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Sujet

METHODE DE QUADRATURE POUR LE P-LAPLACIEN ET APPLICATIONS

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Introduction

This thesis is devoted to the study of boundary value-problems of the type

$$\begin{cases} -(\varphi(u'))' = g(\lambda, u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{0.1}$$

where λ is a real parameter, φ is an odd increasing homeomorphism of R and g is a continuous function.

Note that the differential operator $u \to (\varphi(u'))'$ is not linear, so the problems of the kind of (0.1) are called strongly nonlinear B.V.P's. Also, when $\varphi(u) = u$, $(\varphi(u'))' = u''$ is the classical one dimensional Laplacian and when $\varphi(u) = \varphi_p(u) = |u|^{p-2}u$ with $p > 1$, $(\varphi(u'))' = \left(|u'|^{p-2}u'\right)'$ is the one dimensional p-Laplacian which is linear if and only if $p = 2$.

The p-laplacian operator $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$, $p > 1$, has been widely investigated during the last two decades. Indeed, many physical phenomena are modeled by equations involving this operator. For instance, the p-laplacian operator appears in the study of non-Newtonian fluids. When studying the laws of motion of fluid media Newtonian fluids are those for which the stress tensor τ is proprtional to the velocity gradient $\partial u/\partial x$. But, this relation is satisfied for a limited actual fluid media. The motions of such non-newtonian fluids are studied in rheology (see Astarita and Marruci [28]). Usually the relation between the shear stress τ and the gradient velocity ∇u is sbstitutied in the power rheological law $\tau = \mu|\partial u/\partial x|^{p-2}\partial u/\partial x$ $p > 1$. The constants μ and p are the rheological characteristics of the meduim. Media with $p > 2$ are called dilatant fluids and those with $p > 2$ are called pseudo-plastic.

Also, the p-laplacian operator Δ_p with $p \neq 2$ appears in many others context. It's used in some reaction diffusion problems (see Aris [25]) as well as in flow through porous media (for instance in flow through rock filled dams, see Ahmed and Sunada [13] or Volquer [119]). Also, Δ_p appears in nonlinear elasticity (see E.G. Oden [84]), glaceology (see Pelissier [89]), petrolum extraction (see Schoenawer [101]), Some geometrical interest $p \geq 2$ (see Uhlenbeck [112]).

In chapter 1, we present the quadrature method for the one dimensional p-Laplacian. This method allows us to study the the structure of the solution set of the problem (0.1) when $\varphi(u) = \varphi_p(u) = |u|^{p-2}u$. Besides, this method well known in the litterature as the " time-map approach " will be used in chapters 2, 3, 4 and 5.

In the second, chapter we consider problem (0.1) with $\varphi = \varphi_p$ and $g(\lambda, x) = \lambda f(x)$ for all $x \in \mathbb{R}$ where f is a cubic-like nonlinearity. More precisely, f is odd and admits a positive simple zero α such that $f > 0$ in $(0, \alpha)$ and $f < 0$ in $(\alpha, +\infty)$.

In the chapter 3, we look for the exact number of positive solutions of problem (0.1) when $\varphi = \varphi_p$ and $g(\lambda, x) = \lambda f(x)$ for all $x \in \mathbb{R}$. We examine the cases f is positive in $\mathbb{R}^+$, f vanishes once and f vanishes at least twice.

In the chapter 4, we investigate the case where $\varphi = \varphi_p$ and $g(\lambda, x) = |x|^p - \lambda$ for all $x \in \mathbb{R}$. Note that in this case g is even and superlinear. In the case $\lambda \leq 0$ we give an exact description of the solution set of problem (0.1) and in the case $\lambda > 0$ we give an uniqueness result for the positive and negative solutions and a lower bound for the sign changing solutions.

The chapter 5 deals with the case $\varphi = \varphi_p$ and $g(\lambda, x) = \varphi_\alpha(x) + \lambda\varphi_\beta(x)$ for all $x \in \mathbb{R}$ where $\alpha, \beta > 1$ are real parameters. This problem has been suggested by A. Ambrosetti, H. Brezis and G. Cerami in [17].

In chapter 6, an uniqueness result is given when the inverse function of φ is Lipschitzian and $g(\lambda, x) = \lambda f(x)$ for all $x \in \mathbb{R}^+$ with f is C^1 on $\mathbb{R}^+$ and its derivate is bounded . To prove this result we make use of the Leray-Schauder continuation theorem.

In chapter 7, a complete description of the solution set to problem (0.1) is established when $g(\lambda, x) = f(x)$ for all $x \in \mathbb{R}$ and, φ and f have opposite concavities. This result is obtained by means of the bifurcation theory of P.H. Rabinowitz.

We end this thesis by some concluding remarks and open questions.

Chapter 1

Quadrature method for the one dimensional p-Laplacian and applications

Annales de Mathématiques 6 (1998), pp. 13-21.

1.1 Introduction

In this chapter, we present a method which helps one to study the solution set of boundary-value problems of the type

$$\begin{cases} -\left(\varphi_p(u')\right)' = g(u) & \text{in } (\alpha, \beta) \\ u(\alpha) = u(\beta) = 0 \end{cases}$$

where $p > 1$, $\varphi_p(x) = |x|^{p-2}x$, for all $x \in \mathbb{R}$, and g is a real continuous function.

The operator $u \mapsto \left(\varphi_p(u')\right)'$ is called the p-Laplacaian and is linear if and only if $p = 2$. The quadrature method is well known in the literature. It is also known as time maps approach.

At the begining of time map history, many authors used the one dimensional Laplacian operator, that is, with the one dimensional p-Laplacian and $p = 2$. Starting from the 1960's, we can mention the works by Opial [85], [86], Urabe [113], [114], [115], [116], [117], Pimbley [90], [91], Gavalas [68].

In the early 1970's, Laetsch [76] used the time map to study positive solutions to a class of boundary-value problems with Dirichlet boundary data. Since then many authors have referred to his work. We also want to mention Chafee and Infante [43], and Chafee [44]. Brown and

Budin [34], [35] used the time map approach to study positive solutions to some boundary-value problems. Independently and about the same time, De-Mottoni and Tesei [52] studied positive solutions of some other class of boundary-value problems by means of the same method.

In the early 1980's, Smoller and Wasserman [106] introduced a technique that, in some circumstances, can be used to prove uniqueness of the critical point of time maps. Subsequently their technique has been used by many authors (see for instance, Ammar Khodja [20], Ramaswamy [96], S. H. Wang and Kazarinoff, [126], [127], S. H. Wang and F. P. Lee [128], S. H. Wang [120], [121], [123], and recently by Addou and Benmezaï [9]).

The study of sign-changing solutions by means of time maps was initiated by De-Mottoni and Tesei [53] and independently, some years later, by Shivaji [105].

During the last two decades, time maps have been used in many works. Further to the above mentioned papers, we want to add the following ones: Addou and Ammar Khodja [5], Anuradha, Shivaji and Zhu [22], [23], Anuradha and Shivaji [21], Anuradha, C. Brown and Shivaji [24], K. Brown, Ibrahim and Shivaji [36], Brunovsky and Chow [37], Castro and Shivaji [39], [41], Ding and Zanolin [56], [57], Fernandes [58], Fonda and Zanolin [59], Fonda, Gossez and Zanolin [60], Schaaf [100], Shivaji [103], [104], [105], Smoller, Tromba and Wasserman [110], Smoller and Wasserman [107], [108], [109]. Notice that this list is in alphabetical order, and is not exhaustive. The differential operator in the equations studied in these papers is the p-Laplacian with $p = 2$.

The paper by Guedda and Veron [69] seems to be the first one dealing with time maps approach when the differential operator is the one dimensional $p-$Laplacian with $p > 1$. Next, we mention the papers by Del Pino and Manasevich [51], Manasevich and Zanolin [80] and Manasevich et al. [81].
Notice that time maps were also used when the differential operator generalizes the p-Laplacian: see for instance Arrázola and Ubilla [27], Garcia-Huidobro et al. [62], [63], [64], [65], [66], Garcia-Huidobro and Ubilla [67], and Ubilla [111]. (See also, Huang and Metzen [72]).

1.2 Notations

In order to state the main results let us define for any $k \in \mathbb{N}^*$, the sets

$$S_k^+ = \left\{ \begin{array}{r} u \in C^1([\alpha, \beta]) : u \text{ admits exactly } (k-1) \text{ zero(s) in } (\alpha, \beta) \\ \text{all simple, } u(\alpha) = u(\beta) = 0 \text{ and } u'(\alpha) > 0 \end{array} \right\}$$

$S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$.

Definition 1 Let $u \in C([\alpha, \beta])$ and $x_1 < x_2$ are two consecutive zeros of u. We call I-hump of u the restriction of u to the open interval $I = (x_1, x_2)$. When there is no confusion we say a hump of u.

With this definition in mind, each function in S_k^+ has exactly k humps such that the first one is positive, the second is negative, and so on with alternations.

Let A_k^+ $(k \geq 1)$ be the subset of S_k^+ consisting of the functions u satisfying:

- Every hump of u is symmetrical about the center of the interval of its definition.
- Every positive (resp. negative) hump of u can be obtained by translating the first positive (resp. negative) hump.
- The derivative of each hump of u vanishes once and only once.

Let $A_k^- = -A_k^+$ and $A_k = A_k^+ \cup A_k^-$.

Let B_k^+ $(k \geq 1)$ be the subset of $C^1([\alpha, \beta])$ consisting of the functions u satisfying:

- $u(x) \geq 0$, $\forall x \in [\alpha, \beta]$, and $u(\alpha) = u(\beta) = u'(\alpha) = 0$.
- u admits exactly $(k-1)$ zero(s), all double, in the open interval (α, β).
- If $k > 1$, u is $((\beta - \alpha)/k)-$periodic.
- Every hump of u (necessarily positive) is symmetrical about the center of the interval of its definition.
- The derivative of each hump of u vanishes once and only once.

Let $B_k^- = -B_k^+$ and $B_k = B_k^+ \cup B_k^-$.

1.3 Description of the method

Consider the boundary value problem

$$\begin{cases} -\left(\varphi_p(u')\right)' = g(u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1.1}$$

where p is a real parameter and the nonlinearity g are such that

$$g \in C(\mathbb{R},\mathbb{R}) \text{ and } 1 < p < +\infty \tag{1.2}$$

and $\varphi_p(x) = |x|^{p-2}x$ for all $x \in \mathbb{R}$.

By a solution to problem (1.1) we mean a function $u \in C([0,1])$ satisfying $-\left(\varphi_p(u')\right)' = g(u)$ in $(0,1)$ and $u(0) = u(1) = 0$.

Throughout this chapter p' will designe the conjugate exponent of p $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ while $G(s) = \int_0^s g(t)\,dt$.

The first result in this section which is the key of the method, is

Lemma 2 If u is solution to problem (1.1) then there exists a constant $E \geq 0$ such that for all $x \in [0,1]$, we have the energy equation

$$\left|u'(x)\right|^p + p'G(u(x)) = E^p \tag{1.3}$$

Proof Observe that

$$\left|u'(x)\right|^p = p'\Phi_{p'}\left(\varphi_p\left(u'(x)\right)\right)$$

where $\Phi_{p'}(x) = \int_0^x \varphi_{p'}(t)\,dt$.

Thus,

$$\left(\left|u'\right|^p + p'G(u)\right)' = p'\left(\varphi_p\left(u'\right)\right)' u' + p'g(u)\,u' = 0 \text{ in } (0,1)$$

So the lemma follows. ■

For any $E \geq 0$ and $\nu = \pm$, let

$$\tilde{X}_\nu(E) = \left\{s \in \mathbb{R} : \nu s > 0,\ E^p - p'G(s) = 0 \text{ and } E^p - p'G(\xi) \geq 0\ \forall \xi : 0 < \nu\xi < \nu s\right\}$$

$$X_\nu(E) = \left\{s \in \mathbb{R} : \nu s > 0, \text{ and } E^p - p'G(\xi) > 0\ \forall \xi : 0 < \nu\xi < \nu s\right\}$$

$$r_\nu(E) = \begin{cases} \nu \sup(\nu X_\nu(E)) & \text{if } X_\nu(E) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

We shall also make use of the following sets,

$$\check{D}_\nu = \left\{E \geq 0 : \tilde{X}_\nu(E) \neq \emptyset\right\}$$

$$D_\nu = \{E \geq 0 : 0 < |r_\nu(E)| < +\infty \text{ and } \nu g(r_\nu(E)) \geq 0\}$$

$$D_\nu^* = \{E > 0 : 0 < |r_\nu(E)| < +\infty \text{ and } \nu g(r_\nu(E)) > 0\}$$

$$\check{D}_\nu = \check{D}_+ \cap \check{D}_-, \; D = D_+ \cap D_- \text{ and } D^* = D_+^* \cap D_-^*.$$

Proposition 3 Let u be a nontrivial solution to problem (1.1). If $u'(0) = \nu E$ $\nu = \pm$ and $E \geq 0$ then, the real number E belongs to $\check{D}_+$ or $\check{D}_-$.

Proof Since u is a nontrivial solution, there exists $x_0 \in (0,1)$ such that $u(x_0) \neq 0$. Suppose that $u(x_0) > 0$ and let $x_1 \in (0,1)$ be the point at which u reach its maximum value. Thus, we have

$$u(x_1) = \max_{x \in [0,1]} u(x) \text{ and } u'(x_1) = 0$$

We deduce from (1.3) that $p'G(u(x_1)) = E^p$ and if $\xi \in (0, u(x_1)]$ then there exists $t \in (0, x_1]$ such that $\xi = u(t)$, $E^p - p'G(\xi) = E^p - p'G(u(t)) = |u'(t)|^p$. So, $u(x_1) \in \hat{X}_+(E)$ and $E \in \hat{D}_+$. ■

Proposition 4 Let u be a nontrivial solution to problem (1.1)

(a) If there exist $k \in \mathbb{N}^*$ such that $u \in B_k$, then $u'(0) = 0$ and $0 \in D_+ \cup D_-$.

(b) If there exist $k \in \mathbb{N}^*$ such that $u \in A_k$, then $|u'(0)| \in D_+ \cup D_-$.

Proof (i)-Suppose that $u \in B_k^+$ and let $\rho = \max_{x \in [0,1]} u(x) > 0$. Then, one can verify easily that

$$X_+(0) = \{s \in \mathbb{R} : s > 0 \text{ and } -p'G(\xi) > 0 \; \forall \xi \in (0,s)\} \neq \emptyset;$$

and

$$\rho = \sup X_+(0)$$

Hence $0 \in D_+$.

(ii)-Let z_1 be the first strictly positive zero of u, and suppose that $u > 0$ in $(0, z_1)$. Then if $\rho = \max_{x \in [0,z_1]} u(x) = \max_{x \in [0,1]} u(x)$, we have

$$|u'(0)|^p - p'G(\rho) = 0,$$

$$X_+(u'(0)) = \{s \in \mathbb{R} : s > 0 \text{ and } |u'(0)|^p - p'G(\xi) > 0 \; \forall \xi \in (0,s)\} \neq \emptyset;$$

and

$$r_+(u'(0)) = \rho$$

Moreover $g(\rho) \geq 0$ because if we suppose that $g(\rho) < 0$ then ρ will be a local minimum of u. ■

Theorem 5 Let u be a solution to problem (1.1) such that $u'(0) = \nu E$ where $\nu = \pm$. If E belongs to D_+^* or D_-^* then there exists an integer $k \geq 1$ such that $u \in A_k^\nu$.

This theorem is a consequence of the following four lemmas.

Let u be a solution to problem (1.1) such that $u'(0) = \nu E$ where $\nu = \pm$ and E belongs to D_+^* or D_-^*; then

Lemma 6 u has a finite number of zeros.

Proof Let $Z(u) = \{z \in [0,1] \,/\, u(z) = 0\}$ and suppose that $Z(u)$ is infinite. Since $Z(u)$ is contained in the compact set $[0,1]$, there exists a sequence $(z_n) \subset Z(u)$ converging to $z_* \in Z(u)$. So,

$$u(z_*) = u'(z_*) = \lim_{n \to \infty} \frac{u(z_n) - u(z_*)}{z_n - z_*} = 0.$$

Thus, putting $x = z_*$ in the energy equation (1.3) we get

$$0 = \left| u'(z_*) \right|^p + p' G(u(z_*)) = E^p$$

which is impossible because $E > 0$. ■

Lemma 7 On each hump, u' vanishes once and only once.

Proof Let $z_1 < z_2$ be two consecutive zeros of u and suppose that $u > 0$ in (z_1, z_2). Let $c_0 = \inf \{x \in (z_1, z_2) : u'(x) = 0\}$.

Then, $c_0 > z_1$ because if $c_0 = z_1$ there will exist a sequence $(z_n) \subset \{x \in (z_1, z_2) : u'(x) = 0\}$ which converges to z_1. Then we have

$$u'(z_1) = E = \lim_{n \to \infty} u'(z_n) = 0$$

and this is impossible since $E \in D_+^*$ or $E \in D_-^*$. Thus, $u' > 0$ in (z_1, c_0).

Let $\xi \in (0, u(c_0))$, then there exist $x_\xi \in (z_1, c_0)$ with $u(x_\xi) = \xi$.

It follows from the energy equation (1.3) that

$$0 < \left| u'(c_0) \right|^p = E^p - p' G(u(x_\xi)) = E^p - p' G(\xi)$$

which yields $u(c_0) \in X_+(E)$.

Let $s > u(c_0)$, $s \notin X_+(E)$ because $E^p - p' G(u(c_0)) = 0$ and $u(c_0) < s$.

So any $s \in X_+(E)$ is such that $s \leq u(c_0)$, then $u(c_0) = r_+(E)$.

Also, since $g(u(c_0)) = g(r_+(E)) > 0$, u reachs at c_0 a maximum value.

Now, let us prove that $u(c_0)$ is the maximum value of u on (z_1, z_2). Supose the contrary and let

$$c_1 = \inf \left\{ x \in (c_0, z_2) : u'(x) = 0 \right\}$$

Then, one has $u' < 0$ in (c_0, c_1) and $u(c_1) < u(c_0)$; so there exists $x' \in (z_1, c_0)$ such that $u(x') = u(c_1)$. We deduce from the energy equation that

$$\left| u'\left(x'\right) \right|^p = \left| u'\left(x'\right) \right|^p - \left| u'(c_1) \right|^p = p'\left(G(u(c_1)) - G\left(u\left(x'\right)\right)\right) = 0$$

which contradicts the definition of c_0.

Lemma 8 Each hump of u is symmetrical about the center of its interval of definition.

Proof Let $z_1 < z_2$ be two consecutive zeros of u and suppose that $u > 0$ in (z_1, z_2). Let c_1 the unique critical point of u in (z_1, z_2) at which u reach its maximum value.

The energy equation yields

$$u'(t) = \left\{ p'(G(u(c_1)) - G(u(t))) \right\}^{\frac{1}{p}} \text{ for all } t \in [z_1, c_1]$$

$$u'(t) = -\left\{ p'(G(u(c_1)) - G(u(t))) \right\}^{\frac{1}{p}} \text{ for all } t \in [c_1, z_2].$$

Then,

$$x - z_1 = \int_0^{u(x)} \frac{du(t)}{u'(t)} = \int_0^{u(x)} \frac{du(t)}{\{p'(G(u(c_1)) - G(u(t)))\}^{\frac{1}{p}}} \text{ for all } x \in [z_1, c_1] \tag{1.4}$$

and

$$z_2 - x = -\int_0^{u(x)} \frac{du(t)}{u'(t)} = \int_0^{u(x)} \frac{du(t)}{\{p'(G(u(c_1)) - G(u(t)))\}^{\frac{1}{p}}} \text{ for all } x \in [c_1, z_2]. \tag{1.5}$$

Putting $x = c_1$ in (1.4) and (1.5), he obtain

$$c_1 - z_1 = z_2 - c_1$$

then

$$c_1 = \frac{z_1 + z_2}{2}.$$

Thus the $(z_1, z_2) -$hump of u reaches its maximum value exactly at the center of (z_1, z_2).

Now, if we observe that $x - z_1 = z_2 - (z_2 + z_1 - x)$ for all $x \in [z_1, z_2]$ and replace in (1.4) and (1.5), we get

$$\begin{aligned} x - z_1 &= z_2 - (z_2 + z_1 - x) = \int_0^{u(x)} \frac{du(t)}{\{p'(G(u(c_1)) - G(u(t)))\}^{\frac{1}{p}}} \\ &= \int_0^{u(z_2 + z_1 - x)} \frac{du(t)}{\{p'(G(u(c_1)) - G(u(t)))\}^{\frac{1}{p}}}. \end{aligned}$$

So $u(x) = u(z_2 + z_1 - x)$ for all $x \in [z_1, z_2]$ and u is symmetrical about $\frac{z_1+z_2}{2}$. This completes the proof of the lemma. ■

Lemma 9 Any positive (resp. negative) hump of u is the translation of the first positive (resp. negative) hump of u.

Proof Let $z_1 < z_2 < z_3 < z_4$ be four zeros of u such that $u > 0$ in (z_1, z_2) and in (z_3, z_4).

We begin by showing that the intervals (z_1, z_2) and (z_3, z_4) have the same lengh.

From the previous lemma we deduce that $u\left(\frac{z_1+z_2}{2}\right) = \max\limits_{x \in [z_1, z_2]} u(x)$ and $u\left(\frac{z_3+z_4}{2}\right) = \max\limits_{x \in [z_3, z_4]} u(x)$.

We have $u\left(\frac{z_1+z_2}{2}\right) = u\left(\frac{z_3+z_4}{2}\right) = u_0$, because if $u\left(\frac{z_1+z_2}{2}\right) < u\left(\frac{z_3+z_4}{2}\right)$, then it would exist $y \in \left(z_3, \frac{z_3+z_4}{2}\right)$ such that $u(y) = u\left(\frac{z_3+z_4}{2}\right)$. We deduce from the energy equation that

$$\left|u'(y)\right|^p = E^p - p'G(u(y)) = E^p - p'G\left(u\left(\frac{z_1 + z_2}{2}\right)\right)$$

and this is a contradiction with lemma 7.

Also, we have

$$\begin{aligned} \frac{z_2 - z_1}{2} &= \frac{z_1 + z_2}{2} - z_1 = \int_0^{u_0} \frac{du(t)}{\{p'(G(u_0) - G(u(t)))\}^{\frac{1}{p}}} \\ &= \frac{z_4 - z_3}{2} = \frac{z_3 + z_4}{2} - z_3 \end{aligned}$$

so, $z_2 - z_1 = z_4 - z_3$.

Let $v(x) = u(z_1 + (x - z_3))$ for all $x \in [z_3, z_4]$. One can easly verify that

$$\begin{aligned} v(z_3) &= u(z_4) = 0 \\ v &> 0 \text{ in } (z_3, z_4) \text{ and is symmetrical about } \frac{z_3 + z_4}{2} \end{aligned}$$

also v satisfy

$$\begin{cases} -\left(\varphi_p(v')\right)' = g(v) \text{ in } (z_3, z_4) \\ v(z_3) = v(z_4) = 0 \end{cases}$$

So, for any $x \in \left[z_3, \frac{z_3+z_4}{2}\right]$

$$\begin{aligned} x - z_3 &= \int_0^{u(x)} \frac{du(t)}{\{p'(G(u_0) - G(u(t)))\}^{\frac{1}{p}}} \\ &= \int_0^{v(x)} \frac{dv(t)}{\{p'(G(u_0) - G(v(t)))\}^{\frac{1}{p}}} \end{aligned}$$

This leads to $v(x) = u(x)$ for all $x \in \left[z_3, \frac{z_3+z_4}{2}\right]$ and using the symmetry of u and v we deduce that $v(x) = u(x)$ for all $x \in [z_3, z_4]$ and $u_{[z_3,z_4]}$ is the translation of $u_{[z_1,z_2]}$. ■

Now, we define the time-maps by :

For $\nu = \pm$,

$$\begin{aligned} T_\nu &: D_\nu \to (0, +\infty] \\ E &\to T_\nu(E) = \nu \int_0^{r_\nu(E)} \frac{du}{\{p'(E^p - p'G(u))\}^{\frac{1}{p}}}. \end{aligned}$$

For $E \in D = D_+ \cap D_-$ and $k \in \mathbb{N}^*$

$$\begin{aligned} T_{2k}(E) &= k(T_+(E) + T_-(E)) \\ T_{2k+1}^\nu(E) &= T_{2k}(E) + T_\nu(E). \end{aligned}$$

Theorem 10 Assume that (1.2) holds and let $E \geq 0$ and $\nu = \pm$

(i) The problem (1.1) admits a solution $u_\nu \in A_1^\nu$ with $u_\nu'(0) = \nu E$ if and only if $E \in D_\nu \cap (0, +\infty)$ and $T_\nu(E) = \frac{1}{2}$; in this case the solution is unique.

(ii) The problem (1.1) admits a solution $u_\nu \in A_{2n}^\nu$ ($n \geq 1$) with $u_\nu'(0) = \nu E$ if and only if

$E \in D \cap (0,+\infty)$ and $T_{2n}(E)=\frac{1}{2}$; in this case the solution is unique.

(iii) The problem (1.1) admits a solution $u_\nu \in A^\nu_{2n+1}$ with $u'_\nu(0)=\nu E$ if and only if $E \in D \cap (0,+\infty)$ and $T^\nu_{2n+1}(E)=\frac{1}{2}$; in this case the solution is unique.

(iv) The problem (1.1) admits a solution $u_\nu \in B^\nu_n$ $(n \geq 1)$ if and only if $0 \in D_\nu$ and $nT_\nu(0)=\frac{1}{2}$; in this case the solution is unique.

Proof Assume that u is a solution to problem (1.1) belonging to A_1^+. Thus

$$u'>0 \text{ in } \left[0,\frac{1}{2}\right) \text{ and } u'\left(\frac{1}{2}\right)=0. \tag{1.6}$$

It follows that

$$\sup\left\{x \in [0,\infty) : u'(t)>0 \ \forall t \in [0,x)\right\}=\frac{1}{2}.$$

By the energy equation we have

$$u'(x)=\left\{\left(u'(0)-G(u(x))\right)\right\}^{\frac{1}{p}} \text{ for all } x \in \left[0,\frac{1}{2}\right]. \tag{1.7}$$

Thus,

$$\sup\left\{x \in [0,\infty) : \left(\left(u'(0)\right)^p - p'G(u(t))\right)>0 \ \forall t \in [0,x)\right\}=\frac{1}{2};$$

or equivalently

$$\sup\left\{x \in [0,\infty) : \left(\left(u'(0)\right)^p - p'G(z)\right)>0 \ \forall z \in [0,u(x))\right\}=\frac{1}{2}$$

which implies that

$$\sup\left\{s \geq 0 : \left(\left(u'(0)\right)^p - p'G(\xi)\right)>0 \ \forall \xi \in [0,s)\right\}=\frac{1}{2}.$$

Also, by (1.5) it follows that

$$x=\int_0^{u(x)} \frac{du(t)}{u'(t)}=\int_0^{u(x)} \left\{\left(u'(0)\right)^p - p'G(\xi)\right\}^{-\frac{1}{p}} d\xi \text{ for all } x \in \left[0,\frac{1}{2}\right]. \tag{1.8}$$

Thus, the improper integral in (1.8) is convergent for all $x \in \left[0,\frac{1}{2}\right]$ and in particular, $u'(0)$

is such that the integral

$$\int_0^{r_+(E)} \left\{ \left(u'(0) \right)^p - p' G(\xi) \right\}^{-\frac{1}{p}} d\xi$$

converges and is equal to $\frac{1}{2}$, where for all $E > 0$

$$r_+(E) = \begin{cases} \sup(X_+(E)) & \text{if} \quad X_+(E) \neq \emptyset; \\ 0 & \text{otherwise} \end{cases}$$

and

$$X_+(E) = \left\{ s \in \mathbb{R} : s > 0, \text{ and } E^p - p' G(\xi) > 0 \ \forall \xi : 0 < \xi < s \right\}.$$

So, if $u'(0) = E$, then $E \in D_+ \cap (0, +\infty)$ and $T_+(E) = \frac{1}{2}$.

Conversely, it is possible to assign to each root E_* of the equation $T_+(E) = \frac{1}{2}$ a unique solution to problem (1.1) belonging to A_1^+ and satisfying $u'(0) = E_*$, $\max_{x \in [0,1]} u(x) = u\left(\frac{1}{2}\right) = r_+(E_*)$.

In fact, if $E_* \in D_+ \cap (0, +\infty)$ is such that $T_+(E_*) = \frac{1}{2}$ define the function on $[0, r_+(E_*)]$ by

$$h_+(u) = \int_0^u \left\{ E_*^p - p' G(\xi) \right\}^{-\frac{1}{p}} d\xi.$$

Notice that $h_+(r_+(E_*)) = T_+(E_*) = \frac{1}{2}$ and

$$h_+'(u) = \left\{ E_*^p - p' G(u) \right\}^{-\frac{1}{p}} > 0 \text{ for all } u \in (0, r_+(E_*)).$$

Let u_+ be the inverse function of h_+ defined by

$$u_+(x) = h_+(x) \in [0, r_+(E_*)] \text{ for all } x \in \left[0, \frac{1}{2}\right]$$

and define u on $[0,1]$ by

$$u(x) = \begin{cases} u_+(x) & \text{if } x \in \left[0, \frac{1}{2}\right] \\ u_+(1-x) & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

It is easy to show that the function u is a solution to problem (1.1) belonging to A_1^+ and satisfies $u'(0) = E_*$, $\max_{x \in [0,1]} u(x) = u\left(\frac{1}{2}\right) = r_+(E_*)$. Let us prove the uniqueness. Assume that v is also a solution of the problem (1.1) belonging to A_1^+ and satisfying $v'(0) = E_*$,

$\max_{x\in[0,1]} v(x) = v\left(\frac{1}{2}\right) = r_+(E_*)$. By (1.6), it follows that for all $x \in \left[0,\frac{1}{2}\right]$, we have

$$x = \int_0^{u(x)} \left(E_*^p - p'G(\xi)\right)^{-\frac{1}{p}} d\xi = \int_0^{v(x)} \left(E_*^p - p'G(\xi)\right)^{-\frac{1}{p}} d\xi.$$

Thus

$$\int_{u(x)}^{v(x)} \left(E_*^p - p'G(\xi)\right)^{-\frac{1}{p}} d\xi = 0 \text{ for all } x \in \left[0,\frac{1}{2}\right].$$

Then, $u = v$ in $\left[0,\frac{1}{2}\right]$, and by symmetry it follows that $u = v$ in $[0,1]$. Therefore the theorem is proved.

Theorem 11 Assume that (1.2) holds true and g satisfies the following hypothesis

$$xg(x) > 0 \text{ for all } x \in \mathbb{R}^*$$

Then, if S is the solution set of problem (1.1), $S \subset \{0\} \cup \left(\bigcup_{k\geq 1} A_k\right)$.

Proof We deduce from the added hypothesis in the theorem that $g < 0$ in $(-\infty,0)$ and $g > 0$ in $(0,+\infty)$. So, G defined by $G(x) = \int_0^x g(t)\,dt$ is strictly increasing on $(0,+\infty)$, strictly decreasing on $(-\infty,0)$ and $G(0) = 0$.

Thus, for $E > 0$ the equation in the variable s

$$E^p - p'G(s) = 0$$

has a unique positive solution $s_+(p,E)$ and a unique negative solution $s_-(p,E)$ such that

$$\lim_{E\to 0^+} s_+(p,E) = 0, \quad \lim_{E\to+\infty} s_+(p,E) = +\infty$$

$$\lim_{E\to 0^+} s_-(p,E) = 0, \quad \lim_{E\to+\infty} s_-(p,E) = -\infty.$$

Now for $E > 0$, straighforward computations yield

$$\check{X}_+(E) = \{s_+(p,E)\}, \quad \check{X}_-(E) = \{s_-(p,E)\}$$

$$X_+(E) = (0, s_+(p,E)), \quad X_-(E) = (s_-(p,E), 0).$$

Hence, we deduce that

$$\tilde{D}_{\pm} = D_{\pm} = D^{*}_{\pm} = (0, +\infty)$$

It follows from Theorem 5 that any nontrivial solution of the problem (1.1) necesserily belongs to $\left(\bigcup_{k \geq 1} A_k\right)$ ■

1.4 The locally Lipshitzian case

Consider the boundary value problem

$$\begin{cases} -\left(\varphi_p(u')\right)' = g(u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1.9}$$

where p and g are such that

$$p \in (1,2] \text{ and } g \text{ is locally Lipshitzian.} \tag{1.10}$$

In this section, we will prove the following result

Theorem 12 Assume that (1.10) holds and let S be the solution set of problem (1.9), then

If $g(0) = 0$ then $S \subset \{0\} \cup \left(\bigcup_{k \geq 1} A_k\right)$

If $g(0) \neq 0$ then $S \subset \left(\bigcup_{k \geq 1} B_k\right) \cup \left(\bigcup_{k \geq 1} A_k\right)$

The proof of this theorem is based on a reduction of the equation $-\left(\varphi_p(u')\right)' = g(u)$ into a first order differential equation. Taking $v = \varphi_p(u')$ the previous equation becomes

$$\begin{cases} u' = \varphi_{p'}(v) \\ v' = -g(u) \end{cases} \tag{1.11}$$

Note that the right-handside of system (1.11) is locally Lipshitzian

Proof of the theorem (12)

The proof is devided into several steps. Let u bea nontrivial solution of problem (1.9) :

Step 1: In this first step, we will prove that u has a finite number of zeros. Suppose the contrary and let $(z_n)_n$ be a sequence of zeros of u converging to z_*. Then we have

$$u(z_*) = u'(z_*) = \lim_{n \to \infty} \frac{u(z_n) - u(z_*)}{z_n - z_*} = 0.$$

If $g(0) = 0$ then $(0,0)$ is a solution to the initial value problem

$$\begin{cases} v' = \varphi_{p'}(w) \\ w' = -g(v) \\ v(z_*) = w(z_*) = 0 \end{cases}$$

as well as $\left(u, \varphi_p(u')\right)$. Since the right-handside of the above system is locally Lipshitzian, the uniqueness theorem of Cauchy leads to $\left(u, \varphi_p(u')\right) = (0,0)$. This is impossible since u is a nontrivial solution.

If $g(0) \neq 0$, we can suppose that $g(0) > 0$ (the case $g(0) < 0$ will be treated in the same way). Since g is continuous, there exists a neighboor V_* of z_* in which $\left(\varphi_p(u')\right)' = -g(u) < 0$. Thus, $\varphi_p(u')$ is strictly decreasing in V_* and for all $x \in V_*$, $u'(x) = 0$ implies that $x = z_*$. This means that z_* is an isolated critical point of u. So it is impossible that u admits an infinite number of zeros.

Step 2: In this step we will prove that between two consecutive zeros of u there is a unique critical point.

Let $z_1 < z_2$ two consecutive zeros of u and suppose that $u > 0$ in (z_1, z_2). Define the set

$$Z\left(u'\right) = \left\{x \in [0,1] : u'(x) = 0\right\}$$

and let

$$c_* = \inf\left(Z\left(u'\right) \cap (z_1, z_2)\right).$$

Note that $c_* > z_1$, because if $c_* = z_1$ it would exist a sequence $(c_n)_n$ in $(Z(u') \cap (z_1, z_2))$ converging to z_1 and

$$u(z_1) = u'(z_1) = \lim_{n \to \infty} \frac{u(c_n) - u(z_1)}{c_n - z_1} = 0.$$

So, $\left(u, \varphi_p(u')\right)$ is a solution to the initial value problem with a locally Lipshitzian second member

$$\begin{cases} v' = \varphi_{p'}(w) \\ w' = -g(v) \\ v(z_1) = w(z1) = 0 \end{cases}$$

as well as $(0,0)$. Thus, $u = 0$ in $[z_1, z_2]$ which is impossible since u is a nontrivial solution.

Also, $g(u(c_*)) \neq 0$, this is due to the fact that if $g(u(c_*)) = 0$ then $\left(u, \varphi_p(u')\right)$ will be a solution to the initial value problem with a locally Lipshitzian right-handside

$$\begin{cases} v' = \varphi_{p'}(w) \\ w' = -g(v) \\ v(c_*) = w(c_*) = 0 \end{cases}$$

as well as $(u(c_*), 0)$. Thus, $u(x) = u(c_*) = u(z_1) = 0$ for all $x \in [z_1, z_2]$ which is impossible since $u > 0$ in $[z_1, z_2]$. More precisely, $g(u(c_*)) > 0$ and reachs at c_* a local maximum.

Now, suppose that $(Z(u') \cap (c_*, z_2)) \neq \emptyset$ and let $c_1 = \min(Z(u') \cap (c_*, z_2))$. Then, there exists $c_2 \in (z_1, c_*)$ such that $u(c_2) = u(c_1)$. It follows from the energy equation that

$$\left|u'(c_2)\right|^p = \left|u'(c_2)\right|^p - \left|u'(c_1)\right|^p = p'(G(u(c_1)) - G(u(c_2))) = 0.$$

This is impossible since $c_* = \inf(Z(u') \cap (z_1, z_2))$.

Step 3: In this step, we will prove that each hump of u is symmetrical about the center of its interval of definition. Let z_1 and z_2 be as in the above Step. So, it suffices to show that $u(x) = u(z_1 + z_2 + x)$ for all $x \in [z_1, z_2]$.

Set for all $x \in [z_1, z_2]$ $v(x) = u(z_1 + z_2 - x)$. One can easily verify that $\left(u, \varphi_p(u')\right)$ and $\left(v, \varphi_p(v')\right)$ are solutions to the initial value problem with a locally Lipshitzian right-handside

$$\begin{cases} v' = \varphi_{p'}(w) \\ w' = -g(v) \\ v(z_1) = 0 \\ w(z_1) = u'(z_1). \end{cases}$$

Hence $u = v$ in $[z_1, z_2]$.

Step 4: Let $z_1 < z_2 < z_3 < z_4$ be four zeros of u such that $u > 0$ in (z_1, z_2) and in (z_3, z_4). In this step we will prove that $u_{[z_1,z_2]} = u_{[z_3,z_4]}$ ie: for all $x \in [z_3, z_4]$ $u(x) = u(z_1 + (x - z_3))$.

Denote by v the translation of $u_{[z_1,z_2]}$ to the interval J whose origin is z_3. So, $J = [z_3, z_3 + (z_2 - z_1)]$ and $v(x) = u(z_1 + (x - z_3))$ for all $x \in J$. Hence it suffices to prove that $J = [z_3, z_4]$ and $u(x) = v(x)$ for all $x \in [z_3, z_4]$.

Observe that $\left(v, \varphi_p(v')\right)$ and $\left(u, \varphi_p(u')\right)$ are solutions to the Cauchy problem with a locally

Lipschitzian right-handside

$$\begin{cases} v' = \varphi_{p'}(w) \\ w' = -g(v) \\ v(z_3) = 0 \\ w(z_3) = u'(z_1). \end{cases}$$

So, $u(x) = v(x)$ for all $x \in J \cap [z_3, z_4]$.

Thus, let us prove that $J = [z_3, z_4]$. To do this, it suffices to show that $z_4 = z_3 + (z_2 - z_1)$.

Suppose that $z_4 > z_3 + (z_2 - z_1)$, then $v(z_3 + (z_2 - z_1)) = u(z_2) = 0$. In the other hand, as $z_3 < z_3 + (z_2 - z_1) < z_4$, $u(z_3 + (z_2 - z_1)) > 0$ and this is impossible since $z_3 + (z_2 - z_1) \in [z_3, z_4] \cap J$ and $u(z_3 + (z_2 - z_1)) \neq v(z_3 + (z_2 - z_1))$. So $z_4 = z_3 + (z_2 - z_1)$.

Step 5: conclusion

- If $u'(0) \neq 0$, let $z_1 < z_2 < z_3$ be three consecutive zeros of u. It follows from the energy equation (1.3)

$$\left|u'(z_1)\right| = \left|u'(z_2)\right| = \left|u'(z_3)\right| \neq 0.$$

Hence, if $u'(z_1) > 0$, $u'(2) < 0$ and if $u'(z_1) < 0$, $u'(2) > 0$. This means that if $u > 0$ in (z_1, z_2) then $u < 0$ in (z_2, z_3) and if $u < 0$ in (z_1, z_2) then $u > 0$ in (z_2, z_3). In this case $u \in \left(\bigcup_{k \geq 1} A_k\right)$.

- If $u'(0) = 0$ then $g(0) \neq 0$ otherwise u will be equal to 0 ($\left(u, \varphi_p(u')\right)$ and $(0, 0)$ will be solution to the system (1.11) with the initial conditions $v(0) = w(0) = 0$).

Let $z_1 < z_2 < z_3$ be three consecutive zeros of u; again from the energy equation (1.3) we deduce that

$$u'(z_1) = u(z_1) = u'(z_2) = u(z_2) = u'(z_3) = u(z_3) = 0.$$

Proceeding as in step 4, we prove that the translation of $u_{[z_1, z_2]}$ to the interval $J = [z_2, z_2 + (z_2 - z_1)]$ defined by $v(x) = u(z_1 + (x - z_2))$ are solutions to a Cauchy problem with a locally Lipschitzian right-handside. So, $u = v$ and u and v have the same sign. Thus, $u \in \left(\bigcup_{k \geq 1} B_k\right)$.

1.5 Applications

1.5.1 Eigenvalues problem

For $p > 1$, consider the eigenvalues problem

$$\begin{cases} -\left(\varphi_p(u')\right)' = \lambda \varphi_p(u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1.12}$$

where λ is a real parameter and $\varphi_p(x) = |x|^{p-2}x$ for all $x \in \mathbb{R}$.

Using the quadrature method we will prove the following result

Theorem 13 There exists a sequence $(\lambda_k(p))_{k \geq 1}$ of pseudo-eigenvalues of (1.12) such that

$$0 < \lambda_1(p) < \lambda_2(p) < \ldots\ldots. < \lambda_k(p) < \ldots\ldots..$$

and (1.12) admits a nontrivial solution (λ, u_λ) if and only if there exist an integer $k \geq 1$ such that $\lambda = \lambda_k(p)$ and in this case $u_\lambda \in A_k$. Moreover if S is the solution set to (1.12) then $S \cap (\mathbb{R} \times A_k) = \mathbb{R} \times \{u_\lambda\}$.

Proof Set $g_\lambda(x) = \lambda \varphi_p(x)$. Observe that if $\lambda \leq 0$, G_λ defined by $G_\lambda(x) = \int_0^x g_\lambda(t)\,dt$ is increasing on $(-\infty, 0)$, decreasing on $(0, +\infty)$ and $G_\lambda(0) = 0$. So, $D_\pm = D_\pm = D^*_\pm = \emptyset$. In this case $S = \{(\lambda, 0) : \lambda \in \mathbb{R}\}$.

- If $\lambda > 0$ then $x g_\lambda(x) > 0$ for all $x \in \mathbb{R}^*$. We deduce from Theorem 11 that $S \subset (\mathbb{R} \times \{0\}) \cup \left[\bigcup_{k \geq 1} (\mathbb{R} \times A_k)\right]$.

Also, note that g_λ is an odd function and G_λ is an even function; so, for all $E > 0$, $r_-(E) = -r_+(E)$ and

$$\begin{aligned} T_+(E) &= T_-(E) = \int_0^{r_+(E)} \frac{d\xi}{\{p'(E^p - p'G(\xi))\}^{\frac{1}{p}}} \\ &= [(p-1)\lambda]^{-\frac{1}{p}} \int_0^1 \frac{ds}{\{1 - s^p\}^{\frac{1}{p}}}. \end{aligned}$$

Hence, it follows from Theorem 10 that problem (11) admits a solution (λ, u_λ) belonging to $\mathbb{R} \times A_k$ if and only if

$$\lambda = \lambda_k(p) = (p-1)\left(2k \int_0^1 \frac{ds}{\{1 - s^p\}^{\frac{1}{p}}}\right)^p.$$

■

1.5.2 The Fuçik spectrum

For $p > 1$ consider the boundary value problem

$$\begin{cases} -\left(\varphi_p(u')\right)' = \beta\varphi_p(u^+) - \alpha\varphi_p(u^-) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1.13}$$

where α, β are real parameters, $\varphi_p(x) = |x|^{p-2}x$ for all $x \in \mathbb{R}$, $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$.

Using the quadrature method we will prove

Theorem 14 Problem (1.13) admits a nontrivial solution $u_{\alpha\beta}$ if and only if

(i) $\beta = \lambda_1(p)$ and $\alpha \in \mathbb{R}$; in this case $u_{\alpha\beta} \in A_1^+$.

(ii) $\alpha = \lambda_1(p)$ and $\beta \in \mathbb{R}$; in this case $u_{\alpha\beta} \in A_1^-$.

(iii) $\left(\frac{\lambda_k(p)}{\beta}\right)^{\frac{1}{p}} + \left(\frac{\lambda_k(p)}{\alpha}\right)^{\frac{1}{p}} = 1$; in this case $u_{\alpha\beta} \in A_{2k}^+$ or A_{2k}^-.

(iv) $\left(\frac{\lambda_{k+1}(p)}{\beta}\right)^{\frac{1}{p}} + \left(\frac{\lambda_k(p)}{\alpha}\right)^{\frac{1}{p}} = 1$; in this case $u_{\alpha\beta} \in A_{2k+1}^+$.

(v) $\left(\frac{\lambda_k(p)}{\beta}\right)^{\frac{1}{p}} + \left(\frac{\lambda_{k+1}(p)}{\alpha}\right)^{\frac{1}{p}} = 1$; in this case $u_{\alpha\beta} \in A_{2k+1}^-$.

Moreover if $S_{\alpha\beta}$ is the set of solutions for a fixed α, β then in each case

$$S_{\alpha\beta} = \left\{ t u_{\alpha\beta},\ t \in \mathbb{R}^+ \right\}$$

Proof We have

$$g_{\alpha\beta}(x) = \begin{cases} -\alpha(-x)^{p-1} & \text{for } x \le 0 \\ \beta x^{p-1} & \text{for } x \ge 0 \end{cases}$$

and

$$G_{\alpha\beta}(x) = \begin{cases} \frac{\alpha}{p}|x|^p & \text{for } x \le 0 \\ \frac{\beta}{p}x^p & \text{for } x \ge 0. \end{cases}$$

As in the proof of Theorem 13, if $\alpha, \beta \le 0$ then $\tilde{D}_\pm = D_\pm = D^*_\pm = \emptyset$; in this case $S_{\alpha\beta} = \{0\}$. If $\alpha, \beta > 0$ then $\tilde{D}_\pm = D_\pm = D^*_\pm = (0, +\infty)$ and $S_{\alpha\beta} \subset \{0\} \cup \left(\bigcup_{k \ge 1} A_k\right)$.

Straighforward computations leads to

$$T_{+}(E) = [(p-1)\beta]^{-\frac{1}{p}} \int_0^1 \frac{ds}{\{1-s^p\}^{\frac{1}{p}}}$$

$$T_{-}(E) = [(p-1)\alpha]^{-\frac{1}{p}} \int_0^1 \frac{ds}{\{1-s^p\}^{\frac{1}{p}}}.$$

A straighforward appllication of Theorem 10 implies (i), (ii), (iii), (iv) and (v) of the theorem 14. ■

Remark 1 Notice that if the nonlinearity g is odd, $r_{+} = r_{-}$ and $D_{+} = D_{-}$, so in some chapter in this thesis, in which we use the quadrature method we will change the notations, r_{+}, r_{-} by r and D_{+}, D_{-} by D.

Chapter 2

Multiplicity of solutions for p-Laplacian B.V.P. with cubic-like nonlinearities

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2.1 Introduction

Recently (1995), Ubilla [111] considered a boundary value problem involving the one-dimensional generalized p-Laplacian, explicitly, he considered the problem,

$$-\left(a\left(\left|u'\right|^{p}\right)\left|u'\right|^{p-2}u'\right)' = f(u) \quad \text{in} \quad (0,1)\,; \quad u(0) = u(1) = 0, \tag{2.1}$$

where $p > 1$, $a \in C(\mathbb{R}^+, \mathbb{R})$ is not necessarily equal to 1, and the nonlinearity f is a continuous and odd function such that,

$$\begin{array}{c} \text{there exist constants } \theta \in \left(0, \frac{1}{p}\right) \text{ and } t_0 \geq 0 \text{ such that} \\ \theta t f(t) \geq \int_0^t f(\tau)\, d\tau > 0, \quad \forall |t| \geq t_0. \end{array} \tag{2.2}$$

Under some additional conditions on a, he showed that there exists $k_0 \in \mathbb{N}$ such that Problem (2.1) admits at least two weak solutions in A_k for each integer $k \geq k_0$. One may observe that (2.2) implies that

$$tf(t) > 0, \text{ for } |t| \text{ sufficiently large.} \tag{2.3}$$

In this chapter, we restrict ourselves to $a(t) \equiv 1, \forall t \geq 0$ and we consider an odd function f which satisfies the reversed inequality in (2.3). Under some additional conditions we show (roughly speaking) that the same conclusion as in Ubilla's work still holds.

