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Sujet

Solutions Exactes De Certaines EDP non Linéaires Aux dérivées fractionnaires (1)

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Contents

List of publications

- Various optical solitons to the (1+1)-Telegraph equation with spacetime conformable derivatives ",in a journal : International Journal Of Nonlinear Analysis and Applications IJAA. Ref: Paper No. IJNAA-H-2020-08

Submitted Article

- New exact solutions to the nonlinear conformable time-fractional PHI-four equation And Jimbo Miwa Equation

Abstract

In this thesis, two methods are proposed to address exact soliton solutions to some nonlinear evolution equations (NLEEs) with a given fractional order. The first one is based on the combination of the Generalized Kudryashov and G'/G methods with an auxiliary equation. It is applied to the space-time conformal fractional nonlinear Telegraph equation. The exact solutions including hyperbolic and trigonometric solutions have been derived. 3D and 2D solution trajectories are plotted for different choice of the parameters. The second method is based on the use of the Generalised Kudryashov Method with a new auxiliary equation. It is applied to the search for analytical soliton solutions to some nonlinear evolution equations (NLEEs) with a given fractional order. Application is considered using the space-time conformal fractional nonlinear Phi- four and Jumbo Miwa equations. New soliton solutions are derived. 3D solution trajectories are plotted for different choice of the parameters.

STRUCTURE OF THE THÈSIS

This thesis is structured in three chapters as follows.

Chapter 1

This chapter introduces some special functions basic background, including the Gamma and Beta functions which are quite important in the theory of special functions. Similarly, some basic properties of fractional derivatives are presented with typical examples. Outline on the Katugampola fractional derivative applied both on the local fractional and the conformable fractional derivatives. Some PDEs and ODEs resolution methods are introduced with application examples on nonlinear differential equations.

Chapter 2

This chapter introduces a new method to construct exact analytic solutions of nonlinear PDEs and ODEs. It is applied successfully to find the solutions of the (1+1)-Telegraph equation with space-time conformable derivatives. The proposed method can be used to solve mathematical physics nonlinear problems.

Chapter 3

This chapter describes a new method based on the integration of the Riccati equation

$$(\Psi'(\eta)) = (\Psi(\eta))^2 + A$$

in the formulation of the generalized Kudryashov method. It is tested on the resolution of the (2+2) dimension space-time fractional non linear PHI and Jimbo Miwa equations, in the sense of conformable derivative. Solutions are computed using Maple and Mathematica softwares. Similarly to the former chapter, the proposed approach can be applied effectively to solve mathematical physics non-linear PDEs.

Introduction

Most authors on this topic will cite a particular date as the birthday of so called 'Fractional Calculus'. In a letter dated September 30th, 1695 L'Hopital wrote to Leibniz asking him about a particular notation he had used in his publications for the nth-derivative of the linear function f(x) = x, $\frac{d^n f}{dx^n}$. L'Hopital's posed the question to Leibniz, what would the result be if $n = \frac{1}{2}$. Leibniz's response: "An apparent paradox, from which one day useful consequences will be drawn." In these words fractional calculus was born.

Fractional calculus can be defined as the generalization of classical calculus to orders of integration and differentiation not necessarily integer.

the first serious attempt to give a logical definition for the fractional derivative is due to Liouville who published nine documents in this subject between 1832 and 1837. Independently, Riemann proposed an approach which proved essentially that of Liouville, and it is since she wears the name of "Riemann-Liouville approach. ". Later, other theories appeared like that of Grunwald-Leitnikov, (1867-1872), H.Weyl (1917) and caputo(1967). At that time there were almost no practical applications of this theory, and it is for this reason that it has been considered an abstract science. It was not until 1974 that fractional calculus has been intensively developed thanks to B. Ross who organized the first conference at the University of New Haven under the title "Fractional calculus and its applications".

The goal of fractional calculus is to be able to calculate integrals and derivatives of any arbitrary order whether real or complex. The word derivative of arbitrary order is today what we call fractional derivative

From a mathematical point of view, there are several possible definitions of fractional derivatives, they however have in common to be the derivative in the classical sense if the order is integer.
Fractional derivatives are an excellent instrument to retain the memory of past transformations and hereditary properties of various materials also in certain fields of physics involving diffusion phenomena such as electromagnetism, etc. This is the main advantage of fractional derivatives over to classical models, in which these effects are in fact neglected. The advantages of fractional derivatives appear in the modeling of the electrical and mechanical properties of real materials, hence the interest of this type of fractional analysis.

Engineers realized the importance of fractional order differential equations only in the last three decades when they observed that the description of certain systems is more accurate when the fractional derivative is used. The use of fractional ODEs and PDEs is necessary because many real models give such differential equations.

Chapter 1

Tools, Notions and Methods

This introductory chapter surveys some basic notions of mathematical techniques that are needed throughout the following chapters..

1.1 Gamma Function

The Gamma function which is a type of integral of Euler [27] was introduced by the Swiss mathematician Leonhard Euler (1707-1783), with the aim of generalizing factor values to non integer values i.e. it allows n to take non integer values and even complex values. Later, due to its great importance, it was studied by other eminent mathematicians such as Adrien-MarieLegendre (1752-1833), Carl Friedrich Gauss (1777-1855), Christoph Gudermann (1798-1852), Joseph Liouville (1809 - 1882), Karl Weierstrass (1815-1897), Charles Hermite (1822-1901), as well as many other mathematicians.

Definition :

For any complex number z such that $Re(z) > 0$, we define the gamma function, denoted by the Greek letter Γ, by the Euler integral of the second kind

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad Re(z) > 0$$

Gamma Function Properties

An important property of the Gamma function is the following recurrence relation:

$$\Gamma(z+1) = z\Gamma(z)$$

which can be proved by integration by parts.

The gamma function works as the generalisation of the factorial function

1.2 Fonction Bêta

Beta function is a special type of function, which is also known as Euler integral of the first kind, defined by:

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad Re(p)>0, \quad Re(q)>0$$

property

The function Bêta can be represented by the Gamma function as follows:

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad Re(x)>0, Re(y)>0$$

1.3 Fractional derivative

1.3.1 Introduction

Many mathematicians have looked into the question of Leibniz at L'Hôpital in 1695, concerning the derivative of non-integer order, after many years of research, researchers have presented their approaches to answer this question, in particular Euler (1730) , Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), etc....Different approaches have been used to generalize the notion of derivation to non-integer (fractional) orders. In this part, we present an overview of the fractional Katugampola derivative, the locally conformal fractional derivative and the conformal fractional derivative.

1.4 Katugampola Fractional Derivative

1.4.1 Introduction:

Recently, Katugampola. introduces a new definition of the fractional derivative [24], which obeys classical properties including: linearity, product rule, quotient rule, power rule, chain rule, vanishing derivatives for constant functions. This derivative is defined as follows

1.4.2 Definition

The Katugampola fractional derivative of $f(x)$ of order α, $(0 < \alpha < 1)$ at $x = x_0$ is defined by [24]

for $f : [0, \infty) \to \mathbb{R}$ and $t > 0$. we have :

$$D^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}, \tag{1.4.1}$$

1.4.3 Some Properties Of The Katugampola Fractional Derivative :

Let $f,\ g \in C_{\alpha}(a, b)$,$\alpha \in (0, 1]$ and $t > 0$. then The properties of the Katugampola Fractional Derivative are as follows:

Linearity

Same as integer order derivative, the Katugampola Fractional Derivative is a linear operation

$$D_t^{\alpha}(kf + lg)(t) = kD_t^{\alpha} f(t) + lD_t^{\alpha} g(t), \quad k, l \in \mathbb{R}$$

3. Leibniz rule for Katugampola Fractional Derivative

Leibniz's rule, for the Katugampola Fractional Derivative is given as :

$$\mathcal{D}^{\alpha}(fg) = f\mathcal{D}^{\alpha}(g) + g\mathcal{D}^{\alpha}(f)$$

Proof

It comes directly from the corresponding definition.

$$\mathcal{D}^{\alpha}(fg)(t) = \lim_{\epsilon \to 0} \frac{f(te^{\epsilon t^{-\alpha}})g(te^{\epsilon t^{-\alpha}}) - f(t)g(t)}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{f(te^{\epsilon t^{-\alpha}})g(te^{\epsilon t^{-\alpha}}) - f(t)g(te^{\epsilon t^{-\alpha}}) + f(t)g(te^{\epsilon t^{-\alpha}}) - f(t)g(t)}{\epsilon}$$

$$= \lim_{\epsilon \to 0}[\frac{f(te^{\epsilon t^{-\alpha}}) - f(t)}{\epsilon} \cdot g(te^{\epsilon t^{-\alpha}})] + f(t) \lim_{\epsilon \to 0} \frac{g(te^{\epsilon t^{-\alpha}}) - g(t)}{\epsilon}$$

$$= \mathcal{D}^{\alpha}(f)(t)\lim_{\epsilon\to 0} g(te^{\epsilon t^{-\alpha}}) + f(t)\mathcal{D}^{\alpha}(g)(t)$$
$$= \mathcal{D}^{\alpha}(f)(t)g(t) + f(t)\mathcal{D}^{\alpha}(g)(t)\ .$$

3.Katugampola Fractional Derivative of a Power Function

$$\mathcal{D}^{\alpha}(t^n) = nt^{n-\alpha}, n \in \mathbb{R}.$$

Proof

$$\begin{aligned} D^{\alpha}(t^n) &= \lim_{\epsilon\to 0}\frac{(te^{\epsilon t^{-\alpha}})^n - t^n}{\epsilon} \\ &= t^n \lim_{\epsilon\to 0}\frac{e^{n\epsilon t^{-\alpha}} - 1}{\epsilon} \\ &= nt^{n-\alpha}. \end{aligned}$$

4.Katugampola fractional derivative of a differentiable function

If f is differentiable, then

$$\mathcal{D}^{\alpha}(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$$

.

Proof

$$\mathcal{D}^{\alpha}(f)(t) = \lim_{\epsilon\to 0}\frac{f(te^{\epsilon t^{-\alpha}}) - f(t)}{\epsilon}$$
$$= \lim_{\epsilon\to 0}\frac{f(t + \epsilon t^{1-\alpha} + O(\epsilon^2)) - f(t)}{\epsilon}$$
$$= \lim_{\epsilon\to 0}\frac{f(t+h) - f(t)}{\frac{ht^{\alpha-1}}{1+O(\epsilon)}}, \quad h = \epsilon t^{1-\alpha}(1 + O(\epsilon))$$
$$= t^{1-\alpha}\frac{df}{dt}(t)\ ,$$

From this property we derive the following relation

$$D_t^{\alpha}\left(\frac{1}{\alpha}t^{\alpha}\right) = 1.$$

And from the corresponding definition we get the relation

$$\mathcal{D}^{\alpha}(C) = 0$$

.

5.Katugampola Fractional derivative of a rational function

$$\mathcal{D}^{\alpha}(\frac{f}{g}) = \frac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2}.$$

6. Chain rule for Katugampola Fractional Derivative

The chain rule enables us to find the Katugampola Fractional Derivative of a composite function, which we derive below.

$$D^{\alpha}(f \circ g)(t) = \mathcal{D}^{\alpha} f(g(t)) \mathcal{D}^{\alpha} g(t)$$

.

1.5 Local fractional derivative

1.5.1 Definition

There are many types of fractional derivatives. One of the popular fractional derivatives is the local fractional derivative which is given by [42, 43]

The local fractional derivative of $\phi(x)$ of order α, $(0 < \alpha < 1)$ at $x = x_0$ is defined by

$$\begin{aligned} D^{(\alpha)}\phi(x_0) &= \frac{d^{\alpha}\phi(x_0)}{dx^{\alpha}} \\ &= \lim_{x->x_0} \frac{\Delta^{\alpha}\left(\phi(x) - \phi(x_0)\right)}{(x - x_0)^{\alpha}}, \ \Delta^{\alpha}\left(\phi(x) - \phi(x_0)\right) \simeq \Gamma\left(1+\alpha\right)\left(\phi(x) - \phi(x_0)\right) \\ &= \lim_{x->x_0} \frac{\Gamma\left(1+\alpha\right)\left(\phi(x) - \phi(x_0)\right)}{(x - x_0)^{\alpha}} \end{aligned}$$

1.5.2 Some Properties Of The Local Fractional Derivative:

Let $\phi(x_0)$, $\Psi(x_0) \in C_{\alpha}(a, b)$.

The properties of the LFD are listed as follows :

Linearity

Same as integer order derivative, the LFD is a linear operation

1)

$$D^{(\alpha)}[\phi(x_0) \pm \Psi(x_0)] = D^{(\alpha)}\phi(x_0) \pm D^{(\alpha)}\Psi(x_0)$$

Proof

The linearity of the LFD comes directly from the corresponding definition.

$$
\begin{aligned}
D^{(\alpha)}[\phi(x_0) \pm \Psi(x_0)] &= \Gamma(1+\alpha) \lim_{x->x_0} \frac{([\phi(x) \pm \Psi(x)] - [\phi(x_0) \pm \Psi(x_0)])}{(x-x_0)^{\alpha}} \\
&= \Gamma(1+\alpha) \lim_{x->x_0} \frac{([(\phi(x) - \phi(x_0)) \pm (\Psi(x) - \Psi(x_0)])}{(x-x_0)^{\alpha}} \\
&= \Gamma(1+\alpha) \lim_{x->x_0} \frac{\phi(x) - \phi(x_0)}{(x-x_0)^{\alpha}} \pm \Gamma(1+\alpha) \lim_{x->x_0} \frac{\Psi(x) - \Psi(x_0)}{(x-x_0)^{\alpha}} \\
&= D^{(\alpha)}\phi(x_0) \pm D^{(\alpha)}\Psi(x_0)
\end{aligned}
$$

3. Leibniz rule for LFD

Leibniz's rule, for the LFD is given as :

$$
D^{(\alpha)}[\phi(x_0).\Psi(x_0)] = \phi(x_0).D^{(\alpha)}\Psi(x_0) + \Psi(x_0).D^{(\alpha)}\phi(x_0)
$$

Proof

$$
\begin{aligned}
D^{(\alpha)}[\phi(x_0).\Psi(x_0)] &= \Gamma(1+\alpha) \lim_{x->x_0} \frac{(\phi(x).\Psi(x) - \phi(x_0).\Psi(x_0))}{(x-x_0)^{\alpha}} \\
&= \Gamma(1+\alpha) \lim_{x->x_0} \frac{(\phi(x) - \phi(x_0)).\Psi(x) + [\Psi(x) - \Psi(x_0)].\phi(x_0)}{(x-x_0)^{\alpha}} \\
&= \left\{ \begin{array}{c} \Gamma(1+\alpha) \lim_{x->x_0} \frac{(\phi(x)-\phi(x_0))}{(x-x_0)^{\alpha}} \lim_{x->x_0} \Psi(x) + \\ \phi(x_0)\Gamma(1+\alpha) \lim_{x->x_0} \frac{[\Psi(x)-\Psi(x_0)]}{(x-x_0)^{\alpha}} \end{array} \right\} \\
&= \Psi(x_0).D^{(\alpha)}\phi(x_0) + \phi(x_0).D^{(\alpha)}\Psi(x_0).
\end{aligned}
$$

6. Chain rule for LFD

The chain rule enables us to find LFD of a composite function, which we derive below.

$$
D^{(\alpha)}[\phi \circ \Psi](x_0) = \phi'(\Psi)(x_0).D^{(\alpha)}(\Psi)(x_0).
$$

Proof

$$
\begin{aligned}
D^{(\alpha)}[\phi \circ \Psi](x_0) &= \Gamma(1+\alpha) \lim_{x->x_0} \frac{(\phi(\Psi(x)) - \phi(\Psi(x_0))}{(x-x_0)^{\alpha}} \\
&= \Gamma(1+\alpha) \lim_{x->x_0} \frac{(\phi(\Psi(x)) - \phi(\Psi(x_0))}{\Psi(x) - \Psi(x_0)} \frac{(\Psi(x) - \Psi(x_0))}{(x-x_0)^{\alpha}} \\
&= \lim_{x->x_0} \frac{(\phi(\Psi(x)) - \phi(\Psi(x_0))}{\Psi(x) - \Psi(x_0)} \Gamma(1+\alpha) \lim_{x->x_0} \frac{(\Psi(x) - \Psi(x_0))}{(x-x_0)^{\alpha}} \\
&= \phi'(\Psi)(x_0).D^{(\alpha)}(\Psi)(x_0).
\end{aligned}
$$

