined from (1.5) by

$$P_0 = 0, \quad P_1 = 1, \quad \text{and} \quad P_n = 2P_{n-1} + P_{n-2} \quad \text{for} \quad n \geqslant 2. \tag{1.9}$$

Its first few terms are

$$0, 1, 1, 2, 5, 12, 29, 70, 169, \ldots.$$

Applying Theorem(1.10) yields

$$P_n = \frac{1}{2\sqrt{2}}(\alpha^n - \beta^n) \quad \text{for} \quad n \geqslant 0, \quad \text{where} \quad (\alpha, \beta) = (1+\sqrt{2}, 1-\sqrt{2}).$$

Let us go further in the hierarchy of these sequences by increasing their order, i.e., considering $k \geqslant 2$. This gives rise to new linear recurrence sequences, referred to as generalized sequences. These generalized sequences extend the classical Lucas sequences by adding higher-order recurrence relations and enable a more thorough investigation of their algebraic and arithmetic characteristics.

1.4 Generalized sequences

1.4.1 The k-generalized Fibonacci sequence and the k-generalized Lucas sequence

Definition 1.16. For an integer $k \geqslant 2$, the sequences $U^k = (U_n^{(k)})_{n\geqslant 0}$ and $V^k = (V_n^{(k)})_{n\geqslant 0}$ defined by the following linear recurrence relations of order k

$$U_n^{(k)} = U_{n-1}^{(k)} + U_{n-2}^{(k)} + \ldots + U_{n-k}^{(k)}, \quad \text{for all} \quad n \geqslant 0, \tag{1.10}$$

$$V_n^{(k)} = V_{n-1}^{(k)} + V_{n-2}^{(k)} + \ldots + V_{n-k}^{(k)}, \quad \text{for all} \quad n \geqslant 0, \tag{1.11}$$

with initial conditions

$$U_{-(k-2)}^{(k)} = U_{-(k-3)}^{(k)} = \ldots = U_0^{(k)} = 0, \quad U_1^{(k)} = 1, \tag{1.12}$$

$$V_{-(k-2)}^{(k)} = V_{-(k-3)}^{(k)} = \ldots = V_{(-1)}^{(k)} = 0, \quad V_0^{(k)} = 2 \quad \text{and} \quad V_1^{(k)} = 1, \tag{1.13}$$

are called Fibonacci sequence of order k, Lucas sequence of order k, respectively. They generalize respectively the Fibonacci number sequence and the Lucas number sequence, corresponding to the value $k = 2$.

Definition 1.17. Let denote by $(F_n^{(k)})_{n\geqslant 2}$, $(L_n^{(k)})_{n\geqslant 2}$ the k-generalized Fibonacci sequence and the k generalized Lucas sequence, respectively. The characteristic polynomial of these sequences is

$$P(X) = X^k - \sum_{i=0}^{k-1} X^i = 0.$$

From the following lemmas, we establish estimates for the n-th term of the generalized Fibonacci and Lucas sequences, respectively.

Lemma 1.18. *[11, Lemma 3] If $n < 2^{k/2}$ and $n \geqslant k+2$, then the following estimates hold*

$$F_n^{(k)} = 2^{n-2}(1 + \zeta(n,k)), \quad \textit{where } |\zeta(n,k)| < \frac{2}{2^{k/2}}. \tag{1.14}$$

Lemma 1.19. *[52, Lemma 2.6] For $k \geqslant 2$ and $n \geqslant k+1$, we have*

$$L_n^{(k)} = 3 \times 2^{n-2}(1 + \zeta_2), \quad \textit{where } |\zeta_2| < \frac{1}{2^{k/2}}. \tag{1.15}$$

1.4.2 The k-generalized Pell sequence

Definition 1.20. For an integer $k \geqslant 2$, we define a generalization of the Pell sequence $(P_n^{(k)})_{n\geqslant 2}$ recurrently as follows

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}, \quad \text{for all } n \geqslant 2,$$

with the initial conditions $P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_0^{(k)} = 0$ and $P_1^{(k)} = 1$. Its characteristic polynomial is given by

$$P(X) = X^k - 2X^{k-1} - \cdots - X - 1.$$

We conclude this part with the following lemma that we need later.

Lemma 1.21. *[15, Lemma 2] If $k \geqslant 30$ and $n > 1$ are integers satisfying $n < \varphi^{k/2}$, then*

$$g_k(\alpha)\alpha^n = \frac{\varphi^{2n}}{\varphi+2}(1+\zeta), \quad \textit{where} \quad |\zeta| < \frac{4}{\varphi^{k/2}}.$$

1.5 Linear forms in logarithms

Definition 1.22. Any expression of the form

$$\beta_0 + \beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 + \cdots + \beta_n \log \alpha_n,$$

where the α_i are given non-zero algebraic numbers and the b_i are variables, is called a *linear form in logarithms.*

Remark 1.23. Note that in our applications to Diophantine equations, we are interested in the case when $\beta_0 = 0$ and $\beta_i \in \mathbb{Z}, i = 1, \cdots, n$. In the sequel we write $\beta_i = b_i,\ i = 1, \cdots, n$ and log always represents the principal value of the complex logarithm.

This tool plays a key role through our work. Indeed, many Diophantine problems can be reduced to obtaining lower bounds for linear forms in two or three logarithms. These bounds, often referred to as two-logarithm or three-logarithm bounds, are highly developed. As a result, many Diophantine problems can be solved once they are expressed in this form. Such lower bounds provide a mechanism to limit the size of the variables in a Diophantine equation, ensuring that there are at most finitely many solutions. Moreover, if the lower bound is effective, it may allow us to compute explicit bounds on the variables, enabling us to solve the problem completely by checking a finite number of small cases.

Baker gave in his paper "Linear forms in logarithms of algebraic numbers I, II, III", [4, 6] an effective lower bound on the absolute value of a nonzero linear form in logarithms of algebraic numbers. His result is the following.

Theorem 1.24 (A. Baker, 1975)**.** *Let $\alpha_1, \cdots, \alpha_n$ be algebraic numbers from $\mathbb{C}$ different from $0, 1$. Further, let $b_1, \cdots, b_n$ be rational integers such that*

$$b_1 \log \alpha_1 + \cdots + b_m \log \alpha_n \neq 0.$$

Then

$$|b_1 \log \alpha_1 + \cdots + b_m \log \alpha_n| \geqslant (eB)^{-C},$$

where $B := \max(|b_1|, \cdots, |b_n|)$ and C is an effectively computable constant depending only on n and on $\alpha_1, \cdots, \alpha_n$.

Later, Baker's lower bound was improved initially by: Baker himself, Waldschmidt in 1991, Baker and Wüstholz in 1993 and Matveev in 2000. We will restrict ourselves to those relevant to our work.

Here, we give the result of Baker and Wüstholz.

Theorem 1.25 ([8]). *Let $\Lambda = b_1 \ln \alpha_1 + \ldots + b_n \ln \alpha_n$ be a linear form in logarithms of algebraic numbers $\alpha_1, \ldots, \alpha_n$ with rational integer coefficients $b_1, \ldots, b_n$. If $\Lambda \neq 0$, then*

$$\ln |\Lambda| \geqslant -18(n+1)!n^{n+1}(32d)^{n+2} \ln(2nd)h'(\alpha_1)h'(\alpha_2)h'(\alpha_3) \ln B,$$

where $d = [\mathbb{Q}(\alpha_1, \ldots, \alpha_n : \mathbb{Q})]$, $B = \max\{|b_1|, \ldots, |b_n|\}$, $h'(\alpha) = \max\left(h(\alpha), \frac{|\ln \alpha|}{d}, \frac{1}{d}\right)$, and $h(\alpha)$ is the usual absolute logarithmic height of α.

The following result is due to Matveev. We use a version due to Bugeaud, Mignotte and Siksek.

Theorem 1.26 (Theorem 9.4 in [21]). *Let $\eta_1, \ldots, \eta_s$ be real algebraic numbers and let $b_1, \ldots, b_s$ be nonzero integers. Let $d_{\mathbb{K}}$ be the degree of the number field $\mathbb{Q}(\eta_1, \ldots, \eta_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j \geqslant \max\{d_{\mathbb{K}} h(\eta), |\log \eta|, 0.16\}, \quad \text{for} \quad j = 1, \ldots, s.$$

Assume that

$$B \geqslant \max\{|b_1|, \ldots, |b_s|\}.$$

If $\eta_1^{b_1} \cdots \eta_s^{b_s} - 1 \neq 0$, then

$$|\eta_1^{b_1} \cdots \eta_s^{b_s} - 1| \geqslant \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log B) A_1 \cdots A_s).$$

1.6 The reduction algorithms

As we have seen above, we can have a lower bound for the absolute value of linear forms in logarithms of algebraic numbers as we can obtain an upper bounds for them too.

Usually these bounds are large that we face the inquiry of whether it is possible to reduce them in our applications to Diophantine equations.
Effectively, this is achievable using the following reduction methods.

1.6.1 Baker-Davenport reduction

Our second main tool is a version of the reduction method of Baker and Davenport [7]. We use a slight variant of the version given by Dujella and Pethő [26].

Lemma 1.27. *Let M be a positive integer and let A, B, μ, γ be given real numbers with $A > 0$ and $B > 1$. Assume that p/q is a convergent of the continued fraction of γ such that $q > 6M$. Let*

$$\varepsilon = \|\mu q\| - M \cdot \|\gamma q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < |u\gamma - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leqslant M \quad \text{and} \quad w \geqslant \frac{\log(Aq/\varepsilon)}{\log B}.$$

When $\mu = 0$, the above lemma cannot be applied since then $\varepsilon < 0$. In this case, we use some results from the theory of continued fractions which will be also used in the reduction of these bounds.

1.6.2 Continued fractions

Let introduce the notion of continued fraction by taking an example of one of the most famous irrational numbers: the golden ratio. Let $\phi = \frac{1+\sqrt{5}}{2}$, which is solution of the equation

$$\phi^2 - \phi - 1 = 0.$$

Thus,

$$\phi = 1 + \frac{1}{\phi}.$$

We replace the occurrence of ϕ in the denominator by $1 + \frac{1}{\phi}$, we obtain

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{\phi}}.$$

Again, we replace the occurrence of ϕ in the fraction, we obtain

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\phi}}}.$$

We can continue the process over and over. This leads to write ϕ as a continued fraction

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}.$$

Generally, an expression

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \quad a_{n-1} + \cfrac{1}{a_n}}}}$$

is called a finite simple continued fraction for a given real number α with $a_i \geqslant 1$, for $i = 1, \cdots n$. Its abbreviated notation is

$$[a_0; a_1, a_2, \ldots, a_n].$$

For all $n \geqslant 0$, the rational number

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$$

is called the convergent of α.

Theorem 1.28. *The convergents of a given number α whose continued fraction expansion is given by*

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}}$$

are of the form $\frac{p_n}{q_n}$ where

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad q_n = a_n q_{n-1} + q_{n-2}.$$

under initial conditions

$$p_{-2} = 0, \qquad p_{-1} = 1, \qquad q_{-2} = 1, \qquad q_{-1} = 0.$$

We have already mentioned above that continued fractions are used to reduce the upper bounds for the absolute value of linear forms in logarithms of algebraic numbers. This is achieved using the following elegant property of continued fractions (see [36], pages 30 and 37).

Lemma 1.29. *Let p_i/q_i be the convergents of the continued fraction $[a_0, a_1, \ldots]$ of the irrational number γ. Let M be a positive integer and put $a_L := \max\{a_i | 0 \leqslant i \leqslant N+1\}$ where $N \in \mathbb{N}$ is such that $q_N \leqslant M < q_{N+1}$. If $x, y \in \mathbb{Z}$ with $x > 0$, then*

$$|x\gamma - y| > \frac{1}{(a_L+2)x}, \quad \textit{for all } x < M.$$

1.7 Other useful tools

We end with the following crucial lemmas that will be used in the proofs of our results throughout this thesis.

Lemma 1.30. *[59, Lemma 2.2, page 31] Let $d, x \in \mathbb{R}$ and $0 < d < 1$. If $|x| < d$, then*

$$|\log(1+x)| < \frac{-\log(1-d)}{d}|x|.$$

Lemma 1.31. *[53, Lemma 7] If $m \geqslant 1$, $T > (4m^2)^m$ and $T > y/(\log y)^m$. Then,*

$$y < 2^m T (\log T)^m.$$

Chapter 2

Fermat and Mersenne numbers as product of two k-Fibonacci numbers

The goal of this chapter is to investigate the Fermat and Mersenne numbers having representation as product of two k-Fibonacci numbers. This chapter is based on the paper [34].

2.1 Introduction

A Fermat number is a number of the form $2^n + 1$ obtained by setting $x = 1$ in a Fermat polynomial, the first few of which are 3, 5, 9, 17, 33, ... A Mersenne number is a number of the form $2^n - 1$, where n is a positive integer. The first few Mersenne numbers are 1, 3, 7, 15, 31, 63, 127, 255, ... Some properties of these numbers have been studied. One can cite [22, 37].

Let $k \geqslant 2$ be an integer. By Subsection 1.4.1, we consider a generalization of Fibonacci sequence called the k-generalized Fibonacci sequence $(F_n^{(k)})_{n\geqslant 2-k}$ defined as

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \quad \text{for all } n \geqslant 2, \tag{2.1}$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. If $k = 2$, we obtain the classical Fibonacci sequence. Below, we present the values of these numbers for the first few values of k and $n \geqslant 1$.

k	Name	First non-zero terms
2	Fibonacci	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...
3	Tribonacci	1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, ...
4	Tetranacci	1, 1,2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, ...
5	Pentanacci	1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, 6930, ...
6	Hexanacci	1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, 3840, 7617, ...
7	Heptanacci	1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, ...
8	Octanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, 8080, ...
9	Nonanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 511, 1021, 2040, 4076, 8144, ...
10	Decanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1023, 2045, 4088, 8172, ...

2.2 Motivation and main result

Several authors have worked on problems involving generalized Fibonacci sequences. One can see [11, 12, 13, 14, 17, 19, 22, 45, 46]. In 2020, Bravo and Herrera showed that 3 and

5 are the only Fermat numbers in the k-Fibonacci sequence (see [14]). In [37] and [51], the authors found all the Padovan and Perrin numbers which are also Mersenne numbers and Fermat numbers. Moreover, Bravo and Gómez [12] found all k-Fibonacci numbers which are Mersenne numbers. It is natural to ask if Fermat numbers and Mersenne numbers can be expressed as the product of two k-Fibonacci numbers. So, the aim of this chapter is to determine all Fermat numbers and Mersenne numbers, which are the products of two k-Fibonacci numbers. The main result that we obtain is the following.

Theorem 2.1. *The Diophantine equation*

$$F_n^{(k)} F_m^{(k)} = 2^a \pm 1 \tag{2.2}$$

has no solutions in positive integers n, m, k and a with $3 \leqslant m \leqslant n$ and $k \geqslant 2$.

Note that we make the condition $m \geqslant 3$ because if $m \in \{1, 2\}$ then the Diophantine equation (2.2) becomes $F_n^{(k)} = 2^a \pm 1$ and this equation was already solved in [12, 14].

Theorem 2.1 and remark above enable us to give the following corollary.

Corollary 2.2. There are no Fermat or Mersenne numbers expressible as product of two k-Fibonacci numbers greater than 1.

We will organize this chapter as follows. In Section 2.3, we develop the tools that will be used to prove Theorem 2.1. This proof will be done in the last section in four steps. In the first step, we will setup the problem, and bound a in terms of n, i.e. $a < 2n$. The second step consists in finding an upper bound of n in term of k. In fact, we show that $n < 4.21 \times 10^{28} k^8 \log^5 k$. For the third step, we consider lower values of k, i.e. $2 \leqslant k \leqslant 360$, and solve equation (2.2) in this range. In the final step, we take $k \geqslant 361$ and prove that we have no solution.

2.3 Preliminaries and known results

This section is devoted to collect a few definitions, notations, properties and results which will be used in the proof of Theorem 2.1.

2.3.1 Properties of the k-generalized Fibonacci sequence

In this subsection, we recall some facts and properties of the k-Fibonacci sequence that will be used later. The characteristic polynomial of this sequence is

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

The polynomial $\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root $\alpha(k)$ outside the unit circle (one can see [45, 46, 60]). The other roots are strictly inside the unit circle. Furthermore, Wolfram [60] showed that

$$2(1 - 2^{-k}) < \alpha(k) < 2, \quad \text{for all } k \geqslant 2. \tag{2.3}$$

To simplify the notation, in general, we omit the dependence on k of α. For $s \geqslant 2$, let

$$f_s(x) := \frac{x - 1}{2 + (s + 1)(x - 2)}. \tag{2.4}$$

Bravo et al. [13] proved that

$$1/2 < f_k(\alpha) < 3/4 \quad \text{and} \quad \left| f_k(\alpha^{(i)}) \right| < 1, \quad \text{for } 2 \leqslant i \leqslant k$$

hold, where $\alpha := \alpha^{(1)}, \ldots, \alpha^{(k)}$ are all the zeros of $\Psi_k(x)$. Therefore, the number $f_k(\alpha)$ is not an algebraic integer. In addition, they proved that the logarithmic height of f satisfies

$$h(f_k(\alpha)) < 3\log k, \quad \text{for all } k \geqslant 2. \tag{2.5}$$

In [32], Gomez and Luca proved that

$$h(f_k(\alpha)) < 2\log k, \quad \text{for all } k \geqslant 3. \tag{2.6}$$

With the above notation, Dresden and Du [24] showed that

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)}){\alpha^{(i)}}^{n-1} \tag{2.7}$$

and

$$|e_k(n)| < \frac{1}{2}, \quad \text{where } e_k(n) = F_n^{(k)} - f_k(\alpha)\alpha^{n-1}, \tag{2.8}$$

for all $n \geqslant 2-k$ and $k \geqslant 2$. Furthermore, for $n \geqslant 1$ and $k \geqslant 2$, Bravo and Luca [17] showed that

$$\alpha^{n-2} \leqslant F_n^{(k)} \leqslant \alpha^{n-1}. \tag{2.9}$$

Besides, note that the first $k+1$ non-zero terms in $F^{(k)}$ are powers of two, namely

$$F_1^{(k)} = 1, \quad F_2^{(k)} = 1, \quad F_3^{(k)} = 2, \quad F_4^{(k)} = 4, \cdots, \quad F_{k+1}^{(k)} = 2^{k-1}. \tag{2.10}$$

2.4 The proof of Theorem 2.1

In this section, we will prove Theorem 2.1 in four steps. Let us start with the first step.

2.4.1 An upper bound for a in term of n

First, note that if $3 \leqslant m \leqslant k+1$, then $F_m^{(k)} = 2^{m-2}$. So, the left-hand side of (2.2) is even while the right-hand side is odd which is a contradiction. Therefore, the Diophantine equation (2.2) has no solutions. So, from now, we assume that $n \geqslant m \geqslant k+2$.

Next, we will determine the size of a versus n. By inequalities (2.9), we obtain

$$2^{a-1} < 2^a \pm 1 = F_n^{(k)} F_m^{(k)} < \alpha^{n+m-2} < \alpha^{2n-2}.$$

Then, we deduce that

$$a < (2n-2)\left(\frac{\log\alpha}{\log 2}\right) + 1 = n\left(\frac{2\log\alpha}{\log 2}\right) - \left(\frac{2\log\alpha}{\log 2}\right) + 1.$$

Moreover, using (2.3) and the fact that $3/2 = 2(1-2^{-2}) < \alpha < 2$, for $k \geqslant 2$, we get

$$a < 2n. \tag{2.11}$$

2.4.2 An inequality for n versus k

Now, we will show the following lemma that gives a upper bound for n in term of k.

Lemma 2.3. *If (a, k, m, n) is solution in integers of equation (2.2) with $k \geqslant 2$ and $n \geqslant k + 2$, then the inequality*

$$n < 4.21 \times 10^{28} k^8 \log^5 k \tag{2.12}$$

holds.

Proof. Using estimate (2.7), equation (2.2) can be rearranged as

$$(f_k(\alpha)\alpha^{n-1} + e_k(n))(f_k(\alpha)\alpha^{m-1} + e_k(m)) = 2^a \pm 1, \tag{2.13}$$

i.e.

$$f_k^2(\alpha)\alpha^{n+m-2} - 2^a = -e_k(m)f_k(\alpha)\alpha^{n-1} - e_k(n)f_k(\alpha)\alpha^{m-1} - e_k(n)e_k(m) \pm 1.$$

Thus, we obtain

$$\left|f_k^2(\alpha)\alpha^{n+m-2} - 2^a\right| \leqslant \frac{f_k(\alpha)}{2}\alpha^{n-1} + \frac{f_k(\alpha)}{2}\alpha^{m-1} + \frac{5}{4}.$$

If we divide both sides by $f_k^2(\alpha)\alpha^{n+m-2}$ and using the fact that $f_k(\alpha) > 1/2$, we arrive at

$$|\Gamma_1| \leqslant \frac{1}{\alpha^{m-1}} + \frac{1}{\alpha^{n-1}} + \frac{5}{\alpha^{n+m-2}} < \frac{7}{\alpha^{m-1}}, \tag{2.14}$$

where

$$\Gamma_1 := f_k^{-2}(\alpha) \cdot \alpha^{-(n+m-2)} \cdot 2^a - 1. \tag{2.15}$$

We have $\Gamma_1 \neq 0$, because if we suppose that $\Gamma_1 = 0$, we would get

$$f_k^2(\alpha) = \alpha^{-(n+m-2)} \cdot 2^a$$

and so $f_k^2(\alpha)$ is an algebraic integer, which is impossible. With the goal of applying Theorem 1.26 to Γ_1 given by (2.15), the parameters can be chosen as:

$$(\eta_1, b_1) := (f_k(\alpha), -2), \quad (\eta_2, b_2) := (\alpha, -(n+m-2)), \quad (\eta_3, b_3) := (2, a).$$

The algebraic numbers η_1, η_2, η_3 are elements of the field $\mathbb{K} := \mathbb{Q}(\alpha)$ with degree $d_{\mathbb{K}} = k$. Since $h(\eta_1) < \log(k+1) + \log 4 < 3.6 \log k$, $h(\eta_2) = (\log \alpha)/k < (\log 2)/k$ and $h(\eta_3) = \log 2$, then we can choose

$$A_1 := 3.6k \log k > \max\{kh(\eta_1), |\log \eta_1|, 0.16\},$$

$$A_2 := \log 2 = \max\{kh(\eta_2), |\log \eta_2|, 0.16\},$$

and

$$A_3 := k \log 2 = \max\{kh(\eta_3), |\log \eta_3|, 0.16\}.$$

Finally, the fact that $m \leqslant n$ and the inequality (2.11) imply that we can take $B := 2n$. Therefore, according to Theorem 1.26, it comes that

$$\begin{aligned} |\Gamma_1| &> \exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot k^2(1 + \log k)(1 + \log 2n)(3.6k \log k)(\log 2)(k \log 2)\right) \\ &> 2.48 \cdot 10^{11}(1 + \log k)(k^4 \log k)(1 + \log 2n). \end{aligned}$$

Using the fact $1 + \log k < 2.5 \log k$, which holds for $k \geqslant 2$, we obtain

$$|\Gamma_1| > \exp\left(-6.2 \cdot 10^{11} k^4 \log^2 k(1 + \log 2n)\right). \tag{2.16}$$

The comparison of the lower bound (2.16) and the upper bound (2.14) of $|\Gamma_1|$ gives us

$$(m-1)\log\alpha < 6.21\cdot 10^{11}k^4\log^2 k(1+\log 2n), \tag{2.17}$$

We return to equation (2.2) and we reformulate it as

$$(f_k(\alpha)\alpha^{n-1}+e_k(n))F_m^{(k)} = 2^a \pm 1,$$

i.e.

$$f_k(\alpha)\alpha^{n-1} - \frac{2^a}{F_m^{(k)}} = \frac{\pm 1}{F_m^{(k)}} - e_k(n). \tag{2.18}$$

So, we get

$$\left| f_k(\alpha)\alpha^{n-1} - \frac{2^a}{F_m^{(k)}} \right| \leqslant \frac{1}{F_m^{(k)}} + \frac{1}{2} \leqslant \frac{3}{2}.$$

Dividing through by $f_k(\alpha)\alpha^{n-1}$, we get

$$|\Gamma_2| \leqslant \frac{3}{2f_k(\alpha)\alpha^{n-1}} < \frac{3\alpha}{\alpha^n} < \frac{6}{\alpha^n}, \tag{2.19}$$

where

$$\Gamma_2 := \left(F_m^{(k)} f_k(\alpha)\right)^{-1}\cdot\alpha^{-(n-1)}\cdot 2^a - 1. \tag{2.20}$$

One can see that $\Gamma_2 \neq 0$ by a similar method used to show that $\Gamma_1 \neq 0$. Now, we will apply Theorem 1.26 to Γ_2 by taking

$$(\eta_1, b_1) := (F_m^{(k)} f_k(\alpha), -1), \quad (\eta_2, b_2) := (\alpha, -(n-1)), \quad (\eta_3, b_3) := (2, a).$$

Clearly, $\mathbb{K} := \mathbb{Q}(\alpha)$ contains η_1, η_2, η_3 and has degree $d_{\mathbb{K}} = k$. As calculated before we take

$$A_2 := \log 2, \quad A_3 := k\log 2, \quad \text{and} \quad B := 2n.$$

We need to compute A_1. The estimates (2.5), (2.17), and the properties (1.1)-(1.3) imply that, for all $k \geqslant 2$, we have

$$\begin{aligned} h(\eta_1) &\leqslant h(F_m^{(k)}) + h(f_k(\alpha)) \\ &< (m-1)\log\alpha + \log(k+1) + \log 4 \\ &< 6.22\cdot 10^{11}k^4\log^2 k(1+\log 2n). \end{aligned}$$

On the other hand, since

$$\eta_1 := F_m^{(k)} f_k(\alpha) < \frac{3\alpha^{m-1}}{4} \quad \text{and} \quad \eta_1^{-1} = \frac{1}{F_m^{(k)} f_k(\alpha)} \leqslant 2,$$

then, by (2.17), we get

$$|\log\eta_1| < (m-1)\log\alpha + \log 0.75 < 6.22\cdot 10^{11}k^4\log^2 k(1+\log 2n).$$

Thus, we conclude that

$$\max\{kh(\eta_1), |\log\eta_1|, 0.16\} < 6.22\times 10^{11}k^5\log^2 k\log n := A_1.$$

We apply Theorem 1.26 and compare the resulting inequality with (2.19) to obtain

$$n\log\alpha < 5.66\times 10^{23}k^8\log^3 k\log^2 n,$$

where we have used the facts $1+\log k < 2.5\log k$ and $1+\log(2n) < 2.3\log n$, which hold for $k \geqslant 2$ and $n \geqslant 4$. Therefore, we obtain

$$\frac{n}{\log^2 n} < 1.4 \times 10^{24} k^8 \log^3 k. \tag{2.21}$$

It is easy to check that the inequality

$$\frac{x}{\log^2 x} < A \text{ implies } x < 4A\log^2 A, \text{ whenever } A \geqslant 100. \tag{2.22}$$

Thus, putting $A := 1.4\times 10^{24}k^8\log^3 k$ in inequality (2.22) and using inequality $55.6+8\log k + 3\log\log k < 86.7\log k$, which holds, for all $k \geqslant 2$, we get

$$\begin{aligned} n &< 4(1.4\cdot 10^{24}k^8\log^3 k)\left(\log(1.4\cdot 10^{24}k^8\log^3 k)\right)^2 \\ &< 5.6\cdot 10^{24}k^8\log^3 k)(55.6+8\log k+3\log\log k)^2 \\ &< 4.21\cdot 10^{28}k^8\log^5 k. \end{aligned}$$

This establishes (2.12) and finishes the proof of Lemma 2.3. □

2.4.3 The case $2 \leqslant k \leqslant 360$

In this subsection, we consider $k \in [2, 360]$. Define

$$\Lambda_1 := \log(\Gamma_1 + 1) = a\log 2 - (n+m-2)\log\alpha - 2\log(f_k(\alpha)). \tag{2.23}$$

Suppose that $m \geqslant 10$. So, by the estimate (2.14) and the fact that $\alpha > 1.5$, we have $|\Gamma_1| < 0.19$. Taking $d = 0.19$ in Lemma 1.30, we obtain

$$|\Lambda_1| < \frac{-\log 0.81}{0.19}\cdot|\Gamma_1| < 7.8\cdot\alpha^{-(m-1)}.$$

So, we get

$$\left| a\cdot\frac{\log 2}{\log\alpha} - (n+m) + 2 - \frac{2\log(f_k(\alpha))}{\log\alpha}\right| < 19.3\cdot\alpha^{-(m-1)}. \tag{2.24}$$

In view to apply Lemma 1.27 to Λ_1, we consider

$$\gamma := \frac{\log 2}{\log\alpha}, \quad \mu := 2 - \frac{2\log(f_k(\alpha))}{\log\alpha}, \quad \text{and} \quad (A, B) := (19.3, \alpha).$$

We have $\gamma \notin \mathbb{Q}$ since if we assume the contrary, then there exist coprime integers a and b such that $\gamma = a/b$, then we get $\alpha^a = 2^b$. Let $\sigma \in Gal(\mathbb{K}/\mathbb{Q})$ such that $\sigma(\alpha) = \alpha_i$, for some $i \in \{2, \ldots, k\}$. Applying this to the above relation and taking absolute values we get $1 < 2^a = |\alpha_i| < 1$, which is a contradiction.

