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Dissertation Topic

Integrability of Some Differential Systems, Liénard Systems and LimitCycles

Publicly Defended,on **16 / 12 / 2024** 15h00 , In font of Commitee Members:

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Dedication

I dedicate this doctorate Thesis to those who paved the way
TO my father To my mother
To my brothers and my sisters
To all my family
To everyone I love and everyone who loves me.

Acknowledgments

In the name of Allah, the Most Gracious, the Most Merciful

All praise and thankfulness is due to Allah, i praise him and testify that there is no God but Allah, and that Mohammed is his slave and messenger (peace be upon him and all prophets and messengers)

Prophet Mohammed (Peace be upon him) said "He will not be thankful to Allah, he who would not be thankful to people "

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Finally, it is with great emotion that I express my deep gratitude to my family, especially my parents, my sister and my brothers.

Introduction

The first differential equations appeared towards in the end of the seventeenth century in the works of Isaac Newton, Leibniz, and Bernoulli. They occurred as a natural consequence of the efforts of these great scientists to apply the new ideas of calculus to certain problems in mechanics. Later the theory of integration of differential equations was developed by analysts and mechanicians such as Lagrange, Poisson, Hamilton, Liouville in the eighteenth and nineteenth centuries.

For over 300 years, differential equations have served as the essential tool for describing and analyzing problems in many scientific disciplines.

The importance of differential equations has motivated generations of mathematicians and other scientists to develop methods to study the properties of their solutions. With his memoir on curves defined by a differential equation published in 1886, Henri Poincaré [28] opened the way for an approach to differential equations where priority is no longer given to resolution, but to a more geometric study of the solutions, in particular their properties; this research aims to find the properties of the solutions without really finding the solutions in an explicit way; these are qualitative methods.

In this work we will use the qualitative theory of ordinary differential equations to treat a class of planar differential systems of the form

$$\begin{cases} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{cases}$$

such that $P(x;y)$; and $Q(x;y)$ are polynomials.

One of the main problems in the qualitative theory of differential equations is the study of integrability and limit cycles of planar differential systems and especially polynomial planar differential systems,

see for example [[33], [31], [22], [24], [32]...], and the book by Ye Yanqian et al. [38] devoted only to the study of limit cycles, mainly of quadratic polynomial differential systems. The interest of limit cycles of planar differential systems is due to their important meanings in mathematical models resulting from practice in several branches of science (Physics, Biology, Economics, ...).

It was in 1900 that D. Hilbert [13] posed the famous twenty- three problems. In particular the 16th problem, he poses the question of the number and arrangement of isolated periodic trajectories for polynomial differential systems.

This problem, so far unsolved. Much recent work is devoted to the study of limit cycles, see for example the papers [[33], [22], [10], [26], [23]].

In this work we will study a fairly large class of planar differential systems of the form:

$$\begin{cases} \dot{x} &= x R_m(x,y) + P_n(x,y), \\ \dot{y} &= y R_m(x,y) + Q_n(x,y), \end{cases} \quad (Sh)$$

such that $P_n; Q_n; R_m$ are homogeneous polynomials of degree n and m respectively. First we determine a class of systems (Sh), which has a non-algebraic limit cycle, moreover we determine the exact expression of this limit cycle. This work is published in [Applied Mathematical Sciences, [4]]. The second result obtained in this work, concerns the integrability in the sense of Darboux of the system (Sh), then we study the existence and non-existence of the limit cycles which surround the singular point located at the origin. A very important case of this class is a system with homogeneous nonlinearity and a star node at the origin

$$\begin{cases} \dot{x} &= \lambda x + P_n(x,y), \\ \dot{y} &= \lambda y + Q_n(x,y). \end{cases}$$

This work is published in [Journal of Differential Equations, [8]]. Another result obtained in this thesis concerns quadratic differential systems with a focus at the origin.

It is important to study these questions: **first integral, periodic solution, limit cycle, phase portrait**. The results obtained in this thesis are articulated around these questions.

In the first chapter we presented the more general result indicated for a Pfaffian system see [35]

In the second, we presented certain basic results, concerning the qualitative theory of differential systems, in particular planar polynomial differential systems, for the understanding of the sequence.

We treated a class of planar differential systems homogeneous, this chapter is divided into two parts, in the first part we have determined a class of polynomial differential systems which has a cycle non-algebraic limit, and we devoted the second part to another class of homogeneous planar differential systems, we studied the existence and non-existence of limit cycles which surround the singular point located at the origin, in particular the very interesting case of this class is a system with no homogeneous linearity and a star node at the origin.

To our knowledge it is rare to find in the literature differential systems with a limit cycle not algebraic is explicitly given.

Finally, we studied integrability and phase portraits globals of a class of differential systems which have a limit cycle. What presents our contribution in this result is the description of three new global live portraits

For perspectives, it is convenient to hope to find a class o differential systems which admits an explicitly given non-algebraic limit cycle, and to describe the phase portrait of a differential system which has a focus and a limit cycle.

Chapter 1

Formal and analytic solutions of some singular differential systems

We study here completely integrable Pfaffian systems of the form:

$$dy = \frac{f^1(x_1, x_2, y)}{{x_1}^{p_1+1}}dx_1 + \frac{f^2(x_1, x_2, y)}{{x_2}^{p_2+1}}dx_2 \tag{1.1}$$

with $p_1 > 0$ and $p_2 > 0$. and

where for $i = 1, 2$, f^i is holomorphic in $U^1 \times U^2 \times V$

$$U^i = \{x_i / 0 \ \le |x_i| \le r_i\} \ i = 1, 2..$$

$$V = \{y \in \mathbb{C}^m / \ 0 \le |y_i| < \rho_i\}$$

1.1 Existence of formal solution

This system can be expressed in the form

$$\begin{cases} {x_1}^{p_1+1}\dfrac{dy}{dx_1} &= f_1(x_1, x_2, y); \quad p_1 > 0 \\ {x_2}^{p_2+1}\dfrac{dy}{dx_2} &= f_2(x_1, x_2, y); \quad p_2 > 0 \end{cases}$$

Let's write

$$f_i(x_1, x_2, y) = {f_0}^i(x_1, x_2) + A^i(x_1, x_2)y + R^i(x_1, x_2, y). \tag{1.2}$$

The complete integrability condition of the system

$$dy = \frac{f^1(x_1, x_2, y)}{{x_1}^{p_1+1}}dx_1 + \frac{f^2(x_1, x_2, y)}{{x_2}^{p_2+1}}dx_2 \quad p_1 > 0 \ and \quad p_2 > 0. \tag{1.3}$$

is written as

$${x_1}^{p_1+1}\frac{df_2}{dx_1} = {x_2}^{p_2+1}\frac{df_1}{dx_2} \tag{1.4}$$

i.e.$D_{12} = D_{21}$ where

$$\begin{aligned} D_{12}(x_1, x_2, y) \quad &= {x_2}^{p_2+1}\left(\frac{df_1}{dx_2} + \frac{df_1}{dy}.\frac{dy}{dx_2}\right) \\ &= {x_2}^{p_2+1}\left(\frac{d{f_0}^1}{dx_2} + \frac{A^1}{dx_2}\, y + \frac{dR^1}{dx_2}\right) + \frac{df_1}{dy}\, f_2 \end{aligned}$$

we have $[A^1(0,0), A^2(0,0)] = 0$, indeed:

$$D_{12}(0,0,y) = \left[\frac{d}{dy}(A^1\, y + R^1)\right]_{x_1=x_2=0} f_2(0,0,y)$$

$$\frac{d}{dy}\,(A^1\, y + R^1) = A^1 + \frac{dR^1}{dy} \ \textit{and}\ f_2(0,0,y) = \ A^2(0,0)y + 0(y^2)$$

if $\tilde{\mathrm{D}}_{12}(0,0,y)$ denotes the linear term in y in $D_{12}(0,0,y)$ we see that

$$\tilde{D}_{12}(0,0,y) = A^1(0,0)\, A^2(0,0) \tag{1.5}$$

By a similarity $T \in GL(p, \mathbb{C})$ it is possible to put $A^i(0,0), (i = 1,2,...)$ in the form

$$\tilde{A}^i = \begin{pmatrix} (A^i)_1 & 0 & \cdots & 0 \\ 0 & (A^i)_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & (A^i)_r \end{pmatrix}$$

with for all $s = 1, 2, \cdots, r$

$$(A^i)_s = {\lambda^i}_s I_s\ (N^i)_s$$

where I_s is the identity matrix of the same order as $(A^i)_s$, $(N^i)_s$ a nilpotent matrix in a higher triangular .

Furthermore we can suppose that for all $s = 1, 2, \cdots, r$ there exists $i \in \{1, 2\}$ such as ${\lambda^i}_s \neq {\lambda^i}_{s+1}$.

We also have

$$\left[(A^1)_s, (A^2)_s\right] = \left[(N^1)_s, (N^2)_s\right] = 0.$$

Let us now consider the formal system associated with system (1.1) as follows: We replace

$$f^i(x_1, x_2, y) = \sum_{|p|=0}^{+\infty} \tilde{f}^i_p(x_1, x_2) y^p$$

by the formal series which we will again note $f^i(x_1, x_2, y)$ obtained by replacing the coefficients $\tilde{f}^i_p(x_1, x_2)$ by their asymptotic expansions in S.

Theorem 1.

If one of the matrices $A^i(0,0)$ is non- singular, then the formal system 1.2 admits a formal solution of form

$$\varphi = \sum_{|r|=0}^{+\infty} \varphi_{r_1,r_2}\, {x_1}^{r_1}\, {x_2}^{r_2}$$

Proof 1. *Suppose that $A^1(0,0)$ is non- singular and consider the first equation of system 1.6*

$$x_1^{p_1+1}\frac{dy}{dx_1} = f_0{}^1(x_1,x_2) + A^1(x_1,x_2)\,y + R^1(x_1,x_2,y) \tag{1.6}$$

where $R^1(x_1,x_2,y)$ is of order greater than equal to 2 in y.

By identification we easily find that the coefficients of the formal series φ are given by

$$A^1(0,0)\,\varphi_{10} + (f_0{}^1)_{10} = 0$$

$$A^1(0,0)\,\varphi_{01} + (f_0{}^1)_{01} = 0$$

and more generally

$$(r_1,r_2)\,A^1(0,0)\,\varphi_{r_1,r_2} = \Im_{r_1,r_2}$$

if all these equations are ordered properly, we see that $\Im_{r_1,r_2}$ is a quantity which only contains coefficients known from the data as well as solutions to the equations which precede it.

So this infinite system is solvable.

Let us show that the formal solution φ_* of 1.6 *is also a solution of (1.2).*

We have

$$\begin{aligned} x_1^{p_1+1}\tfrac{d}{dx_1}\left(x_2^{p_2+1}\tfrac{d\varphi}{dx_2}\right) &= x_2^{p_2+1}\tfrac{d}{dx_2}\left(x_2^{p_2+1}\tfrac{d\varphi}{dx_2}\right) \\ &= x_2^{p_2+1}\tfrac{d}{dx_2}\Big(f_1(x_1,x_2,\varphi)\Big) \\ &= x_2^{p_2+1}\tfrac{df_1}{dx_2}(x_1,x_2,\varphi) + \tfrac{df_1}{dy}(x_1,x_2,\varphi)x_2^{p_2+1}\tfrac{d\varphi}{dx_2} \end{aligned}$$

The system 1.1 *being completely integrable, we have*

$$x_2^{p_2+1}\frac{df_1}{dx_2}(x_1,x_2,y) + \frac{df_1}{dy}(x_1,x_2,y)\,f_2(x_1,x_2,y) = x_1^{p_1+1}\frac{df_2}{dx_1}(x_1,x_2,y) + \frac{df_2}{dy}(x_1,x_2,y)\,f_1(x_1,x_2,y)$$

We have this identically in y. And given the hypothesis on the derivability of our asymptotic developments, this last condition is still formally valid, that is to say that the formal system $(I\ddot{y})$ is completely integrable.

So we have :

$$x_2^{p_2+1}\frac{df_1}{dx_2}(x_1,x_2,\varphi) + \frac{df_1}{dy}(x_1,x_2,\varphi)\,f_2(x_1,x_2,\varphi) = x_1^{p_1+1}\frac{df_2}{dx_1}(x_1,x_2,\varphi) + \frac{df_2}{dy}(x_1,x_2,\varphi)\,f_1(x_1,x_2,\varphi)$$

taking into account this equality and the fact that

$$f_1(x_1,x_2,\varphi) = x_1^{p_1+1}\frac{d\varphi}{dx_1}$$

we obtain

$$x_1^{p_1+1}\frac{d}{dx_1}\left(x_2^{p_2+1}\frac{d\varphi}{dx_2}\right) = x_1^{p_1+1}\frac{df_2}{dx_1}(x_1,x_2,\varphi)+\frac{df_2}{dy}(x_1,x_2,\varphi)\,x_1^{p_1+1}\,\frac{d\varphi}{dx_1}-\frac{df_1}{dy}(x_1,x_2,\varphi)\,f_2(x_1,x_2,\varphi)+\frac{df_1}{dy}\,x_2^{p_2+1}\frac{d\varphi}{dx_2}$$

that's to say

$$x_1^{p_1+1}\frac{d}{dx_1}\left(x_2^{p_2+1}\frac{d\varphi}{dx_2} - f_2(x_1,x_2,\varphi)\right) = \frac{df_1}{dy}(x_1,x_2,\varphi)\left(x_2^{p_2+1}\frac{d\varphi}{dx_2} - f_2(x_1,x_2,\varphi)\right)$$

The formal series $v = {x_2}^{p_2+1}\frac{d\varphi}{dx_2} - f_2(x_1, x_2, \varphi)$ *is therefore solution of the linear system*

$$ {x_1}^{p_1+1}\frac{du}{dx_1} = \frac{df_1}{dy}(x_1, x_2, \varphi(x_1, x_2))\, u $$

Or

$$ \frac{df_1}{dy} = A^1(x_1, x_2) = \bigcirc(|\varphi|). $$

But given the hypothesis about $A^1(0,0)$ *the only formal solution of this system is the zero solution. He it follows that formally*

$$ {x_2}^{p_2+1}\frac{d\varphi}{dx_2} = f_2(x_1, x_2, \varphi). $$

So φ *is also solution of the second equation. This ends the proof of Theorem 1.*

*The formal solution given by Theorem*1 *is unique*

i If none of the matrices $(0,0)$ *is regular, the formal system can still have*

has formal solutions.

If the functions $A^i(0,0)$ *satisfy the conditions stated at the beginning of this paragraph, and if in addition both are simultaneously matrices* $A^i(x_1, x_2, y)$ *diagonalizable and both connected, then if* φ *is a formal solution of the Pfaffian system*

$$ dy = \frac{f^1(x_1, x_2, y)}{{x_1}^{p_1+1}}dx_1 + \frac{f^1(x_1, x_2, y)}{{x_2}^{p_2+1}}dx_2 \tag{1.7} $$

which verifies $\varphi(0,0) = 0$*. There exists a solution* ϕ *of (1.1) holomorphic in a sector* $S' \subset S$ *such that* ϕ *is asymptotic to* φ *in the sector* S'*, i.e.*

$$ \phi \overset{S'}{\sim} \varphi $$

If the sector S is large enough, this result remains valid if we only assume that

$$ A^1(0) = \begin{pmatrix} (\lambda^1)_1 & 0 & \cdots & 0 \\ 0 & (\lambda^1)_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & (\lambda^1)_m \end{pmatrix} A^2(0) = \begin{pmatrix} (\lambda^2)_1 & 0 & \cdots & 0 \\ 0 & (\lambda^2)_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & (\lambda^2)_m \end{pmatrix} $$

with $|\lambda^1_k| + |\lambda^k_2| \neq 0$ *for all* $k = 1, 2, ..., m$*, In particular, the above result and the more general result indicated for a Pfaffian system, completely integable of the form*

$$ dy = \left({x_1}^{p_1+1}\right)^{-1} f^1(x_1, x_2, y)dx_1 + \left({x_2}^{p_2+1}\right)^{-1} f^2(x_1, x_2, y)dx_2 $$

Or

$$ {x_1}^{p_1+1} = \begin{pmatrix} {x_1}^{{p_1}^1+1} & 0 & \cdots & 0 \\ 0 & {x_1}^{{p_1}^2+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & {x_1}^{{p_1}^m+1} \end{pmatrix}; \; {x_2}^{p_2+1} = \begin{pmatrix} {x_2}^{{p_2}^2+1} & 0 & \cdots & 0 \\ 0 & {x_2}^{{p_2}^2+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & {x_2}^{{p_2}^m+1} \end{pmatrix} $$

with ${p_i}^j > 0$ *for all* $i = (1, 2)$*,* $j = 1, 2, ...,$

1.1.1 Case of asymptotic developments

Functions $f^1 f^2$ admit uniform asymptotic expansions in x_1 in (x_2, y). Let and U be a polydisk $S = S_1 \times S_2$ a sector $\mathbb{C}^2$ of centered at the origin of $\mathbb{C}^m$. We assume in this paragraph that

1. $f^1 f^2$ are holomorphic in $S_1 \times S_2 \times U$.
2. for $i = 1, 2$

$$f^2(x_1, x_2, y) \simeq \tilde{u} \times \tilde{S_2} \times \tilde{V} \simeq \sum_{m\geq 0} f^i{}_{1,m}(x_2, y){x_1}^m$$

$$f^2(x_1, x_2, y) \simeq \tilde{u} \times \tilde{S_1} \times \tilde{V} \simeq \sum_{m\geq 0} f^i{}_{2,m}(x_2, y){x_2}^m$$

where for all

$$f^i(x_2, y) \simeq u \tilde{\times} V \simeq \sum_{m\geq 0} f^i{}_{1,m,i}(y){x_2}^m$$

$$f^i{}_{i,m}(x_1, y) \simeq u \tilde{\times} V \simeq \sum_{m\geq 0} f^i{}_{2,m,i}(y){x_1}^m$$

The coefficients are not holomorphic in U, given these hypotheses, we have

$$f_i(x_1, x_2, y) = {f_0}^i(x_1, x_2) + A^i(x_1, x_2)\, y + R^i(x_1, x_2, y)$$

with

$$A^i(x_1, x_2) \simeq u \tilde{\times} s_2 \simeq \sum_{m\geq 0} A^i{}_m(x_2){x_1}^m$$

$$A^i{}_m(x_2) \simeq \tilde{s_2} \simeq \sum_{m\geq 0} f^i{}_{2,m,i}{x_2}^m$$

we further assume

3. ${f_{1,0}}^1(0, x_0) = 0(x_2)$
4.$A^1{}_{00}$ invertible
5. if ${\lambda_1}^j (j = 1, 2, ..., m)$ are the eigenvalues of $A^1{}_{00}$

$$|arg(-){\lambda_1}^j - P_1\, arg(x_1)| \leq \frac{3\pi}{2}\ in\ S_1\ \ j = 1, 2, ..., m$$

If the opening of S_1 is strictly greater than a $\frac{\pi}{P_1}$, $\exists$ two strictly positive numbers R_1, R_2 and solution φ of (1) holomorphic in$\sum_1 \times \sum_2$ where $\sum_i = S_i \cap \{|x_i| < R_i\}$ such that

$$\varphi(x_1, x_2) \simeq u \tilde{\times} s_2 \simeq \sum_{m\geq 0} \varphi_m(x_2){x_1}^m$$

with

$$\varphi_m(x_2) \simeq \tilde{s_2} \simeq \sum_{m\geq 0} \varphi_{m,i}{x_2}^m$$

and

$$\varphi_0(x_2) = O(x_2).$$

1.2 Analyticity of formal solutions

We will study here completely integrable Pfaffian systems having the the form $dy = \omega y$ with

$$\omega = \frac{A_1(x_1, x_2)}{{x_1}^{p_1+1}} + \frac{A_2(x_1, x_2)}{{x_2}^{p_2+1}}$$

with $p_1 > 0$ and $p_2 > 0$.

1.2.1 The scalar case

We consider completely integrable systems of the form

$$dy = \left(\frac{a_1(x_1, x_2)}{{x_1}^{p_1+1}} dx_1 + \frac{a_2(x_1, x_2)}{{x_2}^{p_2+1}} dx_2\right) y \tag{1.8}$$

where a_1 and a_2 are formal series or series convergent at the origin.

The condition of complete integrability is

$${x_2}^{p_2+1} \frac{da_1}{dx_2} = {x_1}^{p_1+1} \frac{da_2}{dx_1} \ (CI)$$

we can assume that $p_1 \leq p_2$

The condition (CI) means that

$$a_1(x_1, x_2) = {x_1}^{p_1+1}\left[\int_{h_1}^{x_2} \varphi_1(x_1, t_2) dt_2 + h_1(x_1)\right] + {x_1}^{p_1} {\alpha_1}^{p_1} + \cdots + {\alpha_1}^0$$

$$a_2(x_1, x_2) = {x_2}^{p_2+1}\left[\int_{h_1}^{x_1} \varphi_1(t_1, x_2) dt_1 + h_2(x_2)\right] + {x_2}^{p_2} {\alpha_2}^{p_2} + \cdots + {\alpha_2}^0$$

Proposition 2.

φ is a formal (or convergent) series in arbitrary x_1, x_2. $h_1 (resp.\ h_2)$ a formal (or convergent) series in x_1 $(resp.\ x_2)$ arbitrary.

${\alpha_1}^0, {\alpha_1}^1, ..., {\alpha_1}^{p_1}$ *and* ${\alpha_2}^0, {\alpha_2}^1, ..., {\alpha_2}^{p_2}$ are arbitrary constants.

Proof 2. *Let us set*

$$\frac{\partial a_1}{dx_2} = {x_1}^{p_1+1}\ c_{12}(x_1, x_2)$$

$$\frac{\partial a_2}{dx_1} = {x_2}^{p_2+1}\ c_{21}(x_1, x_2)$$

the condition (CI) is then $c_{12} = c_{21}$*, and*

$$a_1 = {x_1}^{p_1+1} \int_0^{x_2} c_{12}(x_1, t_2)\, dt_2 + \alpha_1(x_1)$$

$$a_2 = {x_2}^{p_2+1} \int_0^{x_1} c_{21}(t_1, x_2)\, dt_1 + \alpha_2(x_2)$$

where α_1 and α_2 are arbitrary series. By setting

$$\alpha_1(x_1) = {\alpha_1}^0 + x_1\tilde{\alpha}_1(x_1)$$

$$\alpha_2(x_2) = {\alpha_2}^0 + x_2\tilde{\alpha}_2(x_2)$$

we can write

$$\alpha_1(x_1, x_2) = x_1{\alpha_1}^1(x_1, x_2) + {\alpha_1}^0$$
$$\alpha_2(x_1, x_2) = x_2{\alpha_2}^1(x_1, x_2) + {\alpha_2}^0$$

with

$${\alpha_1}^1(x_1, x_2) = {x_1}^{p_1} \int_0^{x_2} c_{12}(x_1, t_2)\, dt_2 + \tilde{\alpha}_1(x_1)$$

$${\alpha_2}^1(x_1, x_2) = {x_2}^{p_2} \int_0^{x_1} c_{21}(t_1, x_2)\, dt_1 + \tilde{\alpha}_2(x_2)$$

The condition (CI) is then

$${x_2}^{p_2} \frac{d{\alpha_1}^1}{dx_2} = {x_1}^{p_1} \frac{d{\alpha_2}^1}{dx_1}$$

Then, by proceeding with ${\alpha_1}^1$ and ${\alpha_2}^1$ as we did above with α_1 and α_2, we obtain by induction ($p_1 \leq p_2$), that:

$$\alpha_1(x_1, x_2) = {x_1}^{p_1+1}\, {\alpha_1}^{p_1+1}(x_1, x_2) + {x_1}^{p_1}\, {\alpha_1}^{p_1} + \cdots + {\alpha_1}^0$$
$$\alpha_2(x_1, x_2) = {x_2}^{p_1+1}\, {\alpha_2}^{p_1+1}(x_1, x_2) + {x_2}^{p_1}\, {\alpha_2}^{p_1} + \cdots + {\alpha_2}^0$$

with

$${x_2}^{p_2-p_1} \frac{d{\alpha_1}^{p_1+1}}{dx_2} = \frac{d{\alpha_2}^{p_1+1}}{dx_1}$$

Let us put $\varphi(x_1, x_2) = \dfrac{d{\alpha_1}^{p_1+1}}{dx_2}$, then

$$\frac{d{\alpha_2}^{p_1+1}}{dx_1} = {x_2}^{p_2-p_1}\, \varphi(x_1, x_2),$$

and

$${\alpha_1}^{p_1+1}(x_1, x_2) = \int_{h_2}^{x_2} \varphi(x_1, t_2)dt_2 + h_1(x_1)$$
$${\alpha_2}^{p_1+1}(x_1, x_2) = {x_2}^{p_2-p_1} \int_{h_1}^{x_1} \varphi(t_1, x_2)dt_1 + k(x_1)$$

where h_1 and k are arbitrary series, and

$$\alpha_1(x_1, x_2) = {x_1}^{p_1+1}\Big(\int_{h_2}^{x_2} \varphi(x_1, t_2)dt_2 + h_1(x_1)\Big) + {x_1}^{p_1}\, {\alpha_1}^{p_1} + \cdots + {\alpha_1}^0$$

$$\alpha_2(x_1, x_2) = {x_2}^{p_2+1}\Big(\int_{h_1}^{x_1} \varphi(t_1, x_2)dt_1 + h_2(x_2)\Big) + {x_2}^{p_2}\, {\alpha_2}^{p_2} + \cdots + {\alpha_2}^0$$

which proves Proposition 2.

Theorem 3.

The solutions of system (2.2) are of the form

$$CU\,(x_1, x_2){x_1}^{\rho_1}\ {x_2}^{\rho_2}\ exp\ P_1(\frac{1}{x_1})\ exp\ P_2(\frac{1}{x_2})$$

Where C, ρ_1, ρ_2 are constants

U a formal or convergent series depending on whether

α_1 and α_2 are formal or convergent.

$P_i(\frac{1}{x_i})$ a polynomial of degree p_i in $\frac{1}{x_i}$

Proof 3. *System (2.2) has the following form:*

$$dy = (w_1 + w_2)$$

with

$$w_1 = \Big(\int_{h_2}^{x_2} \varphi(x_1, t_2)dt_2 + h_1(x_1)\Big)dx_1 + \Big(\int_{h_1}^{x_1} \varphi(t_1, x_2)dt_1 + h_2(x_2)\Big)dx_2$$

$$w_2 = \Big(\frac{{\alpha_1}^{p_1}}{x_1} + \cdots + \frac{{\alpha_1}^{0}}{{x_1}^{p_1+1}}\Big)dx_1 + \Big(\frac{{\alpha_2}^{p_2}}{x_2} + \cdots + \frac{{\alpha_2}^{0}}{{x_2}^{p_2+1}}\Big)dx_2$$

and the two systems

$$dz = w_1\, z \tag{1.9}$$

$$dz = w_2\, z \tag{1.10}$$

are separately complement integrable. System (2.4) admits a fundamental matrix of the form :

$$v(x_1, x_2) = exp\left\{\int_{h_1}^{t_1} \Big(\frac{{\alpha_1}^{p_1}}{t_1} + \cdots + \frac{{\alpha_1}^{0}}{{t_1}^{p_1+1}}\Big)dt_1\right\}\ exp\left\{\int_{h_2}^{t_2} \Big(\frac{{\alpha_2}^{p_1}}{t_2} + \cdots + \frac{{\alpha_2}^{0}}{{t_2}^{p_2+1}}\Big)dt_2\right\}$$

By setting $y = Vu$, seen in (2.2), we get

$$du = w_1\, u$$

that's to say

$$\frac{du}{dx_1} = \Big(\int_{h_2}^{x_2} \varphi(x_1, t_2)dt_2 + h_1(x_1)\Big)u_1,$$

$$\frac{du}{dx_2} = \Big(\int_{h_1}^{x_1} \varphi(t_1, x_2)dt_1 + h_2(x_1)\Big)$$

By integration, we obtain

$$u = k\, exp\left\{\int_{h_1}^{x_1} h_1(t_1)dt_1\right\}\ exp\left\{\int_{h_2}^{x_2} h_2(t_2)dt_2\right\}\ exp\ \left\{\int_{h_1}^{x_1}\int_{h_2}^{x_2} \varphi(t_1, t_2)dt_1\, dt_2\right\}$$

where k is a constant.

Eventually

$$u = \tilde{k}\, exp\left\{\int_{h_1}^{x_1} h_1(t_1)dt_1\right\}\ exp\left\{\int_{h_2}^{x_2} h_2(t_2)dt_2\right\}\ exp\ \left\{\int_{h_1}^{x_1}\int_{h_2}^{x_2} \varphi(t_1, t_2)dt_1\, dt_2\right\}\times {x_1}^{{\alpha_1}^{p_1}}\times {x_2}^{{\alpha_2}^{p_2}}\times\Big(p_1(\frac{1}{x_1})\Big)$$

where $\tilde{k}$ is another constant, which is a slightly more precise form than the one given in the proposition. If the data are convergent series, then the series obtained in the result are also convergent.