We consider the boundary value problem,

$$-\left(\varphi_p\left(u'\right)\right)' = \lambda f(u) \quad \text{in } (0,\ 1);\ u(0) = u(1) = 0,$$

where $\varphi_p(y) = |y|^{p-2}y,\ \lambda > 0,\ p > 1$ are two real parameters and $f \in C(\mathbb{R}, \mathbb{R})$ is an odd cubic-like nonlinearity. That is, f has the same shape as the cubic polynomial function $u \mapsto u\left(\alpha^2 - u^2\right)$ for some positive α.

In order to study this problem, we make use of the quadrature method (time map approach) as was used in the previous chapter. Section 2.2 is devoted to the statements of our main results; before proving them in Section 2.4, we set in Section 2.3 some preliminary lemmas. Some comments are the content of Section 2.5.

2.2 Notation and main results

Consider the boundary value problem,

$$-\left(\varphi_p\left(u'\right)\right)' = \lambda f(u) \quad \text{in } (0,\ 1);\ u(0) = u(1) = 0, \tag{2.4}$$

where $\varphi_p(y) = |y|^{p-2}y,\ \lambda,\ p$ are two real parameters such that $\lambda > 0$ and $p > 1$. $f \in C(\mathbb{R}, \mathbb{R})$ is a function satisfying:

(H1) f is odd and there exists some constant $\alpha > 0$ such that,

$$f(x) > 0,\ \forall x \in (0,\ \alpha),\ f(\alpha) = 0 \text{ and } f(x) < 0,\ \forall x \in (\alpha,\ +\infty).$$

(H2) There exist $(m,\ M) \in \left(\mathbb{R}_+^*\right)^2$ and $\delta \in (0,\ \alpha)$ such that,

$$-M \leq \frac{f(\xi) - f(\alpha)}{\xi - \alpha} \leq -m, \quad \forall \xi \in (\alpha - \delta,\ \alpha).$$

(H3) The function $x \mapsto p\int_0^x f(t)\,dt - xf(x)$ is strictly increasing in $(0,\ \alpha)$.

(H4) $\displaystyle\lim_{s \to 0} \frac{f(s)}{\varphi_p(s)} = \sigma \in \mathbb{R}$.

An example of such a function is given by $f(u) = \varphi_p(u)\left(\alpha^2 - u^2\right)$. The method we use (quadrature method) enable us to look for solutions of (2.4) in some prescribed subsets of $C^1([0,\ 1])$ defined in chapter 0.

Recall that the first eigenvalue of the problem,

$$-\left(\varphi_p\left(u'\right)\right)' = \lambda\varphi_p(u) \quad \text{in } (0,\ 1)\,;\ u(0) = u(1) = 0,$$

is given by $\lambda_1(p) = (p-1)\left[2\int_0^1 \{1-\xi^p\}^{-\frac{1}{p}}\,d\xi\right]^p = (p-1)\left(2\pi/\left(p\sin\left(\frac{\pi}{p}\right)\right)\right)^p$ and the other eigenvalues constitute the sequence $\lambda_1(p) < \lambda_2(p) < \cdots < \lambda_k(p) < \cdots,\ \lambda_n(p) = n^p\lambda_1(p)$. (see Chapter 1, Theorem 13).

Denote S_λ the solution set of problem (2.4). The following theorems are the main results of this chapter.

Theorem 15 Assume that $1 < p \le 2$ and f satisfies (H1)-(H4). If $\sigma > 0$, then for each integer $k \ge 1$ and $\lambda > 0$, Problem (2.4) admits at least a solution in A_k if and only if $\lambda > \sigma^{-1}\lambda_k(p)$ and in this case, Problem (2.4) admits exactly two solutions $\{u_i,\ -u_i\}$ in A_i for all $i \in \{1,\ \cdots,\ k\}$. Moreover, there is no additional solution to (2.4) (further to the trivial one) provided f is locally Lipschitzian, more explicitly:

$$S_\lambda = \{0\} \cup \left(\bigcup_{i=1}^{k}\{u_i,\ -u_i\}\right),\quad \sigma^{-1}\lambda_k(p) < \lambda \le \sigma^{-1}\lambda_{k+1}(p).$$

Let $F(s) = \int_0^s f(t)\,dt$ and for any $p > 2$, $J_p = \int_0^\alpha (F(\alpha) - F(\xi))^{-1/p}\,d\xi$. Also, for each integer $k \ge 1$, let $\mu_k(p) := (p-1)(2kJ_p)^p/p$.

Theorem 16 Assume that $p > 2$ and f satisfies (H1)-(H4). Then, for each integer $k \ge 1$, if $\sigma > \lambda_k/\mu_k$ and $\lambda > 0$, Problem (2.4) admits at least a solution in A_k if and only if $\sigma^{-1}\lambda_k(p) < \lambda \le \mu_k(p)$, and in this case $S_\lambda \cap A_k = \{u_k,\ -u_k\}$.

2.3 Some preliminary lemmas

This section is devoted to some preliminary lemmas.

Lemma 17 Consider the equation in $s > 0$,

$$E^p - p'\lambda F(s) = 0 \tag{2.5}$$

where $E > 0$, $\lambda > 0$ are real parameters and $F(s) = \int_0^s f(t)\,dt$. Then,

(i) If $E > E_*(p, \lambda) := (p'\lambda F(\alpha))^{1/p}$, equation (2.5) admits no (positive) zero.

(ii) If $E = E_*(p, \lambda)$, equation (2.5) admits a unique zero $s_* = \alpha$.

(iii) If $E < E_*(p, \lambda)$, equation (2.5) admits, in the open interval $(0, \alpha)$, a unique zero $s_* = s_*(p, \lambda, E)$. Moreover,

(a) The function $E \mapsto s_*(p, \lambda, E)$ is C^1 in $(0, E_*(p, \lambda))$, and

$$\frac{\partial s_*}{\partial E}(p, \lambda, E) = \frac{(p-1)E^{p-1}}{\lambda f(s_*(p, \lambda, E))} > 0, \ \forall E \in (0, E_*(p, \lambda)),$$

(b) $\lim_{E \to 0^+} s_*(p, \lambda, E) = 0$ and $\lim_{E \to E_*} s_*(p, \lambda, E) = \alpha$,

(c) $\lim_{E \to 0^+} (s_*(p, \lambda, E)/E) = ((p-1)/\lambda\sigma)^{1/p}$ and $\lim_{E \to E_*} (s_*(p, \lambda, E)/E) = \alpha(p'\lambda F(\alpha))^{-1/p}$.

Proof Assertions (i), (ii) and the uniqueness part of (iii) follow from a study of the variations of the function $s \mapsto H(p, \lambda, E, s) := E^p - p'\lambda F(s)$. In fact, its minimum value is $H(p, \lambda, E, \alpha) = E^p - p'\lambda F(\alpha)$.

Now, consider the real valued function,

$$(p, \lambda, E, s) \longmapsto H(p, \lambda, E, s)$$

defined in $\Omega = (1, +\infty) \times \mathbb{R}_+^2 \times (0, \alpha)$. One has $H \in C^1(\Omega)$ and

$$\frac{\partial H}{\partial s}(p, \lambda, E, s) = -p'\lambda f(s) < 0 \text{ in } \Omega,$$

and one may observe that $s_*(p, \lambda, E)$ belongs to the open interval $(0, \alpha)$ and satisfies, from its definition,

$$H(p, \lambda, E, s_*(p, \lambda, E)) = 0. \tag{2.6}$$

So, we can make use of the implicit functions theorem to show (in particular) that the function $E \mapsto s_*(p, \lambda, E)$ is $C^1((0, E_*(p, \lambda)), \mathbb{R})$ and obtain the expression of $\frac{\partial s_*}{\partial E}(p, \lambda, E)$ given in (a). Hence, for any fixed $p > 1$ and $\lambda > 0$, the function defined in $(0, E_*(p, \lambda))$ by $E \mapsto s_*(p, \lambda, E)$ is strictly increasing and bounded (by 0 and α). Then, the limit $\lim_{E \to 0^+} s_*(p, \lambda, E) = l_0$ (resp. $\lim_{E \to E_*} s_*(p, \lambda, E) = l_*$) exists as real number and $0 \le l_0 < l_* \le \alpha$.

One may observe that, for any fixed $p > 1$ and $\lambda > 0$, the function,

$$(E, s) \mapsto H(p, \lambda, E, s),$$

is continuous in $[0, E_*(p, \lambda)] \times [0, +\infty)$ and for any $E \in (0, E_*(p, \lambda))$, $s_*(p, \lambda, E)$ satisfies (2.6). So, by passing to the limit in (2.6) as E tends to 0^+ (resp. $E_*(p, \lambda)$) we get,

$$0 = \lim_{E \to 0^+} H(p, \lambda, E, s_*(p, \lambda, E)) = H(p, \lambda, 0, l_0),$$

$$(\text{resp. } 0 = \lim_{E \to E_*} H(p, \lambda, E, s_*(p, \lambda, E)) = H(p, \lambda, E_*(p, \lambda), l_*)).$$

Hence, l_0 (resp. l_*) is a solution, belonging to $[0, \alpha]$, to the following equation in s,

$$H(p, \lambda, 0, s) = 0, \qquad (\text{resp. } H(p, \lambda, E_*, s) = 0).$$

The variations of the function $s \mapsto H(p, \lambda, 0, s)$ (resp. $s \mapsto H(p, \lambda, E_*, s)$) show that $l_0 = 0$ (resp. $l_* = \alpha$).

Now, dividing equation (2.6) by E^p yield

$$F(s_*(p, \lambda, E))/E^p = 1/\left(p'\lambda\right), \tag{2.7}$$

with,

$$F(s_*(p, \lambda, E)) = \int_0^{s_*(p, \lambda, E)} f(t)\,dt = s_*(p, \lambda, E)\int_0^1 f(s_*(p, \lambda, E)\,t)\,dt;$$

so,

$$(s_*(p, \lambda, E)/E)^p = \left(p'\lambda \int_0^1 \frac{f(s_*(p, \lambda, E)\,t)}{(s_*(p, \lambda, E)\,t)^{p-1}} t^{p-1}dt\right)^{-1};$$

then (according to (H4)),

$$\lim_{E \to 0^+} (s_*(p, \lambda, E)/E)^p = \left(p'\lambda\sigma \int_0^1 t^{p-1}dt\right)^{-1} = (p-1)/(\lambda\sigma).$$

Moreover, from (2.7),

$$(s_*(p, \lambda, E)/E)^p = \frac{(s_*(p, \lambda, E))^p}{p'\lambda F(s_*(p, \lambda, E))},$$

then,

$$\lim_{E \to E_*} (s_*(p, \lambda, E)/E)^p = \alpha^p \left(p'\lambda F(\alpha)\right)^{-1},$$

which completes the proof of Lemma 17. ■

At present, we are ready to compute $X(p,\lambda,E)$ for any $E>0$ and $\lambda>0$, as defined in Chapter 1. In fact,

$$X(p,\lambda,E)=\begin{cases}(0,+\infty) & \text{if } E>E_*(p,\lambda)\\ (0,\alpha) & \text{if } E=E_*(p,\lambda)\\ (0,s_*(p,\lambda,E)) & \text{if } 0<E<E_*(p,\lambda),\end{cases}$$

where $s_*(p,\lambda,E)$ is defined in Lemma 17. Then,

$$r(p,\lambda,E):=\sup X(p,\lambda,E)=\begin{cases}+\infty & \text{if } E>E_*(p,\lambda)\\ \alpha & \text{if } E=E_*(p,\lambda)\\ s_*(p,\lambda,E) & \text{if } 0<E<E_*(p,\lambda),\end{cases}$$

and from Lemma 17, one deduces the following limits,

$$\lim_{E\to 0^+} r(p,\lambda,E)=0 \quad \text{and} \quad \lim_{E\to E_*} r(p,\lambda,E)=\alpha \tag{2.8}$$

$$\begin{aligned}&\lim_{E\to 0^+} r(p,\lambda,E)/E=((p-1)/\lambda\sigma)^{1/p} \quad \text{and,}\\ &\lim_{E\to E_*} r(p,\lambda,E)/E=\alpha\,(p'\lambda F(\alpha))^{-1/p}.\end{aligned} \tag{2.9}$$

On the other hand,

$$0<r(p,\lambda,E)<+\infty \quad \text{if and only if} \quad 0<E\le E_*(p,\lambda),$$

and from hypothesis (H1) and the range of $E\mapsto r(p,\lambda,E)$ one deduces,

$$f(r(p,\lambda,E))>0 \quad \text{if and only if} \quad 0<E<E_*(p,\lambda),$$

then,

$$\begin{aligned}D &:= \{E\ge 0 : 0<r(p,\lambda,E)<+\infty \text{ and } f(r(p,\lambda,E))\ge 0\}\\ &= (0,E_*(p,\lambda)].\end{aligned}$$

Define, for $E\in D$, the time map T by,

$$T(p,\lambda,E):=\int_0^{r(p,\lambda,E)}\left\{E^p-p'\lambda F(\xi)\right\}^{-1/p}d\xi,\quad E\in D.$$

A simple change of variables shows that,

$$T(p,\ \lambda,\ E) = r(p,\ \lambda,\ E)\int_0^1 \left\{E^p - p'\lambda F(r(p,\ \lambda,\ E)\xi)\right\}^{-1/p} d\xi, \tag{2.10}$$

which can be written as:

$$T(p,\ \lambda,\ E) = (r(p,\ \lambda,\ E)/E)\int_0^1 \left\{1 - p'\lambda F(r(p,\ \lambda,\ E)\xi)/E^p\right\}^{-1/p} d\xi. \tag{2.11}$$

Lemma 18

$$\lim_{E\to 0^+} T(p,\ \lambda,\ E) = \frac{1}{2}\left(\frac{\lambda_1}{\lambda\sigma}\right)^{1/p} \quad \text{and} \quad \lim_{E\to E_*} T(p,\ \lambda,\ E) = (p'\lambda)^{-1/p} J_p,$$

where

$$J_p := \int_0^\alpha \{F(\alpha) - F(\xi)\}^{-1/p} d\xi \in (0,\ +\infty].$$

Moreover,

$$J_p < +\infty \iff p > 2.$$

Proof To compute the first limit, let us write,

$$F(r(p,\ \lambda,\ E)\xi)/E^p = (r(p,\ \lambda,\ E)/E)^p \xi^p \int_0^1 \frac{f(r(p,\ \lambda,\ E)\xi t)}{(r(p,\ \lambda,\ E)\xi t)^{p-1}} t^{p-1} dt;$$

so,

$$\lim_{E\to 0} F(r(p,\ \lambda,\ E)\xi)/E^p = ((p-1)/\lambda\sigma)\,\xi^p \int_0^1 \sigma t^{p-1} dt = \xi^p/p'\lambda;$$

then,

$$\lim_{E\to 0^+} 1 - p'\lambda F(r(p,\ \lambda,\ E)\xi)/E^p = 1 - \xi^p;$$

hence, from (2.11) we have,

$$\lim_{E\to 0^+} T(p,\ \lambda,\ E) = ((p-1)/\lambda\sigma)^{1/p}\int_0^1 (1-\xi^p)^{-1/p} d\xi = \frac{1}{2}\left(\frac{\lambda_1}{\lambda\sigma}\right)^{1/p}.$$

To compute the second limit, one may pass to the limit as E tends to E_* in (2.11) and make use of (2.9) and the definition of E_* together with (2.8). To prove the last assertion, let us remark that,

$$J_p = \int_0^{\alpha-\delta} (F(\alpha) - F(\xi))^{-1/p} d\xi + \int_{\alpha-\delta}^{\alpha} (F(\alpha) - F(\xi))^{-1/p} d\xi.$$

The first integral converges because the integrand function is continuous on the compact interval $[0, \alpha - \delta]$. To study the convergence of the second integral we claim that,

$$\frac{m}{2}(\alpha - \xi)^2 \leq F(\alpha) - F(\xi) \leq \frac{M}{2}(\alpha - \xi)^2, \quad \forall \xi \in (\alpha - \delta, \alpha).$$

Proof of the claim: Let for any $x > 0$,

$$h_x(\xi) = F(\alpha) - F(\xi) - x(\alpha - \xi)^2, \quad \xi \in (\alpha - \delta, \alpha].$$

We have, for any $\xi \in (\alpha - \delta, \alpha)$,

$$\frac{dh_x}{d\xi}(\xi) = (\alpha - \xi)\left(2x + \frac{f(\xi) - f(\alpha)}{\xi - \alpha}\right),$$

and from (H2), it follows that,

$$\frac{dh_x}{d\xi}(\xi) \geq 0, \ \forall \xi \in (\alpha - \delta, \alpha), \text{ if } 2x = M,$$

and

$$\frac{dh_x}{d\xi}(\xi) \leq 0, \ \forall \xi \in (\alpha - \delta, \alpha), \text{ if } 2x = m.$$

Hence $h_{M/2}$ (resp. $h_{m/2}$) is increasing (resp. decreasing) in $(\alpha - \delta, \alpha)$ and because of,

$$h_x(\alpha) = 0, \ \forall x \in \mathbb{R},$$

it follows that $h_{M/2}(\xi) \leq 0, \ \forall \xi \in (\alpha - \delta, \alpha)$ (resp. $h_{m/2}(\xi) \geq 0, \ \forall \xi \in (\alpha - \delta, \alpha)$) which completes the proof of the claim.

This claim implies that,

$$(M/2)^{-\frac{1}{p}} \int_{\alpha - \delta}^{\alpha} (\alpha - \xi)^{-\frac{2}{p}} d\xi \leq \int_{\alpha - \delta}^{\alpha} (F(\alpha) - F(\xi))^{-1/p} d\xi \leq (m/2)^{-\frac{1}{p}} \int_{\alpha - \delta}^{\alpha} (\alpha - \xi)^{-\frac{2}{p}} d\xi$$

and from the fact,

$$\int_{\alpha - \delta}^{\alpha} (\alpha - \xi)^{-2/p} d\xi < +\infty \iff p > 2,$$

Lemma 18 follows. ■

Lemma 19 The function $E \mapsto T(p, \lambda, E)$ is strictly increasing in D.

Proof From (2.10), we have

$$\frac{\partial T}{\partial E}(p,\ \lambda,\ E) = \frac{\partial r}{\partial E}(p,\ \lambda,\ E)\int_0^1 \{\cdots\}^{-\frac{1}{p}}d\xi + r(p,\ \lambda,\ E)\int_0^1 \frac{\partial}{\partial E}\{\cdots\}^{-\frac{1}{p}}d\xi.$$

Simple computations yield,

$$\frac{\partial T}{\partial E}(p,\ \lambda,\ E) = \frac{E^{p-1}}{f(r(p,\ \lambda,\ E))}\int_0^1 \frac{K(r(p,\ \lambda,\ E)) - K(r(p,\ \lambda,\ E)\,\xi)}{\{E^p - p'\lambda F(r(p,\ \lambda,\ E)\,\xi)\}^{1+\frac{1}{p}}}d\xi \tag{2.12}$$

where $K(x) = pF(x) - xf(x)$. According to hypothesis (H3) the proof of Lemma 19 is complete. ■

2.4 Proof of the main results

2.4.1 Case $1 < p \leq 2$

In this case, for each fixed $\lambda > 0$, the function $E \mapsto T(p,\ \lambda,\ E)$ is strictly increasing in $D = (0,\ E_*(p,\ \lambda))$ and

$$\lim_{E\to 0} T(p,\ \lambda,\ E) = \frac{1}{2}\left(\frac{\lambda_1}{\lambda\sigma}\right)^{1/p}, \quad \text{and} \quad \lim_{E\to E_*} T(p,\ \lambda,\ E) = +\infty.$$

Then, for each integer $k = 1,\ 2,\ \cdots$, the equation in $E \in D$

$$kT(p,\ \lambda,\ E) = \frac{1}{2}$$

admits a (unique) solution in D if and only if,

$$\inf_{E\in D}(kT(p,\ \lambda,\ E)) < \frac{1}{2}. \tag{2.13}$$

We have

$$\inf_{E\in D}(kT(p,\ \lambda,\ E)) = \frac{k}{2}\left(\frac{\lambda_1}{\lambda\sigma}\right)^{1/p},$$

so, condition (2.13) is equivalent to the following

$$\lambda > \frac{1}{\sigma}\lambda_k(p). \tag{2.14}$$

One may observe that the sequence $(\lambda_n(p))_{n\geq 1}$ is strictly increasing; so if (2.14) is satisfied, then we necessarily have

$$\lambda > \sigma^{-1}\lambda_k(p) > \sigma^{-1}\lambda_{k-1}(p) > \cdots > \sigma^{-1}\lambda_1(p).$$

Hence if (2.13) is satisfied for $k = k_0$ then it is also satisfied for all $k \in \{1, \cdots, k_0\}$. This remark completes the proof of Theorem 15. ■

2.4.2 Case $p > 2$

In this case, for each $\lambda > 0$ fixed, the function $E \mapsto T(p, \lambda, E)$ is strictly increasing in $D = (0, E_*(p, \lambda)]$ and,

$$\lim_{E \to 0} T(p, \lambda, E) = \frac{1}{2}\left(\frac{\lambda_1}{\lambda\sigma}\right)^{1/p}, \quad \text{and} \quad \lim_{E \to E_*} T(p, \lambda, E) = \left(p'\lambda\right)^{-1/p} J_p < +\infty$$

where J_p is as defined in Lemma 18. Then, for each integer $k = 1, 2, \cdots$, the equation in $E \in D$,

$$kT(p, \lambda, E) = \frac{1}{2},$$

admits a (unique) solution in D if and only if,

$$\inf_{E \in D}(kT(p, \lambda, E)) < \frac{1}{2} \leq \max_{E \in D}(kT(p, \lambda, E)). \tag{2.15}$$

We have,

$$\inf_{E \in D}\{kT(p, \lambda, E)\} = \frac{k}{2}\left(\frac{\lambda_1}{\lambda\sigma}\right)^{1/p}$$

$$\max_{E \in D}(kT(p, \lambda, E)) = k\left(p'\lambda\right)^{-1/p} J_p,$$

so, condition (2.15) is equivalent to,

$$\frac{1}{\sigma}\lambda_k(p) < \lambda \leq \mu_k.$$

The proof of Theorem 16 is complete. ■

2.5 Comments

The nonlinearity f must be positive at the maximum value of any solution u to the problem if u has at least a positive hump. That is, if we consider $C([0, 1], \mathbb{R})$ endowed with the sup-norm and u is a solution having at least a positive hump, then its norm must belong to the positivity domain of f. Hence, Ubilla in [111] was looking for the solutions with large norm, outside the ball $B(0, t_0]$. In contrast, we were looking for solutions in the ball $B(0, \alpha)$. In that case, there is no possibility to find solutions with positive hump outside $B(0, \alpha]$ since f is negative in $(\alpha, +1)$.

Chapter 3

Exact number of positive solutions for a class of quasilinear boundary value problems

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3.1 Introduction

The study of positive solutions to classes of semilinear boundary value problems, known as positone problems, has been undertaken by several authors over the last thirty years (see for example Ambrosetti & Hess [16]; de-Figueiredo [48]; Korman & Ouyang [75]; Laetsch [76]; P. L. Lions [78]; de-Mottoni & Tesei [52]; Njoku & Zanolin [83]; Rabinowitz [92]; Shivaji [103]; and references therein). Such a study was initiated by Keller & Cohen [74].

Positive solutions for p-Laplace equations with Dirichlet boundary conditions were studied by de-Coster [47]; Huang [71]; Kaper et al. [73]; Manásevich et al.[81].

In this chapter, we shall discuss the exact number of positive solutions for the one-dimensional class of quasilinear boundary value problems of the form :

$$\begin{cases} -\left(\varphi_p(u')\right)' = \lambda f(u), & \text{in } (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{3.1}$$

where $\varphi_p(x) := |x|^{p-2}x$, $p > 1$, $\lambda > 0$ and f is a continuous function which vanishes at least twice, vanishes only once or is strictly positive in $(0, +\infty)$.

Throughout this chapter, the method used is the so-called time mapping approach (quadrature method). This time mapping gives the time needed by a solution of (3.1) to reach its first maximum starting from zero as a function of its initial slope E (as was used, for instance, by de-Mottoni & Tesei [53]; Guedda & Veron [69]; Ubilla [111]; Addou & Ammar Khodja [5], Addou et al. [12], among others) or as a function of its first maximum value ρ (as was used, for instance, by Laetsch, [76]; Shivaji [103]; Manásevich et al. [81], among many others). This method consists in resolving an equivalent scalar equation to (3.1); $T(E) = 1/2$. (or $T(\rho) = 1/2$). Since we are interested in the exact number of positive solutions to (3.1), we need the exact variations of the time-map $E \mapsto T(E)$ (or $\rho \mapsto T(\rho)$) over its entire definition domain. In general, by making use of Rolle's theorem in a way or in an other, we can determine a lower bound of the number of the derivative sign changes of the corresponding time map, but it is more difficult to obtain the exact number, and some numerical calculations are usually necessary to. For example, when encountering this difficulty, de-Mottoni & Tesei [52], reported ,

> $\cdots$ It seems to be rather difficult to improve the results obtained so far by analytical considerations; hence, in order to fill the gap mentioned before, we performed a numerical evaluation of the function $F(\nu, g)\cdots$
>
> $\cdots$ the result is shown in Fig. 5 : there is good evidence for the existence of $\cdots$
>
> $\cdots$ Taking into account this information, the diagram of the solutions of the differential problem takes on a complete aspect, which is shown in Fig. 6.

Also, concerning the same kind of difficulty, the same authors de-Mottoni & Tesei [53] reported

> $\cdots$ Pursuing the analytical investigation of further properties of Z_m is extremely involved; thus numerical computations were performed to establish the following result: $\cdots$

In her thesis, Ramaswamy [96], encountered the same difficulties and reported,

> $\cdots$ En ce point, on rencontre la même difficulté, qu'ont eue de-Mottoni & Tesei [53] : montrer que $T(c, \lambda)$ a exactement un maximum. Du fait que les expressions des dérivées deviennent de plus en plus compliquées $\cdots$ il n'est pas claire de voir comment celle-çi peuvent être simplifiées. Les résultats numériques montrent de toute façon, $T(c, \lambda)$ en tant que fonction de c ne possède qu'un seul maximum. En acceptant ceci, on a que$\cdots$

Brown et al. [36], reported

> ··· The numerical plots of bifurcation curves which we obtained for our various examples were always smooth S-shaped having exactly two bends as shown in Fig 3. We cannot prove, however, that the S-shaped curves discussed above have only two bends. To do so would seem to require a careful analysis of $d^2\lambda/d\rho^2$ and we are unable to find a reasonably tractable formula for $d^2\lambda/d\rho^2$. Smoller and Wasserman discuss in [106], expressions equivalent to $d^2/d\rho^2\,(\lambda(\rho))^{1/2}$ for the special case of cubic nonlinearities.

The idea performed by Smoller and Wasserman can briefly be described as follows. To prove uniqueness of the critical point of a C^2 map S in an interval I, it suffices to show that S is concave (or convex) in an neighborhood of each critical point in I. To this end, one try to find a constant a such that $S'' + aS' < 0$ (resp. $S'' + aS' > 0$) on the interval I. So, if $S'(x_*) = 0$, $S''(x_*) < 0$ (resp. $S''(x_*) > 0$), then at each critical point in I, S attains a local maximum (resp. minimum) value in I. So, S admits at most one maximum (resp. minimum) value in I.

Our first result (Theorem 20) gives sufficient conditions to ensure uniqueness of the minimum value of the corresponding time map. To this end we adapt Smoller and Wasserman's idea to our cases. Next, some applications will follow. To be more specific, the first application (Theorem 21) is concerned with a class of nonlinearities f having at least two successive zeros $\rho_1,\ \rho_2$ in $(0, +\infty)$ and being strictly positive between them. Since we make no sign assumption on f before ρ_1 and after ρ_2, only positive solutions whose sup-norms are between these two zeros are studied. However, we obtain the exact number of this kind of positive solutions for every value of $\lambda > 0$. Some Ambrosetti-Prodi situations are obtained. Theorems 20 and 21 seem to be new even when $p = 2$. The next result, Corollary 24, shows how changes in the sign of f lead to multiple positive solutions of (3.1). This type of problems was much discussed in the particular case $p = 2$, (see Brown & Budin [35]; Hess [70]; P. L. Lions [78], where the authors have got a lower bound of the number of positive solutions). In contrast, Corollary 24 gives the exact number of solutions.

The next result (Theorem 26) concerns a class of nonlinearities f vanishing only once in $(0, +\infty)$ and such that $f(0) > 0$. Existence and uniqueness of positive solution is obtained for all λ in a sub-interval of $(0, +\infty)$. Depending on some cases, this sub-interval is bounded, unbounded or equal to $(0, +\infty)$.

The following result (Theorem 27) is concerned by the nonlinearity $f(u) = \exp(u)$. This problem is well known when $p = 2$. The proof given by de-Mottoni & Tesei [52], (for $p = 2$) can be carried out with some obvious adaptations for all $p : 1 < p \leq 2$. But for $p > 2$ some deep investigations are needed. In fact we adapt the artifice of Smoller and Wasserman described above.

Finally, the last result (Theorem 28) is concerned with the nonlinearity $f(u) = \left(\varphi_q(u)\right)^{\alpha}$ where $q > 1$ and $\alpha > 0$ are real parameters. The proof given by de-Mottoni & Tesei [52] (for $p = q = 2$) can be carried out with some appropriate adaptations. So, the proof of Theorem 28 is given for the sake of completeness.

3.2 Main results

In this section we confine ourselves to state the main results of this chapter. Proofs are postponed to the next one.

Consider the boundary value problem :

$$\begin{cases} -\left(\varphi_p(u')\right)' & = \lambda f(u), \text{ in } (0,1) \\ u(0) & = u(1) = 0 \end{cases} \tag{3.2}$$

where $\lambda > 0$, $p > 1$ are two real parameters and f is specified below.

3.2.1 A time map result

Assume that f is such that

$$D := \{E > 0 : 0 < r(E) < +\infty \text{ and } f(r(E)) > 0\} \neq \emptyset,$$

then $r(E) := \sup X(E)$ (where $X(E)$ is defined in chapter 1) satisfies,

$$E^p - p'\lambda F(r(E)) = 0, \ \forall E \in D. \tag{3.3}$$

So, the time map,

$$T(E) = \int_0^{r(E)} \left\{E^p - p'\lambda F(s)\right\}^{-1/p} ds, \qquad E \in D,$$

may be written from (3.3) as follows,

$$T(E) = \left(p'\lambda\right)^{-1/p} \int_0^{r(E)} \{F(r(E)) - F(s)\}^{-1/p} ds, \qquad E \in D,$$

and one may observe that,

$$T(E) = \left(p'\lambda\right)^{-1/p} S(r(E)), \quad \text{where } S(\rho) = \int_0^\rho \{F(\rho) - F(s)\}^{-1/p}\, ds.$$

Also, from (3.3), one deduce by implicit differentiation that,

$$r'(E) = \frac{(p-1)\,E^{p-1}}{\lambda f(r(E))} > 0, \ \forall E \in D.$$

So, to study the variations of $E \mapsto T(E)$, in D, it suffices to study those of the mapping $\rho \mapsto S(\rho)$ in $\Gamma((0, +\infty))$ where,

$$\Gamma(I) := \{\rho \in I : f(\rho) > 0 \text{ and } F(\rho) > F(s) : \forall s \in [0, \rho)\},$$

for any interval $I \subset (0, +\infty)$.

The following result shows that, under some suitable conditions, S admits a local minimum value at each of its critical points. More precisely :

Theorem 20 Let $f \in C^2(\mathbb{R}_+)$, $p > 1$ and A be an open interval of $\Gamma((0, +\infty))$. Assume that there exist two real numbers α and β such that :

(i) $\alpha(p-1) + \beta + (p-2)(p-1) \geq 0$, and

(ii) $\forall \rho \in A, \forall s \in (0, \rho) : \int_s^\rho \psi(u)\, du < 0$ where,

$$\psi(u) := u^2 f''(u) + \alpha u f'(u) + \beta f(u), \text{ in } (0, \sup A).$$

Then, for all $\bar\rho \in A$, S admits a local minimum value at $\bar\rho$ if and only if $S'(\bar\rho) = 0$.

Theorem 20 will be used to prove Theorem 21.

Remark 2 a) In the particular case $\alpha = -(p-1)$ and $\beta = p(p-1)$ (equality in (i)), we show that S is convex in A. If moreover we assume that $p = 2$, we find a result already proved by Ammar-Khodja ([20] pp. 41-45).

b) It is possible to replace the strict inequality in (ii) by a large one, provided to the large inequality in (i) is replaced by a strict one.

c) The real numbers α and β may depend on A. So, if one can apply this theorem to each component of Γ, then S admits at most one critical point in each one of them (see Corollary 24).

3.2.2 Case where the nonlinearity vanishes at least twice

Let $f \in C^2(\mathbb{R}_+)$ and assume that there exists $(\rho_1, \rho_2) \in \mathbb{R}^2$ $0 < \rho_1 < \rho_2$ such that :

$$f(\rho_1) = f(\rho_2) = 0 \leq f(0), \tag{3.4}$$

$$f > 0 \text{ in } (\rho_1, \rho_2), \tag{3.5}$$

$$\text{and} \int_\rho^{\rho_2} f(s)\,ds > 0, \ \forall \rho \in [0, \rho_2). \tag{3.6}$$

From conditions (3.4)-(3.6), one can easily prove the following,

There exists a unique $\rho_* \in [\rho_1, \rho_2)$ such that :

$$\Gamma([\rho_1, \rho_2]) = (\rho_*, \rho_2).$$

Moreover, $F(\rho_*) = \max_{0 \leq \rho \leq \rho_1} F(\rho)$,

$$F(\rho_2) > F(\rho_*) \geq 0 \text{ and } \rho_* = \rho_1 \text{ iff } F(\rho_1) = \max_{0 \leq \rho \leq \rho_1} F(\rho).$$

The proof of this claim is postponed to Section 3.3.1. Now, assume that,

$$\begin{cases} \exists (\alpha, \beta) \in \mathbb{R}^2 \text{ such that conditions (i) and (ii) of Theorem 20} \\ \text{hold with } A = \Gamma([\rho_1, \rho_2]) = (\rho_*, \rho_2). \end{cases} \tag{3.7}$$

No sign assumption is made on f before ρ_1 and after ρ_2. In fact, we are going to look for positive solutions to problem (3.2) having their maximum values between ρ_1 and ρ_2.

Theorem 21 Assume that $p \in (1, 2]$, $f \in C^2(\mathbb{R}_+)$ satisfies (3.4)-(3.6) and (3.7). Then, for every $\lambda > 0$, all (possibly) positive solutions of (3.2) are in A_1^+ and there maximum values are outside $[\rho_1, \rho_*] \cup \{\rho_2\}$. Moreover, there exists $\lambda_* > 0$ such that :

1. If $\lambda < \lambda_*$, problem (3.2) admits no positive solution whose maximum value belongs to (ρ_1, ρ_2).
2. If $\lambda = \lambda_*$, problem (3.2) admits a unique positive solution u_* satisfying $\rho_* < \max_{[0,1]} u_* < \rho_2$.
3. If $\lambda > \lambda_*$, problem (3.2) admits exactly two positive solutions u_1 and u_2 whose maximum values belong to (ρ_*, ρ_2). Moreover $\rho_* < \max_{[0,1]} u_1 < \max_{[0,1]} u_2 < \rho_2$.

Example 22 Let f satisfying (3.4)-(3.6) and $p > 1$. If there exists a real number $k \in (-\infty, (p-2)) \cup ((p-1), +\infty)$ such that $d^2/dx^2 \left(f(x)/x^k \right) < 0$ in $(0, \rho_2)$ then f verifies (3.7) with $\alpha = -2k$ and $\beta = k(k+1)$.

Example 23 Let f satisfying (3.4)-(3.6) and $p \in (1, 2] \cup [3, +\infty)$. If moreover $f \in C^3(\mathbb{R}^+)$, such that $f''' < 0$ in $(0, \rho_2)$ and $f(0) = 0$, then f verifies (3.7) with $\alpha = -\beta = -2$.

The next result gives an example of a situation described in Assertion (c).on Remark 2.

Corollary 24 Let $p \in (1, 2]$ and suppose that f verifies:

- there exist $0 = \rho_0 < \rho_1 < \cdots < \rho_{2n}$ such that $f(\rho_i) = 0$, $\forall i \in \{0, \cdots, n\}$
- $f < 0$ in $\left(\rho_{2i}, \rho_{2i+1}\right)$, $\forall i \in \{0, \cdots, n-1\}$ and
- $f > 0$ in $\left(\rho_{2i-1}, \rho_{2i}\right)$, $\forall i \in \{1, \cdots, n\}$
- $\int_\rho^{\rho_{2i}} f(t)\, dt > 0$, for all ρ, $0 \leq \rho < \rho_{2i}$ and all $i \in \{1, \cdots, n\}$. Moreover, for all $i \in \{1, \cdots, n\}$, there exists $(\alpha_i, \beta_i) \in \mathbb{R}^2$ such that :
- $\alpha_i (p-1) + \beta_i + (p-2)(p-1) \geq 0$, and
- $\forall \rho \in A_i$, $\int_s^\rho \psi_i(u)\, du < 0$, $\forall s \in (0, \rho)$ where $A_i = (\xi_i, \rho_{2i})$ and $\psi_i(u) := u^2 f''(u) + \alpha_i u f'(u) + \beta_i f(u)$, in (ρ_0, ρ_{2i}).

Then, for all $i \in \{1, \cdots, n\}$, there exists λ_i^* such that for all $\lambda > \lambda_i^*$, the ball $B(0, \rho_{2i})$ of $C([0, 1])$ (with the sup-norm) contains exactly $2i$ positive solutions $u_1, \hat{u}_1, \cdots, u_i, \hat{u}_i$, and moreover $\rho_{2j-1} < \|u_j\|_C < \|\hat{u}_j\|_C < \rho_{2j}$, for all $j = 1, \cdots, i$.

The proof of Corollary 24 follows from an easy repeated application of Theorem 21 to each A_i.

Example 25 The function defined in $\mathbb{R}$ by $f(x) = x \sin(x + \pi)$ verifies the hypothesis of corollary 24 with $\rho_i = i\pi$ and $\alpha_i = -\beta_i = -2$ for all i.

3.2.3 Case where the nonlinearity vanishes only once

We assume throughout this subsection the following conditions :

(H1) $f(0) > 0$ and there exists some constant $\alpha > 0$ such that :

$$f(x) > 0, \ \forall x \in (0, \alpha), \ f(\alpha) = 0 \text{ and } f(x) < 0, \ \forall x \in (\alpha, +\infty).$$

(H2) There exists $(m, M) \in \left(\mathbb{R}_+^*\right)^2$, $\delta \in (0, \alpha]$ and $q > 1$ such that :

$$m \leq \frac{f(\xi)}{\varphi_q(\alpha - \xi)} \leq M, \qquad \forall \xi \in (\alpha - \delta, \alpha).$$

Moreover, we need one of the following conditions :

(H3) There exists $\varepsilon \in (0, \alpha]$ such that the function $x \mapsto H(x) := pF(x) - xf(x)$ is strictly increasing in the open interval $(\alpha - \varepsilon, \alpha)$, where $F(x) := \int_0^x f(t)\, dt$.

(H4) There exists $\varepsilon \in (0, \alpha]$ such that f is C^1 in $(\alpha - \varepsilon, \alpha)$ and $f'(x) < 0$ in $(\alpha - \varepsilon, \alpha)$.

(H5) There exists $\varepsilon \in (0, \alpha]$ such that f is C^1 in $(\alpha - \varepsilon, \alpha)$ and $(p - 1) f(x) - xf'(x) > 0$, in $(\alpha - \varepsilon, \alpha)$.

Theorem 26 Assume that f satisfies (H1), (H2) and at least one of conditions (H3), (H4) or (H5).

Then, there exists $\rho_* \in (\alpha - \varepsilon, \alpha)$ such that :

(a) If $1 < p \leq q$, there exists $\lambda_* = \lambda_*(\varepsilon) \geq 0$ such that, for any $\lambda > \lambda_*$, problem (3.2) admits a unique positive solution in A_1^+ whose sup-norm belongs to (ρ_*, α). Moreover, if $\varepsilon = \alpha$, then, for any $\lambda > 0$, problem (3.2) admits a unique positive solution in A_1^+ (and its sup-norm belongs to $(0, \alpha)$).

(b) If $1 < q < p$, there exist $0 < \lambda_*^- = \lambda_*^-(\varepsilon) < \lambda_*^+ = \lambda_*^+(\varepsilon)$ such that, for any $\lambda \in \left(\lambda_*^-, \lambda_*^+\right)$, problem (3.2) admits a unique positive solution in A_1^+ whose sup-norm belongs to (ρ_*, α). Moreover, if $\varepsilon = \alpha$, then, for any $\lambda \in \left(0, \lambda_*^+\right)$, problem (3.2) admits a unique positive solution in A_1^+ (and its sup-norm belongs to $(0, \alpha)$).

Before going further, let us mention some examples. Let α be a fixed positive number and define $f_1(x) = \varphi_p(\alpha - x)$, $f_2(x) = f_1(x) \exp\left((p - 1)\dfrac{x}{\alpha}\right)$ and $f_3(x) = f_1(x) \exp\left((p - 1)\dfrac{x}{\beta}\right)$ for some $0 < \beta < \alpha$. One can immediately verify that f_i, $i = 1, 2, 3$, satisfies (H1), (H2) with $p = q$. On the other hand f_1 and f_2 satisfy (H4) with any $\varepsilon \in (0, \alpha]$, but f_3 satisfies (H4) if and only if $\varepsilon \in (0, \beta]$.

3.2.4 Case where the nonlinearity is strictly positive

Consider the following boundary value problem :

$$\begin{cases} -\left(\varphi_p(u')\right)' = \lambda \exp(u), \text{ in } (0, 1) \\ u(0) = u(1) = 0 \end{cases} \tag{3.8}$$

where $\varphi_y(x) = |x|^{y-2}x,\ \lambda > 0,\ p > 1$ are real parameters. We are interested in the positive solutions of (3.8). This problem is very familiar when $p = 2$, even in higher dimensions (see for example Amann [14]; Fujita [61]).

Theorem 27 There exists a real number $\lambda_* > 0$ such that :

- If $\lambda > \lambda_*$, problem (3.8) admits no positive solution.
- If $\lambda = \lambda_*$, problem (3.8) admits a unique positive solution, u_*, and this solution is in A_1^+.
- If $0 < \lambda < \lambda_*$, problem (3.8) admits exactly two positive solutions, u_m and u_M, and these solutions are in A_1^+. Moreover, the function $\lambda \mapsto \|u_m\|_0$ (resp. $\lambda \mapsto \|u_M\|_0$) is continuous and increasing (resp. decreasing) and $\lim_{\lambda\to 0}\|u_m\|_0 = 0$ and $\lim_{\lambda\to\lambda_*}\|u_m\|_0 = \|u_*\|_0$ (resp. $\lim_{\lambda\to 0}\|u_M\|_0 = +\infty$ and $\lim_{\lambda\to\lambda_*}\|u_M\|_0 = \|u_*\|_0$).

The second result of this subsection is related to the problem :

$$\begin{cases} -\left(\varphi_p(u')\right)' = \lambda\left(\varphi_q(u)\right)^{\alpha}, & \text{in } (0,1) \\ u(0) = u(1) = 0, & \end{cases} \tag{3.9}$$

where $\varphi_y(x) = |x|^{y-2}x,\ p,\ q > 1$ and $\alpha,\ \lambda > 0$ are real parameters. We are interested in positive solutions of problem (3.9).

Theorem 28

1. If $(p-1)-(q-1)\alpha > 0$ (resp. < 0) then, for each fixed $\lambda > 0$, problem (3.9) admits a unique positive solution u_λ, and this solution is in A_1^+. Moreover the function $\lambda \mapsto u'_\lambda(0)$ is increasing (resp. decreasing) in $(0, +\infty)$ and converges to zero (resp. to infinity) when λ tends to zero, and converges to infinity (resp. to zero) when λ tends to infinity.
2. If $(p-1)-(q-1)\alpha = 0$, then problem (3.9) admits a positive solution if and only if $\lambda = \lambda_1 := (p-1)\left(2\pi/p\sin\left(\frac{\pi}{p}\right)\right)^p$. In this case, if $\bar{u}$ is a positive solution of (3.9), then u is an other solution if and only if $u = \beta\bar{u}$, for some $\beta \in (0, +\infty)$. Moreover, all these solutions are in A_1^+.

Remark 3 In the second case : $(p-1)-(q-1)\alpha = 0$, one has $\left(\varphi_q(u)\right)^{\alpha} = \varphi_p(u)$, as far as $u > 0$. So, the problem is an eigenvalue one.

When $\alpha = 1$, we can state the following :

Corollary 29

1. If $p > q$ (resp. $p < q$) then, for each fixed $\lambda > 0$, problem (3.9) admits a unique positive solution u_λ, and this solution is in A_1^+. Moreover, the function $\lambda \mapsto u'_\lambda(0)$ is increasing (resp. decreasing) in $(0, +\infty)$ and converges to zero (resp. to infinity) when λ tends to zero and converges to infinity (resp. to zero) when λ tends to infinity.

2. If $p = q$, then problem (3.9) admits a positive solution if and only if $\lambda = \lambda_1 := (p-1)\left(2\pi/p \sin\left(\frac{\pi}{p}\right)\right)^p$. In this case, if $\bar{u}$ is positive solution of (3.9), then u is an other solution if and only if $u = \beta\bar{u}$, for some $\beta \in (0, +\infty)$. Moreover, all of these solutions are in A_1^+.

3.3 Proofs

3.3.1 Proof of Claim 3.2.2.

Since F is continuous in $[0, \rho_1]$, there exists $\rho_0 \in [0, \rho_1]$ such that $F(\rho_0) = \max_{0 \le \rho \le \rho_1} F(\rho)$. From (3.6) if follows that $F(\rho_2) > F(\rho_0) \ge F(\rho_1)$. Since F is strictly increasing and continuous in $[\rho_1, \rho_2]$, there exists a unique $\rho_* \in [\rho_1, \rho_2)$ such that $F(\rho_*) = F(\rho_0)$. Moreover $\rho_* = \rho_1$ iff $F(\rho_1) = F(\rho_0)$. So, $F(\rho_*) \ge F(s)$ for all $s \in [0, \rho_1]$ and since F is strictly increasing in $[\rho_1, \rho_*]$ then $F(\rho_*) > F(s)$ for all $s \in [\rho_1, \rho_*)$. It follows that,

$$F(\rho_*) \ge F(s), \quad \forall s \in [0, \rho_*). \tag{3.10}$$

On the other hand, since F is strictly increasing in $[\rho_*, \rho_2]$ then

$$\forall \rho \in (\rho_*, \rho_2) : F(\rho) > F(s), \ \forall s \in [\rho_*, \rho). \tag{3.11}$$

From (3.10) and (3.11), it follows that

$$\forall \rho \in (\rho_*, \rho_2) : F(\rho) > F(s), \ \forall s \in [0, \rho),$$

then,

$$(\rho_*, \rho_2) \subset \Gamma([\rho_1, \rho_2]).$$

On the other hand, since F is strictly increasing in $[\rho_1, \rho_*]$,

$$\forall \rho \in (\rho_1, \rho_*] : F(\rho) \le F(\rho_*) = F(\rho_0),$$

then,

$$(\rho_1, \rho_*] \cap \Gamma([\rho_1, \rho_2]) = \emptyset,$$

and since $f(\rho_1) = 0$ then,

$$[\rho_1, \rho_*] \cap \Gamma([\rho_1, \rho_2]) = \emptyset.$$

On the other hand, one has $F(\rho_*) = \max_{0 \le \rho \le \rho_1} F(\rho) \ge F(0) = 0$ and (3.6) implies that $F(\rho_2) > F(\rho_*)$.

The claim 3.2.2 is completely proved. ■

3.3.2 Proof of Theorem 20

For every $\rho \in A$, we get, after straightforward computations :

$$S(\rho) = \int_0^\rho \{F(\rho) - F(s)\}^{-\frac{1}{p}} ds, \quad S'(\rho) = \frac{1}{p}\int_0^\rho \frac{H(\rho) - H(s)}{\rho\{F(\rho) - F(s)\}^{\frac{p+1}{p}}} ds$$

and

$$S''(\rho) = \left(\frac{p+1}{p^2}\right)\int_0^\rho \frac{(H(\rho) - H(s))^2\, ds}{\rho^2 (F(\rho) - F(s))^{\frac{2p+1}{p}}} + \frac{1}{p}\int_0^\rho \frac{(\Phi(\rho) - \Phi(s))\, ds}{\rho^2 (F(\rho) - F(s))^{\frac{p+1}{p}}}$$

where $H(x) := pF(x) - xf(x)$ and $\Phi(x) = -p(p+1)F(x) + 2pxf(x) - x^2 f'(x)$.