For each $k \in [2, 360]$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k\|\gamma q\| > 0$, where $M_k := \lfloor 8.42\times 10^{28}k^8\log^5 k\rfloor$, which is an upper bound of a from Lemma 2.3. After doing this, we use Lemma 1.27 on inequality (2.24). A computer program with Mathematica revealed that $\min_{2\leqslant k\leqslant 360}\varepsilon_k > 2.23\times 10^{-57}$ and the maximum value of $\frac{\log(Aq/\varepsilon)}{\log B}$ over all $k \in [2, 360]$ is $708.875\ldots$, which is an upper bound of $m-1$ by Lemma 1.27.

Now, we consider $4 \leqslant m \leqslant 708$ and

$$\Lambda_2 := \log(\Gamma_2 + 1) = a\log 2 - (n-1)\log\alpha - \log(f_k(\alpha)F_m^{(k)}). \tag{2.25}$$

Since $n \geqslant 10$, then by (2.19), we have $|\Gamma_2| < 0.11$. Thus, by Lemma 1.30 with $d = 0.11$ we deduce that

$$|\Lambda_2| < \frac{-\log 0.89}{0.11} \cdot |\Gamma_2| < 6.4 \cdot \alpha^{-n}. \tag{2.26}$$

Replacing (5.19) into (2.26) and dividing through by $\log\alpha$, we obtain

$$\left| a \cdot \frac{\log 2}{\log\alpha} - n + 1 - \frac{\log(f_k(\alpha) F_m^{(k)})}{\log\alpha} \right| < 15.8 \cdot \alpha^{-n}. \tag{2.27}$$

To apply Lemma 1.27 to (2.27), this time for $4 \leqslant m \leqslant 708$, we take

$$\gamma := \frac{\log 2}{\log\alpha}, \quad \mu_m := 1 - \frac{\log(f_k(\alpha) F_m^{(k)})}{\log\alpha}, \quad A := 15.8 \quad \text{and} \quad B := \alpha.$$

As seen before $\gamma \notin \mathbb{Q}$. Again, for each $(k, m) \in [2, 360] \times [4, 708]$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k\|\gamma q\| > 0$, where $M_k := \lfloor 8.42 \times 10^{28} k^8 \log^5 k \rfloor$, which is is an upper bound of a from Lemma 2.3. After doing this, we use Lemma 1.27 on inequality (2.27). Again, a program in Mathematica revealed that $\min_{2\leqslant k\leqslant 360, 4\leqslant m\leqslant 708} \varepsilon_k > 0.4999$ and the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$ over all $(k, m) \in [2, 360] \times [4, 708]$ is $377.25\ldots$, which according to Lemma 1.27, is an upper bound of n.

Hence, we deduce that the possible solutions (a, k, n, m) of equation (2.2) for which $k \in [2, 360]$ satisfy $k + 2 \leqslant m \leqslant n \leqslant 377$. Therefore, we use inequality (2.11) to obtain $a \leqslant 754$.

Finally, we used Mathematica to compare $F_n^{(k)} F_m^{(k)}$ and $(2^a \pm 1)$, for $2 \leqslant k \leqslant 360$, $k + 2 \leqslant m \leqslant n \leqslant 377$, and $a \leqslant 754$ to see that equation (2.2) has no solutions in this range.

2.4.4 The case $k > 360$

In this subsection, we analyze the case $k > 360$. For $k > 360$, it is easy to check that

$$m \leqslant n < 4.21 \cdot 10^{28} k^8 \log^5 k < 2^{k/2}.$$

Hence, by Lemma 1.18 and equation (2.2), we get

$$\left|2^{n+m-4} - 2^a\right| \leqslant \frac{2^{n-2}}{2^{k/2}} + \frac{2^{m-2}}{2^{k/2}} + \frac{1}{2^k} + 1.$$

Consequently, the above inequality and the fact that $n \geqslant m > k + 2$ give us

$$\left|1 - 2^{a-n-m+4}\right| < \frac{2}{2^{3k/2}} + \frac{1}{2^{3k}} + \frac{1}{2^{2k}} < \frac{4}{2^{3k/2}}. \tag{2.28}$$

Assume that $a = n + m - 4$, then we have $a \geqslant k$ and equation (2.2) turns into

$$F_n^{(k)} F_m^{(k)} = 2^{n+m-4} \pm 1. \tag{2.29}$$

But from (2.29), we have

$$F_n^{(k)} F_m^{(k)} \leqslant (2^{n-2} - 1)(2^{m-2} - 1) = 2^{n+m-4} - 2^{n-2} - 2^{m-2} + 1. \tag{2.30}$$

By (2.29) and (2.30), we get

$$\pm 1 \leqslant -2^{n-2} - 2^{m-2} + 1 \leqslant -3.$$

Hence, $a \neq n + m - 4$. Since $\min_{x\leqslant 3 \text{ and } x\neq 0} |2^x - 1| = 0.5$, then by (2.28) we obtain $k < 2$, which is impossible because $k > 360$.

Chapter 3

Fermat or Mersenne numbers as products of two k-Pell numbers

This chapter explores the same problem as the previous one, focusing on k-Pell sequences. Specifically, we study Fermat and Mersenne numbers that can be expressed as products of two k-Pell numbers. The material presented in this chapter originates from the paper [55].

3.1 Introduction

The Pell numbers are defined through the recurrence relation given in (1.9). There are several generalizations of the Pell numbers, which are variations or extensions of the original sequence. Let $k \geqslant 2$ be an integer, from the subsection (1.4.2), the generalization of the Pell sequences $P_n^{(k)}$ is defined recurrently as follows

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}, \quad \text{for all } n \geqslant 2,$$

with the initial conditions $P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_0^{(k)} = 0$ and $P_1^{(k)} = 1$.
We note that this generalization is in fact a family of sequences where each new choice of k produces a distinct sequence. For example, the usual Pell sequence $(P_n)_{n\geqslant 0}$ is obtained for $k = 2$. Below we present the values of these numbers for the first few values of k and $n \geqslant 1$.

Table 3.1: default.

k	Name	First non-zero terms
2	Pell	1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, ...
3	3-Pell	1, 2, 5, 13, 33, 84, 214, 545, 1388, 3535, 9003, 22929, 58396, ...
4	4-Pell	1, 2, 5, 13, 34, 88, 228, 591, 1532, 3971, 10293, 26680, 69156, ...
5	5-Pell	1, 2, 5, 13, 34, 89, 232, 605, 1578, 4116, 10736, 28003, 73041, ...

3.2 Motivation and main result

Some mathematicians have studied relations among Fermat/Mersenne numbers and other sequences [33, 34, 35, 48]. In [34], Hernane et al. have determined all Fermat numbers and Mersenne numbers that are products of two k-Fibonacci numbers. Motived by this result, we will find all Fermat and Mersenne numbers that can be written as products of two k-Pell numbers. More precisely, through this chapter, we will study the Diophantine equation presented in the following theorem.

Theorem 3.1. *The Diophantine equations*

$$P_n^{(k)} P_m^{(k)} = 2^a \pm 1 \tag{3.1}$$

have no solutions in positive integers a, k, n, m with $n \geqslant m \geqslant 2$ and $k \geqslant 2$.

Remark 3.2. On can easily notice that if $m = 1$, then $(a, k, m, n) = (1, k, 1, 1)$ is a solution for the Mersenne sequence and $(a, k, m, n) = (2, k, 1, 3)$ is a solution for Fermat sequence.

To establish the aforementioned theorem, we use linear forms in logarithms of algebraic numbers along with a reduction algorithm originally introduced by Baker and Davenport. The subsequent section provides results and definitions that will be applied throughout the remainder of this study.

3.3 Preliminaries and known results

3.3.1 Properties of the k-generalized Pell sequence

In this subsection, we recall some essential facts and properties of the k-Pell sequence, which will be pertinent in subsequent discussions. The characteristic polynomial associated with this sequence is given by

$$\Psi_k(x) = x^k - 2x^{k-1} - \cdots - x - 1.$$

In [16], Bravo et al. demonstrated that $\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root $\alpha(k)$ outside the unit circle. This root is real, positive, and satisfies $\alpha(k) > 1$. The remaining roots are strictly contained within the unit circle. Additionally, in the same paper, they demonstrated that

$$\varphi^2(1 - \varphi^{-k}) < \alpha(k) < \varphi^2, \quad \text{for all } k \geqslant 2, \quad \text{where } \varphi = \frac{1+\sqrt{5}}{2}. \tag{3.2}$$

For simplicity, we generally omit the dependence on k for α. For $k \geqslant 2$, let

$$g_k(x) := \frac{x-1}{(k+1)x^2 - 3kx + k - 1} = \frac{x-1}{k(x^2 - 3x + 1) + x^2 - 1}. \tag{3.3}$$

In [15], Bravo and Hererra proved that the inequalities

$$0.276 < g_k(\alpha) < 0.5 \quad \text{and} \quad \left|g_k(\alpha^{(i)})\right| < 1,\ 2 \leqslant i \leqslant k$$

hold, where $\alpha := \alpha^{(1)}, \ldots, \alpha^{(k)}$ are all the zeros of $\Psi_k(x)$. So, the number $g_k(\alpha)$ is not an algebraic integer. Indeed, if we suppose that $g_k(\alpha)$ is an algebraic integer, then we know that $g_k(\alpha) \neq 0$ and $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(g_k(\alpha)) \in \mathbb{Z}$, where $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(g_k(\alpha))$ is the norm of $g_k(\alpha)$ from $\mathbb{Q}(\alpha)$ to $\mathbb{Q}$. In addition, they proved that the logarithmic height of $g_k(\alpha)$ satisfies

$$h(g_k(\alpha)) < 4k \log \varphi + k \log(k+1), \quad \text{for all } k \geqslant 2. \tag{3.4}$$

With the above notation, Bravo et al. showed in [16] that the inequalities

$$P_n^{(k)} = \sum_{i=1}^{k} g_k(\alpha^{(i)}) {\alpha^{(i)}}^n \quad \text{and} \quad \left|P_n^{(k)} - g_k(\alpha)\alpha^n\right| < \frac{1}{2} \tag{3.5}$$

hold, for all $n \geqslant 1$ and $k \geqslant 2$. Furthermore, for $n \geqslant 1$ and $k \geqslant 2$, it was proved in [16] that

$$\alpha^{n-2} \leqslant P_n^{(k)} \leqslant \alpha^{n-1}. \tag{3.6}$$

Moreover, note that for $n \leqslant k+1$ we have

$$P_n^{(k)} = F_{2n-1}. \tag{3.7}$$

3.4 The proof of Theorem 3.1

The proof of Theorem 3.1 will be done in four steps. We will first start by giving the following precision. If $m = 1$, then $P_m^{(k)} = 1$. In this case, the Diophantine equation (3.1) has been studied in [48]. For the cases $2 \leqslant m \leqslant n \leqslant k+1$, we use the identity (3.7) to rewrite the Diophantine equation (3.1) as

$$F_{2n-1}F_{2m-1} = 2^a \pm 1.$$

According to the main result in [34], we deduce that (3.1) has no solutions in this range. Thus, from now, we assume that $n \geqslant k+2$. Subsequently, we will establish the relationship between the sizes of a and n. By applying inequalities (3.6), we derive

$$2^{a-1} < 2^a \pm 1 = P_n^{(k)}P_m^{(k)} < \alpha^{n+m-2} < \alpha^{2n-2}$$

and so

$$a < (2n-2)\left(\frac{\log\alpha}{\log 2}\right) + 1 = n\left(\frac{2\log\alpha}{\log 2}\right) - \left(\frac{2\log\alpha}{\log 2}\right) + 1.$$

Furthermore, we use (3.2) to obtain

$$a < 3n. \tag{3.8}$$

3.4.1 An inequality for n versus k

Here, we will establish an inequality for n in relation with k, by showing the following lemma.

Lemma 3.3. *If (a,k,m,n) is a solution in integers of equation (3.1) with $k \geqslant 2$ and $n \geqslant k+2$, then we have the following inequality*

$$n < 8.2 \cdot 10^{28} k^9 \log^5 k. \tag{3.9}$$

Proof. Using estimate provided in (3.5), we obtain

$$P_n^{(k)} = g_k(\alpha)\alpha^n + e_k(n), \quad \text{where } |e_k(n)| < \frac{1}{2},$$

thus allowing us to express equation (3.1) as

$$(g_k(\alpha)\alpha^n + e_k(n))(g_k(\alpha)\alpha^m + e_k(m)) = 2^a \pm 1; \tag{3.10}$$

that is,

$$g_k^2(\alpha)\alpha^{n+m} - 2^a = -e_k(m)g_k(\alpha)\alpha^n - e_k(n)g_k(\alpha)\alpha^m - e_k(n)e_k(m) \pm 1.$$

By taking the absolute value and dividing both sides by $g_k^2(\alpha)\alpha^{n+m}$, we get

$$\left|g_k^{-2}(\alpha)\cdot\alpha^{-(n+m)}\cdot 2^a - 1\right| \leqslant \frac{1.82}{\alpha^m} + \frac{1.82}{\alpha^n} + \frac{16.41}{\alpha^{n+m}} < \frac{20.1}{\alpha^m}, \tag{3.11}$$

where we have used the fact that $g_k(\alpha) > 0.276$. Define

$$\Gamma_1 := g_k^{-2}(\alpha) \cdot \alpha^{-(n+m)} \cdot 2^a - 1. \tag{3.12}$$

So, we have

$$|\Gamma_1| < \frac{20.1}{\alpha^m}. \tag{3.13}$$

Note that $\Gamma_1 \neq 0$, otherwise we would get

$$g_k^2(\alpha) = \alpha^{-(n+m)} \cdot 2^a$$

which implies that $g_k^2(\alpha)$ is an algebraic integer, a contradiction.

Now, we will utilize Theorem 1.26 for Γ_1, considering the following parameters:

$$(\eta_1, b_1) := (g_k(\alpha)), -2), \quad (\eta_2, b_2) := (\alpha, -(n+m)), \quad (\eta_3, b_3) := (2, a).$$

As $\eta_1, \eta_2, \eta_3 \in \mathbb{K} := \mathbb{Q}(\alpha)$, we have $d_{\mathbb{K}} = k$. Now, we provide estimates for the usual absolute logarithmic heights of these numbers: $h(\eta_1) < h(g_k(\alpha)) < 4k \log \varphi + k \log(k+1) < 4.4k \log k$, $h(\eta_2) = (\log \alpha)/k < (2 \log \varphi)/k$, and $h(\eta_3) = \log 2$. Therefore, we have the flexibility to choose

$$A_1 := 4.4k^2 \log k > \max\{kh(\eta_1), |\log \eta_1|, 0.16\},$$
$$A_2 := 2 \log \varphi = \max\{kh(\eta_2), |\log \eta_2|, 0.16\},$$

and

$$A_3 := k \log 2 = \max\{kh(\eta_3), |\log \eta_3|, 0.16\}.$$

We previously established that $a < 3n$ and $m \leqslant n$, enabling us to set $B := 3n$. Armed with these considerations, we are now in a position to apply Theorem 1.26 and derive

$$\begin{aligned} |\Gamma_1| &> \exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot k^2(1+\log k)(1+\log 3n)(4.4k^2 \log k)(2\log\varphi)(k \log 2)\right) \\ &> \exp(-1.06 \cdot 10^{12}(k^5 \log^2 k)(1+\log 3n)), \end{aligned}$$

where we have used the fact that $1 + \log k < 2.5 \log k$, which holds for $k \geqslant 2$. Comparing this lower bound with the upper bound of $|\Gamma_1|$ provided by (3.13), we arrive at

$$(m-1)\log\alpha < 1.07 \cdot 10^{12} k^5 \log^2 k(1+\log 3n). \tag{3.14}$$

Now, let us rewrite equation (3.1) in an alternative manner to derive a second linear expression in logarithms. To achieve this, we use estimate (3.5) once again to obtain

$$(P_k(\alpha)\alpha^n + e_k(n))P_m^{(k)} = 2^a \pm 1;$$

that is,

$$g_k(\alpha)\alpha^n - \frac{2^a}{P_m^{(k)}} = \frac{\pm 1}{P_m^{(k)}} - e_k(n). \tag{3.15}$$

As previously done, by taking the absolute value and dividing both sides by $g_k(\alpha)\alpha^n$, we acquire

$$\left|\left(P_m^{(k)} g_k(\alpha)\right)^{-1} \cdot \alpha^{-n} \cdot 2^a - 1\right| \leqslant \frac{3}{2\,|g_k(\alpha)|\,\alpha^n} < \frac{5.5\alpha}{\alpha^n} < \frac{14.4}{\alpha^n}. \tag{3.16}$$

Define

$$\Gamma_2 := \left(P_m^{(k)} g_k(\alpha)\right)^{-1} \cdot \alpha^{-n} \cdot 2^a - 1. \tag{3.17}$$

Let us show that $\Gamma_2 \neq 0$. In fact, $\Gamma_2 = 0$ leads to $g_k(\alpha) = 2^a\alpha^{-n}/P_m^{(k)}$. Taking norms, we get

$$N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(g_k(\alpha)) = \frac{2^{ka}}{\left(P_m^{(k)}\right)^k} N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^{-n}) = \pm\frac{2^{ka}}{\left(P_m^{(k)}\right)^k}.$$

Now, we see that

$$N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(g_k(\alpha)) = \prod_{i=1}^{k} \frac{\alpha_i - 1}{k(\alpha_i^2 - 3\alpha_i + 1) + \alpha_i^2 - 1},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the conjugates of α. The above is a fraction A/B, with A, B integers and

$$A = \prod_{i=1}^{k}(\alpha_i - 1) = (-1)^k \Psi_k(1) = (-1)^k(1 - 2 - \cdots - 1) = \pm k.$$

Hence, we get

$$\frac{k}{B} = \pm \frac{2^{ka}}{\left(P_m^{(k)}\right)^k}.$$

The exponent of 2 in the left is at most the exponent of 2 in the factorization of k, so this exponent is less than $k \leqslant ka$. This shows that in the right, the fraction $2^{ka}/\left(P_m^{(k)}\right)^k$ cannot be reduced (so 2^{ka} cannot be coprime to $P_m^{(k)}$). This shows that $P_m^{(k)}$ is even, which is false since $P_m^{(k)} P_n^{(k)} = 2^a \pm 1$ implies that $P_m^{(k)}$ is odd.

Now, we will apply Theorem 1.26 to Γ_2. In this application, we select the following parameters:

$$(\eta_1, b_1) := (P_m^{(k)} g_k(\alpha), -1), \quad (\eta_2, b_2) := (\alpha, -n), \quad (\eta_3, b_3) := (2, a).$$

Once again, let $\mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. As previously, we can choose $A_2 := 2\log\varphi$ and $A_3 := k\log 2$. Now, we still need to calculate A_1. With the help of the estimates (3.4), (3.14), and the properties (1.1)-(1.3), we obtain

$$\begin{aligned} h(\eta_1) &\leqslant h(P_m^{(k)}) + h(g_k(\alpha)) \\ &< (m-1)\log\alpha + 4k\log\varphi + k\log(k+1) \\ &< 1.08 \cdot 10^{12} k^5 \log^2 k(1 + \log 3n), \end{aligned}$$

for all $k \geqslant 2$. On the other hand, as

$$\eta_1 := P_m^{(k)} g_k(\alpha) < \frac{\alpha^{m-1}}{2} \quad \text{and} \quad \eta_1^{-1} = \frac{1}{P_m^{(k)} g_k(\alpha)} < 4,$$

then, by (3.14), we find

$$|\log \eta_1| < (m-1)\log\alpha + \log 0.5 < 1.08 \cdot 10^{12} k^5 \log^2 k(1 + \log 3n).$$

This implies that

$$\max\{kh(\eta_1), |\log \eta_1|, 0.16\} < 1.08 \times 10^{12} k^6 \log^2 k(1 + \log 3n) := A_1.$$

By choosing $B := 3n$, we can apply Theorem 1.26 to Γ_2 and compare the resulting inequality with (3.16) to derive

$$n \log\alpha < 1.75 \times 10^{24} k^9 \log^3 k \log^2 n,$$

where we utilized the facts $1 + \log k < 2.5\log k$ and $1 + \log(3n) < 2.6\log n$, valid for $k \geqslant 2$ and $n \geqslant 4$. Consequently, we obtain

$$\frac{n}{\log^2 n} < 2.6 \times 10^{24} k^9 \log^3 k. \tag{3.18}$$

Applying the inequality from Lemma 1.31, with $T := 2.6 \times 10^{24} k^9 \log^3 k$, and considering the inequality $56.22 + 9\log k + 3\log\log k < 88.6 \log k$ for all $k \geqslant 2$, we obtain

$$\begin{aligned} n &< 4(2.6 \cdot 10^{24} k^9 \log^3 k) \left(\log(2.6 \cdot 10^{24} k^9 \log^3 k)\right)^2 \\ &< (1.04 \cdot 10^{25} k^9 \log^3 k)(56.22 + 9\log k + 3\log\log k)^2 \\ &< 8.2 \cdot 10^{28} k^9 \log^5 k. \end{aligned}$$

This concludes the proof of Lemma 3.3. □

3.4.2 The small cases $2 \leqslant k \leqslant 630$

In this subsection, we explore the values of k within the interval $[2, 630]$. Define

$$\Lambda_1 := \log(\Gamma_1 + 1) = a\log 2 - (n+m)\log\alpha - 2\log(g_k(\alpha)). \tag{3.19}$$

Assume that $m \geqslant 10$. Utilizing the estimate (3.13) and considering the fact that $\alpha > \varphi$, it follows that $|\Gamma_1| < 0.27$. By setting $d = 0.27$ in Lemma 1.30, we obtain:

$$|\Lambda_1| < \frac{-\log 0.73}{0.27} \cdot |\Gamma_1| < 23.5 \cdot \alpha^{-m}.$$

Hence, we have

$$\left| a \cdot \frac{\log 2}{\log\alpha} - (n+m) - \frac{2\log(g_k(\alpha))}{\log\alpha} \right| < 48.9 \cdot \alpha^{-m}. \tag{3.20}$$

To apply Lemma 1.27 to Λ_1, we consider the parameters

$$\gamma := \frac{\log 2}{\log\alpha}, \quad \mu := 2 - \frac{2\log(g_k(\alpha))}{\log\alpha}, \quad \text{and} \quad (A, B) := (48.9, \alpha).$$

For each $k \in [3, 630]$, we determine a suitable approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ. These convergents satisfy $q_\ell > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k\|\gamma q\| > 0$, where $M_k := \lfloor 2.46 \times 10^{30} k^9 \log^5 k \rfloor$. This value serves as an upper bound for a according to Lemma 3.3. Utilizing Mathematica, we find that q_{114} meets the conditions of Lemma 1.27. Subsequently, we apply Lemma 1.27 to inequality (3.20). A Mathematica program indicates that $\min_{3\leqslant k\leqslant 630} \varepsilon_k > 0.4861$. Moreover, the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$ over all $k \in [3, 630]$ is 150.457, providing an upper bound for m as per Lemma 1.27.

For the case $k = 2$, we have $\alpha = 1 + \sqrt{2}$ and $g_2(\alpha) = \sqrt{2}$. Consequently, inequality (3.20) is transformed into

$$\left| (a-1) \cdot \frac{\log 2}{\log(1+\sqrt{2})} - (n+m) \right| < 48.9 \cdot \alpha^{-m}. \tag{3.21}$$

Let

$$[a_0, a_1, a_2, \ldots] = [0, 1, 3, 1, 2, 6, 1, 2, 11, 2, 2, 1, 1, 18, \ldots]$$

be the continued fraction expression of $\dfrac{\log 2}{\log(1+\sqrt{2})}$ and let p_ℓ/q_ℓ denote its convergent. It is worth noting that in this scenario, we have $a < 2.46 \cdot 10^{30} \cdot 2^9 \cdot \log^5 2 < 2.02 \cdot 10^{32}$. By utilizing Maple, we find that

$$q_{61} < 2.02 \cdot 10^{32} < q_{62}$$

and

$$a_L := \max\{a_i : i = 1, 2, \ldots, 61\} = a_{27} = 100.$$

Thus, based on the properties of continued fractions (refer to Lemma 1.29), we deduce

$$\left|(a-1)\cdot\frac{\log 2}{\log(1+\sqrt{2})}-(n+m)\right|>\frac{1}{(a_L+2)a}.$$

Combining the above inequality with (3.21) and considering the fact that $a<2.02\cdot 10^{32}$, it comes that

$$\alpha^m<48.9\cdot 102\cdot 2.02\cdot 10^{32}.$$

Consequently, we derive $m<95$. Therefore, in all cases, we have $m<151$.

