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**By:** Ouahiba Cherikh,


**Theme**

# Generalized Lyapunov and Hartman type Inequalities and their applications for linear and nonlinear fractional differential equations under different conditions

Publicly defended on December 11, 2025, in front of committee members:

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# *Dedication*

In loving memory of my father

**Mohand Arezki CHERIKH**,

whose constant encouragement and faith in me

made this achievement possible.

**Ouahiba**

# *Thanks*

Thanks First and foremost, **ALLAH** who enlightened me on the path of knowledge. Help me perform this duty and enable me to complete this work... Praise be to **ALLAH**.

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Last but not the least, I would like to thank my family for supporting me spiritually throughout writing this thesis and my life in general.

# *Abstract*

In this thesis, we focus on the study of classes of generalized fractional differential equations that may be subject to various types of conditions.

We first introduce the generalized Caputo-type fractional derivative, referred to as the $\rho$-Caputo derivative, of order $\alpha \in (1,2)$ with $\rho > 0$. This operator has the major advantage of unifying, within a single analytical framework, the classical Riemann–Liouville and Hadamard fractional integrals as well as the Caputo and Caputo–Hadamard fractional derivatives.

We then conduct a detailed analysis of the properties of the Green's function associated with the considered equations. This function plays a key role in deriving sharp estimates and enables the rigorous construction of new integral inequalities. Based on these tools, we establish generalized Lyapunov- and Hartman-type inequalities that are valid for both linear and nonlinear cases. The application of these inequalities leads to significant results, thereby providing effective criteria for the qualitative analysis of solutions.

The obtained results are not merely an extension of existing works; rather, they offer a genuine theoretical consolidation by proposing a unified framework capable of encompassing several particular classes of fractional differential equations that had previously been treated separately in the literature.

**Keywords:** *Generalized Caputo derivatives, Green's Function, fractional differential equations, mixed nonlinearities, generalized Hartman-type inequality, generalized Lyapunov-type inequality.*

# *Résumé*

Dans cette thèse, nous nous intéressons à l'étude de classes d'équations différentielles fractionnaires généralisées pouvant être soumises à diverses conditions. Dans un premier temps, nous présentons la dérivée fractionnaire généralisée de type Caputo, dite $\rho$-Caputo, d'ordre $\alpha \in (1,2)$ avec $\rho > 0$. Cet opérateur présente l'intérêt majeur d'unifier, au sein d'un cadre analytique unique, les intégrales fractionnaires classiques de Riemann–Liouville et de Hadamard, ainsi que les dérivées fractionnaires de Caputo et de Caputo–Hadamard. Nous entamons ensuite une analyse détaillée des propriétés de la fonction de Green associée aux équations considérées. Celle-ci joue un rôle central dans l'obtention d'estimations précises et permet la construction de nouvelles inégalités intégrales. Sur la base de ces outils, nous établissons des inégalités généralisées de type Lyapunov et Hartman, valables pour les équations en question, aussi bien dans le cas linéaire que non linéaire. L'application de ces inégalités conduit à des résultats significatifs, offrant ainsi des critères efficaces pour l'analyse qualitative des solutions. Les résultats obtenus ne constituent pas une simple extension des travaux antérieurs ; ils apportent une véritable consolidation théorique en proposant un cadre unifié capable d'englober plusieurs cas particuliers d'équations différentielles fractionnaires qui, jusqu'à présent, étaient traités séparément dans la littérature.

**Les mots clés:** *Dérivées de Caputo généralisées, Fonction de Green, Équations différentielles fractionnaires, Non-linéarités mixtes, Inégalité généralisée de type Hartman, Inégalité généralisée de type Lyapunov.*

# ملخص

## دراسة بعض الخصائص النوعية لحلول فئات مختلفة من المعادلات التفاضلية المجردة الكسرية

في رسالة الدكتوراه هذه نهتم بدراسة بعض الخصائص النوعية للحل بالنسبة لثلاث فئات من المعادلات التفاضلية المجردة والمعممة والتي يمكن أن تخضع لعدة شروط إبتدائية محلية (على التوالي، غير محلية) في فضاء باناخ، وذلك عن طريق نظرية شبه الزمر باستخدام المؤثرات الخطية المحدودة (على التوالي، غير المحدودة) المولِدة المتناهية الصغر لشبه زمرة.

لهذا الغرض ، أولاً ، نتفحص نموذج كابيتو (ρ-كابيتو) الكسري المعمم من الدرجة α حيث α ينتمي للمجال ]0,1[ و ρ موجب تماما. ثانيًا ، نقوم أيضًا بدراسة نموذج φ-كابيتو بوزن حيث α ينتمي للمجال]0,1[ (على التوالي، α ينتمي للمجال ]1,2[) و φ هي دالة متزايدة تماما قابلة للإشتقاق و العبارة الغير خطية هي دالة تتضمن مؤثرات فولتيرا وفريدهولم التكاملية المعممة.

لقد أثبتنا وجود المؤثر الحل ، بعد ذلك ، قمنا بإشتقاق حلاً خفيفاً بدلالة شبه الزمر لكل مشاكل كوشي المجردة الكسرية الناتجة عن ذلك. النهج المستخدم هو تحويل مشكلة كوشي المجردة إلى مؤثر تكاملي مكافئ بحيث يتم تقليص مشكلة وجود الحلول إلى البحث عن النقاط الثابتة للمؤثر التكاملي. نحن مهتمون بالوجود والوحدانية والتعلق بوسيط وإستقرار أولام-هاير للحل الخفيف. يتم الحصول على النتائج من خلال استخدام نظرية حساب التفاضل والتكامل الكسري ، ونظرية شبه االزمر ، ونظريات النقطة الثابتة ، والعلاقات التراجعية الرتيبة ، وطريقة الحل العلوي والسفلي، ومفهوم قياس كوراتوفسكي لعدم التراص ، ومتراجحة جرونويل ، وتحويل لابلاس والدوال الخاصة.

# *Summary*

# *CHAPTER 1*

# *General introduction*

## Summary

## 1.1 Introduction

The subject of fractional calculus is a generalization of ordinary differentiation and integration on an arbitrary order that can be noninteger. Very recently it was recognized that the fractional calculus arise naturally in various fields of science. In consequence, there are several contributions focusing on the different definitions of fractional derivatives such as Riemann-Liouville, Hadamard, Grünwald-Letnikov, Riesz, Caputo, Marchaud, Weyl, Hilfer, Caputo and Fabrizio, Atangana and Baleanu and others, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the references therein.

In [12], Kiryakova proposed a theory of a generalized fractional calculus and their applications. One of the proposed generalizations of the fractional calculus operators which has wide applications is the $\rho$–fractional operator (generalized fractional operator). This notion is referred to as the fractional integral which combines the Riemann-Liouville and the Hadamard integral into a single form [5, 6, 7]. In 2015, D. Anderson et.al. [13] studied the properties of the Katugampola fractional derivative with potential application in quantum mechanics.

Important property of fractional differential equations is that they are nonlocal : a function's fractional derivative at a particular point is not just influenced by the function's behavior near that point. This novelty arising in fractional but not classical calculus has led to many applications in fields such as control theory and dynamical systems.

In recent contribution, the authors in [7, 2012], introduced a new definition of the fractional derivative. This new derivative has gained widely attention and attracted a large number of scientists in different scientific fields for the exploration of diverse topics.

On the other hand, integral inequalities play a significant role in the study of differential and integral equations. In particular, there has been a continuous interest in the generalized Lyapunov inequality by several authors.

The classical inequality originally due to Lyaponov [14, 15, 1893] states that if a nontrivial solution $x(t)$ of the Hill's linear differential equation

$$x''(t) + p(t)x(t) = 0, \tag{1.1.1}$$

with

$$x(a) = x(b) = 0, \tag{1.1.2}$$

has two zeros $a < b$ and $x(t) \neq 0$, for $t \in (a, b)$ then

$$\int_a^b p(\tau)\, d\tau > \frac{4}{b-a}, \tag{1.1.3}$$

where $p$ is a real valued and continuous on a nontrivial interval of reals The constant $4$ in the above inequality is sharp so that it cannot be replaced by a larger number.

It is worth mentioning that inequality (1.1.3) has found many practical applications in differential equations (oscillation theory, asymptotic theory, stability, disconjugacy, eigenvalue problems, etc.), see [16, 17] and references therein.

It was first noticed by Wintner [18, 1951] and later by seperal other authors that (1.1.3) can be improped by replacing $p(t)$ by the non negative part t,

$$p^{+}(t) = \frac{1}{2}\left(|p(t)| + p(t)\right) = \max\left(p(t), 0\right), \tag{1.1.4}$$

to become

$$\int_a^b p^{+}(\tau)\, d\tau > \frac{4}{b-a}. \tag{1.1.5}$$

In particular, in [19, 1951], Hartman and Wintner proved that if (1.1.1-1.1.2) hape a non-trivial solution, then

$$\int_a^b (b-\tau)(\tau-a)\, p^{+}(\tau)\, d\tau > (b-a). \tag{1.1.6}$$

In fact, for $a \leq \tau \leq b$, by the inequality

$$(b-\tau)(\tau-a) \leq \left(\frac{b-a}{2}\right)^2, \tag{1.1.7}$$

condition (1.1.7) is a generalization of condition (1.1.6). The classical result of Lyapunov is usually connected with the disconjugacy of (1.1.1), i.e., the inequality

$$\int_a^b p^{+}(\tau)\, d\tau \leq \frac{4}{b-a}, \tag{1.1.8}$$

implies that (1.1.1) is disconjugate in $[a, b]$.

Hartman [20] generalized the classical Lyapunov inequality for the linear differential equation

$$(p(t)x'(t))' + q(t)x(t) = 0, \tag{1.1.9}$$

as follows.

**Theorem 1.1.1** ([20]). *Let $p(t) > 0$. If $x(t)$ is a nontrivial solution of (1.1.9) with $x(a) = 0 = x(b)$, where $a, b \in \mathbb{R}$ with $a < b$ and $x(t) \neq 0$ for $t \in (a, b)$, then the following Lyapunov-type inequality holds:*

$$\int_a^b q^+(t)\, dt > \frac{4}{\int_a^b p^{-1}(t)\, dt},$$

*where $q^+(t) := \max\{q(t), 0\}$.*

The results for equation (1.1.9) are worth mentioning due to their contribution to this subject. In [19] it is shown that

$$\int_a^{t_0} q^+(t)\, dt > \frac{1}{t_0 - a}, \qquad \int_{t_0}^b q^+(t)\, dt > \frac{1}{b - t_0},$$

where $t_0 \in (a, b)$ is such that $y'(t_0) = 0$. Hence,

$$\int_b^a q^+(t)\, dt > \frac{1}{t_0 - a} + \frac{1}{b - t_0} = \frac{b - a}{(t_0 - a)(b - t_0)} \geq \frac{4}{b - a}.$$

In [19], the authors obtained

$$\left| \int_a^b q(t)\, dt \right| > \frac{4}{b - a},$$

which implies inequality (1.1.3).

Recently, in [21, 2015] R. P. Agarwal, A. Özbekler have extended the inequality (1.1.5) and (1.1.6) for the second-order non-linear differential equations of the form

$$x''(t) + p(t)x(t)\, x(t)^{\mu-1} + q(t)\, x(t)\, x(t)^{\lambda-1} = f(t),\ t \in (a, b), \tag{1.1.10}$$

where $0 < \lambda < 1 < \mu \leq 2, p, q, f$ are real-valued functions.

**Theorem 1.1.2** (Hartman type inequality)**.** *Let* $x(t)$ *be a nontrivial solution of (1.1.10) satisfying the Dirichlet boundary conditions (1.1.2). If* $x(t) \neq 0$ *in* $(a,b)$*, then the inequality*

$$\left(\int_a^b \frac{p^+(\tau)+q^+(\tau)}{((b-\tau)(\tau-a))^{-1}}d\tau\right)\left(\int_a^b \frac{\mu_0 p^+(\tau)+\lambda_0 q^+(\tau)+|f(\tau)|}{((b-\tau)(\tau-a))^{-1}}d\tau\right) > \left(\frac{b-a}{4}\right)^2, \quad (1.1.11)$$

*holds, where*

$$\mu_0 = (2-\mu)\,2^{2/(\mu-2)}\mu^{\mu/(\mu-2)} > 0 \text{ and } \lambda_0 = (2-\lambda)\,2^{2/(\lambda-2)}\lambda^{\lambda/(\lambda-2)} > 0. \quad (1.1.12)$$

**Theorem 1.1.3** (Lyapunov type inequality)**.** *Let* $x(t)$ *be a nontrivial solution of (1.1.10) satisfying the Dirichlet boundary conditions (1.1.2). If* $x(t) \neq 0$ *in* $(a,b)$*, then the inequality*

$$\left(\int_a^b \left(p^+(\tau)+q^+(\tau)\right)d\tau\right)\left(\int_a^b \left(\mu_0 p^+(\tau)+\lambda_0 q^+(\tau)+|f(\tau)|\right)d\tau\right) > \left(\frac{4}{b-a}\right)^2, \quad (1.1.13)$$

*holds, where the constants* $\mu_0$ *and* $\lambda_0$ *are the same as in (1.1.12).*

There are several ways to introduce the fractional derivatives as a generalization of ordinary derivatives, particularly, the Riemann-Liouville and Caputo ones. We also mention the Hadamard and Caputo-Hadamard Here, we present two definitions of fractional derivatives, the so-called generalized fractional derivatives and Caputo's modification of the generalized fractional derivatives, which we believe are the most appropriate to study fractional differential equations.

## 1.2 Problem statement

Fractional integral inequalities and its generalizations have many applications in numerical quadrature, transform theory, probability, and statistical problems, but the most useful ones are in the study of various properties of solutions such as oscillation theory, disconjugacy and eigenvalue problems.and in establishing uniqueness of solutions in fractional boundary value problems, see [14, 15, 16, 17, 18, 19, 20].

In a recent papers, there are several generalizations and extensions of (1.1.5). This inequality is generalized to fractional differential equation with Hadamard derivative and the Riemann-Liouville fractional derivative which are presented in [22] and by [22, 23, 24] respectively. For other generalizations and extensions of the classical Lyapunov's inequality, we refer to [22, 26, 27] and the references therein.

In this thesis, we aim to obtain a Lyapunov and Hartman type inequality for a different problem from those cited above. Indeed, we will consider the following generalization for linear fractional differential equation,

$$^{c}\mathscr{D}_{a+}^{\alpha,\rho}x(t)+p(t)x(t)=0,\ t\in(a,b), \tag{1.2.1}$$

with the conditions (1.1.2). or

$$x(a)-\beta_0 x'(a)=0=x(b)+\beta_1 x'(b),\beta_0^2+\beta_1^2\neq 0, \tag{1.2.2}$$

where $^{c}\mathscr{D}_{a+}^{\alpha,\rho}$ is Caputo's modification of the generalized fractional derivative, with $1<\alpha\leq 2,\rho>0,\ p\in C([a,b],\mathbb{R}),$ with either Caputo-Riemann-Liouville or Caputo-Hadamard derivatives, involving a generalized derivative operator. The advantage of considering the initial value problem with the generalized derivative is that the obtained results allow us to give results for Riemann-Liouville as well as Hadamard derivative initial value problems as its particular cases.

We also, consider the problem $(1.2.1)$, with $\left(^{c}\mathscr{D}_{a+}^{\alpha,\rho}\right)\left(^{c}\mathscr{D}_{a+}^{\beta,\rho}\right)$ is Caputo's modification of the generalized sequential fractional derivative, with $1<\alpha+\beta\leq 2$. As an application, we obtain a lower bound for the eigenvalues of corresponding equations.

We will consider the following generalization for nonlinear fractional differential equation,

$$^{c}\mathscr{D}_{a+}^{\alpha,\rho}x(t)+p(t)\varphi_{\mu}(x(t))+q(t)\varphi_{\lambda}(x(g(t)))=f(t),\ t\in(a,b), \tag{1.2.3}$$

with the conditions (1.1.2). or the conditions (1.2.2)

The innovations of this thesis can be shown in two points:

Firstly, comparing with the literature, we consider more general nonlinear fractional problems involving the framework of a generalized Caputo fractional type derivative.