1.3 Formal reduction lemmas.

We now assume that system (2) is a formal system, that is to say that the elements of the matrices $A_i(x_1, x_2)$ are formal series. To avoid having to write too many clues, we will note

$$\omega = \frac{A(x_1, x_2)}{{x_1}^{p_1+1}}dx_1 + \frac{B(x_1, x_2)}{{x_2}^{p_2+1}}dx_2 \; p_1 > 0 \; and \; p_2 > 0$$

Total Reduction Lemma:
If

$$A(0,0) = \begin{pmatrix} {A^{11}}_{00} & 0 \\ 0 & {A^{00}}_{22)} \end{pmatrix} and \; B(0,0) = \begin{pmatrix} {B^{11}}_{00} & 0 \\ 0 & {B^{00}}_{22)} \end{pmatrix}$$

Where one of the couple $({A^{11}}_{00}; {A^{22}}_{00}), ({B^{11}}_{00}; {B^{22}}_{00})$ is without common eigenvalues, there exists a unique formal transformation T of the form

$$T = \begin{pmatrix} I & T^{12} \\ T^{21} & I \end{pmatrix}$$

which transforms system (1) into

$$dz = \omega' z$$

with

$$\omega' = \frac{\begin{pmatrix} a^{11} & 0 \\ 0 & a^{22} \end{pmatrix}}{{x_1}^{p_1+1}}dx_1 + \frac{\begin{pmatrix} b^{11} & 0 \\ 0 & b^{22} \end{pmatrix}}{{x_2}^{p_2+1}}dx_2$$

and $a^{ii}(0,0) = {A_{00}}^{ii} \; b^{ii}(0,0) = {B_{00}}^{ii}$

Proof 4. *Let's write*

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}$$

and

$$B = \begin{pmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{pmatrix}$$

and seek a transformation

$$T = \begin{pmatrix} I & T^{12} \\ T^{21} & I \end{pmatrix}, \quad I = identity$$

which puts the system in the following form:

$$A = \begin{pmatrix} A^{11} & 0 \\ 0 & A^{22} \end{pmatrix} B = \begin{pmatrix} B^{11} & 0 \\ 0 & B^{22} \end{pmatrix}$$

Then, an easy calculation gives us to determine T^{12} and T^{21} by the following systems:

$$\begin{cases} {x_1}^{p_1+1}\dfrac{dT^{12}}{dx_1} & = & A^{12} + A^{11}\,T^{12} - T^{12}\,A^{22} - T^{12}\,A^{21}\,T^{12} \\ {x_2}^{p_2+1}\dfrac{dT^{12}}{dx_2} & = & B^{12} + B^{11}\,T^{12} - T^{12}\,B^{22} - T^{12}\,B^{21}\,T^{12} \end{cases}$$

$$\begin{cases} x_1^{p_1+1}\dfrac{dT^{21}}{dx_1} & = & A^{21} + A^{22}\,T^{21} - T^{21}\,A^{11} - T^{21}\,A^{12}\,T^{21} \\ x_2^{p_2+1}\dfrac{dT^{21}}{dx_2} & = & B^{21} + B^{22}\,T^{21} - T^{21}\,B^{11} - T^{21}\,B^{12}\,T^{21} \end{cases}$$

Let us only consider the system in the variable T^{12}, for the other the reasoning is identical. If we arrange T^{12} in a single column $\tilde{T}^{12}$, for example begining with the elements of the first line of T^{12}, we easily see that $\tilde{T}^{12}$ is given by a non-linear and completely integrable Pfaffian System of the following form:

$$x_1^{p_1+1}\frac{dz}{dx_1} = f_0{}^1(x_1,x_2) + C_1(x_1,x_2)z + f_2{}^1(x_1,x_2,z)$$

$$x_2^{p_2+1}\frac{dz}{dx_2} = f_0{}^2(x_1,x_2) + C_2(x_1,x_2)z + f_2{}^2(x_1,x_2,z)$$

where C_1 and C_2 come respectively from

$$A^{11}\,T^{12} - T^{12}\,A^{22},\; B^{11}\,T^{12} - T^{12}\,B^{22}$$

like one of the couple $(A_{11}{}^{00}, A_{22}{}^{00})$ $(B_{11}{}^{00}, B_{22}{}^{00})$ is without common eigenvalues, it results that one of the matrices $C_1(0,0), C_2(0,0)$ is non- singular, then we obtain the existence of the sought solution $\tilde{T}^{12}$. We thus obtain T^{12} and in the same way T^{21}

Partial Reduction Lemma: *If*

$$A(0,0) = \begin{pmatrix} A^{11} & 0 \\ 0 & A^{22} \end{pmatrix} \text{ and } B(0,0) = \begin{pmatrix} B^{11} & 0 \\ 0 & B^{22} \end{pmatrix}$$

Where one of the couples $(A^{11}{}_{00}; A^{22}{}_{00}), (B^{11}{}_{00}; B^{22}{}_{00})$ is without common eigenvalues, there exists a unique formal transformation T of the form

$$T = \begin{pmatrix} I & T^{12} \\ T^{21} & I \end{pmatrix}$$

which transforms system (2) into

$$dz = w^1{}_l\, z$$

with

$$w^1{}_l = \frac{\begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix}}{x_1{}^{p_1+1}}dx_1 + \frac{\begin{pmatrix} b^{11} & b^{12} \\ b^{21} & b^{22} \end{pmatrix}}{x_2{}^{p_2+1}}dx_2$$

and $a^{12}{}_{p_1,p_2} = a^{21}{}_{p_1,p_2} = b^{12}{}_{p_1,p_2} = b^{21}{}_{p_1,p_2} = 0$ for all verifying $P1 + P2 \geq l$

Moreover $a^{ii}(0,0) = A^{ii}{}_{00},\; b^{ii}(0,0) = B^{ii}{}_{00}$ for $i = 1,2$.

Proof 5. *suppose that*

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix};\; B = \begin{pmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{pmatrix}$$

with a $A^{12}{}_{p_1,p_2} = A^{21}{}_{p_1,p_2} = B^{12}{}_{p_1,p_2} = B^{21}{}_{p_1,p_2} = 0$ *for any pair* (p_1, p_2) *satisfying* $P1 + P2 < m < l$*, let us show that there is a unique transformation of the form*

$$T_m = \begin{pmatrix} I & T^{12} \\ T^{21} & I \end{pmatrix}$$

with

$$T^{12} = \sum_{p+q=m} T^{12}{}_{p,q} x_1{}^p\ x_2{}^q \text{ and } T^{21} = \sum_{p+q=m} T^{21}{}_{p,q} x_1{}^p\ x_2{}^q$$

which transforms (2) into a system

$$dz = (\frac{a\ dx_1}{x_1{}^{p_1+1}} + \frac{b\ dx_2}{x_2{}^{p_2+1}}\ z)$$

Where

$$a^{12}{}_{p,q} = a^{21}{}_{p,q} = b^{12}{}_{p,q} = b^{21}{}_{p,q} = 0$$

for any couple (p, q) *verifying* $p + q \geq m$.

To determine T_m *we have*

$$A^{11} + A^{12}T^{21} - a^{11} - T^{12}a^{21} = 0 \tag{1.11}$$

$$B^{11} + B^{12}T^{21} - b^{11} - T^{12}b^{21} = 0 \tag{1.12}$$

$$A^{22} + A^{21}T^{12} - a^{22} - T^{21}a^{12} = 0 \tag{1.13}$$

$$B^{22} + B^{21}T^{12} - b^{22} - T^{21}b^{12} = 0 \tag{1.14}$$

$$\begin{cases} x_1{}^{p_1+1}\dfrac{dT^{12}}{dx_1} & = & A^{11}\ T^{12} + A^{12} - T^{12}\ a^{22} - a^{12} \\ x_2{}^{p_2+1}\dfrac{dT^{12}}{dx_2} & = & B^{11}\ T^{12} + B^{12} - T^{12}\ b^{22} - b^{12} \end{cases} \tag{1.15}$$

$$\begin{cases} x_1{}^{p_1+1}\dfrac{dT^{21}}{dx_1} & = & A^{22}\ T^{21} + A^{21} - T^{21}\ a^{11} - a^{21} \\ x_2{}^{p_2+1}\dfrac{dT^{21}}{dx_2} & = & B^{22}\ T^{21} + B^{21} - T^{21}\ b^{11} - b^{21} \end{cases} \tag{1.16}$$

With equations (1.11), (1.12), (1.13), and (1.14) we obtain:

$$\begin{matrix} a^{11}{}_{r,s} = A^{11}{}_{r,s} & a^{22}{}_{r,s} = A^{22}{}_{r,s} \\ b^{11}{}_{r,s} = B^{11}{}_{r,s} & b^{22}{}_{r,s} = B^{22}{}_{r,s} \end{matrix}$$

for all couples (r, s) *satisfying* $r + s = m$.

To determine the $T^{12}{}_{r,s}$ *, we have the system*

$$\begin{cases} A^{22}{}_{0,0}T^{12}{}_{r,s} - T^{12}{}_{r,s}A^{22}{}_{0,0} & = & -A^{12}{}_{r,s} \\ B^{11}{}_{0,0}T^{12}{}_{r,s} - T^{12}{}_{r,s}B^{22}{}_{0,0} & = & -B^{12}{}_{r,s} \end{cases} \tag{1.17}$$

Let A, B, A' and B' four square matrices which verify

$$[A, A'] = 0, [B, B'] = 0$$

So, if one of the pairs (A, B), or (A', B') is without a common eigenvalue, then the system

$$Ax - xB = C \; A'x - xB' = C'$$

admits one solution and only one if and only if $A'C - CB' = A'C - C'B$

Apply this result to the system(s), this system admits a unique solution if and only if

$$B^{12}{}_{r,s}\, A^{22}{}_{0,0} - A^{11}{}_{0,0}\, B^{12}{}_{r,s} = A^{12}{}_{r,s}\, B^{22}{}_{0,0} - B^{11}{}_{0,0}\, A^{12}{}_{r,s}$$

Now, this condition is:

$$H_{r,s} = 0$$

as $r + s \leq l$ and the system is completely integrable up to order l, it is satisfied.

Reasoning by induction then gives us the result announced above, the transformation sought being

$$T_l \circ .... \circ T_1 \circ T_0$$

which is also polynomial and satisfy $T(0) = id$.

1.4 The Converging case :

System (2) is (CI). We now assume that the matrices A and B are originally holomorphic, they are therefore representable by convergent series:

$$A = \sum_{p+q\geq 0} A_{pq}\, x_1{}^p\, x_2{}^q B = \sum_{p+q\geq 0} B_{pq}\, x_1{}^p\, x_2{}^p$$

Total Reduction Lemma :
If

$$A(0,0) = \begin{pmatrix} A^{11}{}_{00} & 0 \\ 0 & A^{22}{}_{00}) \end{pmatrix} \textit{and } B(0,0) = \begin{pmatrix} B^{11}{}_{00} & 0 \\ 0 & B^{22}{}_{00}) \end{pmatrix}$$

where the two couples $(A^{11}{}_{00}; A^{22}{}_{00}), (B^{11}{}_{00}; B^{22}{}_{00})$ are without eigenvalues like, there exists a trans-unique convergent formation T of the form

$$T = \begin{pmatrix} I & T^{12} \\ T^{21} & I \end{pmatrix}$$

which transforms system (2) into

$$dz = w'z$$

with

$$w' = \frac{\begin{pmatrix} a^{11} & 0 \\ 0 & a^{22} \end{pmatrix}}{x_1{}^{p_1+1}} dx_1 + \frac{\begin{pmatrix} b^{11} & 0 \\ 0 & b^{22} \end{pmatrix}}{x_2{}^{p_2+1}} dx_2$$

and $a^{ii}(0,0) = A^{ii}{}_{00},\; b^{ii}(0,0) = B^{ii}{}_{00}$

Proof 6. *Returning to the proof of the total reduction lemma in the formal case, the additional hypothesis made above leads to the fact that* $c_1(0,0)$ *and* $c_2(0,0)$ *are non-singular and that the formal transformation given by the formal reduction lemma is convergent.*

The partial reduction lemma shows us that the reduction is possible at any order by a polynomial transformation.

From this lemma, we deduce:

If the two matrices A(0, 0) and B(0, 0) have distinct eigenvalues, the Pfaffian System

$$dy = (\frac{A(x_1, x_2)}{{x_1}^{p_1+1}} dx_1 + \frac{B(x_1, x_2)}{{x_2}^{p_2+1}} dx_2) y$$

assumed to be completely integrable, admits a fundamental matrix of the form:

$$U(x_1, x_2)\ x_1^{\Lambda_1}\ x_2^{\Lambda_2}\ exp\ P_1(\frac{1}{x_2})\ exp\ P_2(\frac{1}{x_2})$$

where:$U(x_1, x_2)$ *is a holomorphic matrix with the origin* Λ_1 *and* Λ_2 *of constant diagonal matrices* $P_i(\frac{1}{x_i})$ *polygons in* $\frac{1}{x_i}$ *with matrix confficients.*

Indeed, this result is a consequence of the Total Reduction Lemma which leads to the diagonalization of our Pfaffian System and the results of 1.2.1 on the scalar case.

Chapter 2

First-order Systems of Ordinary Differential Equations

In Chapter 2 certain types of differential equations have been discussed and methods for deriving their solutions have been described. When more general differential equations are considered it is not by any means obvious that they possess solutions. Spending a lot of time trying to solve a differential equation, which does not have a solution, can be very frustrating to say the least. Therefore, we shall give one theorem, which guarantees that a differential equation that satisfies its conditions possesses a solution, and say something about its ramifications. The proof of the theorem is given in the Appendix (3.8 symbolic computation page71 [14]) to this chapter as well as a method for finding the solution.

2.1 Existence and uniqueness

EXISTENCE THEOREM

Let $f(t, y)$ *be a single valued continuous function of t and y in* $t_0 \leq t \leq t_{0+h}, \mid y - y_0 \mid \leq k$ *that satisfies:*

1. $\mid f(t, y) \mid < M$.
2. $\mid f(t, y) - f(t, y') \mid < k \ \mid y - y' \mid$

for any (t, y) *and* (t, y') *that comply with the above inequalities. Then, for* $h < k/M$*, the differential equation*

$$\dot{y} = f(y, y') \tag{2.1.1}$$

possesses one, and only one, continuous solution y(t) in $t_0 \leq t \leq t_{0+h}$ *such that*

$y(t_0) = y_0$*. The constant hdetermines the range of tfor which the solution is valid while the constant k sets a limit on how far* $y(t)$ *deviates from its initial value. It would be ideal if hcould be made as large as desired.*

However, the restriction $h < k/M$ *means that h cannot be increased beyond a certain point without a corresponding increase in k. But, larger h and k may entail an increase in the bound M to meet condition (a) and this increase may be sufficient to prevent any improvement in* k/M*. Nevertheless, it* *may be possible to extend the solution to larger tby taking* $y(t_{0+h})$*as the initial value at* $t = t_{0+h}$ *provided that suitable new* h, k, M *can be found with this starting point.*

Suppose that fsatisfies the conditions of the theorem and, by some means, two continuous solutions $y_1(t), y_2(t)$*of (2.1.1) have been found such that* $y_1(t_0) = y_0$ *and* $y_2(t_0) = y_0$*. Suppose, further, it is known that* $y_1(t)$ *is valid for* $t_0 \leq t \leq t_0 + h_1$ *where as* $y_2(t)$ *holds for* $t_0 \leq t \leq t_0 + h_2$ *with* $h_2 > h_1$*. The uniqueness part of the theorem then says that* $y_1(t) = y_2(t)$ *for* $t_0 \leq t \leq t_0 + h_1$*. The same assertion cannot be made for larger values of t unless it can bc demonstrated that* $y_1(t)$ *can be continued be yond* $t = t_0 + h_1$*.*

For example, $y_1(t) = 1 - t + t_2 ...$ *and* $y_2(t) = 1/(1 + t)$ *are solutions of* $(1 + t)y' = -y$ *which are unity at* $t = 0$ *so long as* $h_1 < 1$*, but the series in* y_1 *is not valid in* $t > 1$*. Generally, it is not difficult to recognise when fis continuous and to assess M. Checking (b) can require more effort but there is one case when (b) holds for sure and that is when*

$$| \frac{d}{dy} f(t, y) | \leq N. \tag{2.1}$$

for $t_0 \leq t \leq y_0 + h, | y - y_0 | \leq k$ *and N finite.*

For then

$$\begin{aligned} | f(t, y) - f(t, y') | &= | \int_{y'}^{y} \tfrac{d}{du} f(t, u) du | \\ &= | \int_0^{y-y'} \tfrac{d}{du} f(t, u - y') du | \\ &\leq \int_0^{|y-y'|} | \tfrac{d}{du} f(t, u - y') | du \\ &\leq N | y - y' | \end{aligned}$$

Example

The differential equation

$$y' = g(t)y^2$$

in which g is continuous clearly has $f(t, y)$ *continuous and, because*

$$\frac{d}{dy} g(t)y^2 = 2\, g(t)y.$$

satisfies the condition (2.1.2) so long as t and yare bounded. Therefore the differential equation has one and only one continuous solution such that $y(t_0) = y_0$*. It remains valid as t increases as long as t and y remain finite.*

Consider, in particular, $y' = y^2$*. The solution of this such that* $y(t_0) = 0$ *is* $y(t) = 0$ *for all t.*

On the other hand, if $y(t_0) = y_0$ *with* $y_0 \neq 0, y(t) = y_0/\{1 + (t_0 + t)y_0\}$ $y_0 < 0$ *this solution holds for all* $t \geq t_0$*.*

In contrast, if y_0 *is positive,* $y(t)$ *becomes unbounded as t approaches* $t_0 + 1/y_0$*; in this case the region of validity is confined to* $t_0 \leq t < t_0 + 1/y_0$*.*

The conditions of Existence Theorem I are sufficient but not necessary. There are differential equations that do not satisfy the conditions but which possess a unique continuous solution. For example,

$$y' = \begin{cases} (1 - 2t)y & (t > 0) \\ (2t - 1)y & (t > 0) \end{cases} \tag{2.2}$$

There may, however, be values of t_0 *or* y_0 *for which the initial value problem*

i *has no solution;*

ii *has a discontinuous solution;*

iii *has more than one continuous solution.*

For instance, the differential equation

$$y y' = -t \tag{2.3}$$

has solution $y_2 + t_2 = C$ *where* C *is a constant. An example when there is no solution is to take* $y = 0$ *at* $t = 0$*. Then* $C = 0$ *and* $y_2 + t_2 = 0$*. This forces* $t = 0$ *and there is no solution for* $t > 0$*.*

An illustration of more than one solution is provided by $y = 0$ *at* $t = t_0 \neq 0$*. Then* $C = {t_0}^2$ *and* $y = \pm({t_0}^2 - t^2)^{1/2}$ *giving two solutions while* $t^2 \leq {t_0}^2$*.*

Thus the initial value $y = 0$ *originates difficulties for (2.1.4). For it,* $f(t, y)$ *in (2.1.1) is* $-t/y$*,which is infinite at* $y = 0$ *for any non-zero t,and so the conditions of Existence Theorem I cannot be met. Thus there is no warranty of a unique continuous solution. Yet, there is no problem if* $y(t0) = y_0$ *with* $y_0 \neq 0$*. Now Existence Theorem I applies and there is the unique continuous solution* $y = ({t_0}^2 + {y_0}^2 - t^2)^{1/2}$ *so long as* ${t_0}^2 + {y_0}^2 \geq t^2$ *Any point* (t_0, y_0) *at which (i), (ii) or (iii) is true is known as a singular point. For example, any point* $(t_0, 0)$ *is a singular point of (2.1.4). At a singular point Existence Theorem I must fail, but the converse is false as (2.1.3) shows. Thus, while places where* f *does not abide by the conditions of Existence Theorem I are candidates for singular points, a special investigation has to be undertaken to check whether or not they are actually singular points.*

The same nomenclature of singular points is used in connection with systems and with the differential equation of order nwhen (i), (ii) or (iii) occurs. Existence Theorems II and III (given in the Appendix $(3.8 symboliccomputationpage 71[14])$ *) are invalid at singular points but the points where their conditions are unsatisfied are not necessarily singular points.*

The linear system

$$y_i' = \sum_{j=1}^{n} = a_{ij}(t)y_j + f_j(t) \; (i = 1, ..., n) \tag{2.4}$$

conforms to Existence Theorem II except at those values of t where a_{ij} *or fiare discontinuous. Apart from these values, the initial value problem has a unique continuous solution. In particular, the linear system with constant coefficients possesses a unique continuous solution except, perhaps, for those t where* f_i *is not continuous.*

2.1.1 Epidemics

The simplest model of the spread of an epidemic in a population stipulates that at time t there are x susceptible individuals and y infected who may transmit the disease. It is assumed that the mixing of these two groups passes on the illness and that, in the short time $\delta t, \mu\, x\, y\, \delta t$ *new infections occur. Also some of those infected will die, or stop mixing, or recover and become immune; suppose that* $v\, y\, \delta\, t$ *disappear in this way. Then, if* $\rho\, \delta\, t$ *new susceptibles arrive in the interval* δt*,*

$$x' = -\mu\, x\, y + \rho, \;\; y' = \mu\, x\, y - v\, y. \tag{2.5}$$

Normally μ, v *and* ρ *are taken as non-negative constants and it is convenient to assume that they are positive. The right-hand sides of* $(2.2.1)$ *vanish for* $x = x_0, y = y_0$ *where*

$$-\mu\, x\, y + \rho = 0,\ \mu\, x\, y - v\, y = 0. \tag{2.6}$$

or $x_0 = v/\mu, y_0 = \rho/v$. *If* $x(t) = x_0, y(t) = y_0$ *then* $x' = 0, y' = 0$ *and the differential equations (2.2.1) are satisfied. In other words,* (x_0, y_0) *is an equilibrium, statein which the numbers of susceptibles and infected do not vary.*

Let us now address the question of whether equilibrium is approached from a nearby state and, if so, in what manner. Put

$$x = x_0(1 + \xi)\ y = y_0(1 + \eta)$$

where ξ *and* η *are so small that their products may be neglected. Then (2.2.1) become*

$$\xi' = -\sigma(\xi + \eta),\ \eta' = v\,\xi$$

where $\sigma = \mu\rho/v$. *Substituting for* ξ *from the second equation we obtain*

$$\eta'' + \sigma\eta' + \sigma\, v\, \eta = 0$$

of which the solution such that $\eta = \eta_0\ \eta' = v\xi_0$, *at* $t = 0$ *is*

$$\eta = \exp^{-\sigma t/2}\left\{\eta_0 \cos\omega t + \frac{1}{\omega}(v\xi_0 + \frac{1}{2}\sigma\eta_0)\sin\omega t\right\}$$

where $\omega_2 = \sigma u - \sigma_2/4$. *Hence*

$$\xi = \exp^{-\sigma t/2}\left\{\xi_0 \cos\omega t - \frac{\sigma}{2\omega}(\xi_0 + 2\eta_0)\sin\omega t\right\}$$

When $\omega_2 > 0$, *i.e.,* $4v > \sigma$ *or* $4v_2 > \mu\rho$,*the population, after a small departure from equilibrium, returns to equilibrium in an oscillatory fashion with exponential decay. If* $\omega_2 < 0$ *or* $4v_2 < \mu\rho$ *the fact that* $|\ \omega\ | < \sigma/2$ *ensures exponential decay again but there is no accompanying oscillation. In either case the population returns* t_o *equilibrium, the approach being more rapid when oscillations are present.*

2.2 The phase plane

If less specific assumptions are made about the mechanism of propagation of epidemics, the most that can be said is that a system of the type

$$x' = f(x, y)\ y' = g(x, y) \tag{2.7}$$

will need to be solved. Any point (x_0, y_0) *such that*

$$f(x_0, y_0) = 0\ g(x_0, y_0) = 0 \tag{2.8}$$

is called a critical pointor fixed pointor equilibrium point. A solution that starts at an equilibrium point never leaves it because x' *and* y' *both vanish there provided that fand gsatisfy the conditions of Existence Theorem II.*

When the solution of (2.3.1) has been found, say $x = h_1(t), y = h_2(t)$*,the point* (x, y) *can be plotted in the* (x, y) *plane at time t. As t varies,* (x, y) *will trace a curve in the* (x, y) *plane. This curve is known as a trajectory and the* (x, y) *plane is called the phase plane. By attaching an arrow to each trajectory the direction in which* (x, y) *moves as t increases can be indicated.*

The phase plane then contains all the information in (2.3.1) except the rate at which the trajectory is traversed. The slope of a trajectory is given from (2.3.1) by

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{g(x, y)}{f(x, y)}. \tag{2.9}$$

A trajectory is vertical at any (x, y) *where* $g(x, y) \neq 0, f(x, y) = 0$ *and horizontal where* $g(x, y) = 0, f(x, y) \neq 0$.

The trajectory corresponding to an equilibrium point reduces to a single point. When fand g satisfy the conditions of Existence Theorem II, the initial value problem has a single continuous solution in the neighbourhood of $t = t_0$.

Therefore, in this case, only one trajectory passes through a given point of the phase plane, i.e., under the conditions of Existence Theorem II, two trajectories do not intersect in general

All these notions can be generalised to a system of nequations. A solution still describes a trajectory, which is now a curve in a space of ndimensions, and we talk of a phase spacerather than a phase plane.

Diagrams are, however, much more difficult to draw.

The same question about behaviour near equilibrium that was asked for epidemics can be raised here. Put

$$x = x_0 + \xi, \quad y = y_0 + \eta.$$

where (x_0, y_0) *is in conformity with (2.3.2). Because of the smallness of* ξ *and* η*,it will be assumed that* $f(x, y)$ *can be approximated by the first terms in its Taylor expansion, namely*

$$\xi\left(\frac{df}{dx}\right) + \eta\left(\frac{df}{dy}\right)$$

where $()_0$ *means calculate the value at* $x = x_0, y = y_0$*. The approximation to (2.3.1) is then*

$$\xi_0 = \xi\left(\frac{df}{dx}\right)_0 + \eta\left(\frac{df}{dy}\right)_0 \tag{2.10}$$

$$\eta_0 = \xi\left(\frac{dg}{dx}\right)_0 + \eta\left(\frac{dg}{dy}\right)_0 \tag{2.11}$$

a linear system with constant coefficients. The behaviour of such systems in the phase plane will be examined in succeeding sections.

The behaviour near equilibrium is a matter of local stability. The larger question of what happens when the initial state is not near equilibrium is one of global stability.

We merely remark that for systems of three or more equations global behaviour is very varied and imperfectly understood.

2.3 Local stability

It has been discovered in the preceding section that local stability reduces to a discussion of

$$\begin{cases} x' = & a\,x + b\,y \\ y' = & c\,x + d\,y \end{cases} \tag{2.12}$$

where a, b, *cand dare real constants. The goal of this section is to determine the trajectories of this system. Let us first remark that, if* α *is a real constant,* $x(t+\alpha), y(t+\alpha)$ *occupies the same points in the phase plane as tvaries as* $x(t), y(t)$ *though at a time a earlier.*

So both points describe the same trajectory despite being different solutions. More than one solution can lie on one trajectory.

In finding the trajectories we shall ignore the degenerate case in which $a\,d = b\,c$ *;should it arise, the equations can be integrated directly without trouble. It would, in any case, be necessary to reconsider the validity of (2.3.4) and (2.3.5) as an adequate prescription for local stability in these circumstances.*

Therefore, from now on, it will be assumed that

$$a\,d \neq b\,c \tag{2.13}$$

so that there is a single critical point at the origin.

According to Section 2.4, the first attempt at a solution is $x = \alpha \exp^{\lambda t}, y = \beta \exp^{\lambda t}$ *For the satisfaction of (2.4.1) we require*

$$(a-\lambda)\alpha + b\beta = 0, \tag{2.14}$$

$$c\alpha + (d-\lambda)\beta = 0 \tag{2.15}$$

These give non-zero α, β *only if*

$$(a-\lambda)\,(d-\lambda) - b\,c = 0$$

or

$$\lambda^2 - (a+d)\lambda + a\,d - b\,c = 0$$

The roots are λ_1, λ_2 *where*

$$2\lambda_1 = a + d + \sqrt{(a-d)^2 + 4bc}$$

$$2\lambda_2 = a + d - \sqrt{(a-d)^2 + 4bc}$$

Several cases have to be studied.

1/ $(a-d)^2 + 4bc > 0$

In this case the values of λ_1 *and* λ_2 *are real and distinct; the special procedure for multiple roots does not have to be called on.*

Assume first that $b \neq 0$. Then, from (2.4.3), we can choose $\alpha = b, \beta = \lambda_1 - a$ corresponding to λ_1 and $\alpha = b, \beta = \lambda_2 - a$ corresponding to λ_2. Consequently

$$x = b(C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}), \tag{2.16}$$

$$y = (\lambda_1 - a)C_1 e^{\lambda_1 t} + (\lambda_2 - a)C_2 e^{\lambda_2 t} \tag{2.17}$$

where C_1 and C_2 are arbitrary constants. These equations may be rearranged to give

$$(\lambda_2 - a)x - by = b(\lambda_2 - \lambda_1)C_1 e^{\lambda_1 t}, \tag{2.18}$$

$$(\lambda_1 - a)x - by = b(\lambda_1 - \lambda_2)C_2 e^{\lambda_2 t}. \tag{2.19}$$

From (2.4.7), $(\lambda_2 - a)x - by$ cannot change sign as t varies. Therefore the trajectory cannot go over the line $(\lambda_2 - a)x = by$. Similarly, from (2.4.8), the trajectory cannot trespass across $(\lambda_1 - a)x = by$.