Next, we fix $m<151$ and we consider

$$\Lambda_2:=\log(\Gamma_2+1)=a\log 2-n\log\alpha-\log(P_m^{(k)}g_k(\alpha)). \tag{3.22}$$

Assuming $n\geqslant 10$, by the estimate (3.16) and considering that $\alpha>\varphi$, we ascertain $|\Gamma_2|<0.12$. Applying $d=0.12$ in Lemma 1.30, we derive

$$|\Lambda_2|<\frac{-\log 0.88}{0.12}\cdot|\Gamma_2|<15.4\cdot\alpha^{-n}.$$

Consequently, we obtain

$$\left|a\cdot\frac{\log 2}{\log\alpha}-n-\frac{\log(P_m^{(k)}g_k(\alpha))}{\log\alpha}\right|<32.1\cdot\alpha^{-n}. \tag{3.23}$$

To apply Lemma 1.27 to Λ_2, we take

$$\gamma:=\frac{\log 2}{\log\alpha},\quad \mu:=-\frac{\log(P_m^{(k)}g_k(\alpha))}{\log\alpha},\quad\text{and}\quad (A,B):=(32.1,\alpha).$$

For each $k\in[2,630]$, we find a suitable approximation of γ, along with a convergent p_ℓ/q_ℓ of the continued fraction of γ. This choice ensures that $q_\ell>6M_k$, and $\varepsilon=\varepsilon(k)=\|\mu q\|-M_k\|\gamma q\|>0$, where $M_k:=\lfloor 2.46\times 10^{30}k^9\log^5 k\rfloor$. This value serves as an upper bound for a according to Lemma 3.3. Utilizing Mathematica, we see that q_{116} satisfies the conditions of Lemma 1.27. Subsequently, we apply Lemma 1.27 to inequality (3.23). A computational analysis with Mathematica indicates that $\min_{2\leqslant k\leqslant 630, 2\leqslant m\leqslant 150}\varepsilon_k>0.4992$, and the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$ over all $k\in[2,630]$ is 159.934. This maximum value serves as an upper bound for n according to Lemma 1.27.

Finally, we used Mathematica to conduct a comparative analysis between $P_n^{(k)}P_m^{(k)}$ and $2^a\pm 1$ across the parameter space defined by $k+2\leqslant n\leqslant 159$, $m\leqslant n$, and $1\leqslant a\leqslant 477$, while ensuring that $a<3n$. This investigation aimed to verify that equation (3.1) exhibits no solutions within these specified ranges.

3.4.3 The large cases $k>630$

In this subsection, we investigate the scenario where $k>630$. For $k>630$, a straightforward verification reveals that

$$m\leqslant n<8.2\cdot 10^{28}k^9\log^5 k<\varphi^{k/2}.$$

Consequently, in accordance with Lemma 1.21, we obtain

$$g_k(\alpha)\alpha^n=\frac{\varphi^{2n}}{\varphi+2}(1+\zeta_1),\quad\text{where}\quad |\zeta_1|<\frac{4}{\varphi^{k/2}} \tag{3.24}$$

and

$$g_k(\alpha)\alpha^m = \frac{\varphi^{2m}}{\varphi+2}(1+\zeta_2), \quad \text{where} \quad |\zeta_2| < \frac{4}{\varphi^{k/2}}. \tag{3.25}$$

By substituting (3.24) and (3.25) into (3.10), we derive

$$\begin{aligned} \frac{\varphi^{2n+2m}}{(\varphi+2)^2} - 2^a &= \frac{\varphi^{2n+2m}}{(\varphi+2)^2}(\zeta_1+\zeta_2+\zeta_1\zeta_2) + \frac{\varphi^{2n}}{\varphi+2}e_k(m) + \frac{\varphi^{2n}}{\varphi+2}e_k(m)\zeta_1 \\ &+ \frac{\varphi^{2m}}{\varphi+2}e_k(n) + \frac{\varphi^{2m}}{\varphi+2}e_k(n)\zeta_2 \pm 1 - e_k(n)e_k(m). \end{aligned}$$

Hence, we obtain

$$\left|\frac{\varphi^{2n+2m}}{(\varphi+2)^2} - 2^a\right| \leqslant \frac{24\varphi^{2n+2m}}{(\varphi+2)^2\varphi^{k/2}} + \frac{\varphi^{2n}}{2(\varphi+2)} + \frac{\varphi^{2n}}{2(\varphi+2)\varphi^{k/2}} + \frac{\varphi^{2m}}{2(\varphi+2)} + \frac{\varphi^{2m}}{2(\varphi+2)\varphi^{k/2}} + \frac{5}{4}.$$

Therefore, the above inequality, along with the conditions $n \geqslant m$ and $n \geqslant k+2$, yields

$$|\Gamma_3| < \frac{24}{\varphi^{k/2}} + \frac{\varphi+2}{2}\left(\frac{1}{\varphi^{2m}} + \frac{1}{\varphi^{k/2+2m}} + \frac{1}{\varphi^{2n}} + \frac{1}{\varphi^{k/2+2n}}\right) < \frac{32}{\varphi^{\min\{k/2,2m\}}}, \tag{3.26}$$

where

$$\Gamma_3 := 1 - (\varphi+2)^2 \cdot 2^a \cdot \varphi^{-2n-2m}.$$

We have $\Gamma_3 \neq 0$, otherwise we would end up with $\varphi^{2n+2m} = (\varphi+2)\cdot 2^a$, implying that $(\varphi+2)\cdot 2^a$ is a unit in $\mathbb{Q}(\sqrt{5})$, which is impossible. Now, we can utilize Theorem 1.26 for Γ_3, considering the following parameters:

$$(\eta_1, b_1) := (2, a), \quad (\eta_2, b_2) := (\varphi+2, 2), \quad (\eta_3, b_3) := (\varphi, -(2n+2m)).$$

It is clear that $\eta_1, \eta_2, \eta_3 \in \mathbb{K} := \mathbb{Q}(\sqrt{5})$ and so $d_{\mathbb{K}} = 2$. We choose $B := 4n$ considering that $n \geqslant m$ and $a < 3n$. The usual absolute logarithmic heights of these numbers are as follows: $h(\eta_1) = \log 2$, $h(\eta_2) = \frac{\log 5}{2}$, and $h(\eta_3) = h(\varphi) = \frac{\log\varphi}{2}$. Consequently, we select $A_1 := 2\log 2$, $A_2 := \log 5$, and $A_3 := \log\varphi$.

Applying Theorem 1.26 to Γ_3, we obtain

$$\begin{aligned} |\Gamma_3| &> \exp\left(-1.4\cdot 30^6 \cdot 3^{4.5}\cdot 4(1+\log 2)(1+\log 4n)(2\log 2)(\log 5)(\log\varphi)\right) \\ &> \exp(-2.92\cdot 10^{12}\log n), \end{aligned}$$

where we have used the fact that $1+\log 4n < 2.8\log n$, for all $n \geqslant 4$. By comparing the resulting inequality with (3.26), we derive

$$\min\{k/2, 2m\} < 6.1\cdot 10^{12}\log n.$$

According to Lemma 3.3 and considering the inequality $66.58+9\log k+5\log\log k < 21\log k$, valid for all $k > 630$, we get

$$\begin{aligned} \min\{k/2, 2m\} &< 6.1\cdot 10^{12}\log(8.2\cdot 10^{28}k^9\log^5 k) \\ &< 6.1\cdot 10^{12}(66.58+9\log k+5\log\log k) \\ &< 1.3\cdot 10^{14}\log k. \end{aligned}$$

If $\min\{k/2, 2m\} = k/2$, the inequality $k < 2.6\cdot 10^{14}\log k$ arises. Solving this inequality and applying Lemma 3.3, we deduce that

$$k < 9.6\cdot 10^{15} \quad \text{and} \quad n < 3.9\cdot 10^{180}. \tag{3.27}$$

If $\min\{k/2, 2m\} = 2m$, so we obtain in this case that

$$m < 6.5 \cdot 10^{13} \log k. \tag{3.28}$$

Now, let us revisit (3.15) and express it differently

$$\frac{\varphi^{2n}}{\varphi+2} - \frac{2^a}{P_m^{(k)}} = \frac{\pm 1}{P_m^{(k)}} - e_k(n) - \frac{\varphi^{2n}\zeta_1}{\varphi+2}.$$

This yields

$$\left|\frac{\varphi^{2n}}{\varphi+2} - \frac{2^a}{P_m^{(k)}}\right| \leqslant \frac{4\varphi^{2n}}{(\varphi+2)\varphi^{k/2}} + \frac{3}{2}.$$

Consequently, we have

$$|\Gamma_4| \leqslant \frac{4}{\varphi^{k/2}} + \frac{3(\varphi+2)}{2\varphi^{2n}} < \frac{4}{\varphi^{k/2}} + \frac{3(\varphi+2)}{2\varphi^{2k+2}} < \frac{9.5}{\varphi^{k/2}}, \tag{3.29}$$

where

$$\Gamma_4 := \frac{\varphi+2}{P_m^{(k)}} \cdot 2^a \cdot \varphi^{-2n} - 1. \tag{3.30}$$

If $\Gamma_4 = 0$, then $(\varphi+2)\varphi^{2n} = P_m^{(k)}/2^a$. We take the conjugate to obtain $(\varphi+2)\varphi^{2n} = (\psi+2)\psi^{2n}$, where $\psi = (1-\sqrt{2})/2$ is the conjugate of φ. This leads to a contradiction since the left-hand side is greater than 3 in absolute value, and the right-hand side is smaller than 2 in absolute value. Therefore, $\Gamma_4 \neq 0$, and we can apply Theorem 1.26 with the following parameters:

$$(\eta_1, b_1) := ((\varphi+2)/P_m^{(k)}, 1), \quad (\eta_2, b_2) := (2, a), \quad (\eta_3, b_3) := (\varphi, 2n).$$

As previously calculated, we set $A_2 := 2\log 2$ and $A_3 := \log\varphi$. Additionally, we choose $B := 4n$. Now, let's estimate $h(\eta_1)$. Utilizing the fact that $P_m^{(k)} < \varphi^{2m-2}$ and the inequality (3.28), we get

$$h(\eta_1) = h(\varphi+2) + h(P_m^{(k)}) < \log 5 + (2m-2)\log\varphi < 3.15 \cdot 10^{13} \log k.$$

Thus, we select $A_1 := 6.3 \cdot 10^{13} \log k$. Consequently, Theorem 1.26 implies that

$$|\Gamma_4| > \exp(-1.15 \cdot 10^{26} \log^2 n), \tag{3.31}$$

where we have used the fact that $1 + \log(4n) < 2.8 \log n$, for all $n \geqslant 4$. Combining (3.29) and (3.31), we obtain

$$k < 4.8 \cdot 10^{26} \log^2 n.$$

Applying Lemma 3.3 and considering that $66.58 + 9\log k + 5\log\log k < 21\log k$ for all $k > 630$, we deduce

$$\begin{aligned} k &< 4.8 \cdot 10^{26} \left(\log(8.2 \cdot 10^{28} k^9 \log^5 k)\right)^2 \\ &< 4.8 \cdot 10^{26} (66.58 + 9\log k + 5\log\log k)^2 \\ &< 2.2 \cdot 10^{29} \log^2 k. \end{aligned}$$

Solving this inequality and applying Lemma 3.3, we get

$$k < 1.3 \cdot 10^{33} \quad \text{and} \quad n < 2.25 \cdot 10^{336}. \tag{3.32}$$

From (3.27) and (3.32), we can deduce that the inequalities (3.32) are consistently satisfied.

The obtained bounds are exceedingly large; thus, we will proceed to reduce them. Let us put

$$\Lambda_3 := \log(\Gamma_3 + 1) = a\log 2 - (2n + 2m)\log\varphi + 2\log(\varphi + 2). \tag{3.33}$$

Suppose that $m \geqslant 20$. In this case, we have $|\Gamma_3| < 0.27$. By taking $d = 0.27$ in Lemma 1.30, we get

$$|\Lambda_3| < \frac{-\log 0.73}{0.27} \cdot |\Gamma_3| < 37.3 \cdot \varphi^{-\min\{k/2, 2m\}}.$$

So, we obtain

$$\left| a \cdot \frac{\log 2}{\log\varphi} - (2n + 2m) + \frac{2\log(\varphi + 2)}{\log\varphi} \right| < 77.6 \cdot \varphi^{-\min\{k/2, 2m\}}. \tag{3.34}$$

To apply Lemma 1.27 to Λ_3, we consider

$$\gamma := \frac{\log 2}{\log\varphi}, \quad \mu := \frac{2\log(\varphi + 2)}{\log\varphi}, \quad \text{and} \quad (A, B) := (77.6, \varphi).$$

Now, the goal is to refine our bounds, which are currently too large, utilizing Lemma 1.27. Let's set $M := 6.75 \times 10^{336}$, an upper bound for a as derived from Lemma 3.3. Applying Lemma 1.27 to (3.34), we aim to obtain an upper bound for k. After a computer search with Maple, we find that q_{658} satisfies the conditions of Lemma 1.27 and provides us with the information that $\min\{k/2, 2m\} < 1632$.

Case 1: $\min\{k/2, 2m\} = k/2$**.** In this case, we get

$$k \leqslant 3264. \tag{3.35}$$

Case 2: $\min\{k/2, 2m\} = 2m$**.** In this case, we obtain that $m \leqslant 816$. Fix $2 \leqslant m \leqslant 3153$ and let's put

$$\Lambda_4 := \log(1 + \Gamma_4) = a\log 2 - 2n\log\varphi + \log((\varphi + 2)/P_m^{(k)}).$$

As $k > 630$, from (4.28), it follows that $|\Gamma_4| < 0.01$. Consequently, applying Lemma 1.30, we derive

$$|\Gamma_4| < -\frac{\log(0.99)}{0.01} \cdot |\Lambda_4| < 9.6 \cdot \varphi^{-k/2}. \tag{3.36}$$

Thus, we obtain

$$\left| a \cdot \frac{\log 2}{\log\varphi} - 2n + \frac{\log((\varphi + 2)/P_m^{(k)})}{\log\varphi} \right| < 20 \cdot \varphi^{-\frac{k}{2}}. \tag{3.37}$$

We apply Lemma 1.27 to Λ_4 with the parameters

$$\gamma := \frac{\log 2}{\log\varphi}, \quad \mu := \frac{\log((\varphi + 2)/P_m^{(k)})}{\log\varphi}, \quad M := 6.75 \cdot 10^{336} \quad \text{and} \quad (A, B) := (20, \varphi).$$

Consider the case where $2m < k/2$, which implies $m < k + 1$. In this scenario, we can substitute $P_m^{(k)}$ with F_{2m-1} in our calculations. By using Maple, we find that q_{660} meets the conditions of Lemma 1.27. Consequently, we find $k < 3284$. Applying this new bound, we find $n < 1.3 \cdot 10^{65}$. Subsequently, we apply Lemma 1.27 again with the same data, but now we set M to be $3.9 \cdot 10^{65}$. With Maple, we see that q_{136} satisfies the conditions of Lemma 1.27, leading to $k < 690$. However, this contradicts our initial assumption that $k > 630$.

Chapter 4

On repdigits as product of k-Fibonacci and k-Lucas numbers

Through this chapter, we will determine all possibilities such that $F_n^{(k)}L_m^{(k)}$ can represent a repdigit. The content discussed in this chapter is drawn from the paper [57].

4.1 Introduction

For an integer $k \geqslant 2$, let $(L_n^{(k)})_{n\geqslant -(k-2)}$ be the k-generalized Lucas sequence following the same recursive pattern as the k-Fibonacci sequence but with initial conditions $L_{-(k-2)}^{(k)} = L_{-(k-3)}^{(k)} = \cdots = L_{-1}^{(k)} = 0$, $L_0^{(k)} = 2$, and $L_1^{(k)} = 1$.

Recall that a positive integer is called a repdigit if it has only one distinct digit in its decimal expansion. In particular, such number has the form $a(10^\ell - 1)/9$, for some $l \geqslant 1$ and $1 \leqslant a \leqslant 9$.

4.2 Motivation and main result

Several problems involving generalized Fibonacci sequences or generalized Lucas sequences and repdigits have been of interest to mathematicians. For instance, Luca [40] and Marques [41] proved that 55 and 44 are the largest repdigits in the sequences $F^{(2)}$ and $F^{(3)}$, respectively. Moreover, Marques conjectured that there are no repdigits with at least two digits, belonging to $F^{(k)}$ for $k > 3$. This conjecture was confirmed by Bravo and Luca in [17]. Also in [40], Luca showed that 11 is the largest repdigit in the sequence $L^{(2)}$. This result was generalized by Bravo and Luca in [20]. For more results see [1, 9, 10, 11, 19, 23, 28, 43, 49] and their references. In [29], Erduvan and Keskin have studied repdigits as product of two Fibonacci and two Lucas numbers. Motivated by these results, we will study the solutions of the Diophantine equation

$$F_n^{(k)}L_m^{(k)} = \frac{a(10^\ell - 1)}{9}. \tag{4.1}$$

We will prove the following results.

Theorem 4.1. *All the solutions of Diophantine equation* (4.1) *in positive integers* $n, m, \ell, k,$ *and* a *with* $0 \leqslant m < n$, $k \geqslant 2$, $\ell \geqslant 2$, *and* $1 \leqslant a \leqslant 9$, *are*

$$(a, k, l, m, n) \in \{(4, 3, 2, 1, 8), (5, 2, 2, 1, 10), (8, 2, 2, 5, 6), (8, 3, 2, 0, 8)\}.$$

Theorem 4.2. *All the solutions of Diophantine equation* (4.1) *in positive integers* n, m, ℓ, k, *and* a *with* $1 \leqslant n \leqslant m$, $k \geqslant 2$, $\ell \geqslant 2$, *and* $1 \leqslant a \leqslant 9$, *are*

$$(a, k, l, m, n) \in \{(1, 2, 2, 5, 1), (1, 2, 2, 5, 2), (2, 4, 2, 5, 1), (2, 4, 2, 5, 2),$$

$$(2, 2, 2, 5, 3), (3, 2, 2, 5, 4), (4, 4, 2, 5, 3), (5, 2, 2, 5, 5), (8, 4, 2, 5, 4)\}.$$

In order to prove our theorems, we will use linear forms in logarithms of algebraic numbers and the reduction method due to Dujella-Pethő. We will start by introducing necessary results and definitions which will be used in the remaining work.

4.3 Preliminaries and known results

4.3.1 Properties of the k-generalized Lucas sequence

The k-Fibonacci and k-Lucas sequences share several properties, such as the characteristic polynomial. These common properties were presented in the first part of subsection 2.3.1. Here, we highlight the specific properties of the k-Lucas sequence that will be used later, alongside those of the k-Fibonacci sequence discussed in the mentioned subsection.

In [20] Bravo and Luca proved that

$$L_n^{(k)} = \sum_{i=1}^{k} (2\alpha^{(i)} - 1) f_k(\alpha^{(i)}) {\alpha^{(i)}}^{n-1} \tag{4.2}$$

and

$$|e'_k(n)| < \frac{3}{2}, \quad \text{where } e'_k(n) = L_n^{(k)} - (2\alpha - 1) f_k(\alpha) \alpha^{n-1}, \tag{4.3}$$

for all $n \geqslant 2 - k$ and $k \geqslant 2$. In addition, for $n \geqslant 1$ and $k \geqslant 2$, it was proved in the same paper that

$$\alpha^{n-1} \leqslant L_n^{(k)} \leqslant 2\alpha^n. \tag{4.4}$$

Furthermore, note that if $2 \leqslant n \leqslant k$, then $L_n^{(k)} = 3 \cdot 2^{n-2}$.

4.4 The proof of Theorem 4.1

In this section, we will prove Theorem 4.1 in four steps. Let us start with the first step.

4.4.1 An upper bound for ℓ in terms of n

We begin our analysis of (4.1), for $2 \leqslant m < n \leqslant k + 1$. In this case, we have $F_n^{(k)} = 2^{n-2}$ and $L_m^{(k)} = 3 \cdot 2^{m-2}$, so the equation (4.1) becomes

$$3 \cdot 2^{n+m-4} = \frac{a(10^\ell - 1)}{9}. \tag{4.5}$$

For any rational number x, let $\nu_2(x)$ denote the 2-adic valuation of x. Since $\nu_2(a(10^\ell - 1)/9) \leqslant 3$, then by comparing the 2-adic valuation on both sides of (4.5), one gets $2 \leqslant m < n \leqslant 7$. In this range, equation (4.5) has no solutions. So, from now, we assume that $n \geqslant k + 2 \geqslant 4$ and $m \geqslant 2$.

We next comment on the size of ℓ versus n. By inequalities (2.9), (4.4), and $10^{\ell-1} < a(10^{\ell}-1)/9$, we obtain

$$10^{\ell-1} < \frac{a(10^{\ell}-1)}{9} = F_n^{(k)} L_m^{(k)} < \alpha^{n-1} 2\alpha^m < 2\alpha^{n+m-1}.$$

Then, we deduce that

$$\ell < \frac{\log 2}{\log 10} + (2n-1)\left(\frac{\log \alpha}{\log 10}\right) + 1 = n\left(\frac{2\log \alpha}{\log 10}\right) - \left(\frac{\log \alpha}{\log 10}\right) + \frac{\log 2}{\log 10} + 1.$$

Moreover, by using the fact that $3/2 = 2(1-2^{-2}) < \alpha < 2$, for $k \geqslant 2$ (see (2.3)), we get

$$\ell < n. \tag{4.6}$$

4.4.2 An inequality for n versus k

Now, we will show the following lemma that gives an upper bound for n in terms of k.

Lemma 4.3. *If (a, k, ℓ, m, n) is a solution in positive integers of equation (4.1) with $k \geqslant 2$ and $n \geqslant k+2$, then we have the following inequality*

$$n < 1.9 \times 10^{30} k^8 \log^5 k. \tag{4.7}$$

Proof. Using estimates (2.7) and (4.2), equation (4.1) can be rearranged as

$$(f_k(\alpha)\alpha^{n-1} + e_k(n))((2\alpha-1)f_k(\alpha)\alpha^{m-1} + e'_k(m)) = a\left(\frac{10^{\ell}-1}{9}\right), \tag{4.8}$$

i.e.,

$$(2\alpha-1)f_k^2(\alpha)\alpha^{n+m-2} - \frac{a10^{\ell}}{9} = -e'_k(m)f_k(\alpha)\alpha^{n-1} - e_k(n)(2\alpha-1)f_k(\alpha)\alpha^{m-1} - e_k(n)e'_k(m) - \frac{a}{9}.$$

Thus, we obtain

$$\left|(2\alpha-1)f_k^2(\alpha)\alpha^{n+m-2} - \frac{a10^{\ell}}{9}\right| \leqslant \frac{3f_k(\alpha)}{2}\alpha^{n-1} + \frac{(2\alpha-1)f_k(\alpha)}{2}\alpha^{m-1} + \frac{7}{4}.$$

If we divide both sides by $(2\alpha-1)f_k^2(\alpha)\alpha^{n+m-2}$ and use the fact that $f_k(\alpha) > 1/2$ and $\alpha > 1.5$, we arrive at

$$|\Gamma_1| \leqslant \frac{3}{2\alpha^{m-1}} + \frac{1}{\alpha^{n-1}} + \frac{7}{2\alpha^{n+m-2}} < \frac{6\alpha}{\alpha^m} < \frac{12}{\alpha^m}, \tag{4.9}$$

where

$$\Gamma_1 := \frac{a}{9(2\alpha-1)f_k^2(\alpha)} \cdot \alpha^{-(n+m-2)} \cdot 10^{\ell} - 1. \tag{4.10}$$

We have $\Gamma_1 \neq 0$, otherwise we would get

$$\frac{a10^{\ell}}{9} = (2\alpha-1)f_k^2(\alpha)\ \alpha^{n+m-2}.$$

Conjugating the above relation by some automorphism of the Galois group of the decomposition field of $\Psi(x)$ over $\mathbb{Q}$ and then taking absolute values, we get that for any $i \geqslant 2$, we have

$$\frac{100}{9} \leqslant \frac{a10^{\ell}}{9} = |(2\alpha_i - 1)| \cdot |f_k(\alpha_i)|^2 \cdot |\alpha_i|^{n+m-2} < 3,$$

which leads to a contradiction. Let us apply Theorem 1.26 to Γ_1 given by (4.10). To this end, we take as parameters

$$(\eta_1, b_1) := (a/(9(2\alpha-1)f_k^2(\alpha)), 1), \quad (\eta_2, b_2) := (\alpha, -(n+m-2)), \quad (\eta_3, b_3) := (10, \ell).$$

The algebraic numbers η_1, η_2, η_3 are elements of the number field $\mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. Since $h(\eta_2) = (\log\alpha)/k < (\log 2)/k$ and $h(\eta_3) = \log 10$, then we can choose

$$A_2 := \log 2 = \max\{kh(\eta_2), |\log \eta_2|, 0.16\}$$

and

$$A_3 := k\log 10 = \max\{kh(\eta_3), |\log \eta_3|, 0.16\}.$$

Next, we compute A_1. Using the estimate (2.5) and the properties (1.1)-(1.3), it follows that for all $k \geqslant 2$

$$\begin{aligned} h(\eta_1) &\leqslant h\left(\frac{a}{9}\right) + h(2\alpha-1) + 2h(f_k(\alpha)) \\ &< \log 9 + \log 3 + 6\log k \\ &< 12\log k. \end{aligned}$$

Hence, we get

$$A_1 := 12k\log k > \max\{kh(\eta_1), |\log \eta_1|, 0.16\}.$$

Finally, the fact that $m < n$ and inequality (4.6), imply that we can take $B := 2n$. Therefore, according to Theorem 1.26, we have

$$\begin{aligned} |\Gamma_1| &> \exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot k^2(1+\log k)(1+\log 2n)(12k\log k)(\log 2)(k\log 10)\right) \\ &> \exp\left(-2.8 \cdot 10^{12}(1+\log k)(k^4\log k)(1+\log 2n)\right). \end{aligned}$$

Using the fact $1 + \log k < 2.5\log k$, which holds for $k \geqslant 2$, we obtain

$$|\Gamma_1| > \exp\left(-7.1 \cdot 10^{12} k^4 \log^2 k(1+\log 2n)\right). \tag{4.11}$$