Let K and L two real numbers and $\rho \in A$. Some simple computations give

$$\rho^2 S''(\rho) + K\rho S'(\rho) + LS(\rho) = \frac{1}{p}\int_0^\rho \frac{(\Psi(\rho) - \Psi(s))}{(F(\rho) - F(s))^{\frac{p+1}{p}}} ds + \left(\frac{p+1}{p^2}\right)\int_0^\rho \frac{(H(\rho) - H(s))^2}{(F(\rho) - F(s))^{\frac{2p+1}{p}}} ds$$

where $\Psi(x) = \Phi(x) + KH(x) + pLF(x)$.

Let $K = \alpha + 2(p-1)$ and $L = \frac{-1}{p}((p-1)\alpha + \beta - (p-1)(2-p))$. Then,

$$\Psi'(x) = -\psi(x), \ \forall x \in (0, \sup A),$$

and from (ii) it follows that $\Psi(\rho) - \Psi(s) > 0$, $\forall \rho \in A$ and $\forall s \in (0, \rho)$. So, $\rho^2 S''(\rho) + K\rho S'(\rho) + LS(\rho) > 0$, for all $\rho \in A$. If $\bar{\rho} \in A$ is such that $S'(\bar{\rho}) = 0$ then $\bar{\rho}^2 S''(\bar{\rho}) + LS(\bar{\rho}) > 0$. Since $L \le 0$ (from (i)) then $S''(\bar{\rho}) > 0$, which implies that S attains a local minimum at $\bar{\rho}$. The proof is complete. ■

3.3.3 Proof of Theorem 21

In order to prove this theorem, we need some lemmas. From claim 3.2.2 we deduce that $F(\rho_2) > F(\rho_*) \geq 0$. So, if we put $E_2 := (p'\lambda F(\rho_2))^{\frac{1}{p}}$ and $E_* := (p'\lambda F(\rho_*))^{\frac{1}{p}}$, for any $p > 1$ and $\lambda > 0$, it follows that $0 \leq E_* < E_2$.

In order to define the time map, we need the following lemma which is concerned with the zeros of the equation in $s \in [0, \rho_2)$:

$$E^p - p'\lambda F(s) = 0, \tag{3.12}$$

with respect to the parameter $E > 0$.

Lemma 30 Assume that (3.4)-(3.6) hold. For $p > 1$ and $\lambda > 0$ fixed, we have :

(a) If $E \in (0, E_*]$ (with $E_* > 0$), equation (3.12) admits at least one zero in $(0, \rho_1]$.

(b) If $E \in (E_*, E_2)$, equation (3.12) admits no zero in $[0, \rho_1]$ and a unique zero $s_* = s_*(p, \lambda, E)$ in (ρ_1, ρ_2). Moreover :

(i) The function $E \mapsto s_*(p, \lambda, E)$ is C^1 in (E_*, E_2) and,

$$\frac{\partial s_*}{\partial E}(p, \lambda, E) = \frac{(p-1)E^{p-1}}{\lambda f(s_*(p, \lambda, E))} > 0, \ \forall E \in (E_*, E_2).$$

(ii) $\lim_{E \to E_*^+} s_*(p, \lambda, E) = \rho_*$ and $\lim_{E \to E_2} s_*(p, \lambda, E) = \rho_2$.

(c) If $E = E_2$, equation (3.12) admits a unique zero s_* in $[0, \rho_2]$: $s_* = \rho_2$.

(d) If $E > E_2$, equation (3.12) admits no zero in $[0, \rho_2]$.

Proof Consider the function $s \mapsto H(p, \lambda, E, s) := E^p - p'\lambda F(s)$. So, H is strictly decreasing in (ρ_1, ρ_2) and since $F(0) = 0$, then for any $E > 0$,

$$H(\rho_*) = H(0) \iff F(\rho_*) = 0 \ (\Leftrightarrow E_* = 0),$$

$$H(\rho_*) < H(0) \iff F(\rho_*) > 0 \ (\Leftrightarrow E_* > 0).$$

Also, from claim 3.2.2 it follows that, for any $E > 0$,

$$H(\rho_*) \leq H(s), \ \forall s \in [0, \rho_*),$$

$$\forall \rho \in (\rho_*,\ \rho_2) : H(\rho) < H(s)\,,\ \forall s \in [0,\ \rho)\,.$$

Then, Assertions (a), (c) (d) and the part of non existence in $[0,\ \rho_1]$ as well as existence and uniqueness in $(\rho_1,\ \rho_2)$ of Assertion (b) follow.

Let us prove (i) and (ii) of (b). To this end, consider the real valued function

$$(p,\ \lambda,\ E,\ s) \longmapsto H(p,\ \lambda,\ E,\ s)$$

defined in $\Omega = (1,\ +\infty) \times \mathbb{R}_+^2 \times (\rho_*,\ \rho_2)$. Then $H \in C^1(\Omega)$ and

$$\frac{\partial H}{\partial s}(p,\ \lambda,\ E,\ s) = -p'\lambda f(s) < 0 \ \text{ in } \Omega,$$

and we notice that $s_*(p,\ \lambda,\ E)$ belongs to the open interval $(\rho_*,\ \rho_2)$ and satisfies (from its definition)

$$H(p,\ \lambda,\ E,\ s_*(p,\ \lambda,\ E)) = 0. \tag{3.13}$$

So, we can make use of the implicit function theorem to show (in particular) that the function $E \longmapsto s_*(p,\ \lambda,\ E)$ is $C^1((E_*,\ E_2),\ \mathbb{R})$ and to obtain the expression of $\dfrac{\partial s_*}{\partial E}(p,\ \lambda,\ E)$ given in (i). Hence, for any fixed $p > 1$ and $\lambda > 0$, the function defined in $(E_*,\ E_2)$ by $E \longmapsto s_*(p,\ \lambda,\ E)$ is strictly increasing and bounded (by ρ_* and ρ_2). Then, the limit $\lim\limits_{E \to E_*^+} s_*(p,\ \lambda,\ E) = l_*$ (resp. $\lim\limits_{E \to E_2^-} s_*(p,\ \lambda,\ E) = l_2$) exists as real number and $\rho_* \le l_* < l_2 \le \rho_2$.

Also, let us observe that, for any fixed $p > 1$ and $\lambda > 0$, the function

$$(E,\ s) \longmapsto H(p,\ \lambda,\ E,\ s)$$

is continuous in $[E_*,\ E_2] \times [\rho_*,\ \rho_2]$ and the function $E \longmapsto s_*(p,\ \lambda,\ E)$ is continuous in $[E_*,\ E_2]$, (we set $s_*(p,\ \lambda,\ E_*) = l_*$ and $s_*(p,\ \lambda,\ E_2) = l_2$) and satisfies (3.13). So, by passing to the limit in (3.13) as E tends to E_* (resp. E_2) one gets :

$$0 = \lim_{E \to E_*^+} H(p,\ \lambda,\ E,\ s_*(p,\ \lambda,\ E)) = H(p,\ \lambda,\ E_*,\ l_*)\,,$$

$$(\text{resp. } 0 = \lim_{E \to E_2^-} H(p,\ \lambda,\ E,\ s_*(p,\ \lambda,\ E)) = H(p,\ \lambda,\ E_2,\ l_2)).$$

Hence, l_* (resp. l_2) is a solution, belonging to $[\rho_*,\ \rho_2]$, to the equation in s :

$$H(p,\ \lambda,\ E_*,\ s) = 0 \ \ (\text{resp. } H(p,\ \lambda,\ E_2,\ s) = 0).$$

But the variations, in $[\rho_*, \rho_2]$, of the function $s \mapsto H(p, \lambda, E_*, s)$ (resp. $s \mapsto H(p, \lambda, E_2, s)$) show that $l_* = \rho_*$ (resp. $l_2 = \rho_2$). The proof of Lemma 30 is complete. ■

With this lemma in mind, we have for any E and $\lambda > 0$, one has

$$\sup X(p, \lambda, E) \begin{cases} \leq \rho_1 & \text{if } E \in (0, E_*] \text{ (with } E_* > 0) \\ = s_*(p, \lambda, E) & \text{if } E \in (E_*, E_2) \\ \geq \rho_2 & \text{if } E \in [E_2, +\infty), \end{cases} \tag{3.14}$$

where

$$X(p, \lambda, E) := \left\{ s > 0 : E^p - p' \lambda F(\xi) > 0, \ \forall \xi, \ 0 < \xi < s \right\}.$$

and $s_*(p, \lambda, E)$ is given by Lemma 30.

Recall that we are interested in positive solutions to problem (3.2) having their maximum values between ρ_1, ρ_2 : no sign assumption on f before ρ_1 and after ρ_2 are made. Moreover $r(p, \lambda, E) := \sup X(p, \lambda, E)$ is the maximum value of the solution to (3.2) (when it exists) with its initial slope E. So, informations on $\sup X(p, \lambda, E)$ collected in (3.14) are sufficient for our purpose.

From (3.14) one deduces that

$$r(p, \lambda, E) \in (\rho_1, \rho_2) \text{ implies that } E \in (E_*, E_2)$$

and from the variations of $E \mapsto r(p, \lambda, E)$ one deduces that :

$$E \in (E_*, E_2) \text{ implies that } r(p, \lambda, E) \in (\rho_*, \rho_2),$$

$$\lim_{E \to E_*^+} r(p, \lambda, E) = \rho_* \quad \text{and} \quad \lim_{E \to E_2^-} r(p, \lambda, E) = \rho_2.$$

and from the definition of $r(p, \lambda, E) := \sup X(p, \lambda, E)$ one gets

$$E^p - p' \lambda F(r(p, \lambda, E)) = 0, \ \forall E \in (E_*, E_2]. \tag{3.15}$$

Then,

$$\begin{aligned} D_1 &:= \{E > 0 : r(p, \lambda, E) \in [\rho_1, \rho_*] \cup \{\rho_2\} \text{ and } f(r(p, \lambda, E)) > 0\} \\ &= \emptyset. \end{aligned}$$

Then, for every $\lambda > 0$, the maximum value of any positive solution of problem (3.2) is neither in $[\rho_1, \rho_*]$ nor equal to ρ_2. So, it remains to look for positive solutions to (3.2) whose maximum value belongs to (ρ_*, ρ_2). From the discussion above, we have,

$$
\begin{aligned}
D_2 &:= \{E > 0 : r(p, \lambda, E) \in (\rho_*, \rho_2) \text{ and } f(r(p, \lambda, E)) > 0\} \\
&= (E_*, E_2).
\end{aligned}
$$

Now we define the time-map T on D_2 by,

$$
T(p, \lambda, E) := \int_0^{r(p, \lambda, E)} \left(E^p - p'\lambda F(s) \right)^{-1/p} ds, \; E \in D_2.
$$

which can be written from (3.15) as follows,

$$
T(p, \lambda, E) = \left(p'\lambda \right)^{-1/p} \int_0^{r(p, \lambda, E)} \{F(r(p, \lambda, E)) - F(s)\}^{-1/p} ds, \; E \in D_2.
$$

One may observe that : $T(p, \lambda, E) = (p'\lambda)^{-1/p} S(r(p, \lambda, E))$ where,

$$
S(\rho) = \int_0^{\rho} \{F(\rho) - F(s)\}^{-1/p} ds.
$$

Then, since the function $E \mapsto r(p, \lambda, E)$ is increasing and continuous from (E_*, E_2) onto (ρ_*, ρ_2), if we put,

$$
J_1 := \left\{ E \in (E_*, E_2) : T(p, \lambda, E) = \frac{1}{2} \right\}
$$

and,

$$
J_2 := \left\{ \rho \in (\rho_*, \rho_2) : \left(p'\lambda \right)^{-1/p} S(p, \lambda, \rho) = \frac{1}{2} \right\}
$$

it follows that,

$$
Card(J_1) = Card(J_2).
$$

So, from now on, we will focus our attention on determining the number of solution(s) of equation $(p'\lambda)^{-1/p} S(\rho) = \frac{1}{2}$ in (ρ_*, ρ_2) instead of $T(E) = \frac{1}{2}$ in (E_*, E_2). To this end we study the variations of S in (ρ_*, ρ_2). First one has :

Lemma 31 (i) $\lim_{\rho \to \rho_*^+} S(\rho) = +\infty$, (ii) $\lim_{\rho \to \rho_2^-} S(\rho) = +\infty$.

Proof This proof is nothing but an adaptation of Theorem 3.7 in Brown & Budin [35]. Let $N := \max\{|f(x)| : x \in [0, \rho_2]\}$ and $M := \max\{|f'(x)| : x \in [0, \rho_2]\}$.

Proof of (i). Suppose first that $\rho_* \in (\rho_1, \rho_2)$. Then there exists $\rho_0 \in [0, \rho_1)$ such that $\max_{[0, \rho_1)} F = F(\rho_0) = F(\rho_*)$. If $\rho_0 \in (0, \rho_1)$, then ρ_0 is a local maximum for F and so $f(\rho_0) = 0$. If $\rho_0 = 0$, then necessarily $f(\rho_0 = 0) = 0$. (In fact, $f(0) > 0$ implies that F is strictly increasing in a right neighborhood of 0, so that $F(0) < \max_{[0,\rho_1]} F)$. Since $f(\rho_0) = 0$ then $|f(x)| \leq M|x - \rho_0|$, for every $x \in (0, \rho_2)$. So, if $\rho \in (\rho_*, \rho_2)$ and $s \in (0, \rho)$, one has,

$$0 < F(\rho) - F(s) \leq |F(\rho) - F(\rho_*)| + |F(\rho_0) - F(s)|.$$

On the one hand, one has,

$$|F(\rho) - F(\rho_*)| = (\rho - \rho_*) f(\eta_1), \quad \text{where } \eta_1 \in (\rho_*, \rho),$$

and then,

$$|F(\rho) - F(\rho_*)| \leq (\rho - \rho_*) N. \tag{3.16}$$

On the other hand,

$|F(\rho_0) - F(s)| = |\rho_0 - s| |f(\eta_2)|$ where η_2 is between ρ_0 and s, that is $|\eta_2 - \rho_0| \leq |\rho_0 - s|$, then $|F(\rho_0) - F(s)| = |\rho_0 - s||\eta_2 - \rho_0| M \leq (\rho_0 - s)^2 M$.

Hence,

$$F(\rho) - F(s) \leq (\rho - \rho_*) N + (\rho_0 - s)^2 M, \ \forall s \in (0, \rho).$$

Then,

$$\int_0^{\rho} \{F(\rho) - F(s)\}^{-\frac{1}{p}} ds \geq \int_0^{\rho_*} \{F(\rho) - F(s)\}^{-\frac{1}{p}} ds \geq \int_0^{\rho_*} \left\{ |\rho - \rho_*| N + |\rho_0 - s|^2 M \right\}^{-\frac{1}{p}} ds.$$

That is,

$$\int_0^{\rho} \{F(\rho) - F(s)\}^{-\frac{1}{p}} ds \geq \int_0^{\rho_*} H_\rho(s)\, ds,$$

where $H_\rho(s) := \left\{ (\rho - \rho_*) N + |\rho_0 - s|^2 M \right\}^{-\frac{1}{p}}$. But as $\rho \to \rho_*^+$, H_ρ is a nondecreasing sequence of measurable functions. Hence by the monotone convergence theorem :

$$\lim_{\rho \to \rho_*^+} \int_0^{\rho_*} H_\rho(s)\, ds = M^{-\frac{1}{p}} \int_0^{\rho_*} |\rho_0 - s|^{-\frac{2}{p}} ds = +\infty \quad (p \in (1, 2]).$$

Suppose now that $\rho_* = \rho_1$. From (3.4), $f(\rho_1) = 0$ that is to say $F'(\rho_1) = 0$. So, it is easy to show that $\int_0^{\rho_1} \{F(\rho_1) - F(s)\}^{-\frac{1}{p}} ds = +\infty$ for $p \in (1, 2]$. On the other hand,

$$\int_0^{\rho} \{F(\rho) - F(s)\}^{-1/p} ds \geq \int_0^{\rho_1} \{F(\rho) - F(s)\}^{-1/p} ds,$$

and by the monotone convergence theorem one gets,

$$\lim_{\rho \to \rho_1^+} \int_0^\infty \{F(\rho) - F(s)\}^{-1/p} \, ds = \int_0^\infty \{F(\rho_1) - F(s)\}^{-1/p} \, ds = +\infty,$$

which completes the proof of (i).

Proof of (ii). Since $f(\rho_2) = 0$ then $|f(\eta)| \leq M|\eta - \rho_2|$, for every $\eta \in (0, \rho_2)$. Moreover, for any $\rho \in (\rho_*, \rho_2)$ and any $s : 0 \leq s \leq \rho$ we have,

$$\begin{aligned} 0 < F(\rho) - F(s) &= |F(\rho) - F(s)| \\ &\leq |F(\rho) - F(\rho_2)| + |F(\rho_2) - F(s)| \\ &\leq |\rho - \rho_2| \sup_{[0, \rho_2]} |f| + |\rho_2 - s||f(\eta)| : \text{ for some } \eta \in (s, \rho_2) \\ &\leq (\rho_2 - \rho) N + (\rho_2 - s) M |\eta - \rho_2|. \end{aligned}$$

Then,

$$0 < F(\rho) - F(s) \leq (\rho_2 - \rho) N + (\rho_2 - s)^2 M,$$

so,

$$\int_0^\rho \{F(\rho) - F(s)\}^{-1/p} \, ds \geq \int_0^\rho \left\{ (\rho_2 - \rho) N + (\rho_2 - s)^2 M \right\}^{-1/p} ds = \int_0^{\rho_2} G_\rho(s) \, ds$$

where $G_\rho(s) := \left\{ (\rho_2 - \rho) N + (\rho_2 - s)^2 M \right\}^{-1/p} \chi_{[0, \rho]}$ and $\chi_{[0, \rho]}$ denotes the characteristic function of $[0, \rho]$. One may observe that $\{G_\rho\}$ is a nondecreasing sequence of measurable functions, then by making use of the monotone convergence theorem we get

$$\begin{aligned} \lim_{\rho \to \rho_2^-} \int_0^\rho \{F(\rho) - F(s)\}^{-1/p} \, ds \geq \lim_{\rho \to \rho_2^-} \int_0^{\rho_2} G_\rho(s) \, ds &= M^{-1/p} \int_0^{\rho_2} (\rho_2 - s)^{-2/p} \, ds \\ &= +\infty \; : (1 < p \leq 2). \end{aligned}$$

Therefore, Lemma 31 is proved. ■

This lemma implies that the function $\rho \mapsto S(\rho)$ admits at least one minimum value (in (ρ_*, ρ_2)). To show the uniqueness of this minimum it suffices to show that S attains a minimum value at each of its critical points. This follows from (3.7) and Theorem 20.

Completion of the proof of Theorem 21: At this stage we have the following picture of the function,

$$\rho \mapsto S(\rho) = \int_0^\rho \{F(\rho) - F(s)\}^{-1/p} \, ds,$$

which is defined in (ρ_*, ρ_2) : $\lim_{\rho \to \rho_*^+} S(\rho) = \lim_{\rho \to \rho_2^-} S(\rho) = +\infty$, and $S(\cdot)$ admits a unique minimum value $S(\bar{\rho}) = \int_0^{\bar{\rho}} \{F(\bar{\rho}) - F(s)\}^{-1/p} \, ds$. So,

- If $(p'\lambda)^{-1/p} S(\bar{\rho}) > 1/2$, problem (3.2) admits no positive solution whose maximum is in (ρ_*, ρ_2).
- If $(p'\lambda)^{-1/p} S(\bar{\rho}) = 1/2$, problem (3.2) admits a unique positive solution whose maximum is in (ρ_*, ρ_2).
- If $(p'\lambda)^{-1/p} S(\bar{\rho}) < 1/2$, problem (3.2) admits exactly two positive solutions whose maximums are (ordered) in (ρ_*, ρ_2).

Theorem 21 is completely proved if we put $\lambda_* = (1/p')\,(2S(\bar{\rho}))^p$. ■

3.3.4 Proof of Theorem 26.

One may observe that if (H1) holds then (H4) $\Rightarrow$ (H5) $\Rightarrow$ (H3). So, to prove Theorem 26 it suffices to assume that (H1), (H2) and (H3) hold.

As in the proof of Theorem 21, in order to define the corresponding time map, we need the following,

Lemma 32 Assume that f satisfies (H1) and consider the equation in $s > 0$,

$$E^p - p'\lambda F(s) = 0, \tag{3.17}$$

where $E > 0$, $\lambda > 0$ are real parameters. Then,

(i) If $E > E_*(p, \lambda) := (p'\lambda F(\alpha))^{1/p}$, equation (3.17) admits no (positive) zero.

(ii) If $E = E_*(p, \lambda)$, equation (3.17) admits a unique zero $s_* = \alpha$.

(iii) If $E < E_*(p, \lambda)$, equation (3.17) admits, in the open interval $(0, \alpha)$, a unique zero $s_* = s_*(p, \lambda, E)$. Moreover,

(a) The function $E \mapsto s_*(p, \lambda, E)$ is C^1 in $(0, E_*(p, \lambda))$, and,

$$\frac{\partial s_*}{\partial E}(p, \lambda, E) = \frac{(p-1)\,E^{p-1}}{\lambda f(s_*(p, \lambda, E))} > 0, \ \forall E \in (0, E_*(p, \lambda)),$$

(b) $\lim_{E \to 0^+} s_*(p, \lambda, E) = 0$ and $\lim_{E \to E_*} s_*(p, \lambda, E) = \alpha$.

The proof of this lemma is essentially the same as that of Lemma 30. So, we omit it.

Now, we are ready to compute for any $E > 0$ and $\lambda > 0$, $X(p, \lambda, E)$ as defined in chapter 1. In fact,

$$X(p, \lambda, E) = \begin{cases} (0, +\infty) & \text{if } E > E_*(p, \lambda) \\ (0, \alpha) & \text{if } E = E_*(p, \lambda) \\ (0, s_*(p, \lambda, E)) & \text{if } E < E_*(p, \lambda) \end{cases}$$

where $s_*(p, \lambda, E)$ is defined in Lemma 32. Then,

$$r(p, \lambda, E) := \sup X(p, \lambda, E) = \begin{cases} +\infty & \text{if } E > E_*(p, \lambda) \\ \alpha & \text{if } E = E_*(p, \lambda) \\ s_*(p, \lambda, E) & \text{if } E < E_*(p, \lambda) \end{cases}$$

and from Lemma 32, one deduce the following limits,

$$\lim_{E \to 0^+} r(p, \lambda, E) = 0 \quad \text{and} \quad \lim_{E \to E_*} r(p, \lambda, E) = \alpha. \tag{3.18}$$

On the other hand,

$$0 < r(p, \lambda, E) < +\infty \quad \text{if and only if} \quad E \leq E_*(p, \lambda),$$

and from hypothesis (H1) and the range of $E \mapsto r(p, \lambda, E)$ one deduces

$$f(r(p, \lambda, E)) > 0 \quad \text{if and only if} \quad 0 < E < E_*(p, \lambda),$$

then

$$\begin{aligned} D &:= \{E > 0 : 0 < r(p, \lambda, E) < +\infty \text{ and } f(r(p, \lambda, E)) > 0\} \\ &= (0, E_*(p, \lambda)). \end{aligned}$$

Now for $E \in D$, we define the time mapping T by,

$$T(p, \lambda, E) := \int_0^{r(p, \lambda, E)} \left[E^p - p'\lambda F(\xi)\right]^{-\frac{1}{p}} d\xi, \quad E \in D, \tag{3.19}$$

which, due to the fact that $r(p, \lambda, E)$ is a solution of (3.17), can be written as,

$$T(p, \lambda, E) = \left(p'\lambda\right)^{-1/p} \int_0^{r(p, \lambda, E)} \{F(r(p, \lambda, E)) - F(\xi)\}^{-1/p} d\xi. \tag{3.20}$$

Moreover, a simple change of variables in (3.19) shows that,

$$T(p,\lambda,E) = r(p,\lambda,E)\int_0^1 \left\{E^p - p'\lambda F(r(p,\lambda,E)\xi)\right\}^{-1/p} d\xi,$$

which can be written as

$$T(p,\lambda,E) = \frac{r(p,\lambda,E)}{E}\int_0^1 \left\{1 - p'\lambda\frac{F(r(p,\lambda,E)\xi)}{E^p}\right\}^{-1/p} d\xi. \tag{3.21}$$

The limits of the time map T are given by,

Lemma 33

$$\lim_{E\to 0^+} T(p,\lambda,E) = 0, \quad \lim_{E\to E_\alpha^-} T(p,\lambda,E) = \begin{cases} +\infty & \text{if } 1 < p \leq q \\ (p'\lambda)^{-1/p} J & \text{if } 1 < q < p \end{cases}$$

where

$$J = \int_0^\alpha \left\{F(\alpha) - F(\xi)\right\}^{-1/p} d\xi.$$

Proof To compute the first limit, we use expression (3.21) of $T(p,\lambda,E)$. By making use of l'Hôspital's rule we get :

$$\lim_{E\to 0}\frac{r(p,\lambda,E)}{E} = \lim_{E\to 0}\frac{r'(p,\lambda,E)}{1} = \lim_{E\to 0}\frac{(p-1)E^{p-1}}{\lambda f(r(p,\lambda,E))} = \frac{0}{f(0)} = 0.$$

In addition, we have :

$$\frac{F(r(p,\lambda,E)\xi)}{E^p} = \frac{r(p,\lambda,E)}{E^p}\xi\int_0^1 f(r(p,\lambda,E)\xi t)\,dt,$$

so,

$$\begin{aligned}\lim_{E\to 0}\frac{F(r(p,\lambda,E)\xi)}{E^p} &= \lim_{E\to 0}\left(\frac{r'(p,\lambda,E)}{pE^{p-1}}\right)\xi\int_0^1 f(r(p,\lambda,E)\xi t)\,dt \\ &= \lim_{E\to 0}\frac{(p-1)\xi}{p\lambda f(r(p,\lambda,E))}\int_0^1 f(r(p,\lambda,E)\xi t)\,dt \\ &= \frac{(p-1)\xi}{p\lambda f(0)}\int_0^1 f(0)\,dt = \frac{\xi}{p'\lambda}.\end{aligned}$$

Then,

$$\lim_{E\to 0}\left(1 - p'\lambda\frac{F(r(p,\lambda,E)\xi)}{E^p}\right) = 1 - p'\lambda\cdot\frac{\xi}{p'\lambda} = 1 - \xi.$$

Hence,

$$\lim_{E\to 0}\int_0^1\left(1-p'\lambda\frac{F(r(p,\lambda,E)\xi)}{E^p}\right)^{-1/p}d\xi=\int_0^1(1-\xi)^{-1/p}d\xi=p'.$$

So,

$$\lim_{E\to 0}T(p,\lambda,E)=0\times p'=0.$$

To compute the second limit, we use expression (3.20) of $T(p,\lambda,E)$. For any E near E_*, we have : $\alpha-\delta<r(p,\lambda,E)<\alpha$, so that :

$$\int_0^{r(p,\lambda,E)}\{F(r(p,\lambda,E))-F(\xi)\}^{-1/p}d\xi=\int_0^{\alpha-\delta}\{\cdots\}^{-1/p}d\xi+\int_{\alpha-\delta}^{r(p,\lambda,E)}\{\cdots\}^{-1/p}d\xi.$$

From (H2) we have,

$$m\varphi_q(\alpha-x)\leq f(x)\leq M\varphi_q(\alpha-x),\ \forall x\in(\alpha-\delta,\alpha),$$

and, since for any E near E_* we have $r(p,\lambda,E)\in(\alpha-\delta,\alpha)$, then $\forall\xi\in(\alpha-\delta,r(p,\lambda,E))$:

$$m\int_\xi^{r(p,\lambda,E)}\varphi_q(\alpha-x)\,dx\leq\int_\xi^{r(p,\lambda,E)}f(x)\,dx\leq M\int_\xi^{r(p,\lambda,E)}\varphi_q(\alpha-x)\,dx,$$

that is,

$$\begin{aligned}\frac{m}{q}\left[(\alpha-\xi)^q-(\alpha-r(p,\lambda,E))^q\right]&\leq F(r(p,\lambda,E))-F(\xi)\\&\leq\frac{M}{q}\left[(\alpha-\xi)^q-(\alpha-r(p,\lambda,E))^q\right];\end{aligned}$$

then,

$$\begin{aligned}\left(\frac{M}{q}\right)^{-\frac{1}{p}}\int_{\alpha-\delta}^{r(E)}\left[(\alpha-\xi)^q-(\alpha-r(E))^q\right]^{-\frac{1}{p}}d\xi&\leq\int_{\alpha-\delta}^{r(E)}\left[F(r(E))-F(\xi)\right]^{-\frac{1}{p}}d\xi\\&\leq\left(\frac{m}{q}\right)^{-\frac{1}{p}}\int_{\alpha-\delta}^{r(E)}\left[(\alpha-\xi)^q-(\alpha-r(E))^q\right]^{-\frac{1}{p}}d\xi.\end{aligned}$$

By passing to the limit in these inequalities as E tends to E_* we find,

$$\left(\frac{M}{q}\right)^{-\frac{1}{p}} \int_{\alpha-\delta}^{\alpha} (\alpha-\xi)^{-\frac{q}{p}}\, d\xi \leq \lim_{E \to E_*} \int_{\alpha-\delta}^{r(E)} \left[F(r(E)) - F(\xi)\right]^{-\frac{1}{p}} d\xi$$

$$\leq \left(\frac{m}{q}\right)^{-\frac{1}{p}} \int_{\alpha-\delta}^{\alpha} (\alpha-\xi)^{-\frac{q}{p}}\, d\xi,$$

and from the well-known fact,

$$\int_{\alpha-\delta}^{\alpha} (\alpha-\xi)^{-\frac{q}{p}}\, d\xi < +\infty \iff q < p,$$

the computation of $\lim_{E \to E_*} T(p, \lambda, E)$ follows. Lemma 33 is completely proved. ■

From (3.20), one may observe that,

$$T(p, \lambda, E) = \left(p'\lambda\right)^{-1/p} S(r(p, \lambda, E)) \tag{3.22}$$

where,

$$S(\rho) := \int_0^{\rho} \{F(\rho) - F(s)\}^{-1/p}\, ds.$$

Since, for each fixed $\lambda > 0$, the function $E \mapsto r(p, \lambda, E)$ is an increasing continuous function from $(0, E_*(p, \lambda))$ onto $(0, \alpha)$, then if we put

$$J_1(p, \lambda) := \left\{E \in (0, E_*(p, \lambda)) : T(p, \lambda, E) = \frac{1}{2}\right\}$$

and

$$J_2(p, \lambda) := \left\{\rho \in (0, \alpha) : \left(p'\lambda\right)^{-1/p} S(\rho) = \frac{1}{2}\right\},$$

it follows that

$$Card J_1(p, \lambda) = Card J_2(p, \lambda).$$

So, from now on, we will focus our attention on determining the number of solution(s) of equation $(p'\lambda)^{-1/p} S(\rho) = 1/2$ in $(0, \alpha)$ instead of $T(p, \lambda, E) = 1/2$ in $(0, E_*(p, \lambda))$.

The key lemma used in the proof of Theorem 26 is the following :

Lemma 34 Assume that f satisfies (H1), (H2) and (H3). Then,

(a) If $\varepsilon = \alpha$, $\dfrac{\partial S}{\partial \rho}(\rho) > 0$, $\forall \rho \in (0, \alpha)$.

(b) If $\varepsilon \in (0, \alpha)$ there exists ρ_*, (ρ_* independent of λ) such that,

$$\frac{\partial S}{\partial \rho}(\rho) > 0, \ \forall \rho \in (\rho_*, \alpha).$$

Proof Some easy computations show that,

$$\frac{\partial S}{\partial \rho}(p, \rho) = \frac{1}{p}\int_0^1 \frac{H(\rho) - H(\rho\xi)}{\{F(\rho) - F(\rho\xi)\}^{1+\frac{1}{p}}} d\xi,$$

where H is defined in (H3).

(a) If $\varepsilon = \alpha$, then (from (H3)) H is strictly increasing in $(0, \alpha)$. So,

$$H(\rho) - H(\rho\xi) > 0, \ \ \forall \rho \in (0, \alpha), \ \forall \xi \in (0, 1).$$

(b) Assume that $\varepsilon \in (0, \alpha)$. From (H1), it follows that F is strictly increasing in $[0, \alpha]$, then,

$$p(F(x) - F(\alpha)) < 0 < xf(x), \ \ \forall x \in (0, \alpha), \text{ and } 0 = F(0) < F(\alpha)$$

then,

$$H(x) < H(\alpha), \qquad \forall x \in [0, \alpha). \tag{3.23}$$

Let $x_0 \in [0, \alpha - \varepsilon]$ be such that $H(x_0) = \sup\{H(x) : 0 \le x \le \alpha - \varepsilon\}$. Then, from (3.23), one gets $H(\alpha - \varepsilon) \le H(x_0) < H(\alpha)$, and then, there exists (a unique) $\rho_* \in [\alpha - \varepsilon, \alpha)$ such that $H(\rho_*) = H(x_0)$. Taking into account the fact that H is strictly increasing in $[\alpha - \varepsilon, \rho_*]$, it follows that $H(x) \le H(\rho_*)$, $\forall x \in [0, \rho_*]$, and since H is strictly increasing in $[\rho_*, \alpha)$, it follows that for any $\rho \in (\rho_*, \alpha)$, one has $H(x) < H(\rho)$, $\forall x \in [0, \rho)$, or equivalently $H(\rho\xi) < H(\rho)$, $\forall \xi \in (0, 1)$. Lemma 34 is completely proved. ■

Now, we can proceed to the

Completion of the proof of Theorem 26:

Case $1 < p \le q$. Assume that $\varepsilon = \alpha$; then for each fixed $\lambda > 0$, the function $\rho \mapsto (p'\lambda)^{-1/p} S(\rho)$ is strictly increasing in $(0, \alpha)$ (lemma 34) and $\lim_{\rho \to 0} (p'\lambda)^{-1/p} S(\rho) = 0$, $\lim_{\rho \to \alpha} (p'\lambda)^{-1/p} S(\rho) = +\infty$. (These limits are obtained from Lemma 33, (3.22) and (3.18)). Then, the equation in $\rho \in (0, \alpha)$: $(p'\lambda)^{-1/p} S(\rho) = 1/2$, admits a unique solution for any $\lambda > 0$.

Now, let us assume that $\varepsilon \in (0, \alpha)$. Then, for each fixed $\lambda > 0$, the function $\rho \mapsto (p'\lambda)^{-1/p} S(\rho)$ is strictly increasing in (ρ_*, α) (Lemma 34). So, the equation in $\rho \in (\rho_*, \alpha)$:

$(p'\lambda)^{-1/p} S(\rho) = 1/2$, admits a (unique) solution in (ρ_*, α) if and only if : $(p'\lambda)^{-1/p} S(\rho_*) < 1/2$, which is equivalent to $\lambda > \lambda_*$ with $\lambda_* = (1/p')(2S(\rho_*))^p$.

Remark 4 If $\varepsilon \in (0, \alpha)$ and $\lambda > \lambda_*$, equation $(p'\lambda)^{-1/p} S(\rho) = 1/2$ admits a unique solution in (ρ_*, α), but we don't know if it admits, or not, any other solution in $(0, \rho_*]$!

Case $1 < q < p$. Assume that $\varepsilon \in (0, \alpha)$ (resp. $\varepsilon = \alpha$). Then, for each fixed $\lambda > 0$, the function $\rho \mapsto (p'\lambda)^{-1/p} S(\rho)$ is strictly increasing in $[\rho_*, \alpha)$ (resp. in $(0, \alpha)$) (Lemma 34) and $\lim_{\rho \to \alpha} (p'\lambda)^{-1/p} S(\rho) = (p'\lambda)^{-1/p} J$, and $\lim_{\rho \to \rho_*} (p'\lambda)^{-1/p} S(\rho) = (p'\lambda)^{-1/p} S(\rho_*)$ (resp. $\lim_{\rho \to 0} (p'\lambda)^{-1/p} S(\rho) = 0$) (from Lemma 33, (3.22) and (3.18)). So, the equation in $\rho \in (\rho_*, \alpha)$ (resp. $\rho \in (0, \alpha)$): $(p'\lambda)^{-1/p} S(\rho) = 1/2$ admits a (unique) solution in (ρ_*, α) (resp. in $(0, \alpha)$) if and only if $(p'\lambda)^{-1/p} S(\rho_*) < 1/2 < (p'\lambda)^{-1/p} J$ (resp. $1/2 < (p'\lambda)^{-1/p} J$) which is equivalent to,

$$\lambda_*^- := \frac{1}{p'}(2S(\rho_*))^p < \lambda < \frac{1}{p'}(2J)^p =: \lambda_*^+,$$

$$(\text{resp. } \lambda < \frac{1}{p'}(2J)^p =: \lambda_*^+.)$$

One may observe that $\lambda_*^- < \lambda_*^+$ from the facts that

- $J = S(\alpha)$,

- The function $\rho \mapsto S(\rho)$ is strictly increasing in $[\rho_*, \alpha)$.

Remark 5 If $\varepsilon \in (0, \alpha)$ and $\lambda_*^- < \lambda < \lambda_*^+$, equation $(p'\lambda)^{-1/p} S(\rho) = 1/2$ admits a unique solution in (ρ_*, α), but we don't know if it admits, or not, any other solution in $(0, \rho_*]$!

The proof Theorem 26 is complete. $\blacksquare$

3.3.5 Proof of Theorem 27.

In this example, we have,

$$f(u) = \exp(u) \text{ and } F(u) = \int_0^u f(s)\,ds = \exp(u) - 1.$$

For any $E > 0$, consider the equation in s,

$$E^p - p'\lambda F(s) = 0.$$

This equation admits a unique solution $s_* = \log\left(1 + (E^p/p'\lambda)\right)$. So, we have

$$X(E) := \left\{ s > 0 : E^p - p'\lambda F(\xi) > 0,\ \forall \xi \in (0, s) \right\} = (0, s_*),$$

and then,

$$r(E, \lambda) := \sup X(E) = s_*.$$

We can observe that,

$$r(E, \lambda) > 0,\ \forall E > 0 \quad \text{ and } \quad f(r(E, \lambda)) > 0,\ \forall E > 0.$$

So,

$$D = \{E > 0 : 0 < r(E, \lambda) < +\infty \text{ and } f(r(E, \lambda)) > 0\} = (0, +\infty).$$

The time-map T is defined on $D = (0, +\infty)$ by,

$$\begin{aligned} T(E, \lambda) &= \int_0^{r(E,\lambda)} \{E^p - p'\lambda F(\xi)\}^{-1/p} d\xi \\ &= (p'\lambda)^{-1/p} \int_0^{r(E,\lambda)} \left\{ \frac{E^p}{p'\lambda} - F(\xi) \right\}^{-1/p} d\xi. \end{aligned}$$

Using the change of variables $t = F(\xi)$ (which is admissible because of $F'(\xi) = \exp(\xi) > 0$) we get

$$\begin{aligned} T(E, \lambda) &= (p'\lambda)^{-\frac{1}{p}} \int_0^{(E^p/p'\lambda)} \left\{ \frac{E^p}{p'\lambda} - t \right\}^{-1/p} (\log(1+t))'\, dt \\ &= (p'\lambda)^{-\frac{1}{p}} \int_0^{(E^p/p'\lambda)} \left\{ \frac{E^p}{p'\lambda} - t \right\}^{-1/p} \frac{dt}{1+t}, \end{aligned}$$

and using the change of variables $t = (E^p/p'\lambda)\, y$, we deduce that

$$T(E, \lambda) = \left[p'\lambda\right]^{-1/p} \left(\frac{E^p}{p'\lambda} \right)^{1-\frac{1}{p}} \int_0^1 \left(1 + \frac{E^p}{p'\lambda} y \right)^{-1} (1 - y)^{-1/p} dy.$$

At this stage, we may notice that,

$$T(E, \lambda) = \left[p'\lambda\right]^{-1/p} R(\mu(E, \lambda)),$$

where,

$$\mu(E,\lambda)=\frac{E^p}{p'\lambda},\quad \text{and}\quad R(\mu)=(\mu)^{1-\frac{1}{p}}\int_0^1\frac{dy}{(1+\mu y)(1-y)^{1/p}}.$$

For each fixed $\lambda>0$, the function $E\mapsto\mu(E,\lambda)$ is an increasing C^1-diffeomorphism of $(0,+\infty)$, and,

$$\frac{\partial T}{\partial E}(E,\lambda)=\left(p'\lambda\right)^{-1/p}\frac{dR}{d\mu}(\mu(E,\lambda))\times\frac{\partial\mu}{\partial E}(E,\lambda).$$

So, to study the variations of $E\longmapsto T(E,\lambda)$, it suffices to study those of R, that is, for each fixed $\lambda>0$, R is strictly increasing (resp. decreasing) on the interval I, if and only if $T(\cdot,\lambda)$ is strictly increasing (resp. decreasing) on $J_\lambda=\mu_\lambda^{-1}(I)$, where μ_λ^{-1} is the inverse function of $\mu(\cdot,\lambda)$, and then R admits a local maximum (resp. minimum) value at μ_* if and only if $T(\cdot,\lambda)$ do so at $\mu_\lambda^{-1}(\mu_*)$.

We may notice that for each fixed $\lambda>0$,

$$\lim_{E\to 0}\mu(E,\lambda)=0\quad\text{and}\quad\lim_{E\to+\infty}\mu(E,\lambda)=+\infty;$$

so, if $\lim_{\mu\to 0}R(\mu)=\lim_{\mu\to+\infty}R(\mu)=0$, then for each fixed $\lambda>0$, the time-map $E\longmapsto T(E,\lambda)$ do so, that is : $\lim_{E\to 0}T(E,\lambda)=\lim_{E\to+\infty}T(E,\lambda)=0$, and then admits at least one maximum value. So, if further R admits a unique maximum value then $T(\cdot,\lambda)$ do so, and then,

$$\max_{E>0}T(E,\lambda)=\left(p'\lambda\right)^{-1/p}\max_{\mu>0}R(\mu)=\left(p'\lambda\right)^{-1/p}R(\mu_*)$$

for some $\mu_*>0$. So, we can conclude that,

- If $(p'\lambda)^{-1/p}R(\mu_*)<1/2$, equation $T(E,\lambda)=1/2$ admits no solution.
- If $(p'\lambda)^{-1/p}R(\mu_*)=1/2$, equation $T(E,\lambda)=1/2$ admits a unique solution.
- If $(p'\lambda)^{-1/p}R(\mu_*)>1/2$, equation $T(E,\lambda)=1/2$ admits exactly two solutions.

Then, if we set $\lambda_*=(1/p')(2R(\mu_*))^p$, Theorem 27 follows. So it remains to prove the following,

Lemma 35 R admits a unique local maximum value and,

$$\lim_{\mu\to 0}R(\mu)=\lim_{\mu\to+\infty}R(\mu)=0.$$

We write $R(\mu)$ as follows,

$$R(\mu) = K(\nu(\mu)), \tag{3.24}$$

where,

$$\nu(\mu) = \frac{1}{1+\mu}; \tag{3.25}$$

since $\nu'(\mu) = (-1)/(1+\mu)^2 < 0$, ν is a decreasing C^1 − diffeomorphism from $(0, +\infty)$ onto $(0, 1)$. So, to study the variations of R it suffices to study those of K. That is, R is strictly increasing (resp. decreasing) on the interval $I \subset (0, +\infty)$ if and only if K is strictly decreasing (resp. increasing) on $\nu(I) = J \subset (0, 1)$, and then R admits a local maximum (resp. minimum) value at μ_* if and only if K admits a local maximum (resp. minimum) value at $\nu(\mu_*)$. Then, we have to prove that K admits at most one maximum value. To this end, we prove that each critical value of K is a maximum one, that is,

$$(\forall \nu_* \in (0, 1)) \; \left(K'(\nu_*) = 0 \Longrightarrow K''(\nu_*) < 0 \right).$$

We have

$$K(\nu) = \int_0^{\frac{1}{\nu}-1} \frac{dt}{(1+t)\left(\frac{1}{\nu} - (1+t)\right)^{\frac{1}{p}}},$$

and performing the change of variables $x = 1 + t$, we find

$$K(\nu) = \nu^{\frac{1}{p}} \int_\nu^1 \frac{dx}{x(1-x)^{\frac{1}{p}}}, \tag{3.26}$$

while some simple computations yield necessiraly

$$K'(\nu) = -\frac{1}{p}\nu^{\frac{1}{p}-1} \left\{ -\int_\nu^1 \frac{dx}{x(1-x)^{\frac{1}{p}}} + \frac{p}{(1-\nu)^{\frac{1}{p}}} \right\} \tag{3.27}$$

$$K''(\nu) = -\frac{1}{p} \left\{ \left(\frac{p-1}{p}\right) \nu^{\frac{1}{p}-2} \int_\nu^1 \frac{dx}{x(1-x)^{\frac{1}{p}}} + \frac{\nu^{\frac{1}{p}-1}}{(1-\nu)^{1+\frac{1}{p}}} + \frac{(2-p)\nu^{\frac{1}{p}-2}}{(1-\nu)^{\frac{1}{p}}} \right\} \tag{3.28}$$

At this stage, one can immediately concludes that for $1 < p \leq 2$, $K''(\nu) < 0$, $\forall \nu \in (0, 1)$. But, for $p > 2$ the sign of $K''(\nu)$ is not known, that is why we perform the artifice of Smoller & Wasserman [106] (described in the introduction of this chapter, page 39). That is, from (3.27) we have the value of the integral,

$$\int_\nu^1 \frac{dx}{x(1-x)^{\frac{1}{p}}} = \frac{p}{(1-\nu)^{\frac{1}{p}}} + pK'(\nu)\nu^{1-\frac{1}{p}},$$

and then, substituting it in (3.28) we find after simple computations,

$$K''(\nu) = \left(-\frac{1}{p}\right)\left\{\frac{\nu^{\frac{1}{p}-2}}{(1-\nu)^{\frac{1}{p}}} + \frac{\nu^{\frac{1}{p}-1}}{(1-\nu)^{1+\frac{1}{p}}} + (p-1)\,\nu^{-1}K'(\nu)\right\}.$$

So, if $\nu_* \in (0,1)$ is such that $K'(\nu_*) = 0$ then,

$$K''(\nu_*) = \left(\frac{-1}{p}\right)\left\{\frac{\nu_*^{\frac{1}{p}-2}}{(1-\nu_*)^{\frac{1}{p}}} + \frac{\nu_*^{\frac{1}{p}-1}}{(1-\nu_*)^{1+\frac{1}{p}}}\right\},$$

which is obviously negative. Then, R admits at most one local maximum value. To show existence, it suffices to show that $\lim_{\mu\to 0} R(\mu) = \lim_{\mu\to+\infty} R(\mu) = 0$.

(i) $\lim_{\mu\to 0} R(\mu) = 0$.

$$0 \leq R(\mu) = \mu^{1-\frac{1}{p}}\int_0^1 \frac{dy}{(1+\mu y)(1-y)^{\frac{1}{p}}} \leq \mu^{1-\frac{1}{p}}\int_0^1 \frac{dy}{(1-y)^{\frac{1}{p}}} = \mu^{1-\frac{1}{p}}p',$$

then (i) follows.

(ii) $\lim_{\mu\to+\infty} R(\mu) = 0$.

a) $$\frac{\mu^{1-\frac{1}{p}}}{(1+\mu y)(1-y)^{1/p}} \leq \frac{1}{y^{1-\frac{1}{p}}(1-y)^{1/p}} \in L^1(0,1),\ \forall\mu > 0$$

b) $$\lim_{\mu\to+\infty} \frac{\mu^{1-\frac{1}{p}}}{(1+\mu y)(1-y)^{1/p}} = 0,\ \forall y \in (0,1),$$

then

$$\lim_{\mu\to+\infty} R(\mu) = \lim_{\mu\to+\infty}\int_0^1 \frac{\mu^{1-\frac{1}{p}}}{(1+\mu y)(1-y)^{\frac{1}{p}}}dy = \int_0^1 \lim_{\mu\to+\infty}\frac{\mu^{1-\frac{1}{p}}}{(1+\mu y)(1-y)^{\frac{1}{p}}}dy = 0.$$

and (ii) follows.