Secondly, using the theory of fractional calculus and Green's functions, we study how Lyapunov- and Hartman-type inequalities can be generalized to nonlinear fractional differential equations.

## 1.3 Research aim and objectives

The main objective of this thesis is to establish and apply generalized Lyapunov- and Hartman-type inequalities to various classes of fractional differential equations, both linear and nonlinear, under different types of boundary or initial conditions.

The specific objectives of this work are as follows:
(**i**) To introduce the fundamental concepts of fractional analysis, including the study of the fractional operators involved, the associated special functions, and the construction of the corresponding Green's functions (Chapters 1 and 2).
(**ii**) To develop new Lyapunov-type inequalities for linear fractional differential equations, using the Green's function method to derive general representations of solutions.
(**iii**) To establish Lyapunov- and Hartman-type inequalities for nonlinear fractional differential equations and investigate their applications. .
(**iv**)To extend these Lyapunov- and Hartman-type inequalities to broader classes of nonlinear fractional differential equations, thereby enlarging their range of applicability.

The main results are presented in Chapters 4 and 5 and constitute a significant generalization of several earlier works in the literature.

## 1.4 Dissertation outline

This dissertation is devoted to several aspects of fractional calculus and its applications to qualitative analysis of fractional differential equations. The structure of this work has been carefully designed so that it can be understood not only by experts in fractional calculus, but also by readers with a standard background in analysis and differential equations. The thesis is organized as follows:

**Chapter 1**: This chapter presents the motivation of the study, the problem statements, and the objectives of the research. An overview of the adopted methodology is given, along with a list of publications and submitted manuscripts. Finally, the general structure of the thesis is outlined.

**Chapter 2**: In this chapter, we introduce the basic concepts and tools employed throughout the dissertation. It is divided into six sections: In section one, we present "Introduction", in section two, we give some " Functions used", in section three, we give "Fractional calculus", in section four, we give ”Recent developments in Lyapunov inequalities for the fractional case”. Finally, in the last section, we present some ”Further reading”.

**Chapter 3**: In this chapter, we study a class of linear fractional differential equation of the form (1.2.1) with the conditions (1.1.2), where $\left({}^{c}\mathscr{D}_{0^+}^{\alpha,\rho}\right)$ is generalized Caputo−type fractional derivative of order $1 < \alpha \leq 2$.

The main objective of this chapter is to present generalized Lyapunov-type inequalities for linear fractional differential equations involving the operator introduced in the previous chapter. These results, originally established by Jarad et al. [28], provide a unified framework that encompasses several classical formulations, particularly those based on the Caputo and Caputo–Hadamard derivatives.

**Chapter 4**: Motivated by the interesting works [28, 29, 30, 31], we consider the nonlinear fractional differential equation given by (1.2.3) and subject to Dirichlet-type boundary conditions (1.1.2).

We obtain generalized Lyapunov and Hartman type inequalities for both linear and nonlinear cases and derive disconjugacy criteria. The obtained results extend and complement known results in the literature.

**Chapter 5**: In this chapter, we investigate various boundary value problems involving generalized Caputo fractional derivatives under nonlocal two-point boundary conditions of the form (1.2.3) and with the conditions (1.2.2).

We derive explicit expressions for the corresponding Green’s functions and study their qualitative properties. Using these properties, we establish Lyapunov-type inequalities for certain fractional boundary value problems in the generalized Caputo framework.

**Conclusion and Perspectives**: The dissertation concludes with a summary of the main contributions and possible directions for further research, followed by the bibliography.

## 1.5 List of publication and manuscript

The work is presented as a series of one published paper and one manuscript in preparation:
(**i**) O. Cherikh, Y. Adjabi, H. Boulares and A. Moumen. *Lyapunov- and Hartman-Type Inequalities for Generalized Caputo Fractional Differential Equations Incorporating Forcing Terms*. Mathe-

matical Methods in the Applied Sciences, Volume 49, Issue1 15 January 2026, Pages 78-95. (First published: 01 October 2025). https://doi.org/10.1002/mma.70135

(**ii**) Generalized Lyapunov type inequality for generalized fractional derivative and Caputo's modification of the generalized fractional derivative and their applications to boundary value problems, with F. Jarad, Y. Adjabi and T. Abdeljawad, To be submitted.

# *CHAPTER 2*

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# *Preliminaries on Fractional Calculus*

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## Summary

# 2.1 Introduction

In this chapter, we begin by presenting the functions used, such as the Gamma, Beta and Mittag-Leffler functions. Secondly, we introduce fractional derivatives, for example generalized fractional derivatives and weighted fractional derivatives. Next, we present the methods used in this thesis, for example the method of Green's method.

# 2.2 Functions used

Here, we give some information, for example gamma function, beta function and Mittag–Leffler functions which play the most important role in the theory of the differentiation of arbitrary order.

## 2.2.1 Gamma function

The Gamma function is considered as an extension to the factorial function to real and complex numbers not only integers [2, 32]:

**Definition 2.2.1.** the Gamma function is given by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt,$$

which converges in the right half of the complex plane $\Re e(x) > 0$.
with $\Gamma(1) = 1$, $\Gamma(0_+) = +\infty$, $\Gamma(x)$ is a monotone and strictly decreasing function for $0 < x \leq 1$.
An important property of the Gamma function $\Gamma(x)$ is the following recurrence relation

$$\Gamma(x+1) = x\Gamma(x),$$

that can be proved by integration by parts

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt = [-e^{-t} t^x]_0^\infty + x \int_0^\infty e^{-t} t^{x-1} dt = x\Gamma(x).$$

Euler's Gamma function generalizes the factorial because $\Gamma(n+1) = n!$, $\forall n \in \mathbb{N}$.

### 2.2.2 Beta function

The Beta function is useful function related to the Gamma functions. It is defined for $x > 0$ and $y > 0$ by the two equivalent identities [32]:

**Definition 2.2.2.** The Beta function (**or Eulerian integral of the first kind**) is given by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

The Beta function is symmetric as will be shown in the next theorem:

**Theorem 2.2.3.** *let* $\Re e(x) > 0$ *and* $\Re e(y) > 0$ *, Then*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y,x)$$

### 2.2.3 Mittag-Leffler function

The Mittag-Leffler function is a generalization of the exponential function and it is one of the most important functions that are related to fractional differential equations.

**Definition 2.2.4.** The exponential function $e^x$, plays a very important role in the theory of differential equations of integer order. The generalization of the exponential function to a single parameter has been introduced by G.M. Mittag-Leffler and designated by the following function :

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+1)}, \ \alpha > 0.$$

The two-parameter Mittag-Leffler function also plays a very important role in the fractional calculus theory. The latter was introduced by Agarwal [33] and is defined by the following series expansion [34]:

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+\beta)}, \ \alpha > 0, \ \beta > 0.$$

Also the function $E_{\alpha,\beta}(z)$ has the integral representation

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{t^{\alpha-\beta}e^t}{t^\beta - z}dt,$$

where the path of integration $\mathscr{C}$ is a loop which starts and ends at $-\infty$, and encircles the circles disc $|t| \leq |z|^{1/\beta}$ in the positive sense: $|\arg(t)| \leq \pi$ on $\mathscr{C}$, we shortly write $E_{\alpha,1}(z) = E_{\alpha}(z)$.

# 2.3 Fractional calculus

Let recall some formulas from the classical fractional calculus [1, 4, 5, 6, 7].

## 2.3.1 History of fractional calculus

**Definition 2.3.1.** The left-sided RiemannLiouville integral operator of order $\alpha > 0$, of a continuous function $f : [a, \infty) \longrightarrow \mathbb{R}$ is given

$$(I^{\alpha}_{a^+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds,$$

**Definition 2.3.2.** The left-sided Riemann-Liouville fractional derivative of order $\alpha \in (n-1, n]$, of a continuous function $f : [a, \infty) \longrightarrow \mathbb{R}$ is given by

$$\left(D^{\alpha}_{a^+} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)ds.$$

**Definition 2.3.3.** The left-sided Caputo fractional derivative of order $\alpha \in (n-1, n]$, of a continuous function $f : [a, \infty) \longrightarrow \mathbb{R}$ is given by

$$\left({}^cD^{\alpha}_{a^+} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds.$$

J.Hadamard in 1892. The Hadamard type fractional integrals and derivatives were introduced in [1] as the following:

**Definition 2.3.4.** The left−sided Hadamard fractional integral of order $\alpha \in (n-1, n]$ has the following form

$$(\mathscr{I}^{\alpha}_{a^+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}.$$

**Definition 2.3.5.** The left−sided Hadamard fractional derivative of order $\alpha \in (n-1, n]$ is given by

$$(\mathscr{D}^{\alpha}_{a^+} f)(t) = \frac{1}{\Gamma(n-\alpha)} \delta^n \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} f(s)\frac{ds}{s},$$

where $\delta = \left(t\frac{d}{dt}\right)$ is the so-called $\delta-$derivative.

**Definition 2.3.6.** The left$-$sided Caputo Hadamard fractional of order $\alpha$, $\Re e(\alpha) \geqslant 0$ is presented as

$$({}^{c}\mathscr{D}^{\alpha}_{a^+}f)(t) = \mathscr{D}^{\alpha}_{a^+}\left[f(s) - \sum_{k=0}^{n-1}\frac{\delta^k f(a^+)}{k!}\left(\ln\frac{s}{a}\right)^k\right](t)$$

**Lemma 2.3.7.** *Let $\alpha \in (n-1,\ n]$ and consider the space*

$$AC^n_{\gamma}[a,\ b] = \left\{f : [a,\ b] \longrightarrow \mathbb{C} \text{ and } \gamma^{n-1}f \in AC[a,\ b],\ \gamma = x^{1-\rho}\frac{d}{dx}\right\},\ AC^1_{\gamma}[a,\ b] = AC[a,\ b].$$

*Then, for all $f \in AC^n_{\delta}[a, b]$, the Caputo Hadamard fractional derivative can be written in the equivalent form*

$$({}^{c}\mathscr{D}^{\alpha}_{a^+}f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{n-\alpha-1}\delta^n f(s)\frac{ds}{s}.$$

In this thesis, the definition of the integer $n$, given by $n = \lfloor\alpha\rfloor + 1$ is not entirely accurate. It should be specified as follows

$$n = \begin{cases} \lfloor\alpha\rfloor + 1, & \text{if } \alpha \notin \{0,1,2,\ldots\}, \\ \alpha, & \text{if } \alpha \in \{0,1,2,\ldots\}. \end{cases}$$

## 2.3.2 Generalized fractional derivatives

Let $0 < a < b$, $c \in \mathbb{R}$ and $1 \leq p < +\infty$. We denote by $X^p_c(a,b)$ the space of complex valued functions $f$ defined on $(a,b)$ such that

$$\int_a^b |t^c f(t)|^p\,\frac{dt}{t} < +\infty,$$

quotiented by the equivalence relation

$$f \sim g \quad \Longleftrightarrow \quad f = g \text{ almost everywhere on } (a,b).$$

The space $X_c^p(a,b)$ is endowed with the norm

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}.$$

For $p = \infty$,

$$\|f\|_{X_c^\infty} = ess \sup_{a \le t \le b} [t^c \mid f(t) \mid].$$

**Definition 2.3.8.** Let $[a,b]$ be a finite interval, $0 \le \epsilon < 1$. Then, we define

$$\mathscr{C}^n_{\gamma,\, \epsilon}[a,\, b] = \left\{ f : [a,\, b] \longrightarrow \mathbb{C} \text{ and } \gamma^{n-1} f \in C[a,\, b],\, \gamma^n f \in \mathscr{C}_{\epsilon,\, \rho}[a,\, b],\, \gamma = x^{1-\rho}\frac{d}{dx} \right\}.$$

endowed with the norm $\|f\|_{\mathscr{C}^n_{\gamma,\, \epsilon}} = \sum_{k=0}^{n-1} \|\gamma^k f\|_C + \|\gamma^n f\|_{\mathscr{C}_{\epsilon,\, \rho}}$.

The convention $\mathscr{C}^n_{\gamma,0}[a,\, b] = \mathscr{C}^n_{\gamma}[a,\, b]$ endowed with the norm $\|f\|_{\mathscr{C}^n_\gamma} = \sum_{k=0}^{n} \|\gamma^k f\|_C$ is used.

**Definition 2.3.9.** [5] The left−sides generalized fractional integrals are defined by, $\alpha \in (n-1,\, n]$

$$\left(\mathscr{I}^{\alpha,\rho}_{a^+}\right) f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau^{1-\rho}},$$

where $\Gamma(.)$ is the Euler gamma function.

**Example 2.3.10.** Let $\alpha \in (0,1], \rho > 0$. Then we have

$$\left(\mathscr{I}^{1-\alpha,\rho}_{a^+}\right)\left((t^\rho - \tau^\rho)^{\alpha-1} \rho^{1-\alpha}\right) = \Gamma(\alpha)$$

In particular

$$\left(\mathscr{I}^{1-\alpha,\rho}_{a^+} 1\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - \tau^\rho)^{\alpha-1} \frac{d\tau}{\tau^{1-\rho}} = \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} (t^\rho - a^\rho)^\alpha.$$

**Definition 2.3.11.** [6] The leftt−sides generalized fractional derivatives of $f$ of order $\alpha > 0$ are defined by

$$\left(\mathscr{D}^{\alpha,\rho}_{a^+}\right) f(t) = \frac{\gamma^n}{\Gamma(n-\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{n-\alpha-1} f(\tau)\frac{d\tau}{\tau^{1-\rho}},$$

where $\gamma = t^{1-\rho}\frac{d}{dt}$.

**Definition 2.3.12.** The left-sided generalized Caputo—type fractional derivative of $f$ of order $\alpha$ is defined by[7]

$$\left({}^{c}\mathscr{D}_{a^+}^{\alpha,\rho}\right) f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{n-\alpha-1} \frac{(\gamma^n f)(\tau) d\tau}{\tau^{1-\rho}}.$$

**Definition 2.3.13.** For $\alpha > 0$ and $\rho > 0$, we have [7]

$$\left({}^{c}\mathscr{D}_{a^+}^{\alpha,\rho}\right) f(t) = \mathscr{D}_{a^+}^{\alpha,\rho} f(t) - \sum_{k=0}^{n-1} \frac{\gamma^k f(a)}{\Gamma(k+1-\alpha)} \left(\frac{t^\rho}{\rho} - \frac{a^\rho}{\rho}\right)^{k-\alpha}.$$

**Lemma 2.3.14.** *[7] (**i**) Let* $f \in AC_\gamma^n[a,b]$ *or* $\mathscr{C}_\gamma^n[a,b]$ *and* $\alpha \in \mathbb{C}$. *Then,*

$$\mathscr{I}_{a^+}^{\alpha,\rho} \left({}^{c}\mathscr{D}_{a^+}^{\alpha,\rho}\right) f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(\gamma^k f)(a)}{k!} \left(\frac{t^\rho}{\rho} - \frac{a^\rho}{\rho}\right)^k.$$

(***ii***) *For* $\alpha > 0;\ \beta > 0;\ 1 \leqslant p < \infty;\ a \in (0,\infty);\ \rho,\ c \in \mathbb{R};\ \rho \geqslant c,$

$$\mathscr{I}_{a^+}^{\alpha,\rho} \mathscr{I}_{a^+}^{\beta,\rho} f = \mathscr{I}_{a^+}^{\alpha+\beta,\rho} f;\ f \in \mathbb{X}_c^p(a,b).$$

(***iii***) *For all* $\alpha \in (n-1,n]$ *and* $\beta \geqslant 0$ *the following relation holds*

$$\mathscr{I}_{a^+}^{\alpha+\beta,\rho} f(t) = \mathscr{I}_{a^+}^{\beta+n,\rho} \mathscr{D}_{a^+}^{n-\alpha,\rho} f(t).$$

**Example 2.3.15.** For $\beta > -1$, one has

$$\mathscr{I}_{a^+}^{\alpha,\rho} \left[(\tau^\rho - a^\rho)^\beta\right] = \frac{\rho^{-\alpha}\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)} (t^\rho - a^\rho)^{\beta+\alpha},$$

and

$$\left({}^{c}\mathscr{D}_{a^+}^{\alpha,\rho}\right) \left[\left(\frac{\tau^\rho}{\rho} - \frac{a^\rho}{\rho}\right)^\beta\right] = \frac{\rho^{1-\beta}\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)} (t^\rho - a^\rho)^{\beta-\alpha}.$$