Comparing the obtained bounds of $|\Gamma_1|$ gives us

$$m\log\alpha < 7.2 \cdot 10^{12} k^4 \log^2 k(1+\log 2n). \tag{4.12}$$

We return to equation (4.1) and we rewrite it as follows

$$(f_k(\alpha)\alpha^{n-1} + e_k(n))L_m^{(k)} = \frac{a(10^{\ell}-1)}{9},$$

i.e.,

$$f_k(\alpha)\alpha^{n-1} - \frac{a10^{\ell}}{9L_m^{(k)}} = -\frac{a}{9L_m^{(k)}} - e_k(n). \tag{4.13}$$

So, we get

$$\left| f_k(\alpha)\alpha^{n-1} - \frac{a10^{\ell}}{9L_m^{(k)}} \right| \leqslant \frac{a}{9L_m^{(k)}} + \frac{1}{2} \leqslant \frac{3}{2}.$$

Dividing through by $f_k(\alpha)\alpha^{n-1}$, we get

$$|\Gamma_2| \leqslant \frac{3}{2f_k(\alpha)\alpha^{n-1}} < \frac{3\alpha}{\alpha^n} < \frac{6}{\alpha^n}, \tag{4.14}$$

where

$$\Gamma_2 := \frac{a}{9L_m^{(k)} f_k(\alpha)} \cdot \alpha^{-(n-1)} \cdot 10^{\ell} - 1. \tag{4.15}$$

We can prove that $\Gamma_2 \neq 0$ by a similar method used to show that $\Gamma_1 \neq 0$. Now, we will apply Theorem 1.26 to Γ_2 by taking

$$(\eta_1, b_1) := (a/(9L_m^{(k)} f_k(\alpha)), 1), \quad (\eta_2, b_2) := (\alpha, -(n-1)), \quad (\eta_3, b_3) := (10, \ell).$$

It is clear that $\mathbb{K} := \mathbb{Q}(\alpha)$ contains η_1, η_2, η_3 and has degree $d_{\mathbb{K}} = k$. As calculated before, we take

$$A_2 := \log 2, \quad A_3 := k \log 10, \quad \text{and} \quad B := 2n.$$

We need to compute A_1. The estimates (2.5), (4.12), and the proprieties (1.1)-(1.3), imply that, for all $k \geqslant 2$, we have

$$\begin{aligned} h(\eta_1) &\leqslant h\left(\frac{a}{9}\right) + h(L_m^{(k)}) + h(f_k(\alpha)) \\ &< \log 9 + \log 2 + m \log \alpha + 3 \log k \\ &< 7.3 \cdot 10^{12} k^4 \log^2 k (1 + \log 2n). \end{aligned}$$

On the other hand, since

$$\eta_1 := \frac{a}{9L_m^{(k)} f_k(\alpha)} < 2 \quad \text{and} \quad \eta_1^{-1} = \frac{9L_m^{(k)} f_k(\alpha)}{a} < \frac{27\alpha^m}{2},$$

then, by (4.12), we get

$$|\log \eta_1| < m \log \alpha + \log 13.5 < 7.3 \cdot 10^{12} k^4 \log^2 k (1 + \log 2n).$$

Thus, we conclude that

$$\max\{kh(\eta_1), |\log \eta_1|, 0.16\} < 7.3 \cdot 10^{12} k^5 \log^2 k (1 + \log 2n) := A_1.$$

Applying Theorem 1.26 and comparing the resulting inequality with (**??**), we obtain

$$n \log \alpha < 2.21 \cdot 10^{25} k^8 \log^3 k \log^2 n,$$

where we have used the facts $1 + \log k < 2.5 \log k$ and $1 + \log 2n < 2.3 \log n$ which hold for $k \geqslant 2$ and $n \geqslant 4$. So, we deduce that

$$\frac{n}{\log^2 n} < 5.5 \cdot 10^{25} k^8 \log^3 k. \tag{4.16}$$

It is easy to check that the inequality

$$\frac{x}{\log^2 x} < A \text{ implies } x < 4A \log^2 A, \text{ whenever } A \geqslant 100. \tag{4.17}$$

Thus, putting $A := 5.5 \cdot 10^{25} k^8 \log^3 k$ in inequality (4.17) and using $59.3 + 8 \log k + 3 \log \log k < 92 \log k$ which holds, for all $k \geqslant 2$, we obtain

$$\begin{aligned} n &< 4(5.5 \cdot 10^{25} k^8 \log^3 k)(\log(5.5 \cdot 10^{25} k^8 \log^3 k))^2 \\ &< (2.21 \cdot 10^{26} k^8 \log^3 k)(59.3 + 8 \log k + 3 \log \log k)^2 \\ &< 1.9 \cdot 10^{30} k^8 \log^5 k. \end{aligned}$$

This gives (4.7) and completes the proof of Lemma 4.3. □

4.4.3 The case $2 \leqslant k \leqslant 440$

For this subsection, we consider $k \in [2, 440]$. Define

$$\Lambda_1 := \log(\Gamma_1 + 1) = \ell \log 10 - (n + m - 2) \log \alpha + \log(a/(9(2\alpha - 1) f_k^2(\alpha))). \tag{4.18}$$

Suppose that $m \geqslant 10$, so by the estimate (4.9) and the fact that $\alpha > 1.5$, we have $|\Gamma_1| < 0.21$. Taking $d = 0.21$ in Lemma 1.30, we obtain

$$|\Lambda_1| < \frac{-\log 0.79}{0.21} \cdot |\Gamma_1| < 13.5 \cdot \alpha^{-m}. \tag{4.19}$$

So, we get

$$\left| \ell \cdot \frac{\log 10}{\log \alpha} - (n + m - 2) + \frac{\log(a/(9(2\alpha - 1) f_k^2(\alpha)))}{\log \alpha} \right| < 33.3 \cdot \alpha^{-m}. \tag{4.20}$$

For all $a \in \{1, \ldots, 9\}$, we apply Lemma 1.27 to Λ_1 with the following parameters

$$\gamma := \frac{\log 10}{\log \alpha}, \quad \mu := \frac{\log(a/(9(2\alpha - 1) f_k^2(\alpha)))}{\log \alpha}, \quad \text{and} \quad (A, B) := (33.3, \alpha).$$

For each $k \in [2, 440]$ and $a \in \{1, \ldots, 9\}$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k \|\gamma q\| > 0$, where $M_k := \lfloor 1.9 \times 10^{30} k^8 \log^5 k \rfloor$, which is an upper bound of ℓ from Lemma 4.3. Using Mathematica, we see that q_{121} satisfies the conditions of Lemma 1.27. After doing this, we use Lemma 1.27 on inequality (4.20). A computer program with Mathematica shows for $k = 417$ and $a = 7$ that $\varepsilon > 0.000064$ and the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$, for $k \in [2, 440]$ and $a \in \{1, \ldots, 9\}$, is 204.861, which is an upper bound of m by Lemma 1.27.

For $2 \leqslant m < 205$, we consider

$$\Lambda_2 := \log(\Gamma_2 + 1) = \ell \log 10 - (n - 1) \log \alpha + \log(a/(9 L_m^{(k)} f_k(\alpha))). \tag{4.21}$$

Suppose that $n \geqslant 10$, so by the estimate (4.14) and the fact that $\alpha > 1.5$, we have $|\Gamma_2| < 0.11$. Taking $d = 0.11$ in Lemma 1.30, we obtain

$$|\Lambda_2| < \frac{-\log 0.89}{0.11} \cdot |\Gamma_2| < 6.4 \cdot \alpha^{-n}.$$

So, we get

$$\left| \ell \cdot \frac{\log 10}{\log \alpha} - (n - 1) - \frac{\log(a/(9 L_m^{(k)} f_k(\alpha)))}{\log \alpha} \right| < 15.8 \cdot \alpha^{-n}. \tag{4.22}$$

For all $a \in \{1, \ldots, 9\}$ and $2 \leqslant m \leqslant 204$, we apply Lemma 1.27 to Λ_2, by fixing

$$\gamma := \frac{\log 10}{\log \alpha}, \quad \mu := \frac{\log(a/(9 L_m^{(k)} f_k(\alpha)))}{\log \alpha}, \quad \text{and} \quad (A, B) := (15.8, \alpha).$$

Again, for each $(k, m) \in [2, 440] \times [2, 204]$ and $a \in \{1, \ldots, 9\}$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k \|\gamma q\| > 0$, where $M_k := \lfloor 1.9 \times 10^{30} k^8 \log^5 k \rfloor$, which is an upper bound of ℓ from Lemma 4.3. With the help of Mathematica, we see again that q_{121} satisfies the conditions of Lemma 1.27. After doing this, we use Lemma 1.27 on inequality (4.22). A computer program with Mathematica revealed for $k = 417$, $a = 7$ and for all $2 \leqslant m \leqslant 204$

that $\varepsilon > 0.000064$ and the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$, for $(k, m) \in [2, 440] \times [2, 204]$ and $a \in \{1, \ldots, 9\}$, is 203.786, which is an upper bound of n by Lemma 1.27.

Hence, we deduce that the possible solutions (a, k, l, m, n) of equation (4.1) for which $k \in [2, 440]$ and $a \in \{1, \ldots, 9\}$ satisfy $m < n \leqslant 203$. Therefore, we use inequality (4.6) to obtain $\ell \leqslant 202$.

Finally, we use Mathematica to compare $F_n^{(k)} L_m^{(k)}$ and $\dfrac{a(10^\ell - 1)}{9}$, for the ranges $k + 2 \leqslant n \leqslant 203$, $m < n$, $1 \leqslant a \leqslant 9$ and $2 \leqslant \ell \leqslant 202$, with $\ell < n$ and check that the only solutions of equation (4.1) are those listed in Theorem 4.1.

4.4.4 The case $k > 440$

In this subsection, we analyze the case $k > 440$. For such k, it is easy to check that

$$m \leqslant n < 1.9 \cdot 10^{30} k^8 \log^5 k < 2^{k/2}.$$

Thus, by Lemmas 1.18 and 1.19, $F_n^{(k)}$and $L_m^{(k)}$ can be respectively rewritten as

$$F_n^{(k)} = 2^{n-2}(1 + \zeta_1), \quad \text{where } |\zeta_1| < \frac{2}{2^{k/2}} \tag{4.23}$$

and

$$L_m^{(k)} = 3 \cdot 2^{m-2}(1 + \zeta_2), \quad \text{where } |\zeta_2| < \frac{1}{2^{k/2}}. \tag{4.24}$$

Substituting (4.23) and (4.24) in (4.1), we obtain

$$3 \cdot 2^{n+m-4}(1 + \zeta_1)(1 + \zeta_2) = \frac{a(10^\ell - 1)}{9}.$$

Hence, we get

$$\left| 3 \cdot 2^{n+m-4} - \frac{a10^\ell}{9} \right| \leqslant \frac{9 \cdot 2^{n+m-4}}{2^{k/2}} + \frac{6 \cdot 2^{n+m-4}}{2^k} + 1.$$

Consequently, the above inequality and the fact that $n \geqslant 4$ give

$$\left| 1 - \frac{a}{27} \cdot 10^\ell \cdot 2^{-(n+m+4)} \right| < \frac{3}{2^{k/2}} + \frac{2}{2^k} + \frac{1}{3 \cdot 2^m} < \frac{5.5}{2^{\min\{k/2, m\}}}. \tag{4.25}$$

Define

$$\Gamma_3 = \frac{a}{27} \cdot 2^{-(n+m-4)} \cdot 10^\ell - 1.$$

We have $\Gamma_3 \neq 0$, because if $\Gamma_3 = 0$ we get $a \cdot 10^\ell = 27 \cdot 2^{n+m-4}$. This implies that 5 divides $27 \cdot 2^{n+m-4}$, which is impossible. Now, we put

$$(\eta_1, b_1) := \left(\frac{a}{27}, 1 \right), \quad (\eta_2, b_2) := (2, -(n + m - 4)), \quad (\eta_3, b_3) := (10, \ell).$$

Then, we obtain

$$h(\eta_1) = \log 27, \quad h(\eta_2) = \log 2, \quad \text{and} \quad h(\eta_3) = \log 10.$$

Note that $\eta_1, \eta_2, \eta_3 \in \mathbb{K} := \mathbb{Q}$. Thus $d_{\mathbb{K}} = 1$. So, we take

$$A_1 := \log 27, \quad A_2 := \log 2, \quad A_3 := \log 10, \quad \text{and} \quad B := 2n.$$

Thus, applying Theorem 1.26, we get

$$|\Gamma_3| > \exp(-1.74 \cdot 10^{12} \log n),$$

where we have used the fact that $1 + \log 2n < 2.3 \log n$, for all $n \geqslant 4$. By comparing the resulting inequality with (5.20), we obtain

$$\min\{k/2, m\} < 2.52 \cdot 10^{12} \log n.$$

By Lemma 4.3 and using the fact that $69.72 + 8 \log k + 5 \log\log k < 21 \log k$, for $k > 440$, we get

$$\begin{aligned} \min\{k/2, m\} &< 2.52 \cdot 10^{12} \log(1.9 \cdot 10^{30} k^8 \log^5 k) \\ &< 2.52 \cdot 10^{12} (69.72 + 8 \log k + 5 \log\log k) \\ &< 5.3 \cdot 10^{13} \log k. \end{aligned}$$

If $\min\{k/2, m\} = k/2$, then we get $k < 1.1 \cdot 10^{14} \log k$. Solving this inequality and using Lemma 4.3 we conclude that

$$k < 4 \cdot 10^{15} \quad \text{and} \quad n < 7.5 \cdot 10^{162}. \tag{4.26}$$

If $\min\{k/2, m\} = m$, we obtain in this case that

$$m < 5.3 \cdot 10^{13} \log k. \tag{4.27}$$

Now, we go back to (4.8) and we rewrite it as

$$2^{n-2} - \frac{a 10^\ell}{9 L_m^{(k)}} = \frac{-a}{9 L_m^{(k)}} - 2^{n-2} \zeta_1.$$

Thus, we obtain

$$\left| 2^{n-2} - \frac{a 10^\ell}{9 L_m^{(k)}} \right| \leqslant \frac{2^{n-1}}{2^{k/2}} + 1.$$

Dividing through by 2^{n-2} and using the fact that $n \geqslant k + 2$, we get

$$|\Gamma_4| \leqslant \frac{1}{2^{n-2}} + \frac{2}{2^{k/2}} < \frac{1}{2^k} + \frac{2}{2^{k/2}} < \frac{3}{2^{k/2}}, \tag{4.28}$$

where

$$\Gamma_4 := \frac{a}{9 L_m^{(k)}} \cdot 2^{-(n-2)} \cdot 10^\ell - 1. \tag{4.29}$$

Note that $\Gamma_4 \neq 0$, since otherwise we would get $\dfrac{a 10^\ell}{9 L_m^{(k)}} = 2^{n-2}$. If $a \in \{1, \ldots, 8\}$ then it is obvious that the left side cannot be an integer. If $a = 9$, then we have $\dfrac{10^\ell}{L_m^{(k)}} = 2^{n-2}$ or in this case $m < k/2$ implies that $m < k$ and so $L_m^{(k)} = 3 \cdot 2^{m-2}$. Then we would get $\dfrac{10^\ell}{3 \cdot 2^{m-2}} = 2^{n-2}$ and this leads to a contradiction. Therefore, $\Gamma_4 \neq 0$. Now, we apply Theorem 1.26 by fixing

$$(\eta_1, b_1) := (a/(9 L_m^{(k)}), 1), \quad (\eta_2, b_2) := (2, -(n-2)), \quad (\eta_3, b_3) := (10, \ell).$$

As calculated before, we take $A_2 := \log 2$ and $A_3 := \log 10$. We take $B := n$. Next, we estimate $h(\eta_1)$. By the fact that $L_m^{(k)} < 2\alpha^m$ and inequality (4.27), we obtain

$$h(\eta_1) \leqslant h(a/9) + h(L_m^{(k)}) < \log 9 + \log 2 + m \log \alpha < 3.7 \cdot 10^{13} \log k.$$

So, we choose $A_1 := 3.7 \cdot 10^{13} \log k$. Therefore, Theorem 1.26 gives

$$|\Gamma_4| > \exp(-1.53 \cdot 10^{25} \log k \log n), \tag{4.30}$$

where we have used the fact that $1 + \log n < 1.8 \log n$, which holds, for $n \geqslant 4$. From (4.28) and (4.30), it results

$$k < 4.42 \cdot 10^{25} \log k \log n.$$

By Lemma 4.3 and using the fact that $69.72 + 8\log k + 5\log\log k < 21 \log k$, for all $k > 440$, we get

$$\begin{aligned} k &< 4.42 \cdot 10^{25} \log k \left(\log(1.9 \cdot 10^{30} k^8 \log^5 k)\right) \\ &< 4.42 \cdot 10^{25} \log k (69.72 + 8\log k + 5 \log\log k) \\ &< 9.3 \cdot 10^{26} \log^2 k. \end{aligned}$$

Solving this inequality and using Lemma 4.3, we obtain

$$k < 4.64 \cdot 10^{30} \quad \text{and} \quad n < 7.2 \cdot 10^{284}. \tag{4.31}$$

Comparing (4.26) and (4.31), we conclude that inequalities (4.31) always hold. The obtained bounds are very large, next we will reduce them. Put

$$\Lambda_3 := \log(\Gamma_3 + 1) = \ell \log 10 - (n + m - 4)\log 2 + \log(a/27). \tag{4.32}$$

Assume that $m \geqslant 10$, then we get $|\Gamma_3| < 0.02$. Taking $d = 0.02$ in Lemma 1.30, we obtain

$$|\Lambda_3| < \frac{-\log 0.98}{0.02} \cdot |\Gamma_3| < 6 \cdot 2^{-\min\{k/2, m\}}.$$

So, we deduce that

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n + m - 4) + \frac{\log(a/27)}{\log 2} \right| < 9 \cdot 2^{-\min\{k/2,m\}}. \tag{4.33}$$

For $a \in \{1, \ldots, 9\}$, we apply Lemma 1.27 to Λ_3 with the data

$$\gamma := \frac{\log 10}{\log 2}, \quad \mu := \frac{\log(a/27)}{\log 2}, \quad \text{and} \quad (A, B) := (9, 2).$$

We now want to reduce our bound, which is too large, by using Lemma 1.27. We take $M := 7.2 \cdot 10^{284}$, which is an upper bound on ℓ by (4.6) and (4.31). Then, we use Lemma 1.27 on inequality (4.33) in order to obtain an upper bound on k. A computer search with Maple shows that q_{575} satisfies the conditions of Lemma 1.27, for all $a \in \{1, \ldots, 9\}$. Thus, the application of Lemma 1.27 yields to the different results presented in the following table.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geqslant$	0.37	0.37	0.25	0.37	0.37	0.25	0.34	0.37	0.13
$\min\{k/2, m\} \leqslant$	956	956	956	956	956	956	956	956	957

According to the obtained results, we find that $\min\{k/2, m\} < 958$ which holds in all cases.

Case 1: $\min\{k/2, m\} = k/2$. In this case, we get

$$k < 1916. \tag{4.34}$$

Case 2: $\min\{k/2, m\} = m$. In this case, we obtain that $m \leqslant 957$. Let $2 \leqslant m \leqslant 957$ and

$$\Lambda_4 := \log(\Gamma_4 + 1) = \ell \log 10 - (n - 2)\log 2 + \log(a/(9L_m^{(k)})).$$

Since $k > 440$, then from (4.28), we have $|\Gamma_4| < 0.01$. Hence by Lemma 1.30, we obtain

$$|\Lambda_4| < -\frac{\log(0.99)}{0.01} \cdot |\Gamma_4| < 3.02 \cdot 2^{-k/2}. \tag{4.35}$$

So, we get

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n-2) + \frac{\log(a/(9L_m^{(k)}))}{\log 2} \right| < 4.4 \cdot 2^{-\frac{k}{2}}. \tag{4.36}$$

For all $a \in \{1, \ldots, 9\}$ and $2 \leqslant m \leqslant 957$, we apply Lemma 1.27 to Λ_4 with the parameters

$$\gamma := \frac{\log 10}{\log 2}, \quad \mu := \frac{\log(a/(9L_m^{(k)}))}{\log 2}, \quad M := 7.2 \cdot 10^{284}, \quad \text{and} \quad (A, B) := (4.4, 2).$$

Note that $m < k/2$. This implies that $m < k$, which holds for $k \geqslant 2$ and we can replace $L_m^{(k)}$ by $3 \cdot 2^{m-2}$ in our calculations. Using again Maple, one can see that q_{575} satisfies the conditions of Lemma 1.27, for all $a \in \{1, \ldots, 9\}$ and $2 \leqslant m \leqslant 957$. Thus, applying Lemma 1.27 gives us the following different results which hold, for all $2 \leqslant m \leqslant 957$.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geqslant$	0.37	0.37	0.25	0.37	0.37	0.25	0.34	0.37	0.13
$k/2 \leqslant$	955	955	955	955	955	955	955	955	956

From the obtained results, we observe that in all cases $k < 1913$. So, in both cases, we have $k < 1916$.

With this new bound, we get $n < 9 \cdot 10^{60}$. Again, we apply Lemma 1.27 with the same above data but this time we take $M := 9 \cdot 10^{60}$. With the help of Maple we see that q_{131} satisfies the conditions of Lemma 1.27, for all $a \in \{1, \ldots, 9\}$. For this application, we obtain the following results.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geqslant$	0.05	0.05	0.37	0.05	0.05	0.37	0.34	0.05	0.32
$\min\{k/2, m\} \leqslant$	217	217	214	217	217	214	214	217	214

After this, we can see that $\min\{k/2, m\} < 218$ which holds in all cases.
As we have done above, we obtain for the first case that $k < 436$ and for the second case, we use again q_{131} which fulfill the conditions of Lemma 1.27, for all $a \in \{1, \ldots, 9\}$ and $2 \leqslant m \leqslant 217$. Then, we obtain the following results which hold for all $2 \leqslant m \leqslant 217$ and give us that $k < 433$ in all cases.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geqslant$	0.05	0.05	0.37	0.05	0.05	0.37	0.34	0.05	0.32
$k/2 \leqslant$	216	216	213	216	216	213	213	216	213

Thus, for both cases it holds that $k < 436$, which contradicts our assumption that $k > 440$. This completes the proof of Theorem 4.1.

4.5 The proof of Theorem 4.2

This section is devoted to show Theorem 4.2. This proof will be similar to that of Theorem 4.1. For the sake of completeness, we will give most of the details.

4.5.1 An upper bound for ℓ in terms of m

We begin our analysis of (4.1) for $2 \leqslant n \leqslant m \leqslant k$. In this case, we have $F_n^{(k)} = 2^{n-2}$ and $L_m^{(k)} = 3 \cdot 2^{m-2}$, so the equation (4.1) turns to

$$3 \cdot 2^{m+n-4} = \frac{a(10^\ell - 1)}{9}. \tag{4.37}$$

For any rational number x, let $\nu_2(x)$ denote the 2-adic valuation of x. Since $\nu_2(a(10^\ell-1)/9) \leqslant 3$, then by comparing the 2-adic valuation on both sides of (4.37) one gets $2 \leqslant n \leqslant m \leqslant 7$. In this range, equation (4.37) has no solutions. So, from now, we assume that $m \geqslant k+1$.

Now, by inequalities (2.9), (4.4), and $10^\ell - 1 \leqslant a(10^\ell-1)/9$, we obtain

$$\ell < m. \tag{4.38}$$

4.5.2 An inequality for m versus k

Here, we will give an inequality for m in terms of k by showing the following lemma.