The exact diagram: In order to study the variations of $\|u\|_0$ with respect to λ it suffices to study those of μ with respect to λ. In fact,

$$\frac{dr}{d\lambda}(\lambda) = \frac{d}{d\lambda}\log(1+\mu(\lambda)) = \left(\frac{1}{1+\mu(\lambda)}\right)\frac{d\mu}{d\lambda}(\lambda) \text{ and } \frac{1}{1+\mu(\lambda)} > 0.$$

Recall that there exists $\mu_* > 0$ and $\lambda_* > 0$ such that,

$$\left(p'\lambda_*\right)^{-1/p} R(\mu) = \frac{1}{2} \Longleftrightarrow \mu = \mu_*, \tag{3.29}$$

and R is strictly increasing (resp. strictly decreasing) in $(0, \mu_*)$ (resp. $(\mu_*, +\infty)$). Moreover, for any fixed λ, $0 < \lambda < \lambda_*$, there exist $\mu_m(\lambda)$ and $\mu_M(\lambda)$ such that,

$$\mu_m(\lambda) < \mu_* < \mu_M(\lambda),$$

and,

$$\left(p'\lambda\right)^{-1/p} R(\mu_m(\lambda)) = \left(p'\lambda\right)^{-1/p} R(\mu_M(\lambda)) = \frac{1}{2}.$$

So,

$$R(\mu_M(\lambda)) = R(\mu_m(\lambda)) = \frac{1}{2}\left(p'\lambda\right)^{1/p}, \tag{3.30}$$

then,

$$\frac{dR}{d\mu}(\mu_m(\lambda))\frac{d\mu_m}{d\lambda}(\lambda) = \frac{dR}{d\mu}(\mu_M(\lambda))\frac{d\mu_M}{d\lambda}(\lambda) = \frac{p'}{2p}\left(p'\lambda\right)^{\frac{1}{p}-1} > 0,$$

so, $\frac{dR}{d\mu}(\mu_m(\lambda))$ (resp. $\frac{dR}{d\mu}(\mu_M(\lambda))$) and $\frac{d\mu_m}{d\lambda}(\lambda)$ (resp. $\frac{d\mu_M}{d\lambda}(\lambda)$) have the same sign, that is,

$$\mu_m(\lambda) < \mu_* \Longrightarrow \frac{dR}{d\mu}(\mu_m(\lambda)) > 0 \Longrightarrow \frac{d\mu_m}{d\lambda}(\lambda) > 0,$$

$$\mu_M(\lambda) > \mu_* \Longrightarrow \frac{dR}{d\mu}(\mu_M(\lambda)) < 0 \Longrightarrow \frac{d\mu_M}{d\lambda}(\lambda) < 0.$$

Moreover, since the function $\lambda \mapsto \mu_m(\lambda)$ (resp. $\mu_M(\lambda)$) is increasing, (resp. decreasing), then there exist $l_m^0 \in [0, \mu_*)$ and $l_m^* \in (0, \mu_*]$ (resp. $l_M^0 \in (\mu_*, +\infty]$ and $l_M^* \in [\mu_*, +\infty)$) such that $\lim_{\lambda \to 0} \mu_m(\lambda) = l_m^0$ and $\lim_{\lambda \to \lambda_*} \mu_m(\lambda) = l_m^*$ (resp. $\lim_{\lambda \to 0} \mu_M(\lambda) = l_M^0$ and $\lim_{\lambda \to \lambda_*} \mu_M(\lambda) = l_M^*$). By passing to the limit as λ tends to 0 in (3.30) one gets : $R\left(l_m^0\right) = 0$, then $l_m^0 = 0$. (Because $\mu = 0$ is the unique solution of the equation $R(\mu) = 0$). Furthermore, by passing to the limit as λ tends to λ_* in (3.30) we get : $R(l_m^*) = \frac{1}{2}(p'\lambda_*)^{1/p}$, (resp. $R(l_M^*) = \frac{1}{2}(p'\lambda_*)^{1/p}$) then from (3.29) it follows that $l_m^* = \mu_* = l_M^*$.

Now, assume that $l_M^0 \neq +\infty$ then by passing to the limit as λ tends to 0 in (3.30) we have that : $R\left(l_M^0\right) = 0$ then $l_M^0 = 0$ which is impossible because $l_M^0 \in (\mu_*, +\infty]$. Then, the exact diagram of positive solutions is now clear.

3.3.6 Proof of Theorem 28.

In this example, we have for $u \geq 0$

$$f(u) = \left[\varphi_q(u)\right]^{\alpha} = u^{(q-1)\alpha}, \text{ and } F(u) = \int_0^u f(t)\,dt = \frac{u^{(q-1)\alpha+1}}{(q-1)\alpha+1}.$$

For any $E > 0$, consider the equation in s

$$E^p - p'\lambda F(s) = 0, \tag{3.31}$$

which is uniquely solvable, in fact : s satisfies (3.31) if and only if,

$$s = s_* := \left(\frac{1}{p'\lambda}\left((q-1)\alpha+1\right)\right)^{1/((q-1)\alpha+1)} E^{p/((q-1)\alpha+1)}.$$

So, we have,

$$X(E) := \left\{s > 0 : E^p - p'\lambda F(\xi) > 0,\ \forall \xi \in (0, s)\right\} = (0, s_*)$$

then,

$$r(E, \lambda) := \sup X(E) = s_* = \left(\frac{1}{p'\lambda}\left((q-1)\alpha+1\right)\right)^{1/((q-1)\alpha+1)} E^{p/((q-1)\alpha+1)}.$$

Let us notice that,

$$r(E, \lambda) > 0,\ \forall E > 0 \text{ and } f(r(E, \lambda)) > 0,\ \forall E > 0,$$

then $D = (0, +\infty)$. So, the time-map T is defined on $(0, +\infty)$ by,

$$T(E) = \int_0^{r(E,\lambda)} \left\{E^p - p'\lambda F(\xi)\right\}^{-1/p} d\xi = \frac{r(E, \lambda)}{E} \int_0^1 \left[1 - \frac{p'\lambda F(\xi r)}{E^p}\right]^{-1/p} d\xi$$

$$T(E) = \frac{r(E, \lambda)}{E} \int_0^1 \left\{1 - \xi^{(q-1)\alpha+1}\right\}^{-1/p} d\xi.$$

The last above-mentioned integral may be calculated (see for instance Lavrentiev & Chabat, [77], Chap. VII, pp. 595-596),

$$\int_0^1 \left\{1 - \xi^{(q-1)\alpha+1}\right\}^{-1/p} d\xi = \frac{1}{(q-1)\alpha+1} B\left(\frac{1}{(q-1)\alpha+1}, 1 - \frac{1}{p}\right),$$

then,

$$T(E) = B\left(1/((q-1)\alpha+1),\, 1-\frac{1}{p}\right)((q-1)\alpha+1)^{-(q-1)\alpha/((q-1)\alpha+1)} \times (p'\lambda)^{-1/((q-1)\alpha+1)}\, E^{((p-1)-(q-1)\alpha)/((q-1)\alpha+1)}.$$

One may observe that the sign of $\partial T/\partial E$ is the same as that of $(p-1)-(q-1)\alpha$, so the following holds :

Discussion:

1. If $(p-1)-(q-1)\alpha > 0$, (resp. < 0) then,

$$\frac{\partial T}{\partial E}(E) > 0,\ (\text{resp. } < 0),\ \forall E > 0,\ \lim_{E\to 0^+} T(E) = 0,\ \ (\text{resp. } +\infty),$$
$$\text{and } \lim_{E\to+\infty} T(E) = +\infty,\ (\text{resp. } 0),$$

in this case, for each fixed $\lambda > 0$, the equation $T(\lambda, E) = 1/2$ admits a unique solution,

$$E(\lambda) = A\lambda^{1/((p-1)-(q-1)\alpha)},$$

where $A > 0$ is a constant, and then,

$$\frac{\partial E}{\partial \lambda}(\lambda) > 0,\ (\text{resp. } < 0)\ \forall \lambda > 0.$$

Therefore, for each fixed $\lambda > 0$, problem (3.9) admits a unique positive solution u_λ such that $u'_\lambda(0) = E(\lambda)$ and the function $\lambda \mapsto u'_\lambda(0)$ is increasing (resp. decreasing) in $(0, +\infty)$ and converges to zero (resp. infinity) when λ tends to zero and converges to infinity (zero) when λ tends to infinity.

2. If $(p-1)-(q-1)\alpha = 0$, then the function $E \mapsto T(E,\lambda)$ is constant, so,

$$T(E,\lambda) = \frac{1}{2} \iff \lambda = \lambda_1 := (p-1)\left(\frac{2\pi}{p\sin\left(\frac{\pi}{p}\right)}\right)^p;$$

if $\lambda \neq \lambda_1$, then problem (3.9) admits no positive solution and if $\lambda = \lambda_1$, then for each $E > 0$, problem (3.9) admits a unique positive solution u_E satisfying $u'_E(0) = E$. Observe that in this case it holds that $\left(\varphi_q(u)\right)^\alpha = \varphi_p(u)$, $\forall u > 0$; so, we can show that any two positive solutions are proportional. The proof is complete. ■

Chapter 4

Boundary value problems for the one dimensional p-Laplacian with even superlinearity

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4.1 Introduction

This chapter is devoted to a study of existence and multiplicity of solutions of the quasilinear two-point boundary value problem

$$\begin{aligned} -\left(\varphi_p(u')\right)' &= f(\lambda, u) \text{ in } (0,1) \\ u(0) &= u(1) = 0, \end{aligned} \tag{4.1}$$

where $\varphi_p(s) := |s|^{p-2}s$ and $f(\lambda, u) := |u|^p - \lambda$. Here, $\left(\varphi_p(u')\right)'$ is the one dimensional p-Laplacian, and $p > 1$.

When the differential operator is linear, i.e., $p = 2$, several existence and multiplicity results, related to superlinear boundary value problems with Dirichlet boundary data, are available in the literature. Let us recall some of them for the one dimensional case.

Lupo et al. [79] have studied the non-autonomous case

$$-u''(x) = u^2(x) - t\sin x, \text{ in } (0,\pi), \quad u(0) = u(\pi) = 0. \tag{4.2}$$

Using a combination of shooting and topological arguments, they show that for any $k \in \mathbb{N}$ there exists $t_k > 0$ such that for all $t \geq t_k$, problem (4.2) admits at least k solutions.

Castro and Shivaji [40], using phase-plane analysis, have considered the problem

$$-u''(x) = g(u(x)) - \rho(x) - t, \quad \text{in} \quad (0, 1), \quad u(0) = u(1) = 0, \tag{4.3}$$

where ρ a continuous function in $[0, 1]$ and $g \in C^1(\mathbb{R})$ satisfies

$$\lim_{s \to -\infty} \frac{g(s)}{s} = M \in \mathbb{R}, \quad \text{and} \quad \lim_{s \to +\infty} \frac{g(s)}{s^{1+\sigma}} = +\infty, \quad \text{with some } \sigma > 0.$$

They show that for $k \in \mathbb{N}$ there exists $t_k(M)$ such that $\lim_{k \to +\infty} t_k(M) = +\infty$, and for all $t > t_k$, problem (4.3) has at least two solutions with k nodes in $(0, 1)$.

The autonomous case has been studied by many authors. Let us mention some of them. Independently of Castro and Shivaji, Ruf and Solimini [97] have considered the problem

$$-u''(x) = g(u(x)) - t, \quad \text{in} \quad (0, \pi), \quad u(0) = u(\pi) = 0 \tag{4.4}$$

where

$$g \in C^1(\mathbb{R}), \quad \limsup_{s \to -\infty} g'(s) < +\infty, \quad \text{and} \quad \lim_{s \to +\infty} g'(s) = +\infty.$$

Using variational methods, they show that for any $k \in \mathbb{N}$ there exists $t_k \in \mathbb{R}$ such that for $t > t_k$ problem (4.4) has at least k distinct solutions.

Prior to the papers mentioned above, Scovel [102] obtained the same result as Ruf and Solimini [97] in the special case where $g(u) = 6u^2$. He has shown that for any integer $k \geq 1$, there exist values $t_1 < \cdots < t_k$ such that for $t > t_k$ problem (4.4) (with $g(u) = 6u^2$) admits at least k distinct solutions.

Independently and prior to Scovel, Ammar Khodja [20] obtained in 1983, a complete description of the solution set of the problem

$$-u''(x) = u^2(x) - \lambda, \quad \text{in} \quad (0, 1), \quad u(0) = u(1) = 0. \tag{4.5}$$

He has detected all the solutions to (4.5) for any value of $\lambda \in \mathbb{R}$, and thus has obtained the exact number of solutions to (4.5) for all λ.. To state his result, denote for any integer $k \geq 1$ the sets

$$S_k^+ = \left\{ u \in C_0^2[0,1] : u'(0) > 0,\ u \text{ admits } k-1 \text{ nodes in } (0,1) \right\}$$

$$S_k^- = -S_k^+ \quad \text{and} \quad S_k = S_k^+ \cup S_k^-.$$

Theorem 36 [20] There exists a sequence $(\lambda_k)_{k \geq 0}$ such that

$$-1 < \lambda_0 < 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$$

and:

(i) If $\lambda < \lambda_0$, problem (4.5) admits no solution.

If $\lambda_0 < \lambda < 0$, problem (4.5) admits exactly two solutions and they are positive.

If $\lambda = \lambda_0$ or $0 \leq \lambda < \lambda_1$, there exists a unique positive solution.

If $\lambda > \lambda_1$, there is no positive solution.

(ii) If $\lambda > 0$, there exists a unique solution in S_1^-.

(iii) If (and only if) $\lambda > \lambda_k$:

- there exist exactly 2 solutions in S_{2k}
- there exists exactly one solution in S_{2k+1}^-

(iv) There exists a sequence $(\mu_k)_{k \geq 1}$ such that

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_k < \lambda_{k+1} < \cdots$$

and such that:

if (and only if) $\mu_k < \lambda < \lambda_{k+1}$, there exist exactly two solutions in S_{2k+1}^+,

if (and only if) $\lambda = \mu_k$ or $\lambda > \lambda_{k+1}$, there exists a unique solution in S_{2k+1}^+.

The objective of this chapter is to extend Ammar Khodja's result to the general quasilinear case $p > 1$. In particular, we will show that if $\lambda \leq 0$ the same result holds for all $p > 1$, but if

$\lambda > 0$ and $p > 2$ the situation is different from that obtained in [20]. So, the behavior of the solution set of problem (4.1) depends not only on the values of λ (as was shown in [20]) but also on those of the parameter p.

These changes in the behavior of the solution set when the parameter p varies is not new in the literature. Guedda and Veron [69] have considered the problem

$$\begin{cases} -\left(\varphi_p(u')\right)' = \lambda\varphi_p(u) - f(u), & \text{in } (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{4.6}$$

where f is a C^1 odd function such that the function $s \mapsto f(s)/s^{p-1}$ is strictly increasing on $(0, +\infty)$ with limit 0 at 0 and $\lim_{s\to+\infty} f(s)/s^{p-1} = +\infty$. They denote by E_λ the solution set of problem (4.6) and show, under some technical assumptions, that when $1 < p \le 2$ the structure of E_λ is exactly the same as in the case $p = 2$, and strictly different in the cases $p > 2$.

This chapter is organized as follows. In Section 4.2 we recall notation introduced in Chapter 0, and state the main results (Theorems 37 and 38). In Section 4.3 we prove Theorem 37 and in Section 4.4, we prove Theorem 38. Finally, we close this chapter by a remark in Section 4.5.

4.2 Main results

The first result is concerned with the case $\lambda \le 0$ and gives the exact number of solutions to Problem (4.1).

Theorem 37 (Cases $\lambda \le 0$) There exists a number $\lambda_* < 0$ such that:

(i) If $\lambda < \lambda_*$, problem (4.1) admits no solution.

(ii) If $\lambda = \lambda_*$, problem (4.1) admits a unique solution and it belongs to A_1^+.

(iii) If $\lambda_* < \lambda < 0$, problem (4.1) admits exactly two solutions and they belong to A_1^+.

(iv) If $\lambda = 0$, beside the trivial solution, problem (4.1) admits a unique solution and it belongs to A_1^+.

The second result is concerned with the case $\lambda > 0$.

Theorem 38 (Cases $\lambda > 0$) For any $p > 1$ there exist two real numbers $J(p) > J_+(p) > 0$ and for all $p > 2$ there exists a positive real number $J_-(p) < J(p)$ such that for all integer $n \geq 1$:

(i) Problem (4.1) admits a solution in B_n^+ if and only if $\lambda = (2nJ(p))^{p^2}$, and in this case, the solution is unique.

(ii) Problem (4.1) admits no solution in $\bigcup_{n \geq 1} B_n^-$.

(iii) Problem (4.1) (with $\lambda > 0$) admits a solution in A_1^+ if and only if $0 < \lambda < (2J(p))^{p^2}$, and in this case, the solution is unique.

(iv) Problem (4.1) admits a solution in A_1^- if and only if ($1 < p \leq 2$ and $\lambda > 0$) or ($p > 2$ and $0 < \lambda < (2J_-(p))^{p^2}$), and in this case, the solution is unique.

(v) Problem (4.1) admits a solution $u_{2n}^{\pm}$ in $A_{2n}^{\pm}$ provided $1 < p \leq 2$ and $\lambda > (2nJ(p))^{p^2}$ or $p > 2$ and

$$\inf\left\{(2nJ(p))^{p^2}, (2n(J_-(p) + J_+(p)))^{p^2}\right\}$$
$$< \lambda < \sup\left\{(2nJ(p))^{p^2}, (2n(J_-(p) + J_+(p)))^{p^2}\right\}$$

(vi) Problem (4.1) admits a solution in A_{2n+1}^+ provided $1 < p \leq 2$ and $\lambda > (2(n+1)J(p))^{p^2}$ or $p > 2$ and

$$\inf\left\{(2(n+1)J(p))^{p^2}, (2((n+1)J_+(p) + nJ_-(p)))^{p^2}\right\}$$
$$< \lambda < \sup\left\{(2(n+1)J(p))^{p^2}, (2((n+1)J_+(p) + nJ_-(p)))^{p^2}\right\}$$

(vii) Problem (4.1) admits a solution in A_{2n+1}^- provided $1 < p \leq 2$ and $\lambda > (2nJ(p))^{p^2}$ or $p > 2$ and

$$\inf\left\{(2nJ(p))^{p^2}, (2((n+1)J_-(p) + nJ_+(p)))^{p^2}\right\}$$
$$< \lambda < \sup\left\{(2nJ(p))^{p^2}, (2((n+1)J_-(p) + nJ_+(p)))^{p^2}\right\}$$

Remark 6 According to Proposition ?? (Chapter 0), if $\lambda > 0$ and $p \in (1, 2]$ then $S \subset \left(\bigcup_{k \geq 1} A_k\right) \cup \left(\bigcup_{k \geq 1} B_k\right)$, where S denotes the solution set of problem (4.1).

Remark 7 The results obtained in [20], for $p = 2$, regarding solutions in A_{2n}, A^-_{2n+1}, and A^+_{2n+1}, are more precise than those stated in Theorem 38, Assertions (v), (vi) and (vii) for $p \neq 2$. In fact, these assertions do not provide the exact number of solutions in A_{2n}, A^-_{2n+1}, and A^+_{2n+1}. The proof given in [20] uses strongly the fact that the nonlinearity $u \mapsto u^2 - \lambda$ is a second degree polynomial function. We were not able to obtain the same degree of accuracy.

4.3 Proof of Theorem 37

Since $\lambda \leq 0$, any solution to (4.1) is positive. In fact, if u is a solution to (4.1) then

$$u'(x) = \varphi_{p'}\left(\varphi_p\left(u'(x)\right)\right), \ \forall x \in (0, 1).$$

Since $x \mapsto \varphi_p(u'(x))$ is decreasing (from $\left(\varphi_p(u')\right)'(x) = -|u(x)|^p + \lambda$, for all $x \in (0, 1)$ and $\lambda \leq 0$) and $\varphi_{p'}$ is increasing, it follows that u' is decreasing. This shows that u is concave, and since $u(0) = u(1) = 0$ it follows that u is positive.

Moreover, the nonlinear term $f(\lambda, u) = |u|^p - \lambda$, satisfies the condition of Theorem 11, so any nontrivial solution is necessarily in A^+_1. Hence, we have only to define the time map T_+. In order to do this, we need the following technical lemma.

4.3.1 Technical Lemmas

Lemma 39 Consider the equation in $s \in \mathbb{R}$:

$$E^p - p' F(\lambda, s) = 0, \tag{4.7}$$

where $p > 1$, $\lambda \leq 0$ and $E \geq 0$ are real parameters and $F(\lambda, s) = \int_0^s f(\lambda, t)\,dt$. Then for any $E > 0$, (resp. $E = 0$) equation (4.7) admits a unique positive zero $s_+ = s_+(p, \lambda, E)$ (resp. a unique zero $s_+ = s_+(p, \lambda, 0) = 0$). Moreover:

(a) The function $E \mapsto s_+(p, \lambda, E)$ is C^1 on $(0, +\infty)$ and

$$\frac{\partial s_+}{\partial E}(p, \lambda, E) = \frac{(p-1)\,E^{p-1}}{f(\lambda, s_+(p, \lambda, E))} > 0$$

for all $p > 1$, all $\lambda \leq 0$, and $E > 0$.

(b) $\lim_{E \to 0^+} s_+(p, \lambda, E) = 0$.

(c) $\lim_{E \to +\infty} s_+(p, \lambda, E) = +\infty$.

Proof For a fixed $p > 1$, $\lambda \leq 0$ and $E \geq 0$, consider the function

$$s \mapsto M(p, \lambda, E, s) := E^p - p' F(\lambda, s) = E^p - p' s \left(\frac{|s|^p}{p+1} - \lambda \right),$$

defined in $\mathbb{R}$ which is strictly decreasing and such that

$$M(p, \lambda, E, 0) = E^p \geq 0, \quad \text{and} \quad \lim_{s \to +\infty} M(s) = -\infty.$$

It is clear that (4.7) admits, for any $E > 0$, a unique positive zero, $s_+ = s_+(p, \lambda, E)$; and if $E = 0$, it admits a unique zero $s_+ = 0$.

Now, for any $p > 1$ and $\lambda \leq 0$, consider the real-valued function

$$(E, s) \mapsto M_+(E, s) := E^p - p' s \left(\frac{s^p}{p+1} - \lambda \right)$$

defined on $\Omega_+ = (0, +\infty)^2$. We have $M_+ \in C^1(\Omega_+)$ and

$$\frac{\partial M_+}{\partial s}(E, s) = -p' f(\lambda, s) = -p'(|s|^p - \lambda) \quad \text{in } \Omega_+,$$

hence

$$\frac{\partial M_+}{\partial s}(E, s) < 0, \quad \text{in } \Omega_+$$

and we can observe that $s_+(p, \lambda, E)$ belongs to the open interval $(0, +\infty)$ and satisfies from its definition

$$M_+(E, s_+(p, \lambda, E)) = 0. \tag{4.8}$$

So, we can make use of the implicit function theorem to show that the function $E \mapsto s_+(p, \lambda, E)$ is $C^1((0, +\infty), \mathbb{R})$ and to obtain the expression for $\frac{\partial s_+}{\partial E}(p, \lambda, E)$ given in (a). Hence, for any fixed $p > 1$ and $\lambda \leq 0$, the function defined in $(0, +\infty)$ by $E \mapsto s_+(p, \lambda, E)$ is strictly increasing and bounded from below by 0 and from above by $+\infty$. Thus the limit $\lim_{E \to 0^+} s_+(p, \lambda, E) = l_0^+$ exists as a real number and the limit $\lim_{E \to +\infty} s_+(p, \lambda, E) = l_{+\infty}$ exists and belongs to $(0, +\infty]$.

Moreover

$$0 \le l_0^+ < l_{+\infty} \le +\infty.$$

Let's notice that, for any fixed $p > 1$ and $\lambda \le 0$, the function $(E, s) \mapsto M_+(E, s)$ is continuous in $[0, +\infty)^2$ and the function $E \mapsto s_+(p, \lambda, E)$ is continuous in $(0, +\infty)$ and satisfies (4.8). So, by passing to the limit in (4.8) as E tends to 0^+ we get:

$$0 = \lim_{E \to 0^+} M_+(E, s_+(p, \lambda, E)) = M_+\left(0, l_0^+\right).$$

Hence, l_0^+ is a zero, belonging to $[0, +\infty)$, of the equation in s : $M_+(0, s) = 0$. By resolving this equation in $[0, +\infty)$ we find: $l_0^+ = 0$. Assertion (b) is proved.

Assume that $l_{+\infty} < +\infty$; then by passing to the limit in (4.8) as E tends to $+\infty$ we find:

$$+\infty = p' l_{+\infty} \left(\frac{(l_{+\infty})^p}{p+1} - \lambda \right) < +\infty,$$

which is impossible. So, $l_{+\infty} = +\infty$ and Lemma 39 is proved. ■

Now, we are ready to compute $X_+(p, \lambda, E)$ as defined in Chapter 1, for any $p > 1$, $\lambda \le 0$ and $E \ge 0$. In fact $X_+(p, \lambda, E) = (0, s_+(p, \lambda, E))$ if $E > 0$ and $X_+(p, \lambda, 0) = \emptyset$. Thus

$$r_+(p, \lambda, E) := \sup X_+(p, \lambda, E) = s_+(p, \lambda, E) \text{ if } E > 0 \text{ and } r_+(p, \lambda, 0) = 0,$$

and since $f(\lambda, s) = |s|^p - \lambda > 0$, $\forall (\lambda, s) \in (-\infty, 0] \times \mathbb{R}$, $(\lambda, s) \neq (0, 0)$ it follows that

$$\begin{aligned} D_+ &:= \{E \ge 0 : 0 < r_+(p, \lambda, E) < +\infty \text{ and } f(\lambda, r_+(p, \lambda, E)) > 0\} \\ &= (0, +\infty). \end{aligned}$$

Before going further in the investigation, we deduce from Lemma 39 the following:

$$\lim_{E \to 0^+} r_+(p, \lambda, E) = 0 \quad \text{and} \quad \lim_{E \to +\infty} r_+(p, \lambda, E) = +\infty, \tag{4.9}$$

$$\frac{\partial r_+}{\partial E}(p, \lambda, E) = \frac{(p-1)\, E^{p-1}}{f(\lambda, r_+(p, \lambda, E))} > 0, \ \forall E \in D_+ = (0, +\infty), \ \forall \lambda \le 0. \tag{4.10}$$

We define, for any $p > 1$, $\lambda \le 0$, and $E \in D_+ = (0, +\infty)$ the time map

$$T_+(p, \lambda, E) := \int_0^{r_+(p, \lambda, E)} \left(E^p - p' F(\lambda, \xi) \right)^{-\frac{1}{p}} d\xi, \quad E \in D_+ \tag{4.11}$$

and a simple change of variables shows that

$$T_+(p, \lambda, E) = r_+(p, \lambda, E) \int_0^1 \left(E^p - p' F(\lambda, r_+(p, \lambda, E)\xi) \right)^{-\frac{1}{p}} d\xi. \tag{4.12}$$

Observe that from the definition of $s_+(p, \lambda, E)$ we have

$$E^p - p' F(\lambda, s_+(p, \lambda, E)) = 0$$

and so, from the definition of $r_+(p, \lambda, E)$, we have $E^p = p' F(\lambda, r_+(p, \lambda, E))$. So, (4.12) may be written as

$$\begin{aligned} &T_+(p, \lambda, E) \\ &= r_+(p, \lambda, E)(p')^{-1/p} \int_0^1 \{F(\lambda, r_+(p, \lambda, E)) - F(\lambda, r_+(p, \lambda, E)\xi)\}^{-1/p} d\xi \end{aligned} \tag{4.13}$$

Equivalently

$$\begin{aligned} &T_+(p, \lambda, E) \\ &= r_+^{1-\frac{1}{p}}(p, \lambda, E)(p')^{-1/p} \int_0^1 \left\{ \frac{r_+^p(p, \lambda, E)\left(1 - \xi^{p+1}\right)}{p+1} - \lambda(1 - \xi) \right\}^{-1/p} d\xi \end{aligned} \tag{4.14}$$

Lemma 40 If $\lambda \leq 0$ then the following hold true

(i) $\lim_{E \to 0^+} T_+(p, \lambda, E) = 0$, if $\lambda < 0$ and $\lim_{E \to 0^+} T_+(p, 0, E) = +\infty$,

(ii) $\lim_{E \to +\infty} T_+(p, \lambda, E) = 0$, $\forall \lambda \leq 0$,

(iii) If $\lambda < 0$, $T_+(p, \lambda, \cdot)$ admits a unique critical point, $E^*(\lambda)$, at which it attains its global maximum value. Moreover,

(a) The function $\lambda \mapsto T_+(p, \lambda, E^*(\lambda))$ is strictly increasing in $(-\infty, 0)$.

(b) $\lim_{\lambda \to -\infty} T_+(p, \lambda, E^*(\lambda)) = 0$.

(c) $\lim_{\lambda \to 0^-} T_+(p, \lambda, E^*(\lambda)) = +\infty$.

(iv) If $\lambda = 0$, $(\partial T_+/\partial E)(p, 0, \cdot) < 0$ in $(0, +\infty)$.

Proof (i) If $\lambda < 0$, we deduce from (4.14)

$$0 \leq T_+(p,\lambda,E) \leq r_+^{1-\frac{1}{p}}(p,\lambda,E)\left(p'\right)^{-\frac{1}{p}} \int_0^1 \{-\lambda(1-\xi)\}^{-\frac{1}{p}} d\xi.$$

So, by passing to the limit as E tends to 0, we find

$$0 \leq \lim_{E \to 0} T_+(p,\lambda,E) \leq \lim_{E \to 0} r_+^{1-\frac{1}{p}}(p,\lambda,E)\left(p'\right)^{-\frac{1}{p}} \int_0^1 \{-\lambda(1-\xi)\}^{-\frac{1}{p}} d\xi = 0.$$

If $\lambda = 0$, then (4.14) yields

$$T_+(p,0,E) = \left(p'\right)^{-\frac{1}{p}} r_+^{-\frac{1}{p}}(p,0,E) \int_0^1 \left(\frac{1-\xi^{p+1}}{p+1}\right)^{-\frac{1}{p}} d\xi,$$

and from (4.9) $\lim_{E \to 0^+} T_+(p,0,E) = +\infty$.

(ii) From (4.14) we have for any $\lambda \leq 0$,

$$0 \leq T_+(p,\lambda,E) \leq r_+^{-1/p}(p,\lambda,E)\left(p'\right)^{-1/p} \int_0^1 \left\{\frac{1-\xi^{p+1}}{p+1}\right\}^{-1/p} d\xi.$$

So, by passing to the limit as E tends to $+\infty$, we deduce

$$\begin{aligned} 0 &\leq \lim_{E \to +\infty} T_+(p,\lambda,E) \\ &\leq \lim_{E \to +\infty} r_+^{-1/p}(p,\lambda,E)\left(p'\right)^{-1/p} \int_0^1 \left\{\frac{1-\xi^{p+1}}{p+1}\right\}^{-1/p} d\xi = 0. \end{aligned}$$

(iii) If $\lambda < 0$, then from (i) and (ii) we deduce that $T_+(p,\lambda,\cdot)$ admits at least one critical point. Here, we are going to prove its uniqueness. From (4.13), let us notice that

$$T_+(p,\lambda,E) = \left(p'\right)^{-1/p} S(p,\lambda,\rho(p,\lambda,E))$$

where $\rho(p,\lambda,E) = r_+(p,\lambda,E)$ and

$$S(p,\lambda,\rho) = \int_0^\rho \{F(p,\lambda,\rho) - F(p,\lambda,\xi)\}^{-1/p} d\xi.$$

On the other hand, observe that for each fixed $\lambda < 0$, the function $E \mapsto \rho(p,\lambda,E)$ is an increasing C^1-diffeomorphism from $(0,+\infty)$ onto itself (Lemma 39, Assertions (a), (b) and (c)), and

$$\frac{\partial T_+}{\partial E}(p,\ \lambda,\ E) = \left(p'\right)^{-1/p} \times \frac{\partial S}{\partial \rho}(p,\ \lambda,\ \rho(p,\ \lambda,\ E)) \times \frac{\partial \rho}{\partial E}(p,\ \lambda,\ E). \tag{4.15}$$

So, to study the variations of $E \mapsto T_+(p,\ \lambda,\ E)$ it suffices to study those of $\rho \mapsto S(p,\ \lambda,\ \rho)$. That is, $S(p,\ \lambda,\ \cdot)$ attains a local maximum (resp. minimum) value at ρ_* if and only if $T_+(p,\ \lambda,\ \cdot)$ does so at $\rho^{-1}_{p,\lambda}(\rho_*)$, where $\rho^{-1}_{p,\lambda}$ is the inverse function of $\rho(p,\ \lambda,\ \cdot)$. From (i) and (ii), it follows that $\lim_{\rho\to 0} S(\rho) = \lim_{\rho\to+\infty} S(\rho) = 0$, that is, S admits at least a maximum value. To prove uniqueness, we first show a priori estimates on the critical points of $S(p,\ \lambda,\ \cdot)$. That is, for each $\lambda < 0$, we look for a compact interval $J(\lambda)$ which contains all possible critical points of $S(p,\ \lambda,\ \cdot)$. Next, we prove that $S(p,\ \lambda,\ \cdot)$ is concave in $J(\lambda)$. We have

$$\frac{\partial S}{\partial \rho}(p,\ \lambda,\ \rho) = \int_0^\rho \frac{H(p,\ \lambda,\ \rho) - H(p,\ \lambda,\ u)}{p\rho\,(F(p,\ \lambda,\ \rho) - F(p,\ \lambda,\ u))^{\frac{p+1}{p}}}du \tag{4.16}$$

where $H(p,\ \lambda,\ u) = pF(p,\ \lambda,\ u) - uf(p,\ \lambda,\ u) = \dfrac{-u^{p+1}}{p+1} - \lambda(p-1)u,\ \forall u > 0$. The variations of $u \mapsto H(p,\ \lambda,\ u)$ can be described as follows. $H(p,\ \lambda,\ \cdot)$ is strictly increasing in $(0,\ \rho_1(p,\ \lambda))$ and strictly decreasing in $(\rho_1(p,\ \lambda),\ +\infty)$ where $\rho_1(p,\ \lambda) = (-\lambda(p-1))^{1/p}$. Moreover, $H(p,\ \lambda,\ 0) = H(p,\ \lambda,\ \rho_2(p,\ \lambda)) = 0$ where $\rho_2(p,\ \lambda) = \left(-\lambda\left(p^2-1\right)\right)^{1/p} > \rho_1(p,\ \lambda)$. So, it follows that:

$$\frac{\partial S}{\partial \rho}(p,\ \lambda,\ \rho) > 0, \quad \forall \rho \in (0,\ \rho_1(p,\ \lambda))$$

and

$$\frac{\partial S}{\partial \rho}(p,\ \lambda,\ \rho) < 0, \quad \forall \rho \in (\rho_2(p,\ \lambda),\ +\infty).$$

That is, we get a priori estimate as follows : $\forall p > 1,\ \forall \lambda < 0,\ \forall \rho_* > 0,$

$$\frac{\partial S}{\partial \rho}(p,\ \lambda,\ \rho_*) = 0 \Rightarrow \rho_* \in J(\lambda) := [\rho_1(p,\ \lambda),\ \rho_2(p,\ \lambda)].$$

Easy computations show that for any $\rho > 0$ and $\lambda < 0$, we have

$$\frac{\partial^2 S}{\partial \rho^2}(p,\ \lambda,\ \rho) = \int_0^1 \frac{(p+1)(H(p,\ \lambda,\ \rho) - H(p,\ \lambda,\ \rho\xi))^2}{p^2\rho\,(F(p,\ \lambda,\ \rho) - F(p,\ \lambda,\ \rho\xi))^{(2p+1)/p}}d\xi$$

$$+ \int_0^1 \frac{p(\Psi(p,\ \lambda,\ \rho) - \Psi(p,\ \lambda,\ \rho\xi))(F(p,\ \lambda,\ \rho) - F(p,\ \lambda,\ \rho\xi))}{p^2\rho\,(F(p,\ \lambda,\ \rho) - F(p,\ \lambda,\ \rho\xi))^{(2p+1)/p}}d\xi$$

where

$$\begin{aligned}\Gamma(p,\lambda,u) &:= -p(p+1)F(p,\lambda,u) + 2puf(p,\lambda,u) - u^2 f'_u(p,\lambda,u)\\ &= \lambda p(p-1)u,\ \forall u>0.\end{aligned}$$

After some substitutions we find

$$\frac{\partial^2 S}{\partial \rho^2}(p,\lambda,\rho) = \int_0^1 \frac{\rho(1-\xi)^2 P(X(\xi))}{p^2 \left(F(p,\lambda,\rho) - F(p,\lambda,\rho\xi)\right)^{\frac{2p+1}{p}}} d\xi$$

where

$$X(\xi) = \begin{cases} p+1 & \text{if } \xi = 1 \\ \dfrac{1-\xi^{p+1}}{1-\xi} & \text{if } \xi \in [0,1) \end{cases}$$

and P is the polynomial function

$$P(X) = \left(\frac{\rho^{2p}}{p+1}\right) X^2 + \frac{(p-1)\left(p^2+2p+2\right)}{(p+1)} \lambda \rho^p X - (p-1)\lambda^2.$$

It's easy to check that $X(\xi) \in [1, p+1]$, for all $\xi \in [0,1]$. In fact, the function $\xi \mapsto h(\xi) := \xi^{p+1}$ is convex in $(0,+\infty)$, and

$$X(\xi) = \frac{h(1)-h(\xi)}{1-\xi} \leq h'(1) = p+1,\ \forall \xi \in (0,1).$$

So, we are interested in the sign of $P(X)$ when $X \in [1, p+1]$. First, its discriminant is $\Delta = \left(\mu(p)/(p+1)^2\right) \lambda^2 \rho^{2p} > 0$ where

$$\mu(p) = (p-1)^2 \left(p^2+2p+2\right)^2 + 4\left(p^2-1\right),$$

and its roots are, for each $\lambda < 0$ and $\rho > 0$,

$$X_1(p,\lambda,\rho) = \frac{\lambda}{2\rho^p}\left(\sqrt{\mu(p)} - (p-1)\left(p^2+2p+2\right)\right) < 0,$$

$$X_2(p,\lambda,\rho) = \frac{-\lambda}{2\rho^p}\left(\sqrt{\mu(p)} + (p-1)\left(p^2+2p+2\right)\right) > 0.$$

It can be verified that $\rho \mapsto X_2(p,\lambda,\rho)$ is decreasing in $(0,+\infty)$ and we deduce, from $H(p,\lambda,\rho_2(p,\lambda)) \equiv 0$, that

$$X_2(p,\lambda,\rho_2(p,\lambda)) = \frac{\sqrt{\mu(p)} + (p-1)\left(p^2+2p+2\right)}{2\left(p^2-1\right)},\ \forall \lambda < 0.$$

Hence, one can deduce that $X_2(p,\ \lambda,\ \rho_2(p,\ \lambda)) > p+1$. (In fact, to prove this it suffices to show that it follows

$$\sqrt{\mu(p)} + (p-1)\left(p^2+2p+2\right) > 2(p+1)\left(p^2-1\right)$$

which is equivalent to proving that $\mu(p) > (p(p-1)(p+2))^2$, and this is (after some simple computations) equivalent to $4(p+1)p^2 > 0$ which is true since $p>1$). Then

$$[1,\ p+1] \subset (X_1(p,\ \lambda,\ \rho),\ X_2(p,\ \lambda,\ \rho)) : \forall\lambda<0,\ \forall\rho\in[\rho_1(p,\ \lambda),\ \rho_2(p,\ \lambda)],$$

hence, $P(X(\xi))<0$, for all $\xi\in[0,\ 1]$, so,

$$\frac{\partial^2 S}{\partial\rho^2}(p,\ \lambda,\ \rho)<0, \qquad \forall\lambda<0,\ \forall\rho\in J(\lambda):=[\rho_1(p,\ \lambda),\ \rho_2(p,\ \lambda)],$$

which proves the uniqueness of the critical point of $S(p,\ \lambda,\ \cdot)$ and of $T_+(p,\ \lambda,\ \cdot)$.

(a) Some easy computations show that

$$\begin{aligned}&\frac{\partial T_+}{\partial E}(p,\ \lambda,\ E)\\ &= (p')^{-1/p}\frac{\partial r_+}{\partial E}(p,\ \lambda,\ E)\int_0^{r_+(p,\lambda,E)}\frac{H(p,\ \lambda,\ r_+)-H(p,\ \lambda,\ \xi)}{pr_+(F(p,\ \lambda,\ r_+)-F(p,\ \lambda,\ \xi))^{\frac{p+1}{p}}}d\xi\end{aligned}\tag{4.17}$$

and that

$$\begin{aligned}\frac{\partial T_+}{\partial\lambda}(p,\ \lambda,\ E) &= (p')^{-\frac{1}{p}}\frac{\partial r_+}{\partial\lambda}(p,\ \lambda,\ E)\times\\ &\times\int_0^{r_+(p,\lambda,E)}\frac{H(p,\ \lambda,\ r_+)-H(p,\ \lambda,\ \xi)}{pr_+(F(p,\ \lambda,\ r_+)-F(p,\ \lambda,\ \xi))^{\frac{p+1}{p}}}d\xi+\\ &+(p')^{-\frac{1}{p}}\int_0^{r_+(p,\lambda,E)}\frac{r_+(p,\ \lambda,\ E)-\xi}{p(F(p,\ \lambda,\ r_+)-F(p,\ \lambda,\ \xi))^{\frac{p+1}{p}}}d\xi\end{aligned}\tag{4.18}$$

and then combining (4.17) and (4.18) we get

$$\begin{aligned}&-\frac{\partial r_+}{\partial\lambda}\frac{\partial T_+}{\partial E}+\frac{\partial r_+}{\partial E}\frac{\partial T_+}{\partial\lambda}\\ &=(p')^{-\frac{1}{p}}\frac{\partial r_+}{\partial E}(p,\ \lambda,\ E)\times\int_0^{r_+(p,\lambda,E)}\frac{r_+(p,\ \lambda,\ E)-\xi}{p(F(p,\ \lambda,\ r_+)-F(p,\ \lambda,\ \xi))^{\frac{p+1}{p}}}d\xi\end{aligned}$$

so,

$$-\frac{\partial r_+}{\partial \lambda}\frac{\partial T_+}{\partial E}+\frac{\partial r_+}{\partial E}\frac{\partial T_+}{\partial \lambda}>0, \qquad \forall E>0,\ \lambda<0. \tag{4.19}$$

Since, $(\partial T_+/\partial E)(p,\ \lambda,\ E^*(\lambda))=0$, using (4.19) and (4.10) one gets

$$\frac{\partial T_+}{\partial \lambda}(p,\ \lambda,\ E^*(\lambda))>0, \qquad \forall \lambda<0.$$

(b) Since $H(p,\ \lambda,\ \cdot)$ is strictly increasing on $(0,\ \rho_1(p,\ \lambda))$, it holds that

$$\frac{\partial T_+}{\partial E}(p,\ \lambda,\ E)>0,\ \forall E\in(0,\ E_1(p,\ \lambda))$$

where $E_1(p,\ \lambda):=(p'F(p,\ \lambda,\ \rho_1(p,\ \lambda)))^{1/p}$. Since $(\partial T_+/\partial E)(p,\ \lambda,\ E^*(\lambda))=0$, it follows that $E^*(\lambda)\ge E_1(\lambda)$, and since

$$F(r_+(E^*(\lambda)))=\frac{(E^*)^p(\lambda)}{p'}\ge\frac{E_1^p(p,\ \lambda)}{p'}=F(\rho_1(p,\ \lambda))$$

and F is continuous and strictly increasing ($\lambda<0$) it follows that

$$r_+(p,\ \lambda,\ E^*(\lambda))\ge\rho_1(p,\ \lambda)=r_+(p,\ \lambda,\ E_1(p,\ \lambda)).$$

From (4.14), we have

$$\begin{aligned}T_+(p,\ \lambda,\ E^*(\lambda)) &\le (p')^{-1/p}r_+^{-1/p}(p,\ \lambda,\ E^*(\lambda))\int_0^1\left(\frac{1-\xi^{p+1}}{p+1}\right)^{-1/p}d\xi\\ &\le (p')^{-1/p}\{\rho_1(p,\ \lambda)\}^{-1/p}\int_0^1\left(\frac{1-\xi^{p+1}}{p+1}\right)^{-1/p}d\xi\end{aligned}$$

and by passing to the limit as λ tends to $-\infty$, we find

$$\begin{aligned}0 &\le \lim_{\lambda\to-\infty}T_+(p,\ \lambda,\ E^*(\lambda))\\ &\le \left(p'\right)^{-1/p}\int_0^1\left(\frac{1-\xi^{p+1}}{p+1}\right)^{-1/p}d\xi\lim_{\lambda\to-\infty}\{-\lambda(p-1)\}^{-1/p^2}=0.\end{aligned}$$

(c) For each $\lambda<0$, we have

$$T_+(p,\ \lambda,\ E^*(\lambda))=\sup_{E>0}T_+(p,\ \lambda,\ E)\ge T_+(p,\ \lambda,\ E_1(\lambda))$$

and from (4.14) and the fact $\rho_1(p, \lambda) = r_+(p, \lambda, E_1(p, \lambda))$ one has

$$T_+(p, \lambda, E_1(\lambda))$$

$$= (-\lambda)^{-1/p^2} \left(p'\right)^{-1/p} (p-1)^{(p-1)/p^2} \int_0^1 \left\{ \frac{p-1}{p+1} \left(1 - \xi^{p+1}\right) + (1-\xi) \right\}^{-1/p} d\xi.$$

So,

$$\lim_{\lambda \to 0^-} T_+(p, \lambda, E^*(\lambda)) \geq \lim_{\lambda \to 0^-} T_+(p, \lambda, E_1(\lambda)) = +\infty.$$

(iv) If $\lambda = 0$, the function $\rho \mapsto S(p, \lambda, \rho)$ is decreasesing strictly on $(0, +\infty)$, since the function $u \mapsto H(p, 0, u) := -u^{p+1}/(p+1)$ does so in $(0, +\infty)$ (See (4.16)). Then, from (4.15) and (4.10) it follows that $(\partial T_+/\partial E)(p, 0, \cdot) < 0$ in $(0, +\infty)$. Therefore, Lemma 40 is proved ■

4.3.2 Completion of the proof of Theorem 37

The proof is an easy consequence of the previous lemmas. In fact, there exists a unique $\lambda^* < 0$ which satisfies $T_+(p, \lambda^*, E^*(\lambda^*)) = \frac{1}{2}$ and the function $\lambda \mapsto T_+(p, \lambda, E^*(\lambda))$ is strictly increasing in $(-\infty, 0)$. So, if $\lambda < \lambda^*$, for any $E > 0$ and $\lambda < 0$,

$$T_+(p, \lambda, E) \leq \sup_{E>0} T_+(p, \lambda, E) = T_+(p, \lambda, E^*(\lambda)) < T_+(p, \lambda^*, E^*(\lambda^*)) = \frac{1}{2}.$$

Thus equation $T_+(p, \lambda, E) = \frac{1}{2}$ admits no solution. If $\lambda = \lambda^*$, then $E^*(\lambda^*)$ is the unique solution of the equation $T_+(p, \lambda^*, E) = \frac{1}{2}$. So, problem (4.1) admits a unique positive solution which is in A_1^+. Finally, if $0 > \lambda > \lambda^*$, then $T_+(p, \lambda, E^*(\lambda)) > T_+(p, \lambda^*, E^*(\lambda^*)) = \frac{1}{2}$. So, equation $T_+(p, \lambda, E) = \frac{1}{2}$ admits exactly two solutions and problem (4.1) admits exactly two positive solutions in A_1^+. If $\lambda = 0$, then $T_+(p, 0, \cdot)$ is strictly decreasing in $(0, +\infty)$ and $\lim_{E \to 0^+} T_+(p, 0, E) = +\infty$, $\lim_{E \to +\infty} T_+(p, 0, E) = 0$. So, equation $T_+(p, 0, E) = (1/2)$ admits a unique solution in $(0, +\infty)$. Theorem 37 is then proved. ■

4.4 Proof of Theorem 38

As for the proof of Theorem 37, we begin this section by giving some preliminary lemmas. In order to define the time-maps we need as usual the following technical lemma.