Derivative of a rational function

$$D^{(\alpha)}[\frac{\phi(x_0)}{\Psi(x_0)}] = \frac{[\Psi(x_0)D^{(\alpha)}(\phi(x_0)) - \phi(x_0)D^{(\alpha)}(\Psi(x_0))]}{(\Psi(x_0))^2}.$$

Proof

$$\begin{aligned}
D^{(\alpha)}[\frac{\phi(x_0)}{\Psi(x_0)}] &= \Gamma(1+\alpha) \lim_{x->x_0} \frac{\left(\frac{\phi(x)}{\Psi(x)} - \frac{\phi(x_0)}{\Psi(x_0)}\right)}{(x-x_0)^\alpha} \\
&= \Gamma(1+\alpha) \lim_{x->x_0} \frac{\left(\frac{\phi(x)\Psi(x_0)-\phi(x_0)\Psi(x)}{\Psi(x)\Psi(x_0)}\right)}{(x-x_0)^\alpha} \\
&= \Gamma(1+\alpha) \lim_{x->x_0} \frac{(\phi(x)\Psi(x_0)-\phi(x_0)\Psi(x))}{\Psi(x)\Psi(x_0)(x-x_0)^\alpha} \\
&= \Gamma(1+\alpha) \lim_{x->x_0} \frac{(\phi(x)-\phi(x_0)).\Psi(x_0) - [\Psi(x)-\Psi(x_0)].\phi(x_0)}{\Psi(x)\Psi(x_0)(x-x_0)^\alpha} \\
&= \Gamma(1+\alpha) \lim_{x->x_0} \frac{(\phi(x)-\phi(x_0))}{(x-x_0)^\alpha} \lim_{x->x_0} \frac{\Psi(x_0)}{\Psi(x)\Psi(x_0)} - \\
&\quad \Gamma(1+\alpha) \lim_{x->x_0} \frac{[\Psi(x)-\Psi(x_0)]}{(x-x_0)^\alpha} \lim_{x->x_0} \frac{\phi(x_0)}{\Psi(x)\Psi(x_0)} \\
&= \frac{[\Psi(x_0)D^{(\alpha)}(\phi(x_0)) - \phi(x_0)D^{(\alpha)}(\Psi(x_0))]}{(\Psi(x_0))^2}.
\end{aligned}$$

1.6 The Conformable Fractional Derivative

1.6.1 Introduction:

Recently, in 2014, Khalil, Horani, Yousef and Sababheh introduced the new fractional derivative called conformable fractional derivative [26]. This derivative is well-behaved and obeys the Leibniz rule and chain rule.

In this part, we are interested in conformal fractional calculus and we give the most important properties of this derivative.

1.6.2 Definition:

The conformable fractional derivative (CFD) of $f(x)$ of order α, $(0 < \alpha < 1)$ at $x = x_0$ is defined by [26, 2, 3, 9, 10, 34]

$$D_t^\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad \alpha \in (0,1), \text{ for all } \quad t > 0 \tag{1.6.1}$$

1.6.3 Some Properties Of The Conformable Fractional Derivative:

Some properties of the Conformable Fractional Derivative are listed as follows

Linearity

Same as integer order derivative, the CFD is a linear operation

$$D_t^\alpha(af+bg) = aD_t^\alpha(f) + bD_t^\alpha(g)$$

Proof:

The linearity of the conformable fractional derivative comes directly from the corresponding definition.

$$\begin{aligned} D_t^\alpha(af+bg) &= \lim_{\epsilon\to 0}\frac{(af+bg)(t+\epsilon t^{1-\alpha})-(af+bg)(t)}{\epsilon} \\ &= \lim_{\epsilon\to 0} a.\frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon} + b.\lim_{\epsilon\to 0}\frac{g(t+\epsilon t^{1-\alpha})-g(t)}{\epsilon} \\ & \quad aD_t^\alpha(f) + bD_t^\alpha(g) \end{aligned}$$

2.Conformable Fractional Derivative of a Power Function

The Conformable fractional derivative of a power function is given by the following formula which follows directly from the definition

$$D_t^\alpha(t^u) = u.t^{u-\alpha}$$

Proof:

2)-From the definition we have:

$$\begin{aligned} D_t^\alpha(t^u) &= \lim_{\epsilon\to 0}\frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon} \\ &= l\lim_{\epsilon\to 0}\frac{(t+\epsilon t^{1-\alpha})^u-(t)^u}{\epsilon} \\ &= \lim_{\epsilon\to 0}\frac{(t)^u[(1+\epsilon t^{-\alpha})^u-1]}{\epsilon} \\ &= \lim_{h\to 0}\frac{(t)^u[(1+h)^u-1]}{h.t^\alpha},\ h=\epsilon t^{-\alpha} \\ &= t^{u-\alpha}.\lim_{h\to 0}\frac{[(1+h)^u-1]}{h} \\ &= t^{u-\alpha}.u \\ &= u.t^{u-\alpha} \end{aligned}$$

3. Leibniz rule for Conformable Fractional Derivative

Leibniz's rule, for the Conformable Fractional Derivative is given as :

$$D_t^\alpha(fg) = gD_t^\alpha(f) + fD_t^\alpha(g)$$

Proof:

$$\begin{aligned}
D_t^\alpha(fg) &= \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha})g(t+\epsilon t^{1-\alpha}) - f(t)g(t)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha})[(g(t+\epsilon t^{1-\alpha}) - g(t))] - g(t)[f(t) - f(t+\epsilon t^{1-\alpha})]}{\epsilon}. \\
&= \lim_{\epsilon\to 0} f(t+\epsilon t^{1-\alpha})\frac{[(g(t+\epsilon t^{1-\alpha}) - g(t))]}{1} + g(t)\frac{[f(t+\epsilon t^{1-\alpha}) - f(t)]}{\epsilon}, \\
&= f.D_t^\alpha(g) + g.D_t^\alpha(f)
\end{aligned}$$

4.Conformable Fractional derivative of a rational function

-

$$D_t^\alpha(\frac{f}{g}) = \frac{gD_t^\alpha(f) - fD_t^\alpha(g)}{g^2}$$

Proof:

$$\begin{aligned}
D_t^\alpha(\frac{f}{g}) &= lim_{\epsilon->0} \frac{\frac{f(t+\epsilon t^{1-\alpha})}{g(t+\epsilon t^{1-\alpha})} - \frac{f(t)}{g(t)}}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha})g(t) - g(t+\epsilon t^{1-\alpha})f(t)}{\epsilon.g(t+\epsilon t^{1-\alpha})g(t)} \\
&= \lim_{\epsilon\to 0} \frac{[f(t+\epsilon t^{1-\alpha}) - f(t)]g(t) - [g(t+\epsilon t^{1-\alpha}) - g(t)]f(t)}{\epsilon.g(t+\epsilon t^{1-\alpha})g(t)} \\
&= \lim_{\epsilon\to 0} \frac{[\frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}]g(t) - [\frac{g(t+\epsilon t^{1-\alpha}) - g(t)}{\epsilon}]f(t)}{g(t+\epsilon t^{1-\alpha})g(t)} \\
&= \lim_{\epsilon\to 0} \frac{[f(t+\epsilon t^{1-\alpha}) - f(t)]}{\epsilon.g(t+\epsilon t^{1-\alpha})} - \lim_{\epsilon\to 0} \frac{[g(t+\epsilon t^{1-\alpha}) - g(t)]f(t)}{\epsilon.g(t+\epsilon t^{1-\alpha})g(t)} \\
&= \lim_{\epsilon\to 0} \frac{[f(t+\epsilon t^{1-\alpha}) - f(t)]g(t)}{\epsilon.g(t+\epsilon t^{1-\alpha})} - \lim_{\epsilon\to 0} \frac{[g(t+\epsilon t^{1-\alpha}) - g(t)]f(t)}{\epsilon.g(t+\epsilon t^{1-\alpha})g(t)} \\
&= \lim_{\epsilon\to 0} \frac{[f(t+\epsilon t^{1-\alpha}) - f(t)]}{\epsilon.g(t+\epsilon t^{1-\alpha})} \lim_{\epsilon\to 0} \frac{g(t)}{g(t+\epsilon t^{1-\alpha})} - \lim_{\epsilon\to 0} \frac{[g(t+\epsilon t^{1-\alpha}) - g(t)]}{\epsilon} \lim_{\epsilon\to 0} \frac{g(t)f(t)}{g(t+\epsilon t^{1-\alpha})} \\
&= \frac{D_t^\alpha(f)(t)}{.g(t)} - \frac{f(t)D_t^\alpha(g)(t)}{(g(t))^2} \\
&= \frac{gD_t^\alpha(f) - fD_t^\alpha(g)}{g^2}
\end{aligned}$$

5.Conformable fractional derivative of a differentiable function

Let f be differentiable function, then

$$D_t^{\alpha}(f)(t) = t^{1-\alpha}.\frac{df}{dt}$$

Proof:

$$\begin{aligned} D_t^{\alpha}(f)(t) &= \lim_{\epsilon\to 0}\frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon} \\ &= t^{1-\alpha}.\lim_{h->0}\frac{f(t+h)-f(t)}{h}, \ h=\epsilon t^{1-\alpha} \\ &= t^{1-\alpha}.\frac{df}{dt}. \end{aligned}$$

6.Conformable fractional derivative of a constant

The Conformable fractional derivative of a constant is zero. :

$$\begin{aligned} D_t^{\alpha}(C)(t) &= t^{1-\alpha}.\frac{dC}{dt} \\ &= t^{1-\alpha}.0 \\ &= 0. \end{aligned}$$

7. Chain rule for CFD

The chain rule enables us to find CFD of a composite function, which we derive below.

$$D_t^{\alpha}(f\circ g)(t) = (f_g)'.D_t^{\alpha}(g(t)).$$

Proof :

$$\begin{aligned} D_t^{\alpha}(f\circ g)(t) &= \lim_{\epsilon\to 0}\frac{(f\circ g)(t+\epsilon t^{1-\alpha})-(f\circ g)(t)}{\epsilon} \\ &= \lim_{\epsilon\to 0}\frac{f(g(t+\epsilon t^{1-\alpha}))-f(g(t))}{\epsilon} \\ &= \lim_{\epsilon\to 0}\frac{f(g(t+\epsilon t^{1-\alpha}))-f(g(t))}{[g(t+\epsilon t^{1-\alpha})-g(t)]}\frac{[g(t+\epsilon t^{1-\alpha})-g(t)]}{\epsilon} \\ &= t^{1-\alpha}.\lim_{h\to 0}\frac{f(g(t+h))-f(g(t))}{[g(t+h)-g(t)]}\frac{[g(t+h)-g(t)]}{h} \\ &= t^{1-\alpha}.f(g(t))'.(g(t))', \ t^{1-\alpha}.(g(t))' = D_t^{\alpha}(g(t)) \\ &= (f_g)'.D_t^{\alpha}(g(t)). \end{aligned}$$

1.7 Some methods of resolutions

This section is devoted to the presentation of some methods for solving nonlinear PDEs and ODEs.

1.8 The Kudryashov Method

Kudryashov's method [7], was developed by Kudryashov, based on a procedure analogous to the first step of the Painlevé test. its goal is to find exact solutions in an analytical form see [7] :

$$u(\eta) = \sum_{i=0}^{N} a_i Q^i(\eta)$$

of the nonlinear equation considered, taking into account the condition made on the function $Q(\eta)$: it must be the solution to the equation

$$Q'_\eta(\eta) = [Q^2(\eta) - Q(\eta)]$$

Description of Kudryashov's method

We present the main steps of the Kudryashov method as follows [7]. For given nonlinear FDEs for a function u of independent variables, (x,t) :

$$P(u, u_x, u_t, u_{xx}, ...) = 0, \tag{1.8.1}$$

- Step 1 :

  We proceed by considering the traveling wave transformation

  $$u(x,t) = u(\eta) \tag{1.8.2}$$

  Equation (1.8.1) can be reduced to the following ordinary differential equation with new variable η

  $$H(u, u', u'', u''', \dots) = 0 \tag{1.8.3}$$

  where the prime stands for differentiation with respect to η.

- Step 2. Assume that the exact solution of Eq.(1.8.3) can be written as

  $$u(\eta) = \sum_{i=0}^{N} a_i Q^i(n) \tag{1.8.4}$$

$a_i\,(i = 0, ..., N)$ are constants to be determined later, again $a_N \neq 0$, and $(Q(\eta)) = \frac{1}{1+Ae^{\eta}}$ is the solution of équation :

$$Q'_{\eta}\,(\eta) = [Q^2\,(\eta) - Q\,(\eta)] \tag{1.8.5}$$

- Step 3 :
  The homogeneous balance principle provides the values of positive integers N from Eq. (1.8.3) by using the highest order derivative and nonlinear term. Substituting Eq. (1.8.4) into Eq. (1.8.3) along with Eq. (1.8.5), we obtain a polynomial in $Q\,(\eta)$. Setting all of the coefficients of the like powers of $Q\,(\eta)$ to zero, we obtain a system of algebraic equations . These equations can be solved by Maple or Mathematica to find the values of $a_i\,(i = 0, \ldots, N)$ and A. Finally, these constants provided the new form traveling wave solutions to the NLEE (1.8.1).

1.8.1 Application of Kudryashov method for The Generalized Ito Integro-Differential Equations

In this subsection, by using the Kudryashov method, we aim to investigate the generalized Ito integro-differential equation, which is given by [7]:

$$q_{tt} + q_{xxxt} + a\left(2q_x q_t + q q_{xt}\right) + a q_{xx} \int_{-\infty}^{x} q_t dx' = 0 \tag{1.8.6}$$

let:

$$q = v_x \tag{1.8.7}$$

Equation (1.8.6) can reduced to

$$v_{ttx} + v_{xxxxt} + a\left(2v_{xx}v_{xt} + v_x v_{xxt} + v_{xxx}v_t\right) = 0 \tag{1.8.8}$$

by the wave variables

$$\eta = kx - ct \tag{1.8.9}$$

Eq.(1.8.8) is converted to the following nonlinear ordinary differential equation

$$c^2 k u''' - ck^4 u^{(5)} - 2ck^3 a\left((u'')^2 + u'u'''\right) = 0$$

where primes denote the derivatives with respect to η.

The previous equation can be rewritten in the form:

$$cu''' - k^3 u^{(5)} - ak^2\left((u')^2\right)'' = 0$$

after integrating twice and letting the constant of integration to be zero, we get

$$cu' - k^3 u''' - ak^2 (u')^2 = 0 \tag{1.8.10}$$

Now, considering the homogeneous balance between the highest order linear term u''' and nonlinear term of the highest order $(u')^2$ in Eq. (1.8.10), we obtain

$$N + 3 = 2N + 2$$

Consequently $N = 1$. So we look for the solution of equation (1.8.10) in the following form

$$u(\eta) = a_0 + a_1 Q(\eta). \tag{1.8.11}$$

using equation (1.8.5) and equation (1.8.11), we obtain

$$\begin{aligned} u' &= -a_1 Q + a_1 Q^2 \\ u'' &= a_1 Q - 3a_1 Q^2 + 2a_1 Q^3 \\ u''' &= -a_1 Q + 7a_1 Q^2 - 12 Q^3 + 6a_1 Q^4. \end{aligned}.$$

Substituting equation (1.8.11) into equation (1.8.10), we obtain the system of algebraic equations in the following form

$$\begin{aligned}
&Q^1 : -ca_1 + k^3 a_1 = 0\\
&Q^2 : ca_1 - 7k^3 a_1 - ak^2 a_1^2 = 0\\
&Q^3 : 12k^3 a_1 + 2ak^2 a_1^2 = 0\\
&Q^4 : -6k^3 a_1 - ak^2 a_1^2 = 0
\end{aligned}$$

Solving the algebraic equations above, this yields:

$$a_1 = \frac{-6k}{a}, \quad c = k^3 \tag{1.8.12}$$

From (1.8.11) and (1.8.12), we obtain the following travelling wave solution of equation (1.8.10)

$$u(\eta) = a_0 + \frac{-6k}{a}\left(\frac{1}{1 + Ae^{\eta}}\right) \tag{1.8.13}$$

where a_0 and A are arbitrary constants. Then the exact solution to equation (1.8.6) is written as

$$q(x,t) = \frac{6k^2 d}{a}\left(\frac{\exp\left(kx - k^3 t\right)}{\left(1 + A\exp\left(kx - k^3 t\right)\right)^2}\right)$$

1.9 Generalized Kudryashov Method

Suppose, the NLEE for $u(x,t)$ is in the following form

$$F(u, u_x, u_t, u_{xx}, ...) = 0, \tag{1.9.1}$$

Here, F is a polynomial in $u(x,t)$ and its various partial derivatives involving the highest order derivatives and nonlinear terms. The procedures of the generalized Kudryashov method [51, 47, 48] are given in briefly as follows

Step 1 :

First, we consider the following wave variable for reducing the NLEE (1.9.1) to an ordinary differential equation (ODE)

$$u(x,t) = u(\eta), \quad \eta = kx - vt \tag{1.9.2}$$

where k is the wave number and v indicates the wave velocity, with $k \neq 0$ and $v \neq 0$. Therefore, the Eq. (1.9.1) moves to the subsequent ODE,

$$P(u, u', u'', ...) = 0, \tag{1.9.3}$$

here prime indicates the derivative with respect to η

Step 2.