Lemma 4.4. *If (a,k,ℓ,n,m) is a solution in positive integers of equation (4.1) with $k \geqslant 2$ and $m \geqslant k+1$, then we have the following inequality*

$$m < 2.6 \times 10^{30} k^8 \log^5 k. \tag{4.39}$$

Proof. Using estimates (2.7) and (4.2), equation (4.1) can be rewritten as

$$(f_k(\alpha)\alpha^{n-1} + e_k(n))((2\alpha-1)f_k(\alpha)\alpha^{m-1} + e'_k(m)) = a\left(\frac{10^\ell-1}{9}\right), \tag{4.40}$$

i.e.,

$$\begin{aligned}(2\alpha-1)f_k^2(\alpha)\alpha^{m+n-2} - \frac{a10^\ell}{9} &= -e'_k(m)f_k(\alpha)\alpha^{n-1} - e_k(n)(2\alpha-1)f_k(\alpha)\alpha^{m-1}\\ &\quad -e'_k(m)e_k(n) - \frac{a}{9}.\end{aligned}$$

Thus, by taking the absolute value and dividing both sides by $(2\alpha-1)f_k^2(\alpha)\alpha^{n+m-2}$, we obtain

$$|\Gamma'_1| \leqslant \frac{3}{2\alpha^{m-1}} + \frac{1}{\alpha^{n-1}} + \frac{7}{2\alpha^{n+m-2}} < \frac{6\alpha}{\alpha^n} < \frac{12}{\alpha^n}, \tag{4.41}$$

where

$$\Gamma'_1 := \frac{a}{9(2\alpha-1)f_k^2(\alpha)} \cdot \alpha^{-(n+m-2)} \cdot 10^\ell - 1. \tag{4.42}$$

As seen previously, we have $\Gamma'_1 = \Gamma_1 \neq 0$. To apply Theorem 1.26 to Γ'_1, we take

$$(\eta_1, b_1) := (a/(9(2\alpha-1)f_k^2(\alpha)), 1), \quad (\eta_2, b_2) := (\alpha, -(n+m-2)), \quad (\eta_3, b_3) := (10, \ell).$$

The algebraic numbers η_1, η_2, η_3 are elements of the number field $\mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. So, as in the previous section, we take

$$A_1 := 12k\log k, \quad A_2 := \log 2, \quad A_3 := k\log 10, \quad \text{and} \quad B := 2m.$$

Now, we apply Theorem 1.26 to Γ'_1 and compare the resulting inequality with (4.41) to obtain

$$n\log\alpha < 7.2 \cdot 10^{12} k^4 \log^2 k(1 + \log 2m). \tag{4.43}$$

We return to equation (4.1) and rewrite it as follows

$$F_n^{(k)}((2\alpha-1)f_k(\alpha)\alpha^{m-1} + e'_k(m)) = \frac{a(10^\ell-1)}{9},$$

i.e.,

$$(2\alpha-1)f_k(\alpha)\alpha^{m-1} - \frac{a10^\ell}{9F_n^{(k)}} = \frac{-a}{9F_n^{(k)}} - e'_k(m). \tag{4.44}$$

So, we get

$$\left|(2\alpha-1)f_k(\alpha)\alpha^{m-1}-\frac{a10^\ell}{9F_n^{(k)}}\right| \leqslant \frac{a}{9F_n^{(k)}}+\frac{3}{2}\leqslant\frac{5}{2}.$$

Dividing through by $(2\alpha-1)f_k(\alpha)\alpha^{m-1}$, we get

$$|\Gamma_2'| \leqslant \frac{5}{2(2\alpha-1)f_k(\alpha)\alpha^{m-1}} < \frac{5\alpha}{2\alpha^m} < \frac{5}{\alpha^m}, \tag{4.45}$$

where

$$\Gamma_2' := \frac{a}{9(2\alpha-1)F_n^{(k)}f_k(\alpha)}\cdot\alpha^{-(m-1)}\cdot 10^\ell - 1. \tag{4.46}$$

We have $\Gamma_2' \neq 0$. Now, we will apply Theorem 1.26 to Γ_2' by fixing

$$(\eta_1, b_1) := (a/(9(2\alpha-1)F_n^{(k)}f_k(\alpha)), 1),\ (\eta_2, b_2) := (\alpha, -(m-1)),\ (\eta_3, b_3) := (10, \ell).$$

It is obvious that $\mathbb{K} := \mathbb{Q}(\alpha)$ contains η_1, η_2, η_3 and $d_{\mathbb{K}} = k$. As calculated before, we take

$$A_2 := \log 2, \quad A_3 := k\log 10, \quad \text{and} \quad B := 2m.$$

We need to compute A_1. The estimates (2.5), (4.43), and the proprieties (1.1)-(1.3) imply that

$$\begin{aligned} h(\eta_1) &\leqslant h\left(\frac{a}{9}\right) + h(2\alpha-1) + h(F_n^{(k)}) + h(f_k(\alpha)) \\ &< \log 9 + \log 3 + (n-1)\log\alpha + \log k \\ &< 7.3\cdot 10^{12}k^4\log^2 k(1+\log 2m), \end{aligned}$$

for $k \geqslant 2$. On the other hand, since

$$\eta_1 := \frac{a}{9(2\alpha-1)F_n^{(k)}f_k(\alpha)} < 1 \quad \text{and} \quad \eta_1^{-1} = \frac{9(2\alpha-1)F_n^{(k)}f_k(\alpha)}{a} < \frac{81\alpha^{(n-1)}}{4},$$

then, by (4.43), we get

$$|\log\eta_1| < (n-1)\log\alpha + \log 20.3 < 7.3\cdot 10^{12}k^4\log^2 k(1+\log 2m).$$

Thus, we conclude that

$$\max\{kh(\eta_1), |\log\eta_1|, 0.16\} < 7.3\cdot 10^{12}k^5\log^2 k(1+\log 2m) := A_1.$$

Applying Theorem 1.26 and comparing the resulting inequality with (4.45) give

$$m\log\alpha < 2.9\cdot 10^{25}k^8\log^3 k\log^2 m,$$

where we have used the facts $1+\log k < 2.5\log k$ and $1+\log 2m < 2.6\log m$, which hold, for $k \geqslant 2$ and $m \geqslant 3$. So, we obtain

$$\frac{m}{\log^2 m} < 7.2\times 10^{25}k^8\log^3 k. \tag{4.47}$$

We use (4.17) by putting $A := 7.2\times 10^{25}k^8\log^3 k$ and the fact that $59.6+8\log k+3\log\log k < 93\log k$, which holds for all $k \geqslant 2$, to obtain

$$\begin{aligned} m &< 4(7.2\times 10^{25}k^8\log^3 k)(\log(7.2\times 10^{25}k^8\log^3 k))^2 \\ &< (3\times 10^{26}k^8\log^3 k)(59.6+8\log k+3\log\log k)^2 \\ &< 2.6\times 10^{30}k^8\log^5 k. \end{aligned}$$

This gives (4.39) and completes the proof of Lemma 4.4. □

4.5.3 The case $2 \leqslant k \leqslant 430$

For this subsection, we consider $k \in [2, 430]$. Define

$$\Lambda_1' := \log(\Gamma_1' + 1) = \ell \log 10 - (n+m-2)\log\alpha + \log(a/(9(2\alpha-1)f_k^2(\alpha))). \tag{4.48}$$

Suppose that $n \geqslant 10$, so by the estimate (4.41) and the fact that $\alpha > 1.5$, we have $|\Gamma_1'| < 0.21$. Taking $d = 0.21$ in Lemma 1.30, we obtain

$$|\Lambda_1'| < \frac{-\log 0.79}{0.21} \cdot |\Gamma_1'| < 13.5 \cdot \alpha^{-n}. \tag{4.49}$$

So, we get

$$\left|\ell \cdot \frac{\log 10}{\log\alpha} - (n+m-2) + \frac{\log(a/(9(2\alpha-1)f_k^2(\alpha)))}{\log\alpha}\right| < 33.3 \cdot \alpha^{-n}. \tag{4.50}$$

For $a \in \{1, \ldots, 9\}$ and $k \in [2, 430]$, we apply Lemma 1.27 to Λ_1'. For this application, we put

$$\gamma := \frac{\log 10}{\log\alpha}, \quad \mu := \frac{\log(a/(9(2\alpha-1)f_k^2(\alpha)))}{\log\alpha}, \quad \text{and} \quad (A, B) := (33.3, \alpha).$$

For each $k \in [2, 430]$ and $a \in \{1, \ldots, 9\}$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k\|\gamma q\| > 0$, where $M_k := \lfloor 2.6 \times 10^{30} k^8 \log^5 k \rfloor$, which is an upper bound of ℓ from Lemma 4.4. After doing this, we use Lemma 1.27 on inequality (4.50). Using Mathematica, we find that q_{121} fulfills the conditions of Lemma 1.27. A computer program with Mathematica reveals for $k = 402$ and $a = 7$ that $\varepsilon > 0.0011$ and the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$, for $k \in [2, 430]$ and $a \in \{1, \ldots, 9\}$, is 200.757. This is an upper bound of n by Lemma 1.27.

For $2 \leqslant n < 201$, we consider

$$\Lambda_2' := \log(\Gamma_2' + 1) = \ell \log 10 - (m-1)\log\alpha + \log(a/(9(2\alpha-1)F_n^{(k)} f_k(\alpha))). \tag{4.51}$$

Suppose that $m \geqslant 10$, then by the estimate (4.45) and the fact that $\alpha > 1.5$, we have $|\Gamma_2'| < 0.09$. Taking $d = 0.09$ in Lemma 1.30, we obtain

$$|\Lambda_2'| < \frac{-\log 0.91}{0.09} \cdot |\Gamma_2'| < 5.3 \cdot \alpha^{-m}.$$

So, we get

$$\left|\ell \cdot \frac{\log 10}{\log\alpha} - (m-1) - \frac{\log(a/(9(2\alpha-1)F_n^{(k)} f_k(\alpha)))}{\log\alpha}\right| < 13.1 \cdot \alpha^{-m}. \tag{4.52}$$

For $a \in \{1, \ldots, 9\}$ and $(k, n) \in [2, 430] \times [2, 200]$, we apply Lemma 1.27 to Λ_2' by taking

$$\gamma := \frac{\log 10}{\log\alpha}, \quad \mu := \frac{\log(a/(9(2\alpha-1)F_n^{(k)} f_k(\alpha)))}{\log\alpha}, \quad \text{and} \quad (A, B) := (13.1, \alpha).$$

Again, for each $(k, n) \in [2, 430] \times [2, 200]$ and $a \in \{1, \ldots, 9\}$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon(k) = \|\mu q\| - M_k\|\gamma q\| > 0$, where $M_k := \lfloor 2.6 \times 10^{30} k^8 \log^5 k \rfloor$, which is an upper bound of ℓ from Lemma 4.4. After doing this, we use Lemma 1.27 on inequality (4.52). Using Mathematica again, we see that q_{123} confirms the conditions of Lemma 1.27. A computer program with

Mathematica proves, for $k = 427$, $a = 1$ and $n = 58$, that $\varepsilon > 0.000019$ and the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$, for $(k, n) \in [2, 430] \times [2, 200]$ and $a \in \{1, \ldots, 9\}$, is 208.792. This is also an upper bound of m by Lemma 1.27.

Hence, we deduce that the possible solutions (a, k, l, m, n) of equation (4.1), for which $k \in [2, 430]$ and $a \in \{1, \ldots, 9\}$, satisfy $n \leqslant m \leqslant 208$. Therefore, we use inequality (4.6) to obtain $\ell \leqslant 207$.

Finally, we use Mathematica to compare $F_n^{(k)} L_m^{(k)}$ and $\dfrac{a(10^\ell - 1)}{9}$, for the ranges $k + 1 \leqslant m \leqslant 208$, $n \leqslant m$, $a \in \{1, \ldots, 9\}$ and $2 \leqslant \ell \leqslant 207$, with $\ell < m$ and check that the only solutions of equation (4.1) are those given in Theorem 4.2.

4.5.4 The case $k > 430$

For $k > 430$, it is easy to check that

$$n \leqslant m < 2.6 \cdot 10^{30} k^8 \log^5 k < 2^{k/2}.$$

Thus, by Lemmas 1.18 and 1.19, $F_n^{(k)}$and $L_m^{(k)}$ can be respectively rewritten as

$$F_n^{(k)} = 2^{n-2}(1 + \zeta_1), \quad \text{where } |\zeta_1| < \frac{2}{2^{k/2}} \tag{4.53}$$

and

$$L_m^{(k)} = 3 \cdot 2^{m-2}(1 + \zeta_2), \quad \text{where } |\zeta_2| < \frac{1}{2^{k/2}}. \tag{4.54}$$

Substituting (4.53) and (4.54) in (4.1) and using the fact that $m \geqslant 3$, we obtain

$$|\Gamma_3'| < \frac{5.5}{2^{\min\{k/2, n\}}}, \tag{4.55}$$

where

$$\Gamma_3' = \frac{a}{27} \cdot 2^{-(n+m-4)} \cdot 10^\ell - 1.$$

One can check that $\Gamma_3' \neq 0$. Thus, we apply Theorem 1.26 by taking

$$(\eta_1, b_1) := \left(\frac{a}{27}, 1\right), \quad (\eta_2, b_2) := (2, -(n + m - 4)), \quad (\eta_3, b_3) := (10, \ell).$$

We take $d_{\mathbb{K}} = 1$, $A_1 := \log 27$, $A_2 := \log 2, A_3 := \log 10$, and $B := 2m$. Thus, applying Theorem 1.26 and comparing the resulting inequality with (4.55), we obtain

$$\min\{k/2, n\} < 3 \cdot 10^{12} \log m.$$

By Lemma 4.4 and using the fact that $70.1 + 8 \log k + 5 \log\log k < 21 \log k$, for all $k > 430$, we get

$$\min\{k/2, n\} < 6.3 \cdot 10^{13} \log k.$$

If $\min\{k/2, n\} = k/2$, then we get $k < 1.3 \cdot 10^{14} \log k$. Solving this inequality and using Lemma 4.4, we conclude that

$$k < 4.7 \cdot 10^{15} \quad \text{and} \quad m < 3.8 \cdot 10^{163}. \tag{4.56}$$

If $\min\{k/2, n\} = n$, then we obtain

$$n < 6.3 \cdot 10^{13} \log k. \tag{4.57}$$

Now, we go back to (4.44) and we rewrite it as

$$3 \cdot 2^{m-2} - \frac{a10^\ell}{9F_n^{(k)}} = \frac{-a}{9F_n^{(k)}} - 3 \cdot 2^{m-2}\zeta_2.$$

Taking the absolute value and dividing through by $3 \cdot 2^{m-2}$, we use the fact that $m \geqslant k+1$ to get

$$|\Gamma_4'| \leqslant \frac{1}{3 \cdot 2^{m-2}} + \frac{1}{2^{k/2}} < \frac{1}{3 \cdot 2^{k-2}} + \frac{1}{2^{k/2}} < \frac{2}{2^{k/2}}, \tag{4.58}$$

where

$$\Gamma_4' := \frac{a}{27F_n^{(k)}} \cdot 2^{-(m-2)} \cdot 10^\ell - 1. \tag{4.59}$$

As above, we show that $\Gamma_4' \neq 0$, since if not we would obtain $\frac{a10^\ell}{27F_n^{(k)}} = 2^{m-2}$ or in this case $F_n^{(k)} = 2^{n-2}$ and so we would obtain $\frac{a10^\ell}{27 \cdot 2^{n-2}} = 2^{m-2}$. This is impossible. Now, we apply Theorem 1.26 by fixing

$$(\eta_1, b_1) := (a/(27F_n^{(k)}), 1), \quad (\eta_2, b_2) := (2, -(m-2)), \quad (\eta_3, b_3) := (10, \ell).$$

As calculated before, we take $A_2 := \log 2$, $A_3 := \log 10$, and $B := m$. Next, we estimate $h(\eta_1)$. As $F_n^{(k)} < \alpha^{n-1}$ and using the inequality (4.57), we obtain

$$h(\eta_1) \leqslant h(a/27) + h(F_n^{(k)}) < \log 27 + (n-1)\log\alpha < 4.4 \cdot 10^{13} \log k.$$

So, we choose $A_1 := 4.4 \cdot 10^{13} \log k$. Therefore, Theorem 1.26 gives

$$|\Gamma_4'| > \exp(-2.1 \cdot 10^{25} \log k \log m), \tag{4.60}$$

where we have used the fact that $1 + \log m < 2\log m$, which holds, for all $m \geqslant 3$. From (5.27) and (5.29), we deduce that

$$k < 6.1 \cdot 10^{25} \log k \log m.$$

By Lemma 4.4 and using the fact that $70.1 + 8\log k + 5\log\log k < 21\log k$, for all $k > 430$, we get

$$\begin{aligned} k &< 6.1 \cdot 10^{25} \log k \left(\log(2.6 \cdot 10^{30} k^8 \log^5 k)\right) \\ &< 6.1 \cdot 10^{25} \log k (70.1 + 8\log k + 5\log\log k) \\ &< 1.3 \cdot 10^{27} \log^2 k. \end{aligned}$$

Solving this inequality and using Lemma 4.4, we obtain

$$k < 6.55 \cdot 10^{30} \quad \text{and} \quad m < 1.6 \cdot 10^{286}. \tag{4.61}$$

From (4.56) and (4.61), we conclude that inequalities (4.61) always hold. The obtained bounds are very large, next we will reduce them. Put

$$\Lambda_3' := \log(\Gamma_3' + 1) = \ell \log 10 - (n+m-4)\log 2 + \log(a/27). \tag{4.62}$$

Assume that $n \geqslant 10$, we get then $|\Gamma_3'| < 0.02$. Taking $d = 0.02$ in Lemma 1.30, we obtain

$$|\Lambda_3'| < \frac{-\log 0.98}{0.02} \cdot |\Gamma_3'| < 6 \cdot 2^{-\min\{k/2, n\}}.$$

So, we get

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n+m-4) + \frac{\log(a/27)}{\log 2} \right| < 9 \cdot 2^{-\min\{k/2, n\}}. \tag{4.63}$$

For $a \in \{1, \ldots, 9\}$, we apply Lemma 1.27 to Λ_3' with the data

$$\gamma := \frac{\log 10}{\log 2}, \quad \mu := \frac{\log(a/27)}{\log 2}, \quad \text{and} \quad (A, B) := (9, 2).$$

We now want to reduce our bound of k, which is too large, by using Lemma 1.27. We take $M := 1.6 \cdot 10^{286}$, which is an upper bound of ℓ by (4.38) and (4.61). After, we use Lemma 1.27 on (4.63) in order to obtain an upper bound of k. A computer search with Maple shows that q_{580} satisfies the conditions of Lemma 1.27, for all $a \in \{1, \ldots, 9\}$. Thus, the application of Lemma 1.27 leads to the different results presented in the following table.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geqslant$	0.45	0.45	0.03	0.45	0.45	0.03	0.44	0.45	0.48
$\min\{k/2, n\} \leqslant$	961	961	965	961	961	965	961	961	961

According to the above results, we see that $\min\{k/2, n\} < 966$ across all cases.

Case 1: $\min\{k/2, n\} = k/2$**.** In this case, we get

$$k < 1932. \tag{4.64}$$

Case 2: $\min\{k/2, n\} = n$**.** In this case, we obtain $n \leqslant 965$. For $2 \leqslant n \leqslant 965$, let

$$\Lambda_4' := \log(\Gamma_4' + 1) = \ell \log 10 - (m - 2) \log 2 + \log(a/(27F_n^{(k)})).$$

Since $k > 430$, then from (4.58), we have $|\Gamma_4'| < 0.01$. Hence by Lemma 1.30, we obtain

$$|\Lambda_4'| < -\frac{\log(0.99)}{0.01} \cdot |\Gamma_4'| < 2.02 \cdot 2^{-k/2}. \tag{4.65}$$

So, we get

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (m - 2) + \frac{\log(a/(27F_n^{(k)}))}{\log 2} \right| < 3 \cdot 2^{-\frac{k}{2}}. \tag{4.66}$$

Again, for $a \in \{1, \ldots, 9\}$ and $2 \leqslant n \leqslant 965$, we apply Lemma 1.27 to Λ_4' with the parameters

$$\gamma := \frac{\log 10}{\log 2}, \quad \mu := \frac{\log(a/(27F_n^{(k)}))}{\log 2}, \quad M := 1.6 \cdot 10^{286}, \quad \text{and} \quad (A, B) := (3, 2).$$

Note that $n < k/2$. This implies that $n \leqslant k + 1$, which holds for $k \geqslant 2$ and we can replace $F_n^{(k)}$ by 2^{n-2} in our calculations. We use again Maple to see that q_{580} satisfies the conditions of Lemma 1.27, for all $a \in \{1, \ldots, 9\}$ and $2 \leqslant n \leqslant 965$. Hence, applying Lemma 1.27 leads us to the following varied results which hold for all $2 \leqslant n \leqslant 965$.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geqslant$	0.45	0.45	0.03	0.45	0.45	0.03	0.44	0.45	0.48
$k/2 \leqslant$	959	959	963	959	959	963	959	959	959

The obtained results reveals across all instances that $k \leqslant 1926$. As a result, in both cases, we have $k < 1932$.

With this new bound, we get $m < 1.25 \cdot 10^{61}$. So, we apply again Lemma 1.27 with the same above data but this time we take $M := 1.25 \cdot 10^{61}$. With the help of Maple, we see that q_{140} satisfies the conditions of Lemma 1.27 for all $a \in \{1, \ldots, 9\}$. In the context of this application which yields to the different results presented in the below table, we find that $\min\{k/2, n\} < 235$.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geqslant$	0.33	0.33	0.46	0.33	0.33	0.46	0.24	0.33	0.23
$\min\{k/2, n\} \leqslant$	233	233	233	233	233	233	233	233	234

For the first case, we obtain that $k < 470$ and for the second case we use again q_{140} which satisfies the conditions of Lemma 1.27 for all $a \in \{1, \ldots, 9\}$ and $2 \leqslant n \leqslant 234$ to obtain the following results which hold for all $2 \leqslant n \leqslant 234$ and reveals that $k \leqslant 464$.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geqslant$	0.33	0.33	0.46	0.33	0.33	0.46	0.24	0.33	0.23
$k/2 \leqslant$	231	231	231	231	231	231	232	231	232

So, in both cases we have $k < 470$. With this new bound, we get $m < 5.5 \cdot 10^{55}$. So, we apply again Lemma 1.27 with the same above data but this time we take $M := 5.5 \cdot 10^{55}$. In this case, q_{128} satisfies the conditions of Lemma 1.27, for all $a \in \{1, \ldots, 9\}$. The third application of Lemma 1.27 leads to the following results from which we observe that $\min\{k/2, n\} < 209$, which holds in all cases.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geqslant$	0.27	0.27	0.18	0.27	0.27	0.18	0.24	0.27	0.09
$\min\{k/2, n\} \leqslant$	207	207	207	207	207	207	207	207	208

The first case gives us that $k < 418$ and for the second case we use again q_{128} which satisfies the conditions of Lemma 1.27, for all $a \in \{1, \ldots, 9\}$ and $2 \leqslant n \leqslant 208$. This application yields to the following results which hold, for all $2 \leqslant n \leqslant 208$ and tell us that $k < 415$.

a	1	2	3	4	5	6	7	8	9
$\varepsilon \geqslant$	0.27	0.27	0.18	0.27	0.27	0.18	0.24	0.27	0.09
$k/2 \leqslant$	205	205	206	205	205	206	205	205	207

Consequently, we deduce that in both cases we have $k < 418$, which contradicts our assumption that $k > 430$. This completes the proof of Theorem 4.2.

Chapter 5

On Equations involving Repdigits and Products of Two k-Fibonacci

In this chapter, we will show that $F_n^{(k)}F_m^{(k)}$ can represent a repdigit, where n and m are two positive integers. This chapter is derived from the paper [56].

5.1 Motivation and main result

In [29], Erduvan and Keskin investigated repdigits as products of two Fibonacci and two Lucas numbers. In this chapter, we search for repdigits which are the product of two k-Fibonacci numbers. Our main result is given by the following theorem.

Theorem 5.1. *The only solution of the Diophantine equation*

$$F_n^{(k)}F_m^{(k)} = \frac{a(10^\ell - 1)}{9} \tag{5.1}$$

in positive integers n, m, ℓ, k, and a with $3 \leqslant m \leqslant n$, $k \geqslant 3$, $\ell \geqslant 2$, and $1 \leqslant a \leqslant 9$, is

$$(a, k, l, m, n) = (8, 3, 2, 3, 8).$$

We set the condition $m \geqslant 3$, as when $m \in \{1, 2\}$ then the Diophantine equation (5.1) transforms into $F_n^{(k)} = \dfrac{a(10^\ell - 1)}{9}$ and this problem was already treated in [17, 40, 41].

5.2 The proof of Theorem 5.1

5.2.1 Preliminary considerations

We start our examination of (5.1), considering the range $2 \leqslant m \leqslant n \leqslant k + 1$. Within this context, we have $F_n^{(k)} = 2^{n-2}$ and $F_m^{(k)} = 2^{m-2}$. Hence, Equation (5.1) transforms into

$$2^{n+m-4} = \frac{a(10^\ell - 1)}{9}. \tag{5.2}$$

For any rational number x, let $\nu_2(x)$ denote the 2-adic valuation of x. Since $\nu_2(a(10^\ell - 1)/9) \leqslant 3$, then by comparing the 2-adic valuation on both sides of (5.2), one gets $2 \leqslant m \leqslant n \leqslant 7$. In this specified range, Equation (5.2) does not possess solutions. Thus, from now, we proceed under the condition $n \geqslant k + 2 \geqslant 5$.

Following this, we will determine the correlation between the sizes of ℓ and n. Using inequalities (2.9) and $10^{\ell-1} < a(10^{\ell}-1)/9$, we obtain

$$10^{\ell-1} < \frac{a(10^{\ell}-1)}{9} = F_n^{(k)} F_m^{(k)} < \alpha^{n+m-2} < \alpha^{2n-2}.$$

Consequently, we get

$$\ell < (2n-2)\left(\frac{\log \alpha}{\log 10}\right) + 1 = n\left(\frac{2\log \alpha}{\log 10}\right) - \left(\frac{2\log \alpha}{\log 10}\right) + 1.$$

Moreover, utilizing (2.3), we get

$$\ell < n. \tag{5.3}$$

5.2.2 An inequality for n versus k

Now, we illustrate the following lemma which provides an inequality relating n to k.

Lemma 5.2. *If (a, k, ℓ, m, n) is a solution in integers of Equation (5.1) with $k \geqslant 3$ and $n \geqslant k+2$, then the inequality*

$$n < 1.64 \times 10^{29} k^8 \log^5 k \tag{5.4}$$

holds.

Proof. Employing estimate (2.7), Equation (5.1) can be expressed as follows:

$$(f_k(\alpha)\alpha^{n-1} + e_k(n))(f_k(\alpha)\alpha^{m-1} + e_k(m)) = a\left(\frac{10^{\ell}-1}{9}\right), \tag{5.5}$$

i.e.,

$$f_k^2(\alpha)\alpha^{n+m-2} - \frac{a10^{\ell}}{9} = -e_k(m)f_k(\alpha)\alpha^{n-1} - e_k(n)f_k(\alpha)\alpha^{m-1} - e_k(n)e_k(m) - \frac{a}{9}.$$

By taking the absolute value, dividing both sides by $f_k^2(\alpha)\alpha^{n+m-2}$, and taking into account the fact that $f_k(\alpha) > 1/2$, along with (2.8), we arrive at

$$\left|\frac{a}{9f_k^2(\alpha)} \cdot \alpha^{-(n+m-2)} \cdot 10^{\ell} - 1\right| \leqslant \frac{1}{\alpha^{m-1}} + \frac{1}{\alpha^{n-1}} + \frac{5}{\alpha^{n+m-2}} < \frac{7}{\alpha^{m-1}}. \tag{5.6}$$

Define

$$\Gamma_1 := \frac{a}{9f_k^2(\alpha)} \cdot \alpha^{-(n+m-2)} \cdot 10^{\ell} - 1. \tag{5.7}$$

Consequently, we obtain

$$|\Gamma_1| < \frac{7}{\alpha^{m-1}}. \tag{5.8}$$

We have $\Gamma_1 \neq 0$, because if we suppose that $\Gamma_1 = 0$, we would get

$$\frac{a10^{\ell}}{9} = f_k^2(\alpha)\alpha^{(n+m-2)}.$$

After applying an automorphism from the Galois group of the decomposition field $\Psi(x)$ over $\mathbb{Q}$ to the above relation and then taking absolute values, we conclude that for any $i \geqslant 2$, we have

$$\frac{100}{9} \leqslant \frac{a10^{\ell}}{9} = |f_k(\alpha_i)|^2 \cdot |\alpha_i|^{n+m-2} < 1,$$

resulting in a contradiction. With the goal of applying Theorem 1.26 to Γ_1 given by (5.7), the parameters can be chosen as:

$$(\eta_1, b_1) := ((a/(9f_k^2(\alpha)), 1), \quad (\eta_2, b_2) := (\alpha, -(n+m-2)), \quad (\eta_3, b_3) := (10, \ell).$$

Since the algebraic numbers η_1, η_2, η_3 are members of $\mathbb{K} := \mathbb{Q}(\alpha)$, it follows that $d_{\mathbb{K}} = k$. Next, we estimate the usual absolute logarithmic heights of η_1 followed by that of η_2 and η_3. Using estimate (2.5) and the properties (1.1) and(1.3), we see that for all $k \geqslant 3$,

$$\begin{aligned} h(\eta_1) &\leqslant h(a/9) + 2h(f_k(\alpha)) \\ &< \log 9 + 4\log k \\ &< 6.1\log k. \end{aligned}$$

$h(\eta_2) = (\log\alpha)/k < (\log 2)/k$ and $h(\eta_3) = \log 10$. Then, we can choose

$$A_1 := 6.1k\log k = \max\{kh(\eta_1), |\log\eta_1|, 0.16\}$$

$$A_2 := \log 2 = \max\{kh(\eta_2), |\log\eta_2|, 0.16\}$$

and

$$A_3 := k\log 10 = \max\{kh(\eta_3), |\log\eta_3|, 0.16\}.$$

Ultimately, given $m < n$ and Inequality (5.3), we can determine that $B := 2n$. Therefore, Theorem 1.26 gives

$$\begin{aligned} |\Gamma_1| &> \exp\left(-1.4\cdot 30^6\cdot 3^{4.5}\cdot k^2(1+\log k)(1+\log 2n)(6.1k\log k)(\log 2)(k\log 10)\right) \\ &> \exp\left(-2.8\cdot 10^{12}k^4\log^2 k(1+\log 2n)\right), \end{aligned}$$

where we have employed the fact that $1+\log k < 2\log k$, which holds for $k \geqslant 3$. Comparing this lower bound with the upper bound of $|\Gamma_1|$ as given in (5.8), we reach

$$(m-1)\log\alpha < 2.9\cdot 10^{12}k^4\log^2 k(1+\log 2n). \tag{5.9}$$

We return to Equation (5.1) and we use again (2.7) to reformulate it as

$$(f_k(\alpha)\alpha^{n-1} + e_k(n))F_m^{(k)} = \frac{a(10^\ell - 1)}{9},$$

i.e.,

$$f_k(\alpha)\alpha^{n-1} - \frac{a10^\ell}{9F_m^{(k)}} = -\frac{a}{9F_m^{(k)}} - e_k(n). \tag{5.10}$$

Following the previous steps, by taking the absolute value and dividing through by $f_k(\alpha)\alpha^{n-1}$, we get

$$\left|\frac{a}{9F_m^{(k)}f_k(\alpha)}\cdot\alpha^{-(n-1)}\cdot 10^\ell - 1\right| \leqslant \frac{3}{2f_k(\alpha)\alpha^{n-1}} < \frac{3\alpha}{\alpha^n} < \frac{6}{\alpha^n}. \tag{5.11}$$

Define

$$\Gamma_2 := \frac{a}{9F_m^{(k)}f_k(\alpha)}\cdot\alpha^{-(n-1)}\cdot 10^\ell - 1. \tag{5.12}$$

Hence, we see that

$$|\Gamma_2| < \frac{6}{\alpha^n}. \tag{5.13}$$

As above, we use the same argument to show that $\Gamma_2 \neq 0$.