These lines are displayed in Figure 2.4.1 as well as the regions to which the trajectory is confined for various choices of C_1, C_2 when $b > 0$ and both λ_1, λ_2 are negative.

Suppose now that both λ_1 and λ_2 are negative. It is evident that $x \to 0, y \to 0$ as $t \to \infty$.Also $\lambda_1 > \lambda_2$ so that $(\lambda_1 - a)x \sim by$ as $t \to \infty$,so long as $C_1 \neq 0$.

Moreover, as $t \to -\infty$, $|x|$ and $|y|$ become large and $(\lambda_2 - a)x \sim by$ provided $C_2 \neq 0$, x approaching $-\infty$ when $b > 0, C_2 < 0$. The trajectories, therefore, have the shape depicted in Figure 2.4.2 for $b > 0$.

The exclusion so far of $C_1 = 0$ or $C_2 = 0$ can be remedied immediately because their trajectories are the dividing straight lines on account of (2.4.7) and (2.4.8). The arrows

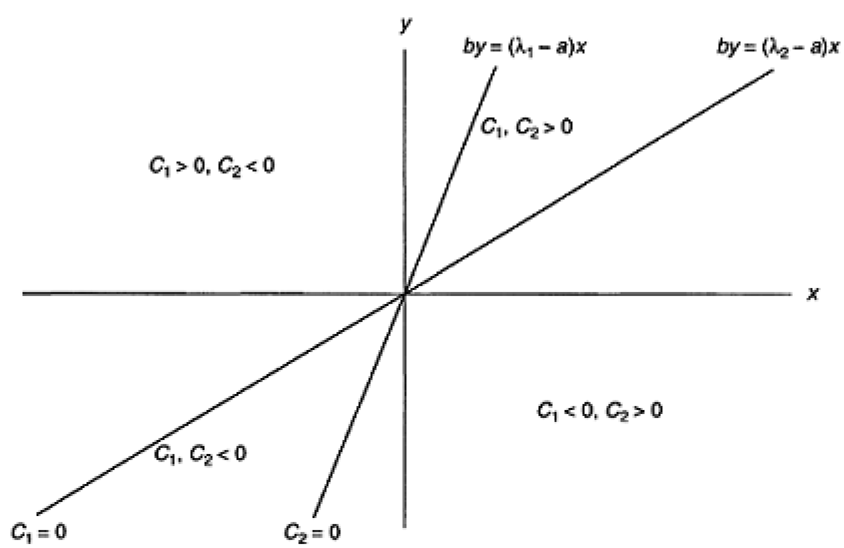

Figure 2.1: Lines that cannot be crossed by trajectories.

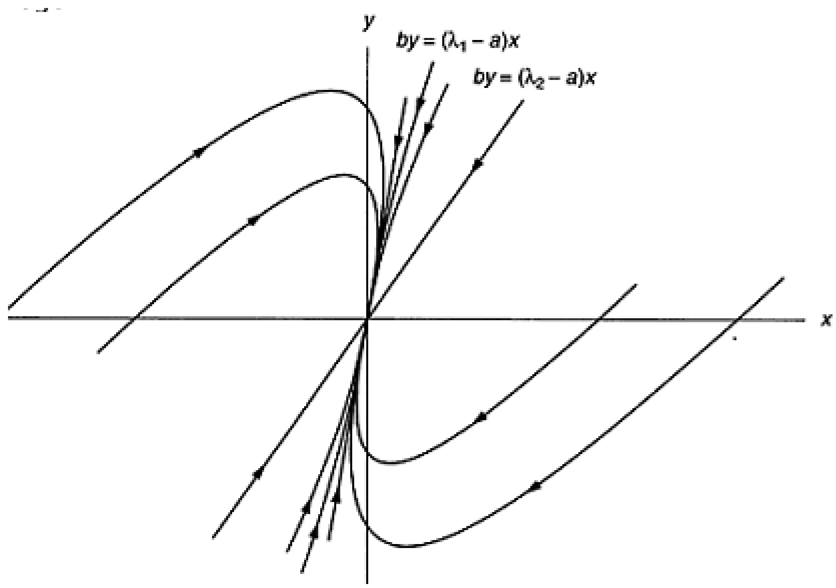

Figure 2.2: The stable node.

on the curves indicate the direction in which (x, y) *moves as* t *increases. A critical point of this type is known as a stable node.*

When λ_1 *and* λ_2 *are both positive, the curves are similar in character to those in Figure 2.4.2 but the directions of the arrows are reversed because* (x, y) *moves away from the origin as* t *increases. We have an unstable node.*

The remaining possibility is that λ_1 *and* λ_2 *have opposite signs so that* $\lambda_1 > 0$ *and* $\lambda_2 < 0$. *From (2.4.7), the magnitude of* $(\lambda_2 - a)x - by$ *increases with time whereas that of* $(\lambda_1 - a)x - by$ *diminishes.*

The origin can never be reached unless $C_1 = 0$ *when the trajectory is a straight line. The trajectory is also a straight line when* $C_2 = 0$, *but now the origin is departed from.*

The behaviour of the trajectories is displayed in Figure 2.4.3. The critical point is called a saddle-point. Clearly a point (x, y) started near the origin cannot stay near it in general and there is no stability.

So far we have assumed that $b \neq 0$. If $b = 0$ we see at once from (2.4.1) that $x = C_1\ e^{at}$ and then

$$y = \frac{C_2 e^{dt} + cC_1 e^{at}}{(a-d)}$$

We remark that $a \neq d$ because $(a-d)^2$ must be positive when $b = 0$. In this case, the dividing lines are $x = 0$ and $(a-d)y = cx$. Apart from this change the pictures are practically unaltered. There is a node if a and d have the same sign (stable if $a < 0$, unstable if $a > 0$) and a saddle-point if a and d have opposite signs.

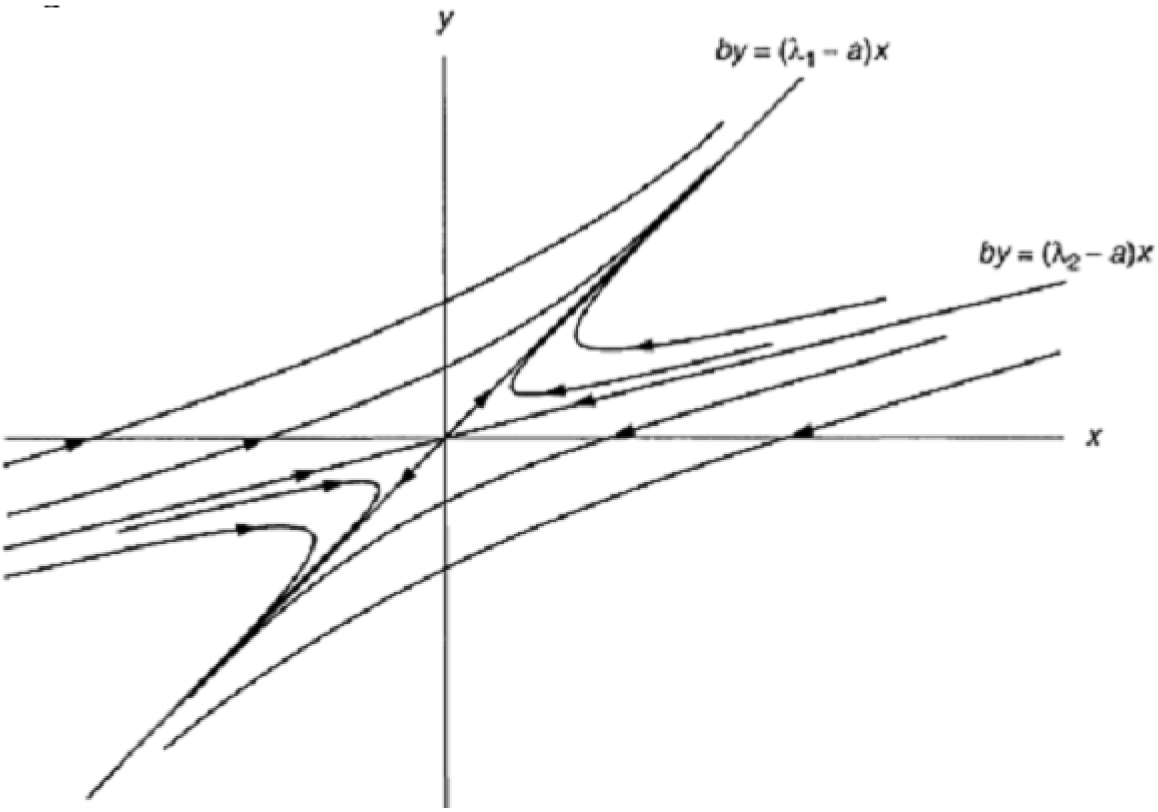

Figure 2.3: The saddle-point.

This possibility can occur only when $bc < 0$; so neither bn or c vanishes and they have opposite signs. The roots λ_1 and λ_2 are still distinct but they are now complex conjugates.

Write $\lambda_2 = \frac{1}{2}(a+d) - i\omega, \lambda_1 = \frac{1}{2}(a+d) + i\omega$ where

$$\omega^2 = -\frac{1}{4}(a-d)^2 - bc.$$

Then

$$x = Ae^{\frac{1}{2}(a+d)t}\cos(\omega t - \alpha) \tag{2.20}$$

where A and α are arbitrary constants. From (2.4.1)

$$y = (A/b)e^{\frac{1}{2}(a+d)t}\{\frac{1}{2}(d-a)\cos(\omega t-\alpha) - \omega\sin(\omega t-\alpha)\}. \tag{2.21}$$

The formulae (2.4.9) and (2.4.10) can be combined to give (2.4.11)

$$cx^2 + (d-a)xy - by^2 = -(\omega^2A^2/b)e^{(a+d)t}. \tag{2.22}$$

Suppose that $a + d = 0$ so that λ_1 and λ_2 are pure imaginary. The equation of a trajectory is given by (2.4.11) as

$$cx^2 + (d-a)xy - by^2 = -\omega^2A^2/b.$$

Rotate the axes by means of the transformation

$$x = X\cos\theta - Y\sin\theta$$

$$y = X\sin\theta + Y\cos\theta$$

where

$$\tan 2\theta = \frac{d-a}{b+c}.$$

The equation of the curve goes over to

$$A'X^2 + C'Y^2 = -\omega^2A^2/b$$

where

$$A' = \frac{1}{2}\left((c-b) + \sqrt{(b+c)^2 + (d-a)^2}\right),$$
$$B' = \frac{1}{2}\left((c-b) - \sqrt{(b+c)^2 + (d-a)^2}\right).$$

Since $A'C' = \omega^2, A'$ and C' have the same sign. Also $A' + C' = c - b$ so that if $b < 0$, which implies $c > 0, A'$ and C' are positive, whereas if $b > 0$, which makes $c < 0, A'$ and C' are negative.

Thus A' and C' have the opposite sign to band the trajectory is an ellipse with semi-axes

$\omega \mid A \mid /(-bA'))1/2$ and $\omega \mid A \mid /(-bC'))1/2$. Typical trajectories are drawn in Figure 2.4.4; the critical point is known as a centre. With regard to the direction of motion on a trajectory, we see from (2.4.1) that when $x = 0$,. Hence, when $b > 0, x$ must be increasing at positive y and so the direction is as shown in Figure 2.4.4; if $b < 0$ the arrows have to be reversed.

The equations (2.4.9) and (2.4.10) make it evident that xand yvary harmonically when $a + d = 0$. *The point* (x, y) *therefore makes continual circuits round the origin and is forever retracing its path. A trajectory started from near the origin never leaves the neighbourhood but never swings into the origin. Therefore, there is stability in the sense that* (x, y) *remains in the vicinity of the critical point, if it is initially near there, but it never attains the critical point.*

It should be observed that any closed trajectory implies periodic motion because it entails there being a fixed T *such that* $x(t + T) = x(t), y(t + T) = y(t)$ *for all* t. *The motion need not, however, have the simple harmonic character mentioned above when the system is more general than (2.4.1).*

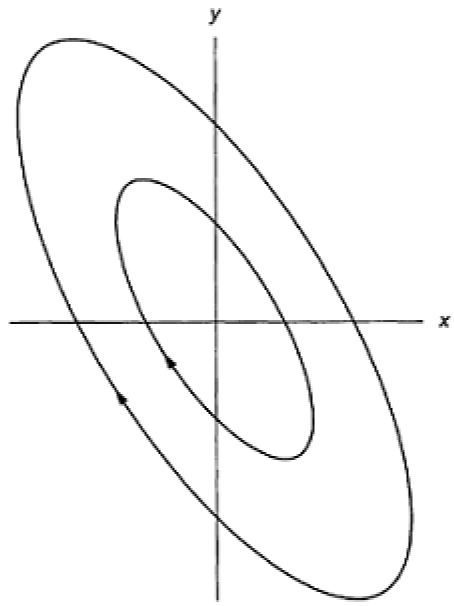

Figure 2.4: The centre.

Turning now to the case when $a + d \neq 0$, *we note that the only difference is the exponential factor in* (2.4.11). *The trajectory may be thought of instantaneously as an ellipse whose axes are changing exponentially.*

The trajectory therefore spirals about the origin as shown in Figure 2.4.5. If $a + d < 0$, *the point* (x, y) *must approach the origin as* $t \to \infty$.

The direction of motion along a trajectory is then that of Figure 2.4.5 *and the critical point is called a stable focus. When* $a + d > 0, (x, y)$ *departs from the origin, the arrows are reversed and we have an unstable focus.*

$$(a - d)^2 + 4bc = 0$$

In this case $\lambda_1 = \lambda_2 = \frac{1}{2}(a + d)$. *However,*

$$(a + d)^2 = (a - d)^2 + 4ad = -4bc + 4ad \neq 0$$

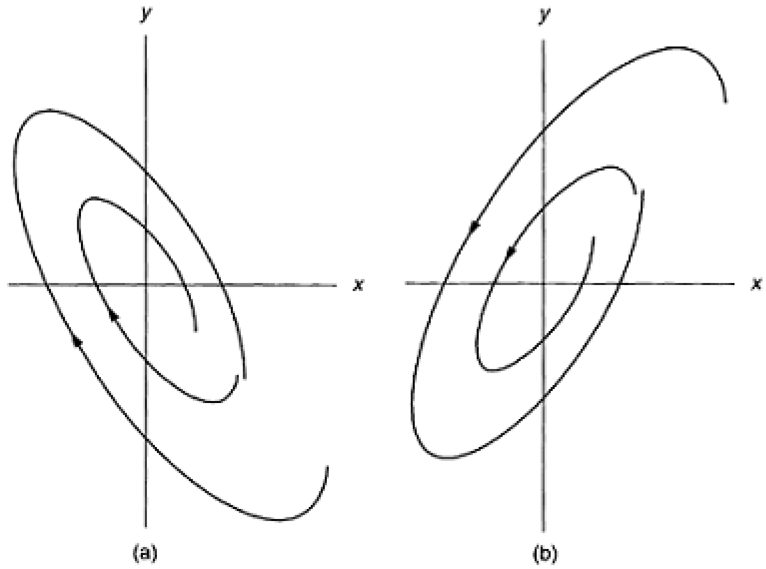

Figure 2.5: The focus when $a + d < 0$; $(a) b > 0$, $(b) b < 0$...

by (3.4.2) so λ_1 and λ_2 are non-zero.

If $b \neq 0$, (2.4.3) and (2.4.4) supply only the single solution $\alpha = b, \beta = \frac{1}{2}(d - a)$.

To find a second solution we try, according to Section 2.4, $x = (\gamma + \alpha t)\, e^{\frac{1}{2}(a+d)t}$, $y = (\delta + \beta t)\, e^{\frac{1}{2}(a+d)t}$ with the result that $y = 0, \delta = 1$. The technique of Section 2.5 leads to the same answer.

Consequently

$$\begin{aligned} x &= b(c_1 + c_2 t)\, e^{\frac{1}{2}(a+d)t}, \\ y &= \left\{c_2 + \tfrac{1}{2}(d - a)(c_1 + c_2 t)\right\}\, e^{\frac{1}{2}(a+d)t}. \end{aligned}$$

The line $by = \frac{1}{2}(d - a)x$ cannot be crossed and the structure of the trajectories is similar to that of Figure 2.4.2 when the dividing lines coalesce. This critical point is therefore also termed a node; it is stable if $a + d < 0$ and unstable if $a + d > 0$.

If $b = 0$, then $a = d$ and $x = C_1\, e^{at}, y = (C_2 + c\, C_1 t)e^{at}$ and the trajectories are not much changed in shape if $c \neq 0$. If, in addition, $c = 0$ the trajectories are the straight lines $y/x =$ constant (see Figure 2.4.6). The critical point is still designated a node, stable if $a < 0$ and unstable if $a > 0$.

It is convenient to summarise these results as follows: if $a\,d \neq b\,c$ the trajectories of (2.4.1) form, when λ_1 and λ_2 are

real, (i) a node if λ_1 and λ_2 have the same sign, (ii) a saddle-point if λ_1 and λ_2 have opposite signs;

purely imaginary, a centre;

complex but not purely imaginary, a focus.

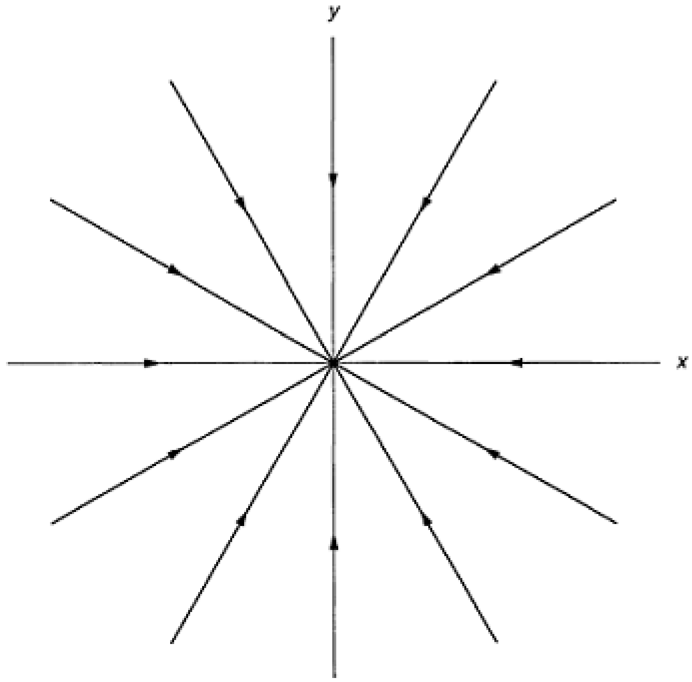

Figure 2.6: The case $b = c = 0, a < 0$.

The reader will note that there can be no oscillation if λ_1 *and* λ_2 *are real; in particular, the system will not be oscillatory if* $bc > 0$.

2.4 Stability

We now want to investigate what general conclusions can be drawn about the behaviour of solutions to (2.3.1) *on the basis of the model of* (2.3.4), (2.3.5) *and the trajectories determined in Section* 2.4. *To fix ideas, we consider a some what generalised model of epidemics in which the number of susceptibles x and y of infected satisfy*

$$x' = h(x, y)\,x \quad y' = k(x, y)\,y.$$

The behaviour near an equilibrium point in which neither x nor y is zero is of interest. So x_0 *and* y_0 *are taken to satisfy*

$$h(x_0, y_0) = 0 \quad k(x_0, y_0) = 0.$$

It then follows that, in the notation of Section 2.3,

$$(\tfrac{df}{dx})_0 = h_1 x_0, \quad (\tfrac{df}{dy})_0 = h_2 x_0,$$
$$(\tfrac{dg}{dx})_0 = k_1 y_0, \quad (\tfrac{dg}{dy})_0 = k_2 y_0,$$

where h_1, h_2, k_1 *and* k_2 *are the values of* $dh/dx, dh/dy, dk/dx$ *and* dk/dy*,respectively, at* (x_0, y_0). *Thus, in the theory of Section* 2.4, $a = h_1\, x_0,\ b = h_2\, x_0,\ c = k_1\, y_0,\ d = k_2\, y_0$.

Since the presence of the infected tends to reduce the number of susceptibles by infection, we expect $h_2 < 0$.

As the number of infected increases there will be less opportunity to affect the susceptibles and so $k_2 < 0$, $k_1 > 0$.

If there is a birth rate of susceptibles, we can suppose $h_1 > 0$, *though* h_1 *will be rather small in comparison with other partial derivatives in most epidemics, because they tend to spread much faster than susceptibles are created.*

Since $ad - bc = (h_1 k_2 - h_2 k_1)x_0 y_0$ *we can be sure that* $a d \neq b c$ *when* h_1 *is small, as suggested above, and the theory of Section* 2.4 *can be applied.*

In order that there can be any kind of oscillation, we must have case (b) of Section 2.4, i.e.,

$$(h_1 x_0 - k_2 y_0)^2 + 4h_2 k_1 x_0 y_0 < 0.$$

The second term on the right-hand side is negative; so the inequality is feasible if the first term is not too large.

Since h_1 *is small, this will be true if* k_2 *is not too large. The oscillations are likely to remain near equilibrium because* $a + d = h_1 x_0 + k_2 y_0$ *is negative on account of the smallness of* h_1. *The critical point will be a stable focus or, possibly, a centre.*

In the absence of oscillations, λ_1 *and* λ_2 *will both be negative because* $ad - bc$ *is positive and the equilibrium will be a stable node.*

Quite a lot of qualitative information about the behaviour of the solution has been obtained without too specific assumptions about hand k. Of course, further conclusions could be drawn if more was known about hand k.

In other problems, the signs of the partial derivatives might be different and the behaviour near equilibrium changed thereby. Some of the unstable critical points might occur.

However, such instability merely means departure from equilibrium and, once this exceeds a certain amount, the model of (2.3.4) *and* (2.3.5) *loses its validity because it assumed motion near equilibrium.*

We then enter the arena of global stability, with the possibility of some kind of stable behaviour away from equilibrium, a matter to be discussed in the next section.

In the foregoing the presence of an equilibrium point has been assumed and it will not be amiss to say a word or two about how the existence of an equilibrium point is verified.

Often it will be done most simply by graphical means but sometimes an analytical argument is helpful. With $\frac{dk}{dx} > 0, k(x, y) = 0$ *can be solved for* x *to give a unique function* $x(y)$ *of* y. *Because*

$$\frac{dx}{dy} = -\frac{dk/dy}{dk/dx}$$

we have $dx/dy > 0$ *when* $dk/dy < 0$. *There can be no infected if there is no population, and so* $x(y) > 0$ *but less than some bound. Thus* $h(x(y), y)$ *is such that*

$$\frac{dh}{dy} = \frac{dh}{dx}\frac{dx}{dy} + \frac{dh}{dy},$$

which will be negative when both dh/dx *and* dh/dy *are. Then* $h(x(y), y)$ *decreases as* y *increases so* *that if $h(x(y), y)$ is positive for small yand negative for large ythere will be one and only one equilibrium point.*

As a final illustration we consider a case in which the trajectories can be traced completely, namely

$$x' = y, \quad y' = \frac{1}{2}(1 - x^2)$$

or, equivalently,

$$x'' + \frac{1}{2}(x^2 - 1) = 0.$$

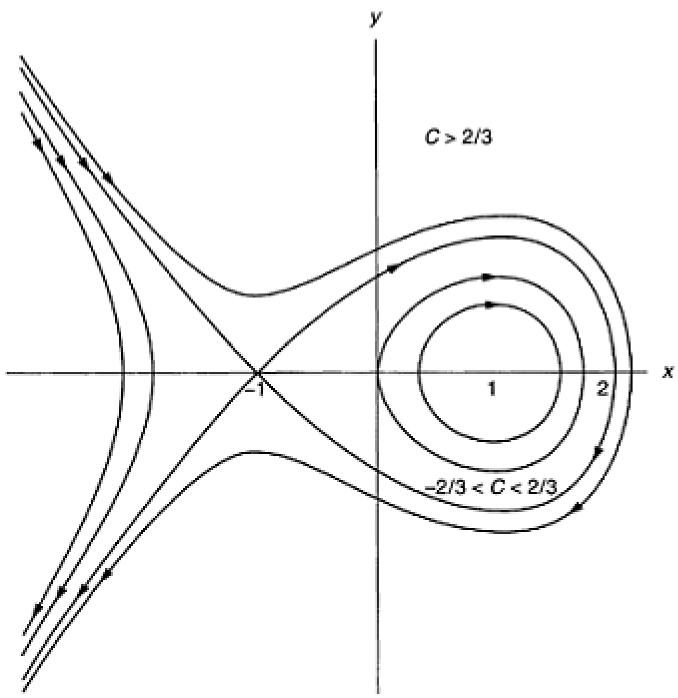

Figure 2.7: Non-linear conservative system.

We have

$$y\frac{dy}{dx} = \frac{1}{2}(1 - x^2)$$

which gives, on integration,

$$y^2 = x - \frac{1}{3}x^3 + C.$$

These trajectories are displayed graphically in Figure 2.5.1. *The critical point $x = 1, y = 0$ is a centre and $x = -1, y = 0$ is a saddle-point.*

If $\frac{-2}{3} < C < \frac{2}{3}$ the closed curves surrounding $x = 1, y = 0$ show that periodic motion is possible with proper initial conditions.

If $C = \frac{2}{3}$ no oscillations are allowed but $x = -1, y = 0$ can be tended to if the initial conditions are appropriate. If $C > \frac{2}{3}$ x and y always approach negative infinity as $t \to \infty$.

A quick idea of the shapes of the trajectories can be obtained as follows. Put $v(x) = \frac{1}{3}x^3 - x$ so that the equation of the trajectories is

$$y^2 = C - v(x).$$

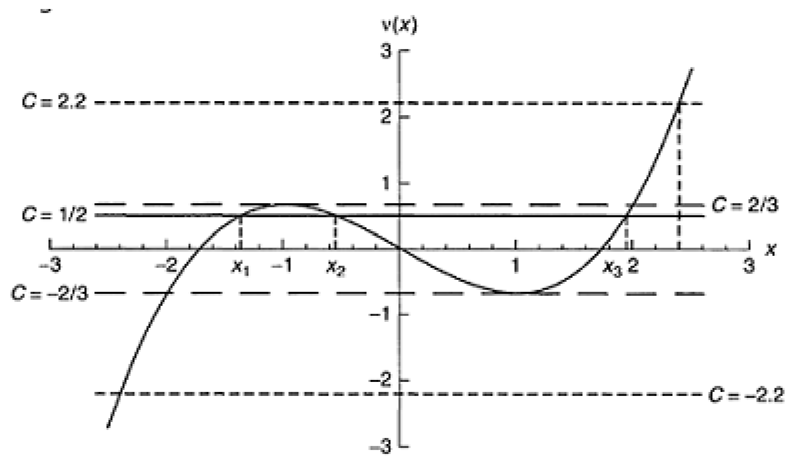

Figure 2.8: Qualitative determination of trajectories.

Since y_2 cannot be negative the only values of x that can occur are those that satisfy $v(x) \leq C$. Such values can be seen easily from a graph of $v(x)$.

In Figure 2.5.2 a graph is displayed together with various possibilities for C. The line on which $C = 2.2$ intersects the curve at $x = 2.4$ approximately and is above the curve for $x < 2.4$. Hence $v(x)$is below 2.2 for $x < 2.4$.

Thus the trajectory starts at $x = -\infty$ with y positive (because x' must be positive at the start) and moves to the right until it crosses the x-axis at $x = 2.4$ approximately.

Thereafter it returns to $x = -\infty$ with y negative. Likewise, when $C = -2.2$, x is restricted to $x < -2.4$; the trajectory goes from $-\infty$ and back again, crossing the x-axis at $x = -2.4$ approximately.

On the other hand, when $C = \frac{1}{2}$ there are three intersections with the curve at x_1, x_2 and x_3. Two trajectories are possible now. On one $x \leq x_1$ and it is similar to the one when $C = -2.2$.

On the other trajectory x is confined to the interval $x_2 \leq x \leq x_3$ so that the trajectory is a closed curve. It is clear from Figure 2.5.2 that trajectories that are closed curves are possible only for $\frac{-2}{3} < C < \frac{2}{3}$.