We assume that the formal solution of the ODE (1.9.3) can be written in the following rational form:

$$u(\eta) = \frac{\sum\limits_{i=0}^{N} a_i Q^i(\eta)}{\sum\limits_{j=0}^{M} b_j Q^j(\eta)} \tag{1.9.4}$$

where $a_i\,(i = 0, ..., N)$, $b_j\,(j = 0, ..., M)$ are constants to be determined later. Again $a_N \neq 0$ and $b_M \neq 0$, and we also note that $Q(\eta)$ is the solution of the auxiliary ODE

$$Q'_\eta(\eta) = Q^2(\eta) - Q(\eta) \tag{1.9.5}$$

The solution of Eq. (1.9.5) has the following form

$$Q(\eta) = \frac{1}{1 + Ae^\eta}$$

herein A being an integrating constant.

Step 3.

The homogeneous balance principle provides the values of positive integers N and M from Eq. (1.9.3) by using the highest order derivative and nonlinear term.

Substituting Eq. (1.9.4) into Eq. (1.9.3) along with Eq. (1.9.5), we obtain a polynomial in $Q(\eta)$. Setting all of the coefficients of the like powers of $Q(\eta)$ to zero, we obtain a system of algebraic equations . These equations can be solved by Maple or Mathematica to find the values of $a_i\,(i = 0, \ldots, N)$, $b_j\,(j = 0, \ldots, M)$, A, k and v. Finally, these constants provided the new form traveling wave solutions to the NLEE (1.9.1).

1.9.1 Application of The Generalized Kudryashov method for the Jimbo-Miwa equation

In this subsection, we will derive the exact solutions to the Jimbo-Miwa (JM) equation by the eminent method The Jimbo-Miwa (JM) equation, which was introduced by Jimbo and Miwa is given as follows [35] :

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0 \tag{1.9.6}$$

First, we will consider the following wave transformation

$$\xi = x + y + z - vt \tag{1.9.7}$$

hither v is a velocity constant to be determined later

Eq. (1.9.6) turns out to be

$$u'''' + 6u'u'' - (2v + 3)u'' = 0 \tag{1.9.8}$$

integrating Eq. (1.9.8), with respect to ξ and letting the constant of integration to be zero, we attain the relation as follows:

$$u''' + 3\left(u'\right)^2 - (2v + 3)u' = 0 \tag{1.9.9}$$

Taking into account the homogeneous balance principle, balancing the highest order derivative u''' with non-linear term $\left(u'\right)^2$ in Eq. (1.9.9), we get:

$$N - M + 3 = 2N - 2M + 2$$

hence

$$N = M + 1.$$

Choosing $M = 1$, we attain $N = 2$, so that the exact solution of Eq.(1.9.9) constitutes as follows:

$$u(\xi) = \frac{a_0 + a_1 Q(\xi) + a_2 Q^2(\xi)}{b_0 + b_1 Q(\xi)} \tag{1.9.10}$$

where a_0, a_1, a_2, b_0 et b_1 are constants to be determined later. Again $a_2 \neq 0$ and $b_1 \neq 0$, and we also note that

$$Q(\xi) = \frac{1}{1 + Ae^{\xi}} \tag{1.9.11}$$

is the solution of the auxiliary ODE :

$$Q_{\xi}(\xi) = Q^2(\xi) - Q(\xi). \tag{1.9.12}$$

Substituting Eq. (1.9.10) along with its first and thierd derivatives into Eq. (1.9.9) and equating the coefficients of like powers of $Q(\xi)$ in the resulting equation, results in

$$Q^8(\xi) : 3b_1^2a_2^2 + 6b_1^3a_2 = 0.$$

$$Q^7(\xi) : 12a_2^2b_0b_1 + 24a_2b_0b_1^2 - 6b_1^2a_2^2 - 12b_1^3a_2 = 0.$$

$$\begin{aligned} Q^6(\xi) : & -6b_1^2a_0a_2 - 24a_2^2b_0b_1 + 3b_1^2a_2^2 + 36a_2b_1^2b_1 - 3a_2b_1^3 + 12a_2^2b_0^2 + \\ & 7b_1^3a_2 - 2va_2b_1^3 - 48a_2b_0b_1^2 + 6a_1a_2b_0b_1 = 0. \end{aligned}$$

$$\begin{aligned} Q^5(\xi) : & -12a_2b_1a_0b_0 - 24a_2^2b_0^2 - b_1^3a_2 + 12a_2^2b_0b_1 - 72a_2b_0^2b_1 \\ & -12a_2b_0b_1^2 + -8va_2b_0b_1^2 - 12a_1a_2b_0b_1 + 12a_1a_2b_0^2 + 2va_2b_1^3 \\ & +12b_1^2a_0a_2 + 24a_2b_0^3 + 3a_2b_1^3 + 28a_2b_0b_1^2 = 0. \end{aligned}$$

$$\begin{aligned} Q^4(\xi) : & \ 3b_1^2a_0^2 - b_1^3a_0 + 12a_2^2b_0^2 - 54a_2b_0^3 + 3a_1^2b_0^2 + 2vb_1^3a_0 + \\ & 3b_1^3a_0 - 6b_1a_0b_0^2 - 6b_1^2a_0a_2 - 2va_1b_0b_1^2 - 10va_2b_0^2b_1 + \\ & 8va_2b_0b_1^2 - 3a_1b_0b_1^2 - 15a_2b_0^2b_1 + 12a_2b_0b_1^2 + 6a_1b_0^2b_1 + \\ & a_1b_0b_1^2 + 41a_2b_0^2b_1 - 4a_2b_0b_1^2 - 6b_1^2a_0b_0 - 24a_1a_2b_0^2 + \\ & 6a_1b_0^3 + 6a_1a_2b_0b_1 - 6a_1b_1a_0b_0 + 24a_2b_1a_0b_0 = 0. \end{aligned}$$

$$\begin{aligned} Q^3(\xi) : & -6b_1^2a_0^2 + b_1^3a_0 - 4va_2b_0^3 - 6a_2b_0^3 + 38a_2b_0^3 - 6a_1^2b_0^2 - 2vb_1^3a_0 - \\ & -3b_1^3a_0 + 12b_1a_0b_0^2 - 4va_1b_0^2b_1 + 2va_1b_0b_1^2 + 10va_2b_0^2b_1 - \\ & -6a_1b_0^2b_1 + 3a_1b_0b_1^2 + 15a_2b_0^2b_1 - 10a_1b_0^2b_1 - a_1b_0b_1^2 - \\ & 5a_2b_0^2b_1 + 10b_1^2a_0b_0 + 12a_1a_2b_0^2 + 4vb_1^2a_0b_0 + 6b_1^2a_0b_0 \\ & -12a_1b_0^3 + 12a_1b_1a_0b_0 - 12a_2b_1a_0b_0 = 0. \end{aligned}$$

$$\begin{aligned} Q^2(\xi) : & -4vb_1^2a_0b_0 + 4va_2b_0^3 - 4b_1^2a_0b_0 - 3a_1b_0^3 + 7a_1b_0^3 - \\ & 8a_2b_0^3 + 3b_1^2a_0^2 + 6a_2b_0^3 + 6a_1b_0^2b_1 - 2va_1b_0^3 + \\ & +3b_1a_0b_0^2 - 6a_1b_1a_0b_0 + 4a_1b_0^2b_1 - 7b_1a_0b_0^2 + \\ & 4va_1b_0^2b_1 + 3a_1^2b_0^2 - 6b_1^2a_0b_0 + 2vb_1a_0b_0^2 = 0. \end{aligned}$$

$$\begin{aligned} Q^1(\xi) : & \ 2va_1b_0^3 + 3a_1b_0^3 - 3b_1a_0b_0^2 + b_1a_0b_0^2 - \\ & -a_1b_0^3 - 2vb_1a_0b_0^2 = 0. \end{aligned}$$

By solving above nonlinear system using symbolic computation package, the following cases are determined.

Case 1 :

$v = -1, a_0 = \frac{b_0}{b_1}(a_1 + 2b_0), a_2 = -2b_1$
we obtain the following solution of Eq. (1.9.6) as follow :

$$u_1(\xi) = \frac{a_1 + 2\left(b_0 - b_1\right) + \left(2b_0 + a_1\right) Ae^{\xi}}{b_1(1 + Ae^{\xi})}$$

where $\xi = x + y + z + t$

Case 2 :

$v = \frac{1}{2},\ a_0 = -\frac{a_1}{2},\ a_2 = 4b_0,\ b_1 = -2b_0$
we obtain the following solution of Eq. (1.9.6) as follow :

$$u_2(\xi) = \frac{-1}{2} \frac{\left(-a_1 - 8b_0 + a_1 A^2 e^{2\xi}\right)}{b_0(-1 + A^2 e^{2\xi})}$$

where $\xi = x + y + z - \frac{1}{2}t$

1.10 The $\frac{G'}{G}$ -Expansion Method

Introduction :

Recently, Wang and Zhang have introduced a simple method which is called the $\frac{G'}{G}$ Expansion Method [38, 6, 1], to research exact solutions for nonlinear evolution equations. Later, Guo and Zhou improved the work done in [38, 6, 1], and they introduced the extended $\frac{G'}{G}$ Expansion Method [21] .

In this section, we describe the algorithm of the expansion method $\frac{G'}{G}$ for finding exact solutions of the nonlinear evolution equations.

Let us consider the nonlinear partial differential equation in the form

$$F(u, u_x, u_t, u_{xx}, ...) = 0, \tag{1.10.1}$$

where the function $u(x, t)$ is unknown and F is a polynomial function with respect to some functions or specified variables, which contains nonlinear terms and highest order derivatives of the $u(x, t)$. The main steps are as follows:

Step 1 :

Using the following ansatz

$$u(x, t) = u(\eta), \quad \eta = x - vt \tag{1.10.2}$$

where v is a constant to be determined later and $v \neq 0$

l'Eq.(1.10.1) is reduced to a nonlinear ordinary differential equation which takes the form:

$$H(u, u', u'', ...) = 0, \tag{1.10.3}$$

where $u' = \frac{du}{d\eta}$.

Step 2.

We suppose that Eq. (1.10.3) has the formal solution

$$u(\eta) = \sum_{i=0}^{N} a_i \left(\frac{G'}{G}\right)^i (n) \tag{1.10.4}$$

where $a_i\,(i = 0, ..., N)$, are arbitrary constants to be determined later, such that $a_N \neq 0$, and $G\,(\eta)$ is the solution of the auxiliary linear second order ordinary differential equation

$$G'' + kG' + lG = 0, \tag{1.10.5}$$

where k, l are constants to be determined later.

Step 3.

We determine the positive integer N in Eq.(1.10.4) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eq. (1.10.3).

We substitute Eq. (1.10.4) into Eq. (1.10.3) along with Eq. (1.10.5). As a result of this substitution, we get a polynomial of $\left(\frac{G'}{G}\right)$ and its derivatives. In this polynomial, we equate the coefficients of same power of $\left(\frac{G'}{G}\right)$ to zero. This procedure yields a system of equations which can be solved to find the unknown parameters. Finally, these constants provided the new form traveling wave solutions to the Eq. (1.10.1)

1.10.1 Constructing of new exact solutions to the mKdV equation by $\left(\frac{G'}{G}\right)$

the purpose of this subsection is to establish the existence of traveling wave solutions for the mKdV equation by the $\left(\frac{G'}{G}\right)$-expansion method

Let us consider the modified Korteweg–de Vries (mKdV) equation

$$u_t - u^2 u_x + \delta u_{xxx} = 0. \tag{1.10.6}$$

The travelling wave variable below

$$u(x,t) = u(\xi), \quad \xi = x - vt, \quad v \neq 0. \tag{1.10.7}$$

permits us converting Eq. (1.10.6) into an ordinary differential equation

$$-vu' - u^2 u' + \delta u''' = 0 \tag{1.10.8}$$

integrating the resulting ordinary differential equation once, one obtains

$$-vu - \frac{u^3}{3} + \delta u'' + c = 0 \tag{1.10.9}$$

where c is an integration constant. balancing the highest derivative term u^3 with the nonlinear term u'' in Eq. (1.10.9) yields

$$3N = N + 2$$

hence the leading order $N = 1$. Therefore, we can write the solution of Eq. (1.10.9) in the form

$$u(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) \tag{1.10.10}$$

where G satisfies (1.10.5), it can be changed into

$$\left(\frac{G'}{G}\right)' = -\left(\frac{G'}{G}\right)^2 - k\left(\frac{G'}{G}\right) - l \tag{1.10.11}$$

where k, l are constants to be determined later.
substitut Eq. (1.10.10) into Eq. (1.10.9) along with Eq. (1.10.11). As a result of this substitution, we get a polynomial of $\left(\frac{G'}{G}\right)$ and its derivatives. In this polynomial, we equate the coefficients of same power of $\left(\frac{G'}{G}\right)$ to zero. This procedure yields a system of equations

$$\left(\frac{G'}{G}\right)^0 : \quad c - va_0 - \frac{1}{3}a_0^3 + \delta a_1 k l = 0$$
$$\left(\frac{G'}{G}\right)^1 : -va_1 - a_1 a_0^2 + \delta a_1 k^2 + 2\delta a_1 l = 0$$
$$\left(\frac{G'}{G}\right)^2 : \quad -a_1^2 a_0 + 3\delta a_1 k = 0$$
$$\left(\frac{G'}{G}\right)^3 : \quad -\frac{1}{3}a_1^3 + 2\delta a_1 = 0$$

Solving this system we obtain

$$a_0 = \pm\tfrac{1}{2}k\sqrt{6\delta}$$
$$a_1 = \pm\sqrt{6\delta},$$
$$v = -\tfrac{1}{2}\delta k^2 + 2\delta l,$$
$$c = 0.$$

Substituting these results in Eq. (1.10.10) we get

$$u(\xi) = \pm\frac{1}{2}k\sqrt{6\delta} \pm \sqrt{6\delta}\left(\frac{G'}{G}\right) \tag{1.10.12}$$

Now, taking the solution set (1.10.12) and the solutions of Eq. (1.10.11) into account; we deduce the traveling wave solutions of Eq. (1.10.6) respectively as follows.