4.4.1 Technical lemmas

Lemma 41 Consider the equation in the variable $s \in \mathbb{R}^*$,

$$E^p - p' F(\lambda, s) = 0 \tag{4.20}$$

where $p > 1$, $\lambda > 0$ and $E \geq 0$ are real parameters. First, if $E = 0$, equation (4.20) admits a unique positive zero $s_+ = s_+(p, \lambda, 0)$ and a unique negative zero $s_- = s_-(p, \lambda, 0)$ such that $|s_\pm| = (\lambda(p+1))^{1/p}$. Moreover, for any $E > 0$, equation (4.20) admits a unique positive zero $s_+ = s_+(p, \lambda, E)$ and this zero belongs to the open interval $\left((\lambda(p+1))^{1/p}, +\infty\right)$. On the other hand,

(i) If $E > E_*(p, \lambda) := \left(\left(\frac{pp'}{p+1}\right)\lambda^{1+\frac{1}{p}}\right)^{1/p}$, equation (4.20) admits no negative zero.

(ii) If $E = E_*(p, \lambda)$, equation (4.20) admits a unique negative zero $s_- = s_-(p, \lambda) = -\lambda^{1/p}$.

(iii) If $0 < E < E_*(p, \lambda)$, equation (4.20) admits, in the open interval $\left(-\lambda^{1/p}, 0\right)$, a unique zero $s_- = s_-(p, \lambda, E)$.

Moreover,

(a) The function $E \mapsto s_\pm(p, \lambda, E)$ is C^1 in $(0, +\infty)$ (resp. $(0, E_*(p, \lambda))$) and

$$\pm\frac{\partial s_\pm}{\partial E}(p, \lambda, E) = \frac{\pm(p-1)E^{p-1}}{f(\lambda, s_\pm(p, \lambda, E))} > 0,$$

for all $p > 1$, for all $\lambda > 0$, and for all $E > 0$ (resp. for all $E \in (0, E_*(p, \lambda))$).

(b) $\lim_{E \to 0^+} s_+(p, \lambda, E) = ((p+1)\lambda)^{1/p}$ and $\lim_{E \to 0^+} s_-(p, \lambda, E) = 0$.

(c) $\lim_{E \to +\infty} s_+(p, \lambda, E) = +\infty$ and $\lim_{E \to E_*} s_-(p, \lambda, E) = -\lambda^{1/p}$.

Proof For any fixed $p > 1$, $\lambda > 0$ and $E \geq 0$, consider the function

$$s \mapsto N(p, \lambda, E, s) := E^p - p' F(\lambda, s) = E^p - p' s\left(\frac{|s|^p}{p+1} - \lambda\right),$$

defined in $\mathbb{R}$. From a study of its variations, it is clear that equation (4.20) admits, if $E = 0$, a unique positive zero s_+ and a unique negative zero s_-. Their values are obtained by solving equation (4.20). Moreover, for any $E > 0$, equation (4.20) admits a unique positive zero, $s_+ = s_+(p, \lambda, E)$, and this zero belongs to the open interval $\left((\lambda(p+1))^{1/p}, +\infty\right)$ (since

$$N\left(p, \lambda, E, (\lambda(p+1))^{\frac{1}{p}}\right) = N(p, \lambda, E, 0) = E^p > 0).$$

Also, Assertions (i) (ii) and (iii) follow readily from the variations of $N(p, \lambda, E, \cdot)$.

Now, for any $p > 1$ and $\lambda > 0$, consider the real valued function

$$(E, s) \mapsto N_{\pm}(E, s) = E^p - p' s \left(\frac{(\pm s)^p}{p+1} - \lambda \right)$$

defined on $\Omega_+ = (0, +\infty) \times \left((\lambda(p+1))^{1/p}, +\infty \right)$ (resp. $\Omega_- = (0, E_*(p, \lambda)) \times \left(-\lambda^{1/p}, 0 \right)$). We have that $N_{\pm} \in C^1(\Omega_{\pm})$ and

$$\frac{\partial N_{\pm}}{\partial s}(E, s) = -p' f(\lambda, s) = -p' (|s|^p - \lambda) \quad \text{in } \Omega_{\pm},$$

hence

$$\pm \frac{\partial N_{\pm}}{\partial s}(E, s) > 0, \quad \text{in } \Omega_{\pm}$$

and we can notice that $s_{\pm}(p, \lambda, E)$ belongs to the open interval $\left((\lambda(p+1))^{1/p}, +\infty \right)$ (resp. $\left(-\lambda^{1/p}, 0 \right)$) and satisfies (from its definition)

$$N_{\pm}(E, s_{\pm}(p, \lambda, E)) = 0. \tag{4.21}$$

So, we can make use of the implicit function theorem to show that the function $E \mapsto s_{\pm}(p, \lambda, E)$ is $C^1((0, +\infty), \mathbb{R})$ (resp. $C^1((0, E_*(p, \lambda)), \mathbb{R})$) and obtain the expression of $\frac{\partial s_{\pm}}{\partial E}(p, \lambda, E)$ given in (a). Hence, for any fixed $p > 1$ and $\lambda > 0$, the function defined in $(0, +\infty)$ (resp. $(0, E_*(p, \lambda))$) by $E \mapsto s_{\pm}(p, \lambda, E)$ is strictly increasing (resp. decreasing) and bounded from below by $(\lambda(p+1))^{1/p}$ (resp. $-\lambda^{1/p}$) and from above by $+\infty$ (resp. by 0). Then, the limit $\lim_{E \to 0^+} s_{\pm}(p, \lambda, E) = l_0^{\pm}$ exists as real number and the limit $\lim_{E \to +\infty} s_+(p, \lambda, E) = l_{+\infty}$ (resp. $\lim_{E \to E_*} s_-(p, \lambda, E) = l_*$) exists and belongs to $\left((\lambda(p+1))^{1/p}, +\infty \right]$ (resp. $\left[-\lambda^{1/p}, 0 \right]$). Moreover

$$-\infty < -\lambda^{1/p} \le l_* < l_0^- \le 0 < (\lambda(p+1))^{1/p} \le l_0^+ < l_{+\infty} \le +\infty.$$

Let us notice that, for any fixed $p > 1$ and $\lambda > 0$, the function

$$(E, s) \mapsto N_{\pm}(E, s)$$

is continuous in $[0, +\infty) \times \left[(\lambda(p+1))^{1/p}, +\infty \right)$ (resp. $[0, E_*(p, \lambda)] \times (-\infty, 0]$) and the function $E \mapsto s_{\pm}(p, \lambda, E)$ is continuous in $(0, +\infty)$ (resp. $(0, E_*(p, \lambda))$) and satisfies $(4.21)_{\pm}$.

So, by passing to the limit in $(4.21)_\pm$ as E tends to 0^+ one gets

$$0 = \lim_{E \to 0^+} N_\pm \left(E,\ s_\pm (p,\ \lambda,\ E)\right) = N_\pm \left(0,\ l_0^\pm\right).$$

Hence, $l_0^\pm$ is a zero, belonging to $\left[(\lambda (p+1))^{1/p},\ +\infty\right)$ (resp. $\left[-\lambda^{1/p},\ 0\right]$), to the equation in the variable s:

$$N_\pm (0,\ s) = 0.$$

By solving this equation in the indicated interval we find : $l_0^+ = ((p+1)\lambda)^{1/p}$ (resp. $l_0^- = 0$). Assertion (b) is proved.

Assume that $l_{+\infty} < +\infty$, then by passing to the limit in $(4.21)_+$ as E tends to $+\infty$, we find

$$+\infty = p' l_{+\infty} \left(\frac{(l_{+\infty})^p}{p+1} - \lambda\right) < +\infty,$$

which is impossible. So, $l_{+\infty} = +\infty$.

To prove that $l_* = -\lambda^{1/p}$, it suffices to pass to the limit in $(4.21)_-$ as E tends to $E_* (p,\ \lambda)$ to get

$$N_- \left(E_* (p,\ \lambda),\ l_*\right) = 0$$

and to solve this equation in the interval $\left[-\lambda^{1/p},\ 0\right]$. (To this end, notice that the function $s \mapsto N_- \left(E_* (p,\ \lambda),\ s\right)$ is strictly increasing in $\left[-\lambda^{1/p},\ 0\right]$ and

$$N_- \left(E_* (p,\ \lambda),\ -\lambda^{1/p}\right) = 0).$$

Therefore, Lemma 41 is proved. ■

Now we are ready to compute $X_\pm (p,\ \lambda,\ E)$ as defined in Chapter 1, for any $p > 1,\ \lambda > 0$ and $E \geq 0$. In fact $X_+ (p,\ \lambda,\ E) = (0,\ s_+ (p,\ \lambda,\ E))$ and

$$X_- (p,\ \lambda,\ E) = \begin{cases} (-\infty,\ 0) & \text{if} \quad E > E_* (p,\ \lambda) \\ (s_- (p,\ \lambda,\ E),\ 0) & \text{if} \quad 0 \leq E \leq E_* (p,\ \lambda), \end{cases}$$

where $s_\pm (p,\ \lambda,\ E)$ is defined in Lemma 41. Then

$$r_+ (p,\ \lambda,\ E) := \sup X_+ (p,\ \lambda,\ E) = s_+ (p,\ \lambda,\ E)$$

and

$$r_-(p,\lambda,E) := \inf X_-(p,\lambda,E) = \begin{cases} -\infty & \text{if } E > E_*(p,\lambda) \\ s_-(p,\lambda,E) & \text{if } 0 \le E \le E_*(p,\lambda). \end{cases}$$

Recall that for any $E \ge 0$, $s_+(p,\lambda,E)$ belongs to $\left[(\lambda(p+1))^{1/p}, +\infty\right)$. Thus

$$0 < r_+(p,\lambda,E) < +\infty \text{ if and only if } E > 0,$$

Also recall that, for any $0 < E \le E_*(p,\lambda)$, $s_-(p,\lambda,E)$ belongs to $\left[-\lambda^{1/p}, 0\right)$ and $s_-(p,\lambda,0) = -((p+1)\lambda)^{1/p}$ so

$$-\infty < r_-(p,\lambda,E) < 0 \text{ if and only if } 0 \le E \le E_*(p,\lambda).$$

We can notice that $f(\lambda, r_+(p,\lambda,E)) = r_+^p(p,\lambda,E) - \lambda > 0$, $\forall E \ge 0$ and

$$f(\lambda, r_-(p,\lambda,E)) = (-r_-(p,\lambda,E))^p - \lambda < 0 \iff E \in (0, E_*(p,\lambda)),$$

so that

$$\begin{aligned} D_+ &:= \{E \ge 0 \mid 0 < r_+(p,\lambda,E) < +\infty \text{ and } f(\lambda, r_+(p,\lambda,E)) > 0\} \\ &= [0, +\infty). \end{aligned}$$

and

$$\begin{aligned} D_- &:= \{E \ge 0 \mid -\infty < r_-(p,\lambda,E) < 0 \text{ and } f(\lambda, r_-(p,\lambda,E)) < 0\} \\ &= (0, E_*(p,\lambda)). \end{aligned}$$

So,

$$D := D_+ \cap D_- = (0, E_*(p,\lambda)).$$

Before going further in the investigation, we deduce from Lemma 41 that

$$\lim_{E \to 0^+} r_+(p,\lambda,E) = ((p+1)\lambda)^{1/p} \quad \text{and} \quad \lim_{E \to 0^+} r_-(p,\lambda,E) = 0, \tag{4.22}$$

$$\lim_{E \to +\infty} r_+(p,\lambda,E) = +\infty \quad \text{and} \quad \lim_{E \to E_*} r_-(p,\lambda,E) = -\lambda^{1/p}, \tag{4.23}$$

$$\frac{\partial r_\pm}{\partial E}(p,\lambda,E) = \frac{(p-1)E^{p-1}}{f(\lambda, r_\pm(p,\lambda,E))}, \ \forall E \in \operatorname{int}(D_\pm), \tag{4.24}$$

$$\pm \frac{\partial r_\pm}{\partial E}(p,\ \lambda,\ E) > 0,\ \forall E \in \operatorname{int}(D_\pm). \tag{4.25}$$

At present, we define, for any $p > 1,\ \lambda > 0$, and $E \in D_\pm$ the time map

$$T_\pm(p,\ \lambda,\ E) := \pm \int_0^{r_\pm(p,\ \lambda,\ E)} \left\{E^p - p' F(\lambda,\ \xi)\right\}^{-1/p} d\xi,\quad E \in D_\pm, \tag{4.26}$$

and a simple change of variables shows that

$$T_\pm(p,\ \lambda,\ E) = \pm r_\pm(p,\ \lambda,\ E) \int_0^1 \left\{E^p - p' F(\lambda,\ r_\pm(p,\ \lambda,\ E)\,\xi)\right\}^{-1/p} d\xi. \tag{4.27}$$

Observe that from the definition of $s_\pm(p,\ \lambda,\ E)$ one has $E^p - p' F(\lambda,\ s_\pm(p,\ \lambda,\ E)) = 0$, and so, from the definition of $r_\pm(p,\ \lambda,\ E)$, one has $E^p = p' F(\lambda,\ r_\pm(p,\ \lambda,\ E))$. So, (4.27) may be written as

$$\begin{aligned} T_\pm(p,\ \lambda,\ E) &= \pm r_\pm(p,\ \lambda,\ E)\,(p')^{-1/p} \times \\ &\int_0^1 \left\{F(\lambda,\ r_\pm(p,\ \lambda,\ E)) - F(\lambda,\ r_\pm(p,\ \lambda,\ E)\,\xi)\right\}^{-1/p} d\xi \end{aligned} \tag{4.28}$$

After some manipulations, we find that

$$\begin{aligned} T_+(p,\ \lambda,\ E) &= (r_+(p,\ \lambda,\ E))^{-1/p}\,(p')^{-1/p} \times \\ &\int_0^1 \left\{\frac{1 - \xi^{p+1}}{p+1} - \lambda \frac{1 - \xi}{(r_+(p,\ \lambda,\ E))^p}\right\}^{-1/p} d\xi,\quad E \in D_+ \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} T_-(p,\ \lambda,\ E) &= (-r_-(p,\ \lambda,\ E))^{1-\frac{1}{p}}\,(p')^{-1/p} \times \\ &\int_0^1 \left\{\lambda(1 - \xi) - \frac{(-r_-(p,\ \lambda,\ E))^p}{p+1}\left(1 - \xi^{p+1}\right)\right\}^{-1/p} d\xi,\quad E \in D_-. \end{aligned} \tag{4.30}$$

Also, let us define for any $E \in D = D_+ \cap D_-$ and $n \in \mathbb{N}$ the time maps:

$$T_{2n}(p,\ \lambda,\ E) := n\left(T_+(p,\ \lambda,\ E) + T_-(p,\ \lambda,\ E)\right),\ E \in D, \tag{4.31}$$

$$T^\pm_{2n+1}(p,\ \lambda,\ E) := T_{2n}(p,\ \lambda,\ E) + T_\pm(p,\ \lambda,\ E),\ E \in D. \tag{4.32}$$

The limits of these time maps are the aim of the following lemmas.

Lemma 42 For any $p > 1$ and $\lambda > 0$, we have $T_+(p, \lambda, E_*(p, \lambda)) = \lambda^{-1/p^2} \times J_+(p)$, where

$$J_+(p) := \left(p'\right)^{-1/p} (p+1)^{1/p} \theta(p) \times \int_0^1 \left\{ p - (\theta(p)\xi)^{p+1} + (p+1)\theta(p)\xi \right\}^{-1/p} d\xi$$

and $\theta(p) > (p+1)^{1/p}$ is the unique positive zero of the equation

$$\theta^{p+1} - (p+1)\theta - p = 0. \tag{4.33}$$

Proof For any $p > 1$, let us consider the function Θ defined in $(0, +\infty)$ by $\Theta(\theta) := \theta^{p+1} - (p+1)\theta - p$. A study of its variations implies that equation (4.33) admits a unique zero in $(0, +\infty)$, denoted by $\theta(p)$, and this zero belongs to $\left((p+1)^{1/p}, +\infty\right)$ (Note that $\Theta\left((p+1)^{1/p}\right) = -p$). Furthermore, recall (Lemma 41) that, for any $\lambda > 0$ and $E > 0$, $r_+(p, \lambda, E)$ is the unique positive solution of equation (4.20). In particular, if $E = E_*(p, \lambda) := \left(\frac{pp'}{p+1} \lambda^{1+\frac{1}{p}}\right)^{1/p}$ then $r_+(p, \lambda, E_*(p, \lambda))$ is the unique positive solution of the following equation in the variable s:

$$s^{p+1} - \lambda(p+1)s - p\lambda^{1+\frac{1}{p}} = 0. \tag{4.34}$$

Easy computations show that $\theta(p)\lambda^{1/p}$ is also a positive solution of (??), and since (??) admits a unique positive solution (which is $r_+(p, \lambda, E_*(p, \lambda))$) it follows that

$$r_+(p, \lambda, E_*(p, \lambda)) = \theta(p)\lambda^{1/p}, \quad \forall p > 1, \ \forall \lambda > 0.$$

Now, from (4.29), simple computations show that $T_+(p, \lambda, E_*(p, \lambda)) = \lambda^{-1/p^2} \times J_+(p)$ where

$$J_+(p) := \left(p'\right)^{-1/p} (p+1)^{1/p} \theta(p) \int_0^1 \left(p - (\theta(p)\xi)^{p+1} + (p+1)\theta(p)\xi \right)^{-1/p} d\xi.$$

Therefore, Lemma 42 is proved. ■

Lemma 43 For any $p > 1$ and $\lambda > 0$, let

$$J(p) := \frac{1}{p} \left(p'\right)^{-1/p} (p+1)^{\frac{p-1}{p^2}} \cdot \frac{\Gamma\left(\frac{p-1}{p^2}\right) \Gamma\left(\frac{p-1}{p}\right)}{\Gamma\left(\frac{(p-1)(p+1)}{p^2}\right)}.$$

and

$$J_-(p) := \left(p'\right)^{-1/p} (p+1)^{1/p} \int_0^1 \left\{ p - (p+1)\xi + \xi^{p+1} \right\}^{-1/p} d\xi.$$

Then it holds:

$$J_-(p) < +\infty \iff p > 2, \tag{4.35}$$

(i) $\displaystyle\lim_{E\to 0^+} T_+(p,\lambda,E) = J(p)\,\lambda^{-1/p^2}$, (ii) $\displaystyle\lim_{E\to 0^+} T_-(p,\lambda,E) = 0$,

(iii) $\displaystyle\lim_{E\to +\infty} T_+(p,\lambda,E) = 0$, (iv) $\displaystyle\lim_{E\to E_*} T_-(p,\lambda,E) = J_-(p)\,\lambda^{-1/p^2}$.

Proof In order to prove the first assertion we first claim that there exists $\varepsilon_0 > 0$ (sufficiently small) such that for any $\xi \in (1-\varepsilon_0, 1)$,

$$\frac{p(p+1)}{4}(1-\xi)^2 \le p - (p+1)\xi + \xi^{p+1} \le p(p+1)(1-\xi)^2 .$$

To prove this claim, for any $x > 0$, let

$$h_x(\xi) := p - (p+1)\xi + \xi^{p+1} - x(1-\xi)^2, \quad \xi \in (0, 1].$$

Simple computations lead to

$$\frac{dh_x}{d\xi}(\xi) = 2(1-\xi)\left(x - \left(\frac{p+1}{2}\right)\left(\frac{1-\xi^p}{1-\xi}\right)\right), \quad \xi \in (0,1).$$

Using l'Hôpital's rule, we get

$$\lim_{\xi\to 1^-}\left(x - \left(\frac{p+1}{2}\right)\left(\frac{1-\xi^p}{1-\xi}\right)\right) = \left(x - \frac{p(p+1)}{2}\right).$$

So, thanks to the continuity properties, there exists $\varepsilon_1 > 0$ (resp. $\varepsilon_2 > 0$) sufficiently small such that

$$\frac{dh_{p(p+1)}}{d\xi}(\xi) > 0, \ \forall \xi \in (1-\varepsilon_1, 1)$$

(resp. $\left(dh_{p(p+1)/4}/d\xi\right)(\xi) < 0, \ \forall \xi \in (1-\varepsilon_2, 1)$).

Notice that : $h_x(1) = 0, \ \forall x > 0$, so that

$$h_{p(p+1)}(\xi) < 0, \ \forall \xi \in (1-\varepsilon_0, 1)$$

(resp. $h_{p(p+1)/4}(\xi) > 0, \ \forall \xi \in (1-\varepsilon_0, 1)$) where $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$. Then the claim is proved.

With this claim we are able to prove easily the first assertion of this lemma. In fact, the integral which appears in the definition of $J_-(p)$ may be rewritten as

$$\int_0^{1-\varepsilon_0}\left(p-(p+1)\,\xi+\xi^{p+1}\right)^{-1/p}d\xi+\int_{1-\varepsilon_0}^{1}\left(p-(p+1)\,\xi+\xi^{p+1}\right)^{-1/p}d\xi.$$

The first integral converges because the integrand function is continuous on the compact $[0,\ 1-\varepsilon_0]$. For the second integral, it follows from the claim

$$A(p)\int_{1-\varepsilon_0}^{1}\frac{d\xi}{(1-\xi)^{\frac{2}{p}}}\le\int_{1-\varepsilon_0}^{1}\left(p-(p+1)\,\xi+\xi^{p+1}\right)^{-1/p}d\xi\le B(p)\int_{1-\varepsilon_0}^{1}\frac{d\xi}{(1-\xi)^{\frac{2}{p}}}$$

where $A(p)=(p\,(p+1))^{-1/p}$ and $B(p)=(p\,(p+1)/4)^{-1/p}$. So, from the well-known fact

$$\int_{1-\varepsilon_0}^{1}\frac{d\xi}{(1-\xi)^{\frac{2}{p}}}<+\infty\iff p>2$$

the first assertion follows.

Proof of (i). From (4.27), we have

$$T_+(p,\ \lambda,\ E)=r_+(p,\ \lambda,\ E)\int_0^1\left(E^p-p'F(\lambda,\ r_+(p,\ \lambda,\ E)\,\xi)\right)^{-1/p}d\xi.$$

Using (4.22) one gets:

$$\begin{aligned}\lim_{E\to0^+}E^p-p'F(\lambda,\ r_+(p,\ \lambda,\ E)\,\xi) &= -p'F\left(\lambda,\ ((p+1)\,\lambda)^{1/p}\,\xi\right)\\ &= p'\,((p+1)\,\lambda)^{1/p}\,\lambda\xi\,(1-\xi^p)\,,\end{aligned}$$

so, some simple computations yield

$$\lim_{E\to0^+}T_+(p,\ \lambda,\ E)=(p+1)^{\frac{p-1}{p^2}}\left(p'\right)^{-1/p}\lambda^{-1/p^2}\int_0^1\xi^{-1/p}\,(1-\xi^p)^{-1/p}\,d\xi.$$

To compute this integral, we can make use of the change of variables $x=\xi^p$ as well as the relationship between the Euler beta and gamma functions (see for instance [77, Chap. VII, no 90, example 2, pp. 595-596]), to obtain:

$$\int_0^1\xi^{-1/p}(1-\xi^p)^{-1/p}\,d\xi=\frac{1}{p}\,\frac{\Gamma\left(\frac{p-1}{p^2}\right)\Gamma\left(\frac{p-1}{p}\right)}{\Gamma\left(\frac{(p-1)(p+1)}{p^2}\right)}.$$

This completes the proof of (i).

Proof of (ii). Consider the expression of $T_-(p, \lambda, E)$ given by (4.30). From (4.22), we find

$$\lim_{E\to 0^+}\int_0^1 \left(\lambda(1-\xi) - \frac{(-r_-(p,\lambda,E))^p}{p+1}\left(1-\xi^{p+1}\right)\right)^{-1/p} d\xi$$

$$= \int_0^1 (\lambda(1-\xi))^{-\frac{1}{p}}\, d\xi = \lambda^{-\frac{1}{p}} \times p'.$$

So, (4.22) and (4.30) yield

$$\lim_{E\to 0^+} T_-(p,\lambda,E) = \left(p'\right)^{-1/p} \times (0)^{1-\frac{1}{p}} \times \lambda^{-1/p} \times p' = 0.$$

This completes the proof of (ii).

Proof of (iii). Consider the expression for $T_+(p, \lambda, E)$ given by (4.29). From (4.22) it holds

$$\lim_{E\to +\infty}\int_0^1 \left(\frac{1-\xi^{p+1}}{p+1} - \lambda\frac{1-\xi}{r_+^p(p,\lambda,E)}\right)^{-1/p} d\xi = \frac{1}{(p+1)^{-1/p}}\int_0^1 \left(1-\xi^{p+1}\right)^{-1/p} d\xi,$$

and this integral may be computed by making use of the change of variables $x = \xi^{p+1}$ to get

$$\int_0^1 \left(1-\xi^{p+1}\right)^{-1/p} d\xi = \frac{1}{p+1}\,\frac{\Gamma\left(\frac{1}{p+1}\right)\Gamma\left(\frac{p-1}{p}\right)}{\Gamma\left(\frac{p-1}{p(p+1)}\right)}.$$

So, (4.22) and (4.29) yield

$$\lim_{E\to +\infty} T_+(p,\lambda,E) = \left(p'\right)^{-1/p} \times 0 \times \frac{1}{(p+1)^{1-\frac{1}{p}}} \times \frac{\Gamma\left(\frac{1}{p+1}\right)\Gamma\left(\frac{p-1}{p}\right)}{\Gamma\left(\frac{p-1}{p(p+1)}\right)} = 0,$$

completing the proof of (iii).

Proof of (iv). Consider the expression for $T_-(p, \lambda, E)$ given by (4.28). We have

$$\begin{aligned}\lim_{E\to E_*}\left(F(\lambda, r_-(p,\lambda,E)) - F(\lambda, r_-(p,\lambda,E)\,\xi)\right) &= \lim_{x\to -\lambda^{1/p}}\left(F(\lambda,x) - F(\lambda,x\xi)\right)\\ &= \frac{\lambda^{1+\frac{1}{p}}}{p+1}\left(p-(p+1)\,\xi+\xi^{p+1}\right)\end{aligned}$$

so that

$$\lim_{E\to E_*} T_-(p,\lambda,E) = \lambda^{1/p}\times\left(p'\right)^{-1/p}\times\left(\frac{\lambda^{1+\frac{1}{p}}}{p+1}\right)^{-1/p}\times\int_0^1\left(p-(p+1)\xi+\xi^{p+1}\right)^{-1/p}d\xi$$

which is the same as

$$\lim_{E\to E_*} T_-(p,\lambda,E) = \lambda^{-1/p^2}\times\left(p'\right)^{-1/p}\times(p+1)^{1/p}\times\int_0^1\left(p-(p+1)\xi+\xi^{p+1}\right)^{-1/p}d\xi.$$

This completes the proof of (iv) and Lemma 43. ■

Lemma 44 For any $p>1$ and $\lambda>0$, it holds

(a) $\lim_{E\to 0^+} T_{2n}(p,\lambda,E) = nJ(p)\,\lambda^{-1/p^2}$,

(b) $\lim_{E\to 0^+} T^+_{2n+1}(p,\lambda,E) = (n+1)\,J(p)\,\lambda^{-1/p^2}$,

(c) $\lim_{E\to 0^+} T^-_{2n+1}(p,\lambda,E) = nJ(p)\,\lambda^{-1/p^2}$,

(d) $\lim_{E\to E_*} T_{2n}(p,\lambda,E) = n\left(J_+(p)+J_-(p)\right)\times\lambda^{-1/p^2}$,

(e) $\lim_{E\to E_*} T^+_{2n+1}(p,\lambda,E) = \left((n+1)\,J_+(p)+nJ_-(p)\right)\times\lambda^{-1/p^2}$,

(f) $\lim_{E\to E_*} T^-_{2n+1}(p,\lambda,E) = \left(nJ_+(p)+(n+1)\,J_-(p)\right)\times\lambda^{-1/p^2}$.

Proof This proof is an immediate consequence of the two preceding lemmas and the definitions (4.31) and (4.32) of the time maps $T_{2n},\ T^{\pm}_{2n+1}$. ■

Lemma 45 For any $p>1,\ \lambda>0$, it holds:

$$\pm\frac{\partial T_\pm}{\partial E}(p,\lambda,E)<0,\quad \forall E\in D_\pm.$$

Proof From (4.28) we find

$$\pm\frac{\partial T_\pm}{\partial E}(p,\lambda,E) = (p')^{-1/p}\times\left\{\frac{\partial r_\pm}{\partial E}(p,\lambda,E)\int_0^1\left(F(\lambda,r_\pm(p,\lambda,E))-F(\lambda,r_\pm(p,\lambda,E)\xi)\right)^{-1/p}d\xi+\right.$$
$$\left.+r_\pm(p,\lambda,E)\int_0^1\frac{\partial}{\partial E}\left(F(\lambda,r_\pm(p,\lambda,E))-F(\lambda,r_\pm(p,\lambda,E)\xi)\right)^{-1/p}d\xi\right\}$$

$$
\begin{aligned}
&= (p')^{-1/p} \times \left(\frac{\partial r_\pm}{\partial E}(p,\lambda,E) \int_0^1 \frac{(F(\lambda, r_\pm(p,\lambda,E)) - F(\lambda, r_\pm(p,\lambda,E)\xi))}{(F(\lambda, r_\pm(p,\lambda,E)) - F(\lambda, r_\pm(p,\lambda,E)\xi))^{1+\frac{1}{p}}} d\xi \right. \\
&\quad - \frac{1}{p} r_\pm(p,\lambda,E) \frac{\partial r_\pm}{\partial E}(p,\lambda,E) \times \\
&\quad \left. \times \int_0^1 \frac{f(\lambda, r_\pm(p,\lambda,E)) - f(\lambda, r_\pm(p,\lambda,E)\xi)\xi}{(F(\lambda, r_\pm(p,\lambda,E)) - F(\lambda, r_\pm(p,\lambda,E)\xi))^{1+\frac{1}{p}}} d\xi \right)
\end{aligned}
$$

then

$$
\begin{aligned}
\pm\frac{\partial T_\pm}{\partial E}(p,\lambda,E) &= \frac{1}{p}(p')^{-1/p} \left(\pm\frac{\partial r_\pm}{\partial E}(p,\lambda,E) \right) \times \\
&\quad \times \int_0^1 \frac{\pm(H(\lambda, r_\pm(p,\lambda,E)) - H(\lambda, r_\pm(p,\lambda,E)\xi))}{(F(\lambda, r_\pm(p,\lambda,E)) - F(\lambda, r_\pm(p,\lambda,E)\xi))^{1+\frac{1}{p}}} d\xi
\end{aligned}
\tag{4.36}
$$

where $H(\lambda, x) = pF(\lambda, x) - xf(\lambda, x) = \frac{-1}{p+1}|x|^p x - (p-1)\lambda x$. Because the function $x \mapsto H(\lambda, x)$ is decreasing for each fixed $\lambda > 0$ (in fact, $\frac{\partial H}{\partial x}(\lambda, x) < 0$), it follows that

$$
\pm(H(\lambda, r_\pm(p,\lambda,E)) - H(\lambda, r_\pm(p,\lambda,E)\xi)) < 0, \ \forall\lambda > 0, \ \forall\xi \in (0,1).
$$

Hence, the integral in (4.36) is negative. So, because of (4.25), the proof of Lemma 45 is completed. ■

4.4.2 Completion of the proof of Theorem 38

The proof is carried out by making use of the quadrature method (Chapter 1). We have to resolve equations of the type $T(E) = \frac{1}{2}$, where T denotes, in each case, the appropriate time map.

Solution in B_n^+. Recall that $r_+(p,\lambda,0) = ((p+1)\lambda)^{1/p}$. Furthermore

$$
T_+(p,\lambda,0) = \int_0^{r_+(p,\lambda,0)} \left(-p'F(p,\lambda,\xi)\right)^{-1/p} d\xi = J(p)\lambda^{-1/p^2},
$$

where $J(p)$ is as defined in Lemma 43. Then problem (4.1) admits a solution in B_n^+ if and only if $nJ(p)\lambda^{-\frac{1}{p^2}} = (1/2)$, that is, if and only if $\lambda = (2nJ(p))^{p^2}$.

Solution in B_n^-. Since $0 \notin D_- = (0, E_*(p,\lambda))$, problem (4.1) admits no solution in $\bigcup_{n \geq 1} B_n^-$.

Solution in A_1^+. Recall that for any $p > 1$ and $\lambda > 0$ the function $E \mapsto T_+(p,\lambda,E)$ is defined in $[0, +\infty)$, strictly decreasing (Lemma 45) and by Lemma 43,

$$\lim_{E\to 0^+} T_+(p,\lambda,E) = J(p)\,\lambda^{-\frac{1}{p^2}}, \qquad \lim_{E\to+\infty} T_+(p,\lambda,E) = 0.$$

Then, equation $T_+(p,\lambda,E) = (1/2)$ in the variable $E \in (0,+\infty)$ admits a solution in $[0,+\infty)$ if and only if $J(p)\,\lambda^{-1/p^2} > 1/2$, that is, if and only if $\lambda < (2J(p))^{p^2}$, and in this case, the solution is unique since the function $T_+(p,\lambda,\cdot)$ is strictly decreasing.

Solution in A_1^-. Case $1 < p \leq 2$. In this case, for each $\lambda > 0$, the function $E \mapsto T_-(p,\lambda,E)$ is defined in $D_- = (0, E_*(p,\lambda))$, strictly increasing (Lemma 45) and

$$\lim_{E\to 0^+} T_-(p,\lambda,E) = 0, \qquad \lim_{E\to E_*} T_-(p,\lambda,E) = +\infty.$$

(Lemma 43, (ii) and Assertion (4.35)). So, the equation $T_-(p,\lambda,E) = (1/2)$ in the variable $E \in D_-$ admits a unique solution in D_- for any $\lambda > 0$.

Case $p > 2$. In this case, for each $\lambda > 0$, the function $E \mapsto T_-(p,\lambda,E)$ is defined in $D_- = (0, E_*(p,\lambda))$, strictly increasing (Lemma 45) and

$$\lim_{E\to 0^+} T_-(p,\lambda,E) = 0, \qquad \lim_{E\to E_*} T_-(p,\lambda,E) = J_-(p)\,\lambda^{-1/p^2} < +\infty.$$

(Lemma 43). So, the equation $T_-(p,\lambda,E) = (1/2)$ in the variable $E \in D_-$ admits a solution in D_- if and only if $(1/2) < J_-(p)\,\lambda^{-1/p^2}$, that is, if and only if $\lambda < (2J_-(p))^{p^2}$, and in this case the solution is unique since $T_-(p,\lambda,\cdot)$ is strictly increasing.

Solution in $A_{2n}^{\pm}$. Case $1 < p \leq 2$. In this case, for each $\lambda > 0$, the function $E \mapsto T_{2n}(p,\lambda,E)$ is defined in $D = (0, E_*(p,\lambda))$, and

$$\lim_{E\to 0^+} T_{2n}(p,\lambda,E) = nJ(p)\,\lambda^{-1/p^2}, \qquad \lim_{E\to E_*} T_{2n}(p,\lambda,E) = +\infty,$$

(Lemma 44 and Lemma 43, Assertion (4.35)). So, the equation $T_{2n}(p,\lambda,E) = (1/2)$ in the variable $E \in D$ admits a solution in D provided that $(1/2) > nJ(p)\,\lambda^{-1/p^2}$, that is, provided that $\lambda > (2nJ(p))^{p^2}$.

Case $p > 2$. In this case, for each $\lambda > 0$, the function $E \mapsto T_{2n}(p,\lambda,E)$ is defined in

$D = (0, E_*(p, \lambda))$, and

$$\lim_{E \to 0^+} T_{2n}(p, \lambda, E) = nJ(p)\,\lambda^{-1/p^2},$$

$$\lim_{E \to E_*} T_{2n}(p, \lambda, E) = n\lambda^{-1/p^2}\left(J_-(p) + J_+(p)\right) < +\infty$$

(Lemma 44 and Lemma 43, Assertion (4.35)). So, the equation $T_{2n}(p, \lambda, E) = (1/2)$ in the variable $E \in D$ admits a solution in D provided that

$$n\lambda^{-1/p^2} \inf\left(J(p),\, J_-(p) + J_+(p)\right) < \frac{1}{2} < n\lambda^{-1/p^2} \sup\left(J(p),\, J_-(p) + J_+(p)\right),$$

that is, provided that

$$\{2n \inf\left(J(p),\, J_-(p) + J_+(p)\right)\}^{p^2} < \lambda < \{2n \sup\left(J(p),\, J_-(p) + J_+(p)\right)\}^{p^2}.$$

Solution in A^+_{2n+1}. Case $1 < p \leq 2$. In this case, for each $\lambda > 0$, the function $E \mapsto T^+_{2n+1}(p, \lambda, E)$ is defined in $D = (0, E_*(p, \lambda))$, and

$$\lim_{E \to 0^+} T^+_{2n+1}(p, \lambda, E) = (n+1)\,J(p)\,\lambda^{-1/p^2}, \qquad \lim_{E \to E_*} T^+_{2n+1}(p, \lambda, E) = +\infty,$$

(Lemma 44 and Lemma 43, Assertion (4.35)). So, the equation $T^+_{2n+1}(p, \lambda, E) = (1/2)$ in the variable $E \in D$ admits a solution in D provided that $(n+1)\,J(p)\,\lambda^{-1/p^2} < (1/2)$, that is, provided that $\lambda > (2(n+1)\,J(p))^{p^2}$.

Case $p > 2$. In this case, for each $\lambda > 0$, the function $E \mapsto T^+_{2n+1}(p, \lambda, E)$ is defined in $D = (0, E_*(p, \lambda))$, and

$$\lim_{E \to 0^+} T^+_{2n+1}(p, \lambda, E) = (n+1)\,J(p)\,\lambda^{-1/p^2},$$

$$\lim_{E \to E_*} T^+_{2n+1}(p, \lambda, E) = \lambda^{-1/p^2}\left((n+1)\,J_+(p) + nJ_-(p)\right) < +\infty,$$

(Lemma 44 and Lemma 43, Assertion (4.35)). So, the equation $T^+_{2n+1}(p, \lambda, E) = (1/2)$ in the variable $E \in D$ admits a solution in D provided that

$$\lambda^{-1/p^2} \inf\left((n+1)\,J(p),\, (n+1)\,J_+(p) + nJ_-(p)\right) < \frac{1}{2} < \lambda^{-1/p^2} \sup\left((n+1)\,J(p),\, (n+1)\,J_+(p) + nJ_-(p)\right),$$

that is, provided that

$$\left\{2\inf\left((n+1)\,J(p)\,,\ (n+1)\,J_+(p)+nJ_-(p)\right)\right\}^{p^2}$$
$$<\lambda<\left\{2\sup\left((n+1)\,J(p)\,,\ (n+1)\,J_+(p)+nJ_-(p)\right)\right\}^{p^2}.$$

Solution in A^-_{2n+1}. Case $1<p\leq 2$. In this case, for each $\lambda>0$, the function $E\mapsto T^-_{2n+1}(p,\lambda,E)$ is defined in $D=(0,E_*(p,\lambda))$, and

$$\lim_{E\to 0^+}T^-_{2n+1}(p,\lambda,E)=nJ(p)\,\lambda^{-1/p^2},\qquad \lim_{E\to E_*}T^-_{2n+1}(p,\lambda,E)=+\infty,$$

(Lemma 44 and Lemma 43, Assertion (4.35)). So, the equation $T^-_{2n+1}(p,\lambda,E)=(1/2)$ in the variable $E\in D$ admits a solution in D provided that $nJ(p)\,\lambda^{-1/p^2}<(1/2)$, that is, provided that $\lambda>(2nJ(p))^{p^2}$.

Case $p>2$. In this case, for each $\lambda>0$, the function $E\mapsto T^-_{2n+1}(p,\lambda,E)$ is defined in $D=(0,E_*(p,\lambda))$, and

$$\lim_{E\to 0^+}T^-_{2n+1}(p,\lambda,E)=nJ(p)\,\lambda^{-1/p^2},$$

$$\lim_{E\to E_*}T^-_{2n+1}(p,\lambda,E)=\lambda^{-1/p^2}\left(nJ_+(p)+(n+1)\,J_-(p)\right)<+\infty,$$

(Lemma 44 and Lemma 43, Assertion (4.35)). So, the equation $T^-_{2n+1}(p,\lambda,E)=(1/2)$ in the variable $E\in D$ admits a solution in D provided that

$$\lambda^{-1/p^2}\inf\left(nJ(p)\,,\ nJ_+(p)+(n+1)\,J_-(p)\right)$$
$$<\frac{1}{2}<\lambda^{-1/p^2}\sup\left(nJ(p)\,,\ nJ_+(p)+(n+1)\,J_-(p)\right),$$

that is, provided that

$$\left\{2\inf\left((nJ(p)\,,\ nJ_+(p)+(n+1)\,J_-(p))\right)\right\}^{p^2}$$
$$<\lambda<\left\{2\sup\left((nJ(p)\,,\ nJ_+(p)+(n+1)\,J_-(p))\right)\right\}^{p^2}.$$

Then the proof of Theorem 38 is complete. ■

4.5 Remark

Theorem 38 shows that for $1<p\leq 2$ (resp. $p>2$) solutions to (4.1) with $k\geq 1$ interior nodes exist for all λ belonging to an interval unbounded from above (resp. a bounded interval). Hence, for $1 < p \leq 2$, if problem (4.1) admits a solution with a prescribed number $k_0 \geq 1$ of nodes for a certain value λ_0 of λ, it steel admits solutions with k_0 nodes for all λ greater than λ_0. In the next chapter we show that this is not the case for $p > 2$ and these changes in the behavior of the solution set as p varies depend strongly on the nonlinearity of the problem.

Chapter 5

Exactness results for a generalized Ambrosetti-Brezis-Cerami problem related to one dimensional elliptic equations

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5.1 Introduction

The combined effects of concave and convex nonlinearities were considered by Ambrosetti, Brezis and Cerami in [17]. They consider the problem

$$\begin{cases} -\Delta u = u^{\alpha-1} + \lambda u^{\beta-1}, & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{5.1}$$

with $1 < \beta < 2 < \alpha$ and $\lambda > 0$. They prove the existence of a constant $\Lambda > 0$ such that a solution u_λ of (5.1) exists if and only if $0 < \lambda \le \Lambda$. Moreover, if the condition $\alpha \le \alpha^*$ holds, then for all $\lambda \in (0, \Lambda)$ problem (5.1) has a second solution $v_\lambda > u_\lambda$, where $\alpha^* := (2N)/(N-2)$ if $N \ge 3$ and $\alpha^* = +\infty$ if $N = 1, 2$. Since then several papers appeared in which concave-convex nonlinearities were involved. We refer the reader to [17]-[29], [88], [118].

At the end of their paper [17], Ambrosetti, Brezis and Cerami suggested the study of the structure of the solution set of the one-dimensional problem

$$\begin{cases} -u'' = |u|^{\alpha-2}u + \lambda|u|^{\beta-2}u, \ a < x < b \\ u(a) = u(b) = 0 \end{cases} \tag{5.2}$$

with $1 < \beta < 2 < \alpha$ and $\lambda > 0$. This study was done by S. Villegas [118] by means of a quadrature method. He has shown that there exist two monotone divergent sequences $\{\varepsilon_n\}$ and $\{L_n\}$; $\varepsilon_n \leq L_n$ satisfying:

i) If $\lambda \in (0, \varepsilon_n)$ then (5.2) has exactly two pairs of opposite solutions with $n+1$ zeros.

ii) If $\lambda \in [\varepsilon_n, L_n)$ then (5.2) has at least two pairs of opposite solutions with $n+1$ zeros.

iii) If $\lambda = L_n$ then (5.2) has at least one pair of opposite solutions with $n+1$ zeros.

iv) If $\lambda > L_n$ then (5.2) has no solutions with $n+1$ zeros.

In the present paper, we consider the p-Laplacian version of problem (5.2), that is, we consider the boundary-value problem

$$\begin{cases} -(\varphi_p(u'))' = f(\lambda, u) \quad \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{5.3}$$

with $\varphi_p(x) = |x|^{p-2}x$ and $f(\lambda, u) = \varphi_\alpha(u) + \lambda\varphi_\beta(u)$ and notice that when $p = 2$, problem (5.3) is reduced to problem (5.2). We are mainly interested in the question: how does the solution set of (5.3) look like when $p \neq 2$ and $1 < \beta < p < \alpha$, $\lambda > 0$? The interest in this question comes from the fact that the structure of the solution set of problem (5.3) depends on $p > 1$ for some examples of second members $f(\lambda, u)$ (see for instance, Guedda and Veron [69], Addou [4]) and does not depend on p for some others (see for instance, Addou and Benmezaï [9], for positive solutions when $f(\lambda, u) = \lambda \exp(u)$). We shall prove that for $f(\lambda, u) = \varphi_\alpha(u) + \lambda\varphi_\beta(u)$, and $1 < \beta < p < \alpha$, Villegas' result holds for problem (5.3) for all $p > 1$ (part of Theorem 46, Assertion (C)).

Notice that Assertions (ii) and (iii), in Villegas' result, do not provide the exact number of solutions with $n+1$ zeros. So, when λ belongs to the range $[\varepsilon_n, L_n]$, the exact number of solutions with $n+1$ zeros has yet to be studied.

The exact number of solutions for problem (5.1) when λ ranges over the whole interval $(0, +\infty)$, was given by Ouyang and Shi [88], but under two restrictions: Ω was taken to be the unit ball in $\mathbb{R}^N$ and the space dimension $N \geq 4$. They proved the existence of some $\ast > 0$ such that problem (5.1) has exactly two solutions for $\lambda \in (0, \star)$, exactly one solution for $\lambda = \star$, and no solution for $\lambda > \star$. Actually their result concerns more general nonlinearities.

Next, the purpose of our investigation is to complete our study by providing the exact number of solutions to (5.3) when $1 < \beta < p < \alpha$, for all $p > 1$ and all $\lambda > 0$. So, in the particular case where $p = 2$, Theorem 46, Assertion (C), completes Villegas' result and solves completely the Ambrosetti-Brezis-Cerami problem [17, Section 6, (d)].

Going on with our study, we were led quite naturally to study what may happens if λ is not necessarily positive or p is not necessarily between α and β. A precise description of the solution set of problem (5.3) for various values of p, α, and β is given

As it is well known, exactness results are difficult to obtain. The difficulties encountered in our study for $\lambda > 0$ and for $\lambda < 0$ are of different kinds. For $\lambda > 0$, we used an idea performed by the authors Addou and Benmezaï in [8], (see the proof of (iii) in [8, pp. 11-13]).

The chapter is organized as follows. The main results are stated in Section 5.2. To prove our results we make use of a quadrature method which is described in Chapter 1. The main results for the case $\lambda > 0$ are proved in Section 5.3, while those for $\lambda < 0$ are proved in Section 5.4.

5.2 Notation and main results

Let, for any integer $k \geq 1$, $S_k^+, S_k^-, S_k, A_k^+, A_k^-$, and A_k the subsets of $C^1([a, b])$ defined in chapter 1.

Denote by $(\lambda_k)_{k \geq 1}$ the eigenvalues of the one dimensional p-Laplacian operator with Dirichlet boundary conditions,

$$\begin{cases} -\left(\varphi_p(u')\right)' = \lambda \varphi_p(u) & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

One has for each integer $k \geq 1$ and $p > 1$, $\lambda_k = k^p \lambda_1$ and

$$\lambda_1 = (p-1)\Big(2\int_0^1 (1-t^p)^{-1/p} dt\Big)^p = (p-1)\Big(\frac{2\pi}{p \sin(\pi/p)}\Big)^p.$$

Now we define some constants we shall use in the statement of the main results. For all p, $\alpha, \beta > 1$, let

$$J(p,\alpha,\beta) := \frac{(p')^{-1/p}}{(\alpha-\beta)}\Big(\frac{\alpha}{\beta}\Big)^{(p-\alpha)/p(\alpha-\beta)} B\Big(\frac{(p-\beta)}{p(\alpha-\beta)}, \frac{p-1}{p}\Big), \tag{5.4}$$

where $B(\cdot,\cdot)$ is the Beta function. Also, for all $\beta > \alpha > 1$, and $p > 1$, let

$$K(p,\alpha,\beta) = \int_0^1 \Big\{\Big(\frac{1-t^\alpha}{\alpha}\Big) - \Big(\frac{1-t^\beta}{\beta}\Big)\Big\}^{-1/p} dt. \tag{5.5}$$

We shall prove (see Lemma 53) that : $K(p, \alpha, \beta) < +\infty$ if and only if $p > 2$.

For all $\lambda \in \mathbb{R}$, denote S_λ the solution set of problem (5.3). The main results of this paper read as follows:

Theorem 46 Let $p, \alpha, \beta > 1$ and $\lambda > 0$.

(A) Assume that one of the following conditions holds:

- (a) $\alpha > p$ and $\beta > p$, or
- (b) $\alpha = p$ and $\beta > p$, or
- (c) $\alpha < p$ and $\beta < p$, or
- (d) $\alpha = p$ and $\beta < p$.