This is an example of a non-linear conservative system. A more general version is

$$x' = y, \; y' = -\frac{d}{dx}V(x)$$

or

$$x'' + \frac{dV(x)}{dx} = 0.$$

The trajectories can be determined as above and are

$$\frac{1}{2}y^2 + V(x) = C.$$

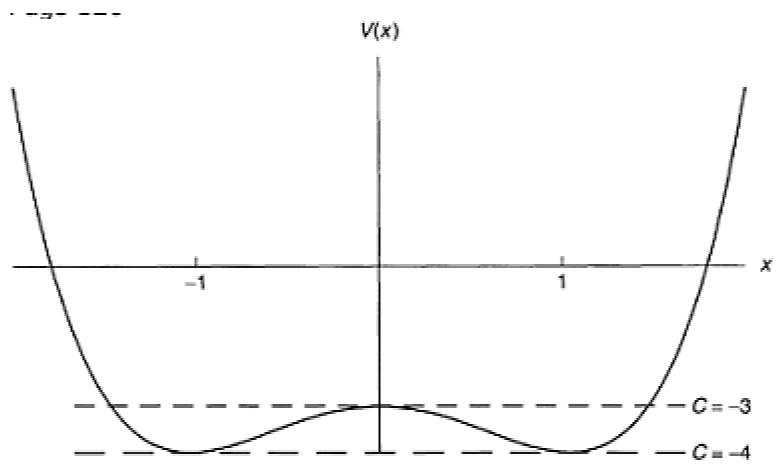

Figure 2.9: Graph of $x^4 - 2x^2 - 3$.

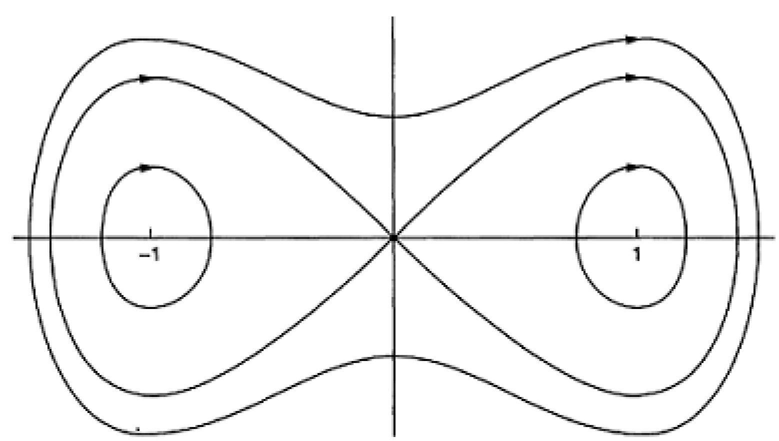

Figure 2.10: The trajectories corresponding to Figure 3.5.3.

The graphical method will give an indication of the shapes of the trajectories. As an illustration take

$$V(x) = x^4 - 2x^2 - 3.$$

Its graph is drawn in Figure 2.5.3. *Evidently, when* $C > -3$ *the trajectory consists of a single closed curve.*

When $-3 > C > -4$, *however, the trajectory has two parts. One is a closed curve surrounding* $x = -1$ *while the other is a closed curve around* $x = 1$. *There is no trajectory for* C *below* -4. *Since the equilibrium points are* $x = 0, x = \pm 1$ *the trajectories are as shown in Figure* 2.5.4. *The points* $x = 1$ *and* $x = -1$ *are centres where as* $x = 0$ *is a saddle-point.*

2.4.1 Stability of Equilibria

Determining the stability of an equilibrium point is straightforward if the equilibrium is hyperbolic.

When this is not the case, this determination becomes more problematic. In this section we develop an alternative method for showing that an equilibrium is asymptotically stable. Due to the Russian mathematician Liapunov, this method generalizes the notion that, for a linear system in canonical form, the radial component r decreases along solution curves.

Liapunov noted that other functions besides r could be used for this purpose. Perhaps more importantly, Liapunov's method gives us a grasp on the size of the basin of attractionof an asymptotically stable equilibrium point.

By definition, the basin of attraction is the set of all initial conditions whose solutions tend to the equilibrium point.

Let $L : O \to \mathbb{R}$ *be a differentiable function defined on an open set* O *in* R^n *that contains an equilibrium point* X^* *of the system* $X' = F(X)$. *Consider the function*

$$L'(X) = DL_X(F(X)).$$

As we have seen, if $\varphi_t(X)$ *is the solution of the system passing through* X *when* $t = 0$, *then we have*

$$L'(X) = \frac{d}{dt}\,|_{t=0}\, L \circ \varphi_t(X)$$

by the chain rule. Consequently, if $L'(X)$ *is negative, then* L *decreases along the solution curve through* X.

We can now state Liapunov's stability theorem:

Theorem *(Liapunov Stability)*

Let X^* *be an equilibrium point for* $X' = F(X)$. *Let* $L : O \to R$ *be a differentiable function defined on an open set* O *containing* X^*. *Suppose further that*

a $L(X^*) = 0$ *and* $L(X) > 0$ *if* $X \neq X^*$;

Figure 2.11: Alexander Lyapunov 19 age old (1857-1918)

b $L' \leq 0$ *in* $O - X^*$;

Then X^* *is stable. Furthermore, if L also satisfies*

c $L' < 0$ *in* $O - X^*$,

then X^* *is asymptotically stable.*

A function L satisfying (a) *and* (b) *is called a Liapunov function for* X^*. *If* (c) *also holds, we call L a strict Liapunov function.*

Note that Liapunov's theorem can be applied without solving the differential equation; all we need to compute is $DL_X(F(X))$. *This is a real plus ! On the other hand, there is no cut-and-dried method of finding Liapunov functions; it is usually amatter of ingenuity or trial and error in each case.*

Sometimes there are natural functions to try. For example, in the case of mechanical or electrical systems, energy is often a Liapunov function.

Example *Consider the system of differential equations in* $\mathbb{R}^3$ *given by*

$$\begin{aligned} x' &= (\epsilon x + 2y)(z+1) \\ y' &= (-x + \epsilon y)(z+1) \\ z' &= -z^3 \end{aligned}$$

where ϵ *is a parameter. The origin is the only equilibrium point for this system.*

The linearization of the system at $(0,0,0)$ *is*

$$\begin{pmatrix} \epsilon & 2 & 0 \\ -1 & \epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues are 0 *and* $\epsilon \pm \sqrt{2}i$. *Hence, from the linearization, we can only conclude that the origin is unstable if* $\epsilon > 0$. *When* $\epsilon \leq 0$, *all we can conclude is that the origin is not hyperbolic.*

When $\epsilon \leq 0$ *we search for a Liapunov function for* $(0,0,0)$ *of the form* $L(x,y,z) = a\,x^2 + b\,y^2 + c\,z^2$, *with* $a, b, c > 0$. *For such an* L, *we have*

$$L' = 2(a\,x\,x' + b\,y\,y' + c\,z\,z'),$$

so that

$$\begin{aligned} L'/2 &= ax(\epsilon x + 2y)\,(z+1) + by(-x + \epsilon y)(z+1) - cz^4 \\ &= \epsilon\,(ax^2 + by^2)\,(z+1) + (2a - b)\,(xy)\,(z+1) - cz^4. \end{aligned}$$

For stability, we want $L \leq 0$; *this can be arranged, for example, by setting* $a = 1, b = 2$, *and* $c = 1$. *If* $\epsilon = 0$, *we then have* $L' = -z^4 \leq 0$, *so the origin is stable. It can be shown (see Exercise 4[5]) that the origin is not asymptotically stable in this case.*

If $\epsilon < 0$, *then we find*

$$L' = \epsilon\,(x^2 + 2y^2)\,(z+1) - z^4$$

so that $L' < 0$ *in the region* O *given by*$z > -1$ *(minus the origin). We conclude that the origin is asymptotically stable in this case, and, indeed, from Exercise 4, that all solutions that start in the region* O *tend to the origin.*

Example. *(The Nonlinear Pendulum)*

Consider a pendulum consisting of a light rod of length l *to which is attached a ball of mass* m.

The other end of the rod is attached to a wall at a point so that the ball of the pendulum moves on a circle centered at this point. The position of the mass at time t *is completely described by the angle* $\theta(t)$ *of the mass from the straight down position and measured in the counter clock wise direction.*

Thus the position of the mass at time t *is given by* $(l\,\sin\theta(t), -l\,\cos\theta(t))$.

The speed of the mass is the length of the velocity vector, which is $l\,d\theta/dt$*, and the acceleration is* $l\,d^2\,\theta/dt^2$*. We assume that the only two forces acting on the pendulum are the force of gravity and a force due to friction.*

The gravitational force is a constant force equal to mg acting in the downward direction; the component of this force tangent to the circle of motion is given by $-\,mg\sin\theta$*.*

We take the force due to friction to be proportional to velocity and so this force is given by $-\,bl\,d\theta/dt$ *for some constant* $b > 0$*. When there is no force due to friction* $(b = 0)$*, we have an ideal pendulum. Newton's law then gives the second-order differential equation for the pendulum*

$$ml\,\frac{d^2\theta}{dt^2} = -bl\frac{d\theta}{dt} - mg\sin\theta.$$

For simplicity, we assume that units have been chosen so that $m = l = g = 1$*. Rewriting this equation as a system, we introduce* $v = d\theta/dt$ *and get*

$$\begin{aligned}\theta' &= v\\ v' &= -bv - \sin\theta.\end{aligned}$$

Clearly, we have two equilibrium points (mod 2π*): the downward rest position at* $\theta = 0, v = 0$*, and the straight-up position* $\theta = \pi, v = 0$*.*

This up ward postion is an unstable equilibrium, both from a mathematical (check the linearization) and physical point of view.

For the downward equilibrium point, the linearized system is

$$Y' = \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix} Y$$

The eigenvalues here are either pure imaginary (when $b = 0$*) or else have negative real parts (when* $b > 0$*). So the downward equilibrium is asymptotically stable if* $b > 0$ *as everyone on earth who has watched a real-life pendulum knows.*

To investigate this equilibrium point further, consider the function $E(\theta, v) = \frac{1}{2}v^2 + 1 - \cos\theta$*. For readers with a background in elementary mechanics, this is the well-knowntotal energy function, which we will describe further [3]. We compute*

$$E' = v\,v' + \sin\theta\,\theta' = -b\,v^2,$$

so that $E' \leq 0$*. Hence E is a Liapunov function. Thus the origin is a stable equilibrium. If* $b = 0$ *(that is, there is no friction), then* $E \equiv 0$*. That is, E is constant along all solutions of the system.*

Hence we may simply plot the level curves of E to see where the solution curves reside. We find the phase portrait shown in $Figure 2.6.[7]$

Note that we donot have to solve the differential equation to paint this picture; knowing the level curves of E (and the direction of the vector field) tells us everything. We will encounter many such (very special) functions that are constant along solution curves later in this chapter.

The solutions encircling the origin have the property that $-\pi < \theta(t) < \pi$ *for all t. Therefore these solutions correspond to the pendulum oscillating about the downward rest position without ever crossing the upward position* $\theta = \pi$*. The special solutions connecting the equilibrium points at* $(\pm\pi, 0)$

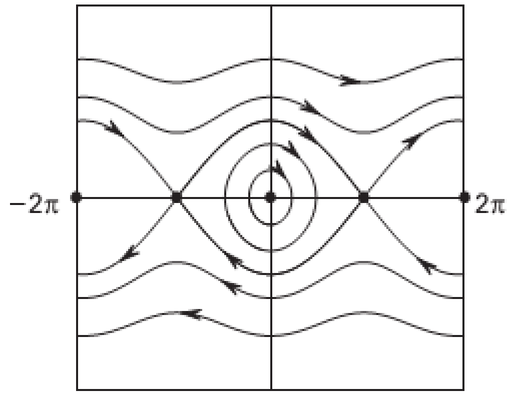

Figure 2.12: The phase portrait for the ideal pendulum.

correspond to the pendulum tending to the upward-pointing equilibrium in both the forward and backward time directions. (You don't often see such motions!) Beyond these special solutions we find solutions for which $\theta(t)$ either increases or decreases for all time; in these cases the pendulum spins forever in the counter clock wise or clock wise direction.

We will return to the pendulum example for the case $b > 0$ later, but first we prove Liapunov's theorem.

Proof :

Let $\delta > 0$ be so small that the closed ball $B_\delta(X^*)$ around the equilibrium point X^* of radius δ lies entirely in O. Let α be the minimum value of L on the boundary of $B_\delta(X^*)$, that is, on the sphere $S_\delta(X^*)$ of radius δ and center X^*.

Then $\alpha > 0$ by assumption. Let $U = \{X \in B_\delta(X^*) \mid L(X) < \alpha\}$ and note that X^* lies in U. Then no solution starting in Ucan meet $S_\delta(X)$ since Lis non increasing on solution curves. Hence every solution starting in U never leaves $B_\delta(X^*)$. This proves that X^* is stable.

Now suppose that assumption (c) in the Liapunov stability theorem holds as well, so that L is strictly decreasing on solutions in $U - X^*$.

Let $X(t)$ be a solution starting in $U - X^*$ and suppose that $X(t_n) \to Z_0 \in B_\delta(X^*)$ for some sequence $t_n \to \infty$. We claim that $Z_0 = X^*$.

To see this, observe that $L(X(t)) > L(Z_0)$ for all $t \geq 0$ since $L(X(t))$ decreases and $L(X(t_n)) \to L(Z_0)$ by continuity of L.If $Z_0 \neq X^*$, let $Z(t)$ be the solution starting at Z_0. For any $s > 0$, we have $L(Z_s) < L(Z_0)$. Hence for any solution $Y(s)$ starting sufficiently near Z_0 we have

$$L(Y_s) < L(Z_0).$$

Setting $Y(0) = X(t_n)$ for sufficiently large n yields the contradiction

$$L(X_{t_n+s}) < L(Z_0).$$

Therefore $Z_0 = X^*$. This proves that X^* is the only possible limit point of the set $\{X(t) \mid t \geq 0\}$ and completes the proof of Liapunov's theorem.

$Figure$2.7[6] makes the theorem intuitively obvious. The condition $L < 0$ means that when a solution crosses a "level surface" $L^{-1}(c)$, it moves inside the set where $L \leq c$ and can never come out again.

Unfortunately, it is sometimes difficult to justify the diagram shown in this figure; why should the sets $L^{-1}(c)$ shrink down to X^*?

Of course, in many cases, Figure 2.7[6] is indeed correct, as, for example, if L is a quadratic function such as $ax^2 + by^2$ with $a, b > 0$. But what if the level surfaces look like those shown in Figure 2.8?

It is hard to imagine such an L that fulfills all the requirements of a Liapunov function; but rather than trying to rule out that possibility, it is simpler to give the analytic proof as above.

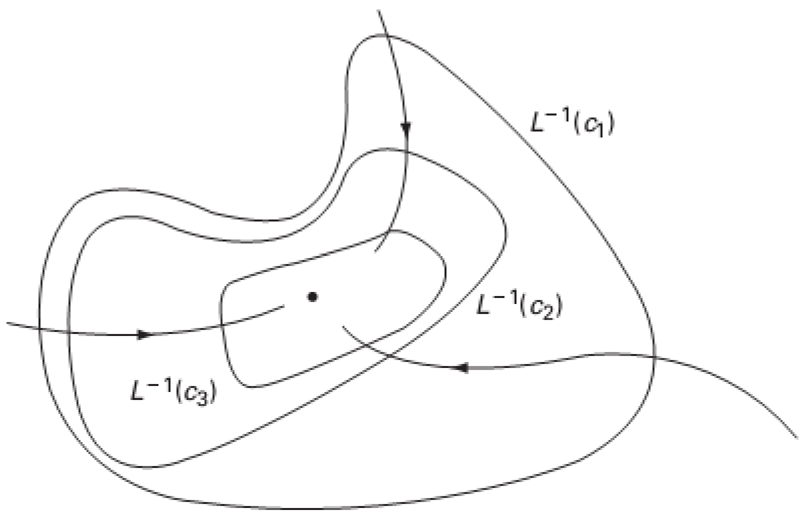

Figure 2.13: Solutions decrease through the level sets $L^{-1}(c_j)$ of a strict Liapunov function.

2.4.2 Classical results on the investigation of the stability properties of flows defined by the solutions of ordinary differential equations via the second method of Liapunov.

The theorems that we shall prove in the sequel are given in the language and technique of differential equations. When not otherwise stated, these theorems will only apply to strong stability properties.

We shall present the Liapunov second method essentially for the case of the autonomous differential equation

$$x' = f(x), \tag{2.23}$$

where $f(x)$ is defined and continuous for all $x \in E$.

From the operational point of view in the second method of Liapunov, the stability properties of closed sets will be characterized by the relative properties of a pair of functions $v = \theta(x)$ *and* $w = \psi(x)$ *connected to the differential system* 3.6.1 *through the relation*

$$\psi(x) =< grad\, \phi(x),\ f(x) >= \sum_{i=1}^{n} \frac{d\phi}{dx_i} f_i(x). \tag{2.24}$$

For a given $\phi(x)$ *the scalar function* $\psi(x)$ *is simply the total time derivative of the scalar function* $v = \theta(x(t))$ *along the solution curves of the differential system 2.6.1; thus*

$$\frac{dv}{dt} = \psi(x). \tag{2.25}$$

For a given $\psi(x)$ *the relation* 3.6.2 *is a linear partial differential equation, which will have a solution* $\phi(x)$ *if integrability conditions are satisfied.*

These integrability conditions can be defined in the following way: given a real-valued function $\psi(x)$ *and a vector* $f(x) \neq 0$*, a vector* $b(x)$ *may be chosen such that*

$$\psi(x) =< b(x), f(x) >, \tag{2.26}$$

Such a vector $b(x)$ *is the gradient of a scalar function if the* $\frac{n(n-1)}{2}$ *integrability conditions :*

$$\frac{db_i(x)}{dx_j} = \frac{db_j(x)}{dx_i} \quad (i = j = 1, ..., n) \tag{2.27}$$

are satisfied.

We shall now first prove a set of theorems which relates the stability properties of a given compact set M *with the sign and uniform boundedness properties of the real valued functions* $v = \phi(x)$ *and* $w = \psi(x)$.

The same theorem holds for the case of sets with a compact neighborhood.

THEOREM *: Let* $v = \phi(x)$ *and* $w = \psi(x)$ *be real-valued functions defined in an open neighborhood* $N(M) \subset E$ *of a* ***compact set*** M*. Assume that*

i $v = \phi(x) \in C^1$,

ii $v = \phi(x)$ *is definite for the set* M*,*

iii $w = \psi(x)$ *is semidefinite for the set* M,

iv *for all* $x \in N(M)$ *with* $\psi(x) \neq 0$ *,sign* $\psi(x) \neq$ *sign* $\phi(x)$

v $\phi(x)$ *and* $\psi(x)$ *satisfy the relation* 2.6.2.

Then the compact set M *is (uniformly)* ***stable***.

Proof *Since the real-valued function $\phi(x)$ is definite for the set M, it follows that there exists a real number $n > 0$ and two strictly increasing function $\alpha(v)$ and $\beta(v)$, with $\alpha(0) = \beta(0) = 0$ such that it is*

$$\alpha(\rho(x, M)) \leq \phi(x) \leq \beta(\rho(x, M)) \quad x \in S[M, \eta] \subset N(M) \tag{2.28}$$

Let $\epsilon > 0$ ($\epsilon \leq \eta$) be given and choose $\delta > 0$ such that

$$\beta(\delta) < \alpha(\epsilon) \tag{2.29}$$

that is, such that

$$0 < \delta < \beta^{-1}(\alpha(\epsilon)) \tag{2.30}$$

where β^{-1} denotes the inverse of the function $\beta(v)$. Obviously $\delta < \epsilon$.

We claim that $\rho(x^0, M) \leq \delta$ implies $\rho(x(x^0, t), M) < \epsilon, t \in R^+$.

In fact, in the set S[M, ϵ]

$$\psi(x) = \phi'(x(x^0, t)) \leq 0 \tag{2.31}$$

which gives

$$\alpha(\rho(x(x^0, t), M)) \leq \phi(x(x^0, t)) \leq \phi(x^0) \leq \beta(\rho(x^0, M)) \leq \beta(\delta). \tag{2.32}$$

If there would exist a $t_1 > t_0$ such that $\rho(x(x^0, t), M) = \epsilon$, then we would have

$$\alpha(\epsilon) \leq \beta(\delta) \tag{2.33}$$

which contradicts the choice of δ in 2.6.8 and proves the theorem.

For sake of completeness and for a better understanding of instability , we shall now state an obvious corollary regarding negative Liapunov stability of a compact set M.

Corollary :

If a compact set M satisfies Theorem 2.6.6 with the condition iv) replaced by

iv′) sign $\psi(x)$ = sign $\phi(x)$ for all $x \in E$ with $\psi(x) \neq 0$, then M is ***negatively stable****.*

Remark :

From the proof of Theorem 2.6.6, it is obvious (as already known for a dynamical system, as shown by Theorem 3.5.24[13]) that a set M which satisfies Theorem 2.6.6 is positively invariant.

Theorem *Let $V = \phi(x)$ and $W = \psi(x)$ be real-valued functions, defined in an open neighborhood $N(M) \subset E$ of a* ***compact set*** *M. Assume that*

i $v = \phi(x) \in C^1$,

ii *$v = \phi(x)$ is definite for the set M,*

iii *$w = \psi(x)$ is definite for the set M,*

iv *sign* $\psi(x) \neq$ *sign* $\phi(x)$,

v $\phi(x)$ *and* $\psi(x)$ *satisfy the condition* 2.6.2.

Then the compact set M is (uniformly) ***asymptotically stable*** *for the system* 2.6.1.

Proof :

In $S[M, \epsilon_0]$, $\epsilon_0 > 0$, *the inequalities* 3.6.7 *are again satisfied and, furthermore, there exist two additional strictly increasing functions* $\omega(v)$ *and* $\gamma(v), \omega(0) = \gamma(0) = 0$, *such that*

$$-\omega\left(\rho(x, M) \leq \psi(x) \leq -\gamma(\rho(x, M))\right). \tag{2.34}$$

From Theorem 3.6.6, *it follows that M is uniformly stable. To prove the theorem we choose* $\delta_0 > 0$ *such that* $\beta(\delta_0) < \alpha(\epsilon_0)$. *Then* $\rho(x^0, M) \leq \delta_0$ *implies that* $\rho(x(x^0, t), M) < \epsilon_0$ *for* $t \in R^+$, *since M is stable.*

We assert that $\rho(x^0, M) \leq \delta$ *implies that:*

$$\lim_{t \to +\infty} \rho(x(x^0, t), M) = 0.$$

For any $x(x^0, t)$ *such that* $\rho(x^0, M) < \delta$ *we set* $\phi(t) = \phi(x(x^0, t))$. *We then have*

$$\psi(x(x^0, t)) = \phi'(t) \leq -\gamma(\rho(x(x^0, t), M)), \quad t \geq t_0.$$

It follows then that

$$\phi(t) - \phi(t_0) \leq -\int_{t_0}^{t} \gamma(\rho(x(x^0, \tau), M))\, d\tau \tag{2.35}$$

Now let $\epsilon > 0$, $(\epsilon < \delta_0)$ *be any positive number. Choose* $\delta > 0$, $(\delta < \epsilon)$ *such that* $\beta(\delta) < \alpha(\epsilon)$, *then if* $\rho(x(x^0, t), M)) = \delta$ *then* $\rho(x(x^0, t), M)) < \epsilon$ *for* $t \geq t_1$.

Now let $\rho(x^0, M)) \leq \delta_0$. *If* $\rho(x^0, M)) \leq \delta$, *then* $\rho(x(x^0, t), M)) < \epsilon$ *for* $t \geq t_0$. *If* $\delta < \rho(x^0, M)) \leq \delta_0$, *then as long as* $\rho(x(x^0, t), M)) > \delta$ *we have*

$$\phi(t) - \phi(t_0) \leq -\int_{t_0}^{t} \gamma(\delta)\, d\tau = -(t - t_0)\, \gamma(\delta)$$

or

$$t - t_0 \leq \frac{\phi(t_0) - \phi(t)}{\gamma(\delta)} \leq \frac{\beta(\delta_0) - \alpha(\delta)}{\gamma(\delta)}. \tag{2.36}$$

Let

$$T(\epsilon) = \frac{\beta(\delta_0) - \alpha(\delta)}{\gamma(\delta)} \tag{2.37}$$

be the maximum time in which the solutions of the system 2.6.1 *remain in the set* $S[M, \delta_0] \setminus S[M, \delta]$. *Since* δ *depends only upon* ϵ *the inequality* 2.6.16 *and, therefore,* 2.6.7 *is violated if* $t > t_0 + T(\epsilon)$.

Hence there exists a t_1, *with* $t_0 \leq t_1 < t_0 + T(\epsilon)$ *such that* $\rho(x(x^0, t_1), M) = \delta$.

Thus $\rho(x(x^0, t_1), M) < \epsilon$ *for* $t \geq t_0 + T(\epsilon)$ *for all* $t_0 > 0$ *and* $\rho(x^0, M) \leq \delta$.

This completes the proof.

Remarks :

In the proof of the theorem no use has been made of the left hand part of the inequality 2.6.16. *By proceeding as before, one can derive the analogue of inequality* 2.6.18 *as follows:*

$$\tau(\epsilon) = \frac{\alpha(\delta_0) - \beta(\delta)}{\omega(\delta)} \leq t - t_0 \tag{2.38}$$

Now $\tau(\epsilon)$ *is the minimum time in which the solution of* 3.6.1 *can cross in the ring* $S[M, \delta_0] \setminus S[M, \delta]$. *By the same argument as in the above proof of Theorem* 3.6.16, *it follows that* 3.6.21 *does not hold for* $t < t_0 + \tau(\epsilon)$. *Thus* $\rho(x(x^0, t_1), M) > \epsilon$ *for* $t \leq t_0 + \tau(\epsilon)$ *for all* $t_0 \geq 0$ *and* $\rho(x^0, M) \geq \delta$.

Thus the solutions $x(x^0, t)$ *have* ***a uniform rate of approach*** *to* M *in* $N(M)$.

From all theorems on asymptotic stability of compact sets it is possible to derive trivial corollaries on the complete instability of such sets by reversing the requirements of the relative sign of the independent variable t, and, therefore, inverting the direction of motion on each trajectory.

For example, from theorem 2.6.15 *it can be deduced that*

Corollary :

If a compact set M *satisfies Theorem* 2.6.15 *with condition iv) replaced by*

iv′) sign $\psi(x)$ = *sign* $\phi(x)$, *then the set* M *is* ***completely unstable.***

We shall now prove the theorem whlch provides sufficient conditions for the instability of a compact set for the differential system 2.6.1.

Theorem :

Let $v = \phi(x)$ *and* $w = \psi(x)$ *be real-valued functions defined in an open non-empty set* $B \subset S(M, n) \subset E$, *where* $\eta > 0$ *and* M *is a* ***compact set***. *Assume*

i $dM \cap dB \neq \emptyset$,

ii $\phi(x) = 0$ *for* $x \in [dB \cap S(M, \eta)], \phi(x) \neq 0$ *for* $x \in [I_B \cap S(M, \eta)]$,

iii $v = \phi(x) \in C^1$,

iv *sign* $\psi(x)$ = *sign* $\phi(x)$, *for* $x \in [I_B \cap S(M, \eta)]$,

v *for all* $x \in B, |\ \phi(x)\ | \leq \beta(\rho(x, M))$ *and* $|\ \psi(x)\ | \geq \gamma(\rho(x, M))$,

vi $\phi(x)$ *and* $\psi(x)$ *satisfy the condition* 2.6.2.

Then the compact set M *is* ***unstable*** *for the system* 2.6.1.

Proof .

Assume that $\phi(x) > 0$ *in* I_B. *For a sufficiently small* $\delta > 0$ *there exists* $x^0 \in I_B, \rho(x^0, M) < \delta$, *such that* $\phi(x^0) > 0$. *Consider the corresponding solution* $x(t) = x(x^0, t)$ *and the values of* $\phi(x)$ *along such solution* $\phi(t) = \phi(x(x^0, t))$.

Integrating $\psi(x) = \phi(x)$ along such solutions and taking into account the condition v) we obtain

$$\phi(t) - \phi(t_0) = \int_{t_0}^{t} \psi(\tau)\, d\tau \geq \int_{t_0}^{t} \gamma(\rho(x(\tau), M))\, d\tau$$

and

$$\phi(t) \geq \gamma(\rho(x(\tau), M)).(t - t_0) + \phi(t_0)$$

If for all $t \geq t_0$, $x(x^0, t) \in I_B$, then $\lim_{t\to+\infty} \phi(t) = +\infty$, which contradicts the hypothesis (v). Hence there exists $t = t_1 > t_0$ for which $x(t_1) \in dB \cap dS(M, \eta)$.

Since, for all $t \geq t_0$ for which $x(t) \in I_B$ $\phi(x) \geq \alpha(\rho(x(x^0, t), M)) > 0$; we cannot have $\phi(t_1) = 0 \in dB$ thus $\rho(x(t_1), M) = \eta$ and the theorem is proved.

It must be pointed out that, from the hypothesis of Theorem 2.6.23, the set B cannot have any compact component which does not contain a component of M. In fact, if there would exist such compact component B_c then there would exist (2.8.25) at least one point $y \in I_{B_c}$ such that grad $\phi(y) = 0$.