**Lemma 2.3.16.** *[7]For any* $f \in \mathscr{C}[a,b]$, *one has*

$$\left\| \mathscr{I}_{a^+}^{\alpha,\rho} f \right\|_{\mathscr{C}} \leq \frac{\rho^{-\alpha}}{\Gamma(1+\alpha)} (b^\rho - a^\rho)^\alpha \|f\|_{\mathscr{C}}.$$

**Lemma 2.3.17.** *[7] Let* $\alpha \geqslant 0$, $\alpha \in (n-1,n]$ *and* $f \in AC_\gamma^n[a,b]$, *where* $0 < a < b < \infty$. *Then*

$$f(t) = f(a) + \frac{\left({}^{c}\mathscr{D}_{a^+}^{\alpha,\rho}\right) f(\xi)}{\Gamma(\alpha+1)} \left(\frac{t^\rho}{\rho} - \frac{a^\rho}{\rho}\right)^\alpha;\ a \leq \xi \leq t \leq b.$$

**Remark 2.3.18.** **Let** $\alpha>0,\beta>0$**,** $\rho>0$ **and** $a>0$ **then**

$$\begin{aligned}\int_a^b\left(\frac{b^\rho-\tau^\rho}{\rho}\right)^\alpha\left(\frac{\tau^\rho-a^\rho}{\rho}\right)^\beta\frac{d\tau}{\tau^{1-\rho}} &= \rho^{-(\alpha+\beta+1)}\left(b^\rho-a^\rho\right)^{\alpha+\beta+1}\int_0^1(1-z)^\alpha z^\beta dz\\ &= \rho^{-(\alpha+\beta+1)}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\left(b^\rho-a^\rho\right)^{\alpha+\beta+1}.\end{aligned}$$

In particular

$$\begin{aligned}\int_a^b\left(\frac{\tau^\rho-a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\frac{d\tau}{\tau^{1-\rho}} &= \left(\frac{b^\rho-a^\rho}{\rho}\right)^{2\alpha-1}\int_0^1(1-z)^{\alpha-1}z^{\alpha-1}dz\\ &= \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2\alpha-1}.\end{aligned}$$

or equivalently,

$$\int_a^b\frac{\tau^{\rho-1}d\tau}{\left(\tau^\rho-a^\rho\right)^{1-\alpha}\left(b^\rho-\tau^\rho\right)^{1-\alpha}}=\rho^{2\alpha-2}\frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2\alpha-1}.$$

Applying the substitution $\tau^\rho=a^\rho+z(b^\rho-a^\rho)$ and invoking the definition of the Beta function yields the evaluation of the inner integral.

Finally, we observe that the limiting case as $\rho\to0$ leads to the Hadamard and Caputo–Hadamard results by noting that

$$\lim_{\rho\to0}\frac{x^\rho-a^\rho}{\rho}=\ln\left(\frac{x}{a}\right),\qquad \lim_{\rho\to0}\frac{b^\rho-x^\rho}{\rho}=\ln\left(\frac{b}{x}\right).$$

Moreover, the case $\rho=1$ yields the classical Riemann–Liouville and Caputo fractional derivatives.

## 2.4 Recent developments in Lyapunov inequalities for the fractional case

Lyapunov-type inequalities are useful tools for studying solutions to boundary value problems. In particular, they can be applied to give nonexistence results for certain homogeneous boundary value problems and to give existence-uniqueness results for corresponding nonhomogeneous boundary value problems. They can also be used to obtain bounds on eigenvalues in certain eigenvalue problems and to consider oscillation and stability criteria of solutions. In addition, the zeros of the solutions, and in particular the distance between consecutive zeros, may be analyzed by means of Lyapunov-type inequalities.

Lyapunov inqualities have been extended and generalized in a variety of directions due to their many applications. For example, see [22, 23, 16], in which Lyapunov inequalities for dierential equations of higher order are considered. See also [36, 37] and [38, 39] for generalizations to fractional BVPs involving the Riemann-Louiville derivative and extensions including fractional BVPs with solutions defined on multivariate domains.

In [22, 23] and [24], Lyapunov inequalities for fractional differential equations involving the continuous Caputo fractional derivative are investigated.

In fractional-order differential equations, a number of recent developments have been made involving Lyapunov inequalities, which are similar to the original Lyapunov inequality in equation (1.1.5). We cite and name a few of these developments here. In [51], boundary value problems involving the equation

$$D^{\beta}_{a+}x(t)+p(t)x(t)=0, \qquad \text{for } 2<\beta\leq 3,$$

are studied and several Lyapunov-type inequalities are derived. Here, $D^{\beta}_{a+}$ denotes the continuous Riemann–Liouville fractional derivative of $x$. The same equation is also studied in [51] under different boundary conditions.

Additional work on Lyapunov inequalities has been done for fractional differential equations in [36, 37, 38, 39], which all involve the Caputo fractional derivative.Green's Functions for Fractional Boundary Value Problems:The Green's function and its properties

## 2.5 Further reading

For other related results, see [35, 40]. In particular, [3, 12] are general references on continuous fractional calculus, while [1, 32] provide an introduction to fractional-order operators with applications in engineering. Moreover, [39, 16] offer a survey and historical account of Lyapunov inequalities in the classical continuous setting, and recent developments are reviewed in [37]. For Lyapunov inequalities in the framework of time scales, see [38], where nonlinear ordinary difference equations are also investigated. Results concerning Lyapunov inequalities in continuous fractional calculus can also be found in the literature.

Other works addressing boundary value problems in continuous fractional calculus are presented in various studies. Discrete fractional boundary value problems are treated in [38], and Lyapunov inequalities in the delta whole-order case are also considered there. For results involving the nabla Riemann–Liouville fractional difference operator, see [37], while linear fractional nabla difference equations are discussed in [39] and also see [27, 55, 56, 57, 58, 59, 60].

# *CHAPTER 3*

# *Generalized fractional Lyapunov type inequality and Caputo's modification and its application*

## Summary

# 3.1 Introduction

In this chapter, we establish a Lyapunov-type inequality for a fractional boundary value problem associated with equations (1.2.1)-(1.1.2). We note that the content presented here is mainly based on Reference [28]. .

In Section 2, we focus on the Green's function associated with the considered problem and establish upper and lower bounds for it. We then present a Lyapunov-type inequality for the linear problem involving Caputo's modification of the generalized fractional derivative, corresponding to equations (1.2.1)-(1.1.2).

In Section 3 is devoted to the sequential case. we derive the Green's function for the sequential case and establish bounds for it. We then prove a Lyapunov-type inequality for the linear sequential Caputo's modification of the sequential generalized fractional derivative corresponding to problem (1.2.1)-(1.1.2).

Finally, we conclude this chapter by presenting an application of the theoretical results obtained for the considered linear problem.

# 3.2 A generalized Lyapunov inequality for some Caputo fractional boundary value problems

In this section, we present new versions of Lyapunov-type integral inequalities, which extend and generalize the results previously established in the literature.

We construct Green's function for the linear boundary value problem (1.2.1)-(1.1.2). Estimate bounds for Green's function, which he used to prove the main theorem in this section.

## 3.2.1 The Green's Function for Linear Equations

**Definition 3.2.1** (Green's function)**.** [32] Let the continuous function $G^b : [a,b] \rightarrow \mathbb{R}$ satisfy the following properties.

**(i)** $\left({}^{c}\mathscr{D}_{a+}^{\alpha,\rho} G^{b}(.,\tau)\right)(t) = -p(t)\, G^{b}(t,\tau)$ for all $t,\tau \in [a,b]$.

**(ii)** $\lim_{\tau\to t-a}\left({}^{c}\mathscr{D}_{a+}^{\alpha-k,\rho}G^{b}(.,\tau)\right)(t)=\delta_{k,1}$, for all $t\in[a,b]$, $k=1,2,...,[\alpha]$.

**(iii)** $\lim_{\tau\to a}\left(\lim_{\tau\to t-a}\left({}^{c}\mathscr{D}_{a+}^{\alpha-k,\rho}G^{b}(.,\tau)\right)(t)\right)=0$, for all $t\in[a,b]$, $k=1,2,...,[\alpha]-1$.

Then, $G^b$ is called the Green's function for the boundary value problem (1.2.1)-(1.1.2).

We begin by writing problem (1.2.1)-(1.1.2) in its equivalent integral form.

**Theorem 3.2.2.** *[28]* $x(t)\in C[a,b]$ *is a solution of (1.2.1-1.1.2) if and only if*

$$x(t)=\int_a^b G^b(t,\tau)p(\tau)x(\tau)d\tau, \tag{3.2.1}$$

*where* $G^b(t,\tau)$ *is the Green's function given by*

$$G^b(t,\tau)=\frac{1}{\Gamma(\alpha)}\begin{cases} G_1^b(t,\tau), & \textit{if } a\le t\le\tau\le b,\\ G_1^b(t,\tau)-(\dfrac{t^\rho-\tau^\rho}{\rho})^{\alpha-1}\tau^{\rho-1} & \textit{if } a\le\tau\le t\le b,\end{cases} \tag{3.2.2}$$

*with*

$$G_1^b(t,\tau)=\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)\left(\frac{b^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\tau^{\rho-1}. \tag{3.2.3}$$

**Proof.** We will show that

$$x(t)=\int_a^b G^b(t,\tau)h(\tau)d\tau, \tag{3.2.4}$$

where $G^b$ given by (3.2.2) is a solution to the problem

$${}^{c}\mathscr{D}_{a+}^{\alpha,\rho}x(t)+p(t)x(t)=0,\ t\in(a,b), \tag{3.2.5}$$

with the conditions (1.1.2).

By applying Lemma 2.3.14-ii, we may reduce (3.2.5)-(1.1.2) to an equivalent integral equation

$$-\mathscr{I}_{a+}^{\alpha,\rho}h(t)+c_0+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}=-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(t^\rho-\tau^\rho)^{\alpha-1}h(\tau)\frac{d\tau}{\tau^{1-\rho}}+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}. \tag{3.2.6}$$

From $x(a)=0$ and (3.2.6), we have $c_0=0$. Consequently the solution of (3.2.5)-(1.1.2) is

$$x(t)=-\mathscr{I}_{a+}^{\alpha,\rho}h(t)+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}=-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(t^\rho-\tau^\rho)^{\alpha-1}h(\tau)\frac{d\tau}{\tau^{1-\rho}}+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}. \tag{3.2.7}$$

By (3.2.7), one has

$$x(b)=-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b(b^\rho-\tau^\rho)^{\alpha-1}h(\tau)\frac{d\tau}{\tau^{1-\rho}}+c_1\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha-1}.$$

And from $x(b)=0$, then we have

$$c_1=\left(\frac{b^\rho-a^\rho}{\rho}\right)^{1-\alpha}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b(b^\rho-\tau^\rho)^{\alpha-1}h(\tau)\frac{d\tau}{\tau^{1-\rho}}.$$

So, the unique solution of problem (3.2.5)-(1.1.2) is

$$x(t)=-\mathscr{I}_{a+}^{\alpha,\rho}h(t)+\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b(b^\rho-\tau^\rho)^{\alpha-1}h(\tau)\frac{d\tau}{\tau^{1-\rho}}. \tag{3.2.8}$$

Hence, equality (3.2.8) becomes

$$x(t)=-\left(\mathscr{I}_{a+}^{\alpha,\rho}q(\tau)x(\tau)\right)(t)+\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b(b^\rho-\tau^\rho)^{\alpha-1}h(\tau)\frac{d\tau}{\tau^{1-\rho}},$$

which can be written as equation (3.2.4) with $G^b(t,\tau)$ given by (3.2.2) This concludes the proof. ■

**Remark 3.2.3.** If we take $\rho=1$ in Theorem 3.2.2, then the Green function given by Theorem 3.2.2 reduces to the Green function for Caputo boundary value problem in [43].

## 3.2.2 Estimates of the Green's function

**Lemma 3.2.4.** *[28]The function $G$ defined in Theorem 3.2.2 satisfies the following property:*

$$\max\{|G(t,\tau)|: a\le t\le b\}\le G(\tau,\tau)\ \textit{for all}\ \tau\in[a,b],$$

*and* $G(\tau,\tau)$ *has a unique maximum* $G_{max}$ *in* $[a,b]$, *given by*

$$G_{max} := \begin{cases} \left(\frac{L-a^{\rho}}{b^{\rho}-a^{\rho}}\right)\left(\frac{b^{\rho}-L}{\rho}\right)^{\alpha-1} L^{\frac{\rho-1}{\rho}} & \text{if } N=0, \\ \frac{\left((1-\alpha\rho)\,a^{\rho}+(2\alpha\rho-1)\,b^{\rho}-M\right)^{\alpha-1}\left((1-(\alpha+2)\,\rho)\,a^{\rho}+(2\rho-1)\,b^{\rho}+M\right)}{\Gamma(\alpha)\,(b^{\rho}-a^{\rho})\,(2N)^{\frac{N}{\rho}}\left((\alpha\rho-1)\,a^{\rho}+(2\rho-1)\,b^{\rho}+M\right)^{\frac{1-\rho}{\rho}}} & \text{if } N\neq 0, \end{cases} \tag{3.2.9}$$

*for all* $\tau\in[a,b]$, *where*

$$L=\left(\frac{(\rho-1)\,a^{\rho}b^{\rho}}{(2\rho+1)\,b^{\rho}-a^{\rho}}\right)^{\frac{1}{\rho}},\; N=(\alpha+1)\,\rho-1 \tag{3.2.10}$$

*and*

$$M=\left(((\alpha\rho-1)\,a^{\rho}+(2\rho-1)\,b^{\rho})^{2}-4\,(1-(\alpha+1)\,\rho)\,(1-\rho)\,a^{\rho}b^{\rho}\right)^{\frac{1}{2}}. \tag{3.2.11}$$

**Proof.** Let us define two functions

$$G_1^b(t,\tau)=G_2^b(t,\tau)-\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}\tau^{\rho-1},\; a\le\tau\le t\le b.$$

We start with the function $G_2^b$. Obviously, $G_2^b$ satisfies the following inequalities:

$$0\le G_2^b(t,\tau)\le G_2^b(\tau,\tau),\; a\le t\le\tau\le b,$$

for $a\le t\le\tau\le b$, we get

$$\frac{\partial G_2^b(\tau,\tau)}{\partial\tau}=0\Longrightarrow\tau\in\{\tau_1,\tau_2\}$$

where

$$\tau_1=\begin{cases}\left(\frac{Q_c+(2b^{\rho}+\alpha a^{\rho})\rho-(a^{\rho}+b^{\rho})}{2w_c}\right)^{\frac{1}{\rho}} & \text{si}\quad \rho>0,\\ \frac{(\alpha-1)a+b}{\alpha} & \text{si}\quad \rho=1,\end{cases}$$

and

$$\tau_2=\begin{cases}(-1)^{\frac{1}{\rho}}\left(\frac{Q_c-(2b^{\rho}+\alpha a^{\rho})\rho+(a^{\rho}+b^{\rho})}{2w_c}\right)^{\frac{1}{\rho}} & \text{si}\quad \rho>0,\\ 0 & \text{si}\quad \rho=1,\end{cases}$$

when $\rho > 0$

$$\tau_c^* = \tau_1 = \left(\frac{Q_c + (2b^\rho + \alpha a^\rho)\rho - (a^\rho + b^\rho)}{2w_c}\right)^{\frac{1}{\rho}},$$

where

$$Q_c = \left(d_2\rho^2 + d_1\right)^{\frac{1}{2}},$$

with

$$\begin{aligned} d_2 &= (\alpha a^\rho)^2 + 4b^{2\rho} - 4(ab)^\rho, \\ d_1 &= -2(a^\rho - b^\rho)(\alpha a^\rho - 2b^\rho)\rho + (a^\rho - b^\rho)^2, \end{aligned}$$

and

$$w_c = (\alpha+1)\rho - 1,$$

$$G^b(\tau_c^*, \tau_c^*) = \left|\frac{(-1)^\alpha\left((L + (a^\rho - 2b^\rho)\alpha\rho + b^\rho - a^\rho)(L + (2b^\rho - (\alpha+2)a^\rho)\rho + a^\rho - b^\rho)\right)^{\alpha-1}}{\Gamma(\alpha)(a^\rho - b^\rho)(2w_c)^{\frac{w_c}{\rho}}(L + (2b^\rho + \alpha a^\rho)\rho - a^\rho - b^\rho)^{\frac{1-\rho}{\rho}}}\right|.$$

This completes the proof of Lemma. ■

#### 3.2.2.0.1 Particular cases

Case 1. As above, in the case $\rho = 1$ we get

$$\tau_c^* = \frac{(\alpha-1)a + b}{\alpha}, \; Q_c = (\alpha-1)a + b, \; w_c = 2(\alpha-1).$$

The corresponding maximum Green's function $G_{max}$ can be written as

$$G_{max} = \left|\frac{\alpha^{-\alpha}}{\Gamma(\alpha)}\left(-\frac{2(\alpha-1)(b-a)}{2}\right)^{\alpha-1}\right|.$$

Case 2. When $\rho = 1, \alpha = 2$

$$\tau_c^* = \frac{a+b}{2}, \; Q_c = a+b, \; w_c = 2.$$

The corresponding maximum Green's function $G_{max}$ can be written as

$$G_{max} = \frac{b-a}{4}.$$

a well-known result.