When : $k^2 - 4l > 0$

$$u_{1.1} = \pm\frac{1}{2}\sqrt{6\delta\left(k^2-4l\right)}\left(\frac{C_1\sinh\frac{1}{2}\sqrt{k^2-4l}\xi + C_2\cosh\frac{1}{2}\sqrt{k^2-4l}\xi}{C_1\cosh\frac{1}{2}\sqrt{k^2-4l}\xi + C_2\sinh\frac{1}{2}\sqrt{k^2-4l}\xi}\right)$$

where: $\xi = x - \left(2\delta l - \frac{1}{2}\delta k^2\right)t$; $\quad C_1, C_2$ are arbitrary constants

When: $k^2 - 4l < 0$

$$u_{1.2} = \pm\frac{1}{2}\sqrt{6\delta\left(4l-k^2\right)}\left(\frac{-C_1\sin\frac{1}{2}\sqrt{4l-k^2}\xi + C_2\cos\frac{1}{2}\sqrt{4l-k^2}\xi}{C_1\cos\frac{1}{2}\sqrt{4l-k^2}\xi + C_2\sin\frac{1}{2}\sqrt{4l-k^2}\xi}\right)$$

where : $\xi = x - \left(2\delta l - \frac{1}{2}\delta k^2\right)t$; $\quad C_1, C_2$ are arbitrary constants

When : $k^2 - 4l = 0$

$$u_{1.3} = \pm\frac{\sqrt{6\delta}C_2}{C_1 + C_2 x}$$

where : $\xi = x - \left(2\delta l - \frac{1}{2}\delta k^2\right) t$; C_1, C_2 are arbitrary constants

1.11 The Exp-function method

Introduction :

The Exp-function method is one of the powerful methods that appear in recently years for finding exact solutions of different types of nonlinear differential equations. It was introduced by He and Wu [22]. In this part we present the main idea of this technique [14, 5], with an example of application

Let us have a general form of nonlinear PDE

$$F(u, u_x, u_t, u_{xx}, ...) = 0, \tag{1.11.1}$$

where F is a polynomial function with respect to the indicated variables

In the following, we give the Basic idea of Exp-function method

Step 1 :

We first unite the independent variables x and t into one wave variable

$$u(x,t) = u(\xi), \quad \xi = kx + wt \tag{1.11.2}$$

where k, w are arbitrary constants and $k \neq 0, w \neq 0$. Eq.(1.11.1) is reduced to an ordinary differential equation,

$$H(u, u', u'', ...) = 0, \tag{1.11.3}$$

where prime denotes the derivative with respect to ξ.

Step 2.

Assume that the solution $u(\xi)$ of Eq. (1.11.3) is in the form

$$U(\xi) = \frac{a_c \exp(c\xi) + \cdots + a_{-d} \exp(-d\xi)}{b_p \exp(p\xi) + \cdots + b_{-q} \exp(-q\xi)} \tag{1.11.4}$$

where $a_n, (n = -d, ..., c), b_m\,(m = -q, ..., p)$, are constants c, d, p and q are positive integers to be determined.

Step 3.

To determine the values of c and p, we balance the linear term of highest order in Eq. (1.11.3) with the highest order nonlinear term. Similarly to determine the values of p and q, we balance the linear term of lowest order in Eq. (1.11.3) with the lowest-order nonlinear term.

Substitute (1.11.4) into Eq. (1.11.3) and equate the coefficients of $\exp(\xi)$ to zero, obtain a system of algebraic equations for $a_n, (n = -d, ..., c)$, $b_m\,(m = -q, ..., p)$, k and w. Then, solve the system with the aid of a computer algebra system such as Mathematica, Maple or Matlab to determine these constants

We finally obtain exact solutions of equation (1.11.3) . From these results and from equation (1.11.2), we deduce the exact solutions of Eq.(1.11.1).

1.11.1 Exact and explicit solutions to the (3 + 1)-dimensional Jimbo–Miwa equation via the Exp-function method

In this part the Exp-function method is applied to the (3 + 1)-dimensional Jimbo–Miwa equation to find new travelling wave solutions.
The (3 + 1)-dimensional Jimbo–Miwa equation has the form [29]

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0 \tag{1.11.5}$$

Using the wave variable

$$u(x, y, z, t) = u(\xi), \quad \xi = kx + my + rz + wt, \tag{1.11.6}$$

Then eqution(1.11.5) reduces to an ordinary dfferential equation

$$k^3 m U^{(4)} + 6k^2 m U' U'' + (2mw - 3kr)U'' = 0 \tag{1.11.7}$$

The two dominant terms of the equation (1.11.7) are $U^{(4)}$ and $U'U''$ so to define c and p, we proceed to the following calculations

As the solution to equation (1.11.7) is sought in the form :

$$U(\xi) = \frac{a_c \exp(c\xi) + \cdots + a_{-d} \exp(-d\xi)}{b_p \exp(p\xi) + \cdots + b_{-q} \exp(-q\xi)}$$

then :

$$U^{(1)} = \frac{k_1 \exp[(c+p)\xi] + \cdots}{k_2 \exp[2p\xi] + \cdots}$$

$$U^{(2)} = \frac{k_3 \exp[(c+3p)\xi] + \cdots}{k_4 \exp[4p\xi] + \cdots}$$

$$U^{(3)} = \frac{k_5 \exp[(c+7p)\xi] + \cdots}{k_6 \exp[8p\xi] + \cdots}$$

$$U^{(4)} = \frac{k_7 \exp[(c+15p)\xi] + \cdots}{k_8 \exp[16p\xi] + \cdots}$$

$$U'U'' = \frac{l \exp[(2c+4p)\xi] + \cdots}{m \exp[6p\xi] + \cdots} = \frac{l \exp[(2c+14p)\xi] + \cdots}{m \exp[16p\xi] + \cdots}$$

By balancing between $U'U''$ et $U^{(4)}$ we find

$$2c + 14p = c + 15p,$$

hence

$$c = p$$

We proceed in the same way to determine d and q

$$U^{(1)} = \frac{\cdots + l_1 \exp[-(d+q)\xi]}{\cdots + k_1' \exp[-2q\xi]}$$

$$U^{(2)} = \frac{\cdots + l_2 \exp[-(d+3q)\xi]}{\cdots + k_2' \exp[-4q\xi]}$$

$$U^{(3)} = \frac{\cdots + l_3 \exp[-(d+7q)\xi]}{\cdots + k_3' \exp[-8q\xi]}$$

$$U^{(4)} = \frac{\cdots + l_4 \exp[-(d+15q)\xi]}{\cdots + k_4' \exp[-16q\xi]}$$

$$U'U'' = \frac{k_9 \exp[-(2d+4q)\xi] + \cdots}{k_{10}' \exp[-6q\xi] + \cdots} = \frac{k_9 \exp[-(2d+14q)\xi] + \cdots}{k_{10} \exp[-16q\xi] + \cdots}$$

By balancing between $U'U''$ and $U^{(4)}$ we obtain

$$-(2d+14q) = -(d+15q)$$

hence

$$d = q$$

in this way we determine the positive values of c, p, q, d
We can assign positive values to c and d, for example for c=1 and d=1, we will have p=q=1 therethe solution takes the form

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)} \tag{1.11.8}$$

Case 1.

$$a_0 = b_0\left(\frac{a_1}{b_1} - k\right) \mp \sqrt{k^2b_0^2 - 4k^2b_{-1}b_1},$$

$$a_{-1} = b_{-1}\left(\frac{a_1}{b_1} - 2k\right), \quad w = \frac{k\left(3r-k^2m\right)}{2m}$$

we obtain the following solution of Eq.(1.11.5) by substituting these results into (1.11.8)

$$u(\xi) = \frac{a_1\exp(\xi) + b_0\left(\frac{a_1}{b_1} - k\right) \mp \sqrt{k^2b_0^2 - 4k^2b_{-1}b_1} + b_{-1}\left(\frac{a_1}{b_1} - 2k\right)\exp(-\xi)}{b_1\exp(\xi) + b_0 + b_{-1}\exp(-\xi)}$$

where : $\xi = kx + my + rz + \frac{k\left(3r-k^2m\right)}{2m}t.$

Case 2.

$$a_0 = 0, \quad a_{-1} = b_{-1}\left(\frac{a_1}{b_1} - 4k\right),$$

$$b_0 = 0, \quad w = \frac{k\left(3r-4k^2m\right)}{2m}$$

so the solution is :

$$u(\xi) = \frac{a_1\exp(\xi) + b_{-1}\left(\frac{a_1}{b_1} - 4k\right)\exp(-\xi)}{b_1\exp(\xi) + b_{-1}\exp(-\xi)}$$

where : $\xi = kx + my + rz + \frac{k\left(3r-4k^2m\right)}{2m}t.$

Case 3.

$$a_0 = b_0\left(\frac{a_{-1}}{b_{-1}} + 2k\right), \quad a_1 = 0, \quad b_1 = 0, \quad w = \frac{k\left(3r-k^2m\right)}{2m}$$

so the solution is :

$$u(\xi) = \frac{b_0\left(\frac{a_{-1}}{b_{-1}} + 2k\right) + a_{-1}\exp(-\xi)}{b_0 + b_{-1}\exp(-\xi)}$$

where : $\xi = kx + my + rz + \frac{k\left(3r-k^2m\right)}{2m}t.$

Note : for other values assigned to p and c, we find other solutions for our equation.

Chapter 2

Various optical solitons to the (1+1)-Telegraphequation with space-time conformable derivatives

Abstract

This thesis presents a new sub-equation method based on an auxiliary equation which is implemented via the well-known generalized Kudryashov method, to construct new traveling waves to the Telegraph equation with time and space conformable derivatives. To illustrate its effectiveness, it was tested for seeking traveling wave solutions to the (1+1)-Telegraph equation with space-time conformable derivatives. With the help of Maple Software we derive some new solitary waves solutions. It can be concluded that the proposed method is an accurate tool for solving several kind of nonlinear evolution equations.

Keywords: (1+1)-Telegraph equation; Generalized Kudryashov method; Conformable derivative; Auxiliary equation; Traveling wave; Optical soliton. 2010 MSC: 35Q20; 35K99; 35P05.

2.1 Introduction

Nowadays, differential equations with conformable derivative order become powerful tool for modeling nonlinear phenomena that are encountered in many fields, such as Physics, Mechanics, Engineering, etc. Finding accurate method for solving such problems has been undertaken by many researchers, [2, 3, 9, 10, 34]. to this end a variety of powerful methods have been presented such as sine-cosine method [39, 44], homotopy perturbation method [31, 17], tanh-sech method [45, 23], homogeneous balance method [16], F-expansion method [31, 28], Exp-function method [5, 14, 22, 30, 29], $\frac{G'}{G}$expansion method [14-16], modified Kudryashov method [36], Kudryashov method [7], the generalized Kudryashov method [51, 47, 48, 35], the double auxiliary equations method [8], and so on.

Our study focusses on presenting a novel technique to solve partial differential equations with time and space conformable derivatives by combining the generalized

Kudryashov method to the auxiliary equation technique. We implement this novel method to seek traveling waves for the space-time non linear Telegraph equation with conformable derivatives.
The remainder of this paper is organized as follows. In section 2, we recall some basic definitions of conformable derivative and some of its useful mathematical properties that will be used throughout the paper. Section 3, deals with the the description of the new method. In section 4, we construct solitary wave solutions for the space-time non linear Telegraph Equation. In section 5, the behavior of the wave solutions of space-time non linear Telegraph Equation are displayed graphically and discussed in detail. In section 6 concluding remarks are given.

.Preliminaries

Following the works by Khalil et al.[26] and Abdeljawad et al.[2, 3, 9, 10, 34], the Conformable derivative of order α with respect to x is defined as the following

$$D_t^\alpha(f)(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \text{ for all } \quad t>0, \quad \alpha \in (0,1). \tag{2.1.1}$$

Here, we recall some useful properties of Conformable derivative

$$D_t^\alpha \left(f \circ g\right)(t) = \left(f_g\right)' . D_t^\alpha \left(g(t)\right).$$

$$D_t^\alpha(fg) = gD_t^\alpha(f) + fD_t^\alpha(g)$$

$$D_t^\alpha(t^u) = u.t^{u-\alpha}$$

$$D_t^\alpha(c) = 0, \text{where c is a constant}$$

$$D_t^\alpha(f)(t) = t^{1-\alpha}.\frac{df}{dt}$$

2.2 Description of the method

We present here the main steps of the proposed method. Consider a general nonlinear equation with conformable derivatives as the following

$$P(u, u_x, u_y, u_z, \ldots, D_t^\delta, D_x^\alpha, D_y^\beta, D_z^\gamma, \ldots) = 0. \tag{2.2.1}$$

Where u is a function of independent variables, $(x, y, z, \ldots, t)$, $D_t^\delta, D_x^\alpha, D_y^\beta$ and D_z^γ are the Conformable derivatives of u with respect to $t, x, y$and z and p is polynomial in u. To seek the traveling wave solutions u of Eq. (2.2.1) explicitly, we will follow the next three steps:

- Step 1. Using the wave transformation

$$U(x, y, z, \ldots, t) = U(\eta), \quad \eta = \frac{kx^{\alpha}}{\alpha} + \frac{ly^{\beta}}{\beta} + \frac{hz^{\gamma}}{\gamma} + \cdots - \frac{vt^{\delta}}{\delta}. \tag{2.2.2}$$

Where k, l, h and v are constants to be determined later, (2.2.2) converts Eq.(2.2.1) to an ordinary differential equation with new variable η

$$H(u, u', u'', u''', \ldots) = 0 \tag{2.2.3}$$

where the prime stands for differentiation with respect to η.

- Step 2. Assume that the exact solution of Eq.(2.2.3) can be written as

$$u(\eta) = \frac{\sum_{i=0}^{N} a_i \left(\frac{h(\eta)}{g(\eta)}\right)^i}{\sum_{j=0}^{M} b_j \left(\frac{h(\eta)}{g(\eta)}\right)^j}. \tag{2.2.4}$$

Here $a_i\,(i = 0, ..., N)$, $b_j\,(j = 0, ..., M)$ are constants to be determined later, ($a_N \neq 0, \quad b_M \neq 0$ and $\left(\frac{h(\eta)}{g(\eta)}\right)$ is the solution of

$$\left(\frac{h(\eta)}{g(\eta)}\right)' = A\left(\frac{h(\eta)}{g(\eta)}\right)^2 + B\left(\frac{h(\eta)}{g(\eta)}\right) + C, \tag{2.2.5}$$

where

$$g(\eta) = \exp(\eta)\,(h(\eta))'$$

.

Eq. (2.2.5) has the following set of solutions

1. Famille 1 : When $\Delta = B^2 - 4AC > 0$,

$$\left(\frac{h(\eta)}{g(\eta)}\right) = \frac{-2C[1 - \tanh(\frac{\sqrt{\Delta}}{2}\eta)\tanh(\frac{\sqrt{\Delta}}{2}k_1)]}{B - B\tanh(\frac{\sqrt{\Delta}}{2}\eta)\tanh(\frac{\sqrt{\Delta}}{2}k_1) - \sqrt{\Delta}[\tanh(\frac{\sqrt{\Delta}}{2}\eta) - \tanh(\frac{\sqrt{\Delta}}{2}k_1)]}. \tag{2.2.6}$$

k_1 is a constant.

2. Famille 2 : When $\Delta = B^2 - 4AC < 0$, then

$$\left(\frac{h(\eta)}{g(\eta)}\right) = \frac{-2C[1 + \tan(\frac{\sqrt{-\Delta}}{2}\eta)\tan(\frac{\sqrt{-\Delta}}{2}k_1)]}{B + B\tan(\frac{\sqrt{-\Delta}}{2}\eta)\tan(\frac{\sqrt{-\Delta}}{2}k_1) + \sqrt{-\Delta}[\tan(\frac{\sqrt{-\Delta}}{2}\eta) - \tan(\frac{\sqrt{-\Delta}}{2}k_1)]}. \tag{2.2.7}$$

k_1 is a constant.

3. Famille 3 : When $\Delta = B^2 - 4AC = 0$ et $AC > 0$, then

$$\left(\frac{h(\eta)}{g(\eta)}\right) = -\frac{C(\eta - k_1)}{\sqrt{AC}(\eta - k_1) - 1}, \tag{2.2.8}$$

   k_1 is a constant.

4. Family 4 :When $C = 0$, $B \neq 0$,

$$\left(\frac{h(\eta)}{g(\eta)}\right) = -\frac{B\exp(B\eta)}{A\exp B\eta + Bk_1}, \tag{2.2.9}$$

   k_1 est une constante.

5. Famille 5 : Pour $B = 0$ et $C = 0$, on a :

$$\left(\frac{h(\eta)}{g(\eta)}\right) = -\frac{1}{A\eta + k_1}, \tag{2.2.10}$$

   k_1 is a constant. .

- Step 3: To compute the positive integer number N and M in Eq. (2.2.4) we balance the highest order linear term and the highest order nonlinear term occurring in Eq.(2.2.3), This completes the determination of the value of N and M.