Hence $\psi(y) = 0$ which contradicts the hypothesis iv). On the other hand, B need not be a region, but it could be formed by a sequence of sets with non-compact closure which satisfy the conditions of the theorem.

From the theorems given it follows that

Theorem :

If there exists a pair of real-valued functions $\phi(x)$ and $\psi(x)$, satisfying the condition 2.6.2, where $\psi(x)$

is definite for a compact set M in the neighborhood $N(M) \subset E$ and $\phi(x) \in C^1$ is such that

$\phi(x) = 0$ for all $x \in M$, then the additional sign properties of the function $\phi(x)$ completely characterize the stability properties of the compact set M.

Proof :

i *If $\phi(x)$ is definite and sign $\phi(x) \neq$ sign $\psi(x)$, then from the theorem (2.6.15) it follows that M is asymptotically stable.*

ii *If $\phi(x)$ is definite and sign $\phi(x) =$ sign $\psi(x)$, then from Corollary (2.6.22) it follows that M is completely unstable.*

i *If $\phi(x)$ is indefinite then Theorem 2.6.23 insures that M is unstable.*

Finally

i *If $\psi(x)$ is definite for M, $\phi(x)$ cannot be semi-definite for M.*

In fact, if $\phi(x)$ is semi-definite the set $M \subset G$ such that if $y \in G, \phi(y) = 0$, is the absolute minimum of the $\phi(x)$, and since $\phi(x) \in C^1$, it follows that for all $y \in G$ grad$\phi(y) = 0$ and, thus, $\psi(y) = 0$ for some $y \notin M$ which contradicts the hypothesis and the theorem is proved.

Notice that Theorem 2.6.24 does not give necessary conditions for the stability of M.

*In fact, there do not always exist real-valued functions $\phi(x)$ and $\psi(x)$ satisfying 2.6.2 and such that $\psi(x)$ is definite for a given **(positively) invariant set**.*

Definition *:*

A real-valued function $v = \phi(x)$ which satisfies one of the stability theorems is called ***Liapunov function***.

Theorem 2.6.15 *and Corollary* 2.6.22 *define only local properties of the compact set M. That is, if Theorem* 2.6.15 *is satisfied, then there exists a sufficiently small* $\delta > 0$, *such that* $S(M, \delta) \subset A(M)$ *where* $A(M)$ *is the region of asymptotic stability of the set M.*

For the practical applications of the stability theorems, local properties are not very useful.

It is, therefore, important to give theorems which provide sufficient conditions for global asymptotic stability or in the case in which the compact set M is not globally asymptotically stable, allow the exact identification of the region of asymptotic stability $A(M)$ *or at least an approximate identification of the set* $d(A(M))$.

Our first concern is to derive a theorem which will provide a sufficient condition for the global asymptotic stability of a compact set M.

Theorem *:*

If the conditions of Theorem 3.6.15 *are satisfied in the whole space E and, in addition,*

vi) $\lim_{\|x\| \to \infty} \phi(x) = \infty.$

Then the compact set M is ***globally asymptotically stable***.

Proof *:*

Along the solutions of the system 3.6.1, *let*

$\phi'(t) = \psi(x(t)) = -\chi(x(t)).$

Assume that $\chi(x(t))$ *is positive definite for M, For all t with* $t_0 \leq t \leq t_1$

$$0 \leq \phi(t) = \phi(t_0) - \int_{t_0}^{t} \chi(x(\tau))\, d\tau. \tag{2.39}$$

We claim that $\chi(x(\tau))$ *is an integrable function in* $[0, +\infty[$. *In fact, from* 2.6.2, 7 *and condition* (*vi*), *it follows that if* $\chi(x(\tau))$ *were not integrable, then* $\lim_{t \to +\infty} \chi(x(\tau)) = -\infty$, *which contradicts the hypothesis on the sign of* $\chi(x(\tau))$ *We shall now prove that*

$$\lim_{t \to +\infty} x(x^0, t) = 0 \ \ for\ all\ x^0 \in E. \tag{2.40}$$

In fact, if this were not true, then there would exist a $\epsilon_1 > 0$ *and a sequence of intervals* $(t_n, t_n + \lambda)$ *with* $t_n \to +\infty,\ \lambda > 0$, *such that*

$$\rho(x(t), M) \geq \epsilon_1 \ for\ t_n \leq t \leq t_n + \lambda; \quad n = 1, 2, \cdots, \lambda > 0$$

But then condition vi) implies that for all $x \in E$ *we have* $\chi(x(\tau)) \geq \epsilon_2$ *for* $t_n \leq t \leq t_n + \lambda; \quad n = 1, 2, \cdots, \lambda > 0$ *which contradicts the integrability of* $\chi(x(\tau))$ *Thus* 2.6.28 *follows. Since the hypothesis of the theorem obviously implies that M is stable, it follows from* 2.6.28 *that M is globally asymptotically stable.Q.E.D.*

Remark :

Theorem 2.6.26 *would be also true if instead of condition vi*), *one simply required that condition* 2.6.28 *be satisfied for all* $x \in E$. *The fact that condition* (*vi*) *is not necessary will be shown by the following theorem which is a trivial corollary of Theorem* 2.6.15.

The Liapunov function commonly used in practice does, however, satisfy the condition vi).

Theorem :

Let $v = \phi(x)$ *and* $\theta = \theta(x)$ *be real-valued functions defined in the whole space* E. *Assume that*

i $v = \phi(x) \in C^1$,

ii $v = \phi(x)$ *is definite for a compact set* M,

iii $\lim_{\|x\|\to+\infty} \phi(x) = \eta > 0$,

iv $\theta = \theta(x)$ *be positive definite for the set* M,

v $\psi(x) = \theta(x)\,(\phi(x) - \eta)$,

vi $\phi(x)$ *and* $\psi(x)$ *satisfy the condition* 2.6.2.

Then the compact set M *is globally asymptotically stable.*

By extending the definition of the function $\phi(x)$ *and* $\psi(x)$ *to an open set* B *with noncompact closure one is able to show the existence in* B *of solutions which tend to infinity and have the so-called global (but not necessarily complete) instability.*

Definition *A compact set* $M \subset E$ *will be called globally unstable for the flow defined by the system of differential equations 2.6. 1 if there is a sequence* $\{x^n\}$ *of points in* $C(M)$, $x^n \to M$, *such that* $\| x(x^n, t) \| \to +\infty$ *as* $t \to +\infty$ *for each* n.

Theorem :

If in Theorem 2.6.23 *the set* $\bar{B}$ *is noncompact, then* M *is globally unstable.*

2.5 Integrable Hamiltonian Systems

(a) In these informal lecture notes we discuss a nunber of integrable Hamiltonian systems which have surfaced recently in very different connections. It is our goal to discuss various aspects underlying the integrability of a system like that of group representation, isospectral deformation and geometrical considerations.

Since this subject is still far from being understood or being systematic we discuss a number of examples which are seemingly disconnected. In fact, there are some rather unexpected connections like between the inverse square potential of Calogero and the Korteweg de Vries equation.

Here we show a surprising new connection between the geodesics on an ellipsoid and Hill's equation with finite gap potential.

(b) The differential equations of mechanics can be written in Hamiltonian from

$$x'_k = \frac{dH}{dy_k}, \quad y'_k = -\frac{dH}{dx_k} \quad (k = 1, 2, ..., n) \tag{2.41}$$

where $x = (x_1, x_2, ..., x_n) \in R^n$, $y = (y_1, y_2, ..., y_n) \in R^n$, *are coordinates in the phase space* R^{2n} *or an open subset of* R^{2n}. *Thus a function H defines a vector field* x_H *defined by*

$$x_H = \sum_{k=1}^{n}(\frac{dH}{dy_k}\frac{d}{dx_k} - \frac{dH}{dx_k}\frac{d}{dy_k}) = \{F, H\}$$

is antisymmetric in F, H. *It is called the Poisson bracket of F and H. The* ***The Hamiltonian systems form a Lie algebra*** *and*

$$[x_H, x_G] = -x_{\{H,G\}}$$

A nonconstant function F is called an integral of x_H *if*

$$x_H\, F = \{F, H\} = 0.$$

In particular,H is an integral. If F is an integral of x_H *then H is an integral of* x_F.

A set of functions $F_1, F_2, ..., F_r$ *are said to be "ininvolution" or to commute, if*

$$\left\{F_k, F_j\right\} = 0 \;\; k, j = 1, 2, ..., r.$$

This implies clearly that the vector fields x_{F_k} *comute.*

If $\phi = \phi(\epsilon_1, ..., \epsilon_r)$, $\psi = \psi(\epsilon_1, ..., \epsilon_r)$ *then*

$\{\phi = \phi(F_1, ..., F_r), \quad \psi = \psi(F_1, ..., F_r)\} = \sum_{k,j} \frac{d\phi}{d\epsilon_k}\frac{d\psi}{d\epsilon_j}\left\{F_k, F_j\right\}.$

Thus,if $F_1, F_2, ..., F_r$ *are in involution, so are any functions of* $F_1, F_2, ..., F_r$.

Definition *: A Hamiltonian system* (1) *, defined in an open domain* $D \subset R^{2n}$ *is called "integrable" if there exist n integrals* $F_1, F_2, ..., F_r$ *in involution with linearly independent gradients, i .e . in D we have*

(i) $\left\{H, F_j\right\} = 0$, ***(ii)*** $\left\{F_k, F_j\right\}$, ***(iii)*** $dF_1, dF_2, ..., dF_r$ *linearly independent.*

Example *1*

$H = \frac{1}{2}\sum \alpha_k({x_k}^2 + {y_k}^2)$ *defines an integrable system in* R^{2n} *with* $F_k = {x_k}^2 + {y_k}^2 \;\; (k = 1, 2, ..., n)$.

Example *2: If* $H = H(y)$ *is independent of x then the system is integrable with* $F_k = Y_k$.

Locally, that is near any point where $dH \neq 0$ *any system is integrable, in fact, in appropriate canonical coordinates H agrees with* Y_1 *which is a special case of Example* 2.

Generally it makes sense to speak of a system being integrable in a domain which E invariant under the flow generated by x_H.

It is highly exceptional for a Hamiltonian system to be integrable globally in an invariant open domain or even locally near a stationary point (where $dH = 0$). However many systems occurring in application are closely approximated by integrable systems.

For example, the n-body problem becomes integrable in the limit when all but one mass tends to zero.

The resulting system is a decoupled system of Kepler problems.

Another example is a system near a stationary point, say $x = y = 0$ where $dH(0) = 0$. Assume that the Taylor expansion of H begins with

$$H = \frac{1}{2}\sum \alpha_k({x_k}^2 + {y_k}^2) + \cdots$$

with real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$. By a theorem of G. D. Birkhoff one can approximate this system locally by an integrable one (see[3], [4]) :

Given any large integer N, assume that $j_1\alpha_1 + \cdots + j_n\alpha_n = 0$, $\mid j_1 \mid + \cdots + \mid j_n \mid \leq N$, j_k integers implies that $j_1 = j_2 = \cdots = j_n = 0$ then there exist canonical variables ${x_1}^*, \cdots, {x_n}^*, {y_1}^*, \cdots, {y_n}^*$ such that

$$H = \phi(F_1, F_2, \cdots, F_n) + R_{N+1}$$

where R_{N+1} vanishes with derivatives of order $\leq N$ at the origin and $F_j = {x_j}^{*2} + {y_j}^{*2}$. Thus if we drop the term R_{N+1} the system is integrable. (To be precise, one has to excise the hyperplanes ${x_k}^* = {y_k}^* = 0$ where the dF_j are linearly dependent.)

On the other hand one can show that in general, even if the $\alpha_1, \alpha_2, \cdots, \alpha_n$ are rationally independent the system is not integrable in any neighborhood of the origin.

(c) The structure of the integrable system is particularly simple. Given the integrals one considers the manifolds N_n defined by

$$F_1 = c_1, \cdots, F_n = c_n$$

with appropriate constants $c_1, c_2, \cdots, c_n$. These manifolds are invariant not only under x_H (because of (i) of definition 1) but also under x_{Fj} (because of (ii)) . Thus $x_{F1}, x_{F2}, \cdots, x_{Fn}$ span the tangent space of N_n.

Since these vector fields comute each component of N_n is topologically a cylinder and in case it is compact, a torus. Thus in the latter case D is foliated by n-dimensional tori.

By a theorem due to Arnold [1], [2] and Jost one can near a compact component of N_n introduce canonical coordinates, called x, y again, such that $H = H(y_1, \cdots, y_n)$, that $y = 0$ corresponds to N_n and that points $(x, y) = (\tilde{x}, y)$ with the integer $(x_j - \tilde{x}_j)(2\pi)^{-1}$ correspond to the same points in D.

The y_k, x_k are called the "action − angle" variables, respectively. In other words, the example 2 is typical.

In Example 1 these tori are given by ${x_k}^2 + {y_k}^2 = c_k$ if the c_k are positive.

The flows generated by the commuting $x_{F1}, x_{F2}, \cdots, x_{Fn}$ are the n rotations in the x_k, y_k plane.

In action-angle variables the differential equations become

$${x_k}^* = H_{y_k}(y),\ {y_k}^* = 0.$$

Thus the differential equations are linear on N_n. If the frequencies $H_{y_1}, \cdots, H_{y_n}$ are rationally independent then the orbits are dense on N_n.

If one solution on $N_n$is periodic then all are. This occurs if and only if $H_{y_k} / j_k = \rho$ is independent of k with some integers $j_1, j_2, \cdots, j_n$. Thus for an integrable system the periodic solutions form $n-1$ dimensional families.

The proof that Hamiltonian systems generally are not integrable is based on the fact that generically the periodic solutions on a fixed energy surface are isolated.

2.5.1 Example of Continuous systems

Volterra system *: We have the system*

$$\begin{cases} \frac{dF}{dt} &= aF - bFS, \\ \frac{dS}{dt} &= -dS + cSF \end{cases}$$

If $x = \log F,\ y = \log S,$

$$\begin{cases} \frac{dx}{dt} &= a - b\,e^y, \\ \frac{dy}{dt} &= -d + c\,e^x \end{cases}$$

Hamiltonian system*, with*

$$H = -ce^x + dx - b\,e^y + ay$$

Epidemic system *:*

Saturation Bernoulli, d'Alembert and Ross

The differential equations are given by

$$\begin{cases} \frac{dx}{dt} &= \dfrac{-\beta\, x(t)\, y(t)}{(1+x)\,(1+y)} \\ \frac{dy}{dt} &= \dfrac{\beta\, x(t)\, y(t)}{(1+x)\,(1+y)} - \mu\, y \end{cases}$$

If $X = \frac{x}{(1+x)} = 1 - \frac{1}{(1+x)},\ Y = \frac{y}{(1+y)}$*, with* $X, Y < 1$*, then*

$$\begin{cases} \frac{dx}{dt} &= -x\,y + 2x^2\,y - x^3\,y \\ \frac{dy}{dt} &= x\,y - 2x\,y^2 + x\,y^3 - \mu\,y + \mu\,y^3 \end{cases}$$

$$\begin{cases} \frac{dx}{dt} &= \frac{-dP}{dx} + \frac{dH}{dy}, \\ \frac{dy}{dt} &= \frac{-dP}{dy} - \frac{dH}{dx} \end{cases}$$

$$P(x,y) = \mu\frac{y^2}{2} - \frac{x^3}{6} + \frac{y^3}{6} - \mu\frac{y^4}{4} - \frac{x^3\,y^2}{2} + \frac{x^2\,y^3}{2} + \frac{x^5}{20} - \frac{y^5}{20}$$

$$H(x,y) = -\frac{y^2\,x}{2} - \frac{x^2\,y}{2} + x^2\,y^2 - \frac{x^3\,y^2}{2} - \frac{x^2\,y^3}{2} + \frac{x\,y^4}{4} + \frac{x^4\,y}{4}$$

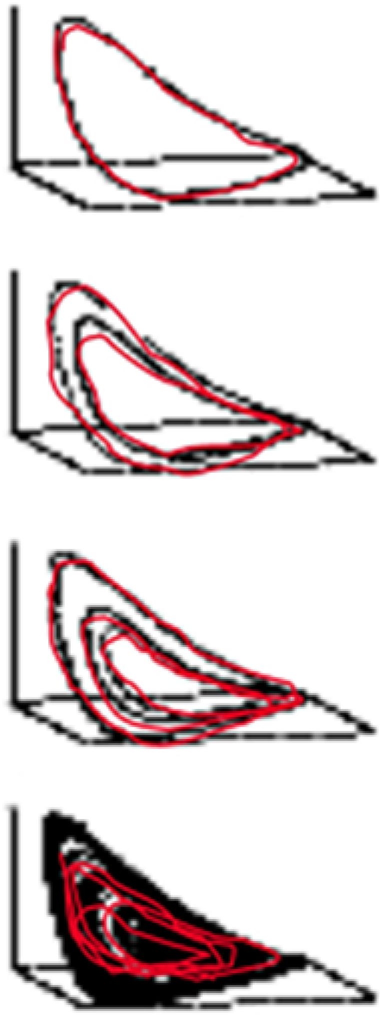

Figure 2.14: A simple Lotka Volterra attractor which a fixed point at (1.1671, 0.740047)

$$\left\{\begin{array}{rcl} \frac{dP}{dx} & = & \frac{-x2}{2} - \frac{3x^2y^2}{2} + xy^3 + \frac{x^4}{4}, \\ \frac{dP}{dy} & = & \frac{y^2}{2} - x^3\,y + \frac{3x^2\,y^2}{2} + \mu y - \mu y^3 - \frac{y4}{4} \\ \frac{dH}{dy} & = & -xy - \frac{x^2}{2} + 2x^2y - x^3 - y\frac{3x^2\,y^2}{2} + xy^3 + \frac{x^4}{4}, \\ \frac{dH}{dx} & = & -\frac{y^2}{2} - xy + 2xy^2 - \frac{3x^2\,y^2}{2} - x\,y^3 + x^3\,y + \frac{y^4}{4} \\ -\frac{dP}{dx} + \frac{dH}{dy} & = & -xy + 2x^2\,y - x^3\,y, \\ -\frac{dP}{dy} - \frac{dH}{dx} & = & xy - 2xy^2 + x\,y^3 - \mu y + \mu y^3 \end{array}\right.$$

When $x \approx y \approx 1, P(X,Y) \approx \frac{\mu}{2}$ *et* $H(x,y) \approx \frac{-1}{2}$ *the system is quasi-conservative, if* $\mu << 1$.

When x *et* $y << 1, P(x,y) \approx \mu\frac{y^2}{2}$ *et* $H(X,Y) \approx 0$, *the system is quasi-dissipative*

Liénard differential system :

A Lienard system consists in two-dimensional ordinary differential equations ($2D\ ODE_s$) *defined by:*

$$\begin{cases} \frac{dx}{dt} & = & y \\ \frac{dy}{dt} & = & -g(x) + y f(x), \end{cases}$$

where $g, f > 0$ *and* $-g' + y f' < 0$

Potential-Hamiltonian decomposable

$$\begin{cases} \frac{dx}{dt} & = & \frac{-dP}{dx} + \frac{dH}{dy} \\ \frac{dy}{dt} & = & \frac{-dP}{dy} - \frac{dH}{dx}, \end{cases}$$

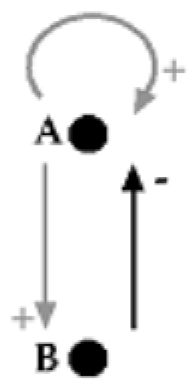

Figure 2.15: Regulon scheme. Grey arrows are activatory (+) whereas the black one is inhibitory (−). A activates the formation of B and its own formation ($self-catalysis$), whereas B inhibits the formation of A.

The regulon can have two stable steady states (due to the presence of the positive loop for the multi-attractority and of the negative one for the stability [31, 32]). *One can note that a Liénard system is a regulon, where* A $(resp.B)$ *is represented by the variable* y $(resp.x)$, *if* $g,\ f > 0$ *and* $-g' + y f' < 0$.

If $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -g + yf$, *where* $f, g(resp.h, l)$ *are polynomial of order* $n, m(resp.p, q)$, *then the Lienard system is PH-decomposable, with :*

$$\begin{cases} \boldsymbol{P} & = & \sum_{k=1,n}(-1)^k f^{(2k-2)} \frac{y^{2k}}{2k!} + \sum_{k=1,m}(-1)^k h^{(2k-2)} \frac{y^{2k-1}}{(2k-1)!} + \sum_{k=1,p}(-1)^k l^{(2k-2)} \frac{x^{2k-2}}{(2k-2)!}. \\ \boldsymbol{H} & = & \int(g+h) + \frac{y^2}{2!} + \sum_{k=1,n}(-1)^k f^{(2k-1)} \frac{y^{2k+1}}{(2k+1)!} + \sum_{k=1,m}(-1)^k h^{(2k-1)} \frac{y^{2k}}{(2k)!} + \sum_{k=1,p}(-1)^{k+1} l^{(2k-1)} \frac{x^{2k-1}}{(2k-1)!}. \end{cases}$$

The set of parameters of P is more amplitude controlling the set of parameters of H is more frequency controlling, In the neighbourhood of a limit-cycle of the Lienard system.

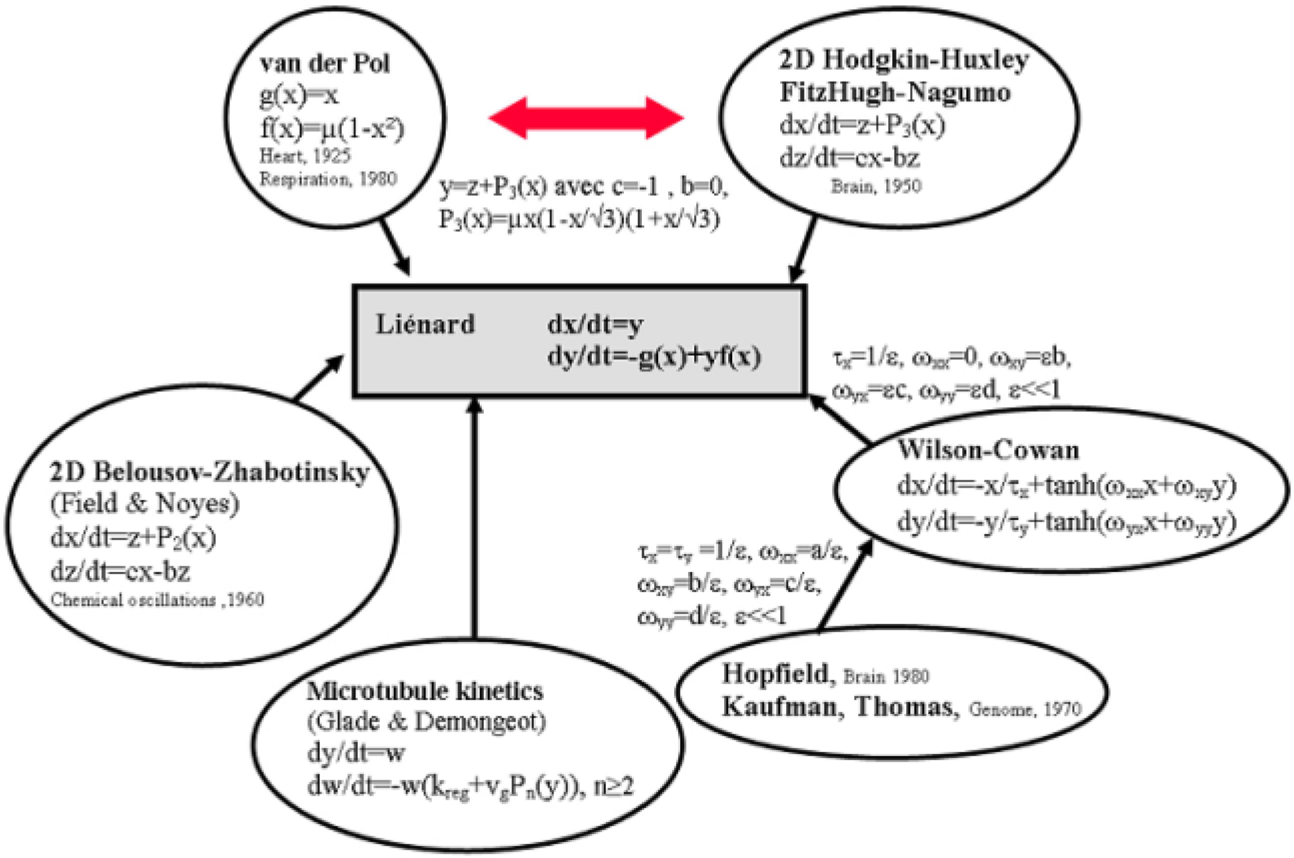

Figure 2.16: An overview of Liénard systems in biological models

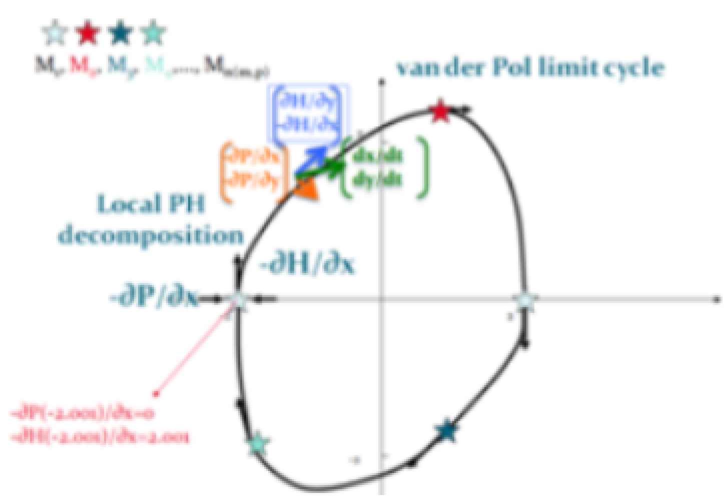

Figure 2.17: JD.,N Glade and Forest lienard systems and potential- Hamiltonian decomposition

Thom's eight system :

A Thom's eight system consists in two-dimensional ordinary differential equations (2D ODEs) de-

fined by:

$$\begin{cases} \frac{dx}{dt} &= y(t) \\ \frac{dy}{dt} &= x(t) * (1 - x(t)^2) - y(t) * (\mu - x(t)^2) \end{cases}$$

Figure 2.18: Thom's eight

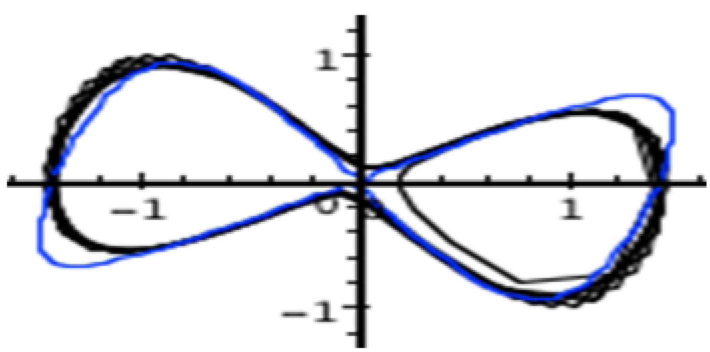

Figure 2.19: $\mu = 0.8$

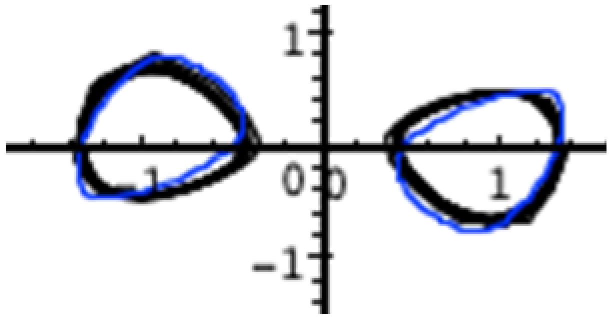

Figure 2.20: $\mu = 0.9$

$$H = \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{24}y^3x - \frac{3}{8}yx^3 + \frac{1}{2}xy + 1.$$

2.6 Limit cycles

We recall that a limit cycle of a real affine polynomial vector fields is an isolated periodic orbit in the set of all periodic orbits of the system.

An algebraic limit cycle of degree r is an oval of an irreducible invariant algebraic curve $f(x, y) = 0$ *of degree r which is a limit cycle of the system.*

In 1958 *Qin Yuan-Xun* [8] *(see also* [4]*) proved that quadratic systems can have algebraic limit cycles of degree 2, moreover when such a limit cycle exists then it is the unique limit cycle of the system. Evdokimenco in* [40˘42] *proved that quadratic systems do not have algebraic limit cycles of degree 3, for two different shorter proofs see* [12, 15].

We provide one of these proofs in what follows

Theorem *:*

Quadratic systems have no algebraic limit cycles of degree3.