Then, for each integer $k = 1, 2, \cdots$, there exists $u_k \in A_k^+$ such that $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(B) Assume that one of the following conditions holds:

- (a) $\alpha > p$ and $\beta = p$, or
- (b) $\alpha < p$ and $\beta = p$.

Then, for each integer $k = 1, 2, \cdots$,

- (i) If $\lambda \geq \lambda_k$, $S_\lambda \cap A_k = \emptyset$.
- (ii) If $0 < \lambda < \lambda_k$, there exists $u_k \in A_k^+$ such that $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(C) Assume that $1 < \beta < p < \alpha$. Then for each integer $k = 1, 2, \cdots$, there exist a real number $\mu_k > 0$ such that

- (i) If $\lambda > \mu_k$, $S_\lambda \cap A_k = \emptyset$.
- (ii) If $\lambda = \mu_k$, there exists $u_k \in A_k^+$ such that $(S_\lambda \cap A_k) = \{u_k, -u_k\}$.
- (iii) If $\lambda \in (0, \mu_k)$, there exist $u_k, v_k \in A_k^+$ such that $u_k \neq v_k$ and $(S_\lambda \cap A_k) = \{u_k, v_k, -u_k, -v_k\}$.

Theorem 47 Let $p, \alpha, \beta > 1$ and $\lambda < 0$.

(A) Assume that one of the following conditions holds:

- (a) $p > 2$,and $1 < \alpha < p < \beta$, or
- (b) $p > 2$,and $1 < \alpha < \beta = p$, or

(c) $p > 2$, and $1 < \alpha < \beta < p$.

Then, there exists an increasing sequence of positive real numbers $(\mu_k)_{k \geq 1}$ such that for each integer $k = 1, 2, \cdots$,

(i) If $\lambda < -\mu_k$, $S_\lambda \cap A_k = \emptyset$.

(ii) If $-\mu_k \leq \lambda < 0$, there exists $u_k \in A_k^+$ such that $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(B) Assume that $1 < \beta < p < \alpha$. Then there exists an increasing sequence of positive real numbers $(\mu_k)_{k \geq 1}$ such that for each integer $k = 1, 2, \cdots$,

(i) If $\lambda \leq -\mu_k$, $S_\lambda \cap A_k = \emptyset$.

(ii) If $-\mu_k < \lambda < 0$, there exists $u_k \in A_k^+$ such that $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(C) Assume that $1 < p \leq 2$, $\lambda < 0$ and one of the following conditions holds:

(a) $1 < \alpha < p < \beta$, or

(b) $1 < \alpha < \beta = p$, or

(c) $1 < \alpha < \beta < p$, or

(d) $1 < p \leq \beta < \alpha$.

Then, for each integer $k = 1, 2, \cdots$, there exists $u_k \in A_k^+$ such that $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(D) Assume that $1 < p \leq 2$, $1 < \alpha = p < \beta$, and $\lambda < 0$. Then for each integer $k = 1, 2, \cdots$,

(i) If $k \geq (\frac{p^2}{\lambda_1})^{1/p}$, $S_\lambda \cap A_k = \emptyset$.

(ii) If $k < (\frac{p^2}{\lambda_1})^{1/p}$, there exists $u_k \in A_k^+$ such that $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(E) Assume that $1 < \beta < \alpha = p$, then for each integer $k = 1, 2, \cdots$,

(i) If $k \leq (2J(p, \alpha = p, \beta))^{-1}$ or $k \geq (p^2/\lambda_1)^{1/p}$, $S_\lambda \cap A_k = \emptyset$.

(ii) If $(2J(p, \alpha = p, \beta))^{-1} < k < (p^2/\lambda_1)^{1/p}$, there exists $u_k \in A_k^+$ such that $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(F) Assume that $2 < p = \alpha < \beta$, then for each integer $k = 1, 2, \cdots$,

(i) If $k < (2K(p, \alpha, \beta))^{-1}$ or $k \geq (p^2/\lambda_1)^{1/p}$, $S_\lambda \cap A_k = \emptyset$.

(ii) If $(2K(p,\alpha,\beta))^{-1} \le k < (p^2/\lambda_1)^{1/p}$, there exists $u_k \in A_k^+$ such that $S_\lambda \cap A_k = \{u_k, -u_k\}$.

(G) Assume that $1 < p \le 2$ and $p < \alpha < \beta$. Then, there exists an increasing sequence $(\mu_k)_{k \ge 1}$ such that $\lim_{k \to +\infty} \mu_k = 0$ and

(i) If $\lambda < \mu_k$, $S_\lambda \cap A_k = \emptyset$.

(ii) If $\lambda = \mu_k$, there exists $u_k \in A_k^+$ such that $(S_\lambda \cap A_k) = \{u_k, -u_k\}$.

(iii) If $\lambda \in (\mu_k, 0)$, there exist $u_k, v_k \in A_k^+$ such that $u_k \ne v_k$ and $(S_\lambda \cap A_k) = \{u_k, v_k, -u_k, -v_k\}$.

(H) Assume that one of the following conditions holds:

(a) $2 < p < \alpha < \beta$ or

(b) $1 < \beta < \alpha < p$.

Then, there exist two increasing sequences $(\mu_k)_{k \ge 1}$ and $(\nu_k)_{k \ge 1}$ such that for all $k \ge 1$, $\mu_k < \nu_k < 0$, $\lim_{k \to +\infty} \mu_k = 0$ and

If $\lambda < \mu_k$, $S_\lambda \cap A_k = \emptyset$.

If $\lambda = \mu_k$, there exists $u_k \in A_k^+$ such that $(S_\lambda \cap A_k) = \{u_k, -u_k\}$.

If $\lambda \in (\mu_k, \nu_k]$, there exist $u_k, v_k \in A_k^+$ such that $u_k \ne v_k$ and $(S_\lambda \cap A_k) = \{u_k, v_k, -u_k, -v_k\}$.

If $\lambda \in (\nu_k, 0)$, there exists $u_k \in A_k^+$ such that $(S_\lambda \cap A_k) = \{u_k, -u_k\}$.

5.3 Proof of Theorem 46

This section is organized as follows. We begin by some lemmas in the first subsection. The first lemma (Lemma 48) is used in order to define the time map, while in Lemma 49 we compute the limits and in Lemma 50 we study the variations of the time map. Next we dedicate a separate subsection to the proof of each assertion of Theorem 46.

5.3.1 Preliminary lemmas

We begin by the following technical Lemma.

Lemma 48 Consider the function defined on $\mathbb{R}^+$ by,

$$s \mapsto G(\lambda, E, s) := E^p - p' F(\lambda, s),$$

where $p, \alpha, \beta > 1, E > 0$ and $\lambda > 0$ are real parameters, and

$$F(\lambda, s) := \int_0^s f(\lambda, t)dt = \frac{1}{\alpha}s^\alpha + \frac{\lambda}{\beta}s^\beta, s \geq 0.$$

For all $\lambda > 0$ and $E > 0$ there exists a unique $s(\lambda, E) > 0$ such that the function $G(\lambda, E, \cdot)$ is strictly positive on $(0, s(\lambda, E))$, vanishes at $s(\lambda, E)$ and is strictly negative on $(s(\lambda, E), +\infty)$. Moreover,

(i) The function $E \mapsto s(\lambda, E)$ is C^1 on $(0, +\infty)$, and

$$\frac{\partial s}{\partial E}(\lambda, E) = \frac{(p-1)E^{p-1}}{f(\lambda, s(\lambda, E))} > 0, \text{ for all } E > 0 \text{ and all } \lambda > 0.$$

(ii) $\lim_{E \to 0^+} s(\lambda, E) = 0$, $\lim_{E \to +\infty} s(\lambda, E) = +\infty$.

Proof For any fixed $p > 1$, $E > 0$ and $\lambda > 0$, consider the function

$$s \mapsto G(\lambda, E, s) := E^p - p'F(\lambda, s),$$

defined on $[0, +\infty)$. One has

$$\frac{\partial G}{\partial s}(\lambda, E, s) = -p'f(\lambda, s).$$

Notice that,

$$f(\lambda, s) = s^{\alpha-1} + \lambda s^{\beta-1} > 0, \quad \text{for all } s > 0 \text{ and all } \lambda \geq 0. \tag{5.6}$$

Thus, the function $G(\lambda, E, \cdot)$ is strictly decreasing on $(0, +\infty)$. On the other hand, one has

$$G(\lambda, E, 0) = E^p > 0 \quad \text{and} \quad \lim_{s \to +\infty} G(\lambda, E, s) = -\infty.$$

Therefore, $G(\lambda, E, \cdot)$ admits a unique positive zero, denoted by $s(\lambda, E)$, and it is strictly positive on $(0, s(\lambda, E))$ and is strictly negative on $(s(\lambda, E), +\infty)$.

Proof of (i). For any $p > 1$ and $\lambda \geq 0$, consider the real-valued function,

$$(E, s) \mapsto G(E, s) := E^p - p'F(\lambda, s),$$

defined on $\Omega = (0, +\infty)^2$. One has $G \in C^1(\Omega)$ and,

$$\frac{\partial G}{\partial s}(E, s) = -p'f(\lambda, s) \quad \text{in } \Omega,$$

hence, according to (5.6), it follows that

$$\frac{\partial G}{\partial s}(E, s) < 0, \quad \text{in } \Omega.$$

Observe that for all $E > 0$ and $\lambda \geq 0$, the couple $(E, s(\lambda, E))$ belongs to Ω and one has

$$G(E, s(\lambda, E)) = 0. \tag{5.7}$$

Thus, we can make use of the implicit function theorem to show that the function $E \mapsto s(\lambda, E)$ is $C^1(\mathbb{R}^+, \mathbb{R}^+)$ and to obtain the expression of $(\partial s/\partial E)(\lambda, E)$ given in (i). Its sign is given by (5.6) and the fact that $s(\lambda, E) > 0$ for all $\lambda \geq 0$ and $E > 0$. Therefore, Assertion (i) is proved.

Proof of (ii). For any fixed $p > 1$ and $\lambda \geq 0$, Assertion (i) of the current lemma implies that the function defined on $(0, +\infty)$ by $E \mapsto s(\lambda, E)$ is strictly increasing. It is bounded from below by 0 and from above by $+\infty$. Thus, the limits $\lim_{E \to 0^+} s(\lambda, E) = l_0$ and $\lim_{E \to +\infty} s(\lambda, E) = l_+$ exist and satisfy

$$0 \leq l_0 < l_+ \leq +\infty.$$

Let us observe that for any fixed $p > 1$ and $\lambda \geq 0$, the function $(E, s) \mapsto G(E, s)$ is continuous on $[0, +\infty)^2$ and the function $E \mapsto s(\lambda, E)$ is continuous on $[0, +\infty)$ and satisfies (5.7). Thus, by passing to the limit in (5.7) as E tends to 0^+, we get

$$0 = \lim_{E \to 0^+} G(E, s(\lambda, E)) = G(0, l_0).$$

Hence, l_0 is a zero, belonging to $[0, +\infty)$, to the equation in the variable s : $G(0, s) = 0$. By solving this equation we find $l_0 = 0$.

Assume that $l_+ < +\infty$, then by passing to the limit in (5.7) as E tends to $+\infty$ we obtain

$$+\infty = p' F(\lambda, l_+) < +\infty,$$

which is impossible. So, $l_+ = +\infty$. Therefore, Lemma 48 is proved ■

Now, for any $p > 1$, $\alpha, \beta > 1$, $\lambda > 0$ and $E > 0$, we compute $X(\lambda, E)$ as defined in Chapter 1. In fact, for all $E > 0$, $X(\lambda, E) = (0, s(\lambda, E))$, where $s(\lambda, E)$ is defined in Lemma 48. Then, $r(\lambda, E) = s(\lambda, E)$ for all $\lambda > 0$ and $E > 0$. Hence, for any $p > 1$, $\lambda > 0$,

$$0 < r(\lambda, E) < +\infty \text{ if and only if } E > 0.$$

Also, for all $E > 0$,

$$f(\lambda, r(\lambda, E)) = \varphi_\alpha(r(\lambda, E)) + \frac{\lambda}{\beta}\varphi_\beta(r(\lambda, E)) > 0.$$

So, $D(\lambda) = (0, +\infty)$ for all $\lambda > 0$.

Before going further in the investigation, from Lemma 48, we deduce that for any fixed $p > 1$ and $\lambda > 0$

$$E^p = p' F(\lambda, r(\lambda, E)), \text{ for all } E > 0 \tag{5.8}$$

$$\frac{\partial r}{\partial E}(\lambda, E) = \frac{(p-1)E^{p-1}}{f(\lambda, r(\lambda, E))} > 0, \text{ for all } E > 0 \text{ and all } \lambda > 0. \tag{5.9}$$

$$\lim_{E \to 0^+} r(\lambda, E) = 0, \quad \lim_{E \to +\infty} r(\lambda, E) = +\infty. \tag{5.10}$$

At present we define, for any $p > 1, \lambda > 0$ and $E > 0$, the time map

$$T(\lambda, E) = \int_0^{r(\lambda,E)} \{E^p - p' F(\lambda, \xi)\}^{-1/p} d\xi, \; E > 0.$$

By (5.8), it follows that

$$T(\lambda, E) = (p')^{-1/p} \int_0^{r(\lambda,E)} \{F(\lambda, r(\lambda, E)) - F(\lambda, \xi)\}^{-1/p} d\xi.$$

Furthermore, a simple change of variables and a substitution yield

$$T(\lambda, E) = (p')^{-1/p} \int_0^1 \{\lambda r^{\beta-p}(\lambda, E)(\frac{1-t^\beta}{\beta}) + r^{\alpha-p}(\lambda, E)(\frac{1-t^\alpha}{\alpha})\}^{-1/p} dt.$$

Let's observe that

$$T(\lambda, E) = S(\lambda, r(\lambda, E)), \text{ for all } \lambda > 0, \; E > 0,$$

where

$$S(\lambda, \rho) = (p')^{-1/p} \int_0^1 \{\lambda \rho^{\beta-p}(\frac{1-t^\beta}{\beta}) + \rho^{\alpha-p}(\frac{1-t^\alpha}{\alpha})\}^{-1/p} dt,$$

for all $\lambda > 0$ and $\rho > 0$.

Beacuase the function $E \mapsto r(\lambda, E)$ is an increasing C^1-diffeomorphism from $(0, +\infty)$ onto itself it follows that if we put, for all $\lambda > 0$,

$$J_1(\lambda) := \{E \in (0, +\infty) : T(\lambda, E) = 1/2\},$$

and

$$J_2(\lambda) := \{\rho \in (0, +\infty) : S(\lambda, \rho) = 1/2\},$$

then

$$Card(J_1(\lambda)) = Card(J_2(\lambda)), \text{ for all } \lambda > 0.$$

Hence, from now on, we will focus our attention on the counting of the number of solution(s) of the equation $S(\lambda, \rho) = 1/2$ in the variable $\rho \in (0, +\infty)$, instead of the equation $T(\lambda, E) = 1/2$ in the variable $E \in (0, +\infty)$. In the next lemma we shall compute the limits of $S(\lambda, \cdot)$ when $\lambda > 0$.

Lemma 49 For all $\lambda > 0$, $p > 1$, it holds

$$\lim_{\rho \to 0^+} S(\lambda, \rho) = 0, \quad \text{if } \alpha < p$$

$$\lim_{\rho \to 0^+} S(\lambda, \rho) = \begin{cases} \frac{1}{2}\lambda_1^{1/p} & \text{if } \alpha = p \text{ and } \beta > p \\ 0 & \text{if } \alpha = p \text{ and } \beta < p \end{cases}$$

$$\lim_{\rho \to 0^+} S(\lambda, \rho) = \begin{cases} +\infty & \text{if } \alpha > p \text{ and } \beta > p \\ \frac{1}{2}(\frac{\lambda_1}{\lambda})^{1/p} & \text{if } \alpha > p \text{ and } \beta = p \\ 0 & \text{if } \alpha > p \text{ and } \beta < p \end{cases}$$

$$\lim_{\rho \to +\infty} S(\lambda, \rho) = \begin{cases} 0 & \text{if } \alpha < p \text{ and } \beta > p \\ \frac{1}{2}(\frac{\lambda_1}{\lambda})^{1/p} & \text{if } \alpha < p \text{ and } \beta = p \\ +\infty & \text{if } \alpha < p \text{ and } \beta < p \end{cases}$$

$$\lim_{\rho \to +\infty} S(\lambda, \rho) = \begin{cases} 0 & \text{if } \alpha = p \text{ and } \beta > p \\ \frac{1}{2}\lambda_1^{1/p} & \text{if } \alpha = p \text{ and } \beta < p \end{cases}$$

$$\lim_{\rho \to +\infty} S(\lambda, \rho) = 0 \quad \text{if } \alpha > p.$$

Proof The proof of this lemma follows from easy computations ∎

Lemma 50 For all $p > 1$, $\alpha, \beta > 1$ and $\lambda > 0$,

1. $S(\lambda, \cdot)$ is strictly decreasing on $(0, +\infty)$ provided that

$$(\alpha > p \text{ and } \beta \geq p) \text{ or } (\alpha = p \text{ and } \beta > p).$$

2. $S(\lambda, \cdot)$ is strictly increasing on $(0, +\infty)$ provided that

$$(\alpha < p \text{ and } \beta \leq p) \text{ or } (\alpha = p \text{ and } \beta < p).$$

3. $S(\lambda, \cdot)$ is strictly increasing on $(0, \rho_1(\lambda))$ and is strictly decreasing on $(\rho_2(\lambda), +\infty)$, provided that

$$(\alpha > p \text{ and } \beta < p) \text{ or } (\alpha < p \text{ and } \beta > p),$$

where

$$\rho_1(\lambda) := ((-\lambda)\frac{(p-\beta)}{(p-\alpha)})^{1/(\alpha-\beta)} < \rho_2(\lambda) := ((-\lambda)\frac{(p-\beta)\alpha}{(p-\alpha)\beta})^{1/(\alpha-\beta)}.$$

Proof For all $p > 1$, $\alpha, \beta > 1$ and $\lambda > 0$, easy computation yields

$$\frac{\partial S}{\partial \rho}(\lambda, \rho) = (p')^{-1/p} \int_0^\rho \frac{H(\lambda, \rho) - H(\lambda, u)}{p\rho(F(\lambda, \rho) - F(\lambda, u))^{1+(1/p)}} du,$$

where

$$H(\lambda, \rho) = pF(\lambda, \rho) - \rho f(\lambda, \rho) = (\frac{p-\alpha}{\alpha})\rho^\alpha + \lambda(\frac{p-\beta}{\beta})\rho^\beta.$$

It follows that,

$$\frac{\partial H}{\partial \rho}(\lambda, \rho) = (p-\alpha)\rho^{\alpha-1} + \lambda(p-\beta)\rho^{\beta-1}.$$

Thus, if ($\alpha > p$ and $\beta \geq p$) or ($\alpha = p$ and $\beta > p$), $H(\lambda, \cdot)$ is strictly decreasing on $(0, +\infty)$ and then, for all $\lambda > 0$ and $\rho > 0$

$$H(\lambda, \rho) - H(\lambda, u) < 0, \text{ for all } u \in (0, \rho),$$

and therefore,

$$\frac{\partial S}{\partial \rho}(\lambda, \rho) < 0, \quad \text{for all } \rho > 0.$$

If ($\alpha < p$ and $\beta \leq p$) or ($\alpha = p$ and $\beta < p$), $H(\lambda, \cdot)$ is strictly increasing on $(0, +\infty)$ and then for all $\rho > 0$, $\lambda > 0$

$$H(\lambda, \rho) - H(\lambda, u) > 0, \quad \text{for all } u \in (0, \rho),$$

and therefore,

$$\frac{\partial S}{\partial \rho}(\lambda, \rho) > 0, \quad \text{for all } \rho > 0.$$

If ($\alpha > p$ and $\beta < p$) or ($\alpha < p$ and $\beta > p$), $H(\lambda, \cdot)$ is strictly increasing on $(0, \rho_1(\lambda))$ and strictly decreasing on $(\rho_1(\lambda), +\infty)$. Moreover, $H(\lambda, \cdot)$ is strictly positive on $(0, \rho_2(\lambda))$ and strictly negative on $(\rho_2(\lambda), +\infty)$ and vanishes at 0 and at $\rho_2(\lambda)$. Therefore, for all $\lambda > 0$ and $\rho \in (0, \rho_1(\lambda))$

$$H(\lambda, \rho) - H(\lambda, u) > 0 \quad \text{for all } u \in (0, \rho)$$

and for all $\rho \in (\rho_2(\lambda), +\infty)$,

$$H(\lambda, \rho) - H(\lambda, u) < 0 \quad \text{for all } u \in (0, \rho).$$

That is, $S(\lambda, \cdot)$ is strictly increasing on $(0, \rho_1(\lambda))$ and is strictly decreasing on $(\rho_2(\lambda), +\infty)$. ■

5.3.2 Proof of Assertion A

Case (a). Assume that $\alpha > p$ and $\beta > p$. By Lemma 49, it follows that for all $\lambda > 0$,

$$\lim_{\rho \to 0^+} S(\lambda, \rho) = +\infty, \text{ and } \lim_{\rho \to +\infty} S(\lambda, \rho) = 0,$$

and by Lemma 50, $S(\lambda, \cdot)$ is strictly decreasing on $(0, +\infty)$. Thus, for each integer $k = 1, 2, \cdots$, the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$, admits a unique solution for all $\lambda > 0$. Therefore, for each integer $k = 1, 2, \cdots$, problem (5.3) admits a unique pair of solutions $\{u_k, v_k\}$ in A_k, for all $\lambda > 0$. Moreover $v_k = -u_k$.

Case (b). Assume that $\alpha = p$ and $\beta > p$. By Lemma 49, it follows that for all $\lambda > 0$,

$$\lim_{\rho \to 0^+} S(\lambda, \rho) = \frac{1}{2}\lambda_1^{1/p}(p), \quad \text{and} \quad \lim_{\rho \to +\infty} S(\lambda, \rho) = 0,$$

and by Lemma 50, $S(\lambda, \cdot)$ is strictly decreasing on $(0, +\infty)$. Thus, for each integer $k = 1, 2, \cdots$, the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$, admits at least a solution in $(0, +\infty)$ if and only if $(k/2)\lambda_1^{1/p}(p) > 1/2$, that is, if and only if,

$$\lambda_1(p) > k^{-p}, \tag{5.11}$$

and in this case, the solution is unique. Notice that for each integer $k = 1, 2, \cdots$, $k^{-p} \leq 1$. So, (5.11) holds provided that

$$\lambda_1(p) > 1, \text{ for all } p > 1. \tag{5.12}$$

In the appendix, we shall prove that (5.12) holds true. Therefore, for each integer $k = 1, 2, \cdots$, problem (5.3) admits a unique pair of solutions $\{u_k, v_k\}$ in A_k, for all $\lambda > 0$. Moreover, $v_k = -u_k$.

The proofs of Cases (c) and (d) are similar and then omitted. Therefore, Assertion A is proved.

5.3.3 Proof of Assertion B

Assume that $\alpha > p$ (resp. $\alpha < p$) and $\beta = p$. By Lemma 49, it follows that for all $\lambda > 0$,

$$\begin{aligned}\lim_{\rho\to 0} S(\lambda,\rho) &= \frac{1}{2}\left(\frac{\lambda_1}{\lambda}\right)^{1/p} \text{ (resp. } =0\text{), and}\\ \lim_{\rho\to+\infty} S(\lambda,\rho) &= 0 \text{ (resp. } =\frac{1}{2}\left(\frac{\lambda_1}{\lambda}\right)^{1/p}\text{),}\end{aligned}$$

and by Lemma 50, $S(\lambda,\cdot)$ is strictly decreasing (resp. increasing) on $(0,+\infty)$. Thus, for each integer $k = 1,2,\cdots$, the equation $kS(\lambda,\rho) = 1/2$, in the variable $\rho > 0$, admits at least a solution in $(0,+\infty)$ if and only if $(k/2)(\lambda_1/\lambda)^{1/p} > 1/2$, that is, if and only if $0 < \lambda < k^p\lambda_1(p) = \lambda_k(p)$, and in this case, the solution is unique. Therefore, for each integer $k = 1,2,\cdots$, problem (5.3) admits a unique pair of solutions $\{u_k, v_k\}$ (resp. admits no solution) in A_k, for all λ satisfying $0 < \lambda < \lambda_k(p)$ (resp. $\lambda \geq \lambda_k(p)$). Moreover, $u_k = -v_k$.

5.3.4 Proof of Assertion C

Assume that $1 < \beta < p < \alpha$. By Lemma 49, it follows that for all $\lambda > 0$,

$$\lim_{\rho\to 0^+} S(\lambda,\rho) = \lim_{\rho\to+\infty} S(\lambda,\rho) = 0.$$

Thus, for all $\lambda > 0$, there exists a unique $M(\lambda) > 0$ such that $M(\lambda) = \sup_{\rho\geq 0} S(\lambda,\rho)$.

Lemma 51 Assume that $1 < \beta < p < \alpha$. Then,

(a) $M(\cdot)$ is continuous on $(0,+\infty)$.

(b) $M(\cdot)$ is strictly decreasing on $(0,+\infty)$.

(c) $\lim_{\lambda\to 0^+} M(\lambda) = +\infty$, and $\lim_{\lambda\to+\infty} M(\lambda) = 0$.

Proof Recall that for all $\lambda > 0$ and $\rho > 0$

$$S(\lambda,\rho) = (p')^{-1/p}\int_0^1 \left(\lambda\rho^{\beta-p}\left(\frac{1-t^\beta}{\beta}\right) + \rho^{\alpha-p}\left(\frac{1-t^\alpha}{\alpha}\right)\right)^{-1/p} dt.$$

For all $\lambda > 0$ and $\rho > 0$, let $\tilde{\rho} = \tilde{\rho}(\lambda, \rho) := \lambda^{1/(\beta - \alpha)}\rho$. Then, $\rho = \lambda^{1/(\alpha - \beta)}\tilde{\rho}$ and a simple substitution yields:

$$S(\lambda, \rho) = \lambda^{(p-\alpha)/p(\alpha-\beta)} S(1, \tilde{\rho}(\lambda, \rho)).$$

Thus,

$$M(\lambda) = \lambda^{(p-\alpha)/p(\alpha-\beta)} \sup_{\rho \geq 0} S(1, \tilde{\rho}(\lambda, \rho)) = \lambda^{(p-\alpha)/p(\alpha-\beta)} \sup_{\tilde{\rho} \geq 0} S(1, \tilde{\rho}).$$

Therefore,

$$M(\lambda) = \lambda^{(p-\alpha)/p(\alpha-\beta)} M(1), \text{ for all } \lambda > 0. \tag{5.13}$$

Assertions (a), (b) and (c) are simple consequences of formula (5.13). Therefore, Lemma 51 is proved ■

By Lemma 51 (or by formula (5.13)), it follows that the function $M(\cdot)$ admits an inverse function $M^{-1}(\cdot)$ defined and strictly decreasing on $(0, +\infty)$ and satisfies:

$$\lim_{y \to 0} M^{-1}(y) = +\infty, \text{ and } \lim_{y \to +\infty} M^{-1}(y) = 0.$$

Therefore, for each integer $k = 1, 2, \cdots$, we define $L_k := M^{-1}(1/2k)$. Thus, $(L_k)_{k \geq 1}$ is a strictly increasing sequence and satisfies $\lim_{k \to +\infty} L_k = +\infty$, and

- $kM(\lambda) < 1/2$, for all $\lambda > L_k$,
- $kM(\lambda) = 1/2$, for all $\lambda = L_k$
- $kM(\lambda) > 1/2$, for all $\lambda \in (0, L_k)$.

Therefore, the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$,

- admits no solution for all $\lambda > L_k$,
- admits at least a solution for $\lambda = L_k$,
- admits at least two solutions for all $\lambda \in (0, L_k)$.

Thus, for each integer $k = 1, 2, \cdots$, problem (5.3),

- admits no solution in A_k, for all $\lambda > L_k$,
- admits at least one pair of solutions $\{u_k, v_k\}$ in A_k, for $\lambda = L_k$. Moreover, $u_k = -v_k$.
- admits at least two pairs of solutions $\{u_k, U_k\} \cup \{v_k, V_k\}$ in A_k, for all $\lambda \in (0, L_k)$. Moreover, $U_k = -u_k$ and $V_k = -v_k$.

At present, let us prove that for each integer $k = 1, 2, \cdots$, there exists $\varepsilon_k \in (0, L_k)$ such that the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$, admits exactly two solutions for all $\lambda \in (0, \varepsilon_k)$. To this end, it suffices to prove that for each integer $k = 1, 2, \cdots$, there exists $\varepsilon_k \in (0, L_k)$ such that for all $\lambda \in (0, \varepsilon_k)$:

$$kS(\lambda, \rho) > 1/2, \text{ for all } \rho \in [\rho_1(\lambda), \rho_2(\lambda)], \tag{5.14}$$

where $\rho_i(\lambda)$, $i = 1, 2$, are defined in Lemma 50. In fact, assume that for all $\lambda \in (0, \varepsilon_k)$, (5.14) holds, then $kS(\lambda, \rho_1(\lambda)) > 1/2$ for all $\lambda > 0$, and by Lemma 49, $\lim_{\rho \to 0} S(\lambda, \rho) = 0$, and by Lemma 50, $kS(\lambda, \cdot)$ is strictly increasing on $(0, \rho_1(\lambda)]$. Thus, for all $\lambda > 0$ there is a unique solution in $(0, \rho_1(\lambda))$ to the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$. Also, by (5.14) it follows that $kS(\lambda, \rho_2(\lambda)) > 1/2$ for all $\lambda > 0$, and by Lemma 49, $\lim_{\rho \to +\infty} kS(\lambda, \rho) = 0$, and by Lemma 50, $kS(\lambda, \cdot)$ is strictly decreasing on $[\rho_2(\lambda), +\infty)$. Thus, for all $\lambda > 0$ there is a unique solution in $(\rho_2(\lambda), +\infty)$ to the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$. On the other hand, (5.14) implies that $kS(\lambda, \rho) \neq 1/2$, for all $\lambda > 0$ and all $\rho \in [\rho_1(\lambda), \rho_2(\lambda)]$. Thus, there is no solution in $[\rho_1(\lambda), \rho_2(\lambda)]$ to the equation $kS(\lambda, \rho) = 1/2$, in the variable $\rho > 0$.

Now, let us prove that for each integer $k = 1, 2, \cdots$, there exists $\varepsilon_k \in (0, L_k)$ such that for all $\lambda > 0$ (5.14) holds. Simple computation shows that for all $\lambda > 0$

$$\begin{aligned} S(\lambda, \rho_1(\lambda)) &= (p')^{-1/p} \lambda^{(p-\alpha)/p(\alpha-\beta)} \int_0^1 \Big\{ \Big(\frac{p-\beta}{\alpha-p}\Big)^{(\beta-p)/(\alpha-\beta)} \Big(\frac{1-t^\beta}{\beta}\Big) \\ &\quad + \Big(\frac{p-\beta}{\alpha-p}\Big)^{(\alpha-p)/(\alpha-\beta)} \Big(\frac{1-t^\alpha}{\alpha}\Big) \Big\}^{-1/p} dt. \end{aligned}$$

It follows that the function $\lambda \mapsto S(\lambda, \rho_1(\lambda))$ is continuous and strictly decreasing on $(0, +\infty)$, and

$$\lim_{\lambda \to 0} S(\lambda, \rho_1(\lambda)) = +\infty, \quad \text{and} \quad \lim_{\lambda \to +\infty} S(\lambda, \rho_1(\lambda)) = 0.$$

Thus, for each integer $k = 1, 2, \cdots$, there exists a unique $\mu_k > 0$ such that

$$kS(\mu_k, \rho_1(\mu_k)) = \frac{1}{2}.$$

Furthermore, the sequence $(\mu_k)_{k \geq 1}$ is strictly increasing and $\lim_{k \to +\infty} \mu_k = +\infty$. It is easy to prove that for each integer $k = 1, 2, \cdots$

$$\mu_k \leq L_k. \tag{5.15}$$

In fact, if the contrary holds, using the fact that the function $\lambda \mapsto kS(\lambda, \rho_1(\lambda))$ is strictly decreasing on $(0, +\infty)$, it follows that

$$\frac{1}{2} = kS(\mu_k, \rho_1(\mu_k)) < kS(L_k, \rho_1(L_k)).$$

But, for each integer $k = 1, 2, \cdots$,

$$kS(L_k, \rho_1(L_k)) \le k \sup_{\rho \ge 0} S(L_k, \rho) = kM(L_k) = \frac{1}{2},$$

a contradiction which proves (5.15).

On the other hand, the function $\lambda \mapsto \rho_2(\lambda)$ is continuous and strictly increasing on $(0, +\infty)$ and

$$\lim_{\lambda \to 0} \rho_2(\lambda) = 0, \text{ and } \lim_{\lambda \to +\infty} \rho_2(\lambda) = +\infty,$$

then, for each integer $k = 1, 2, \cdots$, there exists a unique $\varepsilon_k > 0$ such that

$$\rho_2(\varepsilon_k) = \rho_1(\mu_k).$$

Using the fact that ρ_1, ρ_2 and $(\mu_k)_{k \ge 1}$ are strictly increasing it follows that $(\varepsilon_k)_{k \ge 1}$ is also strictly increasing. Also, using the fact that

$$\lim_{\mu \to +\infty} \rho_1(\mu) = \lim_{\mu \to +\infty} \rho_2(\mu) = \lim_{k \to +\infty} \mu_k = +\infty,$$

it follows that: $\lim_{k \to +\infty} \varepsilon_k = +\infty$.

Furthermore, notice that using the fact that ρ_1 is strictly increasing on $(0, +\infty)$ and $\rho_1 < \rho_2$ on $(0, +\infty)$, it follows that for each integer $k = 1, 2, \cdots$, $\varepsilon_k \in (0, \mu_k)$, and by (5.15), it follows that

$$0 < \varepsilon_k < L_k, \text{ for each integer } k = 1, 2, \cdots \tag{5.16}$$

Now, we believe that for each integer $k = 1, 2, \cdots$, and all $\lambda \in (0, \varepsilon_k)$, (5.14) holds. In fact, let $k = 1, 2, \cdots$, be fixed and $\lambda_0 \in (0, \varepsilon_k)$ and $\bar\rho \in [\rho_1(\lambda_0), \rho_2(\lambda_0)]$.

The variations of ρ_1 and ρ_2 and the fact that $\rho_1 < \rho_2$ on $(0, +\infty)$ imply that there exists $\bar\lambda \in [\lambda_0, \mu_k)$ such that $\rho_1(\bar\lambda) = \bar\rho$. Thus, $kS(\lambda_0, \bar\rho) = kS(\lambda_0, \rho_1(\bar\lambda))$. But $S(\cdot, \rho)$ is decreasing on $(0, +\infty)$. Thus, $kS(\lambda_0, \rho_1(\bar\lambda)) \ge kS(\bar\lambda, \rho_1(\bar\lambda))$.

On the other hand, the function $\lambda \mapsto S(\lambda,\rho_1(\lambda))$ is strictly decreasing on $(0,+\infty)$, thus,

$$\|S(\bar{\lambda},\rho_1(\bar{\lambda}))\| > \|S(\mu_k,\rho_1(\mu_k))\|.$$

Therefore,

$$\|S(\lambda_0,\bar{\rho})\| = \|S(\lambda_0,\rho_1(\bar{\lambda}))\| \geq \|S(\bar{\lambda},\rho_1(\bar{\lambda}))\| > \|S(\mu_k,\rho_1(\mu_k))\| = \frac{1}{2},$$

and hence, (5.14) holds. Therefore, a p-laplacian version of Villegas result can be stated at this point. In fact; we can state that if $1<\beta<p<\alpha$. Then, for each integer $k = 1,2,\cdots$, there exist two real numbers ε_k and L_k such that $\varepsilon_k \leq L_k$ and

(i) If $\lambda > L_k$, $S_\lambda \cap A_k = \emptyset$.

(ii) If $\lambda = L_k$, there exists $u_k \in A_k^+$ such that $(S_\lambda \cap A_k) \supset \{u_k, -u_k\}$.

(iii) If $\lambda \in [\varepsilon_k, L_k)$, there exist $u_k, v_k \in A_k^+$ such that $u_k \neq v_k$ and $(S_\lambda \cap A_k) \supset \{u_k, v_k, -u_k, -v_k\}$.

(iv) If $\lambda \in (0,\varepsilon_k)$, there exist $u_k, v_k \in A_k^+$ such that $u_k \neq v_k$ and $(S_\lambda \cap A_k) = \{u_k, v_k, -u_k, -v_k\}$.

Let us summarize. At this point, we have shown that when $1<\beta<p<\alpha$, then for all $\lambda > 0$,

$$\lim_{\rho\to 0} S(\lambda,\rho) = \lim_{\rho\to+\infty} S(\lambda,\rho) = 0,$$

hence for all $\lambda > 0$, $S(\lambda,\cdot)$ admits at least a critical point; a maximum in $(0,+\infty)$. Next, it was proved that for all $\lambda > 0$, there exist $\rho_1(\lambda)$ and $\rho_2(\lambda)$ such that $0<\rho_1(\lambda)<\rho_2(\lambda)$ and $\frac{\partial S}{\partial\rho}(\lambda,\rho) > 0$ on $(0,\rho_1(\lambda)]$ and $\frac{\partial S}{\partial\rho}(\lambda,\rho) < 0$ on $[\rho_2(\lambda),+\infty)$. So, the critical point belongs necessarily to $(\rho_1(\lambda),\rho_2(\lambda))$. Also, it was proved that the function $\lambda \mapsto M(\lambda) := \sup_{0<\rho<+\infty} S(\lambda,\rho)$, is continuous, strictly decreasing on $(0,+\infty)$ and

$$\lim_{\lambda\to 0^+} M(\lambda) = +\infty, \text{ and } \lim_{\lambda\to+\infty} M(\lambda) = 0.$$

Thus, to complete the proof of Assertion (C), it remains to prove that for all $\lambda > 0$, $S(\lambda,\cdot)$ admits at most one critical point in $(\rho_1(\lambda),\rho_2(\lambda))$. To this end we shall prove that $S(\lambda,\cdot)$ is concave on $(\rho_1(\lambda),\rho_2(\lambda))$ for all $\lambda > 0$. Similar idea was previously used by the authors Addou and Benmezaï in [8, Lemma 7, (iii)].

The derivative of $S(\lambda,\cdot)$ is given by

$$\frac{\partial S}{\partial\rho}(\lambda,\rho) = (p')^{-1/p}\int_0^1 \frac{H(\lambda,\rho)-H(\lambda,u)}{p\rho(F(\lambda,\rho)-F(\lambda,u))^{(p+1)/p}}du$$

where $F(\lambda,\rho) = \int_0^\rho f(\lambda,t)dt = \frac{1}{\alpha}u^\alpha + \frac{\lambda}{\beta}u^\beta$, and

$$H(\lambda,\rho) = pF(\lambda,\rho) - \rho f(\lambda,\rho) = (\frac{p-\alpha}{\alpha})\rho^\alpha + \lambda(\frac{p-\beta}{\beta})\rho^\beta.$$

Easy computations show that for all $\rho > 0$ and $\lambda > 0$

$$\begin{aligned}(p')^{1/p}\frac{\partial^2 S}{\partial\rho^2}(\lambda,\rho) &= \int_0^1 \frac{(p+1)(H(\lambda,\rho)-H(\lambda,\rho\xi))^2}{p^2\rho(F(\lambda,\rho)-F(\lambda,\rho\xi))^{(2p+1)/p}}d\xi \\ &+ \int_0^1 \frac{p(\Psi(\lambda,\rho)-\Psi(\lambda,\rho\xi))(F(\lambda,\rho)-F(\lambda,\rho\xi))}{p^2\rho(F(\lambda,\rho)-F(\lambda,\rho\xi))^{(2p+1)/p}}d\xi,\end{aligned}$$

where

$$\begin{aligned}\Psi(\lambda,\rho) &= -p(p+1)F(\lambda,\rho) + 2p\rho f(\lambda,\rho) - \rho^2 f'_\rho(\lambda,\rho) \\ &= \frac{(p-\alpha)(\alpha-(p+1))}{\alpha}\rho^a + \lambda\frac{(p-\beta)(\beta-(p+1))}{\beta}\rho^\beta.\end{aligned}$$

Some manipulations yield

$$(p')^{1/p}\frac{\partial^2 S}{\partial\rho^2}(\lambda,\rho) = \int_0^1 \frac{(1-\xi^\beta)^2 P(X(\xi))}{p^2\rho(F(\lambda,\rho)-F(\lambda,\rho\xi))^{(2p+1)/p}}d\xi,$$

where $X(1) = \frac{\alpha}{\beta}$ and $X(\xi) = \frac{1-\xi^\alpha}{1-\xi^\beta}$ if $\xi \in [0,1)$, and P is the second degree polynomial function

$$\begin{aligned}P(X) &= \frac{(\alpha-p)\rho^{2\alpha}}{\alpha}X^2 - (\frac{p(\alpha-\beta)^2 + p(\alpha+\beta) - 2\alpha\beta}{\alpha\beta})\lambda\rho^{\alpha+\beta}X \\ &+ \frac{(\beta-p)}{\beta}\lambda^2\rho^{2\beta}.\end{aligned}$$

It can easily be verified that $X(\xi) \in [1,\alpha/\beta]$ for all $\xi \in [0,1]$. In fact, $X(0) = 1$ and $\lim_{\xi\to 1} X(\xi) = \alpha/\beta$ and X is strictly increasing on $(0,1)$ since, $X'(\xi) = \xi^{\beta-1}(1-\xi^\beta)^{-2}(\beta - \alpha\xi^{\alpha-\beta} + (\alpha-\beta)\xi^\alpha)$ and $\beta - \alpha\xi^{\alpha-\beta} > \beta - \alpha > 0$, for all $\xi \in (0,1)$. Thus $X'(\xi) > 0$ for all $\xi \in (0,1)$.

Therefore, we are interested in the sign of $P(X)$ when $X \in [1,\alpha/\beta]$. Its discriminant is

$$d = (Y^2 - 4Z)\frac{\lambda^2\rho^{2(\alpha+\beta)}}{(\alpha\beta)^2},$$

where

$$Y = p(\alpha-\beta)^2 + p(\alpha+\beta) - 2\alpha\beta \tag{5.17}$$

$$
\begin{aligned}
&= p\alpha^2 + (-2(p+1)\beta + p)\alpha + p\beta(\beta+1) \\
Z &= \alpha\beta(\alpha - p)(\beta - p)\,.
\end{aligned}
$$

Notice that by our hypothesis $1 < \beta < p < \alpha$, it follows that $Z < 0$, so that $d > 0$. The roots of P are given by

$$X_1(\lambda,\rho) = \lambda\frac{Y - \sqrt{Y^2 - 4Z}}{2\beta(\alpha - p)\rho^{\alpha-\beta}} \le 0, \text{ and } X_2(\lambda,\rho) = \lambda\frac{Y + \sqrt{Y^2 - 4Z}}{2\beta(\alpha - p)\rho^{\alpha-\beta}}\,.$$

Notice that the function $\rho \to X_2(\lambda,\rho)$ is strictly decreasing on $(0,+\infty)$. So, it would be perfect if

$$X_2(\lambda,\rho_2(\lambda)) > \frac{\alpha}{\beta} \text{ for all } \lambda > 0\,. \tag{5.18}$$

In fact, it follows therefore that,

$$[1,\frac{\alpha}{\beta}] \subset (X_1(\lambda,\rho), X_2(\lambda,\rho)),\ \forall\lambda > 0, \forall\rho \in (\rho_1(\lambda),\rho_2(\lambda)).$$

Hence, $P(X(\xi)) < 0$, for all $\xi \in [0,1]$, so,

$$\frac{\partial^2 S}{\partial\rho^2}(\lambda,\rho) < 0, \forall\lambda > 0, \forall\rho \in (\rho_1(\lambda),\rho_2(\lambda)),$$

which will prove the uniqueness of the critical point of $S(\lambda,\cdot)$, and hence, Theorem 46.

Let us prove the estimates (5.18). Notice that

$$X_2(\lambda,\rho_2(\lambda)) = \frac{Y + \sqrt{Y^2 - 4Z}}{2\alpha(p - \beta)} > \frac{\alpha}{\beta} \iff \beta\sqrt{Y^2 - 4Z} > 2\alpha^2(p - \beta) - \beta Y\,.$$

By separing each side, we find

$$X_2(\lambda,\rho_2(\lambda)) > \frac{\alpha}{\beta} \iff 4\alpha^2\beta(p - \beta)Y - 4\beta^2 Z - 4\alpha^4(p - \beta)^2 > 0\,.$$

Next, we substitute Y and Z as in (5.17) to get

$$X_2(\lambda,\rho_2(\lambda)) > \frac{\alpha}{\beta} \iff 4\alpha(p - \beta)Q_{\beta,p}(\alpha) > 0,$$

where $Q_{\beta,p}(\alpha)$ is the third degree polynomial function defined by

$$Q_{\beta,p}(\alpha) = \alpha^3(\beta(p+1) - p) + \alpha^2\beta(-2(p+1)\beta + p) + \alpha\beta^2(\beta(p+1) + p) - p\beta^3\,.$$

It remains to show that $Q_{\beta,p}(\alpha) > 0$ for all $1 < \beta < p < \alpha$. We can notice that β is a root of $Q_{\beta,p}(\cdot)$ of multiplicity at least two. That is $Q_{\beta,p}(\beta) = Q'_{\beta,p}(\beta) = 0$. Thus there exist two constants A and B such that $Q_{\beta,p}(\alpha) = (\alpha - \beta)^2(A\alpha + B)$. Immediate identification yields: $A = \beta(p+1) - p$ and $B = -p\beta$. It remains to prove that $1 < \beta < p < \alpha \Rightarrow A\alpha + B > 0$. Notice that $1 < \beta < p < \alpha$ implies that $(\beta - 1)p + \beta > \beta$ which implies that $\beta((\beta - 1)p + \beta)^{-1} < 1$ and then $-BA^{-1} < p$ and by $1 < \beta < p < \alpha$ it follows that $A\alpha + B > 0$, which completes the proof of (5.18). Therefore, Theorem 46 is proved.

5.4 Proof of Theorem 47

This section is organized as the previous one.

5.4.1 Preliminary lemmas

Lemma 52 Consider the function defined on $\mathbb{R}^+$ by

$$s \mapsto N(\lambda, E, s) := E^p - p' F(\lambda, s),$$

where $p, \alpha, \beta > 1$, $\lambda < 0$ and $E \in (0, +\infty)$, are real parameters and

$$F(\lambda, s) = \int_0^s f(\lambda, t)dt = \frac{1}{\alpha}s^\alpha + \frac{1}{\beta}s^\beta, \quad s \geq 0.$$

Assume that $(\alpha - \beta) > 0$, then for all $\lambda < 0$ and $E > 0$, the function $N(\lambda, E, \cdot)$ admits a unique positive zero, $s(\lambda, E)$, and is strictly positive on $(0, s(\lambda, E))$.

Assume that $(\alpha - \beta) < 0$, then

(a) If $E > E_* := (p'(-\lambda)^{\alpha/(\alpha-\beta)}(\frac{\beta-\alpha}{\alpha\beta}))^{1/p}$, $N(\lambda, E, \cdot)$ is strictly positive on $(0, +\infty)$.

(b) If $E = E_*$, $N(\lambda, E, \cdot)$ vanishes at $s = (-\lambda)^{1/(\alpha-\beta)}$ and is strictly positive on $(0, (-\lambda)^{1/(\alpha-\beta)}) \cup ((-\lambda)^{1/(\alpha-\beta)}, +\infty)$.

(c) If $0 < E < E_*$, $N(\lambda, E, \cdot)$ admits a first positive zero, $s(\lambda, E) > 0$, and is strictly positive on $(0, s(\lambda, E))$.

Moreover, if $(\alpha - \beta) > 0$ (resp. $(\alpha - \beta) < 0$),

(i) the function $E \mapsto s(\lambda, E)$ is C^1 on $(0, +\infty)$ (resp. $(0, E_*(\lambda))$) and

$$\frac{\partial s}{\partial E}(\lambda, E) = \frac{(p-1)E^{p-1}}{f(\lambda, s(\lambda, E))} > 0, \text{ for all } \lambda < 0 \text{ and } E > 0$$

(resp. $E \in (0, E_*(\lambda))$).

(ii) $\lim_{E\to 0} s(\lambda,E) = ((-\lambda)\frac{\alpha}{\beta})^{1/(\alpha-\beta)}$, (resp. $\lim_{E\to 0} s(\lambda,E) = 0$).