Substituting Eq.(2.2.4) along with Eq.(2.2.5) into Eq. (2.2.3), we calculate all the necessary derivatives $u', u'', \ldots$ As a result of this substitution, we get then a polynomial of $\left(\frac{h(\eta)}{g(\eta)}\right)$, setting the coefficients of the same power of $\left(\frac{h(\eta)}{g(\eta)}\right)$ to zero, we obtain a system of algebraic equations. Solving this system we get the unknown parameters $A, B, C, a_i\,(i = 0, \ldots, N)$, $b_j, (j = 0, \ldots, M)$, $k, l, h, \ldots$ and v. We finally obtain the exact solutions of Eq.(2.2.3). Substituting these results into the solutions of (2.2.5) and using Eq.(2.2.2) we get the exact solutions of Eq. (2.2.1).

2.3 Derivation of new traveling waves to the Telegraph equation

In this section we seek the exact solutions of the Telegraph Equation through above described method. Such equation with time-space conformable derivatives can be written as

$$D_{tt}^{2\alpha}u - D_{xx}^{2\alpha}u + D_t^{\alpha}u + \gamma u + \beta u^3 = 0 \tag{2.3.1}$$

Where α is a parameter describing the order of derivation in conformable sense. When $\alpha = 1$, Eq. (2.3.1) reduces to the nonlinear Telegraph equation. Using the complex transform

$$\eta = \frac{kx^{\alpha}}{\alpha} - \frac{ct^{\alpha}}{\alpha} \tag{2.3.2}$$

Eq.(2.3.1) is reduced to the following ordinary differential equation

$$(c^2 - k^2)u'' - cu' + \gamma u + \beta u^3 = 0 \tag{2.3.3}$$

Balancing the order of u'' and u^3 in Eq. (2.3.3), we obtain $M = 1$ and $N = 2$. Then the solution can be expressed as

$$u(\eta) = \frac{a_0 + a_1\left(\frac{h(\eta)}{g(\eta)}\right) + a_2\left(\frac{h(\eta)}{g(\eta)}\right)^2}{b_0 + b_1\left(\frac{h(\eta)}{g(\eta)}\right)} \tag{2.3.4}$$

Substituting Eq. (2.3.4) and its derivatives using Eq. (2.2.5) into Eq. (2.3.3) collecting the coefficients of each power of $\left(\frac{h(\eta)}{g(\eta)}\right)^i, (i = 0, \ldots, N), (i = 0, \ldots, M)$ and set them to zero, we obtain a system of algebraic equations. Solving this system of algebraic equations with the aid of Maple, we obtain the following sets

- Set 1:

$$a_0 = 0, a_1 = \pm\frac{\sqrt{2}b_1}{4}\sqrt{\frac{-2\gamma(B^2+\Delta-2B\frac{\Delta}{\sqrt{\Delta}})}{\beta\Delta}}, a_2 = \pm\frac{\sqrt{2}b_1}{8C}\sqrt{\frac{-2\gamma(B^2+\Delta-2B\frac{\Delta}{\sqrt{\Delta}})}{\beta\Delta}}(B+\frac{\Delta}{\sqrt{\Delta}}), b_0 = 0,$$

$$k = \pm\sqrt{\frac{9\gamma^2-2\gamma}{4\Delta}}, c = \frac{3\gamma}{\sqrt{4\Delta}}, \Delta = B^2 - 4AC. \tag{2.3.5}$$

Substituting these result into Eq (2.3.4) we have :

$$u_1(\eta) = \pm\frac{1}{4C}\sqrt{\frac{-\gamma(B^2+\Delta-2B\frac{\Delta}{\sqrt{\Delta}})}{\beta\Delta}}\left(2C + (B+\frac{\Delta}{\sqrt{\Delta}})\left(\frac{h(\eta)}{g(\eta)}\right)\right) \tag{2.3.6}$$

where

$$\eta = \pm\sqrt{\frac{9\gamma^2-2\gamma}{4\Delta}}\frac{x^{\alpha}}{\alpha} - \frac{3\gamma}{\sqrt{4\Delta}}\frac{t^{\alpha}}{\alpha}$$

The exact solution $u_1(\eta)$ exists under the constraint condition $\Delta > 0$.

- Set 2:

$$a_0 = \pm\frac{b_0}{2}\sqrt{(\frac{-\gamma}{\beta\Delta})}[B - \frac{\Delta}{\sqrt{\Delta}}], a_1 = \pm\frac{1}{2}\sqrt{(\frac{-\gamma}{\beta\Delta})}[b_1B + 2b_0A - \frac{\Delta b_1}{\sqrt{\Delta}}], a_2 = \pm Ab_1\sqrt{\frac{\gamma}{-\beta\Delta}},$$

$$k = \pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{\Delta}}, c = \frac{3\gamma}{2\sqrt{\Delta}}, \Delta = B^2 - 4AC. \tag{2.3.7}$$

Substituting these result into Eq (2.3.4) we have :

$$u_2\left(\eta\right)=\pm\frac{1}{2}\sqrt{\frac{-\gamma}{\beta\Delta}}\Big(B-\frac{\Delta}{\sqrt{\Delta}}+2A\frac{h(\eta)}{g(\eta)}\Big). \tag{2.3.8}$$

where $\eta=\pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{\Delta}}\frac{x^\alpha}{\alpha}-\frac{3\gamma}{2\sqrt{\Delta}}\frac{t^\alpha}{\alpha}$.

- Set 3:

$$\begin{aligned}&a_0=\pm\tfrac{Cb_1}{4A}\sqrt{\tfrac{A\gamma}{\beta C}},a_1=\pm\tfrac{b_1}{4A}\frac{\sqrt{\frac{\gamma A}{C\beta}}}{\frac{1}{\sqrt{-4AC}}},a_2=\mp\tfrac{b_1}{4}\sqrt{\tfrac{A\gamma}{C\beta}},b_0=0,B=0,\\&k=\pm\tfrac{1}{4}\sqrt{\tfrac{9\gamma^2-2\gamma}{-4AC}},c=\tfrac{3\gamma}{4\sqrt{-4AC}}.\end{aligned} \tag{2.3.9}$$

Substituting these result into Eq (2.3.4) we have :

$$u_3\left(\eta\right)=\mp\frac{1}{4}\sqrt{\frac{A\gamma}{\beta C}}\frac{\left(-\frac{C}{\sqrt{\Delta}}-\frac{h(\eta)}{g(\eta)}+\frac{A}{\sqrt{\Delta}}\left(\frac{h(\eta)}{g(\eta)}\right)^2\right)}{\frac{A}{\sqrt{\Delta}}\left(\frac{h(\eta)}{g(\eta)}\right)}. \tag{2.3.10}$$

where $\eta=\pm\frac{1}{4}\sqrt{\frac{9\gamma^2-2\gamma}{-4AC}}\frac{x^\alpha}{\alpha}-\frac{3\gamma}{4\sqrt{-4AC}}\frac{t^\alpha}{\alpha}$.

- Set 4:

$$\begin{aligned}&a_0=\pm b_0\tfrac{\sqrt{(\frac{-\gamma}{\beta\Delta})}}{4B}[B^2-2\tfrac{\Delta B}{\sqrt{\Delta}}+\Delta],a_1=\pm\tfrac{b_0A\sqrt{(\frac{-\gamma}{\beta\Delta})}[\frac{-\Delta}{\sqrt{\Delta}}+B]}{B},\\&a_2=\pm\tfrac{b_0A^2}{B}\sqrt{(\tfrac{-\gamma}{\beta\Delta})},b_1=\tfrac{2b_0A}{B},\Delta=B^2-4AC,\\&k=\pm\sqrt{\tfrac{9\gamma^2-2\gamma}{16\Delta}},c=\tfrac{3\gamma}{\sqrt{16\Delta}}.\end{aligned} \tag{2.3.11}$$

Substituting these result into Eq (2.3.4) we have :

$$u_4\left(\eta\right)=\pm\frac{1}{4}\sqrt{(\frac{-\gamma}{\beta\Delta})}\left(\frac{[B^2-2\frac{B\Delta}{\sqrt{\Delta}}+\Delta]+A[4B-4\frac{\Delta}{\sqrt{\Delta}}]\left(\frac{h(\eta)}{g(\eta)}\right)+4A^2\left(\frac{h(\eta)}{g(\eta)}\right)^2}{B+2A\left(\frac{h(\eta)}{g(\eta)}\right)}\right) \tag{2.3.12}$$

where $\eta=\pm\sqrt{\frac{9\gamma^2-2\gamma}{16\Delta}}\frac{x^\alpha}{\alpha}-\frac{3\gamma}{\sqrt{16\Delta}}\frac{t^\alpha}{\alpha}$.

- Set 5:

$$a_0 = \pm b_0\sqrt{\tfrac{-\gamma}{\beta}}, a_1 = 0, a_2 = 0, b_1 = \tfrac{2Bb_0}{C}, A = 0,$$
$$k = \pm\tfrac{\sqrt{9\gamma^2-2\gamma}}{2B}, c = \tfrac{-3\gamma}{2B}, BC \neq 0 \tag{2.3.13}$$

Substituting these result into Eq (2.3.4) we have :

$$u_5(\eta) = \frac{\pm\sqrt{\frac{-\gamma}{\beta}}}{1 + \frac{2B}{C}\left(\frac{h(\eta)}{g(\eta)}\right)} \tag{2.3.14}$$

where $\eta = \pm\frac{\sqrt{9\gamma^2-2\gamma}}{2B}\frac{x^\alpha}{\alpha} + \frac{3\gamma}{2B}\frac{t^\alpha}{\alpha}$.

- Set 6:

$$a_0 = \pm b_0\sqrt{\tfrac{-\gamma}{B}}, a_1 = \pm\tfrac{Bb_0}{C}\sqrt{\tfrac{-\gamma}{B}}, , a_2 = 0, b_1 = \tfrac{-b_0B}{C}, A = 0,$$
$$k = \pm\tfrac{\sqrt{9\gamma^2-2\gamma}}{2B}, c = \tfrac{3\gamma}{2B}. \tag{2.3.15}$$

Substituting these result into Eq (2.3.4) we have :

$$u_6(\eta) = \pm\sqrt{\frac{-\gamma}{\beta}}\left(\frac{C + B\frac{h(\eta)}{g(\eta)}}{-C + B\frac{h(\eta)}{g(\eta)}}\right) \tag{2.3.16}$$

where $\eta = \pm\frac{\sqrt{9\gamma^2-2\gamma}}{2B}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{2B}\frac{t^\alpha}{\alpha}$.

- Set 7:

$$a_0 = \pm\tfrac{\sqrt{2}b_0}{2(B^2-\Delta)}\left(\sqrt{\tfrac{-2\gamma(B^2+\Delta+2B\frac{\Delta}{\sqrt{\Delta}})}{\beta}}(B - \tfrac{\Delta}{\sqrt{\Delta}})\right), a_1 = \pm\tfrac{\sqrt{2}b_0}{4C}\left(\sqrt{\tfrac{-2\gamma(B^2+\Delta+2B\frac{\Delta}{\sqrt{\Delta}})}{\beta}}\right)$$
$$a_2 = 0, b_1 = \tfrac{b_0}{2C}\left(B - 3\tfrac{\Delta}{\sqrt{\Delta}}\right)$$
$$k = \pm\tfrac{1}{2}\sqrt{\tfrac{9\gamma^2-2\gamma}{\Delta}}, c = \tfrac{3\gamma}{2\sqrt{\Delta}}, \Delta = B^2 - 4AC. \tag{2.3.17}$$

Substituting these result into Eq (2.3.4) we have :

$$u_7(\eta) = \pm\sqrt{\frac{-\gamma(B^2+\Delta+2B\frac{\Delta}{\sqrt{\Delta}})}{\beta}}\left(\frac{2BC-2\frac{\Delta C}{\sqrt{\Delta}}+(-\Delta+B^2)\left(\frac{h(\eta)}{g(\eta)}\right)}{(B^2-\Delta)\left(2C+(B-3\frac{\Delta}{\sqrt{\Delta}})\left(\frac{h(\eta)}{g(\eta)}\right)\right)}\right) \tag{2.3.18}$$

where $\eta = \pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{\Delta}}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{2\sqrt{\Delta}}\frac{t^\alpha}{\alpha}$.

- Set 8:

$$\begin{aligned} &a_0 = \pm b_0\sqrt{\tfrac{-\gamma}{\beta}}, a_1 = \pm\tfrac{Ab_0}{B}\sqrt{\tfrac{-\gamma}{\beta}}, a_2 = 0, C = 0, \\ &k = \pm\tfrac{1}{2}\sqrt{\tfrac{9\gamma^2-2\gamma}{B^2}}, c = \tfrac{-3\gamma}{2B}. \end{aligned} \tag{2.3.19}$$

Substituting these result into Eq (2.3.4) we have :

$$u_8(\eta) = \pm b_0\sqrt{\frac{-\gamma}{\beta}}\left(\frac{1+\frac{A}{B}\left(\frac{h(\eta)}{g(\eta)}\right)}{b_0+b_1\left(\frac{h(\eta)}{g(\eta)}\right)}\right) \tag{2.3.20}$$

where $\eta = \pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{B^2}}\frac{x^\alpha}{\alpha} + \frac{3\gamma}{2B}\frac{t^\alpha}{\alpha}$.

- Set 9:

$$\begin{aligned} &a_0 = \pm\tfrac{Cb_1}{B}\sqrt{\tfrac{-\gamma}{\beta}}, a_1 = 0, , a_2 = 0, b_0 = 0, A = 0, \\ &k = \pm\tfrac{\sqrt{9\gamma^2-2\gamma}}{2B}, c = \tfrac{-3\gamma}{2B}. \end{aligned} \tag{2.3.21}$$

Substituting these result into Eq (2.3.4) we have :

$$u_9(\eta) = \frac{\pm\frac{C}{B}\sqrt{\frac{-\gamma}{\beta}}}{\left(\frac{h(\eta)}{g(\eta)}\right)}. \tag{2.3.22}$$

where $\eta = \pm\frac{\sqrt{9\gamma^2-2\gamma}}{2B}\frac{x^\alpha}{\alpha} + \frac{3\gamma}{2B}\frac{t^\alpha}{\alpha}$.

- Set 10:

$$a_0=\pm\frac{1}{2}\sqrt{-\frac{2\gamma[2C^2b_1^2+2b_0b_1C\frac{\Delta}{\sqrt{\Delta}}+b_0^2\Delta+2b_0^2CA-2BCb_0b_1-Bb_0^2\frac{\Delta}{\sqrt{\Delta}}]}{\beta\Delta}},$$

$$a_1=\pm\sqrt{2}\sqrt{-\frac{\gamma(2C^2b_1^2+2\frac{b_0b_1C\Delta}{\sqrt{\Delta}}+b_0^2\Delta+2b_0^2CA-2b_0b_1BC-\frac{B}{\sqrt{\Delta}}b_0^2\Delta)}{\beta\Delta}}\frac{(B+\frac{\Delta}{\sqrt{\Delta}})}{4C},\ a_2=0,$$

$$k=\pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{\Delta}},\ c=\frac{3\gamma}{2\sqrt{\Delta}},\ \Delta=B^2-4AC. \tag{2.3.23}$$

Substituting these result into Eq (2.3.4) we have :

$$u_{10}\left(\eta\right)=\pm\frac{\left(2C+(B+\frac{\Delta}{\sqrt{\Delta}})\left(\frac{h(\eta)}{g(\eta)}\right)\right)}{4C(b_0+b_1\left(\frac{h(\eta)}{g(\eta)}\right))}\sqrt{\frac{-\gamma([B^2+\Delta-2B\frac{\Delta}{\sqrt{\Delta}}]b_0^2+4C^2b_1^2+4b_0b_1(\frac{C\Delta}{\sqrt{\Delta}}-BC))}{\beta\Delta}} \tag{2.3.24}$$

where $\eta=\pm\sqrt{\frac{9\gamma^2-2\gamma}{4\Delta}}\frac{x^\alpha}{\alpha}-\frac{3\gamma}{\sqrt{4\Delta}}\frac{t^\alpha}{\alpha}$

- Set 11:

$$a_0=\frac{Ca_1}{B},\ a_2=0,\ b_0=0,\ b_1=\pm a_1\sqrt{\frac{-\beta}{\gamma}},\ A=0,$$
$$k=\pm\frac{\sqrt{9\gamma^2-2\gamma}}{2B},\ c=\frac{3\gamma}{2B}. \tag{2.3.25}$$

Substituting these result into Eq (2.3.4) we have :

$$u_{11}\left(\eta\right)=\pm\frac{\frac{C}{B}+\left(\frac{h(\eta)}{g(\eta)}\right)}{\sqrt{\frac{-\beta}{\gamma}}\left(\frac{h(\eta)}{g(\eta)}\right)} \tag{2.3.26}$$

where $\eta=\pm\frac{\sqrt{9\gamma^2-2\gamma}}{2B}\frac{x^\alpha}{\alpha}-\frac{3\gamma}{2B}\frac{t^\alpha}{\alpha}$.