### 3.2.3 Lyapunov-type inequality

**Theorem 3.2.5.** *( see [28]) If a nontrivial continuous solution of the problem* $(1.2.1)$-$(1.1.2)$ *exists, then for Lemma* 3.2.4 *the Lyapunov-type inequality is*

$$\int_a^b |p(\tau)|\,d\tau \geq \left| \frac{\Gamma(\alpha)(a^\rho - b^\rho)(2w_c)^{\frac{w_c}{\rho}}\left(Q_c + (2b^\rho + \alpha a^\rho)\rho - a^\rho - b^\rho\right)^{\frac{1-\rho}{\rho}}}{(-1)^\alpha \left((Q_c + (a^\rho - 2b^\rho)\alpha\rho + b^\rho - a^\rho)(Q_c + (2b^\rho - (\alpha+2)a^\rho)\rho + a^\rho - b^\rho)\right)^{\alpha-1}} \right|. \tag{3.2.12}$$

*In particular, for* $\rho = 1$ *and* $\alpha = 2$ *in* $(3.2.12)$ *gives the standard Lyapunov inequality for problem* $(1.2.1)$-$(1.1.2)$.

## 3.3 A generalized Lyapunov inequality for some generalized sequential fractional boundary value problems

In this section, we consider a generalized sequential fractional differential equation subject to Dirichlet-type boundary conditions and perform an analysis aimed at deriving a Lyapunov-type inequality.

We adopt the definition of Caputo's modification of the generalized fractional derivative given by

$$\left({}^c\mathscr{D}_{a+}^{\alpha,\rho}\right)\left({}^c\mathscr{D}_{a+}^{\beta,\rho}x\right)(t) + h(t) = 0,\ 1 < \alpha+\beta \leq 2,\ t \in (a,b). \tag{3.3.1}$$

The corresponding Green's function is also presented.

### 3.3.1 The Green's Function for Linear Equations

**Theorem 3.3.1.** *Let* $0 < \alpha, \beta \leq 1$ *be such that* $1 < \alpha+\beta \leq 2$ *and* $p \in C[a,b]$ *for some* $a < b$. *Then,* $x \in C[a,b]$ *is a solution of the fractional boundary value problem* $(3.3.1)-(1.2.1)$ *if, and* *only if, $x$ satisfies the integral equation (3.2.2) where*

$$G^b(t,\tau) = \frac{1}{\Gamma(\alpha+\beta)} \begin{cases} G_2^b(t,\tau) - (\frac{t^\rho - \tau^\rho}{\rho})^{\alpha+\beta-1}\tau^{\rho-1} & a \le \tau \le t \le b, \\ G_2^b(t,\tau), & a \le t \le \tau \le b, \end{cases} \tag{3.3.2}$$

*where*

$$G_2^b(t,\tau) = \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^\beta \left(\frac{b^\rho - \tau^\rho}{\rho}\right)^{\alpha+\beta-1} \tau^{\rho-1}.$$

**Proof.** Taking the generalized fractional integral $\left(\mathscr{I}_{a+}^{\alpha,\rho}\right)$ a to both side of the equation (3.3.1)

$$\left(\mathscr{I}_{a+}^{\alpha,\rho}\right)\left({}^c\mathscr{D}_{a+}^{\alpha,\rho}\right)\left({}^c\mathscr{D}_{a+}^{\beta,\rho}x\right)(t) = -\left(\mathscr{I}_{a+}^{\alpha,\rho}\right)h(t), \tag{3.3.3}$$

and using Lemma 2.3.14-ii, the fractional differential equation (3.4.1) can be written as

$$\left({}^c\mathscr{D}_{a+}^{\beta,\rho}x\right)(t) = -\left(\mathscr{I}_{a+}^{\alpha,\rho}\right)h(t) + c_0. \tag{3.3.4}$$

Taking the generalized fractional integral $\left(\mathscr{I}_{a+}^{\beta,\rho}\right)$ a to both side of the equation (3.4.2)

$$\left(\mathscr{I}_{a}^{\beta,\rho}\right)\left({}^c\mathscr{D}_{a+}^{\beta,\rho}x\right)(t) = \left(\mathscr{I}_{a}^{\beta,\rho}\right)\left(c_0 - \left(\mathscr{I}_{a+}^{\alpha,\rho}\right)h(t)\right).$$

Consequently the general solution of (3.3.1) is

$$x(t) = c_1 + \left(\mathscr{I}_{a}^{\beta,\rho}\right)\left(c_0 - \left(\mathscr{I}_{a+}^{\alpha,\rho}\right)h(t)\right) = c_1 + \left(\mathscr{I}_{a}^{\beta,\rho}\right)c_0 - \left(\mathscr{I}_{a}^{\alpha+\beta,\rho}\right)h(t). \tag{3.3.5}$$

Since $x(a) = 0$ and (3.3.5), we get $c_1 = 0$. Consequently the solution of (3.3.1) is

$$x(t) = \left(\mathscr{I}_{a}^{\beta,\rho}\right)\left(c_0 - \left(\mathscr{I}_{a+}^{\alpha,\rho}\right)h(t)\right) \tag{3.3.6}$$

And from $x(b) = 0$ and (3.3.6), we have

$$c_0 = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \int_a^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{-\alpha} h(s)\,\frac{ds}{s^{1-\rho}}.$$

Then for the function $x(t)$ we get

$$x(t)=\frac{1}{\Gamma(\alpha+\beta)}\int_a^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\beta-1}\int_a^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{-\alpha}h(s)\frac{ds}{s^{1-\rho}}\frac{d\tau}{\tau^{1-\rho}}. \tag{3.3.7}$$

This ends the proof. ∎

## 3.3.2 Estimates of the Green's function

The remainder of this section is essentially devoted to determining the maximum of $G^b(t,\tau)$ for $(t,\tau)\in[a,b]^2$.

**Lemma 3.3.2.** *The Green function $G^b$ defined in Theorem 3.3.1 satisfies the following property*

$$\max_{\tau,t\in[a,b]}\left|G^b(t,\tau)\right|\leq\left|\frac{-(-1)^\alpha(Q_a)^\beta(Q_b)^{(\alpha+\beta-1)}}{\Gamma(\alpha+\beta)(2w_c)^{\frac{w_c}{\rho}}\left(Q_c+((\alpha+\beta)a^\rho+b^\rho(\beta+1))\rho-(a^\rho+b^\rho)\right)^{\frac{1-\rho}{\rho}}}\right|, \tag{3.3.8}$$

*where*

$$Q_a=\frac{Q_c+(b^\rho(\beta+1)+(3\beta-\alpha)a^\rho)\rho+(a^\rho-b^\rho)}{(a^\rho-b^\rho)},$$

*and*

$$Q_b=\frac{Q_c+((1-3\beta-2\alpha)b^\rho+(\alpha+\beta)a^\rho)\rho+(b^\rho-a^\rho)}{\rho},$$

*for all $(t,\tau)\in[a,b]^2$, with equality if and only if*

$$t=\tau=\left(\frac{Q_c+((\alpha+\beta)a^\rho+(\beta+1)b^\rho)\rho-(a^\rho+b^\rho)}{2w_c}\right)^{\frac{1}{\rho}}, \tag{3.3.9}$$

*where*

$$Q_c=\left(d_2\rho^2+d_1\right)^{\frac{1}{2}}, \tag{3.3.10}$$

*with*

$$\begin{aligned} d_2 &= \left(((\alpha+\beta)a^\rho)^2+((\beta+1)b^\rho)^2+2\left(\beta^2+(\alpha-3)\beta-\alpha\right)(ab)^\rho\right),\\ d_1 &= -2(a^\rho-b^\rho)((\alpha+\beta)a^\rho-(\beta+1)b^\rho)\rho+(a^\rho-b^\rho)^2,\end{aligned}$$

*and*

$$w_c=(2\beta+\alpha-1)\rho-1. \tag{3.3.11}$$

**Proof.** Let us start by defining a function

$$G_1^b(t,\tau) = \left[\left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^\beta \left(\frac{b^\rho - \tau^\rho}{\rho}\right)^{\alpha+\beta-1} - \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha+\beta-1}\right]\tau^{\rho-1},\ a \le \tau \le t \le b, \quad (3.3.12)$$

when $a \le \tau \le t \le b$, we have

$$\frac{\partial G_1^b(t,\tau)}{\partial t} = \frac{\partial G_1^b(t,\tau)}{\partial \tau} = 0, \quad (3.3.13)$$

from which follows that

$$[((\alpha+\beta)\rho-1)\tau^\rho - b^\rho(\rho-1)]t^{\rho+1} + \beta\rho\left(\tau^{\rho+1} - b^\rho\tau\right)t^\rho - a^\rho[((\alpha+\beta)\rho-1)\tau^\rho - b^\rho(\rho-1)]t = 0,$$

provided $\tau \le t \le b$.

$G_2^b$ is obviously nonnegative. Moreover

$$G_2^b(t,\tau) \le \left(\frac{\tau^\rho - a^\rho}{b^\rho - a^\rho}\right)^\beta \left(\frac{b^\rho - \tau^\rho}{\rho}\right)^{\alpha+\beta-1} = G_2^b(\tau,\tau),\ a \le \tau \le b,$$

Taking

$$g^\rho(\tau) = \left(\frac{\tau^\rho - a^\rho}{b^\rho - a^\rho}\right)^\beta \left(\frac{b^\rho - \tau^\rho}{\rho}\right)^{\alpha+\beta-1}.$$

So, we are left to find $\max_{\tau\in[a,b]} g^\rho(\tau)$, we have

$$\frac{dg^\rho(\tau)}{d\tau} = \frac{(-1)^{2\beta+\alpha-1}\left(U\rho^{2-\alpha-\beta} - V\rho^{1-\alpha-\beta}\right)}{(a^\rho - b^\rho)^\beta(\tau^\rho - a^\rho)^{1-\beta}(\tau^\rho - b^\rho)^{2-\alpha-\beta}},$$

where

$$U = \left[(2\beta+\alpha)\tau^{3\rho-2} - ((\alpha+\beta)a^\rho + (\beta-1)b^\rho)\tau^{2\rho-2} + a^\rho b^\rho \tau^{\rho-2}\right],$$

and

$$V = \left[\tau^{3\rho-2} - (a^\rho + b^\rho)\tau^{2\rho-2} + a^\rho b^\rho \tau^{\rho-2}\right],$$

$$\frac{dg^\rho(\tau)}{d\tau} = 0 \Longrightarrow \tau \in \{\tau_1, \tau_2\},$$

where

$$\tau_1 = \begin{cases} \left(\frac{Q_c+((\alpha+\beta)a^\rho+(\beta+1)b^\rho)\rho-(a^\rho+b^\rho)}{2w_c}\right)^{\frac{1}{\rho}} & \text{si} \quad \rho>0, \\ \frac{(\alpha-1)a+\beta(a+b)}{2\beta+\alpha-1} & \text{si} \quad \rho=1, \end{cases}$$

and

$$\tau_2 = \begin{cases} (-1)^{\frac{1}{\rho}}\left(\frac{Q_c-[((\alpha+\beta)a^\rho+(\beta+1)b^\rho)\rho-(a^\rho+b^\rho)]}{2w_c}\right)^{\frac{1}{\rho}} & \text{si} \quad \rho>0, \\ 0 & \text{si} \quad \rho=1, \end{cases}$$

from which follows that $\frac{dg^\rho(\tau)}{d\tau}=0$ if, and only if,

$$\tau_c^* = \tau_1 = \left(\frac{Q_c+((\alpha+\beta)\,a^\rho+(\beta+1)\,b^\rho)\,\rho-(a^\rho+b^\rho)}{2w_c}\right)^{\frac{1}{\rho}}, \rho>0, \tag{3.3.14}$$

where

$$g^\rho(\tau) = \begin{cases} <0 & \text{if} \quad \tau>\tau_c^*, \\ =0 & \text{if} \quad \tau=\tau_c^*, \\ >0 & \text{if} \quad \tau<\tau_c^*. \end{cases}$$

We conclude that,

$$\max_{t,\tau\in[a,b]} G_2^b(t,\tau) = g^\rho(\tau_c^*),$$

where

$$g^\rho(\tau_c^*) = \left|\frac{-(-1)^\alpha (Q_a)^\beta (Q_b)^{(\alpha+\beta-1)}}{(2w_c)^{\frac{w_c}{\rho}}\left(Q_c+((\alpha+\beta)\,a^\rho+b^\rho(\beta+1))\,\rho-(a^\rho+b^\rho)\right)^{\frac{1-\rho}{\rho}}}\right|.$$

■

## 3.3.3 Lyapunov-type inequality

**Theorem 3.3.3.** *If the fractional boundary value problem* $(3.3.1)-(1.2.1)$ *has a nontrivial continuous solution, then*

$$\int_a^b |p(\tau)|\,d\tau \geq \left|\frac{\Gamma(\alpha+\beta)\,(2w_c)^{\frac{w_c}{\rho}}\left(Q_c+((\alpha+\beta)\,a^\rho+b^\rho(\beta+1))\,\rho-(a^\rho+b^\rho)\right)^{\frac{1-\rho}{\rho}}}{-(-1)^\alpha (Q_a)^\beta (Q_b)^{(\alpha+\beta-1)}}\right|. \tag{3.3.15}$$

**Proof.** By Lemma 3.3.2 and from (3.2.2), it follows that if $x$ is a nontrivial continuous solution of the (3.3.1)

$$|x(t)| \leq \int_a^b |G^\rho(t,\tau)p(\tau)||x(\tau)|\, d\tau. \tag{3.3.16}$$

Let $B = C[a,b]$ be a Banach space endowed a norm

$$\|x\|_\infty = \max_{t\in[a,b]} |x(t)|, x \in B. \tag{3.3.17}$$

Hence, from (3.3.16) and (3.3.17) we get

$$\|x\|_\infty \leq \max_{t\in[a,b]} \left| \int_a^b G^b(t,\tau)p(\tau)\, d\tau \right| \|x\|_\infty,$$

or equivalently,

$$\max_{t\in[a,b]} \left| \int_a^b G^b(t,\tau)p(\tau)\, d\tau \right| \geq 1. \tag{3.3.18}$$

Using the properties of Green's function $G^b(t,\tau)$ particularly, $G^\rho_{max}$ in Lemma 3.3.2 gives the inequality

$$\int_a^b |p(\tau)|\, d\tau > \frac{1}{G^\rho_{max}}, \tag{3.3.19}$$

called the Lyapunov-type inequality for (3.3.1). ■

In the case $\rho = 1$ we have

$$\tau_r^* = \frac{a+b}{2},\ Q_r = (a+b)(\alpha-1),\ w_r = 2(\alpha-1).$$

The corresponding maximum Green's function $G_{max}$ can be written as

$$G_{max} = \frac{1}{\Gamma(\alpha)} \left(\frac{b-a}{4}\right)^{\alpha-1}.$$

In the integer case $\alpha = 2$ and $\rho = 1$ we get the classical Lyapunov inequality.