(iii) $\lim_{E\to+\infty} s(\lambda,E) = +\infty$, (resp. $\lim_{E\to E_*} s(\lambda,E) = (-\lambda)^{1/(\alpha-\beta)}$).

Proof The proof is the same as that of Lemma 48 and is omitted ■

Assume that $\alpha - \beta > 0$ (resp. $\alpha - \beta < 0$) then for any $p > 1$ and $E \in (0,+\infty)$ (resp. $E \in (0, E_*(\lambda))$ we compute $X(\lambda,E)$ as defined in Section ??. We derive from Lemma 52, for the case where $\alpha - \beta > 0$, $X(\lambda,E) = (0, s(\lambda,E))$ and for the case where $\alpha - \beta < 0$

$$X(\lambda,E) = \begin{cases} (0,+\infty) & \text{if } E > E_*(\lambda) \\ (0,(-\lambda)^{1/(\alpha-\beta)}) & \text{if } E = E_*(\lambda) \\ (0,s(\lambda,E)) & \text{if } 0 < E < E_*(\lambda), \end{cases}$$

where $s(\lambda,E)$ is defined in Lemma 52. Then, for $\alpha - \beta > 0$

$$r(\lambda,E) := \sup X(\lambda,E) = s(\lambda,E), \quad \text{for all } \lambda < 0 \text{ and } E > 0$$

and for $\alpha - \beta < 0$,

$$r(\lambda,E) = \begin{cases} +\infty & \text{if } E > E_*(\lambda) \\ (-\lambda)^{1/(\alpha-\beta)} & \text{if } E = E_*(\lambda) \\ s(\lambda,E) & \text{if } 0 < E < E_*(\lambda). \end{cases}$$

Hence, for $p > 1$ and $\lambda < 0$,

$$0 < r(\lambda,E) < +\infty \text{ if and only if } \begin{cases} E \in (0,+\infty), & \text{if } \alpha - \beta > 0, \\ 0 < E \le E_*(\lambda) & \text{if } \alpha - \beta < 0. \end{cases}$$

Also,

$$f(\lambda, r(\lambda,E)) > 0 \text{ if and only if } \begin{cases} E \in (0,+\infty), & \text{if } \alpha - \beta > 0, \\ 0 < E < E_*(\lambda) & \text{if } \alpha - \beta < 0. \end{cases}$$

So, for all $\lambda < 0$,

$$D(\lambda) = \begin{cases} (0,+\infty) & \text{if } \alpha - \beta > 0, \\ (0, E_*(\lambda)] & \text{if } \alpha - \beta < 0. \end{cases}$$

Before going further in the investigation, we deduce from Lemma 52 that for any $p > 1$ and

$\lambda < 0$,

$$\begin{aligned} E^p &= p' F(\lambda, r(\lambda, E)), \ \text{for all } E \in D(\lambda) \qquad (5.19)\\ \frac{\partial r}{\partial E}(\lambda, E) &= \frac{(p-1)E^{p-1}}{f(\lambda, r(\lambda, E))} > 0, \ \text{for all } E \in D(\lambda)\\ \lim_{E \to 0^+} r(\lambda, E) &= \begin{cases} ((-\lambda)(\alpha/\beta))^{1/(\alpha-\beta)} & \text{if } \alpha - \beta > 0 \\ 0 & \text{if } \alpha - \beta < 0 \, . \end{cases}\\ \lim_{E \to +\infty} r(\lambda, E) &= +\infty \ \text{if } \alpha - \beta > 0 \, ,\\ \lim_{E \to E_*} r(\lambda, E) &= (-\lambda)^{1/(\alpha-\beta)} \ \text{if } \alpha - \beta < 0 \, . \end{aligned}$$

At present, we define for any $p > 1$, $\lambda < 0$ the time-map

$$T(\lambda, E) = \int_0^{r(\lambda,E)} \{E^p - p' F(\lambda, \xi)\}^{-1/p} d\xi, \ E \in D(\lambda) \, .$$

By (5.19), it follows that

$$T(\lambda, E) = (p')^{-1/p} \int_0^{r(\lambda,E)} \{F(\lambda, r(\lambda, E)) - F(\lambda, \xi)\}^{-1/p} d\xi, \ E \in D(\lambda) \, .$$

Furthermore, a simple change of variable and a substitution yield

$$T(\lambda, E) = (p')^{-1/p} \int_0^1 \{\lambda r^{\beta-p}(\lambda, E)(\frac{1-t^\beta}{\beta}) + r^{\alpha-p}(\lambda, E)(\frac{1-t^\alpha}{\alpha})\}^{-1/p} dt.$$

Let's observe that

$$T(\lambda, E) = S(\lambda, r(\lambda, E)), \quad \text{for all } \lambda < 0 \text{ and } E \in D(\lambda),$$

where

$$S(\lambda, \rho) = (p')^{-1/p} \int_0^1 \{\lambda \rho^{\beta-p}(\lambda, E)(\frac{1-t^\beta}{\beta}) + \rho^{\alpha-p}(\lambda, E)(\frac{1-t^\alpha}{\alpha})\}^{-1/p} dt,$$

for all $\lambda < 0$ and all $\rho \in R(\lambda)$ where, for all $\lambda < 0$, $R(\lambda)$ is the range of the function $E \mapsto r(\lambda, E)$, defined on $D(\lambda)$, that is, for $\alpha - \beta > 0$, $R(\lambda) := (((-\lambda)(\alpha/\beta))^{1/(\alpha-\beta)}, +\infty)$, and for $\alpha - \beta < 0$, $R(\lambda) := (0, (-\lambda)^{1/(\alpha-\beta)}]$ and $R(\lambda) := (0, (-\lambda)^{1/(\alpha-\beta)})$ otherwise.

Due to the fact that the function $E \mapsto r(\lambda, E)$ is an increasing and continuous function from $D(\lambda)$ onto $R(\lambda)$, it follows, that if we put

$$J_1(\lambda) := \{E \in D(\lambda) : T(\lambda, E) = \frac{1}{2}\}, \lambda < 0, \ \text{and}$$

$$J_2(\lambda) := \{ \in R(\lambda) : S(\lambda,\rho) = \frac{1}{2}\}, \lambda < 0,$$

then,

$$\text{Card} J_1(\lambda) = \text{Card} J_2(\lambda), \text{ for all } \lambda < 0.$$

Hence, from now on, we will focus our attention on counting the number of solution(s) of the equation $S(\lambda,\rho) = 1/2$ in the variable $\rho \in R(\lambda)$ instead of the equation $T(\lambda, E) = 1/2$ in the variable $E \in D(\lambda)$.

Lemma 53 Assume that $\lambda < 0$. Then

(a) If $\alpha - \beta > 0$,

$$\lim_{\rho \to ((-\lambda)(\alpha/\beta))^{1/(\alpha-\beta)}} S(\lambda,\rho) = \begin{cases} (-\lambda)^{\frac{(p-\alpha)}{p(\alpha-\beta)}} J(p,\alpha,\beta) & \text{if } 1 < \beta < p \\ +\infty & \text{if } p \leq \beta. \end{cases}$$

where $J(p,\alpha,\beta)$ is defined in (5.4).

(b) If $\alpha - \beta > 0$,

$$\lim_{\rho \to +\infty} S(\lambda,\rho) = \begin{cases} 0 & \text{if } p \leq \beta \\ \frac{1}{2}(\frac{\lambda_1}{p^2})^{1/p} & \text{if } p > \beta \text{ and } \alpha = p \\ 0 & \text{if } p > \beta \text{ and } \alpha > p \\ +\infty & \text{if } p > \beta \text{ and } \alpha < p \end{cases}$$

(c) If $\alpha - \beta < 0$,

$$\lim_{\rho \to 0} S(\lambda,\rho) = \begin{cases} 0 & \text{if } p > \alpha \\ \frac{1}{2}(\frac{\lambda_1}{p^2})^{1/p} & \text{if } p = \alpha \\ +\infty & \text{if } p < \alpha \end{cases}$$

(d) If $\alpha - \beta < 0$,

$$\lim_{\rho \to (-\lambda)^{1/(\alpha-\beta)}} S(\lambda,\rho) = (-\lambda)^{\frac{(p-\alpha)}{p(\alpha-\beta)}} \int_0^1 \{(\frac{1-t^\alpha}{\alpha}) - (\frac{1-t^\beta}{\beta})\}^{-1/p} dt$$

and

$$\int_0^1 \{(\frac{1-t^\alpha}{\alpha}) - (\frac{1-t^\beta}{\beta})\}^{-1/p} dt < +\infty \iff p > 2.$$

Proof Assume that $\alpha - \beta > 0$. Easy computation yields

$$\lim_{\rho \to ((-\lambda)(\alpha/\beta))^{1/(\alpha-\beta)}} S(\lambda,\rho) = (p')^{-1/p} (\frac{-\lambda}{\beta})^{(p-\alpha)/p(\alpha-\beta)} \alpha^{(p-\beta)/p(\alpha-\beta)} \times$$

$$\leq \int_0^1 t^{-\beta/p}(1 - t^{\alpha-\beta})^{-1/p}dt.$$

It can be shown that for all $p > 1$ and $1 < \beta < \alpha$

$$\int_0^1 t^{-\beta/p}(1 - t^{\alpha-\beta})^{-1/p}dt = \begin{cases} +\infty & \text{if } \beta \geq p \\ \frac{1}{\alpha-\beta}B(\frac{(p-\beta)}{p(\alpha-\beta)}, \frac{p-1}{p}) & \text{if } 1 < \beta < p. \end{cases}$$

In fact, in the case where $\beta \geq p$, we use the estimates

$$t^{-\beta/p}(1 - t^{\alpha-\beta})^{-1/p} \geq t^{-\beta/p}, \text{ for all } t \in (0,1),$$

and in the case where $1 < \beta < p$, we use the change of variable $x = t^{\alpha-\beta}$, as in Lavrentiev and Chabat [77, pp. 595-596]. Therefore Assertion (a) is proved.

Also, in the case where $\alpha - \beta > 0$ and $\beta - p \geq 0$

$$\begin{aligned} \lim_{\rho\to+\infty}(\lambda\rho^{\beta-p}(\frac{1-t^\beta}{\beta}) + \rho^{\alpha-p}(\frac{1-t^\alpha}{\alpha}))^{-1/p} &= \\ \lim_{\rho\to+\infty}\rho^{(p-\beta)/p}(\lambda(\frac{1-t^\beta}{\beta}) + \rho^{\alpha-\beta}(\frac{1-t^\alpha}{\alpha}))^{-1/p} &= 0 \cdot 0 = 0. \end{aligned}$$

and an easy discussion shows that for $\alpha - \beta > 0$,

$$\begin{aligned} &\lim_{\rho\to+\infty}(\lambda\rho^{\beta-p}(\frac{1-t^\beta}{\beta}) + \rho^{\alpha-p}(\frac{1-t^\alpha}{\alpha}))^{-1/p} \\ &= \begin{cases} 0 & \text{if } \beta \geq p \\ 0 & \text{if } \beta < p \text{ and } \alpha > p \\ +\infty & \text{if } \beta < p \text{ and } \alpha < p \\ ((1-t^p)/p)^{-1/p} & \text{if } \beta < p \text{ and } \alpha = p \end{cases} \end{aligned}$$

Therefore, Assertion (b) is proved.

In the case where $\alpha - \beta < 0$ and $\lambda < 0$,

$$\begin{aligned} &\lim_{\rho\to 0}\int_0^1(\lambda\rho^{\beta-p}((1-t^\beta)/\beta) + \rho^{\alpha-p}((1-t^\alpha)/\alpha))^{-1/p}dt \\ &= \lim_{\rho\to 0}\rho^{(p-\alpha)/p}\int_0^1(\lambda\rho^{\beta-\alpha}(1-t^\beta)/\beta + (1-t^\alpha)/\alpha)^{-1/p}dt \\ &= \lim_{\rho\to 0}\rho^{(p-\alpha)/p}\int_0^1((1-t^\alpha)/\alpha)^{-1/p}dt. \end{aligned}$$

Notice that for all $\alpha > 1$

$$\int_0^1 (1 - t^\alpha)^{-1/p} dt = \frac{1}{\alpha} B(\frac{1}{\alpha}, \frac{p-1}{p}) < +\infty .$$

This follows by making use of the change of variable $x = t^\alpha$, see Lavrentiev and Chabat [77, pp. 595-596]. Assertion (c) follows.

Also, in the case $\alpha - \beta < 0$ and $\lambda < 0$, easy computation shows

$$\begin{aligned} \lim_{\rho \to (-\lambda)^{1/(\alpha-\beta)}} S(\lambda, \rho) &= (p')^{-1/p} (-\lambda)^{(p-\alpha)/p(\alpha-\beta)} \times \\ &\int_0^1 ((1 - t^\alpha)/\alpha) - ((1 - t^\beta)/\beta))^{-1/p} dt . \end{aligned}$$

By making use of L'Hopital's rule twice, we compute

$$\lim_{t \to 1^-} \frac{(1 - t^\alpha)/\alpha - (1 - t^\beta)/\beta}{(1 - t)^2} = \frac{\beta - \alpha}{2} > 0 .$$

So, the integral $\int_0^1 ((1 - t^\alpha)/\alpha) - ((1 - t^\beta)/\beta))^{-1/p} dt$ is convergent if and only if the integral $\int_0^1 (1 - t)^{-2/p} dt$ is. Therefore, Assertion (d) follows from the well known fact that

$$\int_0^1 (1 - t)^{-2/p} dt < +\infty \text{ if and only if } p > 2 .$$

Therefore, Lemma 53 is proved ■

Lemma 54 Assume that $p > 1$, $\lambda < 0$, and $\alpha \neq \beta$, $\alpha, \beta > 1$.

1. If one of the following conditions holds:

   (a) $\beta < p \leq \alpha$

   (b) $\beta = p < \alpha$

   (c) $p < \beta < \alpha$

   then, $S(\lambda, \cdot)$ is strictly decreasing on $R(\lambda)$.

2. If one of the following conditions holds:

   (a) $\alpha \leq p < \beta$

   (b) $\alpha < p = \beta$

   (c) $\alpha < \beta < p$

then, $S(\lambda, \cdot)$ is strictly increasing on $R(\lambda)$.

3. If one of the following conditions holds:

   (a) $p < \alpha < \beta$

   (b) $\beta < \alpha < p < \alpha + \beta$

   then, $S(\lambda, \cdot)$ is strictly decreasing on $(\inf R(\lambda), \rho_1]$ and is strictly increasing on $[\rho_2, \sup R(\lambda))$, where $\rho_1(\lambda)$ and $\rho_2(\lambda)$ are defined in Lemma 50.

4. If $\beta < \alpha < p$ and $\alpha + \beta \leq p$, then $S(\lambda, \cdot)$ is strictly increasing on $[\rho_2, +\infty)$.

Proof This Lemma follows by a similar discussion as in the proof of Lemma 50. So,its proof is omitted ■

Note that the third and the fourth assertions of Lemma 54 above do not provide the exact variations of the map $S(\lambda, \cdot)$ over its entire definition domain, which are necessary for the process of showing the exactness part in the main result. They are the aim of the following pioneer lemma.

Lemma 55 Assume that one of the following conditions holds:

(c1) $1 < p \leq 2$, and $p < \alpha < \beta$,

(c2) $2 < p < \alpha < \beta$,

(c3) $1 < \beta < \alpha < p$.

Then, for all $\lambda < 0$, there exists an interior point $\rho^*(\lambda) \in int(R(\lambda))$ such that $S(\lambda, \cdot)$ is strictly decreasing on $(\inf R(\lambda), \rho^*(\lambda))$ and then strictly increasing on $(\rho^*(\lambda), \sup R(\lambda))$.

Proof We shall prove Lemma 55 in two steps.

Step 1: Existence

If

$$p < \alpha < \beta \text{ or } 1 < \beta < \alpha < p < \alpha + \beta \tag{5.20}$$

the existence of $\rho^*(\lambda)$ follows immediatly from Lemma 54, Assertion 3.

Also, if

$$1 < \beta < \alpha < \alpha + \beta \leq p, \tag{5.21}$$

then, according to Lemma 54, Assertion 4, existence follows after proving that for all $\lambda < 0$, $S(\lambda, \cdot)$ is strictly decreasing on a right neighborhood of $\inf R(\lambda)$. (Notice that in this case $\rho_1(\lambda) \leq \inf R(\lambda)$). In fact, we shall prove:

Lemma 56 If $1 < \beta < \alpha < \alpha + \beta \leq p$ and $\lambda < 0$ then

$$\frac{\partial S}{\partial \rho}\Big(\lambda, \big(\frac{-\lambda\alpha}{\beta}\big)^{1/(\alpha-\beta)}\Big) = -\infty.$$

Proof. The derivative of $S(\lambda, \cdot)$ is given by

$$\frac{\partial S}{\partial \rho}(\lambda, \rho) = (p')^{-1/p} \int_0^1 \frac{H(\lambda,\rho) - H(\lambda, \rho u)}{p\rho(F(\lambda,\rho) - F(\lambda,\rho u))^{1+\frac{1}{p}}} du.$$

where

$$F(\lambda,\rho) = \frac{1}{\alpha}\rho^\alpha + \frac{\lambda}{\beta}\rho^\beta, \rho > 0 \quad \text{and} \quad H(\lambda,\rho) = (\frac{p-\alpha}{\alpha})\rho^\alpha + \lambda(\frac{p-\beta}{\beta})\rho^\beta, \rho > 0.$$

Simple computations yield

$$\begin{aligned}\frac{\partial S}{\partial \rho}\Big(\lambda, \big(-\lambda\frac{\alpha}{\beta}\big)^{1/(\alpha-\beta)}\Big) &= \frac{(p')^{-1/p}}{p}\Big(\frac{\beta}{-\lambda}\Big)^{\frac{p+\alpha}{p(\alpha-\beta)}} \alpha^{\frac{p+\beta}{p(\beta-\alpha)}} \\ &\times \int_0^1 \frac{(p-\alpha)(1-u^\alpha) - (p-\beta)(1-u^\beta)}{(u^\beta - u^\alpha)^{1+\frac{1}{p}}} du.\end{aligned}$$

The improper integral has two singularities; at 0 and at 1. Then we write

$$\frac{\partial S}{\partial \rho}\Big(-1, \big(-\lambda\frac{\alpha}{\beta}\big)^{1/(\alpha-\beta)}\Big) = \frac{(p')^{-1/p}}{p}\Big(\frac{\beta}{-\lambda}\Big)^{\frac{p+\alpha}{p(\alpha-\beta)}} \alpha^{\frac{p+\beta}{p(\beta-\alpha)}} (I_0 + I_1)$$

where

$$I_0 = \int_0^{1/2} \frac{(p-\alpha)(1-u^\alpha) - (p-\beta)(1-u^\beta)}{(u^\beta - u^\alpha)^{1+\frac{1}{p}}} du,$$

$$I_1 = \int_{1/2}^1 \frac{(p-\alpha)(1-u^\alpha) - (p-\beta)(1-u^\beta)}{(u^\beta - u^\alpha)^{1+\frac{1}{p}}} du.$$

In what follows we shall prove that $I_0 = -\infty$ and $I_1 \in \mathbb{R}$.

First, we write I_0 as follows:

$$I_0 = \int_0^{1/2} \frac{-(\alpha-\beta) - [(p-\alpha)u^\alpha - (p-\beta)u^\beta]}{(1-u^{\alpha-\beta})^{1+\frac{1}{p}} u^{\beta(1+\frac{1}{p})}} du.$$

Let us observe that in a right neighborhood of 0,

$$\frac{-(\alpha-\beta)-[(p-\alpha)u^{\alpha}-(p-\beta)u^{\beta}]}{(1-u^{\alpha-\beta})^{1+\frac{1}{p}}u^{\beta(1+\frac{1}{p})}} \sim \frac{-(\alpha-\beta)}{u^{\beta(1+\frac{1}{p})}},$$

and by (5.21), it follows that $\beta(1+\frac{1}{p})>1$ and $-(\alpha-\beta)<0$, so

$$\int_0^{1/2}\frac{-(\alpha-\beta)}{u^{\beta(1+\frac{1}{p})}}du = -\infty,$$

and therefore $I_0=-\infty$.

Next, we write I_1 as follows:

$$I_1=\int_{1/2}^{1}\frac{(p-\alpha)(1-u^{\alpha})-(p-\beta)(1-u^{\beta})}{u^{\beta(1+\frac{1}{p})}(\frac{1-u^{\alpha-\beta}}{1-u})^{1+\frac{1}{p}}(1-u)^{1+\frac{1}{p}}}du.$$

Applying Taylor's Theorem to the function N, defined by

$$N(u)=(p-\alpha)(1-u^{\alpha})-(p-\beta)(1-u^{\beta}),$$

we find that

$$\begin{aligned} I_1 &= \int_{1/2}^{1}\frac{[(p-\alpha)\alpha-(p-\beta)\beta](1-u)}{u^{\beta(1+\frac{1}{p})}(\frac{1-u^{\alpha-\beta}}{1-u})^{1+\frac{1}{p}}(1-u)^{1+\frac{1}{p}}}du \\ &+\int_{1/2}^{1}\frac{-\frac{1}{2}[(p-\alpha)\alpha(\alpha-1)-(p-\beta)\beta(\beta-1)](1-u)^2+o((1-u)^2)}{u^{\beta(1+\frac{1}{p})}(\frac{1-u^{\alpha-\beta}}{1-u})^{1+\frac{1}{p}}(1-u)^{1+\frac{1}{p}}}du. \end{aligned}$$

Notice that $\lim_{u\to 1^-}\frac{1-u^{\alpha-\beta}}{1-u}=(\alpha-\beta)\in\mathbb{R}$ then, in a left neighborhood of 1,

$$u^{\beta(1+\frac{1}{p})}(\frac{1-u^{\alpha-\beta}}{1-u})^{1+\frac{1}{p}}(1-u)^{1+\frac{1}{p}} \sim (\alpha-\beta)(1-u)^{1+\frac{1}{p}}.$$

Next, we have to distinguish between two cases.

Case $[(p-\alpha)\alpha-(p-\beta)\beta]\neq 0$. In this case the integrand function in I_1 is equivalent in a left neighborhood of 1 to the function $u\mapsto \frac{(p-\alpha)\alpha-(p-\beta)\beta}{(\alpha-\beta)(1-u)^{1/p}}$ and since $p>1$ it follows that $\int_0^{1/2}\frac{(p-\alpha)\alpha-(p-\beta)\beta}{(\alpha-\beta)(1-u)^{1/p}}du\in\mathbb{R}$ and therefore $I_1\in\mathbb{R}$.

Case $[(p-\alpha)\alpha-(p-\beta)\beta]=0$. In this case the integrand function in I_1 is equivalent in a left neighborhood of 1 to the function $u\mapsto(-\frac{1}{2})(p-\alpha)\alpha(1-u)^{1-\frac{1}{p}}$ which is a continuous function on the compact interval $[\frac{1}{2},1]$ then $\int_{1/2}^{1}(-\frac{1}{2})(p-\alpha)\alpha(1-u)^{1-\frac{1}{p}}du\in\mathbb{R}$. Then, in this case $I_1\in\mathbb{R}$ too. Therefore Lemma 56 is proved.

Step2: Uniqueness

First, we point out that Step 1 shows a little bit general result than the existence. In fact, for all $\lambda < 0$, it was proved that $\rho^*(\lambda)$ exists and belongs necessarily to $(\rho_1(\lambda), \rho_2(\lambda))$ if (5.20) holds and belongs to $(\inf R(\lambda), \rho_2(\lambda))$ if (5.21) holds. So, to prove uniqueness, we shall restrict ourselves to $(\rho_1(\lambda), \rho_2(\lambda))$ (resp. to $(\inf R(\lambda), \rho_2(\lambda))$). That is, we shall prove that $S(\lambda, \cdot)$ admits at most one critical point in $(\rho_1(\lambda), \rho_2(\lambda))$ (resp. in $(\inf R(\lambda), \rho_2(\lambda))$). To this end we shall prove that for all $\lambda < 0$, $S(\lambda, \cdot)$ is convex in a neighborhood of each of its critical points lying in $(\rho_1(\lambda), \rho_2(\lambda))$ (resp. in $(\inf R(\lambda), \rho_2(\lambda))$). Similar idea was previously used in [4].

This step is followed by two lemmas. The first one is technical but the second one is the key step.

Lemma 57 Let $p, \alpha, \beta > 1$. If $\alpha \neq \beta$, then $(\frac{\beta}{\alpha})^{1/(\alpha-\beta)} < 1$. If one of the following conditions holds: (a) $p < \alpha < \beta$ or (b) $\beta < \alpha < p$, then for all $\lambda < 0$, the function defined on the interval $[0, \rho_2(\lambda)]$ by $\rho \mapsto \Psi(\lambda, \rho) := (p - \alpha)\rho^\alpha + \lambda(p - \beta)\rho^\beta$, is strictly decreasing on $[0, \rho_1(\lambda)(\frac{\beta}{\alpha})^{1/(\alpha-\beta)}]$ and is strictly increasing on $[\rho_1(\lambda)(\frac{\beta}{\alpha})^{1/(\alpha-\beta)}, \rho_2(\lambda)]$. Moreover, $\Psi(\lambda, 0) = \Psi(\lambda, \rho_1(\lambda)) = 0$, for all $\lambda < 0$.

Proof. The proof is very easy and therefore omitted. For all $\lambda < 0$, let $\rho_*(\lambda) := \max\{\rho_1(\lambda), \inf R(\lambda)\}$.

Lemma 58 Let $p, \alpha, \beta > 1$. Assume that one of the following conditions holds:

(a) $p < \alpha < \beta$, or

(b) $\beta < \alpha < p$.

Then for all $\lambda < 0$,

$$\frac{\partial^2 S}{\partial \rho^2}(\lambda, \rho) + (\frac{p+1}{\rho})\frac{\partial S}{\partial \rho}(\lambda, \rho) > 0, \text{ for all } \rho \in (\rho_*(\lambda), \rho_2(\lambda)).$$

Proof. Notice that for all $\lambda < 0$, $(\rho_*(\lambda), \rho_2(\lambda)) \subset R(\lambda) =: domS(\lambda, \cdot)$. The second derivative of $S(\lambda, \cdot)$ is given by

$$\begin{aligned} \frac{\partial^2 S}{\partial \rho^2}(\lambda, \rho) &= (p')^{-1/p} \int_0^1 \frac{(p+1)(H(\lambda, \rho) - H(\lambda, \rho u))^2}{p^2 \rho (F(\lambda, \rho) - F(\lambda, \rho u))^{(2p+1)/p}} du \\ &\quad + (p')^{-1/p} \int_0^1 \frac{\Phi(\lambda, \rho) - \Phi(\lambda, \rho u)}{p\rho(F(\lambda, \rho) - F(\lambda, \rho u))^{(p+1)/p}} du, \end{aligned}$$

where

$$\begin{aligned} \Phi(\lambda, \rho) &= -p(p+1)F(\lambda, \rho) + 2p\rho f(\lambda, \rho) - \rho^2 f'_\rho(\lambda, \rho) \\ &= \frac{(p-\alpha)(\alpha-(p+1))}{\alpha}\rho^a + \lambda \frac{(p-\beta)(\beta-(p+1))}{\beta}\rho^\beta . \end{aligned}$$

It follows that

$$
\begin{aligned}
&(p')^{1/p} p\rho \left\{ \rho \frac{\partial^2 S}{\partial \rho^2}(\lambda,\rho) + (p+1)\frac{\partial S}{\partial \rho}(\lambda,\rho) \right\} \\
= &\int_0^{\rho} \frac{\Psi(\lambda,\rho) - \Psi(\lambda,\xi)}{(F(\lambda,\rho) - F(\lambda,\xi))^{(p+1)/p}} d\xi \\
&+ \left(\frac{p+1}{p}\right) \int_0^{\rho} \frac{(H(\lambda,\rho) - H(\lambda,\xi))^2}{(F(\lambda,\rho) - F(\lambda,\xi))^{(2p+1)/p}} du
\end{aligned}
$$

where

$$
\Psi(\lambda,\rho) := \Phi(\lambda,\rho) + (p+1)H(\lambda,\rho) = \rho \frac{\partial H}{\partial \rho}(\lambda,\rho) = (p-\alpha)\rho^{\alpha} + \lambda(p-\beta)\rho^{\beta}
$$

By Lemma 57, it follows that for all $\lambda < 0$ and all $\rho \in (\rho_*(\lambda), \rho_2(\lambda))$,

$$
\Psi(\lambda,\rho) - \Psi(\lambda,\xi) > 0, \text{ for all } \xi \in (0,\rho).
$$

Therefore

$$
\int_0^{\rho} \frac{\Psi(\lambda,\rho) - \Psi(\lambda,\xi)}{(F(\lambda,\rho) - F(\lambda,\xi))^{(p+1)/p}} d\xi > 0, \text{ for all } \lambda < 0 \text{ and all } \rho \in (\rho_*(\lambda), \rho_2(\lambda))
$$

Lemma 58 is proved, which ends the proof of Lemma 55 ■

5.4.2 Completion of the proof of Theorem 47

Notice that if one of the hypothesis of Assertions A-F holds then $S(\lambda,\cdot)$ is monotonic; the proofs follow by an elementary discussion as in Assertion A or B of Theorem 46. Therefore the proofs of Assertions A-F are omitted.

Concerning the remaining assertions, the same ideas performed for Assertion C of Theorem 46 still apply. For this, it suffices to use Lemma 55 and the following

Lemma 59 By Lemma 55, let $m(\lambda) := \inf_{\rho \in R(\lambda)} S(\lambda,\rho)$, $\forall \lambda < 0$. Then,

(a) $m(\cdot)$ is continuous on $(-\infty, 0)$

(b) $m(\cdot)$ is strictly decreasing on $(-\infty, 0)$

(c) $\lim_{\lambda \to -\infty} m(\lambda) = +\infty$ et $\lim_{\lambda \to 0} m(\lambda) = 0$

Lemma 60 The function $\lambda \to l(\lambda) := \lim_{\rho \to \inf R(\lambda)} S(\rho,\lambda)$ is either infinite on the whole set $(-\infty, 0)$ or satisfies Assertions (a), (b), and (c) of Lemma 59.

Lemma 61 The function $\lambda \mapsto L(\lambda) := \lim_{\rho \to \sup R(\lambda)} S(\lambda,\rho)$ is (independently of $\ell(\lambda)$) either infinite on the whole set $(-\infty, 0)$ or satisfies Assertions (a), (b), and (c) of Lemma 59.

Proof of Lemma 59.

Recall that for all $\lambda < 0$ and $\rho \in R(\lambda)$,

$$S(\lambda,\rho) = (p')^{-1/p}\int_0^1 \{\lambda\rho^{\beta-p}(\frac{1-t^\beta}{\beta}) + \rho^{\alpha-p}(\frac{1-t^\alpha}{\alpha})\}^{-1/p} dt$$

For all $\lambda < 0$ and $\rho \in R(\lambda)$, let $\tilde\rho = \tilde\rho(\lambda,\rho) := (-\lambda)^{1/(\beta-\alpha)}\rho$. Then, $\rho = (-\lambda)^{1/(\alpha-\beta)}\tilde\rho$ and a simple substitution yields

$$S(\lambda,\rho) = (-\lambda)^{(p-\alpha)/p(\alpha-\beta)} S(-1, \tilde\rho(\lambda,\rho)).$$

Thus,

$$\begin{aligned} m(\lambda) &= (-\lambda)^{(p-\alpha)/p(\alpha-\beta)} \inf_{\rho \in R(\lambda)} S(-1, \tilde\rho(\lambda,\rho)) \\ &= (-\lambda)^{(p-\alpha)/p(\alpha-\beta)} \inf_{\rho \in R(-1)} S(-1, \rho) \end{aligned}$$

So,

$$m(\lambda) = (-\lambda)^{(p-\alpha)/p(\alpha-\beta)} m(-1), \forall \lambda < 0$$

Therefore, the lemma isproved ◇

Proof of Lemma 60.

By Lemma 53, it follows in the case where $1 < \beta < \alpha < \alpha+\beta \le p$ that $\lim_{\rho \to \inf R(\lambda)} S(\lambda,\rho) = (-\lambda)^{(p-\alpha)/p(\alpha-\beta)} J(p,\alpha,\beta) < +\infty$, where $J(p,\alpha,\beta)$ is defined in (5.4). Hence, Lemma 60 is proved in this case. The other cases are similar or easier.

The proof of Lemma 61 is similar to that of Lemma 60.

5.5 Remarks

By the study of Problem (5.3) with $\lambda \in \mathbb{R}$, we can deduce the structure of the solution set of a quite general problem

$$\begin{aligned} -(\varphi_p(u'))' &= \mu\varphi_\alpha(u) + \lambda\varphi_\beta(u) \quad \text{in } (0,1) \\ u(0) &= u(1) = 0, \end{aligned} \tag{5.22}$$

when $p, \alpha, \beta > 1$, $\mu > 0$ and $\lambda \in \mathbb{R}$. In fact, if v is a solution of (5.3) for $\lambda = \lambda_0 \in \mathbb{R}$ and $\alpha \neq p$, then for all $\mu_0 > 0$, the function $u := \mu_0^{1/(p-\alpha)} v$ is a solution to (5.22) with $\mu = \mu_0$ and $\lambda = \lambda_0 \mu_0^{(\beta-p)/(\alpha-p)}$. (If $\alpha = p$ and $\beta \neq p$, similar change of variable works). Conversely, if u is a solution of (5.22) with $\mu = \mu_0 > 0$, $\lambda = \lambda_0 \in \mathbb{R}$ and $\alpha \neq p$, then $v := \mu_0^{1/(\alpha-p)} u$ is a solution to (5.3) with $\lambda = \lambda_0 \mu_0^{(\beta-p)/(\alpha-p)}$.

The structure of the solution set of problem (5.22) when $\mu < 0$ and $\lambda \in \mathbb{R}$ can be deduced from that of the problem

$$\begin{aligned} -(\varphi_p(u'))' &= -\varphi_\alpha(u) + \lambda \varphi_\beta(u) \quad \text{in } (0,1) \\ u(0) &= u(1) = 0, \end{aligned}$$

which is not treated here. However, upon completing our paper, the work [55] by Díaz and Hernández appeared. Positive solutions to problem (5.22) with $\mu < 0$, $\lambda > 0$ and $1 < \alpha < \beta \leq p$ are treated there.

After completing this work, e-mail correspondence between the first author (I. Addou) and Professor Pedro Ubilla from Chile, revels that simultaneously and independently of the present authors, Professors J. Sánchez and P. Ubilla from Chile, were studying problem (5.3) with $\lambda > 0$. That is to say, they resolved the p-Laplacian version of the Ambrosetti-Brezis-Cerami problem. To do so, they provide essentially the same proof as that of Theorem 46 above by making use of the same idea performed in [8, Lemma 7, (iii)]. Their work was presented, under the title: "The exact number of positive solutions for an elliptic equation with concave and convex nonlinearities" by Professor P. Ubilla at the "USA-Chile Workshop on Nonlinear Analysis" meeting which held in Valparaíso in Chile on 17-21, January 2000. It was also published in a volume of this journal (see [105]).

Also, after submitting this work for publication, e-mail correspondence between the first author (I. Addou) and Professor Shin-Hwa Wang from Taiwan (R. O. China) revels that he wrote (independently of J. Sanchez and P. Ubilla and independently of the present authors) a paper [122] in which he resolves (for $p = 2$) the Ambrosetti-Brezis-Cerami problem [17, Sect. 6, (d)] (among many other interesting things), by making use of the quadrature technique. To deal with the difficult step (uniqueness of the maximum of the time map), he used an interesting argument which is comparable to that of [8, Lemma 7, (iii)] and used by him previously in [122, Proof of Theorem 7]. (See, also [106]).

5.6 Appendix

In the process of our proofs, we have used the fact that

$$\lambda_1(p) = (p - 1)\left(\frac{2\pi}{p\sin(\pi/p)}\right)^p > 1, \forall p > 1.$$

In this appendix we shall prove the following:

(A1) $\lambda_1(p) > 1, \forall p \in (1,2)$.

(A2) $\lambda_1(2) = \pi^2$.

(A3) $\lambda_1(p) > 4, \forall p \in (2, +\infty)$.

(A4) $\lim_{p \to 1^+} \lambda_1(p) = 2$, and $\lim_{p \to +\infty} \lambda_1(p) = +\infty$.

Proof of (A1). Observe that for all $p \in (1,2)$, $\lambda_1(p)$ can be written as

$$\lambda_1(p) = \left(\frac{2}{p}\right)^p (p - 1)^{(1-p)} \left(\frac{\pi}{\frac{\sin(\pi/p) - \sin(\pi/1)}{p-1}}\right)^p. \tag{5.23}$$

The function $p \mapsto \theta(p) := (\frac{2}{p})^p$ is strictly decreasing on $(1,2]$ and $\theta(2) = 1$. Thus,

$$\left(\frac{2}{p}\right)^p > 1, \quad \forall p \in (1,2). \tag{5.24}$$

The function $p \mapsto K(p) := (p - 1)^{(1-p)}$ is strictly increasing on $(1, 1 + \exp(-1)]$ and is strictly decreasing on $[1 + \exp(-1), 2)$. Thus, $K(p) > \min\{K(2),$
$\lim_{p \to 1^+} K(p)\} = 1$, for all $p \in (1,2)$. Therefore,

$$(p - 1)^{(1-p)} > 1, \forall p \in (1,2). \tag{5.25}$$

Notice that for all $c \in (1,2)$: $c^2 > 1 > -\cos(\pi/c) > 0$. Then, $\frac{\pi}{-(\pi^2/c)\cos(\pi/c)} > 1$, for all $c \in (1,2)$. Therefore,

$$\left(\frac{\pi}{-(\pi^2/c)\cos(\pi/c)}\right)^p > 1, \ \forall (p,c) \in (1,2)^2. \tag{5.26}$$

On the other hand, for all $p \in (1,2)$, there exists $c = c_p \in (1,p) \subset (1,2)$, such that

$$\frac{\sin(\pi/p) - \sin(\pi/1)}{p - 1} = -\frac{\pi}{c^2}\cos\frac{\pi}{c}.$$

Therefore, for all $p \in (1,2)$, there exists $c = c_p \in (1,p) \subset (1,2)$, such that

$$\left(\frac{\pi}{\frac{\sin(\pi/p) - \sin(\pi/1)}{p-1}}\right)^p = \left(\frac{\pi}{-(\pi^2/c)\cos(\pi/c)}\right)^p$$

and by (5.26), it follows that

$$\left(\frac{\pi}{\frac{\sin(\pi/p) - \sin(\pi/1)}{p-1}}\right)^p > 1, \ \forall p \in (1,2). \tag{5.27}$$

Now, by (5.24), (5.25), (5.27), and (5.23), Assertion (A1) follows.

Proof of (A2). Easy computation.

Proof of (A3). Observe that for all $p > 2$, $\lambda_1(p)$ can be written as

$$\lambda_1(p) = (p - 1) \cdot 2^p \cdot \left(\frac{\sin(\pi/p) - \sin(0)}{(\pi/p) - 0}\right)^{-p}. \tag{5.28}$$

It is clear that for all $p > 2$

$$(p - 1) > 1 \text{ and } 2^p > 2^2 = 4. \tag{5.29}$$

On the other hand, for all $p > 2$, there exists $c = c_p$ such that

$$0 < \frac{\sin(\pi/p) - \sin(0)}{(\pi/p) - 0} < \cos c < 1.$$

Therefore,

$$\left(\frac{\sin(\pi/p) - \sin(0)}{(\pi/p) - 0}\right)^{-p} > 1, \forall p > 2. \tag{5.30}$$

Now, by (5.29), (5.30), and (5.28), Assertion (A3) follows.

Proof of (A4). It is clear that

$$\lim_{p \to 1^+} \theta(p) = 2, \quad \lim_{p \to 1^+} K(p) = 1, \quad \lim_{p \to 1^+} \pi^p = \pi, \text{ and}$$

$$\lim_{p \to 1}\left(\frac{\sin(\pi/p) - \sin(0)}{p - 1}\right)^p = \left(-\frac{\pi}{1^2}\cos\frac{\pi}{1}\right)^1 = \pi.$$

Therefore, using expression (5.23) of $\lambda_1(p)$, it follows that $\lim_{p \to 1^+} \lambda_1(p) = 2$. The computation of the second limit is straightforward, which completes the proof of Assertion (A4).

Chapter 6

Uniqueness result for a strongly nonlinear O.D.E's

To appear in Maghreb Mathematical Revue.

6.1 Introduction :

During the last two decades, the study of the existence and the number of solutions to boundary value problems of the type:

$$\begin{cases} -(\varphi(u'))' = F(\lambda, x, u), & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

where λ is a real parameter, φ is an odd strictly increasing homeomorphism of R, has been the subject of several works. For most of them, the authors attempted to generalize some results established for $\varphi(u) = u$ to the case where φ is any odd strictly increasing homeomorphism of R. Several results have been obtained in the case $\varphi(x) = |x|^{p-2}x$, $p>1$, in which the operator $u \to \left(|u'|^{p-2}u'\right)'$ is called the one dimensional p-Laplacian (see [9], [8], [33], [49], [50], [69] and [111]). However, in the general case, only few results are obtained.

In this paper, we give an existence and uniqueness result of positive solution to the boundary value problem

$$\begin{cases} -(\varphi(u'))' = \lambda f(u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{6.1}$$

where $\lambda > 0$ is a real parameter; the functions φ and f satisfy the following assumptions:

$$\varphi \text{ is an odd strictly increasing homeomorphism of } \mathbb{R}, \tag{6.2}$$

$$\begin{aligned} &\text{there exists } m_\varphi > 0 \text{ such that :} \\ &\varphi(x) - \varphi(y) \geq m_\varphi (x - y) \text{ for all } x \geq y \text{ in } \mathbb{R}^+, \end{aligned} \tag{6.3}$$

$$f \in C^1\left(\mathbb{R}^+, (0, +\infty)\right) \text{ and } f \text{ is strictly increasing,} \tag{6.4}$$

$$\text{there exists } M_f > 0 \text{ such that } f'(x) < M_f \text{ for all } x \in \mathbb{R}^+ \tag{6.5}$$

By a solution of problem (7.2), we mean a pair $(\lambda, u) \in (0, +\infty) \times C^1([0,1])$ such that : $-(\varphi(u'))' = \lambda f(u)$ in $(0,1)$ and $u(0) = u(1) = 0$.

Let $F(\lambda, \cdot) : C^1([0,1]) \to L^1$ be the Nemitski operator associated to λf and $K : L^1 \to C^1([0,1])$ be the operator defined in [64], where $K(h) = u$ with u is the unique solution of the problem

$$\begin{cases} -(\varphi(u'))' = h \text{ in } (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

Thus, problem (7.2) is equivalent to the nonlinear equation:

$$u = T(\lambda, u)$$

where $T = K \circ F$ is completely continuous (for a detailed proof see G. Huidobro et al [64]).

Hence, it follows from the Leray-Schauder continuation theorem that there exists a component $\mathcal{C} \subset [0, +\infty) \times C^1([0,1])$ of solutions to problem (7.2) which is unbounded in $[0, +\infty) \times C^1([0,1])$.

Let

$$\lambda_0 = \sup \left\{ \lambda > 0 \, / \text{ there exists } u \in C^1([0,1]) \text{ such that } (\lambda, u) \text{ is a solution of (7.2)} \right\}$$

and

$$\lambda_1 = \mu_1 \left(\frac{M_f}{m_\varphi}, [0,1] \right)$$

where $\mu_1(q, [0,1])$ is the first eigenvalue of the problem:

$$\begin{cases} -u'' = \mu q u \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

with q measurable, bounded, positive in $(0,1)$.

We recall that the above problem has an increasing sequence of eigenvalues $(\mu_k)_{k\geq 1}$ such that $\lim_{k\to+\infty}\mu_k = +\infty$. Moreover the eigenfunction ϕ_k associated to μ_k has exactly $(k-1)$ simple zeros.

Denote by $\lambda^* = \min(\lambda_0, \lambda_1)$, then our purpose is to prove the following result :

Theorem 62 Assume that assumptions (6.2) $-$ (6.5) hold true. Then for any $\lambda \in (0, \lambda^*)$, there exists a unique $u \in C^1([0,1])$, $u>0$ in $(0,1)$ such that (λ, u) is a solution of problem (7.2).

Remark 8 We can replace hypothesis (6.3) by the following assumption:

$$\varphi \in C^1(\mathbb{R}) \text{ and } \varphi' \geq m_\varphi > 0.$$

Remark 9 A similar result have been obtained by A.Ambrosetti (see Theorem 1 in [15]) for the semilinear problem

$$\begin{cases} -\Delta u = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \delta\Omega \end{cases}$$

where $\lambda > 0$, Ω is an open bounded domain of $\mathbb{R}^n$ and $f \in C^1(\Omega \times \mathbb{R}^+, (0, +\infty))$.

6.2 Preliminary results:

In this section, we give some results we need in the sequel.

Consider the boundary value problem :

$$\begin{cases} -(\varphi(u'))' = g(u) & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{6.6}$$

where φ satisfies assumption (6.2) and g is any function in $C(\mathbb{R}^+, (0, +\infty))$.

We define a solution of problem (6.6) to be a function $u \in C^1([0,1])$ satisfying $-(\varphi(u'))' = g(u)$ in $(0,1)$ and $u(0) = u(1) = 0$.

Let u be a solution of problem (6.6), then there exists a real constant $C \geq 0$ such that:

$$\Phi\left(\varphi\left(u'(x)\right)\right) + G(u(x)) = C \text{ for all } x \in [0,1] \tag{6.7}$$

where $G(s) := \int_0^s g(t)\,dt$, $\Phi(s) := \int_0^s \psi(s)\,dt$ and $\psi := \varphi^{-1}$.

Observe that Φ is even, $\Phi(0) = 0$ and $\Phi(s) > 0$ for all $s \neq 0$.

We now define some subsets of $C^1([0,1])$ which will be used in the rest of this paper.

Let S_1^+ and A_1^+ be the subsets defined in Chapter 1. Then

Lemma 63 Any solution of problem (6.6) is necessarily in A_1^+.

Proof Let u be a solution of problem (6.6). We begin our proof by showing that u' vanishes once and only once. To this end, we will prove that any critical point x_* of u remains in $(0,1)$ and at x_*, u reaches a maximum value.

Suppose that $u'(0) = 0$, then for $x = 0$ in (6.7), we get $C = 0$. So, for any $x \in [0,1]$, $G(u(x)) = -\Psi(\varphi(u'(x))) \leq 0$. Since G is strictly increasing in $\mathbb{R}^+$ and $G(0) = 0$, we have $u(x) = 0$ for any $x \in [0,1]$. This is impossible because 0 is not a solution of problem (6.6).

Using the same arguments, we get $u'(1) \neq 0$.

Now, let $x_* \in (0,1)$ be a critical point of u. It follows from (6.7) that $(\varphi(u')) = -g(u) < 0$ in $(0,1)$. This implies that $\varphi(u')$ is strictly decreasing in $(0,1)$. So, $\varphi(u') > 0$ in $(0,x_*)$ and $\varphi(u') < 0$ in $(x_*,1)$. As φ is odd strictly increasing, we get $u' > 0$ in $(0,x_*)$ and $u' < 0$ in $(x_*,1)$; thus, u reaches a maximum value at x_* and $u > 0$ in $(0,1)$.

Let $\rho = u(x_*) = \max u(x)$.

From equation (6.7), it follows that:

$$u'(x) = \psi\left(\Psi_+^{-1}(G(\rho) - G(u(x)))\right) \text{ for any } x \in [0,x_*] \tag{6.8}$$

$$u'(x) = -\psi\left(\Psi_+^{-1}(G(\rho) - G(u(x)))\right) \text{ for any } x \in [x_*,1] \tag{6.9}$$

where Ψ_+^{-1} is the inverse of the function Ψ on $[0,+\infty)$. Then:

$$x = \int_0^{u(x)} \frac{du(t)}{u'(t)} = \int_0^{u(x)} \frac{du(t)}{\psi\left(\Psi_+^{-1}(G(\rho) - G(u(t)))\right)} \text{ for any } x \in [0,x_*] \tag{6.10}$$

$$1 - x = -\int_0^{u(x)} \frac{du(t)}{u'(t)} = \int_0^{u(x)} \frac{du(t)}{\psi\left(\Psi_+^{-1}(G(\rho) - G(u(t)))\right)} \text{ for any } x \in [x_*,1] \tag{6.11}$$

Putting $x = x_*$ in (6.10) and (6.11), we get

$$x_* = \int_0^{\rho} \frac{du(t)}{\psi\left(\Psi_+^{-1}(G(\rho) - G(u(t)))\right)} = 1 - x_*$$

which yields

$$x_* = \frac{1}{2}.$$

As for the symmetry of u about $\frac{1}{2}$, it suffices to show that $u(1-x) = u(x)$ for any $x \in [0,1]$. This becomes very easy if we observe that $x = 1-(1-x)$; then from (6.10) and (6.11), we get:

$$\begin{aligned} x = 1-(1-x) &= \int_0^{u(x)} \frac{du(t)}{\psi\left(a_+^{-1}\left(G(\rho)-G(u(t))\right)\right)} \\ &= \int_0^{u(1-x)} \frac{du(t)}{\psi\left(a_+^{-1}\left(G(\rho)-G(u(t))\right)\right)}; \end{aligned}$$

so, we obtain $u(1-x) = u(x)$ for any $x \in [0,1]$. Therefore the lemma is proved. ■

Lemma 64 If $u_1 \neq u_2$ are two solutions of problem (6.6), then $u_1 \neq u_2$ are ordered ($u_1<u_2$ in $(0,1)$ or $u_1<u_2$ in $(0,1)$).