- Set 12:

$$a_0=\pm\sqrt{\frac{-\gamma[(\Delta+2CA-B\frac{\Delta}{\sqrt{\Delta}})b_0^2+2Cb_0b_1(\frac{\Delta}{\sqrt{\Delta}}-B)+2C^2b_1^2]}{2\beta\Delta}},$$
$$a_1=\pm\sqrt{\frac{-2\gamma[(\Delta+2CA-B\frac{\Delta}{\sqrt{\Delta}})b_0^2+2Cb_0b_1(\frac{\Delta}{\sqrt{\Delta}}-2B)+2C^2b_1^2]}{\beta\Delta}}\frac{(B+\frac{\Delta}{\sqrt{\Delta}})}{4C},,\ a_2=0, \tag{2.3.27}$$
$$k=\pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{\Delta}},\ c=\frac{3\gamma}{2\sqrt{\Delta}}.$$

Substituting these result into Eq (2.3.4) we have :

$$u_{12}(\eta) = \pm\sqrt{\frac{-\gamma[(\Delta + B^2 - 2B\frac{\Delta}{\sqrt{\Delta}})b_0^2 + 4b_0b_1C(\frac{\Delta}{\sqrt{\Delta}} - B) + 4C^2b_1^2]}{\beta\Delta}}\left(\frac{2C + (B + \frac{\Delta}{\sqrt{\Delta}})\left(\frac{h(\eta)}{g(\eta)}\right)}{4C(b_0 + b_1\left(\frac{h(\eta)}{g(\eta)}\right))}\right) \tag{2.3.28}$$

where $\eta = \pm\frac{1}{2}\sqrt{\frac{9\gamma^2-2\gamma}{\Delta}}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{2\sqrt{\Delta}}\frac{t^\alpha}{\alpha}$.

- Set 13:

$$\begin{aligned} &a_0 = 0, a_2 = 0, b_0 = \frac{B}{A\gamma}(b_1\gamma + a_1\sqrt{-\beta\gamma}), C = 0, \\ &k = \pm\frac{1}{2B}\sqrt{9\gamma^2 - 2\gamma}, c = \frac{3\gamma}{2B}. \end{aligned} \tag{2.3.29}$$

Substituting these result into Eq (2.3.4) we have :

$$u_{13}(\eta) = \frac{a_1\left(\frac{h(\eta)}{g(\eta)}\right)}{\frac{B}{A\gamma}(b_1\gamma + a_1\sqrt{-\beta\gamma}) + b_1\left(\frac{h(\eta)}{g(\eta)}\right)}. \tag{2.3.30}$$

where $\eta = \pm\frac{1}{2B}\sqrt{9\gamma^2 - 2\gamma}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{2B}\frac{t^\alpha}{\alpha}$.

In particular, the new exact solution of the Telegraph Equation with conformable space-time derivatives (2.3.1) with the help of Eq.(2.2.6) to Eq. (2.2.10) as follows:

When $C = 0, B \neq 0$,

$$\begin{aligned} &u_{1.1}(\eta) = \pm\frac{1}{4C}\sqrt{\frac{-\gamma(B^2+\Delta-2B\frac{\Delta}{\sqrt{\Delta}})}{\beta\Delta}}\left(2C + (B + \frac{\Delta}{\sqrt{\Delta}}) - \frac{B\exp(B\eta)}{A\exp B\eta + Bk_1}\right), \\ &\eta = \pm\sqrt{\frac{9\gamma^2-2\gamma}{4\Delta}}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{\sqrt{4\Delta}}\frac{t^\alpha}{\alpha} \end{aligned} \tag{2.3.31}$$

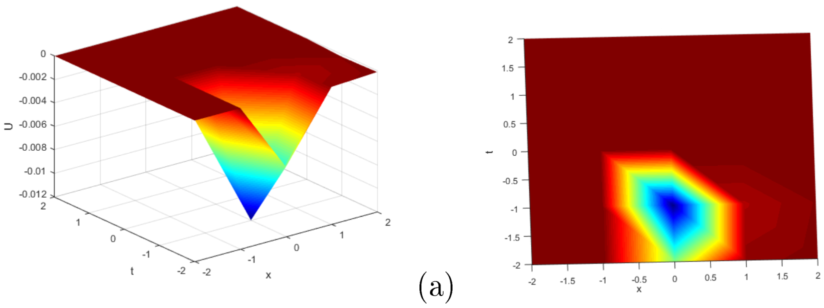
The solution $u_{1.1}$ exists under the constraint condition $\Delta > 0$.
When $\Delta > 0$,

$$\begin{aligned} &u_{1.2}(\eta) = \pm\frac{1}{4C}\sqrt{\frac{-\gamma(B^2+\Delta-2B\frac{\Delta}{\sqrt{\Delta}})}{\beta\Delta}}\left(2C + (B + \frac{\Delta}{\sqrt{\Delta}})\frac{-2C[1-\tanh(\frac{\sqrt{\Delta}}{2}\eta)\tanh(\frac{\sqrt{\Delta}}{2}k_1)]}{B-B\tanh(\frac{\sqrt{\Delta}}{2}\eta)\tanh(\frac{\sqrt{\Delta}}{2}k_1)-\sqrt{\Delta}[\tanh(\frac{\sqrt{\Delta}}{2}\eta)-\tanh(\frac{\sqrt{\Delta}}{2}k_1)]}\right), \\ &\eta = \pm\sqrt{\frac{9\gamma^2-2\gamma}{4\Delta}}\frac{x^\alpha}{\alpha} - \frac{3\gamma}{\sqrt{4\Delta}}\frac{t^\alpha}{\alpha} \end{aligned} \tag{2.3.32}$$

The other solutions can be derived in an analogous way.

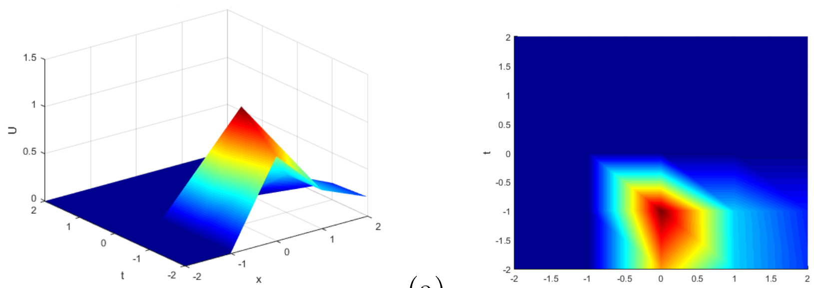
2.3.1 Graphical illustrations and discussion

For the sake of briefness, we present here the profiles of some obtained solutions. The figures has been plotted by the Matlab software. In Figs.1-5, the 3D profiles and the contour plots of those solutions are given for some selected parameters satisfying the above mentioned calculations for each case. We underline that the resulting solutions were substituted in the studied equation to verify their correctness of the method. All computations were done using Maple17 software.



(a) (b)

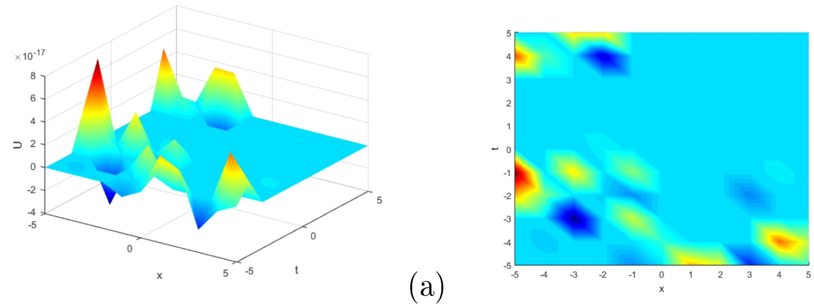
Figure 2.1: (a)Profile of the solution $u_1(x, y)$ for $A = 2$, $B = 5$, $C = 2$, $\alpha = 0.5$, $\beta = -1$, $\gamma = 1$ and $w_1 = 1$. (b) Contour plots corresponding to $u_1(x, y)$.



(a) (b)

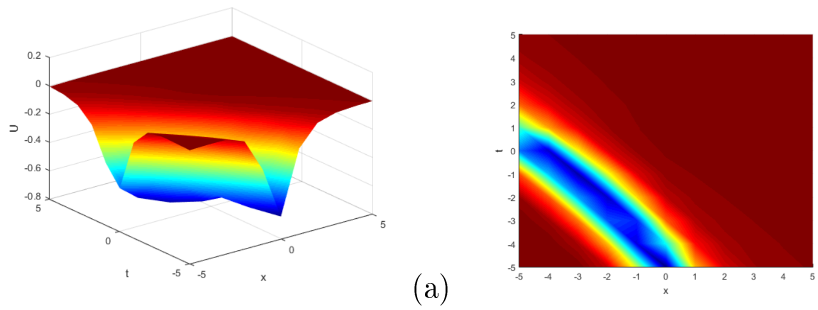
Figure 2.2: Profile of the solution $u_2(x, y)$ for $A = -2; B = -1; C = 5; \alpha = \frac{1}{2}; \beta = -\frac{1}{3}; \gamma = \frac{1}{2}$ and $w_1 = 0$. (b) Contour plots corresponding to $u_2(x, y)$.

The others solutions are also hyperbolic, trigonometric, exponential or rational solutions. They have similar profiles to the formers.



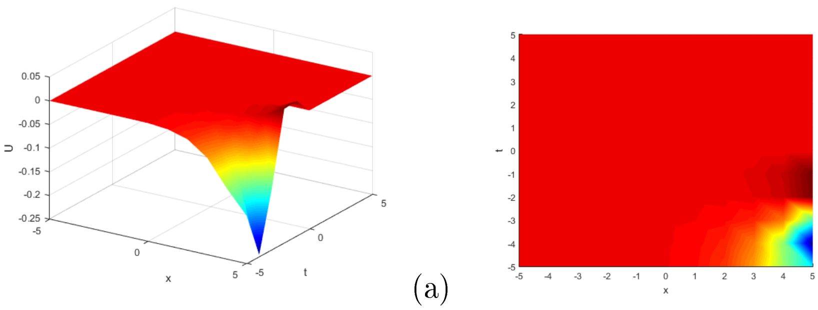
(a) (b)

Figure 2.3: Profile of the solution $u_3(x,y)$ for $A=1$, $B=0$, $C=-8$, $\alpha=\frac{1}{3}$, $\beta=-2$, $\gamma=3$ and $w_1=8$. (b) Contour plots corresponding to $u_3(x,y)$.



(a) (b)

Figure 2.4: Profile of the solution $u_4(x,y)$ for $A=3$, $B=4$, $C=1$, $\alpha=0.9$, $\beta=0.25$, $\gamma=1$ and $w_1=1$. (b) Contour plots corresponding to $u_4(x,y)$.



(a) (b)

Figure 2.5: Profile of the solution $u_5(x,y)$ for $A=0$, $B=-2$, $C=3$, $\alpha=0.9$, $\beta=3$, $\gamma=-1$ and $w_1=5$. (b) Contour plots corresponding to $u_5(x,y)$.

2.4 Conclusion

In the present paper, a new method was proposed using the auxiliary equation and the general Kudryashov method. Its accuracy has been tested by applying it successfully to the Telegraph equation with time-space conformable derivatives. It can be seen through this study, that this method is a powerful mathematical technique for finding traveling wave solutions for the partial differential equations. This method could be applied to the nonlinear evolution equations which arising in many fields of science. We derived various solitary waves for the (1+1)-Telegraph equation with space-time conformable derivatives, namely : hyperbolic, trigonometric, exponential and rational solutions. Some of the results have already been reported in the literature and some of them are new. The outcomes of this work would be beneficial to understand the behaviors of wave propagation in nonlinear science with beta derivatives or M-derivatives.

2.5 Contribution and outlook

-Contribution

The contribution made in this modest work is the presentation of a new method which allows us to solve certain nonlinear PDEs and ODEs of any order and this through a combination between the Kudryashov method, the method $\frac{G'}{G}$ and some auxiliary equation. The use of the proposed method has shown its worth and effectiveness.

- Outlook

The results of this modest work constitute the bases of a work to be continued and improved for a much more in-depth study which could be the subject of research work. Thus, the future perspectives are initially:
1 -Application of the proposed method to solve PDEs of fractional order, with other fractional derivatives.

2 - Looking for other analytical methods that are more powerful and easier to apply in order to find many new solutions for nonlinear PDEs and ODEs of any order.

"all's well That ends well"

Chapter 3

New exact solutions of the nonlinear conformable space-time-fractional PHI-four And Jimbo Miwa Equations

3.0.1 Introduction:

As we all know, looking for the exact solutions of the non-linear evolution equations (NLEE) has attracted many researchers. Therefore, to calculate the exact and solitary solutions of NLPDE, the researchers introduced various kinds of methods, Such as sine-cosine method [39, 44], homotopy perturbation method [20, 11], tanh-sech method [45, 23], homogeneous balance method [16], F-expansion method [31, 28], Exp-function method [5, 14, 31], $\frac{G'}{G}$expansion method [14-16], Kudryashov method [7], the generalized Kudryashov method [51, 47, 48], the double auxiliary equations method [8], and so on.

In this chapter, we investigate the analytical solutions of the non-linear PHI and JIMBO MIWA equations by using a new technique

3.1 Description of the method

In this section, we outline the main steps of the new technique to solve nonlinear partial differential equations of fractional order.

Now, we consider the following general nonlinear FPDE:

$$P(u, u_x, u_y, u_z, \dots, D_t^\alpha, D_x^\alpha, D_y^\alpha, D_z^\alpha, \dots) = 0 \tag{3.1.1}$$

where $D_t^\alpha, D_x^\alpha, D_y^\alpha$ and D_z^α are the Conformable derivatives of u with respect to t, x, y and z. , and P is a polynomial of u.

We present the three steps of the method as follows

- Step 1. Using the wave transformation

$$U(x, y, z, \dots, t) = U(\eta), \eta = \frac{kx^\alpha}{\alpha} + \frac{ly^\alpha}{\alpha} + \frac{hz^\alpha}{\alpha} + \dots - \frac{vt^\alpha}{\alpha} \tag{3.1.2}$$

Where k, l, h and v are constants to be determined later, Eq.(3.1.2) converts Eq.(3.1.1) to an ordinary differential equation for new variable η

$$H(u, u', u'', u''', \dots) = 0 \tag{3.1.3}$$

where $"\prime" = \frac{d}{d\eta}$.

- Step 2. Suggest that the exact solutions of Eq.(3.1.3) can be written as following form:

$$u(\eta) = \frac{\sum_{i=0}^{N} a_i (\Psi(\eta))^i}{\sum_{j=0}^{M} b_j (\Psi(\eta))^j} \tag{3.1.4}$$

Here $a_i\,(i = 0, ..., N)$, $b_j\,(j = 0, ..., M)$ are constants to be determined later, $a_N \neq 0, b_M \neq 0$ and $\psi(\eta)$ is solution of equation:

$$(\Psi'(\eta)) = (\Psi(\eta))^2 + A \tag{3.1.5}$$

How have the following solutions:

Case1: When $A > 0$

- $\Psi(\eta) = \sqrt{A}\tan(\sqrt{A}\eta)$
- $\Psi(\eta) = -\sqrt{A}\cot(\sqrt{A}\eta)$

Case2: When $A < 0$

- $\Psi(\eta) = -\sqrt{-A}\tanh(\sqrt{-A}\eta)$

- $\Psi(\eta) = -\sqrt{-A}\coth(\sqrt{-A}\eta)$

Case3: When $A = 0$

- $\Psi(\eta) = -\frac{1}{\eta+c}$, c constant

- Step 3: To compute the positive integer number N and M in Eq.(3.1.4) we balance the highest order linear term and the highest order nonlinear term occurring in Eq.(3.1.3) This completes the determination of the value of N and M.