**3.3.3.0.1 Particular cases** The Lyapunov-type inequality obtained for $\rho = 1$, obtained for $\rho = 1$ and $\alpha = 1$ both considered as fractional versions, are presented in this section.

In the case $\rho = 1$ we have

$$\tau_c^* = \frac{(\alpha-1)a+\beta(a+b)}{2\beta+\alpha-1}, Q_c = (\beta+\alpha-1)a+b\beta,\ w_c = 2\beta+\alpha-2.$$

The corresponding maximum Green's function $G_{max}$ can be written as

$$G_{max} = \left| -(-1)^{\alpha+\beta} \frac{((a-b)(\alpha+\beta-1))^{(\alpha+\beta-1)}}{\Gamma(\alpha+\beta)(2\beta+\alpha-1)^{(2\beta+\alpha-1)}} \beta^{\beta} \right|,$$

with

$$Q_a = \frac{(\beta+\alpha-1)a+b\beta+(b(\beta+1)-(3\beta+\alpha)a)+(a-b)}{(a-b)} = -2\beta,$$

and

$$Q_b = (\beta+\alpha-1)a+b\beta+((1-3\beta-2\alpha)b+(\alpha+\beta)a)+(b-a) = 2(a-b)(\alpha+\beta-1).$$

In the case $\alpha = 1$ and $\rho = 1$,

$$\tau_c^* = \frac{a+b}{2}.$$

The corresponding maximum Green's function $G_{max}$ can be written as

$$G_{max} = \left| -\frac{(-1)^{1+\beta}}{\Gamma(1+\beta)} \left(\frac{b-a}{4}\right)^{\beta} \right|.$$

Observe that when $\alpha = \beta = 1$, then Theorem 3.4.2 reduces to Theorem of classical Lyapunov inequality.

## 3.4 Application

First, we derive a Lyapunov-type inequality for boundary value problems of the form

$$\begin{gathered}\left({}^{c}\mathscr{D}^{\alpha,\rho}_{a+}\left(\frac{\tau^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(1-m)}y(\tau)\right)(t)+q(t)y(t)=0, m>0,\\ y(a)=y(b)=0,\ t\in(a,b),\alpha\in(1,2],\end{gathered}\tag{3.4.1}$$

where $g(t)\in C([a,b])$ such that $\left(\frac{\tau^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(1-m)}g(t)\in C([a,b])$, $q:[a,b]\to\mathbb{R}$ is a continuous function.

The boundary value prolems $(3.4.1)$ can be reduced to $(1.1.2)-(1.2.1)$ by a change of

$$y(t)=\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(m-1)}x(t)\text{ and }q(t)=\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(1-m)}p(t).\tag{3.4.2}$$

For $x(t)$ and $p(t)$ in $(3.4.2)$, Theorem 3.2.5, yields to the following Corollary.

**Corollary 3.4.1.** *If a nontrivial continuous solution of the problem*

$$\begin{gathered}\left({}^{c}\mathscr{D}^{\alpha,\rho}_{a+}\left(\frac{\tau^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(1-m)}y(\tau)\right)(t)+q(t)y(t)=0, m>0,\\ y(a)=y(b)=0,\ t\in(a,b),\alpha\in(1,2],\end{gathered}\tag{3.4.3}$$

*exists, then*

$$\int_a^b\left|\left(\frac{\tau^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(m-1)}q(\tau)\right|d\tau>$$

$$\frac{\Gamma(\alpha)(b^{\rho}-a^{\rho})(2w_c)^{\frac{w_c}{\rho}}\left(Q_c+(2b^{\rho}+\alpha a^{\rho})\rho-a^{\rho}-b^{\rho}\right)^{\frac{1-\rho}{\rho}}}{\left((2b^{\rho}-a^{\rho})\alpha\rho+a^{\rho}-b^{\rho}-Q_c\right)^{\alpha-1}\left(Q_c+(2b^{\rho}-(\alpha+2)a^{\rho})\rho+a^{\rho}-b^{\rho}\right)}.\tag{3.4.4}$$

Secondly, we present explicit solutions to fractional differential equations involving the generalized fractional derivative.

$${}^{c}\mathscr{D}^{\alpha,\rho}_{a+}\left(\left(\frac{\tau^{\rho}-a^{\rho}}{\rho}\right)^{\alpha(1-m)}y(\tau)\right)(t)=-\lambda y(t),\ \alpha>0,\ m>0,\ \lambda\neq0.\tag{3.4.5}$$

## Real zeros for generalized Mittag-Leffler function

We now turn our attention to the eigenvalue problem.

$$\begin{gathered}\mathscr{D}_{a+}^{\alpha,\rho}\left(\left(\frac{\tau^{\rho}}{\rho}\right)^{\alpha(1-m)}y(\tau)\right)(t)=-\lambda y(t),\alpha\in(1,2],m>0,\ \lambda\neq 0,\\ y(0)=y(1)=0,\ t\in[0,1].\end{gathered}\tag{3.4.6}$$

**Theorem 3.4.2.** *The fractional eigenvalue problem* (3.4.6) *has an infinite number of eigenvalues, and they are roots of the Mittag-Leffler type equation*

$$E_{\alpha,m,m-1/\alpha}\left(-\lambda\left(\frac{1}{\rho}\right)^{m\alpha}\right)=0$$

*and the corresponding eigenfunctions are given by*

$$y(t)=\left(\frac{t^{\rho}}{\rho}\right)^{\alpha\left(m-\frac{1}{\alpha}\right)}E_{\alpha,m,m-1/\alpha}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{m\alpha}\right).\tag{3.4.7}$$

**Proof.** Using Theorem 3.3.1, the solution of (3.4.6) can be obtained as

$$y(t)=c_2\left(\frac{t^{\rho}}{\rho}\right)^{\alpha\left(m-\frac{2}{\alpha}\right)}E_{\alpha,m,m-2/\alpha}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{m\alpha}\right)+c_1\left(\frac{t^{\rho}}{\rho}\right)^{\alpha\left(m-\frac{1}{\alpha}\right)}E_{\alpha,m,m-1/\alpha}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{m\alpha}\right).$$

Since $y(0)$ we have $c_2=0$. Now

$$y(1)=c_1\left(\frac{1}{\rho}\right)^{\alpha m-1}E_{\alpha,m,m-1/\alpha}\left(-\lambda\left(\frac{1}{\rho}\right)^{m\alpha}\right)=0,$$

where $c_1$ is an arbitrary real constant, we get

$$E_{\alpha,m,m-1/\alpha}\left(-\lambda\left(\frac{1}{\rho}\right)^{m\alpha}\right)=0.$$

The eigenfunctions of the problem (3.4.6) then has the form

$$y(t)=\left(\frac{t^{\rho}}{\rho}\right)^{\alpha m-1}E_{\alpha,m,m-1/\alpha}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{m\alpha}\right),$$

where $-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{m\alpha}$ are zeros of the generalized Mittag-Leffler function. ■

**Corollary 3.4.3.** *In particular, if $m = 1$, the fractional eigenvalue problem $(3.4.6)$ has an infinite number of eigenvalues, and they are roots of the Mittag-Leffler type equation*

$$E_{\alpha,1,1-1/\alpha}\left(-\lambda\left(\frac{1}{\rho}\right)^{\alpha}\right) = \Gamma(\alpha)\, E_{\alpha,\alpha}\left(-\lambda\left(\frac{1}{\rho}\right)^{\alpha}\right) = 0$$

*and the corresponding eigenfunctions are given by*

$$y(t) = \left(\frac{t^{\rho}}{\rho}\right)^{\alpha m-1} E_{\alpha,1,1-1/\alpha}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right),$$

For $[a, b] = [0, 1]$ and $q(t) = -\lambda$ in $(3.3.5)$, Theorem 3.2.5, yields to the following Corollary.

**Corollary 3.4.4.** *Let $\lambda$ be the smallest eigenvalue of $(3.4.6)$. Then*

$$\int_0^1 \left|\lambda\left(\frac{1}{\rho}\right)^{\alpha(m-1)}\right| d\tau = \left|\lambda\left(\frac{1}{\rho}\right)^{\alpha(m-1)}\right| > \frac{\Gamma(\alpha)\left(Q_r + (\alpha\rho - 1)\right)^{\frac{1-\rho}{\rho}} (2w_r)^{\frac{w_r}{\rho}} (\rho)^{\alpha-1}}{\left[\left(Q_r + (\alpha\rho - 1)\right)\left(Q_r + ((2-3\alpha)\rho + 1)\right)\right]^{\alpha-1}}. \quad (3.4.8)$$

**Corollary 3.4.5.** *If $(3.4.8)$ is does not hold then the eigenfunctions*

$$y(t) = \left(\frac{t^{\rho}}{\rho}\right)^{\alpha m-1} E_{\alpha,m,m-1/\alpha}\left(-\lambda\left(\frac{t^{\rho}}{\rho}\right)^{m\alpha}\right),\ t \in [0, 1], \quad (3.4.9)$$

*of the eigenvalue problem $(3.4.6)$ has no real zeros.*

**Corollary 3.4.6.** *The generalized Mittag-Leffler function $E_{\alpha,m,\beta}(z)$ has no real zeros for*

$$|z| \leq \frac{\Gamma(\alpha)\left(Q_r + (\alpha\rho - 1)\right)^{\frac{1-\rho}{\rho}} (2w_r)^{\frac{w_r}{\rho}} (\rho)^{\alpha-1}}{\left[\left(Q_r + (\alpha\rho - 1)\right)\left(Q_r + ((2-3\alpha)\rho + 1)\right)\right]^{\alpha-1}}. \quad (3.4.10)$$

# *CHAPTER 4*

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# *Nonlinear generalized Caputo fractional differential equations with a forcing term and zero Dirichlet boundary conditions in (a,b)*

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## Summary

## 4.1 Introduction

In this chapter, we establish generalized Hartman- and Lyapunov-type inequalities for linear and nonlinear generalized Caputo fractional differential equations subject to Dirichlet-type boundary conditions. These inequalities provide important results concerning the disconjugacy of fractional differential equations.

The inequalities we propose extend and enrich existing results by systematically addressing particular cases of fractional differential equations. The problem defined by equations (1.2.3-1.1.2) is more general than those previously studied in the literature. It is worth emphasizing that this type of equation presents significant analytical challenges due to its complex structure. In particular, Lyapunov-type inequalities have not yet been fully developed for forced generalized nonlinear Caputo fractional differential equations. This gap in the literature is mainly due to the complexity introduced by the nonlinear terms and the presence of parameters in the fractional operators.

Moreover, by employing classical analytical techniques and following the approach used by Agarwal et al. [30], who studied a Riemann-Liouville fractional differential equation with Dirichlet boundary conditions, we are able to construct suitable generalizations.

Recently, the authors in [30] obtained the following Hartman- and Lyapunov-type inequalities for Riemann-Liouville fractional differential equations of the form (1.2.3-1.1.2).

**Theorem 4.1.1** (Hartman type inequality)**.** *Let $x(t)$ be a nontrivial solution of (1.2.3) satisfying the Dirichlet boundary conditions (1.1.2). If $x(t) \neq 0$ in $(a,b)$, then the inequality*

$$\begin{aligned}\left(\int_a^b \frac{p^+(\tau)+q^+(\tau)}{((b-\tau)(\tau-a))^{1-\alpha}}d\tau\right)\left(\int_a^b \frac{\mu_0 p^+(\tau)+\lambda_0 q^+(\tau)+|f(\tau)|}{((b-\tau)(\tau-a))^{1-\alpha}}d\tau\right)\\ > \left(\frac{\Gamma(\alpha)}{2}(b-a)^{\alpha-1}\right)^2,\end{aligned} \tag{4.1.1}$$

*holds, where the constants $\mu_0$ and $\lambda_0$ are the same as in (1.1.12).*

**Theorem 4.1.2** (Lyapunov type inequality). *Let $x(t)$ be a nontrivial solution of (1.2.3) satisfying the Dirichlet boundary conditions (1.1.2). If $x(t) \neq 0$ in $(a,b)$, then the inequality*

$$\left(\int_a^b p^+(\tau) + q^+(\tau)\,d\tau\right)\left(\int_a^b \left(\mu_0 p^+(\tau) + \lambda_0 q^+(\tau) + |f(\tau)|\right)d\tau\right) > \left(\frac{\Gamma(\alpha)}{2}\left(\frac{2}{b-a}\right)^{\alpha-1}\right)^2, \tag{4.1.2}$$

*holds, where the constants $\mu_0$ and $\lambda_0$ are defined as before.*

We remark that both inequalities (4.1.1) and (4.1.2) reduce to (1.1.5) and (1.1.6) respectively, when $\alpha = 2$.

For some other related results on Hartman and Lyapunov type inequalities were obtained for different fractional boundary value problems. In this direction, we refer in particular to, S.B. Eliason [44], Ferreira [22, 23], R. P. Agarwal, A. Özbekler [**?**, **?**, 31, **?**], Abdeljawad et al. [45], Jarad et al. [28] and the references given therein.

The main objective of this work is to establish several generalizations of Hartman- and Lyapunov-type inequalities for forced generalized Caputo fractional differential equations with mixed nonlinearities. In particular, we consider the following generalization of the Caputo fractional boundary value problem (1.2.3)-(1.1.2).

For the continuous real valued functions, $p, q, f$ and $g$, with $p(t) > 0$, $q(t) > 0$ and $g$ strictly increasing and $g(t) < t$ on $[a, +\infty)$ and $\mu, \lambda > 0$ being reals, we will consider the following generalization for Caputo boundary value problem

$${}^{c}\mathscr{D}_{a+}^{\alpha,\rho} x(t) + H(t,x) = f(t),\ t \in (a,b), \tag{4.1.3}$$

with the conditions (1.1.2), where

$$H(t,x) = p(t)x(t)|x(t)|^{\mu-1} + q(t)x(g(t))|x(g(t))|^{\lambda-1}, \tag{4.1.4}$$

whose special cases contain the well-known equations of Emden–Fowler-type and half-linear equations.

It is clear that the two special cases of (1.2.3) are the second order forced sub-linear equation

$${}^{c}\mathscr{D}_{a+}^{\alpha,\rho} x(t) + q(t)x(g(t))|x(g(t))|^{\lambda-1} = f(t),\ 0 < \lambda < 1\ t \in (a,b), \tag{4.1.5}$$

and the second order forced super-linear equation

$$ {}^{c}\mathscr{D}_{a+}^{\alpha,\rho}x(t)+p(t)x(t)|x(t)|^{\mu-1}=f(t),\ 1<\mu\leq 2,\ t\in(a,b). \tag{4.1.6}$$

Further, we note that letting $\rho=1,\ \lambda\longrightarrow 1^{-}$ and $\mu\longrightarrow 1^{+}$ in (1.2.3) results in

$$ {}^{c}\mathscr{D}_{a+}^{\alpha}x(t)+v(t)x(t)=f(t),\ 1<\alpha\leq 2,\ t\in(a,b), \tag{4.1.7}$$

where

$$v(t)=p(t)+q(t). \tag{4.1.8}$$

and hence, our results extend the classical Lyapunov [15] and Hartman [20] results.

Using the generalized fractional integrals, an attempt is made to establish certain new fractional integral inequalities, related to the nonlinear problem (1.2.3-1.1.2). The best fractional polynomial inequalitie has powers $(2,\alpha)$ and is of the form

$$P_{(A,B,\alpha)}(y)=Ay^{2}-By^{\alpha}+C_{(A,B,\alpha)}>0, \tag{4.1.9}$$

where $A$ is positive, and $B,y$ are nonnegative.

We will need to prove the following lemma for the next result.

**Lemma 4.1.3.** *[30]If $A$ is positive, and $B$, $y$ are nonnegative, then for any $\alpha\in(0,2)$ we have*

$$P_{(A,B,\alpha)}(y)=Ay^{2}-By^{\alpha}+\Theta_{\alpha}A^{\alpha/(\alpha-2)}B^{-2/(\alpha-2)}\geq 0, \tag{4.1.10}$$

*where $\Theta_{\alpha}=-(2-\alpha)\alpha^{\frac{-\alpha}{\alpha-2}}2^{\frac{2}{\alpha-2}}$. Moreover, equality is attained for the $B=y=0$.*

**Proof.** Let

$$F(y)=Ay^{2}-By^{\alpha},\qquad y\geq 0, \tag{4.1.11}$$

where $A>0$ and $B\geq 0$. Clearly, when $y=0$ or $B=0$, inequality (4.1.10) is obvious.