Proof *:*

Let $f = 0$ *be an invariant algebraic curve of degree 3 of a real affine polynomial vector field of degree* 2*. If the cubic curve f=0 has multiple points, then it is rational (its genus is 0) and there is no oval in it. If* $f = 0$ *has no multiple points, Equation in the proof of Proposition* 2.2 *implies* $h' = 2^2 = 4$ *According to Theorem* 2.3*, the system has a rational first integral and thus no limit cycle.*

The first class of algebraic limit cycles of degree 4 was given in 1966 by Yablonskii [13]*. The second class was found in 1973 by Filiptsov* [23]*. Recently, two new classes has been found and in* [25] *the authors proved that there are no other algebraic limit cycles of degree* 4 *for quadratic systems. The uniqueness of these limit cycles has been proved in* [21]*. We summarize all these results into the following theorem, for a proof see the mentioned papers.*

Theorem :

The following statements hold.

(a) After an affine change of variables and a rescaling of the time variable the only quadratic systems having an algebraic limit cycle of degree 2 *are*

$$\begin{cases} x' &= -y(ax+by+c)-(x^2+y^2-1), \\ y' &= x(ax+by+c) \end{cases} \tag{2.42}$$

with $c^2+4(b+1)<0$ *and* $a^2+b^2<c^2$. *This system possesses the irreducible invariant algebraic curve*

$$x^2+y^2-1=0$$

of degree 2. *This algebraic limit cycle is the unique limit cycle of system .*

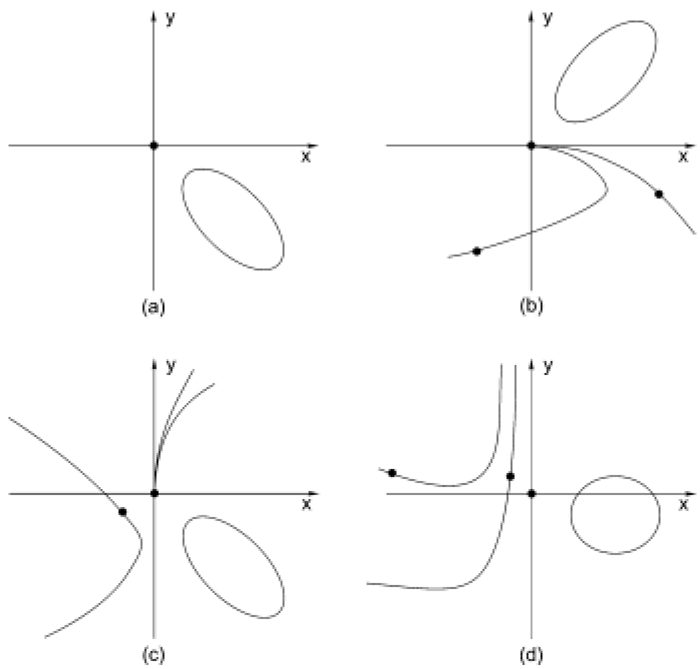

Figure 2.21: Algebraic limit cycles of degree 4.

(b) *There are no quadratic systems having algebraic limit cycles of degree* 3.

(c) *After an affine change of variables the only quadratic systems having an algebraic limit cycle of degree* 4 *are*

(c.1) ***Yablonskii's system***

$$\begin{cases} x' &= -4abc\,x-(a+b)y+3(a+b)c\,x^2+4xy, \\ y' &= (a+b)bc\,x-4abc\,y+(4abc^2-3(a+b)^2/2+4ab)\,x^2+8(a+b)cxy+8y^2, \end{cases} \tag{2.43}$$

with $abc \neq 0, a \neq b, ab > 0$ *and* $4c^2(a-b)^2 + (3a-b)(a-3b) < 0$. *This system possesses the irreducible invariant algebraic curve*

of degree 4 having two components, an oval(the algebraic limit cycle) and an isolated point(a singular point), see Figure $1(a)$.

(c.2) ***Filiptsov's system***

$$\begin{cases} x' &= 6(1+a)x + 2y - 6(2+a)x^2 + 12xy, \\ y' &= 15(1+a)y + 3a(1+a)x^2 - 2(9+5a)xy + 16y^2, \end{cases} \tag{2.44}$$

with $0 < a < \frac{3}{13}$. *This system possesses the irreducible invariant algebraic curve*

$$3(1+a)(ax^2+y)^2 + 2y^2(2y - 3(1+a)x) = 0$$

of degree 4 having two components, one is an oval and the other is homeomorphic to a straight line. This last component contains three singular points of the system, see Figure$1(b)$.

(c.3) ***The system***

$$\begin{cases} x' &= 5x + 6x^2 + 4(1+a)xy + ay^2 \\ y' &= x + 2y + 4xy + (2+3a)y^2, \end{cases} \tag{2.45}$$

with $\frac{(-71+17\sqrt{17})}{32} < a < 0$ *possesses the irreducible invariant algebraic curve*

$$x^2 + x^3 + x^2y + 2axy^2 + 2axy^3 + a^2y^4 = 0$$

of degree 4 having three components, one is an oval and each of the other two is homeomorphic to a straight line. Each one of these last two components contains one singular point of the system, see Figure$1(c)$

(c.4) ***The system***

$$\begin{cases} x' &= 2(1 + 2x - 2ax^2 + 6xy), \\ y' &= 8 - 3a - 14ax - 2axy - 8y^2, \end{cases} \tag{2.46}$$

with $0 < a < \frac{1}{4}$ *possesses the irreducible invariant algebraic curve*

$$\frac{1}{4} + x - x^2 + ax^3 + xy + x^2y^2 = 0, \tag{2.47}$$

of degree 4 having three components, one is an oval and each of the other two is homeomorphic to a straight line. Each one of these last two components contains one singular point of the system, see Figure$1(d)$.

(d) *Quadratic systems* $(13), (14), (16), (17)$ *and* (18) *have a unique limit cycle, the algebraic one.*

We note that the algebraic limit cycle of Filiptsov's system is born in a Hopf bifurcation at the singular point $(4, 48/13)$ *when* $a = \frac{3}{13}$. *Then, when a decreases the algebraic limit cycle increases its size and ends having infinite size at the curve* $y^2(3 - 6x + 4y) = 0$ *when* $a = 0$.

We note that the algebraic limit cycle of system (c.3) *is born in a Hopf bifurcation at the singular point* $\left(\frac{9-\sqrt{17}}{8}, \frac{-(5+3\sqrt{17})}{8}\right)$ *when* $a = \frac{-71+17\sqrt{17}}{32.}$ *Then, when a increases the algebraic limit cycle increases its size and ends having infinite size at the curve* $x^2(1+x+y)=0$ *when* $a=0$.

We note that the algebraic limit cycle of system (c.4) *is born in a Hopf bifurcation at the singular point* $2, \frac{-1}{4}$ *when* $a = \frac{1}{4}$

Then, when a decreases the algebraic limit cycle increases its size and ends having infinite size at the irreducible curve $\frac{1}{4} + x - x^2 + xy + x^2y^2 = 0$ *when* $a = 0$

The uniqueness of limit cycles for planar differential systems is a classical problem which, in general, does not have an easy solution, for more details see the books of Ye Yanqian [14] *and Zhang Zhifen* [15].

Now following the paper [25] *we apply a change of variables to the known quadratic systems having an algebraic limit cycle, which preserves the degree of the system, but increases the degree of the algebraic curve. For this purpose we use the birational transformation*

$$(x,\, y) \to (\frac{x}{y^2},\ \frac{1}{y}), \tag{2.48}$$

after an appropriate translation. In fact, this transformation is an involution.

If the system is of the form

$$\begin{cases} x' &=& \alpha x + \beta y + 2e\,x^2 + bxy + cy^2, \\ y' &=& \gamma x + \delta y + e\,xy + f\,y^2, \end{cases} \tag{2.49}$$

then it is easy to see that we can apply the transformation above and still remain in the class of quadratic systems.

As a simple example, we show that the example of Yablonskii with an algebraic limit cycle of degree 4 *can be obtained from the well-known example of an algebraic limit cycle of degree* 2 *due to Qin Yuan-Xun* [18].

Proposition *:*

The system of Yablonskii (54) *with its irreducible invariant algebraic curve* (35) *can be transformed into the system*

$$\begin{cases} x' &=& -3\,(a+b)cx + 4abc\,x^2 - 4y + (a+b)xy, \\ y' &=& \left(a(b+4bc^2) - (\frac{3a^2+3b^2}{2})x + 2(a+b)cy + ab(a+b)x^2 + 4abcxy + 2(a+b)y^2\right), \end{cases} \tag{2.50}$$

with the invariant algebraic curve

$$-\frac{(a-b)^2}{4ab} + ab\left(x - \frac{a+b}{2ab}\right)^2 + (y+c)^2 = 0. \tag{2.51}$$

by the transformation $(x,y) \to \left(\frac{1}{x}, \frac{y}{x^2}.\right)$

The algebraic curves (55) *and* (62) *both give limit cycles when* $abc \neq 0, a \neq b, ab > 0,$ *and* $4c^2(a-b)^2 + (3a-b)(a-3b) < 0$ *, and the transformation maps the one limit cycle onto the other.*

We now apply the birational transformation to system (58) *having the irreducible invariant algebraic curve* (59) *which defines an algebraic limit cycle of degree* 4 *for* $0 < a < \frac{1}{4}$, *and we get a quadratic system having an algebraic limit cycle of degree* 5.

Theorem :

System

$$\begin{cases} x' &= 28x - \frac{12}{\alpha+4}y^2 - 2(\alpha^2-16)(12+\alpha)x^2 + 6(3\alpha-4)xy, \\ y' &= (32-2\alpha^2)x + 8y - (\alpha+12)(\alpha^2-16)xy + (10\alpha-24)y^2, \end{cases} \tag{2.52}$$

has an irreducible algebraic invariant curve of degree 5 *given by*

$$\begin{aligned} x^2 + (16-\alpha^2)\,x^3 + (\alpha-2)x^2y + \tfrac{1}{(4+\alpha)^2}y^4 + \tfrac{6}{(4+\alpha)^2}y^5 - \tfrac{2}{4+\alpha}xy^2 \\ + \tfrac{(\alpha-4)(12+\alpha)}{4}x^2y^2 + \tfrac{(8-\alpha)}{(4+\alpha)}xy^3 = 0. \end{aligned} \tag{2.53}$$

For $\alpha \in \frac{3\sqrt{7}}{2}, 4$ *the curve* (64) *contains an algebraic limit cycle of degree* 5.

Proof .

Let $a = 16 - \alpha^2$. *When we make the change of coordinates*

$$(x, y) = \left(\frac{u}{v^2} - \frac{1}{\alpha+4}, \frac{1}{v} + \frac{\alpha-2}{2}\right) \tag{2.54}$$

multiply by v, *and replace* (u, v) *again with* (x, y), *system* (8) *becomes* (13). *The curve* (14) *is obtained from* (9) *by means of the same change of coordinates and multiplication by* v^6. *The irreducibility of* (14) *follows from the irreducibility of* (9). *Since the curve* (9) *contains an algebraic limit cycle for* $a \in (0, \frac{1}{4})$ *one may easily check, that the above oval does not intersect the singular line of the transformation* (15), *so the theorem follows.*

In a similar way we have the following result.

Theorem :

System

$$\begin{cases} x' &= 28(\beta-30)\beta x + y + 168\beta^2x^2 + 3xy, \\ y' &= 16\beta(\beta-30)(14(\beta-30)\beta x + 5y + 84\beta^2x^2) + 24(17\beta-6)\beta xy + 6y^2, \end{cases} \tag{2.55}$$

has an irreducible algebraic invariant curve of degree 6 *given by*

$$\begin{aligned} &-7y^3 + 3(\beta-30)^2\beta y^2 + 18(\beta-30)(-2+\beta)\beta xy^2 + 27(\beta-2)^2\beta x^2y^2 \\ &24(\beta-30)^30\beta^2xy + 144(\beta-30)(\beta-2)^2\beta^2x^3y + 48(\beta-30)^4\beta^3x^2 \\ &+576(\beta-30)^2(-2+\beta)^2\beta^3x^4 - 432(\beta-2)^2\beta^2(3+2\beta)x^4y \\ &-3456(\beta-30)(-2+\beta)^2\beta^3(3+2\beta)x^5 \\ &+3456(\beta-2)^2\beta^3(12+\beta)(3+2\beta)x^6 \\ &+24(\beta-30)^2\beta^2(9\beta-4)x^2y + 64(\beta-30)^3\beta^3(9\beta-4)x^3 = 0 \end{aligned} \tag{2.56}$$

For $\beta \in (\frac{3}{2}, 2)$ *the curve* [7] *contains an algebraic limit cycle of degree* 6

Proof .

Let $a = \frac{(4-\beta^2)}{7}$. *When we make the change of coordinates*

$$(x, y) = \left(\frac{v + 4u\beta(-30 + 3u(-2 + \beta) + \beta)}{12u^2\beta(\beta^2 - 4)}, \frac{30 - \beta - u(8 + 3\beta)}{14u}\right), \tag{2.57}$$

multiply by $\frac{-21\beta u}{2}$, *and replace* (u, v) *again with* (x, y), *system* (8) *becomes* (16). *The curve* (17) *is obtained from* (19) *by means of the same change of coordinates and multiplication by* $2016\beta^2(\beta^2-4)^2u^6$.

The irreducibility of (17) *is now obvious. Since the curve* (9) *contains an algebraic limit cycle for* $a \in (0, \frac{1}{4})$, *the theorem follows in a way similar to the last part of Theorem* 3.4.

After these results some natural questions are:

OPEN QUESTION.

1/ Does there exist a chain of rational transformations like the ones above which give examples of quadratic polynomial systems with algebraic limits cycles of arbitrary degree?

2/ What is the maximum degree of all algebraic limit cycles for quadratic systems ?

Chapter 3

Integrability and a Limit Cycle Solver for a Generalization of Polynomial Liénard Differential Systems

Abstract *In this paper, we study the periodic orbits of the second-order differential equation. We find a way of study the integrability liénard systems in the plane. In this article such problem is formulated in the more general framework of Poincarée Bendixson structures, which include Hamiltonian systems as a particular case. We deal with the analyticity of the second integral of any (possibly degenerate) center of an analytic planar differential system. A concrete example as application is given.*

Keywords: *Integrating factors, Limit cycles, Ordinary differential equations.*

3.1 Introduction

Hereby, we have presented a systematic algorithm for building and integrating factors of the form $(x, y), \mu(x, \dot{y})$ *or* $\mu(y, \dot{y})$ *concerning the second-order ODEs. The algorithm can determine the existence and explicit form of the integrating factors themselves without solving any differential equation, except for a linear ODE in one sub-case of the* $\mu(x, y)$ *problem. Examples of ODEs which dont have point symmetries are shown to be solved algorithmically*

Besides all that, it is has been shown a comparison between this implementation and other computer algebra ODE solvers in tackling nonlinear examples from Kamke's book. The study of nonlinear systems is one of an academic and industrial points of view and which affects on many fields, such as hydrodynamics, aeronautics, civil engineering, transport, acoustics music, nuclear engineering and others.

A limit cycle is an isolated periodic solution that occurs in planar differential systems commonly by modeling both the technological and natural sciences. Most of the early of history in the theory of limit cycles in the plane was stimulated by practical problems.
For example, the differential equation derived by Rayleigh in 1877 [15]and related to the oscillation of a violin string, is given by

$$\ddot{x} + \epsilon\left(\frac{1}{3}(\dot{x})^2 - 1\right)\dot{x} + x = 0.$$

So we get:

where $\ddot{x} = \frac{d^2x}{dt^2}$ *and* $\dot{x} = \frac{dx}{dt}$.

Let $\dot{x} = y$, *then this differential equation can be written as a system of first-order autonomous differential equations in the plane*

$$\dot{x} = y, \qquad \dot{y} = -x - \epsilon \left(\frac{y^2}{3} - 1 \right) y. \tag{3.1}$$

A phase portrait is shown in Figure 3.1.

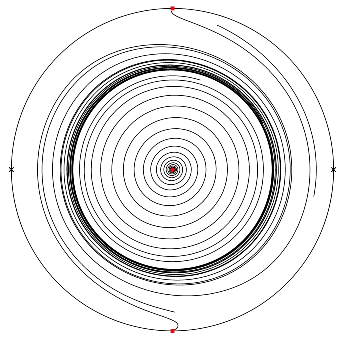

Figure 3.1: Periodic behavior in the Rayleigh system 3.1 when $\epsilon = 0.1$.

Following the invention of the triode vacuum tube which was able to produce stable self-excited oscillations of constant amplitude, Van Der Pol [6] obtained the following differential equation to describe this phenomenon:

$$\ddot{x} + \epsilon\,(x^2 - 1)\dot{x} + x = 0.$$

Which can be written as a planar system of the form:

$$\dot{x} = y, \quad \dot{y} = -x - \epsilon(x^2 - 1)y. \tag{3.2}$$

A phase portrait is shown in Figure 3.2.

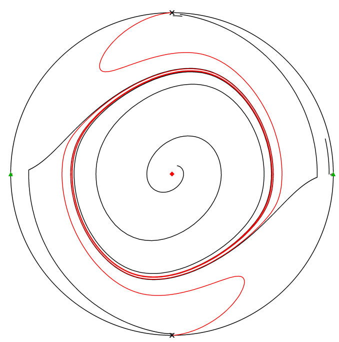

Figure 3.2: Periodic behavior for system when $\epsilon = 0.5$

The basic model of a cell membrane is what a resistor and capacitor is in parallel. The equations used to model the membrane are a variation of the Van Der Pol equation. The famous Fitzhugh-Nagumo *oscillator [7, 13, 20] used to model the action potential of a neuron is a two-variable simplification of the Hodgkin-Huxley equations [5]. The Fitzhugh-Nagumo model creates quite accurate action potentials and models of the qualitative behavior of the neurons. The differential equations are given by:*

$$\dot{u} = -u(u-\theta)\,(u-1) - v + w, \quad \dot{v} = \epsilon\,(u - \gamma v),$$

where u is a voltage, v is the recovery of voltage, θ is a threshold, γ is a shunting variable, and w is a constant voltage. For certain parameter values, the solution demonstrates a slow collection and fast release of voltage; this kind of behavior has been labeled integrating and fire. Note that, for biological systems, neurons cannot collect voltage immediately after firing and they need to be at rest. Oscillatory behavior for the Fitzhugh-Nagumo system is shown in Figure 3.3. Mathematica command lines for producing Figure 3. When $w = w(t)$ is a periodic external input, the system becomes non autonomous and can display chaotic behavior [8].

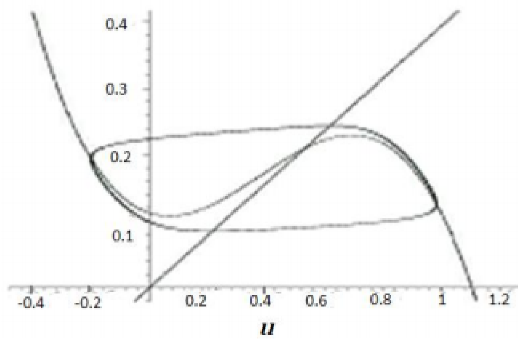

Figure 3.3: A limit cycle for Fitzhugh-Nagumo oscillator. In this case, $\gamma = 2.54, \theta = 0.14, w = 0.112$, and $\epsilon = 0.01$. The dashed curves are the isoclines, where the trajectories cross horizontally and vertically.

Perhaps the most famous class of differential equations that generalize 3.2 *are those of first investigated by Liénard in 1928 [4],*

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

or in the phase plane

$$\dot{x} = y, \qquad \dot{y} = -g(x) - f(x)\,y. \tag{3.3}$$

This system can be used to model mechanical systems, where $f(x)$ is known as the damping term and $g(x)$ is called the restoring force or stiffness.

Equation 3.3 *is also used to model resistor-inductor-capacitor circuits (see Chapter 1 Dynamical Systems with Applications using Mathematica @) with nonlinear circuit elements.*

Possible physical interpretations for the limit cycle behavior of certain dynamical systems are listed as follows:

- *For predator-prey and epidemic models, the populations oscillate in phase with one another and the systems are robust (see examples in Chapter 3, and Exercise 8 in Chapter 7)(Dynamical Systems with Applications using Mathematica @).*

- *Periodic behavior is present in integrating and fire neurons (see Figure 3).*
- *For mechanical systems, examples include the motion of simple nonlinear pendula, wing rock oscillations in aircraft flight dynamics [10], and surge oscillations in axial flow compressors [2].*
- *For periodic chemical reactions, examples include both of Landolt Clock reaction and Belousov-Zhabotinski reaction.*
- *For electrical or electronic circuits, it is possible to construct simple electronic oscillators (Chua's circuit, for example) using a nonlinear circuit element; a limit cycle can be observed if the circuit is connected to an oscilloscope.*

In 2000, Johan studied this oscillator by studying its fixed points. He used the Poincaré-Bendixson theorem to prove the existence of the limit cycle of the oscillator autonomous of Rayleigh then it studied the stability of this limit cycle by using the technic of Poincaré-Bendixson.

Our work will look for cycle the systems by another variant of the theorem of Poincaré-Bendixson, perturbing plane Hamiltonian system. After that, the study will focus on the stability of the found limit cycle by using general method which is called multiple scales that which treats the systems all conditions according to the literature. Then, we will prove that the systems show a bifurcation of PAH. Finally, our study will simulate the equation of this oscillator to verify our results.

Limit cycles are common solutions for all types of dynamical systems. Sometimes it becomes necessary to prove the existence and uniqueness of a limit cycle, as the following.

3.2 Main result

As a main result, we shall prove the following theorem.

Theorem *A technic for solving standard form:*

$$\ddot{y} + 2\dot{p}(x)\dot{y} + [\ddot{p}(x) + \dot{p}^2(x)]y = q(x).$$

p is function in the class 2, $q(x)r(x)$ is integrating.

compute $r(x) = e^{p(x)}$.

The solution is given by:

$$\begin{aligned} y &= \frac{1}{r(x)}\left[\int_0^x \int_0^t q(s)\, r(s)\, ds\, dt \ + k_1 x + k_2 \right] \\ &= e^{-p(x)}\left[\int_0^x \int_0^t q(s)\, e^{p(s)}\, ds\, dt \ + k_1 x + k_2 \right], \quad /k_1, k_2 \in \mathbb{R}. \end{aligned} \tag{3.4}$$

Where the variables k_1, k_2 according to the initial conditions.

Proof *: We star by:*

$$\ddot{y} + 2\dot{p}(x)\dot{y} + [\ddot{p}(x) + \dot{p}^2(x)]y = q(x).$$

Multiplicate whith $r(x)$:

$$r(x)\ddot{y} + 2\dot{p}(x)r(x)\dot{y} + [\ddot{p}(x) + \dot{p}^2(x)]r(x)y = q(x)r(x).$$

We get

$$\Big[r(x)\ddot{y} + \dot{p}(x)r(x)\dot{y}\Big] + \Big[\dot{p}(x)r(x)\dot{y} + [\ddot{p}(x) + \dot{p}^2(x)]r(x)y\Big] = q(x)r(x).$$

Which implies

$$\dot{r}(x) = \dot{p}(x)\ r(x),$$

we obtain:

$$\Big[r(x)\ddot{y} + \dot{r}(x)\dot{y}\Big] + \Big[\dot{r}(x)\dot{y} + [\ddot{p}(x)\, r(x) + \ \dot{p}^2(x)\, r(x)]y\Big] = q(x)r(x).$$

$$\Big[r(x)\ddot{y} + \dot{r}(x)\dot{y}\Big] + \Big[\dot{r}(x)\dot{y} + [\ddot{p}(x)\, r(x) + \ \dot{p}(x)\, \dot{r}(x)]y\Big] = q(x)r(x).$$

It follows that

$$\Big[r(x)\ddot{y} + \dot{r}(x)\dot{y}\Big] + \Big[\dot{r}(x)\dot{y} + [\dot{p}(x)r(x)\dot{]}y\Big] = q(x)r(x).$$

$$\Big[r(x)\ddot{y} + \dot{r}(x)\dot{y}\Big] + \Big[\dot{r}(x)\dot{y} + [r(\dot{x})\dot{]}y\Big] = q(x)r(x),$$

by simplification, we get

$$\Big[r(x)\dot{y}\dot{\Big]} + \Big[\dot{r}(x)\dot{y} + [r(\ddot{x})]y\Big] = q(x)r(x).$$

$$\Big[r(x)\dot{y}\dot{\Big]} + \Big[\dot{r}(x)y\dot{\Big]} = q(x)r(x).$$

$$\Big[r(x)\dot{y} + \dot{r}(x)y\dot{\Big]} = q(x)r(x).$$

$$\Big[(r(x)\, y\dot{)}\dot{\Big]} = q(x)r(x),$$

which implies

$$\Big[r(x)\, y\ddot{\Big]} = q(x)r(x). \tag{3.5}$$

The equivalent standard form has the equation 3.5.

To solve the standard form simply solve equation 3.5 .

$$\Big[r(x)\, y\dot{\Big]} = \int_0^x q(t)r(t)\, dt + k_1, \qquad k_1 \in \mathbb{R}$$

$$\Big[r(x)\, y\Big] = \int_0^x \int_0^t q(s)r(s)\, ds\ dt\ + k_1\, x + k_2, \qquad k_1, k_2 \in \mathbb{R}.$$

Solution

$$y = \frac{1}{r(x)} \int_0^x \int_0^t q(s)r(s)\,ds\,dt + k_1 x + k_2, \qquad k_1, k_2 \in \mathbb{R}.$$

Theorem *: Apply to standard equation:*

$$\ddot{x} + \mu\,(x^{2n} - a)\dot{x} + x = 0.$$

We have:

$$f(x) = \mu\,(x^{2n} - a), \quad g(x) = x\,, n \in N^* \quad , a > 0.$$

Hence:

$$\begin{aligned} F(x) = \int_0^x f(u)\,du &= \mu(\tfrac{1}{2n+1}x^{2n+1} - ax) \\ &= \mu\tfrac{1}{2n+1}x \quad (x^{2n} - a\,(2n+1)), \end{aligned}$$

and $R = \sqrt{a\,(2n+1)}$.

So there exists R unique stable limit cycle.

Proof *Step 1. p(1) is true ? For Van der Pol equation* $n = 1$

$$\ddot{x} + \mu\,(x^2 - 1)\dot{x} + x = 0.$$

We have:

$$f(x) = \mu\,(x^2 - a), \quad g(x) = x\,, n \in N \quad , a > 0.$$

Hence:

$$\begin{aligned} F(x) = \int_0^x f(u)\,du &= \mu(\tfrac{1}{3}x^3 - x) \\ &= \mu\tfrac{1}{3}x \quad (x^2 - 3), \end{aligned}$$

and $R = \sqrt{3}$. *So there exists R unique stable limit cycle.*

$P(1)$ *is true.*

Step 2. $p(n) \to p(n+1)$ *for all* $n \geq 1$*?*

We have given an any integer $n \geq 1$*, It is assumed that for this integer n the properties P(n) are true (This is the hypothesis of recurrence) and we will prove that for P(n+1) ?*

$$f(x) = \mu\,(x^{2(n+1)} - a), \quad g(x) = x\,, n \in N^* \quad , a > 0.$$

Hence,

$$\begin{aligned} F(x) = \int_0^x f(t)\,dt &= \mu(\tfrac{1}{2n+3}x^{2n+3} - ax) \\ &= \mu\tfrac{1}{2n+3}x \quad (x^{2n+2} - a\,(2n+3)), \end{aligned}$$

We obtain: $R' = \sqrt{a\,(2n+3)} = \sqrt{a\,(2n'+1)}$ $\quad n' = n+1$ *is the hypothesis .*
P(n+1) is true.

We obtain by reccurence R unique stable limit cycle.

3.3 Example

3.3.1 The Fixed limit cycles points and existence of the nonlinear system

Study of fixed points

To make out a general study of this system, let us put equation 3.6

$$\ddot{y} + 2\dot{p}(x)\dot{y} + [\ddot{p}(x) + \dot{p}^2(x)]y = q(x). \tag{3.6}$$

into the form of system of first-order differential equations. Indeed, we put $\dot{y} = x$*.*

We have

$$\ddot{y} = \dot{x} = -2\dot{p}(x)\,x - [\ddot{p}(x) + \dot{p}^2(x)]\,y + q(x),$$

we will take

$$p(x) = (\frac{1}{2}x^2 - 5x)\,\mu, \qquad q(x) = -10x + x^2 - 4x^3 - 2.$$

We obtain:

$$\begin{cases} \dot{y} &= x \\ \dot{x} &= (-\frac{39}{4}\mu - 10)\,x - \mu y - 2\mu x^2 - \mu^2(x-5)^2 y + x^2 - 4x^3 - 2 \end{cases}$$

When we look for the fixed points of this dynamic system, we will find the origin $X = 0$ *and* $Y = 0$ *that is the only fixed point of this dynamic system.*

As we said in previous lines, we are looking for fixed point stability by studying the nature of the eigenvalues the Jacobian matrix of this dynamic system.