Now, we will apply Theorem 1.26 to Γ_2 by fixing the following parameters:

$$(\eta_1, b_1) := (a/(9F_m^{(k)}f_k(\alpha)), 1), \quad (\eta_2, b_2) := (\alpha, -(n-1)), \quad (\eta_3, b_3) := (10, \ell).$$

Once more, considering $\mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. As before, we can select

$$A_2 := \log 2 \quad \text{and} \quad A_3 := k \log 10.$$

Now, We still need to determine A_1. Using the estimates (2.6), (5.9), along with properties (1.1)-(1.3), we deduce that, for all $k \geqslant 3$, we obtain

$$\begin{aligned} h(\eta_1) &\leqslant h\left(\frac{a}{9}\right) + h(F_m^{(k)}) + h(f_k(\alpha)) \\ &< \log 9 + (m-1)\log\alpha + 2\log k \\ &< 3 \times 10^{12} k^4 \log^2 k (1 + \log 2n). \end{aligned}$$

Conversely, given that

$$\eta_1 := \frac{a}{9F_m^{(k)} f_k(\alpha)} < 2 \quad \text{and} \quad \eta_1^{-1} := \frac{9F_m^{(k)} f_k(\alpha)}{a} < \frac{27\alpha^{m-1}}{4},$$

then, by (5.9), we derive

$$|\log \eta_1| < (m-1)\log\alpha + \log 6.75 < 3 \times 10^{12} k^4 \log^2 k(1 + \log 2n).$$

This indicates that

$$\max\{kh(\eta_1), |\log \eta_1|, 0.16\} < 3 \times 10^{12} k^5 \log^2 k(1 + \log 2n) := A_1.$$

Opting for $B := 2n$, we can apply Theorem 1.26 to Γ_2 and then compare the resulting inequality with (5.11) to obtain

$$n \log \alpha < 6.1 \times 10^{24} k^8 \log^3 k \log^2 n,$$

where we have used the inequalities $1 + \log k < 2\log k$ and $(1 + \log 2n) < 2.1 \log n$, which are valid for $k \geqslant 3$ and $n \geqslant 5$. For smaller values of n (i.e., $n = 1, 2, 3, 4$), the inequality $(1 + \log 2n) < 2.1 \log n$ does not hold as the left-hand side exceeds the right-hand side in these cases. Therefore, we find

$$\frac{n}{\log^2 n} < 1.1 \times 10^{25} k^8 \log^3 k. \tag{5.14}$$

It is evident that the inequality

$$\frac{x}{\log^2 x} < A \text{ implies } x < 4A\log^2 A, \text{ whenever } A \geqslant 100, \tag{5.15}$$

which is derived from [53, Lemma 7] for $m = 2$. Hence, substituting $A := 1.1 \times 10^{25} k^8 \log^3 k$ in Inequality (5.15) and applying the inequality $57.7 + 8\log k + 3\log\log k < 61\log k$, valid for all $k \geqslant 3$, we obtain

$$\begin{aligned} n &< 4(1.1 \times 10^{25} k^8 \log^3 k)(\log(1.1 \times 10^{25} k^8 \log^3 k))^2 \\ &< (4.4 \times 10^{25} k^8 \log^3 k)(57.7 + 8\log k + 3\log\log k)^2 \\ &< 1.64 \times 10^{29} k^8 \log^5 k. \end{aligned}$$

On this, the proof of Lemma 5.2 is complete. □

5.2.3 The case $3 \leqslant k \leqslant 430$

Within this subsection, our focus lies on examining the small values of k, specifically in the range $[3, 430]$. Define

$$\Lambda_1 := \log(\Gamma_1 + 1) = \ell \log 10 - (n + m - 2)\log\alpha + \log(a/(9f_k^2(\alpha))). \tag{5.16}$$

Assume that $m \geqslant 10$. With the help of estimate (5.8) and the use of the fact that $\alpha > 1.75$, we get $|\Gamma_1| < 0.05$. Putting $d = 0.05$ in Lemma 1.30, we obtain

$$|\Lambda_1| < \frac{-\log 0.95}{0.05} \cdot |\Gamma_1| < 14.4 \cdot \alpha^{-m}. \tag{5.17}$$

Thus, we get

$$\left| \ell \cdot \frac{\log 10}{\log \alpha} - (n + m - 2) + \frac{\log(a/(9f_k^2(\alpha)))}{\log \alpha} \right| < 25.8 \cdot \alpha^{-m}. \tag{5.18}$$

We apply Lemma 1.27 to Λ_1, for each $a \in \{1, \cdots, 9\}$ and $k \in [3, 430]$, by taking as parameters

$$\gamma := \frac{\log 10}{\log \alpha}, \quad \mu_{(k,a)} := \frac{\log(a/(9f_k^2(\alpha)))}{\log \alpha}, \quad \text{and} \quad (A, B) := (25.8, \alpha).$$

For each $k \in [3, 430]$ and $a \in \{1, \ldots, 9\}$, we find a reliable approximation of γ. Additionally, we obtain a convergent p_i/q_i of the continued fraction of γ, satisfying the condition $q_i > 6M_k$ and $\varepsilon = \varepsilon_{(k,a)} = \|\mu_{(k,a)} q_i\| - M_k \|\gamma q_i\| > 0$. Here, $M_k := \lfloor 1.64 \times 10^{29} k^8 \log^5 k \rfloor$, representing an upper bound for ℓ as derived from Lemma 5.2. Using Mathematica we determine that q_{185} fulfills the conditions specified in Lemma 1.27. After completing this step, Lemma 1.27 is applied to Inequality (5.18). By employing a computer program with Mathematica, it was determined for $k = 430$ and $a = 9$ that $\varepsilon > 1.02 \times 10^{-36}$ and the highest value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$ across all $k \in [3, 430]$ and $a \in \{1, \ldots, 9\}$ is 425.623. This value serves as an upper bound of m as dictated by Lemma 1.27.

Note that for the remaining values of k and a, it is observed that the corresponding results yield significantly smaller values compared to the chosen upper bound for m of 425.623.

Let's set $3 \leqslant m < 426$ and consider

$$\Lambda_2 := \log(\Gamma_2 + 1) = \ell \log 10 - (n - 1)\log\alpha + \log(a/(9F_m^{(k)} f_k(\alpha))). \tag{5.19}$$

Assuming $n \geqslant 10$, with the given estimate (5.13) and considering $\alpha > 1.75$, it follows that $|\Gamma_2| < 0.03$. Substituting $d = 0.03$ in Lemma 1.30, we derive

$$|\Lambda_2| < \frac{-\log 0.97}{0.03} \cdot |\Gamma_2| < 6.1 \cdot \alpha^{-n}.$$

Hence, we get

$$\left| \ell \cdot \frac{\log 10}{\log \alpha} - (n - 1) - \frac{\log(a/(9F_m^{(k)} f_k(\alpha)))}{\log \alpha} \right| < 11 \cdot \alpha^{-n}. \tag{5.20}$$

In view to apply Lemma 1.27 to Λ_2, for all $a \in \{1, \cdots, 9\}$ and $3 \leqslant m \leqslant 425$, we consider

$$\gamma := \frac{\log 10}{\log \alpha}, \quad \mu_{(k,m,a)} := \frac{\log(a/(9F_m^{(k)} f_k(\alpha)))}{\log \alpha}, \quad \text{and} \quad (A, B) := (11, \alpha).$$

Once more, considering each pair $(k, m) \in [3, 430] \times [3, 425]$, and $a \in \{1, \ldots, 9\}$, we find a reliable approximation of γ and a convergent p_i/q_i of the continued fraction of γ ensuring $q_i > 6M_k$ and $\varepsilon = \varepsilon_{(k,m,a)} = \|\mu_{(k,m,a)} q_i\| - M_k \|\gamma q_i\| > 0$, where $M_k := \lfloor 1.64 \times 10^{29} k^8 \log^5 k \rfloor$, serving as an upper bound of ℓ obtained from Lemma 5.2. Again, we use Mathematica to verify that q_{179} satisfies the conditions of Lemma 1.27. Next, applying Lemma 1.27 on Inequality (5.20). Our Mathematica computation revealed for $k = 409$, $a = 9$, and $m = 3$ that $\varepsilon > 2.22 \times 10^{-34}$ and the highest value attained by $\dfrac{\log(Aq/\varepsilon)}{\log B}$ across all $(k, m) \in [3, 430] \times [3, 425]$ and $a \in \{1, \ldots, 9\}$ is 425.634, serving as an upper bound for n by Lemma 1.27. The other upper bounds for n obtained with the remaining values of k, m, and a fall substantially below the established upper bound for n of 425.634.

Therefore, we conclude that the possible solutions (a, k, l, m, n) of Equation (5.1), where $k \in [3, 430]$ and $a \in \{1, \cdots, 9\}$, adhere to $m \leqslant n \leqslant 425$. Hence, utilizing Inequality (5.3), we derive $\ell \leqslant 424$.

Finally, we use Mathematica to manage a comparative analysis between $F_n^{(k)} F_m^{(k)}$ and $\dfrac{a(10^\ell - 1)}{9}$ over the intervals $k + 2 \leqslant n \leqslant 425$, $m \leqslant n$, and $2 \leqslant \ell \leqslant 424$, where $\ell < n$, confirming that within this defined range the solutions to Equation (5.1) are exclusively those enumerated in Theorem 5.1.

5.2.4 The case $k > 430$

In the following subsection, we undertake an examination of the large values of k, precisely when $k > 430$. A simple verification for $k > 430$ affirms that

$$m \leqslant n < 1.64 \times 10^{29} k^8 \log^5 k < 2^{k/2}.$$

As a result, following Lemma 1.18, we derive

$$F_n^{(k)} = 2^{n-2}(1 + \zeta_1), \quad \text{where } |\zeta_1| < \frac{2}{2^{k/2}}. \tag{5.21}$$

and

$$F_m^{(k)} = 2^{m-2}(1 + \zeta_2), \quad \text{where } |\zeta_2| < \frac{2}{2^{k/2}}. \tag{5.22}$$

By substituting (5.21) and (5.22) into (5.1), we achieve

$$2^{n+m-4} - \frac{a10^\ell}{9} = 2^{n+m-4}\left(-\zeta_1 - \zeta_2 - \zeta_1\zeta_2\right) - \frac{a}{9}.$$

Hence, we obtain

$$\left|2^{n+m-4} - \frac{a10^\ell}{9}\right| \leqslant 2^{n+m-4}\left(|\zeta_1| + |\zeta_2| + |\zeta_1\zeta_2|\right) + \frac{a}{9} \leqslant \frac{2^{n+m-2}}{2^{k/2}} + \frac{2^{n+m-2}}{2^k} + 1.$$

Consequently, dividing through by 2^{n+m-4} and using the fact that $n \geqslant m$ give us

$$\begin{aligned} \left|1 - \frac{a}{9} \cdot 10^\ell \cdot 2^{-(n+m+4)}\right| &< \frac{4}{2^{k/2}} + \frac{4}{2^k} + \frac{1}{2^{2m-4}} \\ &< \frac{8.5}{2^{\min\{k/2, m-2\}}}. \end{aligned} \tag{5.23}$$

Define

$$\Gamma_3 = \frac{a}{9} \cdot 2^{-(n+m-4)} \cdot 10^\ell - 1.$$

Thus, we derive

$$|\Gamma_3| < \frac{8.5}{2^{\min\{k/2, m-2\}}}. \tag{5.24}$$

We have $\Gamma_3 \neq 0$, because assuming $\Gamma_3 = 0$ leads to $a \cdot 10^\ell = 9 \cdot 2^{n+m-4}$. Consequently, this would imply that 5 divides $9 \cdot 2^{n+m-4}$, which is an impossibility. We are now in a position to apply Theorem 1.26 to Γ_3, taking into account the following parameters:

$$(\eta_1, b_1) := (a/9, 1), \quad (\eta_2, b_2) := (2, -(n+m-4)), \quad (\eta_3, b_3) := (10, \ell).$$

Then, the usual absolute logarithmic heights of these numbers are given by

$$h(\eta_1) = \log 9, \quad h(\eta_2) = \log 2, \quad \text{and} \quad h(\eta_3) = \log 10.$$

Observing that η_1, η_2, η_3 belong to $\mathbb{K} := \mathbb{Q}$, we ascertain $d_{\mathbb{K}} = 1$. Consequently, we opt for:

$$A_1 := \log 9, \; A_2 := \log 2, \; A_3 := \log 10.$$

Finally, we choose $B := 2n$ and we apply Theorem 1.26 to Γ_3, which gives us

$$\begin{aligned} |\Gamma_3| &> \exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot (1 + \log 2n)(\log 9)(\log 2)(\log 10)\right) \\ &> \exp\left(-1.1 \cdot 10^{12} \log n\right), \end{aligned}$$

where we have used the fact that $1 + \log 2n < 2.1 \log n$, for all $n \geqslant 5$. By comparing the resulting inequality with (5.24), we obtain

$$\min\{k/2, m-2\} < 1.6 \cdot 10^{12} \log n.$$

As specified by Lemma 5.2 and considering that $67.3 + 8 \log k + 5 \log\log k < 20 \log k$, valid for all $k > 430$, we get

$$\begin{aligned} \min\{k/2, m-2\} &< 1.6 \cdot 10^{12} \log(1.64 \cdot 10^{29} k^8 \log^5 k) \\ &< 1.6 \cdot 10^{12}(67.3 + 8 \log k + 5 \log\log k) \\ &< 3.2 \cdot 10^{13} \log k. \end{aligned}$$

If $\min\{k/2, m-2\} = k/2$, then we get $k < 6.4 \cdot 10^{13} \log k$. Solving this inequality and applying Lemma 5.2, we establish that

$$k < 2.3 \cdot 10^{15} \quad \text{and} \quad n < 7.12 \cdot 10^{159}. \tag{5.25}$$

If $\min\{k/2, m-2\} = m-2$, we obtain in this case that

$$m < 3.21 \cdot 10^{13} \log k. \tag{5.26}$$

Returning now to (5.10) and we proceed to rephrase it as follows

$$\frac{a 10^\ell}{9 F_m^{(k)}} - 2^{n-2} = 2^{n-2}\zeta_1 + \frac{a}{9 F_m^{(k)}},$$

thus we obtain

$$\left| \frac{a 10^\ell}{9 F_m^{(k)}} - 2^{n-2} \right| \leqslant \frac{2^{n-1}}{2^{k/2}} + 1.$$

Consequently, dividing through by 2^{n-2} and using the fact that $n \geqslant k+2$ lead to

$$|\Gamma_4| \leqslant \frac{2}{2^{k/2}} + \frac{1}{2^{n-2}} < \frac{2}{2^{k/2}} + \frac{1}{2^k} < \frac{3}{2^{k/2}}, \tag{5.27}$$

where

$$\Gamma_4 := \frac{a}{9F_m^{(k)}} \cdot 2^{-(n-2)} \cdot 10^{\ell} - 1. \tag{5.28}$$

We must ensure that $\Gamma_4 \neq 0$. Otherwise, we would derive the equation $\dfrac{a10^{\ell}}{9F_m^{(k)}} = 2^{n-2}$. If $a \in \{1, \ldots, 8\}$ then it is evident that the expression on the left cannot yield an integer value. For the case where $a = 9$, we encounter $\dfrac{10^{\ell}}{F_m^{(k)}} = 2^{n-2}$. In this case as $m - 2 < k/2$, it implies that $m \leqslant k + 1$, consequently leading to $F_m^{(k)} = 2^{m-2}$. Substituting this into the equation results in $\dfrac{10^{\ell}}{2^{m-2}} = 2^{n-2}$, which inevitably leads to a contradiction. Consequently, $\Gamma_4 \neq 0$. Now, we apply Theorem 1.26 to Γ_4 by setting

$$(\eta_1, b_1) := (a/(9F_m^{(k)}), 1), \quad (\eta_2, b_2) := (2, -(n-2)), \quad (\eta_3, b_3) := (10, \ell).$$

As previously calculated, we define $A_2 := \log 2$, $A_3 := \log 10$, and $B := n$. Subsequently, we proceed to estimate $h(\eta_1)$. Utilizing the fact that $F_m^{(k)} < \alpha^{m-1}$ and the inequality (5.26), we derive

$$h(\eta_1) \leqslant h(a/9) + h(F_m^{(k)}) < \log 9 + (m-1)\log\alpha < 2.23 \cdot 10^{13} \log k.$$

Consequently, we set $A_1 := 2.23 \cdot 10^{13} \log k$. Thus, according to Theorem 1.26, we obtain

$$|\Gamma_4| > \exp(-8.67 \cdot 10^{24} \log k \log n), \tag{5.29}$$

where we have used the fact that $1 + \log n < 1.7 \log n$, for all $n \geqslant 5$. By considering both (5.27) and (5.29), it follows that

$$k < 2.51 \cdot 10^{25} \log k \log n.$$

According to Lemma 5.2 and using the fact that $67.4 + 8\log k + 5\log\log k < 20\log k$, for all $k > 430$, we achieve

$$\begin{aligned} k &< 2.51 \cdot 10^{25} \log k \left(\log(1.64 \cdot 10^{29} k^8 \log^5 k)\right) \\ &< 2.51 \cdot 10^{25} \log k (67.3 + 8\log k + 5\log\log k) \\ &< 5.1 \cdot 10^{26} \log^2 k. \end{aligned}$$

Solving this inequality and applying Lemma 5.2, we obtain

$$k < 2.5 \cdot 10^{30} \quad \text{and} \quad n < 4.21 \cdot 10^{281}. \tag{5.30}$$

From (5.25) and (5.30), it is evident that (5.30) consistently remains valid. However, the resulting bounds are exceedingly large. Therefore, our subsequent step involves their reduction.

Let's put

$$\Lambda_3 := \log(\Gamma_3 + 1) = \ell \log 10 - (n + m - 4)\log 2 + \log(a/9). \tag{5.31}$$

Assume that $m \geqslant 10$, then it results $|\Gamma_3| < 0.04$. Setting $d = 0.04$ in Lemma 1.30, we obtain

$$|\Lambda_3| < \frac{-\log 0.96}{0.04} \cdot |\Gamma_3| < 8.7 \cdot 2^{-\min\{k/2, m-2\}}.$$

Consequently, we have

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n + m - 4) + \frac{\log(a/9)}{\log 2} \right| < 12.6 \cdot 2^{-\min\{k/2, m-2\}}. \tag{5.32}$$

We apply Lemma 1.27 to Λ_3, for $a \in \{1, \dots, 8\}$, with the following parameters

$$\gamma := \frac{\log 10}{\log 2}, \quad \mu := \frac{\log(a/9)}{\log 2}, \quad \text{and} \quad (A, B) := (12.6, 2).$$

We aim to diminish our excessively large bounds utilizing Lemma 1.27.
Setting $M := 4.21 \cdot 10^{281}$ as an upper bound on ℓ by (5.3) and (5.30), we employ Lemma 1.27 on Inequality (5.32) to derive an upper bound on k. After conducting a computer search with Maple, we confirm that q_{571} satisfies the conditions of Lemma 1.27 for $a \in \{1, \dots, 8\}$. As a result, applying Lemma 1.27 leads to the results shown in Table 5.1.

Table 5.1: First reduction.

a	1	2	3	4	5	6	7	8
$\varepsilon \geqslant$	0.19	0.19	0.41	0.19	0.19	0.41	0.25	0.19
$\min\{k/2, m-2\} \leqslant$	947	947	945	947	947	945	946	947

From the obtained results, it follows that $\min\{k/2, m-2\} < 948$, a condition that holds in all cases.

For the case $a = 9$, it follows that

$$\Lambda_3 := \log(\Gamma_3 + 1) = \ell \log 10 - (n + m - 4) \log 2. \tag{5.33}$$

Therefore, Inequality (5.32) becomes

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n + m - 4) \right| < 12.6 \cdot 2^{-\min\{k/2, m-2\}}. \tag{5.34}$$

Considering the continued fraction expression of $\frac{\log 10}{\log 2}$, represented as

$$[a_0, a_1, a_2, \dots] = [3, 3, 9, 2, 2, 4, 6, 2, 1, 1, 3, 1, 18, 1, \dots].$$

Let p_s/q_s denote its convergent. Recall that $\ell < 4.21 \cdot 10^{281}$. Using Maple, we determine that

$$q_{567} < 4.21 \cdot 10^{281} < q_{568}$$

and

$$a_L := \max\{a_i : i = 1, 2, \dots, 568\} = a_{135} = 5393.$$

Thus, from the known properties of the continued fractions (refer to Lemma 1.29), we obtain that

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n + m - 4) \right| > \frac{1}{(a_L + 2)\ell}. \tag{5.35}$$

Putting the above inequality together with (5.34) and using the fact that $\ell < 4.21 \cdot 10^{281}$, we get

$$2^{\min\{k/2, m-2\}} < 12.6 \cdot 5395 \cdot 4.21 \cdot 10^{281}.$$

Hence, we obtain $\min\{k/2, m-2\} < 952$. So, in all cases, we have $\min\{k/2, m-2\} < 952$. Let's now continue the procedure of reduction with each case individually in order to achieve the reduced bound on k.

Case 1: If $\min\{k/2, m-2\} = k/2$, it results

$$k < 1904. \tag{5.36}$$

Case 2: If $\min\{k/2, m-2\} = m-2$, then we obtain that $m \leqslant 953$. Set $3 \leqslant m \leqslant 953$ and we put

$$\Lambda_4 := \log(\Gamma_4 + 1) = \ell \log 10 - (n-2)\log 2 + \log(a/(9F_m^{(k)})).$$

Since $k > 430$, then by (5.34), we have $|\Gamma_4| < 0.01$. Thus, applying Lemma 1.30 with $d = 0.01$, we obtain

$$|\Lambda_4| < -\frac{\log(0.99)}{0.01} \cdot |\Gamma_4| < 3.02 \cdot 2^{-k/2}. \tag{5.37}$$

So, we get

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n-2) + \frac{\log(a/(9F_m^{(k)}))}{\log 2} \right| < 4.4 \cdot 2^{-\frac{k}{2}}. \tag{5.38}$$

For all $a \in \{1, \ldots, 8\}$ and $3 \leqslant m \leqslant 953$, we apply Lemma 1.27 to Λ_4 by considering

$$\gamma := \frac{\log 10}{\log 2}, \quad \mu := \frac{\log(a/(9F_m^{(k)}))}{\log 2}, \quad M := 4.21 \cdot 10^{281}, \quad \text{and} \quad (A, B) := (4.4, 2).$$

The fact that $m-2 < k/2$ implies that $m \leqslant k+1$, for $k \geqslant 2$ and we can replace $F_m^{(k)}$ by 2^{m-2} in our calculations. Utilizing Maple once more, it follows that q_{571} meets the conditions of Lemma 1.27, for all $a \in \{1, \ldots, 8\}$ and $3 \leqslant m \leqslant 953$. Hence, the application of Lemma 1.27 yields a range of the following distinct results, all of which remain valid for $3 \leqslant m \leqslant 953$.

Table 5.2: First reduction for second case.

a	1	2	3	4	5	6	7	8
$\varepsilon \geqslant$	0.19	0.19	0.41	0.19	0.19	0.41	0.25	0.19
$k/2 \leqslant$	945	945	944	945	945	944	945	945

Analyzing the obtained results as shown in Table 5.2, we note that in all instances $k < 1891$. When $a = 9$, then inequality (5.38) becomes

$$\left| \ell \cdot \frac{\log 10}{\log 2} - (n+m-4) \right| < 4.4 \cdot 2^{-\frac{k}{2}}. \tag{5.39}$$

According to (5.35) and (5.39), we deduce that $k < 1901$. Therefore, in all cases we have $k < 1904$.

With this refined bound, we deduce $n < 1.5 \cdot 10^{40}$. Subsequently, We once again employ Lemma 1.27 with the same dataset but with a revised upper bound of $M := 1.5 \cdot 10^{40}$. Utilizing Maple, we confirm that q_{120} satisfies the conditions stipulated in Lemma 1.27 for all $a \in \{1, \ldots, 8\}$. The results of this application are presented in Table 5.3.

Table 5.3: Second reduction.

a	1	2	3	4	5	6	7	8
$\varepsilon \geqslant$	0.11	0.11	0.45	0.11	0.11	0.45	0.18	0.11
$\min\{k/2, m-2\} \leqslant$	191	191	189	191	191	189	190	191

These obtained results affirm that $\min\{k/2, m-2\} < 192$, hold in all cases.
For $a = 9$, we find that $\min\{k/2, m-2\} < 143$. Thus, we deduce that $\min\{k/2, m-2\} < 192$, valid in all cases.