On the other hand, if $B>0$, then $F$ attains its minimum at

$$y_{0}=\left(\frac{\alpha A^{-1}B}{2}\right)^{1/(2-\alpha)},$$

and the minimum value is

$$F_{min} = -(2-\alpha)\,\alpha^{\frac{\alpha}{2-\alpha}}\,2^{\frac{2}{\alpha-2}}\,A^{-\frac{\alpha}{2-\alpha}}\,B^{\frac{2}{2-\alpha}}.$$

Thus, inequality (4.1.10) follows immediately. Note that if $B > 0$, then inequality (4.1.10) is strict. ■

The chapter is organized as follows. Section 2, we prove a generalized Hartman and Lyapunov type inequalities and related disconjugacy for the linear generalized Caputo fractional derivatives problem (1.2.1-1.1.2). Section 3, we derive a generalized Hartman and Lyapunov type inequalities and related disconjugacy for the nonlinear generalized Caputo fractional derivatives of problem (4.2.17-1.1.2). The generalized Hartman and Lyapunov type inequalities for (4.3.25-1.1.2) are stated and proved in Section 4. We will end this work by presenting an application of the theorems obtained for the nonlinear problem (1.2.3-1.1.2).

## 4.2 Generalized Hartman and Lyapunov type inequalities (1.2.1-1.1.2)

### 4.2.1 Lyapunov type inequality for the problem (1.2.1-1.1.2)

We establish a new some versions generalized Hartman and Lyapunov type integral inequality, which generalize previous result. In [28], Lyapunov established the following striking inequality:

**Theorem 4.2.1.** *[28]If a nontrivial continuous solution of the problem (1.2.1-1.1.2) exists, then the Lyapunov-type inequality is*

$$\int_a^b |p(\tau)|\,d\tau > \frac{1}{G_{max}}, \tag{4.2.1}$$

*where $G_{max}$ is defined in (3.2.9) and in particular, for $\alpha = 2$ and $\rho = 1$ in (1.2.1) gives the standard Lyapunov inequality for (1.1.1-1.1.2).*

**Theorem 4.2.2.** *If a nontrivial continuous solution of the problem (1.2.1-1.1.2) exists, then the Lyapunov-type inequality is*

$$\int_a^b p^+(\tau)\,d\tau > \frac{1}{G_{max}}, \tag{4.2.2}$$

*where $G_{max}$ is defined in (3.2.9) and in particular, for $\alpha = 2$ and $\rho = 1$ in (1.2.1) gives the standard*

*Lyapunov inequality for (1.1.1-1.1.2).*

**Proof.** By Theorem 3.2.2 and from (3.2.1), it follows that if $x$ is a nontrivial continuous solution of the (1.2.1)

$$|x(t)| \leq \int_a^b |G_{max}(t,\tau)p(\tau)||x(\tau)|d\tau. \tag{4.2.3}$$

Let $\mathfrak{B} = C([a,b])$ be a Banach space endowed a norm

$$\|x\|_\infty = \max_{t\in[a,b]} |x(t)|, x \in B. \tag{4.2.4}$$

Hence, from (4.2.3) and (4.2.4), we get

$$\|x\|_\infty \leq \max_{t\in[a,b]} \left| \int_a^b G_{max}(t,\tau)p(\tau)\,d\tau \right| \|x\|_\infty,$$

or equivalently,

$$\max_{t\in[a,b]} \left| \int_a^b G_{max}(t,\tau)p(\tau)\,d\tau \right| \geq 1. \tag{4.2.5}$$

Using the properties of Green's function $G_{max}(t,\tau)$ particularly, $G_{max}$ in Lemma 3.2.4 gives the inequality

$$\int_a^b |p(\tau)|d\tau > \frac{1}{G_{max}}, \tag{4.2.6}$$

By replacing $|p(\tau)|$ by (1.1.4), the non negative part of $p(\tau)$ becomes as in (4.2.1).

Using the fact that

$$\int_a^b |p(\tau)|d\tau \geq \int_a^b p^+(\tau)\,d\tau \geq \int_a^\kappa p^+(\tau)\,d\tau > \frac{1}{G_{max}^\kappa},$$

for all $\kappa \in [a,b]$ where $G_{max}^b := G_{max}$.

The key idea is to apply Sturm's oscillation theory, which guarantees the existence of a nontrivial solution to the problem

$$\left({}^c\mathscr{D}_{a+}^{\alpha,\rho}x\right)(\tau) + p^+(\tau)x(\tau) = 0, \qquad \text{with } x(a) = x(\kappa) = 0,\ \tau \in [a,\kappa],\ \text{for some } a < \kappa < b. \tag{4.2.7}$$

This solution vanishes at a point $\kappa < b$, indicating that oscillatory behavior is induced by $p^+$, even though $p(\tau)$ may be negative over a portion of the interval. Integrating over $[a, b]$, we obtain

$$\int_a^b p^+(\tau)\, d\tau \geq \int_a^\kappa p^+(\tau)\, d\tau. \tag{4.2.8}$$

Now, since the solution vanishes at $\kappa$, the Lyapunov inequality (4.2.1) applied on $[a, \kappa]$ yields

$$\int_a^\kappa p^+(\tau)\, d\tau > \frac{1}{G^\kappa_{max}}. \tag{4.2.9}$$

Combining the previous estimates gives

$$\int_a^b |p(\tau)|\, d\tau \geq \int_a^b p^+(\tau)\, d\tau > \frac{1}{G^\kappa_{max}}.$$

Now, we claim that

$$\frac{1}{G^\kappa_{max}} > \frac{1}{G^b_{max}}.$$

In fact, from (3.2.9), we have

and

$$\begin{aligned} \frac{G^\kappa_{max}}{G^b_{max}} &= \left(\frac{L_\kappa}{L_b}\right)^{\frac{\rho-1}{\rho}} \frac{(L_\kappa - a^\rho)(b^\rho - a^\rho)}{(L_b - a^\rho)(\kappa^\rho - a^\rho)} \left(\frac{(\kappa^\rho - L_\kappa)}{(b^\rho - L_b)}\right)^{\alpha-1} \\ &\leq \left(\frac{L_\kappa}{L_b}\right)^{\frac{2\rho-1}{\rho}} \left(\frac{b}{\kappa}\right)^\rho \left(\frac{(\kappa^\rho - L_\kappa)}{(b^\rho - L_b)}\right)^{\alpha-1} \end{aligned} \tag{4.2.10}$$

with $0 < a < \kappa < b$.

We consider three cases.

Case 1. $N = 0$

$(\alpha+1)\rho-1=0$. From (3.2.9),we have

$$\begin{aligned}\frac{L_\kappa^\rho}{L_b^\rho} &= \frac{(\rho-1)\,a^\rho\kappa^\rho}{(2\rho+1)\,\kappa^\rho-a^\rho}\frac{(2\rho+1)\,b^\rho-a^\rho}{(\rho-1)\,a^\rho b^\rho} \\ &= \frac{\kappa^\rho}{b^\rho}\frac{(2\rho+1)\,b^\rho-a^\rho}{(2\rho+1)\,\kappa^\rho-a^\rho} \\ &= \frac{(2\rho+1)\,\kappa^\rho b^\rho-\kappa^\rho a^\rho}{(2\rho+1)\,\kappa^\rho b^\rho-b^\rho a^\rho} \\ &\leq \frac{\kappa^\rho a^\rho}{b^\rho a^\rho}<1.\end{aligned}$$

and

$$\begin{aligned}\left(\frac{(\kappa^\rho-L_\kappa)}{(b^\rho-L_b)}\right)^{\alpha-1} &= \left(\frac{\kappa^\rho}{b^\rho}\frac{((2\rho+1)\,b^\rho-a^\rho)\,((2\rho+1)\,\kappa^\rho-\rho a^\rho)}{((2\rho+1)\,\kappa^\rho-a^\rho)\,((2\rho+1)\,b^\rho-\rho a^\rho)}\right)^{\alpha-1} \\ &\leq \left(\frac{\kappa^\rho}{b^\rho}\frac{((2\rho+1)\,b^\rho-a^\rho)}{((2\rho+1)\,\kappa^\rho-a^\rho)}\right)^{\alpha-1} \\ &= \left(\frac{\kappa^\rho a^\rho-(2\rho+1)\,\kappa^\rho b^\rho}{b^\rho a^\rho-(2\rho+1)\,\kappa^\rho b^\rho}\right)^{\alpha-1} \\ &= \left(\frac{\kappa^\rho}{b^\rho}\right)^{\alpha-1}.\end{aligned}$$

By substituting into the inequality (4.2.10), we obtain

$$\frac{G_{\max}^\kappa}{G_{\max}^b}\leq\left(\frac{\kappa^\rho}{b^\rho}\right)^{2-\alpha},$$

or

$$\frac{1}{\kappa^{\rho(2-\alpha)}G_{\max}^\kappa}\geq\frac{1}{b^{\rho(2-\alpha)}G_{\max}^b}.$$

Case 2. $N \neq 0$ $(\alpha+1)\rho-1 \neq 0$. From (3.2.11), we have

$$\begin{aligned}
\left(\frac{M_\kappa}{M_b}\right)^2 &= \frac{\left(\left((\alpha\rho-1)\,a^\rho+(2\rho-1)\,\kappa^\rho\right)^2-4\left(1-(\alpha+1)\,\rho\right)(1-\rho)\,a^\rho\kappa^\rho\right)}{\left(\left((\alpha\rho-1)\,a^\rho+(2\rho-1)\,b^\rho\right)^2-4\left(1-(\alpha+1)\,\rho\right)(1-\rho)\,a^\rho b^\rho\right)} \\
&\leq \frac{\kappa^\rho}{b^\rho}\frac{(2\rho-1)^2\kappa^\rho-4\left(1-(\alpha+1)\,\rho\right)(1-\rho)\,a^\rho}{(2\rho-1)^2 b^\rho-4\left(1-(\alpha+1)\,\rho\right)(1-\rho)\,a^\rho} \\
&\leq \left(\frac{\kappa^\rho}{b^\rho}\right)^2.
\end{aligned}$$

and also from (3.2.10),we have

$$\begin{aligned}
\frac{L_\kappa^\rho}{L_b^\rho} &= \frac{(\alpha\rho-1)\,a^\rho+(2\rho-1)\,\kappa^\rho+M_\kappa}{(\alpha\rho-1)\,a^\rho+(2\rho-1)\,b^\rho+M_b} \\
&\leq \frac{b^\rho}{\kappa^\rho}\frac{(2\rho-1)\,\kappa^\rho+M_\kappa}{(2\rho-1)\,b^\rho+M_b} \\
&= \frac{(2\rho-1)\,\kappa^\rho b^\rho+b^\rho M_\kappa}{(2\rho-1)\,\kappa^\rho b^\rho+\kappa^\rho M_b} \\
&\leq \frac{b^\rho M_\kappa}{\kappa^\rho M_b}<1
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{(\kappa^\rho-L_\kappa)}{(b^\rho-L_b)}\right)^{\alpha-1} &= \left(\frac{2N\kappa^\rho-\left((\alpha\rho-1)\,a^\rho+(2\rho-1)\,\kappa^\rho+M_\kappa\right)}{2Nb^\rho-\left((\alpha\rho-1)\,a^\rho+(2\rho-1)\,b^\rho+M_b\right)}\right)^{\alpha-1} \\
&\leq \left(\frac{2N\kappa^\rho b^\rho-\left((2\rho-1)\,\kappa^\rho b^\rho+b^\rho M_\kappa\right)}{2N\kappa^\rho b^\rho-\left((2\rho-1)\,\kappa^\rho b^\rho+\kappa^\rho M_b\right)}\right)^{\alpha-1} \\
&\leq \left(\frac{b^\rho M_\kappa}{\kappa^\rho M_b}\right)^{\alpha-1}.
\end{aligned}$$

Replacing in the inequality (4.2.10), we get

$$\frac{G_{\max}^\kappa}{G_{\max}^b} \leq \left(\frac{\kappa^\rho}{b^\rho}\right)^{2-\alpha}\cdot\left(\frac{M_\kappa}{M_b}\right)^{\alpha-1},$$

or

$$\frac{1}{\kappa^{2-\alpha}M_\kappa^{\alpha-1}G_{\max}^\kappa} \geq \frac{1}{b^{2-\alpha}M_b^{\alpha-1}G_{\max}^b}.$$

Case 3. $N \neq 0$ with $\rho = 1$ . From (3.2.9), we have

$$L_\kappa - L_b = \frac{\kappa - b}{\alpha} < 0$$

and

$$\left(\frac{(\kappa - L_\kappa)}{(b - L_b)}\right)^{\alpha-1} = \left(\frac{\alpha\kappa - ((\alpha-1)a+\kappa)}{\alpha b - ((\alpha-1)a+b)}\right)^{\alpha-1} \leq \left(\frac{\kappa}{b}\right)^{\alpha-1}.$$

We obtain

$$\frac{G^{\kappa}_{max}}{G^{b}_{max}} \leq \left(\frac{b}{\kappa}\right)^{2-\alpha},$$

or

$$\frac{1}{\kappa^{2-\alpha}G^{\kappa}_{max}} \geq \frac{1}{b^{2-\alpha}G^{b}_{max}},$$

which completes the proof of the claim.

■

**Corollary 4.2.3** (Disconjugacy)**.** *if*

$$\left(\int_a^b p^+(\tau)\,d\tau\right) \leq \frac{1}{G_{max}}, \tag{4.2.11}$$

*then (1.2.1) is disconjugate in* $[a,b]$.

## 4.2.2 Hartman type inequality for the problem (4.1.7-1.1.2)

**Theorem 4.2.4.** *Let* $p : [a,b] \longrightarrow \mathbb{R}^+$ *be a real nontrivial Lebesgue integrable function. Assume that* $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ *is a concave and nondecreasing function. If the fractional boundary value problem (4.1.7-1.1.2) has a nontrivial solution* $x$*, then*

$$\int_a^b \frac{p^+(\tau)\,\tau^{\rho-1}d\tau}{(\tau^\rho - a^\rho)^{1-\alpha}(b^\rho - \tau^\rho)^{1-\alpha}} > \left(\frac{\Gamma(\alpha)}{2}\right)^2 \left(\frac{\rho}{b^\rho - a^\rho}\right)^{2-2\alpha}. \tag{4.2.12}$$

**Proof.** By Theorem 3.2.2, a solution $x \in C([a,b])$ to (4.1.7-1.1.2) has the expression

$$x(t) \leq \int_a^b G_{max}(t,\tau)p(\tau)x(\tau)\,d\tau. \tag{4.2.13}$$

From this, for any $a \leq t \leq b$, we obtain

$$|x(t)| \leq \int_a^b |G_{max}(t,\tau)||p(\tau)x(\tau)|d\tau. \tag{4.2.14}$$

We get that

$$\begin{aligned} |x(t)| &\leq \int_a^b G_{max}(\tau,\tau)|p(\tau)||x(\tau)|d\tau \\ &\leq \frac{1}{\Gamma(\alpha)}(\rho(b^\rho - a^\rho))^{1-\alpha}\int_a^b \frac{|p(\tau)||x(\tau)|\tau^{\rho-1}}{(\tau^\rho - a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}d\tau. \end{aligned} \tag{4.2.15}$$

Then we get

$$\int_a^b \frac{p^+(\tau)\tau^{\rho-1}}{(\tau^\rho - a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}d\tau > \Gamma(\alpha)\left(\frac{\rho}{b^\rho - a^\rho}\right)^{1-\alpha}.$$

The proof is complete. ■

**Corollary 4.2.5** (Disconjugacy)**.** *if*

$$\left(\int_a^b \left(\frac{\tau^\rho - a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}\tau^{\rho-1}p^+(\tau)d\tau\right) \leq \Gamma(\alpha)\left(\frac{\rho}{b^\rho - a^\rho}\right)^{1-\alpha}. \tag{4.2.16}$$

*Then (4.1.7) is disconjugate in* $[a,b]$.