  Substituting Eq.(3.1.4) into Eq.(3.1.3) and equating the coefficients of the same powers of $\Psi(\eta)$ to zero, we obtain a system of algebraic equations, solving this system to get the unknown parameters $a_i\,(i = 0, ..., N)$, $b_j\,(j = 0, ..., M)$, $k, l, h, ...$ and v. We obtain the exact solution of Eq. (3.1.1)

3.2 Travelling wave solutions of the PHI-four equation

In this section, we Consider the PHI-four equation of the form

$$D_{tt}^{2\alpha}u - D_{xx}^{2\alpha}u + m^2u + \lambda u^3 = 0 \tag{3.2.1}$$

α is a parameter describing the order of the fractional space and time derivative. Let $\lambda \neq 0$ Using the fractional complex transform

$$\eta = \frac{kx^\alpha}{\alpha} - \frac{ct^\alpha}{\alpha}; \quad k \neq 0, \quad c \neq 0 \tag{3.2.2}$$

,

Eq.(3.2.1) is reduced to an ODE

$$(c^2 - k^2)u'' + m^2u + \lambda u^3 = 0 \tag{3.2.3}$$

Balancing the highest order derivative u'' with the non linear term u^3 in Eq.(3.2.3), we get $N = M + 1$. If we choose $M = 1$ then $N = 2$, hence the exact solution of Eq.(3.2.3) is given as follows:

$$u(\eta) = \frac{a_0 + a_1(\Psi(\eta)) + a_2(\Psi(\eta))^2}{b_0 + b_1(\Psi(\eta))} \tag{3.2.4}$$

Substituting Eq.(3.2.4) and its derivatives using Eq.(3.1.5) into Eq.(3.2.3), collecting the coefficients of each power of $(\Psi(\eta))^i$, $(i = 0, \ldots, N)$, $(i = 0, \ldots, M)$, and set them to zero, we obtain a system of algebraic equations. Solving this system of algebraic equations with the aid of Maple, we obtain the following families of solutions as:

Family 1

$$a_0 = \pm A\frac{mb_1}{\sqrt{A\lambda}}; a_1 = \mp\frac{mb_0}{\sqrt{A\lambda}}; a_2 = 0; c = \pm\sqrt{-\frac{1}{2}(\frac{-2Ak^2+m^2}{A})}$$

Then solution of Eq. (3.2.3) is:

$$u_1(\eta) = \mp\frac{m}{\sqrt{A\lambda}}\frac{-Ab_1+b_0\left(\Psi(\eta)\right)}{b_0+b_1\left(\Psi(\eta)\right)}$$

hence the exact solutions of Eq.(3.2.1) are:

- $u_{1,1}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{A}}\frac{t^\alpha}{\alpha}\right) = \frac{\frac{m}{\sqrt{A\lambda}}[-Ab_1-b_0\sqrt{A}\cot(\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{A}}\frac{t^\alpha}{\alpha}\right)\sqrt{A})]}{b_0-b_1\sqrt{A}\cot(\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{A}}\frac{t^\alpha}{\alpha}\right))\sqrt{A}}; A>0$

- $u_{1,2}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{A}}\frac{t^\alpha}{\alpha}\right) = \frac{m}{\sqrt{A\lambda}}\frac{-Ab_1+b_0\left(\sqrt{A}\tan(\sqrt{A}\eta)\right)}{b_0+b_1\left(\sqrt{A}\tan(\sqrt{A}\eta)\right)}; A>0$

- $u_{1,3}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{A}}\frac{t^\alpha}{\alpha}\right) = \frac{m}{\sqrt{A\lambda}}\frac{Ab_1+b_0\sqrt{-A}\tanh\left(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{A}}\frac{t^\alpha}{\alpha}\right)\right)}{-b_0+b_1\sqrt{-A}\tanh\left(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{A}}\frac{t^\alpha}{\alpha}\right)\right)}; A<0$

- $u_{1,4}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{A}}\frac{t^\alpha}{\alpha}\right) = \frac{m}{\sqrt{A\lambda}}\frac{Ab_1+b_0\sqrt{-A}\coth\left(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{A}}\frac{t^\alpha}{\alpha}\right)\right)}{-b_0+b_1\sqrt{-A}\coth\left(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{A}}\frac{t^\alpha}{\alpha}\right)\right)}; A<0$

Family 2

$$a_0 = mb_1\sqrt{\frac{A}{\lambda}}; a_1 = 0; a_2 := 0; b_0 = 0; c := \pm\sqrt{\frac{2Ak^2-m^2}{2A}}$$

Then solution of Eq. (3.2.3) is:

$$u_2(\eta) = \frac{m\sqrt{\frac{A}{\lambda}}}{Q(\eta)}$$

hence the exact solutions of Eq.(3.2.1) are:

- $u_{2.1}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right) = \frac{m\sqrt{\frac{A}{\lambda}}}{\sqrt{A}\tan\left(\sqrt{A}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right)\right)}; \ A > 0$

- $u_{2.2}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right) = \frac{m\sqrt{\frac{A}{\lambda}}}{-\sqrt{A}\cot\left(\sqrt{A}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right)\right)}; \ A > 0$

- $u_{2.3}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right) = \frac{m\sqrt{\frac{A}{\lambda}}}{-\sqrt{-A}\tanh\left(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right)\right)}; \ A < 0$

- $u_{2.4}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right) = \frac{m\sqrt{\frac{A}{\lambda}}}{-\sqrt{-A}\coth\left(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right)\right)}; \ A < 0$

Family 3

$$a_0 = 0; a_1 = \pm b_0\frac{\sqrt{-2\lambda(c^2-k^2)}}{\lambda}; a_2 = 0; A = 0; m = 0;$$

Then solution of Eq. (3.2.3) is:

$$u_3(\eta) = \pm\frac{b_0\sqrt{-2\lambda(c^2-k^2)}Q(\eta)}{\lambda(b_0 + b_1Q(\eta))}.$$

Hence, the exact solutions of Eq.(3.2.1) are:

- $u_{3.1}(k\frac{x^\alpha}{\alpha} - c\frac{t^\alpha}{\alpha}) = \pm\frac{b_0\sqrt{-2\lambda(c^2-k^2)}}{\lambda\left(b_0(k\frac{x^\alpha}{\alpha} - c\frac{t^\alpha}{\alpha} + c) - b_1\right)}, \ b_0, b_1 \in \mathbb{R}$

Family 4

$$a_0 = 0; a_1 = \pm\frac{mb_0}{\sqrt{A\lambda}}; a_2 = \pm\frac{mb_1}{\sqrt{A\lambda}}; c = \pm\sqrt{\frac{2Ak^2-m^2}{2A}}.$$

Then, the solution of Eq. (3.2.3) is:

$$u_4(\eta) = \frac{m}{\sqrt{A\lambda}}Q(\eta)$$

Thus, the exact solutions of Eq.(3.2.1) are:

- $u_{4.1}(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}) = \frac{m}{\sqrt{A\lambda}}\sqrt{A}\tan(\sqrt{A}(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha})); A > 0$

- $u_{4.2}(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}) = -\frac{m}{\sqrt{A\lambda}}\sqrt{A}\cot(\sqrt{A}(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha})); A > 0$

- $u_{4.3}(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}) = -\frac{m}{\sqrt{A\lambda}}\sqrt{-A}\tanh(\sqrt{-A}(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha})), A < 0$

- $u_{4.4}(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}) = -\frac{m}{\sqrt{A\lambda}}\sqrt{-A}\coth(\sqrt{-A}(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha})), A < 0$

- $u_{4.5}(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}) = -\frac{m}{\sqrt{A\lambda}}\frac{1}{k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}+c}; A = 0; c$ constant

Family 5

$$a_1 = 0; a_2 = \frac{a_0}{A}; b_0 = 0; b_1 = \pm\frac{a_0\sqrt{-2\lambda}}{m\sqrt{A}}; c = \pm\sqrt{\frac{4Ak^2+m^2}{4A}}.$$

Then solution of Eq. (3.2.3) is:

$$u_5(\eta) = \pm\frac{m(A+\Psi^2(\eta))}{A\sqrt{2}}\sqrt{\frac{-\lambda}{A}}\Psi(\eta).$$

Hence the exact solutions of Eq.(3.2.1) are:

- $u_{5.1}(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{4Ak^2+m^2}{4A}}\frac{t^\alpha}{\alpha}) = \frac{m}{\sqrt{-2A\lambda}\sin\left(\sqrt{A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{4Ak^2+m^2}{4A}}\frac{t^\alpha}{\alpha}\right)\right)\cos\left(\sqrt{A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{4Ak^2+m^2}{4A}}\frac{t^\alpha}{\alpha}\right)\right)}; A > 0$

- $u_{5.2}(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{4Ak^2+m^2}{4A}}\frac{t^\alpha}{\alpha}) = \frac{m}{\sqrt{2\lambda}\sinh\left(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{4Ak^2+m^2}{4A}}\frac{t^\alpha}{\alpha}\right)\right)\cosh\left(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{4Ak^2+m^2}{4A}}\frac{t^\alpha}{\alpha}\right)\right)}; A < 0$

Family 6

$$a_1 = 0; a_2 = -\frac{a_0}{A}; b_0 = 0; b_1 = \pm\frac{2a_0\sqrt{\lambda}}{m\sqrt{A}}; c = \pm\sqrt{\frac{8Ak^2-m^2}{8A}}.$$

Then, the solution of Eq. (3.2.3) is:

$$u_6(\eta) = \pm\frac{m(A-\Psi^2(\eta))}{2A\sqrt{\frac{\lambda}{A}}\Psi(\eta)}$$

Hence the exact solutions of Eq.(3.2.1) are:

- $u_{6.1}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{8Ak^2-m^2}{8A}}\frac{t^\alpha}{\alpha}\right) = \frac{m\left(-1+\cot^2\left(\sqrt{A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{8Ak^2-m^2}{8A}}\frac{t^\alpha}{\alpha}\right)\right)\right)}{2\sqrt{\lambda}\cot\left(\sqrt{A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{8Ak^2-m^2}{8A}}\frac{t^\alpha}{\alpha}\right)\right)} \; ; A > 0$

- $u_{6.2}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{8Ak^2-m^2}{8A}}\frac{t^\alpha}{\alpha}\right) = \frac{m\left(-1+\tan^2\left(\sqrt{A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{8Ak^2-m^2}{8A}}\frac{t^\alpha}{\alpha}\right)\right)\right)}{2\sqrt{\lambda}\tan\left(\sqrt{A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{8Ak^2-m^2}{8A}}\frac{t^\alpha}{\alpha}\right)\right)} \; ; A > 0$

- $u_{6.3}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{8Ak^2-m^2}{8A}}\frac{t^\alpha}{\alpha}\right) = \frac{m\left(1+\tanh^2\left(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{8Ak^2-m^2}{8A}}\frac{t^\alpha}{\alpha}\right)\right)\right)}{2\sqrt{-\lambda}\tanh\left(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{8Ak^2-m^2}{8A}}\frac{t^\alpha}{\alpha}\right)\right)}$; $A < 0$

- $u_{6.4}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{8Ak^2-m^2}{8A}}\frac{t^\alpha}{\alpha}\right) = \frac{m\left(1+\coth^2\left(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{8Ak^2-m^2}{8A}}\frac{t^\alpha}{\alpha}\right)\right)\right)}{2\sqrt{-\lambda}\coth\left(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{8Ak^2-m^2}{8A}}\frac{t^\alpha}{\alpha}\right)\right)}$; $A < 0$

Family 7

$$a_0 = \mp\frac{mb_1A}{\sqrt{A\lambda}}; a_1 = \pm\frac{mb_0}{\sqrt{A\lambda}}; a_2 = 0; ; c = \pm\sqrt{\frac{2Ak^2-m^2}{2A}}.$$

Then, the solution of Eq. (3.2.3) is:

$$u_7(\eta) = \pm\frac{m(-Ab_1 + b_0\Psi(\eta))}{\sqrt{A\lambda}(b_0 + b_1\Psi(\eta))}$$

Hence the exact solutions of Eq.(3.2.1) are:

- $u_{7.1}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right) = \pm\frac{m(Ab_1+b_0\sqrt{A}\cot(\sqrt{A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right)))}{\sqrt{A\lambda}(-b_0+b_1\sqrt{A}\cot(\sqrt{A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right)))}$; $A > 0$

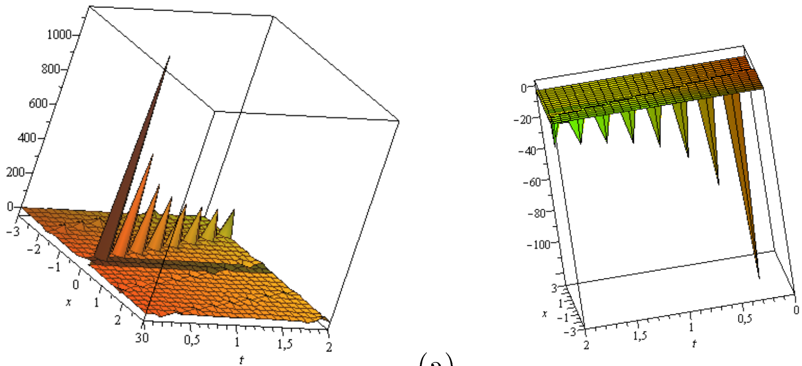
- $u_{7.2}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right) = \pm\frac{m(-Ab_1+b_0\sqrt{A}\tan(\sqrt{A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right)))}{\sqrt{A\lambda}(b_0+b_1\sqrt{A}\tan(\sqrt{A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right)))}$; $A > 0$

- $u_{7.3}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right) = \pm\frac{m(Ab_1+b_0\sqrt{-A}\tanh(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right)))}{\sqrt{A\lambda}(-b_0+b_1\sqrt{-A}\tanh(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right)))}$; $A < 0$

- $u_{7.4}\left(k\frac{x^\alpha}{\alpha} \pm \sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right) = \pm\frac{m(Ab_1+b_0\sqrt{-A}\coth(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right)))}{\sqrt{A\lambda}(-b_0+b_1\sqrt{-A}\coth(\sqrt{-A}\left(k\frac{x^\alpha}{\alpha}\pm\sqrt{\frac{2Ak^2-m^2}{2A}}\frac{t^\alpha}{\alpha}\right)))}$; $A < 0$

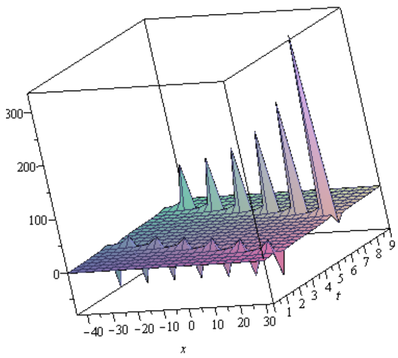
3.2.1 Graphical illustrations

In this section, some of our obtained traveling wave solutions of PHI four Equation, by choosing suitible values of the paramet are represented in the following figures with the help of software Maple :



(a) (b)

Figure 3.1: Profile of the solution $u_{6.1}(\eta)$ for $m = -2; \lambda = 2; k = 22; A = 22$ and . (b) Profile of the solution $u_{6.3}(\eta)$ for $m = -11; \lambda = -11; k = 44; A = -11.$.