As previously, for the first case, we ascertain that $k < 384$, while for the second case, we once again employ q_{120}, meeting the conditions specified in Lemma 1.27 for all $a \in \{1, \ldots, 8\}$ and $3 \leqslant m \leqslant 193$. Then, we derive the subsequent results, valid for all $3 \leqslant m \leqslant 193$.

Table 5.4: Second reduction for second case.

a	1	2	3	4	5	6	7	8
$\varepsilon \geqslant$	0.11	0.11	0.45	0.11	0.11	0.45	0.18	0.11
$k/2 \leqslant$	189	189	187	189	189	187	188	189

The obtained results from Table 5.4 state that $k < 380$ for all cases. When $a = 9$, we find that $k < 283$.
Consequently, in all cases, we establish that $k < 384$. However, this contradicts our assumption that $k > 430$. This achieves the proof of Theorem 5.1.

Chapter 6

Almost Repdigits in k-Pell sequences

In this chapter, we work on the problem of finding all the k-Pell numbers which have the property that all their digits are equal except for at most one digit, called *Almost Repdigits.* This chapter builds upon the findings from the paper [54]

6.1 Motivation and main result

Recently, several authors have studied and solved Diophantine equations involving repdigits and k-Pell sequences [15, 30, 31, 43, 50]. In [3], A. Altassan and M. Alan introduced the concept of an "almost repdigit" defined as a positive integer where all digits are identical except for at most one. These numbers take the form

$$a\left(\frac{10^{d_1}-1}{9}\right)+(b-a)10^{d_2}, \quad 0 \leqslant d_2 < d_1, \quad \text{and} \quad 0 \leqslant a, b \leqslant 9.$$

Furthermore, in their paper, they determine all almost repdigits within the k-generalized Fibonacci sequence $(F_n^{(k)})_{n \geqslant -(k-2)}$. Motived by their result, we aim to prove the following result.

Theorem 6.1. *The Diophantine equation*

$$P_n^{(k)} = a\left(\frac{10^{d_1}-1}{9}\right)+(b-a)10^{d_2}, \quad 0 \leqslant d_2 < d_1, \quad \textit{and} \quad 0 \leqslant a, b \leqslant 9 \tag{6.1}$$

has only the following solution $P_8^{(3)} = 545$, $P_7^{(4)} = 228$ *and* $P_7^{(5)} = 232$ *when* $P_n^{(k)}$ *has at least three digits.*

Remark 6.2. To exclude trivial cases, the theorem is stated for numbers with at least three digits, as all integers with one or two digits are trivially almost repdigits. Therefore, we also assume $d_1 \geqslant 3$ and $n > 5$.

6.2 The proof of Theorem 6.1

Initially, we can establish some relationships between the variables that will be beneficial for our forthcoming analysis. To do this, we combine the following inequalities

$$10^{d_1-2} < a\left(\frac{10^{d_1}-1}{9}\right)+(b-a)10^{d_2} \leqslant 2\times 10^{d_1}$$

with (6.1), (3.2), and (3.6) to obtain

$$10^{d_1-2} \leqslant a\left(\frac{10^{d_1}-1}{9}\right)+(b-a)10^{d_2} = P_n^{(k)} \leqslant \alpha^{n-1} < \varphi^{2n-2}$$

and

$$\varphi^{n-2} < \alpha^{n-2} \leqslant P_n^{(k)} = a\left(\frac{10^{d_1}-1}{9}\right)+(b-a)10^{d_2} \leqslant 2\times 10^{d_1}.$$

We deduce that

$$d_1 < (n-1)\frac{2\log\varphi}{\log 10}+2 < 0.42n+1.6 < n \tag{6.2}$$

and

$$0.2n-0.7 < (n-2)\frac{\log\varphi}{\log 10}-\frac{\log 2}{\log 10} < d_1, \tag{6.3}$$

for all $n > 5$.

For the case $2 \leqslant n \leqslant k+1$, we use the identity (3.7) and so the Diophantine equation (6.1) transforms into

$$F_{2n-1} = a\left(\frac{10^{d_1}-1}{9}\right)+(b-a)10^{d_2}. \tag{6.4}$$

According to the main result in [3], we deduce that the only solution for (6.4) is $F_{13} = 233$. Thus, from now, we assume that $n \geqslant k+2$.

6.2.1 An inequality for n versus k

Here, we will establish an inequality for n in relation with k, by showing the following lemma.

Lemma 6.3. *If (a, b, k, d_1, d_2, n) is a solution in positive integers of equation (6.1) with $k \geqslant 2$ and $n \geqslant k+2$, then we have the following inequality*

$$n < 6.21\cdot 10^{31}k^8\log^5 k. \tag{6.5}$$

Proof. Using estimate provided in (3.5), we obtain

$$P_n^{(k)} = g_k(\alpha)\alpha^n + e_k(n), \quad \text{where } |e_k(n)| < \frac{1}{2},$$

then, we can express equation (6.1) as

$$g_k(\alpha)\alpha^n + e_k(n) = a\left(\frac{10^{d_1}-1}{9}\right)+(b-a)10^{d_2}, \tag{6.6}$$

that is,

$$g_k(\alpha)\alpha^n - \frac{a10^{d_1}}{9} = -e_k(n) - a/9 - (b-a)10^{d_2}. \tag{6.7}$$

By taking the absolute value and dividing both sides by $a10^{d_1}/9$, we get

$$|\Gamma_1| \leqslant \frac{9/2}{10^{d_1}} + \frac{1}{10^{d_1}} + \frac{|b-a|\,(9/a)}{10^{d_1-d_2}} < \frac{78}{10^{d_1-d_2}}, \tag{6.8}$$

where

$$\Gamma_1 := g_k(\alpha)9/a\cdot\alpha^n\cdot 10^{-d_1} - 1. \tag{6.9}$$

Define

$$\Lambda_1 := n\log\alpha - d_1\log 10 + \log\left(g_k(\alpha)9/a\right). \tag{6.10}$$

Then, from (6.8)

$$|e^{\Lambda_1}-1| = |\Gamma_1| < \frac{78}{10^{d_1-d_2}}. \tag{6.11}$$

Note that $\Lambda_1 \neq 0$ since $\Gamma_1 \neq 0$, otherwise we would get

$$\frac{a10^{d_1}}{9} = g_k(\alpha)\cdot\alpha^n.$$

After applying an automorphism from the Galois group of the decomposition field $\Psi(x)$ over $\mathbb{Q}$ to the above relation and then taking absolute values, we conclude that for any $i \geqslant 2$, we have

$$100 < \frac{10^3}{9} \leqslant \frac{a10^{d_1}}{9} = |g_k(\alpha_i)|\cdot|\alpha_i|^n < 1,$$

which is a contradiction. So, we distinguish the following two cases.

If $\Lambda_1 > 0$, then from (6.11) and using the fact that $x \leqslant e^x - 1$ for all $x \in \mathbb{R}$ we obtain

$$|\Lambda_1| = \Lambda_1 \leqslant e^{\Lambda_1} - 1 = |e^{\Lambda_1}-1| < \frac{78}{10^{d_1-d_2}}.$$

If $\Lambda_1 < 0$, then $1 - e^{\Lambda_1} = |e^{\Lambda_1}-1| < \frac{78}{10^{d_1-d_2}} < \frac{1}{2}$, which is valid for $d_1 - d_2 > 3$. From this, we get $e^{\Lambda_1} > \frac{1}{2}$ and so $e^{|\Lambda_1|} = e^{-\Lambda_1} < 2$. As a result, we deduce that

$$0 < |\Lambda_1| \leqslant e^{|\Lambda_1|} - 1 = e^{|\Lambda_1|}|e^{\Lambda_1}-1| < \frac{156}{10^{d_1-d_2}}.$$

In both cases, we have

$$0 < |\Lambda_1| < \frac{156}{10^{d_1-d_2}}.$$

Now, in view to apply Theorem 1.25 to Λ_1, we consider the following parameters:

$$(\eta_1, b_1) := (\alpha, n), \quad (\eta_2, b_2) := (10, -d_1), \quad (\eta_3, b_3) := (g_k(\alpha)9/a, 1),$$

where $\eta_1, \eta_2, \eta_3 \in \mathbb{K} := \mathbb{Q}(\alpha)$, with $d_{\mathbb{K}} = k$. By (6.2), we take $B := n$.
Since $h(\eta_1) = (\log\alpha)/k < 2\log\varphi/k$ and $h(\eta_2) = \log 10$. For $h(\eta_3)$, we use the properties (1.1)-(1.3) and the estimate (3.4). Then, it follows that

$$\begin{aligned} h(\eta_3) = h(g_k(\alpha)9/a) &< 4\log\varphi + \log(k+1) + \log 9 \\ &< 7.6\log k, \end{aligned}$$

for all $k \geqslant 2$. Consequently, we have the flexibility to select

$$h'(\eta_1) := \max\left\{h(\eta_1), \frac{|\log\eta_1|}{k}, \frac{1}{k}\right\} = \frac{1}{k},$$

$$h'(\eta_2) := \max\left\{h(\eta_2), \frac{|\log\eta_2|}{k}, \frac{1}{k}\right\} = \log 10,$$

and

$$h'(\eta_3) := \max\left\{h(\eta_3), \frac{|\log\eta_3|}{k}, \frac{1}{k}\right\} = 7.6\log k.$$

With these considerations in mind, we are now equipped to apply Theorem 1.25 and derive

$$\begin{aligned} \log(|\Gamma_1|) &\geqslant -18\cdot 4!\cdot 3^4\cdot(32k)^5\cdot(7.6\log k)\cdot(1/k)\cdot\log 10\cdot\log(6k)\log n \\ &> -2.1\cdot 10^{13}k^4\log(6k)\log n. \end{aligned}$$

Comparing this lower bound with the upper bound of $|\Lambda_1|$ provided by (6.11) and using the fact that $\log(6k) < 3.6 \log k$ which is valid for all $k \geqslant 2$, we arrive at

$$(d_1 - d_2)\log 10 < 8 \cdot 10^{13} k^4 \log^2 k \log n. \tag{6.12}$$

Now, let us express equation (6.1) in another manner to derive a second linear expression in logarithms. To achieve this, we use estimate (3.5) once again to obtain

$$g_k(\alpha)\alpha^n - 10^{d_1}\left((a/9) + (b-a)10^{d_2-d_1}\right) = -a/9 - e_k(n). \tag{6.13}$$

As previously done, by taking the absolute value, dividing both sides by $g_k(\alpha)\alpha^n$ and using the fact that $g_k(\alpha) > 0.276$, we obtain

$$|\Gamma_2| \leqslant \frac{3}{2g_k(\alpha)\alpha^n} < \frac{5.5}{\alpha^n}, \tag{6.14}$$

where

$$\Gamma_2 := g_k^{-1}(\alpha)\left((a/9) + (b-a)10^{d_2-d_1}\right) \cdot \alpha^{-n} \cdot 10^{d_1} \cdot -1. \tag{6.15}$$

Define

$$\Lambda_2 = -n \log \alpha + d_1 \log 10 + \log\left(g_k^{-1}(\alpha)((a/9) + (b-a)10^{d_2-d_1})\right). \tag{6.16}$$

Using the similar argument that has been used above for Λ_1 to show that Λ_2 is not zero too and following the same way with $|\Gamma_2| = |e^{\Lambda_2} - 1|$ to achieve

$$0 < |\Lambda_2| < \frac{11}{\alpha^n}. \tag{6.17}$$

Now, we will apply Theorem 1.25 to Λ_2. For this application, we select the following parameters:

$$(\eta_1, b_1) := (\alpha, -n), \quad (\eta_2, b_2) := (10, d_1),$$

and

$$(\eta_3, b_3) := (g_k^{-1}(\alpha)\left((a/9) + (b-a)10^{d_2-d_1}\right), 1).$$

Once again, let $\mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. As previously, we can choose $h'(\eta_1) := \dfrac{1}{k}$, and $h'(\eta_2) := \log 10$. Now, we still need to calculate $h'(\eta_3)$. With the help of the estimates (3.4), (6.12), and the proprieties (1.1)-(1.3), we obtain

$$\begin{aligned} h(\eta_3) &\leqslant h(g_k(\alpha)^{-1}) + h((a/9) + (b-a)10^{d_2-d_1}) \\ &\leqslant 4\log\varphi + \log(k+1) + h(a/9) + h(b-a) + h(10^{d_2-d_1}) + \log 2 \\ &\leqslant 4\log\varphi + \log(k+1) + \log(144) + |d_2 - d_1| \log 10 \\ &< 8.1 \times 10^{13} k^4 \log^2 k \log n, \end{aligned}$$

for all $k \geqslant 2$. This implies that

$$h'(\eta_3) := \max\left\{h(\eta_3), \frac{|\log \eta_3|}{k}, \frac{1}{k}\right\} < 8.1 \times 10^{13} k^4 \log^2 k \log n.$$

By choosing $B := n$, we can apply Theorem 1.25 to Λ_2 and compare the resulting inequality with (6.17) to have

$$n \log \alpha < 7.9 \times 10^{26} k^8 \log^3 k \log^2 n.$$

Consequently, we obtain

$$\frac{n}{\log^2 n} < 1.64 \times 10^{27} k^8 \log^3 k. \tag{6.18}$$

Using Lemma 1.31, with $T := 1.64 \times 10^{27} k^8 \log^3 k$, and considering the inequality $62.7 + 8\log k + 3\log\log k < 97\log k$, for all $k \geqslant 2$, we get

$$\begin{aligned} n &< 4(1.64 \cdot 10^{27} k^8 \log^3 k)\left(\log(1.64 \cdot 10^{27} k^8 \log^3 k)\right)^2 \\ &< (6.6 \cdot 10^{27} k^8 \log^3 k)(62.7 + 8\log k + 3\log\log k)^2 \\ &< 6.21 \cdot 10^{31} k^8 \log^5 k. \end{aligned}$$

This concludes the proof of Lemma 6.3. □

6.2.2 The case $2 \leqslant k \leqslant 650$

In this subsection, we analyze the case $2 \leqslant k \leqslant 650$. As seen previously, we put

$$\Lambda_1 := n\log\alpha - d_1\log 10 + \log(g_k(\alpha)9/a), \tag{6.19}$$

where

$$|\Lambda_1| < \frac{156}{10^{d_1-d_2}}. \tag{6.20}$$

Thus, we obtain

$$\left| n \cdot \frac{\log\alpha}{\log 10} - d_1 + \frac{\log(g_k(\alpha)9/a)}{\log 10} \right| < \frac{156}{\log 10} \cdot 10^{-(d_1-d_2)}. \tag{6.21}$$

To apply Lemma 1.27 to Λ_1, for all $a \in \{1,\dots,9\}$, let us take

$$\gamma := \frac{\log\alpha}{\log 10}, \quad \mu_{(k,a)} := \frac{\log(g_k(\alpha)9/a)}{\log 10}, \quad \text{and} \quad (A,B) := (67.8, 10).$$

For each $k \in [2,650]$ and $a \in \{1,\dots,9\}$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon_{(k,a)} = \|\mu_{(k,a)} q_\ell\| - M_k\|\gamma q_\ell\| > 0$, where $M_k := \lfloor 6.21 \times 10^{31} k^8 \log^5 k \rfloor$, which is an upper bound of n from Lemma 6.3. Using Mathematica, we see that q_{134} satisfies the conditions of Lemma 1.27. After doing this, we use Lemma 1.27 on inequality (6.21). A computer program with Mathematica shows for $k = 485$ and $a = 5$ that $\varepsilon_{(k,a)} > 0.00003327$ and the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$, for $k \in [2,650]$ and $a \in \{1,\dots,9\}$, is 64.4623, which is an upper bound of $d_1 - d_2$ by Lemma 1.27.

Now, we consider $1 \leqslant d_1 - d_2 \leqslant 64$ and

$$\Lambda_2 = -n\log\alpha + d_1\log 10 + \log\left(g_k^{-1}(\alpha)((a/9) + (b-a)10^{d_2-d_1})\right), \tag{6.22}$$

where

$$|\Lambda_2| < \frac{11}{\alpha^n}. \tag{6.23}$$

So, we deduce that

$$\left| -n \cdot \frac{\log\alpha}{\log 10} + d_1 + \frac{\log\left(g_k^{-1}(\alpha)((a/9) + (b-a)10^{d_2-d_1})\right)}{\log 10} \right| < \frac{11}{\log 10} \cdot \alpha^{-n}. \tag{6.24}$$

To apply Lemma 1.27, for each $d_1 - d_2 \in \{1,\dots,64\}$, $a \in \{1,\dots,9\}$, and $b \in \{0,\dots,9\}$, we fix

$$\gamma := \frac{\log\alpha}{\log 10}, \quad \mu_{(k,d_1-d_2,a,b)} := \frac{\log\left(g_k^{-1}(\alpha)((a/9) + (b-a)10^{d_2-d_1})\right)}{\log 10},$$

and

$$(A, B) := (4.8, \alpha).$$

Again, for each $k \in [2, 650]$, $d_1 - d_2 \in \{1, \ldots, 64\}$, $a \in \{1, \ldots, 9\}$ and $b \in \{0, \ldots, 9\}$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon_{(k,d_1-d_2,a,b)} = \|\mu_{(k,d_1-d_2,a,b)}q_\ell\| - M_k\|\gamma q_\ell\| > 0$, where $M_k := \lfloor 6.31 \times 10^{31} k^8 \log^5 k \rfloor$, which is an upper bound of n from Lemma 6.3. We use Mathematica to see that q_{137} satisfies the conditions of Lemma 1.27. After doing this, we use Lemma 1.27 on inequality (6.24). A computer program in Mathematica reveales for $k = 550$, $d_1 - d_2 = 36$, $a = 7$, and $b = 5$, that $\varepsilon_{(k,d_1-d_2,a,b)} > 2.20 \times 10^{-6}$ and the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$, for $k \in [2, 650]$, $d_1 - d_2 \in \{1, \ldots, 64\}$, $a \in \{1, \ldots, 9\}$ and $b \in \{0, \ldots, 9\}$, is 159.911, which is an upper bound of n by Lemma 1.27.

Finally, we use Mathematica to compare $P_n^{(k)}$ and $a\left(\dfrac{10^{d_1}-1}{9}\right)+(b-a)10^{d_2}$, for the ranges $2 \leqslant k \leqslant 650$, $k+2 \leqslant n \leqslant 159$, $1 \leqslant a \leqslant 9$ and $0 \leqslant b \leqslant 9$, with $0 \leqslant d_2 < d_1 < 0.42n + 1.6$ and check that the only solutions of equation (6.1) are those listed in Theorem 6.1.

6.2.3 The case $k > 650$

In this subsection, we treat the case $k > 650$. For this, we prove the following result.

Lemma 6.4. *If (a, b, k, d_1, d_2, n) is a solution in positive integers of equation (6.1) with $k \geqslant 2$ and $n \geqslant k + 2$, then we have the following inequality*

$$k < 5.2 \times 10^{35} \quad \textit{and} \quad n < 1.3 \times 10^{327}. \tag{6.25}$$

Proof. If $k > 650$, the following inequalities are valid

$$n < 6.21 \cdot 10^{31} k^8 \log^5 k < \varphi^{k/2}.$$

At this point, we use the estimation from Lemma 1.21 in (6.7) to obtain

$$\left|\frac{\varphi^{2n}}{\varphi+2} - \frac{a10^{d_1}}{9}\right| \leqslant \frac{\varphi^{2n}}{\varphi+2}\cdot\frac{4}{\varphi^{k/2}} + \frac{3}{2} + 8\cdot 10^{d_2}.$$

By dividing both sides by $a10^{d_1}/9$, we obtain

$$\begin{aligned}
|\Gamma_3| &< \frac{\varphi^{2n}}{\varphi+2}\cdot\frac{4}{\varphi^{k/2}}\frac{9}{a10^{d_1}} + \frac{27}{2\cdot 10^{d_1}} + \frac{72}{10^{d_1-d_2}} \\
&< \frac{36}{2^{k/2}} + \frac{14}{10^{d_1}} + \frac{72}{10^{d_1-d_2}} \\
&< \frac{2^6}{2^{k/2}} + \frac{2^4}{10^{d_1}} + \frac{2^7}{10^{d_1-d_2}} \\
&< \frac{1/4}{2^{k/2-8}} + \frac{1/4}{2^{d_1\frac{\log 10}{\log 2}-6}} + \frac{1/4}{2^{(d_1-d_2)\frac{\log 10}{\log 2}-9}}.
\end{aligned}$$

Note that the above estimate was obtained as follows

$$\begin{aligned}
\frac{\varphi^{2n}}{\varphi+2}\cdot\frac{9}{a10^{d_1}} &\leqslant \frac{1}{|1+\zeta|}\left(\frac{9P_n^{(k)}}{a10^{d_1}} + \frac{9}{2a10^{d_1}}\right) \\
&\leqslant \frac{1}{0.999}\left(1 + \frac{1.5}{10^{d_1}} + \frac{72}{10^{d_1-d_2}}\right) \\
&\leqslant \frac{1}{0.999}\left(1 + \frac{1}{10^3} + \frac{72}{10}\right) < 9.
\end{aligned}$$

Consequently, we have

$$|\Gamma_3| := |\varphi^{2n} \cdot 10^{-d_1} \cdot 9/a(\varphi+2) - 1| < \frac{1}{2^\lambda}, \tag{6.26}$$

where

$$\lambda := \min\left\{(k/2) - 8, (d_1 - d_2)\frac{\log 10}{\log 2} - 9\right\}.$$

In order to apply Theorem 1.25, let us define

$$\Lambda_3 := 2n\log\varphi - d_1\log 10 + \log(9/a(\varphi+2)).$$

Note that the left-hand side of (6.26) is not zero, since if not, we would get

$$\frac{a}{9}(\varphi+2)10^{d_1} = \varphi^{2n}. \tag{6.27}$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we get

$$\frac{a}{9}(\overline{\varphi}+2)10^{d_1} = \overline{\varphi}^{2n}, \tag{6.28}$$

where $\overline{\varphi} = (1-\sqrt{5})/2$. Putting (6.27) and (6.28) together we obtain that $F_{2n} = a10^{d_1}/9$ implies that $a = 9$ and so $F_{2n} = 10^{d_1}$. Or, there are no powers of 10 in the Fibonacci sequence. Hence we ensure that $\Gamma_3 \neq 0$ and so $\Lambda_3 \neq 0$.

From (6.26), we deduce that

$$|e^{\Lambda_3} - 1| = |\Gamma_3| < \frac{1}{2^\lambda}. \tag{6.29}$$

If $\Lambda_3 > 0$, then $|e^{\Lambda_3} - 1| = e^{\Lambda_3} - 1$ and so from (6.29), we obtain

$$0 < \Lambda_3 \leqslant e^{\Lambda_3} - 1 = |e^{\Lambda_3} - 1| < \frac{1}{2^\lambda}.$$

If $\Lambda_3 < 0$, then $1 - e^{\Lambda_3} = |e^{\Lambda_3} - 1| < \frac{1}{2^\lambda} < \frac{1}{2}$, which holds, for $\lambda > 1$. So, we get $e^{|\Lambda_3|} < 2$. Consequently, we have

$$0 < |\Lambda_3| \leqslant e^{|\Lambda_3|} - 1 = e^{|\Lambda_3|}|e^{\Lambda_3} - 1| < \frac{2}{2^\lambda}.$$

So, in all cases we have

$$0 < |\Lambda_3| < \frac{2}{2^\lambda}. \tag{6.30}$$

Now, we are allowed to apply Theorem 1.25 by taking the following parameters

$$(\eta_1, b_1) := (\varphi^2, n), \quad (\eta_2, b_2) := (10, -d_1), \quad (\eta_3, b_3) := (9/a(\varphi+2), 1).$$

The positive real numbers η_1, η_2 and η_3 belong to $\mathbb{K} = \mathbb{Q}(\varphi) = \mathbb{Q}(\sqrt{5})$. Therefore, we take $d_{\mathbb{K}} = 2$. Next, we can also take $B := n$.