This Jacobian matrix is defined at the origin by:

$$\mathcal{M} = \begin{pmatrix} 0 & 1 \\ (-\frac{39}{4}\mu - 10) & -\mu \end{pmatrix}.$$

The eigenvalues λ *of the Jacobian are giving by*

$$\lambda^2 + \mu\lambda + \frac{39}{4}\mu + 10 = 0.$$

Discussion

– If $\mu \in]-\infty, -1[\cup]40, +\infty[$, $\delta > 0$ *and the Jacobian admits two eigenvalues* $\lambda_1 = \frac{-\mu-\sqrt{\mu^2-39\mu+40}}{2}$ *and* $\lambda_2 = \frac{-\mu+\sqrt{\mu^2-39\mu+40}}{2}$

- *If* $\mu > 40$, *then* λ_1 *and* λ_2 *are negative. So origin is an attractive node and is stable.*
- *If* $\mu < -1$, *then* $\lambda_1 < 0$ *and* $\lambda_2 > 0$. *So origin is an saddle point.*

– If $\mu = -1$ *or* $\mu = 40, \delta = 0$ *and therefore* $\lambda_1 = \lambda_2 = -\frac{\mu}{2}$. *It comes:*

- λ_1 *and* λ_2 *for negative* $\mu = 40$ *and therefore the origin is an attractive node.*
- λ_1 *and* λ_2 *for positive* $\mu = -1$ *and therefore the origin is an repulsive node.*

– Let us now consider the case $\mu \in]-1, 40[\rightarrow \delta < 0$ *and* $\lambda_\pm = \frac{-\mu\pm i\sqrt{\mu^2-39\mu+40}}{2}$ *The eigenvalues are complex so the origin, here, is a focus or a center.*

- *In addition,for* $-1 < \mu < 0$, $Re(\lambda_+) = Re(\lambda_-) > 0$ *in this case, the origin is an repulsive focus.*
- *For* $0 < \mu < 40$, $Re(\lambda_+) = Re(\lambda_-) < 0$ *in this case, the origin is an attractive focus.*

– Finally, if $\mu = 0$, $\lambda_\pm = \pm\sqrt{10}\, i$, *then the origin is a center.*

Existence of Periodic Orbits

In this part, we will apply the Poincaré-Bendixson theorem in the case of autonomous perturbations of a plane Hamiltonian system (J. BRIC- MONT [2]).

Equation **??** *of the nonlinear system is a perturbed plane system.*

The system

$$\begin{cases} \dot{y} &= x \\ \dot{x} &= (-\frac{39}{4}\mu - 10)\, x - \mu y - 2\mu x^2 - \mu^2(x-5)^2 y + x^2 - 4x^3 - 2 \end{cases} \tag{3.7}$$

The unperturbed system (harmonic oscillator) defined by $\mu = 0$ *is Hamiltonian whose Hamiltonian is*

$H = \frac{1}{2}x^2 - (-10x + x^2 - 4x^3 - 2)y$. *Except for equilibrium* $(0,0)$, *the solutions of this Hamiltonian system are minimal period* $T_r = 2\pi$ *all periodic and is given by*

$$\gamma_r : \vec{X}(t,r) = (r\cos t, -r\sin t)^T, r > 0 \tag{3.8}$$

We calculate Melnikov integral along γ_r

$$M(r) = \int_0^{T_r} f(\vec{X}(t,r)) \wedge g(\vec{X}(t,r)) \quad dt \tag{3.9}$$

with $f(\vec{X}(t,r)) = (x, -y)^T$ *and* $g(\vec{X}(t,r)) = (0, -4x^3 - \; y\, x^2 + (\frac{-79}{4} + 10y)\, x - 2)^T$ *where* f *and* g *are of class* C^∞.

$M(r) = \int_0^{2\pi} (4r^4 \sin^4 t \; + \; r^4 \cos t \; \sin^3 t - (\frac{79}{4} - 10r \; \cos t) \, r^2 \sin^2 t + 2r \sin t) \; dt.$

We find that:

$$M(r) = \frac{1}{4} \pi r^2 (16 \, r^2 - 79) \tag{3.10}$$

Note that: $\sqrt{\frac{79}{16}}$ *is the only value r reviser* $M(\sqrt{\frac{79}{16}}) = 0$ *and* $\grave{M}(\sqrt{\frac{79}{16}}) = -2\pi \sqrt{\frac{79}{16}} \neq 0$. *It, therefore, follows that all the hypothesies theorem of Poincaré Bendixson in the case of autonomous perturbations of a Hamiltonian system are verified by the nonlinear system. We deduce that the nonlinear system possessed for μ sufficiently small, a limit cycle which tends the circle whose center is the origin and radius* $\sqrt{\frac{79}{16}}$ *when μ tends to* 0.

Study of the stability of the limit cycle

In this section, we will use the multiple scales method to study the stability of the limit cycle of the nonlinear system.Which it is not the only method. However, this suggests that the most general method of a hierarchy of time scales is able to rectify the lack of synchronization between the reference linear system and the nonlinear system considered. Thus the aim of this method is to make the approximation of the solution uniform in time, like as one corrects a lack of calendar by the introduction of the leap years following an algorithm that is more complicated considering long periods Paul Manneville .

Let us define the basic formulas of the multiple scaling's method.

$$\frac{d}{dt} = d_{t_0} + \mu \, d_{t_1} + \mu^2 \, d_{t_2} + \ldots \tag{3.11}$$

$$\frac{d^2}{dt^2} = d_{t_0{}^2} + 2\mu \, d_{t_0} d_{t_1} + \mu^2 \, (d_{t_1{}^2} + 2 d_{t_0} \, d_{t_2}) + \ldots \tag{3.12}$$

$$X = X_0(t_0, t_1, t_2, \ldots) + \mu \, X_1(t_0, t_1, t_2, \ldots) + \ldots \tag{3.13}$$

$t_0 = t, \; t_1 = \mu t, \; t_2 = \mu^2 t, \ldots and \frac{d}{d_{t_i}} = d_{t_i}$

Thus, when we replace 3.11, 3.12 *and* 3.13 *in* 3.6, *we find:*

$$\begin{aligned} &[d_{t_0{}^2} + 2\mu d_{t_0} \, d_{t_1} + \mu^2 (d_{t_1{}^2} + 2 d_{t_0} \, d_{t_2}) \ldots](x_0 + \mu x_1 + \mu^2 x_2 + \ldots) \\ &+ (x_0 + \mu x_1 + \mu^2 x_2 + \ldots) = \mu[1 - (d_{t_0 x_0})^2 - 2\mu (d_{t_0} x_0)(d_{t_0} x_1 + d_{t_1} x_0) + \\ &\ldots][d_{t_0} + \mu d_{t_1} + \mu^2 d_{t_2} + \ldots](x_0 + \mu x_1 + \mu^2 x_2 + \ldots) \end{aligned} \tag{3.14}$$

We are getting:

At order μ^0

$$d_{t_0{}^2} x_0 + x_0 = 0. \tag{3.15}$$

In order μ^1

$$d_{t_0{}^2} x_1 + x_1 = [1 - (d_{t0} \, x_0)^2] d_{t0} x_0 - 2 d_{t0} \, d_{t1} x_0. \tag{3.16}$$

The solutions of equation 3.15

$$x_0 = A_0(t_1, t_2, \ldots) \; \cos(t_0 + \phi_0(t_1, t_2, \ldots)). \tag{3.17}$$

Using 3.16 , 3.17 *becomes*

$$\begin{aligned} d_{t_0{}^2}x_1 + x_1 = \quad & [-A_0 + \tfrac{79}{16}{A_0}^3 + 2(d_{t1}A_0)]\ \sin(t_0+\phi_0)+ \\ & 2A_0(d_{t1}\phi_0)\cos(t_0+\phi_0) - \tfrac{1}{4}{A_0}^3\sin 3(t_0+\phi_0) \end{aligned} \tag{3.18}$$

Let us cancel the resonant terms, so we have:

$$-A_0 + \frac{79}{16}{A_0}^3 + 2d_{t1}A_0 = 0 \tag{3.19}$$

$$2A_0 d_{t1}\ \phi_0 = 0 \tag{3.20}$$

The integration of equation 3.19 *gives*

$$\phi_0 = \phi(t_2, t_3, \ldots). \tag{3.21}$$

When we move t has the aid of $t_1 = \mu t$, *equation* 3.20 *becomes:*

$$\frac{dA_0}{dt} = \frac{1}{2}\mu A_0(1 - \frac{79}{16}{A_0}^2). \tag{3.22}$$

As a result, there is no correction in the phase and the phase is a function. We study the stability of the limit cycle using equation 3.22*. Indeed, the amplitude of the limit cycle is*

$$A_{0s} = \sqrt{\frac{79}{16}}. \tag{3.23}$$

The stability of this cycle limit is related to the sign of $\frac{dA_0}{dt}$.

Thus, for $(1 - \frac{79}{16}{A_0}^2) < 0$ *that is to say* ${A_0}^2 > A_{0s}^2$, $\frac{dA_0}{dt} < 0$*. Therefore, any orbit of amplitude which is greater than when the limit cycle always falls towards the latter.*

For $(1 - \frac{79}{16}{A_0}^2) > 0$ *that is to say* ${A_0}^2 < A_{0s}^2$, $\frac{dA_0}{dt} > 0$ *so any orbit characterized by an lower than the stable limit cycle that will die on the latter.*

Therefore, the limit cycle centered on the origin of amplitude $A_{0_s} = \sqrt{\frac{79}{16}}$ *is stable . It should be noted that this result is the same regardless of the method perturb used.*

3.3.2 Search for the bifurcation of Poincaré-Andronov- Hopf forthe nonlinear system

The Poincaré-Andronov-Hopf (PAH) called the bifurcation of Hopf (because it is the simple form of the Hopf bifurcation) that is the appearance or the disappearance of a periodic orbit by a local change of stability property of a fixed point of a dynamic system. This local birth or death of a periodic self-excited oscillation of an equilibrium point occurs when one control parameter passes through a critical value. In general, in an differential equation , a Hopf bifurcation occurs when a pair of conjugate eigenvalues of the linearized system at a fixed point becomes purely imaginary. This implies that a Hopf bifurcation cannot occur in systems of dimensions more than one (in dimension two, it is a bifurcation of PAH). The search for the bifurcation of Hopf in general and PAH in particular is not equivalent to the search for stable limit cycles.

Firstly, a few bifurcations of Hopf (for example, subcritical) do not imply the existence of stables.

Secondly, there may be limit cycles unrelated to bifurcations of Hopf. In this section we will apply the Poincaré-Andronov-Hopf theorem (PAH) page 286-412 [33]. For this section, we will seek to check if the nonlinear system to fulfill the assumptions of this theorem. Consider the following equation of the nonlinear system:

$\ddot{y} + \mu[1 + (x^2 - 10x + 25)\mu]y = 2(5 - x)\mu\, y - 4x^3 + x^2 + 5x - 2$ *or* μ *is a control parameter. We have seen above that this equation is written by decomposition as the system* 3.6. *Thus, this system is in the form*

$$\begin{cases} \dot{y} &= f(\mu, x, y) \\ \dot{x} &= g(\mu, x, y) \end{cases} \tag{3.24}$$

with $f(\mu, X, Y) = x$ *and* $g(\mu, X, Y) = (-\frac{39}{4}\mu - 10)\ x - \mu y - 2\mu x^2 - \mu^2(x-5)^2 y + x^2 - 4x^3 - 2.$

1. *The functions f and g are polynomials, so they are differentiable from class* C^∞.

2. *For* $X = 0$ *and* $Y = 0, f(\mu, 0, 0) = 0$ *and* $g(\mu, 0, 0) = 0$.

3. *Based on the study of the fixed points of the nonlinear system made in subsection 2.2, we have for* $-1 < \mu < 40$ *two complex conjugate roots of the Jacobian dynamic system defined at the origin which is the only fixed point of this system.*

   These eigenvalues are:

   $\lambda_\pm = \frac{-\mu \pm i\sqrt{\mu^2 - 39\mu - 40}}{2}$ *so* $\lambda_\pm = \alpha(\mu) \pm i\beta(\mu)$ *with*

   $\alpha(\mu) = \frac{-\mu}{2}$ *and* $\beta(\mu) = \frac{\sqrt{\mu^2 - 39\mu - 40}}{2}$.

   Thus, we have:

   $\mu < 0 \Longrightarrow \alpha(\mu) > 0, \mu > 0 \Longrightarrow \alpha(\mu) < 0, \grave{\alpha}(0) = \frac{d\alpha}{d\mu} = \frac{-1}{2} \neq 0$ *and* $\beta(0) = \sqrt{10} \neq 0$

4. *Still in subsection 2.2, we have shown that the origin is a center and according to the study of stability, origin is an repulsive focus.*

   These four points show that the four hypotheses of the PAH theorem are satisfied by the equation of the nonlinear system.

   Under the PAH theorem, the nonlinear system is such that:

   -For $\mu < 0$, *the origin is a stable stationary state.*

   -For $\mu > 0$, *the origin is an unstable stationary state surrounded by a limit cycle, the diameter of which is of the order of* $\sqrt{\mu}$. *It follows that* $\mu = 0$ *is a bifurcation value of the nonlinear system. In other words, when* μ *passes through 0, the origin loses its stability.*

   This loss of stability is accompanied by the birth of a stable limit cycle whose radius grows as $\sqrt{\mu}$ *since* $\mu > 0$, *the trajectories move away from the focus at a proportional distance to* $\sqrt{\mu}$ *and wind around this stable limit cycle. In this case the bifurcation is supercritical and therefore there is a gentle excitation of self-oscillations. This result is justified by the commentary by V. Arnold [38] on the PAH theorem in his book [31]. To confirm our results, we simulated the equation of this system in using P4.*

We notice that the numerical simulations analytical predictions. We also note the presence of a limit cycle stable and we thus have a bifurcation of Poincaré-Andronov-Hopf supercritic for $\mu = 0$. *The presence of the bifurcation of PAH shows that a stable equilibrium of the nonlinear system will destabilize in the space of the control parameters by following the diagram of this bifurcation by passing an eigenvalue through the axis of pure imaginary without the system presenting a chaotic or catastrophic activity.*

3.3.3 The integral curves for the system

If $\mu \to 0$ $\lambda_\pm = \sqrt{10}i$, *the origin is a center(Linear) and limit cycles (Nonlinear).*

Where $\mu \to 0$ *is the integrating factor of system* 3.7 *corresponding to the first integral H.*

Moreover, the function 3.9 *is a displacement function and is the corresponding first order Melnikov function of the system* 3.7.

$$\begin{aligned} y &= \tfrac{1}{r(x)}\Big[\int_0^x \int_0^t q(s)\, r(s)\, ds\, dt \ + k_1 x + k_2\Big] \\ &= e^{-p(x)}\Big[\int_0^x \int_0^t q(s)\, e^{p(s)}\, ds\, dt \ + k_1 x + k_2\Big], \quad /k_1, k_2 \in \mathbb{R}. \end{aligned} \tag{3.25}$$

Where the variables k_1, k_2 *according to the initial conditions.*

The origin $X = 0$ *and* $Y = 0$ *is the only fixed point of this dynamic system then* $k_2 = 0, k_1 \in \mathbf{R}$.

We compute this integral using maple and obtain

$$f_1(x) = \frac{-1}{5}((x^5) - \frac{5}{12}x^4 + \frac{25}{3}x^3 + 5x^2 - 5x) \quad k_1 = 1.$$

You can think of the solution in the preceding example as coordinates of a point $(x(t), y(t))$ *in two-dimensional space* $\mathbb{R}$. *As the independent variable t changes, the points trace out a curve in the* $x - y$ *plane called a trajectory(wxMaxima) . The positive direction of the curve is the direction, it takes as t increases. Figure 4*

A solution $y(x)$ *satisfying the boundary conditions* $y(0) = 0$ *and* $x(0) = 0$.

Note

Phase portrait is the set formed by all the integral curves.

Conclusion

The study of the nonlinear system has allowed us to show that it possesses a cycle limit. To study the stability of this limit cycle, we used the method of averaging and the multiple scales method. It appears from the application of these two methods that the amplitude limit cycle $\sqrt{\frac{79}{16}}$ *of nonlinear system is stable.*

We have also proved the existence of a bifurcation of Poincaré-Andronov- Hopf for this oscillator and determined analytically and verified by numerical simulation. The value of the control parameter μ

is obtained for which this bifurcation. That, it results the nonlinear system possesses properties which satisfy the hypotheses of this bifurcation.

Thus, the presence of this supercritical bifurcation of PAH justifies that nonlinear system will not be able to present any values of the control parameter a chaotic or catastrophic aspect.

Chapter 4

Dynamics of the polynomial differential systems

Abstract *:*

In this paper, we study the bifurcation of limit cycles for the following Lienard systems

$$\dot{x} = y, \qquad \dot{y} = -f_m(x)y - g_n(x),$$

where, $f_m(x)$ and $g_n(x)$ respectively are polynomials of degree m and n, $g_n(0) = 0$.

We prove that, if $m = 5$ and $g_n(x) = x$, then there always exist Lienard systems of the above form such that they have a limit cycle.

Keywords: *Limit cycles , The bifurcation set.*

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4.1 Introduction

In general, several free parameters. By using a method introduced in a previous paper, we obtain a sequence of algebraic approximations to the bifurcation sets, in the parameter space.

Each algebraic approximation represents an exact lower bound to the bifurcation set.

The method is perturbative. It is not necessary to have a small or a large parameter in order to obtain these results.

We consider the following problem

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= \epsilon(1 - x^2)y - x. \end{cases}$$

(See [1]).

Liénard system

In 1926, German Van Der Pol proposed the differential equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0 \quad \epsilon > 0 \tag{4.1}$$

4.2 Main result

Does the Lienard's system with $m > 5$ and $m + 1 < n < 2m$ can have an algebric limit cycle ?

In this part, by developing the main ideas we prove the next results which gives a positive solution to the problem opened above.

The first case

We choose $m = 5$ and $n = 1$ then $f_m(x) = a_0x^5 + a_1$, $g_n(x) = x$.

The system becomes :

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -(a_0x^5 + a_1)y - x. \end{cases}$$

First approximation the general solution is

$$x = C\,cos(t - \alpha).$$

(See [2])

Where C and α are arbitrary, we study how the presence of the term with ϵ affects the solution

$$x = Ccost.$$

Being given what should be a first reasonable approximation, we try

$$x = Acos(\omega t).$$

Where ω is a constant which is close to 1 (See [2]) .

$$\begin{cases} \dot{x} &= -A\omega\sin(\omega t) \\ \ddot{x} &= -A\omega^2\cos(\omega t). \end{cases}$$

by replacing this in the coming differential equation

$$\ddot{x} + (a_0x^5 + a_1)\dot{x} + x = 0, \tag{4.2}$$

We find that $-A\omega^2\cos(\omega t) - (a_0A^5\cos(\omega\,t)^5 + a_1)(A\omega\sin(\omega t)) + A\cos(\omega t) = 0$,

$$\begin{aligned} A(1-\omega^2)\cos(\omega t) &= a_0A^6\omega\cos(\omega t)^5\sin(\omega t) + a_1A\omega\sin(\omega t), \\ &= A\omega\Big(a_0A^5\cos(\omega t)^5\sin(\omega t) + a_1A\sin(\omega t)\Big). \end{aligned}$$

According to the simplification, we find the second member equal at:

$$= A\omega\Big(a_0\tfrac{A^5}{32}\sin(6\omega t) + \tfrac{3}{16}a_0A^5\sin(4\omega t) + \tfrac{15}{32}a_0A^5\sin(2\omega t) + a_1 sin(\omega t).\Big)$$

This equation can be satisfactory for all t only if the coefficients of the different terms sinusoidal disappear .

The term $cos(\omega t)$ *disappear if* $\omega = 1$ *and the coefficient of* $\sin(\omega t)$ *is zero if we take* $a_1 = 0$.
Thus we choose:

$$\omega = 1, \quad a_1 = 0.$$

We see that the choice of A is arbitrary that signify the system doesn't admit an isolated closed curve (limited cycle) (See [9]). So, we search for another more efficient method :

Passage in polar coordinates *We take :*

$$\begin{cases} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{cases}$$

According to the study,was:

$$\begin{cases} \dot{r} &= (x\dot{x} + y\dot{y})/r. \\ \dot{\theta} &= (x\dot{y} - y\dot{x})/r^2. \end{cases}$$

(See [11,13]) We have:

$$\begin{aligned} x\dot{x} + y\dot{y} &= xy - (a_0x^5 + a_1)y^2 - xy, \\ &= -(a_0x^5 + a_1)y^2, \\ &= -r^2(a_0r^5\cos(\theta)^5 + a_1)\sin(\theta)^2. \end{aligned}$$

Then

$$\begin{aligned} \dot{r} &= -r \quad (a_0r^5\cos(\theta)^5 + a_1) \quad \sin(\theta)^2, \\ &= -r \quad (a_0r^5\cos(\theta)^5\sin(\theta)^2 + a_1\sin(\theta)^2), \\ &= -r \quad \left(a_0r^5(\tfrac{1}{16}\cos(5\theta) + \tfrac{5}{16}\cos(3\theta) + \tfrac{5}{8}\cos(\theta))\sin^2(\theta) + a_1\tfrac{1-\cos(2\theta)}{2}\right), \\ \dot{r} &= \tfrac{3}{64}a_0r^6\cos(5\theta) + \tfrac{1}{64}a_0r^6\cos(7\theta) + a_1\tfrac{r}{2}\cos(2\theta) - \tfrac{5}{64}a_0r^6\cos(\theta) - a_1\tfrac{r}{2} + \tfrac{1}{64}a_0r^6\cos(3\theta). \end{aligned}$$

We have :

$$\begin{aligned} x\dot{y} - y\dot{x} &= -(a_0x^5 + a_1)\,x\,y - x^2 - y^2, \\ &= -(a_0x^5 + a_1)\,x\,y - (x^2 + y^2), \\ &= -r^2((a_0r^5\cos(\theta)^5 + a_1)\cos(\theta)\sin(\theta) + 1). \end{aligned}$$

We have:

$$\begin{aligned} \dot{\theta} &= -\Big((a_0r^5\cos(\theta)^5 + a_1)\sin(\theta)\ \cos(\theta) + 1\Big), \\ &= -\Big(a_0r^5(\tfrac{1}{16}\cos(5\theta) + \tfrac{5}{16}\cos(3\theta) + \tfrac{5}{8}\cos(\theta) + a_1)\tfrac{\sin(2\theta)}{2} + 1\Big), \\ &= a_0r^5(\tfrac{1}{32}\cos(5\theta)\sin(2\theta) - \tfrac{5}{32}\cos(3\theta)\sin(2\theta) - \tfrac{5}{16}\cos(\theta)\sin(2\theta) - a_1\tfrac{\sin(2\theta)}{2} - 1), \\ \dot{\theta} &= -\tfrac{1}{64}a_0r^5\sin(7\theta) - \tfrac{5}{64}a_0r^5\sin(5\theta) - \tfrac{5}{64}a_0r^5\sin(3\theta) + \tfrac{15}{64}a_0r^5\sin(\theta) - \tfrac{a_0r^5a_1}{2}\sin(2\theta) - 1. \end{aligned}$$

So the system becomes :

$$\begin{cases} \dot{r} &= \tfrac{3}{64}a_0r^6\cos(5\theta) + \tfrac{1}{64}a_0r^6\cos(7\theta) + a_1\tfrac{r}{2}\cos(2\theta) - \tfrac{5}{64}a_0r^6\cos(\theta) - a_1\tfrac{r}{2} + \tfrac{1}{64}a_0r^6\cos(3\theta). \\ \dot{\theta} &= -\tfrac{1}{64}a_0r^5\sin(7\theta) - \tfrac{5}{64}a_0r^5\sin(5\theta) - \tfrac{5}{64}a_0r^5\sin(3\theta) + \tfrac{15}{64}a_0r^5\sin(\theta) - \tfrac{a_0r^5a_1}{2}\sin(2\theta) - 1. \end{cases}$$

We estimate the conditions starting from the existence of the limited cycle (th Bendixon)(See [9,6,3,10]) , so that the hypothesis doesn't change $a_0, a_1 \neq 0$ *.*

Perturbation of the system *We take the equation :*

$$\ddot{x} + (a_0x^5 + a_1)\dot{x} + x = 0.$$

Development in the neighbourhood limit $(0,0)$*. (See [9])*

For the value of $\epsilon = 0$*, we find the exact solution.*

The idea is to extend the solution in power serial ϵ*:*

$$x(t) = x_0(t) + x_1(t)\epsilon + x_2(t)\epsilon^2 + 0(\epsilon^3). \tag{4.3}$$

So:

$\dot{x}(t) = \dot{x}_0(t) + \dot{x}_1(t)\epsilon + \dot{x}_2(t)\epsilon^2 + 0(\epsilon^3).$

$\ddot{x}(t) = \ddot{x}_0(t) + \ddot{x}_1(t)\epsilon + \ddot{x}_2(t)\epsilon^2 + 0(\epsilon^3).$

$$\begin{aligned}
(a_0x^5 + a_1) &= a_0(x_0(t) + x_1(t)\epsilon + x_2(t)\epsilon^2 + 0(\epsilon^3))^5 + a_1,\\
&= a_0\Big((x_0(t) + x_1(t)\epsilon)^5 + 5(x_0(t) + x_1(t)\epsilon)^4 x_2\epsilon^2 + 10(x_0(t) + x_1(t)\epsilon)^3 {x_2}^2\epsilon^4\\
&\quad +10(x_0(t) + x_1(t)\epsilon)^2 {x_2}^3\epsilon^6 + 5(x_0(t) + x_1(t)\epsilon){x_2}^4\epsilon^8 + {x_2}^5\epsilon^{10}\Big) + a_1,\\
&= a_0\Big((x_0(t) + x_1(t)\epsilon)^5 + 5(x_0(t) + x_1(t)\epsilon)^4 x_2\epsilon^2\Big) + a_1,\\
&= a_0\Big({x_0}^5(t) + 5{x_0}^4(t)x_1(t)\epsilon + 10{x_0}^3(t){x_1}^2(t)\epsilon^2 + 10{x_0}^2(t){x_1}^3(t)\epsilon^3\\
&\quad +5x_0(t){x_1}^4(t)\epsilon^4 + {x_1}^5(t)\epsilon^5 + 5{x_0}^4(t)x_2(t)\epsilon^2\Big) + a_1,\\
&= a_0\Big({x_0}^5(t) + 5{x_0}^4(t)x_1(t)\epsilon + 10{x_0}^3(t){x_1}^2(t)\epsilon^2 + 5{x_0}^4(t)x_2(t)\epsilon^2\Big) + a_1.
\end{aligned}$$

Is replaced in equation :

$\ddot{x}(t) + x(t) = (\ddot{x}_0(t) + x_0(t)) + (\ddot{x}_1(t) + x_1(t))\epsilon + (\ddot{x}_2(t) + x_2(t))\epsilon^2 + 0(\epsilon^3).$

$a_1\dot{x}(t) = a_1\dot{x}_0(t) + a_1\dot{x}_1(t)\epsilon + a_2\dot{x}_2(t)\epsilon^2 + 0(\epsilon^3).$

$$\begin{aligned}
a_0x^5(t)\dot{x}(t) &= a_0{x_0}^5(t)\dot{x}_0(t) + 5a_0{x_0}^4(t)x_1(t)\dot{x}_0(t)\epsilon + 10a_0{x_0}^3(t){x_1}^2(t)\dot{x}_0(t)\epsilon^2+\\
&\quad 5a_0{x_0}^4(t)x_2(t)\dot{x}_0(t)\epsilon^2 + a_0{x_0}^5(t)\dot{x}_1(t)\epsilon + 5a_0{x_0}^4(t)x_1(t)\dot{x}_1(t)\epsilon^2 + a_0{x_0}^5(t)\dot{x}_2(t)\epsilon^2.
\end{aligned}$$

we find :

$$\Big(x_0 + \ddot{x}_0 + a_1\dot{x}_0 + a_0{x_0}^5\dot{x}_0\Big) + \Big(x_1 + \ddot{x}_1 + a_1\dot{x}_1 + a_0{x_0}^5\dot{x}_1 + 5a_0{x_0}^4x_1\dot{x}_0\Big)\epsilon + \Big(x_2 + \ddot{x}_2 + a_1\dot{x}_2 + 10a_0{x_0}^3{x_1}^2\dot{x}_0 + 5a_0{x_0}^4x_2\dot{x}_0 + 5a_0{x_0}^4x_1\dot{x}_1 + a_0{x_0}^5\dot{x}_2\Big)\epsilon^2 + 0(\epsilon^3) = 0.$$

since we want a solution that is valid for all little values of ϵ*. we cancel each of the coefficient of*ϵ^n*.*

For $n = 0, 1, 2, \cdots$

For $n = 0, 1$ *and 2, we obtained:*

$$\begin{aligned}
x_0 + \ddot{x}_0 + a_1\dot{x}_0 + a_0{x_0}^5\dot{x}_0 &= 0 \qquad (4.4)\\
x_1 + \ddot{x}_1 + a_1\dot{x}_1 + a_0{x_0}^5\dot{x}_1 + 5a_0{x_0}^4x_1\dot{x}_0 &= 0 \qquad (4.5)\\
x_2 + \ddot{x}_2 + a_1\dot{x}_2 + 10a_0{x_0}^3{x_1}^2\dot{x}_0 + 5a_0{x_0}^4x_2\dot{x}_0 + 5a_0{x_0}^4x_1\dot{x}_1 + a_0{x_0}^5\dot{x}_2 &= 0 \qquad (4.6)
\end{aligned}$$

As the equation is autonomous we can choose the instant for corresponding $t = 0$ *to any point of the limited cycle.*

Thus, we can choose the initial condition $\dot{x}(0) = 0$ *without loss of generality from developing* (5) *we obtain the initial conditions :*

$$\dot{x}_0(0) = \dot{x}_1(0) = \dot{x}_2(0) = \cdots = 0. \tag{4.7}$$

EQS (6) *and* (9) *given*

$$x_{0h}(t) = \alpha\cos(\theta), \qquad \textit{homogeneous solution.} \tag{4.8}$$

Where α *is yet determined .*

we find the particular solution equal at:

$$x_{0p}(t) = A_0\cos(6\theta) + B_0\cos(4\theta) + R_0\cos(2\theta) + M_0\cos(\theta).$$

We need of all terms (are not periodic) A_0, B_0, R_0, M_0 *to be removed .*

by substituting (10) in (7) and by using trigonometric identities.