(a)

Figure 3.2: Profile of the solution $u_{7.2}(\eta)$ for $m = 1; \lambda = 1; k = 1; A = 1; b_0 = 1; b_1 = 1$

3.3 Travelling wave solutions of the Nonlinear Time Fractional JM equation

In this section, we Consider the Time Fractional Jimbo Miwa equation of the form

$$v_{xxxy} + 3v_y v_{xx} + 3v_x v_{xy} + 2D_t^{\alpha} v_y - 3v_{xz} = 0 \tag{3.3.1}$$

where $0 < \alpha \leq 1$,

which is a model of nonlinear physic describes the dynamics of a certain (3+1)-dimensional waves in physics. Transformation the NLFPD equation into the ODE with integer order by using the following conformable fractional derivative

$$v\left(x,y,z,t\right)=v\left(\eta\right),\eta=x+y+z-c\frac{t^{\alpha}}{\alpha} \tag{3.3.2}$$

Where c is a constant to be determined later, yields:

$$v'''' + 6v'v'' - (2c+3)v'' = 0 \tag{3.3.3}$$

Integrating equation (3.3.3) with respect to η once and neglecting the constant of integration, we have:

$$v''' + 3(v')^2 - (2c+3)v' = 0 \tag{3.3.4}$$

by homogenenous balance between $(v')^2$ and v''' in Eq. (3.3.4), we attain $N-M+3=2N-2M+2$ hence $N=M+1$. If we choose $M=1$then $N=2$. Consequently, Eq. (3.3.4) has the formal solution

$$v(\eta)=\frac{a_0+a_1\Psi\left(\eta\right)+a_2\Psi^2\left(\eta\right)}{b_0+b_1\Psi\left(\eta\right)}, \tag{3.3.5}$$

where $\Psi\left(\eta\right)$ is solution of

$$(\Psi'(\eta)) = (\Psi(\eta))^2 + A$$

Substituting Eq.(3.3.5) along with its derivatives into Eq.(3.3.4) and equating the coefficients of like powers of $\Psi(\eta)$ and set them to zero, we obtain a system of algebraic equations. Solving this system of algebraic equations using symbolic computation package, we obtain the following families of solutions as:

Family 1:

$$a_0=0; a_2=-2b_1; b_0=0; A=-\frac{1}{2}c-\frac{3}{4}$$

Then solution of Eq.(3.3.4) is:

$$v_1(\eta)=-\frac{-a_1+2b_1\psi(\eta)}{b_1}, \text{where } a_1 \textit{ and } b_1 \text{are free parameters.}$$

hence the exact solution of Eq. (3.3.1) is:

- If $A>0$, $(c<\frac{-3}{2})$ then $v_{1,1}(\eta)=\frac{a_1\sqrt{-2c-3}-2b_1\cot(\frac{\eta\sqrt{-2c-3}}{2})c-3b_1\cot(\frac{\eta\sqrt{-2c-3}}{2})}{b_1\sqrt{-2c-3}}$
- If $A>0$, $(c<\frac{-3}{2})$ then $v_{1,2}(\eta)=\frac{a_1\sqrt{-2c-3}+2b_1\tan(\frac{\eta\sqrt{-2c-3}}{2})c+3b_1\tan(\frac{\eta\sqrt{-2c-3}}{2})}{b_1\sqrt{-2c-3}}$
- If $A<0$, $(c>\frac{-3}{2})$ then $v_{1,3}(\eta)=\frac{a_1\sqrt{2c+3}+2b_1\tanh(\frac{\eta\sqrt{2c+3}}{2})c+3b_1\tanh(\frac{\eta\sqrt{2c+3}}{2})}{b_1\sqrt{2c+3}}$
- If $A<0$, $(c>\frac{-3}{2})$ then $v_{1,4}(\eta)=\frac{a_1\sqrt{2c+3}-2b_1\coth\frac{\eta\sqrt{2c+3}}{2})c-3b_1\coth\frac{\eta\sqrt{2c+3}}{2})}{b_1\sqrt{2c+3}}$

- If $A = 0$, $(c = \frac{-3}{2})$ then $v_{1,5}(\eta) = \frac{2a_1\eta - 3a_1 + 4b_1}{(2\eta - 3)b_1}$,

where $\eta = x + y + z - c\frac{t^\alpha}{\alpha}$.

Family 2:

$$A := 0; a_2 = 0; b_0 = 0; a_0 = \frac{-(2c+3)b_1}{3}.$$

Then, the solution of Eq.(3.3.4) is:

$$v_2(\eta) = \frac{-2\,b_1c - 3\,b_1 + 3\,a_1\psi\,(\eta)}{3b_1\psi\,(\eta)}, \text{where } a_1;\ b_1 \ \textit{and}\ c \text{ are free parameters.}$$

Hence the exact solution of Eq. (3.3.1) is:

- $v_{2,1}(\eta) = \frac{2b_1c\eta + 2b_1c^2 + 3b_1\eta + 3b_1c + 3a_1}{3b_1}$

Where $\eta = x + y + z - c\frac{t^\alpha}{\alpha}$

Family 3:

$$a_2 = 0; a_1 = \frac{2cb_1^2 + 3b_1^2 + 2b_1a_0 - 4b_0^2}{2b_0}; A = -\frac{c}{2} - \frac{3}{4}.$$

Then solution of Eq.(3.3.4) is:

$$v_3(\eta) = \frac{2a_0b_0 + 2\psi\,(\eta)\,c{b_1}^2 + 3\psi\,(\eta)\,{b_1}^2 + 2\psi\,(\eta)\,b_1a_0 - 4\psi\,(\eta)\,{b_0}^2}{2b_0\,(b_0 + b_1\psi\,(\eta))}$$

where a_0, b_0 and b_1 are free parameters, hence the exact solution of Eq. (3.3.1) is:

- If $A > 0$, $v_{3,1}(\eta) = \frac{\left(-\sqrt{-2c-3}c{b_1}^2 - \frac{3}{2}\sqrt{-2c-3}{b_1}^2 - \sqrt{-2c-3}b_1a_0 + 2\sqrt{-2c-3}{b_0}^2\right)\cot\left(\frac{1}{2}\eta\sqrt{-2c-3}\right) + 2a_0b_0}{2b_0\left(b_0 - \frac{1}{2}\sqrt{-2c-3}\cot\left(1/\eta\sqrt{-2c-3}\right)b_1\right)}$

- If $A > 0$, $v_{3,2}(\eta) = \frac{\left(\sqrt{-2c-3}c{b_1}^2 + \frac{3}{2}\sqrt{-2c-3}{b_1}^2 + \sqrt{-2c-3}b_1a_0 - 2\sqrt{-2c-3}{b_0}^2\right)\tan\left(\frac{1}{2}\eta\sqrt{-2c-3}\right) + 2a_0b_0}{2b_0\left(b_0 + \frac{1}{2}\tan\left(\frac{1}{2}\eta\sqrt{-2c-3}\right)\sqrt{-2c-3}b_1\right)}$

- If $A < 0$, $v_{3,3}(\eta) = \frac{\left(-\sqrt{2c+3}c{b_1}^2 - \frac{3}{2}\sqrt{2c+3}{b_1}^2 - \sqrt{2c+3}b_1a_0 + 2\sqrt{2c+3}{b_0}^2\right)\tanh\left(\frac{1}{2}\eta\sqrt{2c+3}\right) + 2a_0b_0}{2b_0\left(b_0 - \frac{1}{2}\tanh\left(\frac{1}{2}\eta\sqrt{2c+3}\right)\sqrt{2c+3}b_1\right)}$

- If $A < 0$, $v_{3,4}(\eta) = \frac{\left(-\sqrt{2c+3}c{b_1}^2 - \frac{3}{2}\sqrt{2c+3}{b_1}^2 - \sqrt{2c+3}b_1a_0 + 2\sqrt{2c+3}{b_0}^2\right)\coth\left(\frac{1}{2}\eta\,\sqrt{2c+3}\right) + 2a_0b_0}{2b_0\left(b_0 - \frac{1}{2}\coth\left(\frac{1}{2}\eta\,\sqrt{2c+3}\right)\sqrt{2c+3}b_1\right)}$

- If $A = 0$, $v_{3,5}(\eta) = \frac{2a_0b_0\eta - 3a_0b_0 - 2b_1a_0 + 4{b_0}^2}{(2b_0\eta - 3b_0 - 2b_1)b_0}$

Where $\eta = x + y + z - c\frac{t^\alpha}{\alpha}$

Family 4

$$a_0 = \frac{ca_2}{8} + \frac{3a_2}{16}; b_1 = -\frac{a_2}{2}; A := -\frac{c}{8} - \frac{3}{16}; b_0 = 0.$$

Then solution of Eq.(3.3.4) is:

$v_4(\eta) = -\dfrac{2ca_2 + 3a_2 + 16a_1Q(\eta) + 16a_2(Q(\eta))^2}{8a_2Q(\eta)}$.where a_1 *and* a_2 are free parameters.

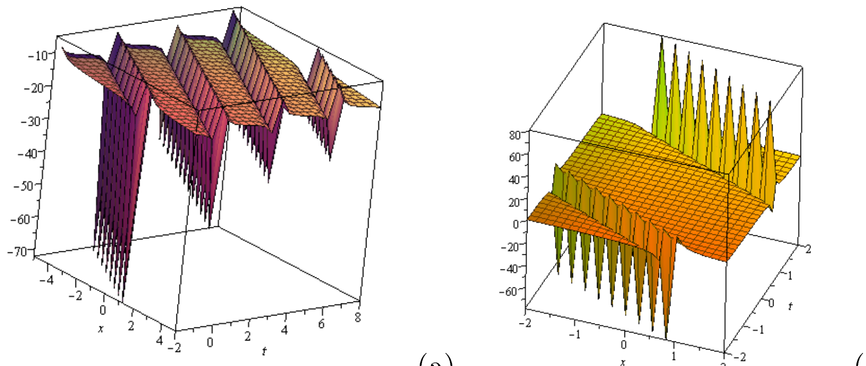
Hence the exact solution of Eq. (3.3.1) is:

- If $A > 0$, $v_{4,1}(\eta) = \frac{2ca_2+3a_2-4a_1\sqrt{-2c-3}\cot\left(1/4\eta\sqrt{-2c-3}\right)+a_2(-2c-3)\left(\cot\left(\frac{1}{4}\eta\sqrt{-2c-3}\right)\right)^2}{2a_2\sqrt{-2c-3}\cot\left(1/4\eta\sqrt{-2c-3}\right)}$
- If $A > 0$, $v_{4,2}(\eta) = -\frac{2\,ca_2+3a_2+16\,a_1\tan\left(\eta\sqrt{A}\right)\sqrt{A}+16a_2\left(\tan\left(\eta\sqrt{A}\right)\right)^2A}{8a_2\tan\left(\eta\sqrt{A}\right)\sqrt{A}}$
- If $A < 0$, $v_{4,3}(\eta) = -\frac{-2ca_2-3\,a_2+16a_1\tanh\left(\eta\sqrt{-A}\right)\sqrt{-A}+16a_2\left(\tanh\left(\eta\sqrt{-A}\right)\right)^2A}{8a_2\tanh\left(\eta\sqrt{-A}\right)\sqrt{-A}}$
- If $A < 0$, $v_{4,4}(\eta) = \frac{2ca_2+3a_2-16a_1\coth\left(\eta\sqrt{-A}\right)\sqrt{-A}-16a_2\left(\coth\left(\eta\sqrt{-A}\right)\right)^2A}{8a_2\coth\left(\eta\sqrt{-A}\right)\sqrt{-A}}$
- If $A = 0, (c = -\frac{3}{2})$r then: $v_{4,5}(\eta) = -2\frac{2a_1\eta-3a_1-2a_2}{a_2(2\eta-3)}$

Where $\eta = x + y + z - c\frac{t^\alpha}{\alpha}$

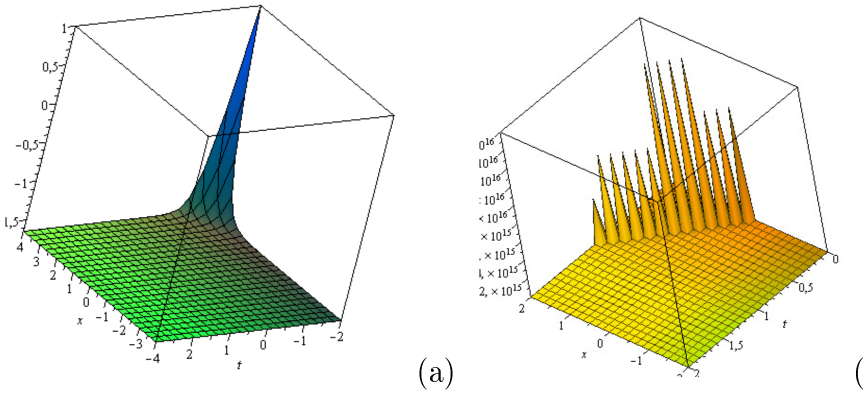
3.3.1 Graphical illustrations

In this section, some of our obtained traveling wave solutions of JM Equation, by choosing suitible values of the paramet are represented in the following figures with the help of software Maple:



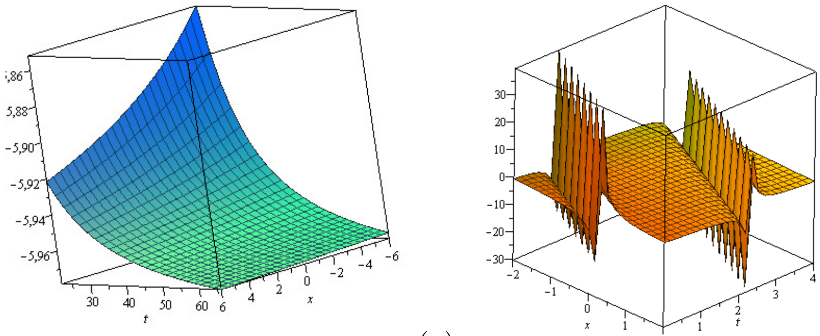
(a) (b)

Figure 3.3: Profile of the solution (JM equation) $v(1,1)(\eta)$ for $b_1 = 1; c = -2; a_1 = -12; y = 12; z = -3$ and . (b) Profile of the solution (JM equation) $v(1,2).eps(\eta)$ for $b_1 = -1; c = -2; a_1 = -1; y = -1, z = 1$.



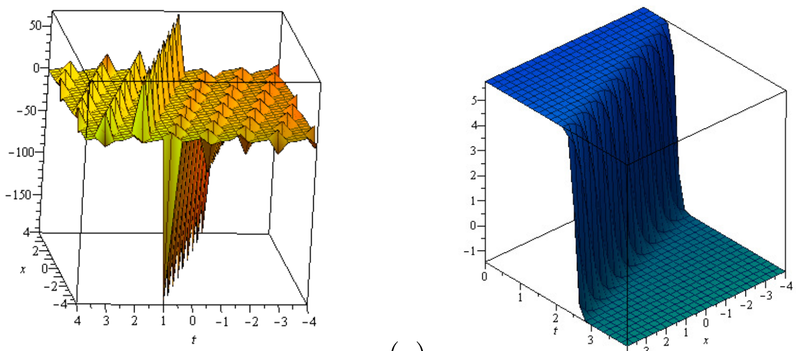
(a) (b)

Figure 3.4: Profile of the solution (JM equation) $v(1,3)(\eta)$ for $b_1 = -4; c = 2; a_1 = -4; y = -4; z = -4$ and . (b) Profile of the solution (JM equation) $v(1,4).eps(\eta)$ for $b_1 = 1; c = 2; a_1 = -1; y = 1; z = -1$.



(a) (b)

Figure 3.5: Profile of the solution (JM equation) $v(1,5)(\eta)$ for $b_1 = 1; c = -3/2; a_1 = -6; y = -6; z = -6$ and . (b) Profile of the solution (JM equation) $v(3,1).eps(\eta)$ for $b_0 = -9; b_1 = 0; c = -2; a_0 = 2; y = 3; z = 3$.



(a) (b)

Figure 3.6: Profile of the solution (JM equation) $v(3,2)(\eta)$ for $b_0 = 1; b_1 = 1; c = -3; a_0 = 0; y = 0; z = 0$ and . (b) Profile of the solution (JM equation) $v(3,3).eps(\eta)$ for $b_0 = 3; b_1 = 1; c = 5; a_0 = 0; y = 2; z = 7$.

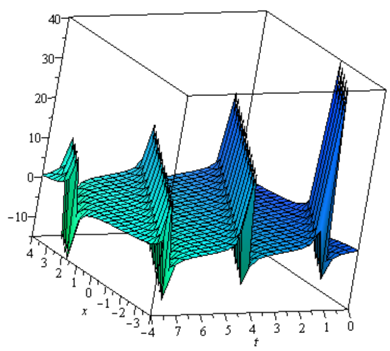


Figure 3.7: solution $v_(3,4)(\eta)$ of JM equation for $b_0 = 12; b_1 = 1; c = -2; a_0 = 0; y = -12; z = -12$.

3.4 Conclusion

The new technique has successfully been applied to extract some new solutions for PHI-four and Jimbo Miwa equations. The acquired solutions are rational; trigonometric and hyperbolic solutions. Our study show that the proposed technique is a robust analytical appliance for solving NPDEs arising in natural science.

Finally, the presented method can be applied for further physical models emerging in applied science and new physics.

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