Since $h(\eta_1) = \log\varphi$, $h(\eta_2) = \log 10$ and $h(\eta_3) \leqslant h(a/9) + h(\varphi+2) \leqslant \log 9 + (\log 5)/2$. Then, we take

$$h'(\eta_1) := \frac{1}{2} = \max\left\{h(\eta_1), \frac{|\log\eta_1|}{2}, \frac{1}{2}\right\},$$

$$h'(\eta_2) := \log 10 = \max\left\{h(\eta_2), \frac{|\log\eta_2|}{2}, \frac{1}{2}\right\}$$

and

$$h'(\eta_3) := 3.5 > \max\left\{h(\eta_3), \frac{|\log \eta_3|}{2}, \frac{1}{2}\right\}.$$

Thus, the application of Theorem 1.25 leads to obtain

$$\begin{aligned} \log(|\Lambda_3|) &\geqslant -18 \cdot 4! \cdot 3^4 \cdot (64)^5 \cdot (\log 12) \cdot (1/2) \cdot \log 10 \cdot 3.5 \cdot \log n \\ &> -3.77 \cdot 10^{14} \log n. \end{aligned} \tag{6.31}$$

Comparing (6.30) and (6.31), then we obtain

$$\lambda < 5.44 \times 10^{14} \log n.$$

Recall that by Lemma 6.3, we have $n < 6.21 \cdot 10^{31} k^8 \log^5 k$. By considering that $73.21 + 8\log k + 5 \log\log k < 21 \log k$, which is valid for all $k > 650$, we deduce

$$\begin{aligned} \lambda &< 5.44 \times 10^{14} \log(6.21 \cdot 10^{31} k^8 \log^5 k) \\ &< 5.44 \times 10^{14}(73.21 + 8\log k + 5\log\log k) \\ &< 5.44 \times 10^{14} \cdot 21 \log k \\ &< 1.15 \times 10^{16} \cdot \log k. \end{aligned}$$

If $\lambda = k/2 - 8$, then we obtain $k < 2.31 \times 10^{16} \log k$. Solving this inequality and applying Lemma 6.3, we get

$$k < 9.6 \times 10^{17} \quad \text{and} \quad n < 5.5 \times 10^{183}.$$

If $\lambda = (d_1 - d_2)\frac{\log 10}{\log 2} - 9$, then we obtain

$$d_1 - d_2 < 3.5 \times 10^{15} \log k.$$

In this case, we need to return to equation (6.1) and rewrite it differently by using again the estimate from Lemma 1.21 in (6.13) and hence we obtain

$$\left|\frac{\varphi^{2n}}{\varphi+2} - \frac{a10^{d_1}}{9} - (b-a)10^{d_2}\right| \leqslant \frac{3}{2} + \frac{\varphi^{2n}}{\varphi+2} \cdot \frac{4}{\varphi^{k/2}}.$$

Then, by dividing both sides by $\dfrac{\varphi^{2n}}{\varphi+2}$ and using the fact that $n \geqslant k+2$, we derive

$$|\Gamma_4| < \frac{3}{2} \cdot \frac{\varphi+2}{\varphi^{2n}} + \frac{4}{\varphi^{k/2}} < \frac{4}{\varphi^{k/2}} + \frac{3(\varphi+2)}{2\varphi^{2k+4}} < \frac{9.5}{\varphi^{k/2}}.$$

Consequently, we have

$$|\Gamma_4| := \left|(\varphi+2)((a/9) + (b-a)10^{d_2-d_1}) \cdot 10^{d_1} \cdot \varphi^{-2n} - 1\right| < \frac{9.5}{\varphi^{k/2}}. \tag{6.32}$$

For another application of Theorem 1.25, we define

$$\Lambda_4 := -2n \log \varphi + d_1 \log 10 + \log((\varphi+2)((a/9) + (b-a)10^{d_2-d_1})). \tag{6.33}$$

Note that $\Lambda_4 \neq 0$ since $\Gamma_4 \neq 0$ and as done with $|\Lambda_3|$, we obtain

$$0 < |\Lambda_4| < \frac{19}{\varphi^{k/2}}. \tag{6.34}$$

For applying Theorem 1.25, we take

$$(\eta_1, b_1) := (\varphi^2, -n), \quad (\eta_2, b_2) := (10, d_1),$$

and

$$(\eta_3, b_3) := ((\varphi+2)((a/9)+(b-a)10^{d_2-d_1}, 1).$$

Once again, we take $\mathbb{K} = \mathbb{Q}(\varphi) = \mathbb{Q}(\sqrt{5})$ with $d_{\mathbb{K}} = 2$ and $B := n$. Then, as above we can take $h'(\eta_1) = 1/2$ and $h'(\eta_2) = \log 10$. For $h'(\eta_3)$, let us compute $h(\eta_3)$ as follows

$$\begin{aligned} h(\eta_3) &= h((\varphi+2)((a/9)+(b-a)10^{d_2-d_1}) \\ &\leqslant h(\varphi+2)+h(a/9)+h(b-a)+|d_2-d_1|\log 10+\log 2 \\ &< (\log 5)/2+\log 9+\log 8+(d_1-d_2)\log 10+\log 2 \\ &< 8.1\times 10^{15}\log k. \end{aligned}$$

This implies that

$$h'(\eta_3) := 8.1\times 10^{15}\log k > \max\left\{h(\eta_3), \frac{|\log\eta_3|}{2}, \frac{1}{2}\right\}.$$

Therefore, applying Theorem 1.25 and making the resulting inequality with (6.34) yields

$$k/2(\log\varphi) < 8.72\times 10^{29}\log k\log n.$$

Using Lemma 6.3 and the inequality $73.21 + 8\log k + 5\log\log k < 21\log k$, which is valid for all $k > 650$, we get

$$k < 7.62\times 10^{31}\log^2 k.$$

Solving this inequality and applying Lemma 6.3, we obtain

$$k < 5.2\times 10^{35} \quad \text{and} \quad n < 1.3\times 10^{327}. \tag{6.35}$$

This completes the proof of Lemma 6.4. □

6.2.4 Reducing the bound on k

This subsection is devoted to reduce upper bounds, obtained in Lemma 6.4, which are too large. Put

$$\Lambda_3 := 2n\log\varphi - d_1\log 10 + \log((9/a)(\varphi+2)).$$

From (6.30), we derive

$$0 < \left| n\cdot\frac{2\log\varphi}{\log 10} - d_1 + \frac{\log((9/a)(\varphi+2))}{\log 10}\right| < \frac{2}{\log 10}\cdot 2^{-\lambda}. \tag{6.36}$$

Then, for all $a \in \{1,\dots,9\}$, we apply Lemma 1.27 with the following data

$$\gamma := \frac{2\log\varphi}{\log 10}, \quad \mu_a := \frac{\log((9/a)(\varphi+2))}{\log 10}, \quad \text{and} \quad (A,B) := (\frac{2}{\log 10}, 2).$$

Moreover, we take $M := 1.3\times 10^{327}$, which serves as an upper bound on n according to (6.35) in view to reduce it by applying Lemma 1.27 on inequality (6.36). We use Mathematica to see that q_{649} fulfills the conditions specified in Lemma 1.27. Then, for each $a \in \{1,\dots,9\}$, we find

$$0.00007162 < \varepsilon_a = \|\mu_a q_{649}\| - M\cdot\|\gamma q_{649}\|.$$

Consequently, from Lemma 1.27, we obtain

$$\lambda < 1117.$$

Let's proceed with the reduction process for each case of λ individually to achieve the reduced bound on k.

If $\lambda = k/2 - 8$. Then, $k < 2250$.

If $\lambda = (d_1 - d_2)\frac{\log 10}{\log 2} - 9$. Then $d_1 - d_2 < 339$. Let us fix $1 \leqslant d_1 - d_2 \leqslant 338$ and put

$$\Lambda_4 := -2n \log \varphi + d_1 \log 10 + \log((\varphi + 2)((a/9) + (b - a)10^{d_2 - d_1}).$$

From (6.34), we obtain

$$0 < \left| -n \cdot \frac{2 \log \varphi}{\log 10} + d_1 + \frac{\log((\varphi + 2)((a/9) + (b - a)10^{d_2 - d_1})}{\log 10} \right| < \frac{19}{\log 10} \cdot \varphi^{-(k/2)}.$$

For this application of Lemma 1.27, we take

$$\gamma := \frac{2 \log \varphi}{\log 10}, \quad \mu_{(a,b,d_1-d_2)} := \frac{\log((\varphi + 2)((a/9) + (b - a)10^{d_2 - d_1})}{\log 10},$$

and

$$(A, B) := (\frac{19}{\log 10}, \varphi).$$

Using Mathematica, we find again that $q = q_{649}$ satisfies the conditions of Lemma 1.27 and thus we obtain

$$6.36 \times 10^{-6} < \varepsilon_{(9,3,73)} \leqslant \varepsilon_{(a,b,d_1-d_2)}.$$

By Lemma 1.27, we derive

$$k < 3236.$$

With this new upper bound on k, we obtain $n < 2.6 \times 10^{64}$. We now proceed as before but with $M := 2.6 \times 10^{64}$ and $q = q_{157}$. Thus, we find that

$$0.0074 < \varepsilon_a = \|\mu_a q_{157}\| - M \cdot \|\gamma q_{157}\|.$$

Then, by Lemma 1.27 we obtain that

$$\lambda < 241.$$

Thus, if $\lambda = k/2 - 8$, then $k < 498$, whereas $\lambda = (d_1 - d_2)\frac{\log 10}{\log 2} - 9$ yields

$$d_1 - d_2 < 76.$$

We work on Λ_4 as we did previously but with q_{157}. Thus, we find that

$$7.48 \times 10^{-6} < \varepsilon_{(4,2,64)} \leqslant \varepsilon_{(a,b,d_1-d_2)}.$$

From Lemma 1.27, we find

$$k < 730.$$

With this improved bound on k, we apply again Lemma 1.27 with $M := 6.3 \times 10^{58}$ and $q = q_{144}$. By repeating the same previous steps, we find $k < 656$.

Once again, for another application of Lemma 1.27. this time with $M := 2.5 \times 10^{58}$ and $q = q_{143}$, we achieve $k < 648$, which contradicts our assumption that $k > 650$. This completes the proof of Theorem 6.1.

6.3 The k-Pell Numbers as powers of 10

If $a = 0$, then equation (6.1) becomes

$$P_n^{(k)} = b10^{d_2}. \tag{6.37}$$

It's obvious that $b \neq 0$. We will use the previous bounds on the variables with minor adjustments. Therefore, we will also adhere the precious notations.

From (6.37), Λ_2 as presented in (6.16), is valid as follows

$$|\Lambda_2'| := |-n\log\alpha + d_2\log 10 + \log(b.g_k^{-1}(\alpha))| < \frac{4}{\alpha^n}.$$

For the same reason as above, $\Lambda_2' \neq 0$. This allows us to apply Theorem 1.25 on $\Lambda_2' \neq 0$ with

$$(\eta_1, b_1) := (\alpha, -n), \quad (\eta_2, b_2) := (10, d_2), \quad (\eta_3, b_3) := (bg_k^{-1}(\alpha), 1).$$

Then, we get

$$\begin{aligned} h(\eta_3) &\leqslant h(b) + h(g_k(\alpha)^{-1}) \\ &\leqslant \log 9 + 4\log\varphi + \log(k+1) \\ &< 7.6\log k. \end{aligned}$$

Thus, we take

$$h'(\eta_1) = \frac{1}{k}, \quad h'(\eta_2) = \log 10, \quad \text{and} \quad h'(\eta_3) = 7.6\log k.$$

By utilizing the bound provided in (3.6) along with (6.37), we find that $d_2\log 10 < (n-1)\log\varphi$, implying that $d_2 < n-1$. Applying Theorem 1.25, similarly to our approach for Λ_2, we obtain that

$$n < 1.65 \times 10^{14} k^4 \log^2 k \log n.$$

Using Lemma 1.31, with $T := 1.65 \times 10^{14} k^4 \log^2 k$, and considering the inequality $32.74 + 4\log k + 2\log\log k < 51\log k$, we obtain

$$n < 1.72 \times 10^{18} k^4 \log^4 k. \tag{6.38}$$

6.3.1 The case $2 \leqslant k \leqslant 650$

Considering again

$$|\Lambda_2'| := |-n\log\alpha + d_2\log 10 + \log(b.g_k^{-1}(\alpha))| < \frac{4}{\alpha^n}.$$

Thus, we obtain

$$\left| -n \cdot \frac{\log\alpha}{\log 10} + d_2 + \frac{(bg_k^{-1}(\alpha))}{\log 10} \right| < 1.8 \cdot \alpha^n. \tag{6.39}$$

Then, for all $b \in \{1, \ldots, 9\}$, we apply Lemma 1.27 by taking

$$\gamma := \frac{\log\alpha}{\log 10}, \quad \mu_{(k,b)} := \frac{\log(bg_k^{-1}(\alpha))}{\log 10}, \quad (A, B) := (1.8, \alpha).$$

For each $k \in [2, 650]$ and $b \in \{1, \ldots, 9\}$, we find a good approximation of γ and a convergent p_ℓ/q_ℓ of the continued fraction of γ such that $q_\ell > 6M_k$ and $\varepsilon = \varepsilon_{(k,b)} = \|\mu_{(k,b)}q_\ell\| - M_k\|\gamma q_\ell\| > 0$, where $M_k := \lfloor 1.72 \times 10^{18} k^4 \log^4 k \rfloor$, which is an upper bound of n from (6.38). Using Mathematica, we see that q_{76} meets the conditions of Lemma 1.27. After doing this, we use

Lemma 1.27 on inequality (6.39). A computer program in Mathematica helps to see that, for $k = 474$ and $b = 9$, $\varepsilon_{(k,b)} > 0.000014$ and the maximum value of $\dfrac{\log(Aq/\varepsilon)}{\log B}$, for $k \in [2, 650]$ and $b \in \{1, \ldots, 9\}$, is 91.2789, which is an upper bound of n by Lemma 1.27.

We deduce that the same bound holds for the case $a = 0$. Consequently, we use again Mathematica to ensure that equation (6.37) has no solutions in this case.

6.3.2 The case $k > 650$

From (6.33), we define

$$|\Lambda_4'| := -2n \log \varphi + d_2 \log 10 + \log(b(\varphi + 2)),$$

where,

$$0 < |\Lambda_4'| < \frac{12}{\varphi^{k/2}}. \tag{6.40}$$

To apply Lemma 1.27 to Λ_4', let us consider

$$(\eta_1, b_1) := (\varphi^2, -n), \quad (\eta_2, b_2) := (10, d_2), \quad (\eta_3, b_3) := (b(\varphi + 2), 1).$$

Then, $h(\eta_3) = h(b(\varphi + 2)) \leqslant h(b) + h(\varphi + 2) < 3.1$. Thus, we take

$$h'(\eta_1) = \frac{1}{2}, \quad h'(\eta_2) = \log 10 \quad \text{and} \quad h'(\eta_3) = 3.1.$$

Therefore, we find

$$k < 7 \times 10^{17} \quad \text{and} \quad n < 1.2 \times 10^{96}. \tag{6.41}$$

As we did before, we reduce these bounds and let put again

$$|\Lambda_4'| := -2n \log \varphi + d_2 \log 10 + \log(b(\varphi + 2)).$$

From (6.40), we have

$$0 < \left| -n \cdot \frac{2 \log \varphi}{\log 10} + d_2 + \frac{\log(b(\varphi + 2)}{\log 10} \right| < \frac{12}{\log 10} \cdot \varphi^{-(k/2)}.$$

for each $b \in \{1, \ldots, 9\}$, we apply Lemma 1.27 by setting

$$\gamma := \frac{2 \log \varphi}{\log 10}, \quad \mu_b := \frac{\log(b(\varphi + 2))}{\log 10}, \quad \text{and} \quad (A, B) := (\frac{12}{\log 10}, \varphi).$$

For the first application of Lemma 1.27, by using Mathematica we find that q_{209} satisfies the conditions requiered in Lemma 1.27 which gives $0.0245732 \leqslant \varepsilon_b = \|\mu_b q_{209}\| - M \cdot \|\gamma q_{209}\|$ and then $k < 974$. Thus, we obtain a reduced bound for (6.41). We repeat the same reduction algorithm with $M = 3.4 \times 10^{33}$ and q_{79} and as a result we obtain $0.123356 \leqslant \varepsilon_b = \|\mu_b q_{79}\| - M \cdot \|\gamma q_{79}\|$and $k < 358$. So, in all cases, we have $k < 650$ which is a contradiction.

Conclusion

In this thesis, we have investigated several Diophantine equations involving generalized sequences. Specifically, we constructed Diophantine equations using Fibonacci, Lucas, and Pell sequences. These equations were examined in connection with Fermat numbers, Mersenne numbers, and repdigits, considering various distinct cases.

The resolution of these equations relied primarily on Baker's method, mainly based on linear forms in logarithms of algebraic numbers. A central tool in our approach was the Baker-Davenport reduction method, which played a crucial role in refining upper bounds and facilitating the determination of integer solutions.

By combining these techniques with careful analysis of the properties of the generalized sequences, we were able to achieve significant progress in understanding the interplay between Diophantine equations and the algebraic and arithmetic structures of the sequences involved. This work not only addresses specific questions in the field but also highlights the potential of these methods for broader applications in number theory.

A natural perspective arising from this thesis is to explore whether similar studies can be conducted with other families of sequences. One could investigate generalized forms of other well-known sequences, and analyze their interplay with Fermat numbers, Mersenne numbers, or repdigits. Extending the methods and techniques developed in this thesis, such as those involving linear forms in logarithms and the Baker-Davenport reduction, could provide valuable insights into the solvability of new classes of Diophantine equations.

Bibliography

[1] K. N. Adédji, A. Filipin and A. Togbé, *Fibonacci and Lucas numbers as products of three repdigits in base g*, Rend. Circ. Mat. Palermo, II. Ser (2023), 4003–4021. Zbl 1532.11022, https://doi.org/10.1007/s12215-023-00878-4

[2] S. Alaca and K. S. Williams, *Introductory algebraic number theory*, Cambridge University Press, New York, 2004.

[3] A. Altassan and M. Alan, *Almost Repdigit k-Fibonacci Numbers with an Application of k-Generalized Fibonacci Sequences*, Mathematics, (2023), 11(2), 455

[4] A. Baker, *Linear forms in logarithms of algebraic numbers, I.* Mathematika J. Pure Appl. Math. **13**, 204-216 (1966).

[5] A. Baker, *Linear forms in logarithms of algebraic numbers, II.* Mathematika J. Pure Appl. Math. **14**, 102-107 (1967).

[6] A. Baker, *Linear forms in logarithms of algebraic numbers, II.* Mathematika J. Pure Appl. Math. **14**, 220-228 (1967).

[7] A. Baker and H. Davenport, *The equations* $3x^2 - 2 = y^2$ *and* $8x^2 - 7 = z^2$, Quart. J. Math. Oxford Ser. (2) **20** (1969), 129–137.

[8] A. Baker and G. Wüstholz, *Logarithmic forms and group varieties*, J. Reine Angew. Math, **442** (1993), 19-62.

[9] D. Bednařík and E. Trojovská, *Repdigits as product of Fibonacci and Tribonacci numbers*, Mathematics. **8** (10), 1720 (2020). https://doi.org/10.3390/math8101720

[10] K. Bhoi and P. K. Ray, *Repdigits as difference of two Fibonacci or Lucas numbers*, Mat. Stud. **56** (2021), 124-132. Zbl 1495.11029, https://doi.org/10.30970/ms.56.2.124-132

[11] J. J. Bravo, C. A. Gómez and F. Luca, *A Diophantine equation in k-Fibonacci numbers and repdigits*, Colloq. Math. **152** (2018), 299–315. Zbl 1407.11026, https://doi.org/10.4064/cm7149-6-2017

[12] J. J. Bravo, C. A. Gómez and F. Luca, *Mersenne k-Fibonacci numbers*, Glas. Mat. **51** (2016), 307–319.

[13] J. J. Bravo, C. A. Gómez and F. Luca, *Powers of two as sums of two k-Fibonacci numbers*, Miskolc Math. Notes **17** (2016), 85–100. Zbl 1389.11041, https://doi.org/10.18514/MMN.2016.1505

[14] J. J. Bravo and J. L. Herrera, *Fermat k-Fibonacci and k-Lucas numbers*, Mathematica Bohemica **145.1** (2020), 19-32.

[15] J. J. Bravo and J. L. Herrera, *Repdigits in generalized Pell sequences*, Archivum Mathematicum 2020; 56(4): 249–262. doi: 10.5817/AM2020-4-249

[16] J. J. Bravo, J. L. Herrera and F. Luca, *On a generalization of the Pell sequence*, Math. Bohem. 2021; 146(2): 199–213. doi: 10.21136/MB.2020.0098-19

[17] J.J. Bravo and F. Luca, *On a conjecture about repdigits in k-generalized Fibonacci sequences*, Publ. Math. Debrecen. **82** (2013), 623-639. Zbl 1274.11035, https://doi.org/10.5486/PMD.2013.5390

[18] J. J. Bravo and F. Luca, *Powers of two in generalized Fibonacci sequences*, Rev. Colombiana Mat. **46** (2012), 67–79. Zbl 1353.11020

[19] J.J. Bravo and F. Luca, *Repdigits as sums of two k-Fibonacci numbers*, Monatsh. Math. **176** (2015), 31–51. Zbl 1390.11034, https://doi.org/10.1007/s00605-014-0622-6

[20] J. J. Bravo and F. Luca, *Repdigits in k-Lucas sequences*, Proc. Indian Acad. Sci. Math. Sci. **124 (2)** (2014), 141–154, Zbl 1382.11019, https://doi.org/10.1007/s12044-014-0174-7

[21] Y. Bugeaud, M. Maurice, S. Siksek, *Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers*, Ann. Math. **163** (2006), 969-1018. Zbl 1113.11021, MR2215137, https://doi.org/10.4007/annals.2006.163.969

[22] P. Catarino, H. Campos, P. Vasco, *On the Mersenne sequence*, Ann. Math. Inform. **46** (2016), 37-53.

[23] P. Coufal, P. Trojovský, *Repdigits as product of terms of k-Bonacci sequences*, Mathematics. **9** (6), 682 (2021). https://doi.org/10.3390/math9060682

[24] G. Dresden and Z. Du, *A simplified Binet formula for k-generalized Fibonacci numbers*, J. Integer Sequences **17** (2014), Article 14.4.7.

[25] A. Dujella, *Number Theory*, Školska knjiga, Zagreb, (2021)

[26] A. Dujella, A. Pethő, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford Ser. (2) **49** (1998), no. 195, 291-306. Zbl 0911.11018

[27] F. Erduvan and R. Keskin, *Fibonacci and Lucas numbers as products of two repdigits*, Turk. J. Math. **43** (2019), 2142– 2153. Zbl 1440.11016

[28] F. Erduvan, R. Keskin and Z. Şiar, *Fibonacci or Lucas Numbers Which are Products of Two Repdigits in Base b*, Bull Braz Math Soc, New Series. **57** (2021), 1025–1040. Zbl 1484.11063, https://doi.org/10.1007/s00574-02100243-y

[29] F. Erduvan and R. Keskin, *Repdigits as products of two Fibonacci or Lucas numbers*, Proc. Indian Acad. Sci. Math. Sci. **2020**, 130, 28. Zbl 1440.11016, https://doi.org/10.1007/s12044-020-0554-0

[30] B. Faye and F. Luca, *Pell and Pell-Lucas numbers with only one distinct digits*, Ann. of Math. 2015; 45: 55–60.

[31] M. N. Faye, S. E Rihane and A. Togbé, *On Repdigits Which are Sums or Differences of Two $k-$Pell Numbers*, Mathematica Slovaca (2023); 73(6): 1409-1422.

[32] C. A. Gómez and F. Luca, *Power of two-classes in k-generalized Fibonacci sequences*, Revista Colombiana de Matemáticas. **48**(2) (2014), 219–234.

[33] A. Gueye, S. E. Rihane and A. Togbé, *Coincidence Between k-Fibonacci Numbers and Products of Two Fermat Numbers*, Bull Braz Math Soc, New Series 2022; 53: 541–552. doi: 10.1007/s00574-021-00269-2

[34] M. O. Hernane, S. E. Rihane, S. Seffah and A. Togbé, *On Fermat and Mersenne numbers expressible as product of two k-Fibonacci numbers*, Boletín de la Sociedad Matemática Mexicana. 28 (2), 36 (2022). doi: 10.1007/s40590-022-00429-4

[35] B. Kafle, S.E Rihane and A. Togbé, *A note on Mersenne Padovan and Perrin numbers*, The Notes on Number Theory and Discrete Mathematics 2021; 27 (1): 161–170. doi: 10.7546/nntdm.2021.27.1.161-170

[36] A. Ya. Khinchin, *Continued Fractions.* 3rd ed. Noordhoff. Groningen. 1963.

[37] M. Krízek, F. Luca, L. Somer, *17 Lectures on Fermat Numbers: From Number Theory to Geometry*, CMS Books in Mathematics, Vol. 9, Springer, New York, 2001.

[38] K. Liptai, G. K. Panda and L. Szalay, *A balancing problem on a binary recurrence and its associate*, Fibonacci Quart. **54** (3) (2016), 235-240.

[39] F. Luca, *Effective methods for Diophantine equations*, Bordeaux, January 23, 2009.

[40] F. Luca, *Fibonacci and Lucas numbers with only one distinct digit*, Port. Math. **57** (2000), 243–254.

[41] D. Marques, *On k-generalized Fibonacci numbers with only one distinct digit*, Util. Math. **98** (2015), 23–31. Zbl 1369.11014

[42] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers, II*, Izv. Math. **64(6)** (2000) 1217–1269. Zbl 1013.11043, https://doi.org/10.1070/im2000v064n06abeh000314

[43] D. Meguedmi, S. E. Rihane and A. Togbé, *Generalization of a theorem of Adegbindin, Luca and Togbé*, RACSAM **116** (2022), Article number: 36. Zbl 1480.11024, https://doi.org/10.1007/s13398-021-01177-2

[44] M. Mignotte, *Intersection des images de certaines suites récurrentes linéaires*, Theoret. Comput. Sci. 1978; 7(1): 117–122. doi: 10.1016/0304-3975(78)90043-9

[45] E. P. Jr. Miles, *Generalized Fibonacci numbers and associated matrices*, Amer. Math. Monthly. **67** (1960), 745âĂŞ-752. Zbl 0103.27203, https://doi.org/10.1080/00029890.1960.11989593

[46] M. D. Miller, *Mathematical notes: on generalized Fibonacci numbers*, Amer. Math. Monthly. **78** (1971), 1108–1109. Zbl 0236.30004, https://doi.org/10.1080/00029890.1971.11992952

[47] I. Niven, H. S. Zuckermann and H. L. Mongomery, *Introduction to the theory of numbers*, Springer New York, (1991).

[48] B. V. Normenyo, S. E. Rihane and A. Togbé, *Fermat and Mersenne numbers in k-Pell sequence*, Matematychni Studii 2021; 56(2): 115-123. doi: 10.30970/ms.56.2.115-123

[49] S. E. Rihane, *k-Fibonacci and k-Lucas Numbers as Product of Two Repdigits*, Results Math **76** (2021), Article number: 208. Zbl 1495.11035, https://doi.org/10.1007/s00025-021-01526-y

[50] S. E. Rihane, *k-Pell numbers as product of two repdigits*, Mediterr. J. Math. (2022). https://doi.org/10.1007/s00009-022-01983-x.

[51] S. E. Rihane, C. A. Adegbindin and A. Togbé, *Fermat Padovan and Perrin numbers*, J. Int. Seq. 23.6, Article 20.6.2 (2020).

[52] S. E. Rihane and A. Togbé, *On the intersection between k-Lucas sequences and some binary sequences* , Period Math Hung **84** (2022), 125–145. Zbl 1499.11077, https://doi.org/10.1007/s10998-021-00387-w

[53] S. G. Sánchez and F. Luca, *Linear combinations of factorials and S-units in a binary recurrence sequences*, Ann. Math. Québec 2014; 38: 169–188. doi: 10.1007/s40316-014-0025-z

[54] S. Seffah, S. E. Rihane and A. Togbé, *Almost Repdigits in k-Pell numbers.*

[55] S. Seffah, S. E. Rihane and A. Togbé, *Fermat and Mersenne numbers as products of two k-Pell numbers*, https://doi.org/10.1007/s13226-025-00806-5

[56] S. Seffah, S. E. Rihane and A. Togbé, *On equations involving repdigits and products of two k-Fibonacci*, accepted for publication.

[57] S. Seffah, S. E. Rihane and A. Togbé, *On repdigits as product of k-Fibonacci and k-Lucas numbers.*(2025). https://doi.org/10.21136/MB.2025.0035-24

[58] M. Waldshmidt, *Diophantine Approximation on Linear Algebraic Groups: Transcendence Properties of the Exponential Function in Several Variables*, Springer-Verag Berlin Heidelberg, 2000.

[59] B. M. M. de Weger, *Algorithms for Diophantine equations*, PhD, Eindhoven University of Technology, Eindhoven, the Netherlands. (1989). Zbl 0687.10013

[60] D. A. Wolfram, *Solving generalized Fibonacci recurrences*, Fibonacci Quart. **36(2)** (1998), 129–145. Zbl 0911.11014