**Proof.** Suppose, to the contrary, that (4.1.7) is not disconjugate on $[a,b]$ Then, by definition, there exists a real solution $x$ of (4.1.7) with $x$ which is nontrivial and such that $x(a) = 0$ and $x$ has a generalized zero $b$ in $[a,b]$. We will have $a < b$ and either $x(b) = 0$ or $x(a) \times x(b) < 0$. Therefore, applying Theorems 4.2.2 and 3.2.1, we obtain

$$\left(\int_a^b \left(\frac{\tau^\rho - a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}\tau^{\rho-1}p^+(\tau)d\tau\right) > \Gamma(\alpha)\left(\frac{\rho}{b^\rho - a^\rho}\right)^{1-\alpha},$$

which contradicts to the condition of the theorem. ■

We note that the results obtained in this subsection generalize the results by R. P. Agarwal, A. Özbekler (see [30]).

In the following section, we wish to consider the fractional differential equation

$$ {}^{c}\mathscr{D}_{a+}^{\alpha,\rho}x(t)+v(t)h(x(t))=f(t), t\in(a,b), \tag{4.2.17}$$

assuming that there exist nonnegative numbers $A, B$, and $\mu\in(0,2)$ such that

$$|h(x(t))|\leq A|x^{\mu}(t)|+B, \text{ for all } x. \tag{4.2.18}$$

where $A, B$ and $\mu$ are positive constants

# 4.3 Generalized Hartman and Lyapunov type inequalities (4.2.17-1.1.2)

We establish a new some versions generalized Hartman and Lyapunov type integral inequality, which generalize previous result.

## 4.3.1 Hartman type inequality for the problem (4.2.17-1.1.2)

**Theorem 4.3.1.** *Let $h:[a,b]\longrightarrow\mathbb{R}^{+}$ be a real nontrivial Lebesgue integrable function. If the fractional boundary value problem (4.2.17-1.1.2) has a nontrivial solution $x$. Assume that there exist nonnegative numbers $A, B$, and $\mu$ such that (4.2.18) holds, if $x(t)>0$ in $(a,b)$ then the inequality*

$$\left(\int_a^b\left(\frac{\tau^\rho-a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\tau^{\rho-1}p^{+}(\tau)\,d\tau\right)\left(\int_a^b\frac{p^{+}(\tau)(A\mu_0+B)+f^{-}(\tau)\tau^{\rho-1}d\tau}{(\tau^\rho-a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\right) > \frac{\Gamma(\alpha)}{2(A(B+A\mu_0))^{\frac{1}{2}}}\left(\frac{\rho}{b^\rho-a^\rho}\right)^{1-\alpha}, \tag{4.3.1}$$

*holds, where the constant $\mu_0$ is defined in (1.1.12).*

**Proof.** Let $x(t)$ be a positive solution of (4.2.17) with (1.1.2) where where $a<b$ are consecutive zeros. Then by using Theorem 3.2.2, $x(t)$ can be expressed as

$$x(t)=\int_a^b G(t,\tau)\left[p(\tau)h(x(\tau))-f(\tau)\right]d\tau. \tag{4.3.2}$$

By (4.3.2), we get that

$$|x(t)| \leq \int_a^b G(\tau,\tau)\,|p(\tau)\,h(x(\tau)) - f(\tau)|\,d\tau, \tag{4.3.3}$$

where

$$f^-(\tau) = \max(-f(\tau), 0). \tag{4.3.4}$$

It follows from $x(a) = 0 = x(b)$ and $x$ is not identically zero on $[a,b]$ one can choose $c \in (a,b)$ such that $M = |x(c)| = \max_{t\in(a,b)}(|x(t)|)$.

By assumption, we have, $M = |x(c)| = \max_{t\in(a,b)}(|x(t)|)$, $c \in (a,b)$. From (4.3.4), we observe that

$$\begin{aligned} |x(c)| &\leq \frac{1}{\Gamma(\alpha)}(\rho(b^\rho - a^\rho))^{1-\alpha}\int_a^b \frac{p^+(\tau)[A|x^\mu(\tau)| + B] + f^-(\tau)\tau^{\rho-1}}{(\tau^\rho - a^\rho)^{1-\alpha}(b^\rho - \tau^\rho)^{1-\alpha}}d\tau \\ &\leq P_A|x^\mu(c)| + P_B + F_0, \end{aligned} \tag{4.3.5}$$

where

$$P_A = \frac{A}{\Gamma(\alpha)}\left(\frac{\rho}{b^\rho - a^\rho}\right)^{\alpha-1}\int_a^b \left(\frac{\tau^\rho - a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho - \tau^\rho}{\rho}\right)^{\alpha-1}\tau^{\rho-1}p^+(\tau)\,d\tau \tag{4.3.6}$$

and

$$P_B = \frac{B}{A}P_A, \tag{4.3.7}$$

$$F_0 = \frac{1}{\Gamma(\alpha)}\left(\frac{\rho}{b^\rho - a^\rho}\right)^{\alpha-1}\int_a^b \left(\frac{\tau^\rho - a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} f^-(\tau)\tau^{\rho-1}d\tau. \tag{4.3.8}$$

On the other hand, (4.1.10) in Lemma 4.1.3 with $A = B = 1$, implies that

$$|x^\mu(c)| \leq x^2(c) + \mu_0, \tag{4.3.9}$$

we have

$$|x(c)| \leq \frac{1}{\Gamma(\alpha)}(\rho(b^\rho - a^\rho))^{1-\alpha}\int_a^b \frac{|p^+(\tau)|\left|Ax^2(c) + A\mu_0 + B\right| + f^-(\tau)}{(\tau^\rho - a^\rho)^{1-\alpha}(b^\rho - \tau^\rho)^{1-\alpha}\tau^{1-\rho}}d\tau. \tag{4.3.10}$$

Using these inequalities and (4.3.5), we find the following quadratic inequality

$$P_A x^2(c) - |x(c)| + (P_A \mu_0 + P_B + F_0) > 0. \tag{4.3.11}$$

But this is only possible when

$$P_A (P_A \mu_0 + P_B + F_0) > \frac{1}{4},$$

which is the same as (4.3.1). This completes the proof of Theorem 4.3.1. ■

**Remark 4.3.2.** For $p \in L^1([a,b], \mathbb{R}^+)$, if $h(x(t)) = x(t)$ (linear case), $A = \mu = 1$ and $B = 0$, we obtain

$$\left(\int_a^b \frac{p^+(\tau)\tau^{\rho-1}}{(\tau^\rho - a^\rho)^{1-\alpha}(b^\rho - \tau^\rho)^{1-\alpha}} d\tau\right) \left(\int_a^b \frac{p^+(\tau)\mu_0 + f^-(\tau)\tau^{\rho-1}}{(\tau^\rho - a^\rho)^{1-\alpha}(b^\rho - \tau^\rho)^{1-\alpha}} d\tau\right) > \Gamma(\alpha)\left(\frac{\rho}{b^\rho - a^\rho}\right)^{1-\alpha}.$$

The proof of the following theorem can be obtained easily by the same method used in above theorem, with a slight modification. Hence it is omitted.

**Theorem 4.3.3.** *Let $h : [a,b] \longrightarrow \mathbb{R}^+$ be a real nontrivial Lebesgue integrable function. If the fractional boundary value problem (4.2.17-1.1.2) has a nontrivial solution $x$. Assume that there exist nonnegative numbers $A, B$, and $\mu$ such that (4.1.10) holds, if $x(t) < 0$ in $(a,b)$ then the inequality*

$$\left(\int_a^b \left(\frac{\tau^\rho - a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} \tau^{\rho-1} p^+(\tau) d\tau\right) \left(\int_a^b \frac{p^+(\tau)(A\mu_0 + B) + f^+(\tau)\tau^{\rho-1} d\tau}{(\tau^\rho - a^\rho)^{1-\alpha}(b^\rho - \tau^\rho)^{1-\alpha}}\right) > \frac{\Gamma(\alpha)}{2(A(B + A\mu_0))^{\frac{1}{2}}}\left(\frac{\rho}{b^\rho - a^\rho}\right)^{1-\alpha}, \tag{4.3.12}$$

*holds, where the constant $\mu_0$ is defined in (1.1.12).*

**Proof.** Let $x(t)$ be a negative solution of (4.2.17-1.1.2) where where $a < b$ are consecutive zeros In fact, if $x(t) < 0$ for $t \in (a,b)$, then we can consider $-x(t)$ as a positive solution of (4.2.17-1.1.2)

Let $M = |x(c)| = \max_{t \in (a,b)}(|x(t)|)$, $c \in (a,b)$.

Then by (4.2.17-4.2.18) and (4.3.2), we have

$$\begin{aligned}|x(c)| &= \int_a^b G(t,\tau)[p(\tau)h(x(\tau))+f(\tau)]d\tau \\ &\le \frac{1}{\Gamma(\alpha)}(\rho(b^\rho-a^\rho))^{1-\alpha}\int_a^b \frac{p^+(\tau)[A|x^\mu(\tau)|+B]+f^+(\tau)\tau^{\rho-1}}{(\tau^\rho-a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}d\tau \\ &\le P_A|x^\mu(c)|+P_B+F_0,\end{aligned}$$

where $P_A$ and $P_B$ are defined in (4.3.6) and (4.3.7) , and

$$\tilde{F}_0=\frac{1}{\Gamma(\alpha)}\left(\frac{\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\int_a^b\left(\frac{\tau^\rho-a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}f^+(\tau)\tau^{\rho-1}d\tau.$$

Now repeating the same steps as in Theorem 4.3.1 , we obtain (4.3.12) which completes the proof of Theorem 4.3.3 .

From Theorems 4.3.1 and 4.3.3 , the next result immediately follows. ■

**Theorem 4.3.4.** *Let $h:[a,b]\longrightarrow\mathbb{R}^+$ be a real nontrivial Lebesgue integrable function. If the fractional boundary value problem (4.2.17-1.1.2) has a nontrivial solution $x$. Assume that there exist nonnegative numbers $A, B$, and $\mu$ such that (4.2.18) holds, if $x(t)\neq 0$ in $(a,b)$ then the inequality*

$$\left(\int_a^b\left(\frac{\tau^\rho-a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\tau^{\rho-1}p^+(\tau)d\tau\right)\left(\int_a^b\frac{p^+(\tau)(A\mu_0+B)+|f(\tau)|\tau^{\rho-1}d\tau}{(\tau^\rho-a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\right) > \frac{\Gamma(\alpha)}{2(A(B+A\mu_0))^{\frac{1}{2}}}\left(\frac{\rho}{b^\rho-a^\rho}\right)^{1-\alpha} \tag{4.3.13}$$

*holds, where the constant $\mu_0$ is defined in (1.1.12).*

**Corollary 4.3.5** (Disconjugacy)**.** *if*

$$\left(\int_a^b\left(\frac{\tau^\rho-a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\tau^{\rho-1}p^+(\tau)d\tau\right)\left(\int_a^b\frac{p^+(\tau)(A\mu_0+B)+|f(\tau)|\tau^{\rho-1}d\tau}{(\tau^\rho-a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\right) \le \frac{\Gamma(\alpha)}{2(A(B+A\mu_0))^{\frac{1}{2}}}\left(\frac{\rho}{b^\rho-a^\rho}\right)^{1-\alpha} \tag{4.3.14}$$

*then Eq. (4.2.17) is disconjugate in $[a,b]$.*

Since (4.2.17) for all $t \in (a,b)$, Ineq.'s (4.3.1), (4.3.12) and (4.3.13) in Theorems 4.3.1-4.3.3 and Theorem 4.3.4 immediately imply the following Lyapunov type inequalities for (4.2.17).

### 4.3.2 Lyapunov-type inequality for the problem (4.2.17-1.1.2)

**Theorem 4.3.6.** *Let $h : [a,b] \longrightarrow \mathbb{R}^+$ be a real nontrivial Lebesgue integrable function. Assume that there exist nonnegative numbers $A, B$, and $\mu \in (0,2)$ such that (4.2.18) holds. If the fractional boundary value problem (4.2.17-1.1.2) has a nontrivial positive solution $x$, then*

$$\left(\int_a^b p^+(\tau)\, d\tau\right)\left(\int_a^b p^+(\tau)(A\mu_0 + B) + f^-(\tau)\,\tau^{\rho-1} d\tau\right) > \frac{(A(B+A\mu_0))^{\frac{-1}{2}}}{2G_{max}}, \tag{4.3.15}$$

*holds*.

**Proof.** By virtue of Theorem 3.2.2 and from (3.2.1), it follows that if $x$ is a nontrivial continuous solution of the Eq. (4.2.17)

$$|x(t)| \leq \int_a^b |G_{max}(t,\tau)||p(\tau)\, h(x(\tau))| + f^-(\tau)\, d\tau. \tag{4.3.16}$$

Using the properties of Green's function $G(t,\tau)$ particularly, $G_{max}$ in Lemma 3.2.4, gives the inequality

$$|x(t)| \leq G_{max} \int_a^b |p(\tau)[A|x^{\mu}(\tau)| + B]| + f^-(\tau)\, d\tau. \tag{4.3.17}$$

By assumption, we have $M = |x(c)| = \max_{t\in(a,b)}(|x(t)|)$, $c \in (a,b)$.

$$|x(c)| \leq G_{max} \int_a^b |p(\tau)[A|x^{\mu}(c)| + B]| + f^-(\tau)\, d\tau. \tag{4.3.18}$$

From (4.3.18) we observe that

$$|x(c)| \leq P_A |x^{\mu}(c)| + P_B + F_0$$

where

$$P_B = \frac{B}{A} P_A,\ P_A = A G_{max} \int_a^b p^+(\tau)\, d\tau \tag{4.3.19}$$

Using the facts

$$|x(c)|^{\mu} \leq x^2(c) + \mu_0,$$

we have

$$P_A x^2(c) - |x(c)| + (P_A \mu_0 + P_B + F_0) > 0 \tag{4.3.20}$$

But this is only possible when

$$P_A (P_A \mu_0 + P_B + F_0) > \frac{1}{4}$$

and

$$\left(\int_a^b p^+(\tau)\, d\tau\right)^2 > \frac{1}{4(G_{max})^2 (A(B + A\mu_0))} \tag{4.3.21}$$

which is the same as (4.3.15). This completes the proof of Theorem 4.3.6. ■

**Theorem 4.3.7.** *Let $h : [a,b] \longrightarrow \mathbb{R}^+$ be a real nontrivial Lebesgue integrable function. Assume that there exist nonnegative numbers $A, B$, and $\mu \in (0,2)$ such that (4.2.18) holds. If the fractional boundary value problem (4.2.17-1.1.2) has a nontrivial negative solution $x$, then*

$$\left(\int_a^b p^+(\tau)\, d\tau\right) \left(\int_a^b \left(p^+(\tau)(A\mu_0 + B) + f^+(\tau)\right) \tau^{\rho-1} d\tau\right) > \frac{(A(B + A\mu_0))^{\frac{-1}{2}}}{2G_{max}} \tag{4.3.22}$$

*holds.*

**Theorem 4.3.8.** *Let $h : [a,b] \longrightarrow \mathbb{R}^+$ be a real nontrivial Lebesgue integrable function. Assume that there exist nonnegative numbers $A, B$, and $\mu \in (0,2)$ such that (4.2.18) holds. If $x(t) \neq 0$, then*

$$\left(\int_a^b p^+(\tau)\, d\tau\right) \left(\int_a^b \left(p^+(\tau)(A\mu_0 + B) + |f(\tau)|\right) \tau^{\rho-1} d\tau\right) > \frac{(A(B + A\mu_0))^{\frac{-1}{2}}}{2G_{max}} \tag{4.3.23}$$

*holds.*

**Corollary 4.3.9** (Disconjugacy). *if*

$$\begin{aligned} &\left(\int_a^b p^+(\tau)\tau^{\rho-1} d\tau\right) \left(\int_a^b p^+(\tau)(A\mu_0 + B) + |f(\tau)|\tau^{\rho-1} d\tau\right) \\ &\qquad \leq \frac{\Gamma(\alpha)}{2(A(B + A\mu_0))^{\frac{1}{2}}} \left(\frac{\rho}{b^{\rho} - a^{\rho}}\right)^{1-\alpha} \end{aligned} \tag{4.3.24}$$

*then (4.2.17) is disconjugate in $[a,b]$.*

In the third section, we shall present a new result for forced fractional differential equations with mixed non linearities

$$ {}^{c}\mathscr{D}_{a+}^{\alpha,\rho}x(t)+p(t)x(t)|x(t)|^{\mu-1}+q(t)x(g(t))|x(g(t))|^{\lambda-1}=f(t),\ t\in(a,b), \tag{4.3.25}$$

with $p(t)>0$, $q(t)>0$, $,f\in C([a,b],\mathbb{R})$ and $g$ strictly increasing and $g(t)\leq t$ on $[a,+\infty)$.