We obtain:

$$\ddot{x}_1 + x_1 = -(a_1 + a_0\alpha^5\cos^5(\theta))\dot{x}_1 + 5a_0\alpha^5\cos^4(\theta)\sin(\theta)x_1. \tag{4.9}$$

$$\ddot{x}_1 + (a_1 + a_0\alpha^5\cos^5(\theta))\dot{x}_1 + (1 - 5a_0\alpha^5\cos^4(\theta)\sin(\theta))x_1 = 0. \tag{4.10}$$

this is thus a differential equation of second order with variable coefficients , equation that is relatively easy to solve in the general case.

Be the equation

$$\ddot{x}_1 + (a_1 + a_0\alpha^5\cos^5(\theta))\dot{x}_1 + (1 - 5a_0\alpha^5\cos^4(\theta)\sin(\theta))x_1 = 0. \tag{4.11}$$

Supposing that the function x_1 *that satisfies the differential equation* $x_1 = \exp(k\,t)$ *where* k *can be a complex number.*

So we have:

$k^2\exp(k\,t) + (a_1 + a_0\alpha^5\cos^5(\theta))k\ \exp(k\,t) + (1 - 5a_0\alpha^5\cos^4(\theta)\sin(\theta))\exp(k\,t) = 0,$

or

$k^2 + (a_1 + a_0\alpha^5\cos^5(\theta))k\ + (1 - 5a_0\alpha^5\cos^4(\theta)\sin(\theta)) = 0,$

this last relation is thus the quadratic equation auxiliary of the differential equation (polynomial characteristic)(See [9]).

It has to solutions / roots that we will notice in the general case : k_1, k_2*.*

Immediately comes that :

$$k_{1,2} = \frac{-(a_1 + a_0\alpha^5\cos^5(\theta)) \mp \sqrt{(a_1 + a_0\alpha^5\cos^5(\theta))^2 - 4\,(1 - 5a_0\alpha^5\cos^4(\theta)\sin(\theta))}}{2}.$$

So

$$x_1 = s_1 \exp(k_1 t) + s_2 \exp(k_2 t).$$

If we take $a_0 = 0$ *or* $\alpha = 0$ *we find that* x_1 *is a periodic solution, but it's contradictory to the hypothesis or to the trivial solution* $x = 0$.

Bifurcation system *We have the following system:*

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -(a_0 x^5 + a_1)y - x. \end{cases}$$

We search equilibrium points for this system:

$$\begin{cases} \dot{x} &= 0 \\ \dot{y} &= 0. \end{cases}$$

The system accepts one equilibrium point which is : $(0, 0)$.

After that, we look for the Jacobean:

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ -(5a_0 x^4 y) - 1 & -(a_0 x^5 + a_1) \end{pmatrix}$$

So, at the equilibrium point :

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -a_1 \end{pmatrix}$$

We search for the polynomial characteristic :

$$\lambda^2 + a_1\lambda + 1 = 0.$$

$$\delta = {a_1}^2 - 4.$$

We find three cases δ*:*

If $\delta = 0$ *then* $a_1 = \pm 2$*, thus the equilibrium point is a node.*

If $\delta > 0$ *then* $a_1 > 2$ *, thus* λ *it accepts two values of different signs , the equilibrium point is a saddle point.*

The interesting case is $\delta < 0$ *then the value of* λ *is complex and in this case , we have:*

$$\lambda = -a_1 \pm \sqrt{{a_1}^2 - 4}.$$

If $a_1 = 0$ *the equilibrium points is centre in the linear case for the non linear part it's a paasage of polar*

coordinated because linear is topologically equivalent with the non linear but the centre can be a limited cycle.

If $0 < a_1 < 2$ *the equilibrium point is an unstable focus (positive real part).*

If $0 < a_1$ *the equilibrium point is a steady focus (negative real part).*

In the non linear case, we can use lyaponov's criterium:

The system linear predicts centres when the setting is equal to 0.

In the non linear system, we can see that those centres aren't in fact conserved.

To determine the steadiness of the origin , we consider positive defined function $v(x, y)$.

I take some examples of $v(x, y)$ *but they aren't efficient.*

The second case

we choose :

$f_m(x) = a_0 x^5 + a_1 x^4 + a_2, \qquad g_n(x) = x.$

The system becomes:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -(a_0 x^5 + a_1 x^4 + a_2)\, y - x \end{cases}$$

In, first approximation the general solution is:

$$x = C\, cos(t - \alpha).$$

Where C and α *are arbitrary.*

We study how the presence of the term with ϵ *affects the solution:*

$$x = C cost.$$

By giving what can be a first reasonable approximation , we try

$$x = A cos(\omega t).$$

Where ω *is a constant close to* 1.

$$\begin{cases} \dot{x} &= -A\omega \sin(\omega t) \quad \textit{We replace in the following differential equation} \\ \ddot{x} &= -A\omega^2 \cos(\omega t). \end{cases}$$

$$\ddot{x} + (a_0 x^5 + a_1 x^4 + a_2)\dot{x} + x = 0. \tag{4.12}$$

We find that:

$-A\omega^2 \cos(\omega t) - (a_0 A^5 \cos(\omega\, t)^5 + a_1 A^4 \cos(\omega t)^4 + a_2)(A\omega \sin(\omega t)) + A\cos(\omega t) = 0,$

$$\begin{aligned} A(1-\omega^2)\cos(\omega t) &= a_0 A^6 \omega \quad \cos(\omega t)^5 \sin(\omega t) + a_1 A^5 \omega \quad \cos(\omega t)^4 \sin(\omega t) + a_2 A\omega \quad \sin(\omega t), \\ &= A\omega\Big(a_0 A^5 \cos(\omega t)^5 + a_1 A^4 \cos(\omega\, t)^4 + a_2\Big)\sin(\omega t). \end{aligned}$$

After the simplification, we find the second member equal at:

$$\begin{aligned} = \quad & a_0 \tfrac{A^6\, \omega}{32} \quad \sin(6\omega\, t) + a_1 \tfrac{A^5 \omega}{16} \sin(5\omega\ t) + a_0 \tfrac{A^6 \omega}{8} \sin(4\omega\ t) \\ & + 3a_1 \tfrac{A^5 \omega}{16} \sin(3\omega\ t) + a_0 \tfrac{A^6 \omega}{32} \sin(2\omega\ t) + (a_1\ \tfrac{A^5 \omega}{8} + a_2 A\omega)\sin(\omega\ t). \end{aligned}$$

This equation can be satisfied for all t only if the coefficients of the different sinusoidal terms disappear .

The term cos(ωt) disappears if $\omega = 1$*, and the coefficient of* $A = \sqrt[4]{-8\frac{a_2}{a_1}}$ *is zero as if we take* $a_1.a_2 < 0, a_1 \neq 0$.

Thus, we choose:

$$\omega = 1, \quad A = \sqrt[4]{-8\frac{a_2}{a_1}}.$$

This lets yet the term that containing $\sin(6\omega t), \sin(5\omega t), \sin(4\omega t), \sin(3\omega t), \sin(2\omega t)$ *to disappear because we have fixed ω and A.*

Thus, we find that the system admits a limited cycle of radius $A = \sqrt[4]{-8\frac{a_2}{a_1}}$ *as* $a_1.a_2 < 0, a_1 \neq 0$.

3^{rd} case

we choose

$f_m(x) = a_0x^5 + a_1x^4 + a_2x^3 + a_3, \qquad g_n(x) = x.$

The system becomes:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -(a_0x^5 + a_1x^4 + a_2x^3 + a_3)\, y - x \end{cases}$$

In first approximation the general solution is

$$x = C\,cos(t - \alpha).$$

Where C and α are arbitrary.

We study how the presence of the term with ϵ affects the solution

$$x = Ccost.$$

By giving what can be a first reasonable approximation , we try

$$x = Acos(\omega t).$$

Where ω is a constant close to 1.

$$\begin{cases} \dot{x} &= -A\omega\sin(\omega t) \quad \text{It replaces the following differential equation} \\ \ddot{x} &= -A\omega^2\cos(\omega t). \end{cases}$$

$$\ddot{x} + (a_0x^5 + a_1x^4 + a_2x^3 + a_3)\dot{x} + x = 0. \tag{4.13}$$

We find that:

$-A\omega^2\cos(\omega t) - (a_0A^5\cos(\omega t)^5 + a_1A^4\cos(\omega t)^4 + a_2A^3\cos(\omega t)^3 + a_3))(A\omega\sin(\omega t)) + A\cos(\omega t) = 0,$

$$\begin{aligned} A(1-\omega^2)\cos(\omega t) &= a_0A^6\omega \cos(\omega t)^5\sin(\omega t) + a_1A^5\omega \cos(\omega t)^4\sin(\omega t) \\ &\quad + a_2A^4\omega\cos(\omega t)^3 \sin(\omega t) + a_3A\omega \sin(\omega t), \\ &= A\omega\Big(a_0A^5\cos(\omega t)^5 + a_1A^4\cos(\omega t)^4 + a_2A^3\cos(\omega t)^3 + a_3\Big)\sin(\omega t). \end{aligned}$$

After the simplification, we find the second member equal at:

$$= \quad a_0\frac{A^6\,\omega}{32}\sin(6\omega\,t) + a_1\frac{A^5\omega}{16}\sin(5\omega\,t) + (a_0\frac{A^6\omega}{8} + a_2\frac{A^4\omega}{8})\sin(4\omega\,t)$$
$$+3a_1\frac{A^5\omega}{16}\sin(3\omega\,t) + (a_0\frac{A^6\omega}{32} + a_2\frac{A^4\omega}{4})\sin(2\omega\,t) + (a_1\,\frac{A^5\omega}{8} + a_3A\omega)\sin(\omega\,t).$$

This equation can be satisfied for all t only if the coefficients of the different sinusoidal terms disappear
.

The term $cos(\omega t)$ *disappears if* $\omega = 1$*, and the coefficient of* $\sin(\omega t)$ *is zero if we take* $A = \sqrt[4]{-8\frac{a_3}{a_1}}$ *as* $a_1.a_3 < 0, a_1 \neq 0.$

So, we choose:

$$\omega = 1, \quad A = \sqrt[4]{-8\frac{a_3}{a_1}}.$$

This lets yet the term that containing $\sin(6\omega t), \sin(5\omega t), \sin(4\omega t), \sin(3\omega t), \sin(2\omega t)$ *to disappear because we have fixed* ω *and* A*.*

Thus, we find that the system admits a limited cycle of radius $A = \sqrt[4]{-8\frac{a_3}{a_1}}$ *as* $a_1.a_3 < 0, a_1 \neq 0.$

4^{th} case

We choose

$f_m(x) = a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4, \qquad g_n(x) = x.$

The system becomes:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -(a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4)\,y - x \end{cases}$$

In first approximation, the general solution is

$$x = C\,cos(t - \alpha).$$

Where C and α *are arbitrary. (See [2]) We study how the presence of the term whith* ϵ *affects the solution :*

$$x = Ccost.$$

By giving what can be a first reasonable approximation , we try:

$$x = Acos(\omega t).$$

Where ω *is a constant close to* 1.

$$\begin{cases} \dot{x} &= -A\omega\sin(\omega t) \quad \textit{It replaces the following differential equation} \\ \ddot{x} &= -A\omega^2\cos(\omega t). \end{cases}$$

$$\ddot{x} + (a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4)\dot{x} + x = 0. \tag{4.14}$$

We find that:

$-A\omega^2\cos(\omega t) - (a_0A^5\cos(\omega t)^5 + a_1A^4\cos(\omega t)^4 + a_2A^3\cos(\omega t)^3 + a_3A^2\cos(\omega t)^2 + a_4)(A\omega\sin(\omega t)) + A\cos(\omega t) = 0,$

$$\begin{aligned} A(1-\omega^2)\cos(\omega t) &= a_0A^6\omega \quad \cos(\omega t)^5\sin(\omega t) + a_1A^5\omega \quad \cos(\omega t)^4\sin(\omega t) \\ &\quad + a_2A^4\omega\cos(\omega t)^3 \quad \sin(\omega t) + a_3A^3\omega\cos^2(\omega t)\sin(\omega t) + a_4A\omega \quad \sin(\omega t), \\ &= A\omega\Big(a_0A^5\cos(\omega t)^5 + a_1A^4\cos(\omega t)^4 + a_2A^3\cos(\omega t)^3 \\ &\quad + a_3A^2\cos^2(\omega t) + a_4\Big)\sin(\omega t). \end{aligned}$$

After the simplification, we find the second member equal at :

$$\begin{aligned} &= a_0\tfrac{A^6\omega}{32} \quad \sin(6\omega t) + a_1\tfrac{A^5\omega}{16}\sin(5\omega t) + (a_0\tfrac{A^6\omega}{8} + a_2\tfrac{A^4\omega}{8})\sin(4\omega t) \\ &\quad + (3a_1\tfrac{A^5\omega}{16} + a_3\tfrac{A^3\omega}{4})\sin(3\omega t) + (a_0\tfrac{A^6\omega}{32} + a_2\tfrac{A^4\omega}{4})\sin(2\omega t) + (a_1 \tfrac{A^5\omega}{8} + a_3\tfrac{A^3\omega}{4} + a_4A\omega)\sin(\omega t). \end{aligned}$$

This equation can be satisfied for all t only if the coefficients of the different sinusoidal terms disappear .

The term in $cos(\omega t)$ disappears if $\omega = 1$, and the coefficient of $\sin(\omega t)$ is zero if we take

$$A\omega \quad (a_1\frac{A^4}{8} + a_3\frac{A^2}{4} + a_4) = 0.$$

$A\omega \neq 0$ in order not to fall in trivial case.

$a_1A^4 + 2a_3A^2 + 8a_4 = 0$. We put $y = A^2$, so

$$\delta = 4a_3{}^2 - 32a_1\, a_4.$$

If $\delta = 0 \to a_3{}^2 = 8a_1\, a_4$, then $y = \frac{-a_3}{a_1}$ so the system accepts a limited cycle of radius $A = \sqrt{\frac{-a_3}{a_1}}$ as $a_1 \neq 0, a_1.a_3 < 0$.

If $\delta > 0 \to a_3{}^2 > 8a_1\, a_4$, we find two solutions:

$$y_{1,2} = \frac{-4a_3 \pm \sqrt{4a_3{}^2 - 32a_1\, a_4}}{2a_1}.$$

So, the system accepts a limited cycle only if $A_{1,2} = \sqrt{y_{1,2}}$ provided that $y_{1,2} > 0$.

If $\delta < 0$, it accepts any limited cycle.

5^{th} case

We choose

$f_m(x) = a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5, \qquad g_n(x) = x.$

The system becomes:

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -(a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5)\, y - x \end{cases}$$

In first approximation the general solution is:

$$x = C\,cos(t - \alpha).$$

Where C and α are arbitrary.

We study how the presence of the term with ϵ affects the solution:

$$x = Ccost.$$

By giving what can be a first reasonable approximation , we try:

$$x = Acos(\omega t).$$

Where ω is a constant close to 1.

$$\begin{cases} \dot{x} = -A\omega \sin(\omega t) & \textit{It replaces in the following differential equation} \\ \ddot{x} = -A\omega^2 \cos(\omega t). \end{cases}$$

$$\ddot{x} + (a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5)\dot{x} + x = 0. \tag{4.15}$$

we find that:

$-A\omega^2 \cos(\omega t) - (a_0A^5\cos(\omega\, t)^5 + a_1A^4\cos(\omega t)^4 + a_2A^3\cos(\omega t)^3 + a_3A^2\cos(\omega\, t)^2 + a_4A\cos(\omega t))(A\omega \sin(\omega t)) + A\cos(\omega t) = 0,$

$$\begin{aligned} A(1 - \omega^2)\cos(\omega t) &= a_0A^6\omega \quad \cos(\omega t)^5 \sin(\omega t) + a_1A^5\omega \quad \cos(\omega t)^4 \sin(\omega t) \\ &\quad + a_2A^4\omega\cos(\omega\, t)^3 \quad \sin(\omega t) + a_3A^3\omega\cos^2(\omega t)\sin(\omega t) + a_4A^2\cos(\omega\, t)\omega \quad \sin(\omega\, t) + \\ &\quad a_5A\ \sin(\omega\, t), \\ &= A\omega\Big(a_0A^5\cos(\omega t)^5 + a_1A^4\cos(\omega\, t)^4 + a_2A^3\cos(\omega\, t)^3 \\ &\quad + a_3A^2\cos^2(\omega t) + a_4A\cos(\omega\, t) + a_5\Big)\sin(\omega t). \end{aligned}$$

After the simplification, we find the second member equal at:

$$\begin{aligned} &= a_0\frac{A^6\,\omega}{32} \quad \sin(6\omega\, t) + a_1\frac{A^5\omega}{16}\sin(5\omega\, t) + (a_0\frac{A^6\omega}{8} + a_2\frac{A^4\omega}{8})\sin(4\omega\, t) \\ &\quad + (3a_1\frac{A^5\omega}{16} + a_3\frac{A^3\omega}{4})\sin(3\omega\, t) + (a_0\frac{A^6\omega}{32} + a_2\frac{A^4\omega}{4}a_4\frac{A^2\,\omega}{2})\sin(2\omega\, t) \\ &\quad + (a_1\ \frac{A^5\omega}{8} + a_3\frac{A^3\omega}{4} \\ &\quad + a_5A\omega)\sin(\omega\, t). \end{aligned}$$

This equation can be satisfied for all t only if the coefficients of the different sinusoidal terms disappear .

The term in $cos(\omega t)$ disappears if $\omega = 1$, and the coefficient of $\sin(\omega t)$ *is zero if we take:*

$$A\omega \quad (a_1\frac{A^4}{8} + a_3\frac{A^2}{4} + a_5) = 0.$$

$A\omega \neq 0$

in order not to fall in trivial case.

$a_1 A^4 + 2a_3 A^2 + 8a_5 = 0.$

We obtain $y = A^2$, *then*

$$\delta = 4{a_3}^2 - 32a_1\, a_5.$$

If $\delta = 0 \rightarrow {a_3}^2 = 8a_1\, a_5$, *so* $y = \frac{-a_3}{a_1}$ *then the system has a limit cycle radius* $A = \sqrt{\frac{-a_3}{a_1}}$ *as* $a_1 \neq 0, a_1.a_3 < 0.$

If $\delta > 0 \rightarrow {a_3}^2 > 8a_1\, a_5$, *we find two solutions:*

$$y_{1,2} = \frac{-4a_3 \pm \sqrt{4{a_3}^2 - 32a_1\, a_5}}{2a_1}.$$

So, the system accepts a limit cycle only radius $A_{1,2} = \sqrt{y_{1,2}}$ *provided that* $y_{1,2} > 0$.

If $\delta < 0$, *it accepts any limited cycle.*

Conclusion *In this paper, we proved the system can have at most* 2 *limit cycles. If* $f(x)$ *is an odd polynomial of degree* 5 *then the probabilities that the Lieard equation for* $f(x)$ *has at least* 2 *periodic solutions is greater than* 47, 23 *and that it has no periodic solution is greater than* 34.54 .

Conclusion

In this work we were interested in the qualitative study of planar polynomial differential systems, it is important for a differential system of know whether or not it admits a periodic solution, moreover if this periodic solution is isolated, we speak by definition of a limit cycle. On the other hand, the calculation of the first integral of a planar differential system completely determines the phase portrait of the system. For models resulting from practice, it

Keywords : *Polynomial planar differential systems, Periodic solutions, Cycles algebraic limits, Non-algebraic limit cycles, Homogeneous differential systems, First integral, Darboux integrability.*

Annexe (P5, P4)(Polynomial Planar Phase Portraits)

In this section we present a computer program based on the theory of planar polynomial differential systems.

This program is carried out to draw the phase portrait of any polynomial differential system on the plane obtained by the Poincare compactification.

The new version of P5 changed the symbolic language from REDUCE to MAPLE, and now it is implemented more easily in any system, WINDOWS, UNIX, or MACINTOSH OS-X, i.e. where MABLE is available.

The P4 is a tool that can be used in the study of a differential system planar polynomial depending on the user's choice, it draws phase portraits on the Poincare disk, or on any rectangle in the plane, or space in one of the four maps used in the compactification.

Now we will see how the P4 works. First P4 verified if the vector field has a continuous set of singular points in the plane , if the two polynomial components of the vector field have a factor common.

If they have a common factor, we divide the vector field by this common factor and we study the new vector field.

Sometimes, if the vector field is too large, the algebraic calculation software used (i.e., REDUCE or MABLE) cannot find this common factor.

In In these cases the P4 will not work correctly.

Maxima 5.25.1 Manual

Maxima is a computer algebra system, implemented in Lisp.

Maxima is derived from the Macsyma system, developed at MIT in the years 1968 through 1982 as part of Project MAC.

MIT turned over a copy of the Macsyma source code to the Department of Energy in 1982; that version is now known as DOE Macsyma.

A copy of DOE Macsyma was maintained by Professor William F. Schelter of the University of Texas from 1982 until his death in 2001.

In 1998, Schelter obtained permission from the Department of Energy to release the DOE Macsyma source code under the GNU Public License, and in 2000 he initiated the Maxima project at SourceForge to maintain and develop DOE Macsyma, now called Maxima.

Bibliography

[1] A.A. Andronov, *Les cycles limites de Poincaré et la théorie des oscillations auto-entretenues*, C. R. Acad. Sci. Paris 89 (1929) 559-561.

[2] A. Agarwal and N. Ananthkrishnan, *Bifurcation analysis for onset and ces-sation of surge in axial flow compressors*, Internat. J. Turbo Jet Engines, 17-3 (2000), 207-217.

[3] A. Buica, J. Giné, J. Llibre, *Bifurcation of limit cycles from a polynomial degenerate center*, Adv. Nonlinear Stud. 10 (2010) 597-609.

[4] A. Bendjeddou, R. Benteri and T. Salhi, *Explicit Non-Algebraic Limit Cycle for Polynomial Systems of Degree Seven*, Applied Mathematical Sciences, Vol. 3, 2009, no. 13, 613-622.

[5] A. Liénard, *Étude des oscillations entrenues*, Rev. Gén. Électricité, 23 (1928),946-954.

[6] A. L. Hodgkin and A. F. Huxley, *A qualitative description of membrane current and its application to conduction and excitation in nerve*, J. Physiol.,117 (1952), 500-544 ; reproduced in Bull. Math. Biol., 52 (1990), 25-71.

[7] Anco, S. C. Bluman, **G. W.** , *Integrating factors and first integrals for ordinary differential equations* , University of British Columbia 1997

[8] B. van der Pol, *On relaxation-oscillations* , Phil. Mag. 2 (1926) 978-992.

[9] C. C. Rocsoreanu, A. Georgeson, and N. Giurgiteanu, *The Fitzhugh-Nagumo Model : Bifurcation and Dynamics*, Kluwer, Dordrecht, Netherlands, 2000.

[10] C. Du and Y. Liu, *General center conditions and bifurcation of limit cycles for a quasi-symmetric seventh degree system*, Comput. Math. Appl. 56 (2008), 2957-2969.

[11] C. Liu, G. Chen and J. Yang, *On the hyperelliptic limit cycles of Lienard systems*, Puplished 20 April 2012.

[12] D. B. Owens, F. J. Capone, R. M. Hall, J. M. Brandon, and J. R. Cham-bers, *Transonic free-to-roll analysis of abrupt wing stall on military aircraft*, J. Aircraft , 41-3 (2004), 474-484

[13] D. Hilbert, *Mathematische Problem (lecture)*, Second Internat. Congress Math. Paris, 1900, Nachr. Ges. Wiss. Gettingen Math.Phys. Kl. 1900, pp 253-297

[14] D.S.JONES B.D.SLEEMAN.*Differential Equations and Mathematical Biology* This edition published in the Taylor and Francis e-Library, 2005.

[15] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Universitext, Springer-Verlag, Berlin, 1996.

[16] F. Feneniche, Med-Salem Rezaoui, *Integrability and a Limit Cycle Solver for a Generalization of Polynomial Liénard Differential Systems*, Journal of Applied Computer Science and Mathematics 41 – 47, No. 26, Suceava

[17] F. Feneniche, Med-Salem Rezaoui, *Dynamics of the polynomial differential systems* Turkish Journal of Computer and Mathematics Education

[18] J. Bricmont, *Introduction á la dynamique non linéaire*, Unité de la physique théorique et mathématique académique 2009-2010.

[19] J. le Bourlot, *Introductions aux systémes dynamique dissipatifs* , P6.P7.P11.observatoire de paris 2011-2012.

[20] J. Nagumo, S. Arimoto, and S. Yoshizawa, *An active pulse transmission line simulating 1214-nerve axons*, Proc. IRL, 50 (1970), 2061-2070.

[21] J.Quinet, J.Fazekas, *Cours élémentaire de Mathématiques supérieures Tome 4 page 81-94* Dunod, Bordas, paris,1998, ISBN 2-04-010079-2.

[22] J. Rayleigh, *The Theory of Sound* , Dover, New York, 1945.

[23] J. Chavarriga, H. Giacomini and J. GinÈ, *On a new type of bifurcation of limit cycles for a planar cubic system*, Nonlinear Analysis 36 (1999), 139-149.

[24] J. Llibre and G. Rodriguez, *Configuration of limit cycles and planar polynomial vector fields*, J. Differential Equations 198 (2004), 374-380.

[25] J. Sotomayor and R. Paterlini, *Quadratic vector fields with finitely many periodic orbits*, Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math., 1007, Springer, Berlin, 1983, pp 753-766.

[26] H. Chen, Y. Li, *Bifurcation and stability of periodic solutions of Duffing equations* , Nonlinearity 21 (2008) 2485-2503.

[27] H. Giacomini, S.Neukirch, *Algebraic approximations to bifurcation curves of limit cycles for the lienard equation* , Chao-dyn /9707002 v130 Jun 1997.

[28] H. Giacomini, J. Llibre and M. Viano, *On the nonexistence, existence, and uniquennes of limit cycles*, Nonlinearity 9 (1996), 501-516.

[29] H. Poincare, *"Memoire sur les courbes definies par une equation differentielle," J. Mathematiques*, 7 (1881), 375-422 ; Ouevre (1880-1890), Gauthier-Villar, Paris.

[30] L. Forest, N. Glade, J. Demongeot, *Lienard systems and potentiel Hamiltonian decomposition appliations in biology*, Hal -00346975, version 1-13 Dec 2008.

[31] L.Perko, *Differential equation and dynamical systems* , ISBN 0-387-95116-4 WWW.Springer-ny.com.

[32] M. Carbonell, B. Coll and J. Llibre, *Limit cycles of polynomial systems with homogeneous nonlinearities*, J. of Math. Anal. and Appl. 142 (1989), 573-590.

[33] Morris W. Hirsch, Stephen Smale, Robert L. Devaney *Differential equations, dynamical systems, and an introduction to chaos* This book is printed on acid-free paper 2004, Elsevier (USA)

[34] N.G. *Lloyd Limit cycles of certain polynomial differential systems*, in Non-linear Functional Analysis and its Applications, S.P. Singh, ed NATO AS1 Series C, Vol. 173, Reidel, Dordrecht, The Netherlands, 1986, pp. 317-326.

[35] R. Asheghi and H. R. Z. Zangeneh, *Bifurcations of limit cycles for a quintic Hamiltonian system with a double cuspidal loop*, Comput. Math. Appl. 59 (2010), 1409- 1418.

[36] R. Fitzhugh, *Impulses and physiological states in theoretical models of nerve membranes*, J. Biophys., 1182 (1961), 445-466.

[37] R.Gérard and Sibuya :*Etude de certains systémes de Pfaff avec singulaités*, Lecture Notes in Math.No712, Springer-Verlag,131-288,(1979).

[38] V.I.ARNOLD, *Méthodes mathématiques de la mécanique classique*. Mir, Moscou 1976

[39] W. Xu, C. Li, *Limit cycles of some polynomial Liénard systems*, J. Math. Anal. Appl. 389 (2012) 367-378.

[40] Ye Yanqian and others, *Theory of Limit Cycles*, Transl. Math. Mono-graphs 66, Amer. Math. Soc., Providence, 1984.