Proof Let u_1 and u_2 be two solutions of the lemma.

We have

- either $u_1'(0) = u_2'(0)$
- or $u_1'(0) \neq u_2'(0)$.

Assume that the first situation holds. We deduce from equation (6.7) :

$$G\left(u_1\left(\frac{1}{2}\right)\right) = a\left(\varphi\left(u_1'(0)\right)\right) = G\left(u_2\left(\frac{1}{2}\right)\right) = a\left(\varphi\left(u_2'(0)\right)\right).$$

Since G is strictly increasing, we get $u_1\left(\frac{1}{2}\right) = u_2\left(\frac{1}{2}\right)$.

Let $\rho = u_1\left(\frac{1}{2}\right) = u_2\left(\frac{1}{2}\right)$, then (6.10) written for u_1 and u_2 gives:

$$\begin{aligned} x &= \int_0^{u_1(x)} \frac{du(t)}{\psi\left(a_+^{-1}\left(G(\rho)-G(u_1(t))\right)\right)} \\ &= \int_0^{u_2(x)} \frac{du(t)}{\psi\left(a_+^{-1}\left(G(\rho)-G(u_2(t))\right)\right)} \text{ for any } x \in \left[0,\frac{1}{2}\right]. \end{aligned}$$

Hence, $u_1(x) = u_2(x)$ for all $x \in \left[0,\frac{1}{2}\right]$. But u_1 and u_2 are in A_1^+; then using the fact that u_1 and u_2 are symmetrical about $\frac{1}{2}$, it follows $u_1(x) = u_2(x)$ for all $x \in [0,1]$, which is impossible.

Now, suppose that $u_1'(0) < u_2'(0)$. Since u_1 and u_2 are symmetrical about $\frac{1}{2}$, we will prove that $u_1(x) = u_2(x)$ for all $x \in \left(0,\frac{1}{2}\right]$.

Let $A = \left\{x \in \left(0,\frac{1}{2}\right], u_1(x) = u_2(x)\right\}$. Assume $A \neq \emptyset$ and let $x_0 = \inf A$ and $u = u_1 - u_2$.

Then $x_0 > 0$, because if $x_0 = 0$ and (x_n) is a sequence such that $\lim\limits_{n\to+\infty} x_n = x_0$, we get :

$$0 < u_2'(0) - u_1'(0) = \lim_{n\to+\infty} \frac{u_2(x_n) - u_1(x_n)}{x_n} = 0$$

which is impossible.

Thus, let (y_n) be a sequence in $(0, x_0)$ such that $\lim\limits_{n\to+\infty} y_n = x_0$. We get:

$$u'(x_0) = \lim_{n\to+\infty} \frac{u(y_n) - u(x_0)}{y_n - x_0} = \lim_{n\to+\infty} \frac{u(y_n)}{y_n - x_0} \leq 0$$

and then,

$$0 \leq u_2(x_0) \leq u_1(x_0).$$

Using again (6.7), we obtain:

$$\begin{aligned} &\Psi(\varphi(u_1'(0))) - \Psi(\varphi(u_1'(x_0))) = G(u_1(x_0)) \\ &= G(u_1(x_0)) = \Psi(\varphi(u_2'(0))) - \Psi(\varphi(u_2'(x_0))) \end{aligned}$$

and then

$$0 > \Psi\left(\varphi\left(u_1'(0)\right)\right) - \Psi\left(\varphi\left(u_2'(0)\right)\right) = \Psi\left(\varphi\left(u_1'(x_0)\right)\right) - \Psi\left(\varphi\left(u_2'(x_0)\right)\right) \geq 0$$

which is impossible, therefore $A = \emptyset$ ■.

Now to any homeomorphism φ of R satisfying (6.2), we associate the operator :

$$A_\varphi : C^1([0,1]) \to C^1([0,1])$$

defined as follows:

$$A_\varphi(u)(x) = \int_0^x \varphi\left(u'(t)\right) dt \text{ for any } u \in C^1([0,1]) \text{ and } x \in [0,1].$$

Then the following result holds true:

Lemma 65 If $u \in A_1^+$ then $v = A_\varphi(u) \in A_1^+$. Moreover if (λ, u) is a solution of (7.2), then (λ, v) is a solution of the boundary value problem

$$\begin{cases} -v''' = \lambda f(A_\psi(v)) \text{ in } (0,1) \\ v(0) = v(1) = 0 \end{cases}$$

where $\psi = \varphi^{-1}$.

Proof Let $u \in A_1^+$ and $v = A_\varphi(u)$. It is clear that $v'(x) = \varphi(u'(x))$. So, v' vanishes only once.

As u is symmetrical about $\frac{1}{2}$, we deduce :

$$u(1-x) = u(x) \text{ and } u'(1-x) = -u'(x) \text{ for any } x \in [0,1].$$

Since φ is odd, we get:

$$\varphi\left(u'(1-x)\right) = -\varphi\left(u'(x)\right) \text{ for any } x \in [0,1].$$

So, for any $x \in [0,1]$, we have

$$\int_x^{1-x} \varphi\left(u'(t)\right) dt = \int_x^{\frac{1}{2}} \varphi\left(u'(t)\right) dt + \int_{\frac{1}{2}}^{1-x} \varphi\left(u'(t)\right) dt.$$

Using the change of variables $s = 1-t$, we get:

$$\begin{aligned}\int_x^{1-x} \varphi\left(u'(t)\right) dt &= \int_x^{\frac{1}{2}} \varphi\left(u'(t)\right) dt + \int_{\frac{1}{2}}^{x} -\varphi\left(u'(1-s)\right) ds \\ &= \int_x^{\frac{1}{2}} \varphi\left(u'(t)\right) dt + \int_{\frac{1}{2}}^{x} \varphi\left(u'(s)\right) ds = 0.\end{aligned}$$

Finally, we obtain:

$$\begin{aligned}v(1-x) &= \int_0^{1-x} \varphi\left(u'(t)\right) dt = \int_0^{x} \varphi\left(u'(t)\right) dt + \int_x^{1-x} \varphi\left(u'(t)\right) dt \\ &= \int_0^{x} \varphi\left(u'(t)\right) dt = v(x).\end{aligned}$$

The fact that v satisfies the equation in Lemma 65 can be checked by straighforward computations. ■

6.3 Proof of Theorem 62:

Suppose the contrary, and let $\lambda \in (0, \lambda^*)$, $u_1, u_2 \in C^1([0,1])$ such that (λ, u_1) and (λ, u_2) are two solutions of problem (7.2).

Let $v_i = A(u_i)$ for $i = 1, 2$ and $w = v_2 - v_1$.

We deduce from Lemmas 63 and 65 that u_i and v_i lie in A_1^+ for $i = 1, 2$.

Moreover, from Lemma 64, we can suppose that $u_1 < u_2$. Since f is strictly increasing and φ is odd strictly increasing, we have:

$$\left(\varphi\left(u_2'\right) - \varphi\left(u_1'\right)\right)' = -\lambda\left(f(u_2) - f(u_1)\right) < 0 \text{ in } (0,1)$$

and so,

$$\varphi\left(u_2'(t)\right) > \varphi\left(u_1'(t)\right) \text{ for all } t \in \left[0, \tfrac{1}{2}\right).$$

Then,

$$v_2(x) = \int_0^x \varphi\left(u_2'(t)\right) dt > \int_0^x \varphi\left(u_1'(t)\right) dt = v_1(x) \text{ for all } x \in \left(0, \tfrac{1}{2}\right].$$

Finally, since for $i = 1, 2$ v_i is symmetrical about $\frac{1}{2}$, it follows that $v_2 > v_1$ in $(0,1)$ and $w = v_2 - v_1$ lies in A_1^+.

Now observe that w is a solution of the problem

$$\begin{cases} -w'' = \lambda q(x) w \text{ in } (0,1) \\ w(0) = w(1) = 0 \end{cases}$$

where

$$q(x) = \begin{cases} q_f(x)\, q_\varphi(x) & \text{if } x \in (0,1) \\ \dfrac{f'(0)}{m_\varphi} & \text{if } x = 0, 1 \end{cases}$$

$$q_f(x) = \frac{f(A_\psi(v_2)(x)) - f(A_\psi(v_1)(x))}{A_\psi(v_2)(x) - A_\psi(v_1)(x)} = \frac{f(u_2(x)) - f(u_1(x))}{u_2(x) - u_1(x)}$$

and

$$q_\varphi(x) = \frac{A_\psi(v_2)(x) - A_\psi(v_1)(x)}{v_2(x) - v_1(x)} = \frac{u_2(x) - u_1(x)}{v_2(x) - v_1(x)}.$$

On one hand, assumptions (6.4) and (6.5) lead to

$$q_f(x) < M_f \text{ for all } x \in (0,1).$$

On the other hand, assumption (6.3) gives

$$\psi(x) - \psi(y) \leq \frac{1}{m_\varphi}(x - y) \text{ for all } x \geq y.$$

Then

$$q_\varphi(x) = \frac{\int_0^x (\psi(v_2(t)) - \psi(v_1(t)))\,dt}{v_2(x) - v_1(x)} \leq \frac{1}{m_\varphi} \text{ for all } x \in \left[0, \tfrac{1}{2}\right].$$

As q_φ is symmetrical about $\frac{1}{2}$, we get

$$q_\varphi(x) \leq \frac{1}{m_\varphi} \text{ for all } x \in (0,1).$$

Finally, we get

$$q(x) < \frac{M_f}{m_\varphi} \text{ for all } x \in [0,1].$$

Since w lies in A_1^+, $\lambda = \mu_1(q,[0,1])$. Now, if we use Proposition 1.12A in [48], we obtain

$$\lambda = \mu_1(q,[0,1]) > \mu_1\left(\frac{M_f}{m_\varphi},[0,1]\right) = \lambda_1 \geq \lambda^*$$

which is impossible, because $\lambda \in (0,\lambda^*)$. Thus, the theorem is proved. ■

Chapter 7

Complete desciption of the set of solutions to a strongly nonlinear O.D.E's.

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7.1 Introduction

The purpose of this paper is to give a complete description of the set of solutions to the boundary value problem

$$\begin{cases} -(\varphi(u'))' = f(u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{7.1}$$

where φ is an odd increasing homeomorphism of R and f is an odd function of $C(\mathrm{R}, \mathrm{R})$.

By a solution of (7.1), we mean a function $u \in C^1([0,1])$ satisfying $(\varphi(u'))' = -f(u)$ in $(0,1)$ and the Dirichlet conditions $u(0) = u(1) = 0$.

Note that the differential operator $u \to (\varphi(u'))'$ is linear if and only the function $x \to \varphi(x)$ is linear, hence the ODE in (7.1) is said strongly nonlinear.

This work is motivated by the previous ones done in [64], [63], [65], [45] and essentially by [67].

In [67] García-Huidobro & Ubilla study problem (7.1) under the following hypothesis on the functions f and φ

$$\lim_{x \to 0} \frac{\varphi(\sigma x)}{\varphi(x)} = \sigma^{q-1} \text{ for some } q > 1 \text{ and for all } \sigma \in (0,1),$$

$$\lim_{x \to +\infty} \frac{\varphi(\sigma x)}{\varphi(x)} = \sigma^{p-1} \text{ for some } p > 1 \text{ and for all } \sigma \in (0,1),$$

$$\lim_{x \to 0} \frac{f(x)}{\varphi(x)} = a \text{ and } \lim_{x \to +\infty} \frac{f(x)}{\varphi(x)} = A.$$

Using time-maps approach they give a multiplicity result when a and A lie in some resonance intervals.

In this work we will replace the growing conditions on φ and f at 0 and $+\infty$ by global conditions on the convexity of φ and f. These new conditions which will play significant role in the proof of existence of solutions as well as in the proof of uniqueness of these solutions in some areas of $C^1([0,1])$, can appear very restrictive. However we think that this condition is usual, indeed this kind of assumption is often met in the literature when an exactitude result is aimed (see [15], [42] and [76]).

Our strategy is as follows:

In a first stage, we locate the possible solutions of problem (7.1) in some subsets A_k^ν, (where for $k \in \mathbb{N}^*$ and $\nu = +, -$ A_k^ν is defined in section 2) of $C^1([0,1])$ and we give some properties of these solutions. An immediate consequence of these results is: $u \in A_k^+$ is solution to problem (7.1) if and only if u is a positive solution to the problem

$$\begin{cases} -(\varphi(u'))' = f(u) \text{ in } \left(0, \frac{1}{2k}\right) \\ u(0) = u'\left(\frac{1}{2k}\right) = 0. \end{cases} \tag{7.2}$$

Then we associate to problem (7.2) the auxiliary Sturm-Liouville problem

$$\begin{cases} -v''(x) = f\left(\int_0^x \psi(v'(t))\,dt\right) \text{ in } \left(0, \frac{1}{2k}\right) \\ v(0) = v'\left(\frac{1}{2k}\right) = 0. \end{cases} \tag{7.3}$$

such that u is positive solution to problem (7.2) if and only if $v(x) = \int_0^x \varphi(u'(t))\,dt$ is a positive solution to the auxiliary Sturm-Liouville problem (7.3). Thus we are driven to investigate a nonlinear Sturm-Liouville problem for which after addition of a linear part containing a real parameter existence of a positive solution will be proved by the use of Rabinowitz global bifurcation theory (see [93], [94] and [95]).

At the end, we will use assumptions (7.5) and (7.7) to prove uniqueness of the solution in each subset A_k^{ν}.

The chapter is organized as follows: The section 7.2 is devoted to the statement of the main results and some necessary notations. In section 7.3, we expose some preliminary results we need in the proof of the principal results. In the last section, we give the proofs of main results.

7.2 Notations and main results

In the following we denote by $E = C^1([a,b])$ with its norm $\|u\|_1 = \|u\|_0 + \|u'\|_0$

Let, for any integer $k \geq 1$ A_k^-, A_k^+ and A_k the subsets defined in Chapter 1.

We recall that the boundary value problem:

$$\begin{cases} -u'' = \lambda u \text{ in } (a,b) \\ u(a) = u'(b) = 0 \end{cases}$$

has an increasing sequence of eigenvalues $(\mu_k([a,b]))_{k \geq 1}$ with $\mu_k([a,b]) = \dfrac{(2k-1)^2\pi^2}{4(b-a)^2}$.

We will use in this work the so called Jensen inequality given by:

$$F\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \geq \frac{1}{b-a}\int_a^b F(u(t))\,dt$$

where $F : \mathbb{R} \to \mathbb{R}$ is a concave function and u is a function in $C([a,b])$.

Moreover if $b-a<1$ and $F(0)=0$ then

$$F\left(\int_a^b u(t)\,dt\right) \geq \int_a^b F(u(t))\,dt \tag{7.4}$$

Let S be the set of solutions to problem (7.1), then our main results are :

Theorem 66 (Superlinear case) :

Suppose the functions φ and f satisfy the following conditions:

$$\varphi \text{ is concave on } \mathbb{R}^+, \tag{7.5}$$

$$\lim_{x \to 0} \frac{f(x)}{\varphi(x)} = 0 \text{ and } \lim_{x \to +\infty} \frac{f(x)}{\varphi(x)} = +\infty, \tag{7.6}$$

$$\text{the function } s \to \frac{f(s)}{s} \text{ is increasing on } (0,+\infty) \tag{7.7}$$

Then

$$S \subset \{0\} \cup \left(\bigcup_{k \geq 1} A_k \right)$$

and for each integer $k \geq 1$ there exists $u_k \in A_k^+$ such that

$$S \cap A_k = \{u_k, -u_k\}.$$

Theorem 67 (Sublinear case) :

Suppose the functions φ and f satisfy the following conditions:

$$\varphi \text{ is convex on } \mathbb{R}^+, \tag{7.8}$$

$$\lim_{x \to 0} \frac{f(x)}{\varphi(x)} = +\infty \text{ and } \lim_{x \to +\infty} \frac{f(x)}{\varphi(x)} = 0, \tag{7.9}$$

$$f \text{ is increasing and concave in } \mathbb{R}^+. \tag{7.10}$$

Then

$$S \subset \{0\} \cup \left(\bigcup_{k \geq 1} A_k \right)$$

and for each integer $k \geq 1$ there exists $u_k \in A_k^+$ such that

$$S \cap A_k = \{u_k, -u_k\}.$$

Remark 10 The above theorems give a complete description of the solution set of the problem (7.1), indeed the theorems state that there is no solution except the trivial solution and those belonging to $\bigcup_{k \geq 1} A_k$, and in each $A_k^{\pm}$ there is exactly one solution.

Remark 11 Hypothesis (7.7) is similar to (3-3) assumed in [32]. To obtain the exact number of solutions to the boundary value problem

$$\begin{cases} -u'' = \lambda u + f(u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{7.11}$$

according λ in a resonance interval, the author assumed that the function $s \to \frac{f(s)}{s}$ and $s \to \frac{-f(-s)}{s}$ are increasing on $(0, +\infty)$.

Note that, hypothesis (7.7) implies that f is increasing, and if f is convex then hypothesis (7.7) is satisfied.

In the sublinear case, hypothesis (7.10) implies that the function $s \to \frac{f(s)}{s}$ is decreasing on

$(0,+\infty)$.

7.3 Some preliminary results:

In this section we give some lemmas which will be crucial for the proof of our main results.

Consider the boundary value problem

$$\begin{cases} -(\varphi(u'))' = g(u) \text{ in } (a,b) \\ u(a) = u(b) = 0 \end{cases} \tag{7.12}$$

where φ is an odd increasing homeomorphism of $\mathbb{R}$ and g is a function in $C(\mathbb{R},\mathbb{R})$ satisfying

$$xg(x) > 0 \text{ for all } x \in \mathbb{R}^*. \tag{7.13}$$

We define a solution of problem (7.12) to be a function $u \in E$ satisfying $(\varphi(u'))' = -g(u)$ in (a,b) and $u(a) = u(b) = 0$.

If u is a solution to problem (7.12), then there exists a real constant $C \geq 0$ such that

$$\Phi\left(\varphi\left(u'(x)\right)\right) + G(u(x)) = C \text{ for all } x \in [a,b] \tag{7.14}$$

where $G(x) = \int_0^x g(t)\,dt$, $\Phi(x) = \int_0^x \psi(t)\,dt$ with $\psi = \varphi^{-1}$.

Note that Φ the Legendre transform of the convex function Ψ where $\Psi(s) = \int_0^s \varphi(t)\,dt$, is even, $\Phi(0) = 0$ and $\Phi(s) > 0$ for all $s \neq 0$.

Then, the first result in this section is:

Lemma 68 Suppose that hypothesis (7.13) holds true. If u is a nontrivial solution to problem (7.12), then there exists an integer $k \geq 1$ such that $u \in A_k$.

Proof Let u be a nontrivial solution to problem (7.12). We begin the proof by showing that $u'(a) \neq 0$.

Let us suppose the contrary. Then, if we put $x = a$ in equation (7.14), we get $C = 0$. Thus, for any $x \in [0,1]$, $G(u(x)) = -\Phi(\varphi(u'(x))) \leq 0$. Since G is strictly positive on $\mathbb{R}^*$ and $G(0) = 0$, $u(x) = 0$ for all $x \in [a,b]$. This is impossible since u is a nontrivial solution.

Now, let us show that u has a finite number of zeros. Suppose the contrary and let (z_n) be the infinite sequence of zeros of u and z_* an accumulate point of (z_n). Then we have

$$u(z_*) = u'(z_*) = \lim_{n\to\infty} \frac{u(z_n) - u(z_*)}{z_n - z_*} = 0.$$

Again, putting $x = z_*$ in equation (7.14) we get the same contradiction as above.

Let z_1 and z_2 two consecutive zeros of u, and suppose that $u > 0$ in (z_1, z_2) and y_* is a critical point of u in (z_1, z_2). It follows from equation (7.12) that $(\varphi(u'))' = -g(u)$ in (z_1, z_2). Since φ is an increasing odd homeomorphism of R, $u' > 0$ in (z_1, y_*), $u' < 0$ in (y_*, z_2) and $u'(y_*) = 0$. Thus y_* is the unique critical point of u at which u reach its maximum value.

Let

$$\rho = u(y_*) = \max_{x \in (z_1, z_2)} u(x).$$

It follows from equation (7.14) that

$$u'(t) = \psi\left(\Phi_+^{-1}(G(\rho) - G(u(t)))\right) \text{ for all } t \in [z_1, y_*] \tag{7.15}$$

and

$$u'(t) = -\psi\left(\Phi_+^{-1}(G(\rho) - G(u(t)))\right) \text{ for all } t \in [y_*, z_2] \tag{7.16}$$

where Φ_+^{-1} is the inverse of Φ on R$^+$. Then

$$x - z_1 = \int_0^{u(x)} \frac{du(t)}{u'(t)} = \int_0^{u(x)} \frac{du(t)}{\psi\left(\Phi_+^{-1}(G(\rho) - G(u(t)))\right)} \text{ for all } x \in [z_1, y_*] \tag{7.17}$$

and

$$z_2 - x = -\int_0^{u(x)} \frac{du(t)}{u'(t)} = \int_0^{u(x)} \frac{du(t)}{\psi\left(\Phi_+^{-1}(G(\rho) - G(u(t)))\right)} \text{ for all } x \in [y_*, z_2]. \tag{7.18}$$

Putting $x = y_*$ in equations (7.17) and (7.18), we get

$$y_* - z_1 = \int_0^{\rho} \frac{du(t)}{\psi\left(\Phi_+^{-1}(G(\rho) - G(u(t)))\right)} = z_2 - y_*$$

which yields

$$y_* = \frac{z_1 + z_2}{2}.$$

For the symmetry of the $(z_1, z_2)-$hump of u about $\frac{z_1 + z_2}{2}$, it suffices to show that for all $x \in [z_1, z_2]$ $u(z_1 + z_2 - x) = u(x)$. This becomes very easy if we observe that $x = (z_1 + z_2) - (z_1 + z_2 - x)$ and make use of equations (7.17) and (7.18), then we get: in each of the cases

$x \in \left[z_1, \frac{z_1+z_2}{2}\right]$ or $x \in \left[\frac{z_1+z_2}{2}, z_2\right]$

$$
\begin{aligned}
x - z_1 &= z_2 - (z_1 + z_2 - x) = \int_0^{u(x)} \frac{du(t)}{\psi\left(a_+^{-1}\left(G(\rho) - G(u(t))\right)\right)} \\
&= \int_0^{u(z_1+z_2-x)} \frac{du(t)}{\psi\left(a_+^{-1}\left(G(\rho) - G(u(t))\right)\right)}
\end{aligned}
$$

which leads to $u(z_1 + z_2 - x) = u(x)$ for all $x \in [z_1, z_2]$[1].

It remains to show that if $z_3 < z_4$ are two consecutive zeros of u and $u > 0$ in $[z_3, z_4]$, then $u_{[z_3, z_4]}$ is the translation of $u_{[z_1, z_2]}$.

To do this it suffices to prove that $u(z_3 + (x - z_1)) = u(x)$ for all $x \in [z_1, z_2]$.

Putting respectively $x = \frac{z_1+z_2}{2}$ and $x = \frac{z_3+z_4}{2}$ in equation (7.14) we deduce

$$
C = G\left(u\left(\frac{z_1+z_2}{2}\right)\right) = G\left(u\left(\frac{z_3+z_4}{2}\right)\right).
$$

Since G is strictly increasing on $(0, +\infty)$, $u\left(\frac{z_1+z_2}{2}\right) = u\left(\frac{z_3+z_4}{2}\right)$.

Making use of equations (7.17) and (7.18), we get :

$$
\begin{aligned}
z_4 - \frac{z_3+z_4}{2} &= \frac{z_4 - z_3}{2} \\
&= \int_0^{u\left(\frac{z_3+z_4}{2}\right)} \frac{du(t)}{\psi\left(a_+^{-1}\left(G\left(u\left(\frac{z_3+z_4}{2}\right)\right) - G(u(t))\right)\right)} \\
&= \int_0^{u\left(\frac{z_1+z_2}{2}\right)} \frac{du(t)}{\psi\left(a_+^{-1}\left(G\left(u\left(\frac{z_1+z_2}{2}\right)\right) - G(u(t))\right)\right)} \\
&= z_2 - \frac{z_1+z_2}{2} = \frac{z_2 - z_1}{2}
\end{aligned}
$$

which yields $z_3 + (z_2 - z_1) = z_4$.

If we set $v(x) = u(z_3 + (x - z_1))$ for all $x \in [z_1, z_2]$, then we have

$$
\begin{aligned}
v(z_1) &= u(z_3) = 0 \\
v(z_2) &= u(z_4) = 0.
\end{aligned}
$$

[1] We have $\int_0^a f(t)\,dt = \int_0^b f(t)\,dt$ with $f > 0$.

Observe that u and v are solutions of the problem

$$\begin{cases} -(\varphi(w'))' = g(w) \text{ in } (z_1, z_2) \\ w(z_1) = w(z_2) = 0. \end{cases}$$

So, for any $x \in \left[z_1, \frac{z_1+z_2}{2}\right]$, we have:

$$\begin{aligned} x - z_1 &= \int_0^{u(x)} \frac{du(t)}{\psi\left(a_+^{-1}\left(G\left(u\left(\frac{z_1+z_2}{2}\right)\right) - G(u(t))\right)\right)} \\ &= \int_0^{v(x)} \frac{dv(t)}{\psi\left(a_+^{-1}\left(G\left(v\left(\frac{z_1+z_2}{2}\right)\right) - G(v(t))\right)\right)} \end{aligned}$$

which leads to $v(x) = u(x)$ for all $x \in \left[z_1, \frac{z_1+z_2}{2}\right]$.

Using the symmetry of the function u we deduce that $v(x) = u(x)$ for all $x \in [z_1, z_2]$. This completes the proof of the lemma. ■

Lemma 69 Suppose that hypothesis (7.13) holds true and g is odd. If $u \in A_k^+$ (resp. A_k^-) is solution to problem (7.12) with $k \geq 2$ then the first negative (resp. positive) hump of u is a translation of the first negative (resp. positive) of $(-u)$.

Proof Let $u \in A_k^+$ be a solution to problem (7.12) and $(z_i)_{i=0}^{i=k}$ the finite sequence of zeros of u such that $0 = z_0 < z_1 < z_2 < \cdots < z_k = 1$.

Since the positive (resp. negative) humps of u are translations of the first positive (resp. negative) hump one, it suffices to prove that $u_{[z_1,z_2]}$ is a translation of $-u_{[0,z_1]}$.

Let us prove that the two humps have the same length.. Putting $x = \frac{z_1}{2}$ and $x = \frac{z_1+z_2}{2}$ in (7.14), we get

$$C = G\left(u\left(\frac{z_1}{2}\right)\right) = G\left(u\left(\frac{z_1+z_2}{2}\right)\right).$$

Since G is even and increasing in $\mathbb{R}^+$

$$u\left(\frac{z_1}{2}\right) = -u\left(\frac{z_1+z_2}{2}\right).$$

Set $\rho = u\left(\frac{z_1}{2}\right) = -u\left(\frac{z_1+z_2}{2}\right)$, as in the proof of Lemma 6

$$\frac{z_1}{2} = \int_0^{\rho} \frac{ds}{\psi\left(a_+^{-1}(G(\rho) - G(s))\right)}$$

and

$$\frac{z_2 - z_1}{2} = \frac{z_1 + z_2}{2} - z_1 = \int_{u\left(\frac{z_1+z_2}{2}\right)}^{0} \frac{ds}{\psi\left(a_+^{-1}\left(G\left(u\left(\frac{z_1+z_2}{2}\right)\right) - G(s)\right)\right)}$$
$$= \int_0^{-u\left(\frac{z_1+z_2}{2}\right)} \frac{ds}{\psi\left(a_+^{-1}\left(G\left(u\left(\frac{z_1+z_2}{2}\right)\right) - G(s)\right)\right)}$$
$$= \int_0^{\rho} \frac{ds}{\psi\left(a_+^{-1}\left(G(\rho) - G(s)\right)\right)} = \frac{z_1}{2}$$

which leeds to

$$z_2 - z_1 = z_1.$$

Setting $v(x) = -u(z_1 + x)$ for all $x \in [0, z_1]$ and arguing as in the proof of Lemma 6, we get $u(x) = v(x) = -u(z_1 + x)$ for all $x \in [0, z_1]$. So the lemma is proved ∎

Lemma 70 Suppose that hypothesis (7.13) holds true. If $u_1 \neq u_2$ are two positives solutions of problem (7.12), then $u_1$and u_2 are ordered, namely $u_1 < u_2$ in (a, b) or $u_1 < u_2$ in (a, b).

Proof See the proof of Lemma 64 ∎.

7.4 Proof of the main results

Since the function f is odd and satisfies hypothesis (7.13), it leads from lemma 68 any non trivial solution to problem (7.1) belongs to $\bigcup_{k \geq 1} A_k$.

7.4.1 Existence of solutions :

From lemmas 68 and 69, to get a solution belonging to A_k^+ (resp. A_k^-) to problem (7.1) it suffices to prove that the problem

$$\begin{cases} -\left(\varphi\left(u'(x)\right)\right)' = f(u(x)) \text{ in } \left(0, \frac{1}{2k}\right) \\ u(0) = u'\left(\frac{1}{2k}\right) = 0. \end{cases} \tag{7.19}$$

admits a positive (resp. negative) solution.[2]

Set $f^+ = \max(f, 0)$ and consider the boundary value problem

$$\begin{cases} -v''(x) = f^+\left(\int_0^x \psi(v'(t))\, dt\right) \text{ in } \left(0, \frac{1}{2k}\right) \\ v(0) = v'\left(\frac{1}{2k}\right) = 0. \end{cases} \tag{7.20}$$

[2] Any positive solution of (7.19) is concave. to see that one can use (7.14).

Observe that if v is a positive solution to problem (7.20) if and only if $u(x) = \int_0^x \psi(u'(t))\,dt$ is a positive solution to the problem (7.19)[3].

Hence, we are driven to look for positive solutions to the problem

$$\begin{cases} -v''(x) = f^+\left(\int_0^x \psi(v'(t))\,dt\right) \text{ in } (0,a) \\ v(0) = v'(a) = 0 \end{cases} \tag{7.21}$$

where $a \in (0,1)$.

Consider the boundary value problem

$$\begin{cases} -v''(x) = \lambda v(x) + f^+(u(x)) \text{ in } (0,a) \\ v(0) = v'(a) = 0. \end{cases} \tag{7.22}$$

where λ is a real parameter and $u(x) = \int_0^x \psi(v'(t))\,dt.$

We mean by a solution of problem (7.22) a pair $(\lambda, v) \in \mathbb{R} \times C^1([0,a])$ satisfying $-v''(x) = \lambda v(x) + f^+\left(\int_0^x \psi(v'(t))\,dt\right)$ $x \in (0,a)$ and the boundary conditions $v(0) = v'(a) = 0$.

Existence in the superlinear case:

Let $\varepsilon > 0$, we deduce from assumption (7.6) existence of $\delta > 0$ such that

$$\text{for all } x \in \mathbb{R},\ |x| < \delta \text{ implies } |f(x)| < \varepsilon|\varphi(x)| = \varepsilon\varphi(|x|).$$

Since ψ is an odd increasing function on $\mathbb{R}^+$, we have for $v \in C^1([0,a])$ and for all $x \in [0,a]$

$$\left|\int_0^x \psi(v'(t))\,dt\right| \le \int_0^x |\psi(v'(t))|\,dt = \int_0^x \psi(|v'(t)|)\,dt \le \psi(\|v\|_1).$$

Thus, if $\eta := \varphi(\delta)$ then for all $v \in C^1([0,a])$

$$\|v\|_1 < \eta \text{ implies } \left|\int_0^x \psi\left(v'(t)\right)dt\right| \le \delta \text{ for all } x \in [0,a]$$

[3] Any solution of (7.20) is concave.

then

$$\left|f\left(\int_0^x \psi(v'(t))\,dt\right)\right| = f\left(\left|\int_0^x \psi(v'(t))\,dt\right|\right)$$
$$\leq \varepsilon\varphi\left(\left|\int_0^x \psi(v'(t))\,dt\right|\right)$$
$$\leq \varepsilon\varphi(\psi(\|v\|_1))$$
$$\leq \varepsilon\|v\|_1 .$$

which means $f(u) = o(\|v\|_1)$ and $f^+(u) = o(\|v\|_1)$.

However, Rabinowitz global bifurcation theory (see [93] and [94]) states: the pair $(\lambda_1, 0)$ is a bifurcation point for a component $\mathcal{S}_1^+ \subset \mathbb{R} \times \overset{\circ}{S}{}_1^+$ of positive solutions to Problem (7.22) which is unbounded in $\mathbb{R} \times C^1([0,a])$ where $\lambda_1 = \mu_1([0,a])$ and

$$\overset{\circ}{S}{}_1^+ = \left\{ v \in C^1([0,a]) : v(0) = v'(a) = 0 \text{ and } v > 0 \text{ in } (0,a) \right\}.$$

Thus, to prove existence of a positive solution to problem (7.21) it suffices to show the following

Theorem 71 $\mathcal{S}_1^+$ *crosses* $\{0\} \times C^1([0,a])$.

Before proving theorem 5, we need the following lemma:

Lemma 72 *If* $(\lambda, v) \in \mathcal{S}_1^+$ *then* $\lambda < \lambda_1$.

Proof Let Φ be the first positive eigenfunction of

$$\begin{cases} -\Phi'' = \lambda_1 \Phi \text{ in } (0,a) \\ \Phi(0) = \Phi'(a) = 0. \end{cases}$$

Multiplying (7.22) by Φ and integrating on $(0,a)$ we get:

$$-\int_0^a v''\Phi = \lambda \int_0^a v\Phi + \int_0^a f^+(u)\Phi.$$

Then, two integrations by parts give

$$(\lambda_1 - \lambda)\int_0^a v\Phi = \int_0^a f^+(u)\Phi > 0$$

so that

$$\lambda < \lambda_1.$$

$\blacksquare$

Proof of theorem 71

Suppose the contrary, and let $(\lambda_n, v_n) \subset S_1^+$ an unbounded sequence in $\mathbb{R} \times C^1([0,a])$ and set $u_n(x) = \int_0^x \psi(v_n'(t))\,dt$. An immediate consequence of Lemma 72 is: $0 < \lambda_n < \lambda_1$ and (v_n) is unbounded in $C^1([0,a])$.

First Let us prove that v_n is unbounded with the respect of the C^0 norm. Suppose the contrary. Since $-v_n'' = \lambda_n v_n + f(u_n)$ and v_n'' is unbounded[4] with the respect of the C^0 norm, u_n is unbounded with the respect of the C^0 norm on $[0,a]$.

Let for any $R > 0$ $J_n = \{x \in [0,a] : \varphi(u_n(x)) \geq R\}$.

We claim that there exist $R_0 > 0$ such that $l(J_n) \leq \dfrac{1}{2a}$. This is due to:

Denote by θ_n the real number belonging to $[0,a]$ such that $\varphi(u_n(\theta_n)) = R$ and let Φ_n and $\lambda_{1,n}$ be respectively the first positive eigenfunction and the first eigenvalue of the problem

$$\begin{cases} -v'' = \lambda v \text{ in } (\theta_n, a) \\ v(\theta_n) = v'(a) = 0. \end{cases}$$

Multiplying (7.22) by Φ_n and integrating between θ_n and a we get

$$\int_{\theta_n}^a -v_n''\Phi_n = \lambda_n \int_{\theta_n}^a v_n\Phi_n + \int_{\theta_n}^a f^+(u_n)\Phi_n.$$

After two integrations by parts we obtain:

$$\lambda_{1,n}\int_{\theta_n}^a v_n\Phi_n \geq \lambda_n \int_{\theta_n}^a v_n\Phi_n + \int_{\theta_n}^a f^+(u_n)\Phi_n. \tag{7.23}$$

We deduce from hypothesis (7.6) that $\lim\limits_{x \to +\infty} \dfrac{f^+(\psi(x))}{x} = +\infty$, so for $M = \dfrac{\pi^2}{a^2}$ there exists $R_0 > 0$ such that

$$x \geq R_0 \text{ implies } f^+(\psi(x)) \geq Mx.$$

Thus, we deduce from (7.23):

$$\begin{aligned} (\lambda_{1,n} - \lambda_n)\int_{\theta_n}^a v_n\Phi_n &\geq \int_{\theta_n}^a (f^+ \circ \psi)(\varphi(u_n))\Phi_n \\ &\geq M\int_{\theta_n}^a \varphi(u_n)\Phi_n. \end{aligned} \tag{7.24}$$

[4] Otherwise v_n' will be bounded on $[0,a]$ with the respect of the C^0 norm, and then v_n with the respect of the C^1 norm.

Since φ is concave, Jensen inequality (7.4) leads to

$$\varphi(u_n(x)) \geq v_n(x) \text{ for all } x \in [\theta_n, a].$$

Thus, we deduce from (7.24):

$$(\lambda_{1,n} - (\lambda_n + M)) \int_{\theta_n}^{a} v_n \varphi_n \geq 0;$$

then

$$\frac{\pi^2}{(a - \theta_n)^2} \geq (\lambda_n + M)$$

finally

$$l(J_n) = \left(\frac{1}{2} - \theta_n\right) \leq \frac{1}{2a}. \tag{7.25}$$

Now let us return to the equation satisfied by u_n. We have

$$-\left(\varphi\left(u_n'\right)\right)' = \lambda_n v_n + f^+(u_n) \text{ in } (0,a)$$

Multiplying by u' and integrating over $[x,a]$, we get

$$a\left(\varphi\left(u_n'(x)\right)\right) = F^+(\rho_n) - F^+(u_n(x)) + \lambda_n \int_x^a v_n u_n' \text{ for all } x \in [0,a]$$

where $\rho_n = u_n(a)$ and $F^+(x) = \int_0^x f^+(t)\,dt$.

Then, as in the proof of Lemma 6 we obtain

$$\theta_n = \int_0^{R_0} \frac{du_n(t)}{u_n'(t)} = \int_0^{R_0} \frac{ds}{\psi\left(a_+^{-1}\left(F^+(\rho_n) - F^+(s) + \lambda_n \int_x^{\frac{1}{2}} v_n u_n'\right)\right)} \leq \int_0^{R_0} \frac{ds}{\psi\left(a_+^{-1}(F^+(\rho_n) - F^+(s))\right)}. \tag{7.26}$$

Thus, on one hand, since $\frac{1}{\psi\left(a_+^{-1}(F^+(\rho_n) - F^+(s))\right)}$ is bounded in $[0, R_0]$ and $\lim_{n\to\infty} \rho_n = +\infty$.

$$\lim_{n\to+\infty} \theta_n = \lim_{n\to+\infty} \int_0^{R_0} \frac{ds}{\psi\left(a_+^{-1}(F^+(\rho_n) - F^+(s))\right)} = 0$$

and on the other hand it follows from (7.25) $\theta_n \geq \frac{1}{2a}$ which is impossible and v_n is unbounded in $C^0([0,a])$.

Now arguing as above, let for any $R > 0$ $J_n = \{x \in [0,a] : v_n(x) \geq R\}$, $R_0 > 0$ such that $l(J_n) \leq \frac{1}{2a}$ and θ_n the real number belonging to $[0,a]$ such that $v_n(\theta_n) = R_0$.

Thus, in one hand

$$R_0 = \int_0^{\theta_n} v_n'(t)\,dt \geq \frac{1}{2a} v_n'(\theta_n) \tag{7.27}$$

and on the other hand,

$$\begin{aligned} v_n\left(\frac{1}{2}\right) &= \int_0^a v_n'(t)\,dt = \int_0^{\theta_n} v_n'(t)\,dt + \int_{\theta_n}^a v_n'(t)\,dt \\ &\leq R_0 + \frac{1}{2a} v_n'(\theta_n) \end{aligned} \tag{7.28}$$

which is impossible because from (7.27) we deduce that $v_n'(\theta_n)$ is bounded and (7.28) leads to $v_n'(\theta_n)$ is unbounded. This completes the proof of theorem 5. ■

Existence in the sublinear case:

Let $\varepsilon > 0$, we deduce from hypothesis (7.9) existence of $\chi > 0$ such that

$$x > \chi \text{ implies } f^+(x) < \varepsilon\varphi(x).$$

Note that since ψ is concave and increasing, and f is increasing

$$\begin{aligned} f^+\left(\int_0^x \psi(v'(t))\,dt\right) &\leq f^+(\psi(v(x))) \\ &\leq f^+(\psi(\|v\|_1)) \text{ for all } x \in [0,a]. \end{aligned}$$

Thus if $\eta = \varphi(\chi)$, then for all $v \in C^1([0,a])$ and for all $x \in [0,a]$

$$\|v\|_1 > \eta \text{ implies } f^+\left(\int_0^x \psi\left(v'(t)\right)dt\right) < \varepsilon\|v\|_1$$

and $f^+\left(\int_0^x \psi(v'(t))\,dt\right) = o(\|v\|_1)$

Therefore, Rabinowitz Global Bifurcation Theory states (see [95]): the pair $(\lambda_1, +\infty)$ is a bifurcation point for a component $S_1^+ \subset \mathbb{R} \times \hat{S}_1^+$ of positive solutions to (7.22) such that:

If Ω is a neighborhood of $(\lambda_1, +\infty)$ whose projection on $\mathbb{R}$ is bounded and whose projection on $C^1([0,a])$ is bounded away from 0 then either

1. $S_1^+ \cap \Omega$ is bounded in $\mathbb{R} \times C^1([0,a])$, in which a case $S_1^+ \cap \Omega$ meets $\mathbb{R} \times \{0\}$ or

2. $S_1^+ \cap \Omega$ is unbounded in $\mathbb{R} \times C^1([0,a])$. Moreover if $S_1^+ \cap \Omega$ has a bounded projection on $\mathbb{R}$ then $S_1^+ \cap \Omega$ meets $(\mu_k([0,a]), +\infty)$ with $k \geq 2$.

Thus, to prove existence of a positive solution to problem (7.21) it suffices to show the following

Theorem 73 S_1^+ crosses $\{0\} \times C^1([0,a])$.

Proof of theorem 73:

To prove theorem 73, it suffices to prove that if Ω is as above, then $S_1^+ \cap \Omega$ dos not meet $(\mu_k([0,a]), +\infty)$ with $k \geq 2$ and does not meet $\mathbb{R}^+ \times \{0\}$.

Let φ be the first positive eigenfunction of

$$\begin{cases} -\varphi'' = \lambda_1 \varphi \text{ in } (0,a) \\ \varphi(0) = \varphi'(a) = 0. \end{cases}$$

and $(\lambda, v) \in S_1^+$. Arguing as in the proof of Lemma 72 we get

$$\lambda < \lambda_1$$

which means that $S_1^+ \cap \Omega$ don't meet $(\mu_k([0,a]), +\infty)$ with $k \geq 2$.

Now suppose that (λ_n, v_n) is a sequence in S_1^+ converging[5] to $(\lambda^*, 0)$ with $\lambda_n > 0$. Multiplying (7.22) by φ and integrating on $(0,a)$ we get

$$(\lambda_1 - \lambda_n) \int_0^a v_n \varphi = \int_0^a f^+(u_n) \varphi$$

where $u_n(x) = \int_0^x \psi(v_n'(t))\, dt.$

Using the concavity of f we get

$$(\lambda_1 - \lambda_n) \int_0^a v_n(t) \varphi(t)\, dt \geq \int_0^a \left(\int_0^t f^+\left(\psi\left(v_n'(s)\right)\right) \varphi(s)\, ds \right) dt.$$

We deduce from hypothesis (7.9) that $\lim_{x \to 0} \dfrac{f^+(\psi(x))}{x} = +\infty$ and for $M = \dfrac{\pi^2}{a^2}$, there exist

[5] $v_n \to 0$ with the respect of the C^1 norm.

$\delta > 0$ such that

$$0 \le x < \delta \text{ implies } f^+(\psi(x)) > Mx.$$

Hence, For n large enough

$$f^+\left(\psi\left(v_n'(s)\right)\right) \ge M v_n'(s)$$

and

$$(\lambda_1 - \lambda_n - M)\int_0^a v_n(t)\,\phi(t)\,dt \ge 0.$$

This is impossible since

$$\lambda_1 - \lambda_n - M < \lambda_1 - M < 0.$$

which completes the proof of Theorem 73.

7.4.2 Uniqueness in $A_k^{\pm}$

We will expose in this paragraph the proof of uniqueness in $A_k^{\pm}$ in the superlinear case. The other case will be treated similarly.

We deduce from Lemma 68 and Lemma 69 that: to show uniqueness of the solution to problem (7.1) in each $A_k^{\pm}$, it suffices to show that the boundary value problem

$$\begin{cases} -(\varphi(u'))' = f(u) \text{ in } (a,b) \\ u(a) = u(b) = 0 \end{cases} \tag{7.29}$$

has a unique solution in A_1^+.

Now, if u and v are two solutions in A_1^+ to problem (7.29), then we have

$$\int_a^b -\left(\varphi\left(u'\right)\right)' v + \left(\varphi\left(v'\right)\right)' u = \int_a^b f(u)\,v - f(v)\,u$$

or

$$2\int_a^{\frac{a+b}{2}} \left(\frac{\varphi(u')}{u'} - \frac{\varphi(v')}{v'}\right) u'v' = \int_a^b \left(\frac{f(u)}{u} - \frac{f(v)}{v}\right) uv. \tag{7.30}$$

First we deduce from Lemma 70 that u and v are ordered and from assumption (7.7) that f is increasing in $\mathbb{R}^+$. Then, if we suppose $u \lneq v$ in $(0,1)$ we get $(\varphi(u') - \varphi(v'))' = -(f(u) - f(v)) < 0$ in $\left(a, \frac{a+b}{2}\right)$, namely $u' < v'$ in $\left(a, \frac{a+b}{2}\right)$.

In one hand, it follows from assumption (7.7) that

$$\int_a^b \left(\frac{f(u)}{u} - \frac{f(v)}{v} \right) uv < 0. \tag{7.31}$$

In the other hand, the concavity of φ involve that the function $s \to \frac{\varphi(s)}{s}$ is decreasing on $(0, +\infty)$, then

$$\int_a^{\frac{a+b}{2}} \left(\frac{\varphi(u')}{u'} - \frac{\varphi(v')}{v'} \right) u'v' > 0. \tag{7.32}$$

Inequalities (7.31) and (7.32) contradict equation (7.32); so uniqueness of the solution to problem (7.29) is proved. ■

Concluding remarks and open questions

Through this thesis, we have attempted to give a complete description of the solution set of the boundary value problem

$$\begin{cases} -(\varphi(u'))' = g(\lambda, u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{7.33}$$

where λ is a real parameter, φ an odd increasing homeomorphism of $\mathbb{R}$ and g is a continuous real function.

In the chapters 2, 3, 4 and 5 we used the quadrature method established in chapter 1, for solving several of the type of (7.33) when for all $x \in \mathbb{R}$ $\varphi(x) = \varphi_p(x) = |x|^{p-2}x$. Therefore through the application of this method we have noted the following two remarks :

- The quadrature method is very descriptive in the sence that it detect solutions and their qualitative proprieties (number of zeros, number of critical points, symmetry.....)
- To get multiplicity (resp. exact multiplcity) of solutions, we are led to compute the a lower bound (resp. the exact number) of the critical points of the time maps.

Also, we have seen that the quadrature method allows us to detct solutions of (7.33) in the subsets of $C^1([0,1])$: A_k, and B_k; this don't means by any way that there isn't other solutions of (7.33) (see for instance Addou [4], Guedda and Veron [69]).

When φ isn't necessarily equal to φ_p, few works have been done. So, the study of this kind of boundary value problems remines an open question. For instance, the structure of the solution set of the pseudo eigenvalue-problem

$$\begin{cases} -(\varphi(u'))' = \lambda\varphi(u) \text{ in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

is unknown.

Observe also that in this work, the problems studied are autonomous; so the nonautonomous case with other kinds of boundary value conditions remain open.

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