# 4.4 Generalized Hartman and Lyapunov type inequalities for (4.3.25-1.1.2)

We establish a some versions generalized Hartman and Lyapunov type integral inequality.

## 4.4.1 Hartman type inequality for the problem (4.3.25-1.1.2)

**Theorem 4.4.1** (Hartman type inequality). *Let* $x(t)$ *be a nontrivial solution of (4.3.25) satisfying the Dirichlet boundary conditions (1.1.2). If* $x(t)>0$ *in* $(a,b)$*, Let* $|x|$ *be maximized at a point* $c\in(a,b)$*, then the inequality*

$$\left(\int_a^b\frac{\left(p^{+}(\tau)+q^{+}\right)(\tau)\tau^{\rho-1}d\tau}{(\tau^{\rho}-a^{\rho})^{1-\alpha}(b^{\rho}-\tau^{\rho})^{1-\alpha}}\right)\left(\int_a^b\frac{\left(\mu_0p^{+}+\lambda_0q^{+}+f^{-}\right)(\tau)\tau^{\rho-1}d\tau}{(\tau^{\rho}-a^{\rho})^{1-\alpha}(b^{\rho}-\tau^{\rho})^{1-\alpha}}\right) > \left(\frac{\Gamma(\alpha)}{2}\right)^2\left(\frac{\rho}{b^{\rho}-a^{\rho}}\right)^{2-2\alpha} \tag{4.4.1}$$

*holds, where the constants* $\mu_0$ *and* $\lambda_0$ *are defined in (1.1.12).*

**Proof.** First, we note that the solution of the generalized fractional differential equation (4.3.25) of order $\alpha\in(0,2]$, satisfying the Dirichlet boundary conditions (1.1.2) can be represented by (3.2.2) where $G(t,\tau)$ is defined by (3.2.1).

Now let $x(t)$ be a positive solution of (4.3.25-1.1.2), $x(t)$ can be expressed as

$$x(t)=\int_a^b G(t,\tau)\left[p(\tau)x^{\mu}(\tau)+q(\tau)x^{\lambda}(g(\tau))-f(\tau)\right]d\tau \tag{4.4.2}$$

By assumption, we have, $|x(c)|=\max_{t\in(a,b)}(|x(t)|), c\in(a,b)$. Then letting $t=c$ in (4.4.2).

Since $g(\tau) \le \tau$ we have $x(g(\tau)) \le M$ and by Theorem 3.2.2, Theorem 3.2.2 and (4.4.2), we have

$$\begin{aligned} x(c) &= \int_a^b G(t,\tau)\left[p(\tau)x^{\mu}(\tau)+q(\tau)x^{\lambda}(g(\tau))-f(\tau)\right]d\tau \qquad (4.4.3)\\ &\le \int_a^b G(\tau,\tau)\left[p^{+}x^{\mu}(\tau)+q^{+}x^{\lambda}(g(\tau))+f^{-}(\tau)\right]d\tau\\ &\le \int_a^b \left[\frac{1}{\Gamma(\alpha)}\left(\frac{\tau^{\rho}-a^{\rho}}{b^{\rho}-a^{\rho}}\right)^{\alpha-1}\left(\frac{b^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}\tau^{\rho-1}\right]\left[p^{+}x^{\mu}+q^{+}x^{\lambda}+f^{-}\right](\tau)\,d\tau\\ &\le \frac{1}{\Gamma(\alpha)}\left(\frac{\rho}{b^{\rho}-a^{\rho}}\right)^{\alpha-1}\int_a^b\left[\left(\frac{\tau^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{b^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}\tau^{\rho-1}\right]\left[p^{+}x^{\mu}+q^{+}x^{\lambda}+f^{-}\right](\tau)\,d\tau\\ &\le P_0x^{\mu}(\tau)+Q_0x^{\lambda}(\tau)+F_0, \end{aligned}$$

where

$$f^{-}(\tau)=\max(-f(\tau),0).$$

Using the facts

$$x^{\mu}(c)\le x^{2}(c)+\mu_0 \text{ and } x^{\lambda}(c)\le x^{2}(c)+\lambda_0.$$

where

$$P_0=\frac{1}{\Gamma(\alpha)}\left(\frac{\rho}{b^{\rho}-a^{\rho}}\right)^{\alpha-1}\int_a^b\left(\frac{\tau^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{b^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}\tau^{\rho-1}p^{+}(\tau)\,d\tau, \qquad (4.4.4)$$

$$Q_0=\frac{1}{\Gamma(\alpha)}\left(\frac{\rho}{b^{\rho}-a^{\rho}}\right)^{\alpha-1}\int_a^b\left(\frac{\tau^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{b^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}\tau^{\rho-1}q^{+}(\tau)\,d\tau \qquad (4.4.5)$$

and

$$F_0=\frac{1}{\Gamma(\alpha)}\left(\frac{\rho}{b^{\rho}-a^{\rho}}\right)^{\alpha-1}\int_a^b\left(\frac{\tau^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}\left(\frac{b^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}\tau^{\rho-1}f^{-}(\tau)\,d\tau. \qquad (4.4.6)$$

On the other hand, (4.1.10), with $A=B=1$, implies that

$$x^{\mu}(c)\le x^{2}(c)+\mu_0,\; x^{\lambda}(g(c))\le x^{2}(c)+\lambda_0$$

Using these inequalities and (4.4.3) we find the following quadratic inequality

$$(P_0+Q_0)x^{2}(c)-x(c)+\mu_0P_0+\lambda_0Q_0+F_0>0.$$

But this is only possible when

$$(P_0 + Q_0)(\mu_0 P_0 + \lambda_0 Q_0 + F_0) > \frac{1}{4}, \tag{4.4.7}$$

which is the same as (4.4.1). This completes the proof of Theorem 4.4.1.

**Theorem 4.4.2** (Hartman type inequality)**.** *Let $x(t)$ be a nontrivial solution of (4.3.25) satisfying the Dirichlet boundary conditions (1.1.2). If $x(t) < 0$ in $(a,b)$, then the inequality*

$$\begin{aligned}\left(\int_a^b \frac{(p^+(\tau)+q^+)(\tau)\tau^{\rho-1}d\tau}{(\tau^\rho - a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\right)\left(\int_a^b \frac{(\mu_0 p^+ + \lambda_0 q^+ + f^+)(\tau)\tau^{\rho-1}d\tau}{(\tau^\rho - a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\right)\\ > \left(\frac{\Gamma(\alpha)}{2}\right)^2 (\rho(b^\rho - a^\rho))^{2\alpha-2},\end{aligned} \tag{4.4.8}$$

*holds, where the constants $\mu_0$ and $\lambda_0$ are the same as in (1.1.12).*

■

**Proof.** Let $x(t)$ be a negative solution of (4.3.25). In fact, if $x(t) < 0$ for $t \in (a,b)$ then we can consider $-x(t)$ as a positive solution of

$${}^c\mathscr{D}_{a+}^{\alpha,\rho} x(t) + p(t)x(t)|x(t)|^{\mu-1} + q(t)x(t)|x(t)|^{\lambda-1} = -f(t). \tag{4.4.9}$$

Then using Theorem 3.2.2 and (4.4.9), $x(t)$ can be expressed as

$$x(t) = \int_a^b G(t,\tau)\left[p(\tau)x(\tau)^\mu + q(\tau)x(\tau)^\lambda + f(\tau)\right]d\tau. \tag{4.4.10}$$

Let $|x(c)|$ be maximized at a point $c \in (a,b)$, i.e., $M = |x(c)| = \max_{t\in(a,b)}(|x(t)|)$ for some $c \in (a,b)$. by (3.2.2) and (4.4.10), we have

$$\begin{aligned} x(c) &= \int_a^b G(c,\tau)\left[p(\tau)x^\mu(\tau) + q(\tau)x^\lambda(\tau) + f(\tau)\right]d\tau \\ &\le \int_a^b G(\tau,\tau)\left[p^+x^\mu(\tau) + q^+x^\lambda(\tau) + f^+(\tau)\right]d\tau \\ &\le \int_a^b \left[\frac{1}{\Gamma(\alpha)}\left(\frac{\tau^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1}\left(\frac{b^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\tau^{\rho-1}\right]\left[p^+x^\mu(\tau) + q^+x^\lambda(\tau) + f^+(\tau)\right]d\tau \end{aligned}$$

$$\begin{aligned} x(c) &\leq \frac{1}{\Gamma(\alpha)}\left(\frac{\rho}{b^{\rho}-a^{\rho}}\right)^{\alpha-1}\int_a^b\left[(\tau^{\rho}-a^{\rho})^{\alpha-1}(b^{\rho}-\tau^{\rho})^{\alpha-1}\tau^{\rho-1}\right]\left[p^{+}x^{\mu}(\tau)+q^{+}x^{\lambda}(\tau)+f^{+}(\tau)\right]d\tau \\ &\leq P_0x^{\mu}(\tau)+Q_0x^{\lambda}(\tau)+\tilde{F}_0, \end{aligned}$$

where $P_0$ and $Q_0$ are defined in (4.4.4) and (4.4.5), and

$$\tilde{F}_0=\frac{1}{\Gamma(\alpha)}\left(\frac{\rho}{b^{\rho}-a^{\rho}}\right)^{\alpha-1}\int_a^b(\tau^{\rho}-a^{\rho})^{\alpha-1}(b^{\rho}-\tau^{\rho})^{\alpha-1}\tau^{\rho-1}f^{+}(\tau)\,d\tau. \tag{4.4.11}$$

With a similar argument to the proof of Theorem 4.4.1, we obtain (4.4.8) which completes the proof of Theorem 4.4.2. From Theorems 4.4.1 and 4.4.2, the next result immediately follows. ■

**Theorem 4.4.3** (Hartman type inequality)**.** *Let $x(t)$ be a nontrivial solution of (4.4.9) satisfying the Dirichlet boundary conditions (1.1.2). If $x(t)$ is not identically zero on $(a,b)$, then the inequality*

$$\begin{aligned} &\left(\int_a^b\frac{(p^{+}(\tau)+q^{+})(\tau)\tau^{\rho-1}d\tau}{(\tau^{\rho}-a^{\rho})^{1-\alpha}(b^{\rho}-\tau^{\rho})^{1-\alpha}}\right)\left(\int_a^b\frac{(\mu_0p^{+}+\lambda_0q^{+}+|f|)(\tau)\tau^{\rho-1}d\tau}{(\tau^{\rho}-a^{\rho})^{1-\alpha}(b^{\rho}-\tau^{\rho})^{1-\alpha}}\right) \\ &\qquad > \left(\frac{\Gamma(\alpha)}{2}\right)^2\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{2\alpha-2} \end{aligned} \tag{4.4.12}$$

*holds, where the constants $\mu_0$ and $\lambda_0$ are the same as in (1.1.12).*

**Proof.** Let $x(t)$ be a nontrivial solution of (4.4.9-1.1.2). Since either $x(t)<0$ or $x(t)>0$ for $t\in(a,b)$ and $f^{\pm}(t)\leq|f(t)|$ by Theorems 4.4.1 and 4.4.2, we obtain (4.4.12). ■

**Remark 4.4.4.** When $\alpha=2$ and $\rho=1$, then Theorem 4.4.3 coincides with [**?**, Theorem 2.4].

**Corollary 4.4.5** (Disconjugacy)**.** *If*

$$\begin{aligned} &\left(\int_a^b\frac{(p^{+}(\tau)+q^{+})(\tau)\tau^{\rho-1}d\tau}{(\tau^{\rho}-a^{\rho})^{1-\alpha}(b^{\rho}-\tau^{\rho})^{1-\alpha}}\right)\left(\int_a^b\frac{(\mu_0p^{+}+\lambda_0q^{+}+|f|)(\tau)\tau^{\rho-1}d\tau}{(\tau^{\rho}-a^{\rho})^{1-\alpha}(b^{\rho}-\tau^{\rho})^{1-\alpha}}\right) \\ &\qquad \leq \left(\frac{\Gamma(\alpha)}{2}\right)^2\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{2\alpha-2}, \end{aligned} \tag{4.4.13}$$

*then (4.4.9) is disconjugate in $[a,b]$, where the constants $\mu_0$ and $\lambda_0$ are the same as in (1.1.12).*

## 4.4.2 Lyapunov type inequality for the problem (4.4.9-1.1.2)

**Theorem 4.4.6** (Lyapunov type inequality). *Let $x(t)$ be a nontrivial solution of (4.4.9) satisfying the Dirichlet boundary conditions (1.1.2). If $x(t) > 0$ in $(a, b)$, then the inequality*

$$\left(\int_a^b (p^+(\tau) + q^+(\tau))\, d\tau\right) \left(\int_a^b [\mu_0 p^+(\tau) + \lambda_0 q^+(\tau) + f^-(\tau)]\, d\tau\right) > \left(\frac{\Gamma(\alpha)}{2}\right)^2 \left(-\frac{\rho}{a^\rho - b^\rho}\right)^{2-2\alpha}, \tag{4.4.14}$$

*holds, where the constants $\mu_0$ and $\lambda_0$ are the same as in (1.1.12).*

**Proof.** First, we note that the solution of the generalized fractional differential equation (4.4.9) of order $\alpha \in (0, 2]$, satisfying the conditions (1.1.2) can be represented by (3.2.1) and (3.2.2).

Let $x(c) = \max_{t\in(a,b)} (x(t))$ .Then by (4.4.10) and (3.2.9), we have

$$\begin{aligned} x(c) &= \int_a^b G(c,\tau) \left[p(\tau)x^\mu(\tau) + q(\tau)x^\lambda(\tau) - f(\tau)\right] d\tau \\ &\le \int_a^b G(\tau,\tau) \left[p(\tau)x^\mu(\tau) + q(\tau)x^\lambda(\tau) - f(\tau)\right] d\tau \\ &\le G_{max} \int_a^b \left[p^+x^\mu(\tau) + q^+x^\lambda(\tau) + f^-(\tau)\right] d\tau \\ &\le G_{max} \int_a^b \left[p^+x^\mu(\tau) + q^+x^\lambda(\tau) + f^-(\tau)\right] d\tau \\ &\le P_0x^\mu(\tau) + Q_0x^\lambda(\tau) + F_0, \end{aligned} \tag{4.4.15}$$

where

$$f^-(\tau) = \max(\_f(\tau), 0)$$

and

$$P_0 = G\int_a^b p^+(\tau)\, d\tau,\ Q_0 = G_{max}\int_a^b q^+(\tau)\, d\tau,\ F_0 = G_{max}\int_a^b f^-(\tau)\, d\tau.$$

On the other hand, (4.1.10) in Lemma 2.3.16 with $A = B = 1$, implies that

$$x^\mu(c) \le x^2(c) + \mu_0,\ x^\lambda(c) \le x^2(c) + \lambda_0.$$

Using these inequalities and (4.4.15) we find the following quadratic inequality

$$(P_0+Q_0)\,x^2(c)-x(c)+\mu_0P_0+\lambda_0Q_0+F_0>0 \tag{4.4.16}$$

But this is only possible when

$$(P_0+Q_0)(\mu_0P_0+\lambda_0Q_0+F_0)>\frac{1}{4}$$

which is the same as (4.4.14). This completes the proof of Theorem 4.4.6. ■

**Theorem 4.4.7** (Lyapunov type inequality)**.** *Let $x(t)$ be a nontrivial solution of (4.4.9) satisfying the Dirichlet boundary conditions (1.1.2). If $x(t)<0$ in $(a,b)$, then the inequality*

$$\left(\int_a^b\left[p^+(\tau)+q^+(\tau)\right]d\tau\right)\left(\int_a^b\left[\mu_0p^+(\tau)+\lambda_0q^+(\tau)+f^+(\tau)\right]d\tau\right)>\left(\frac{G_{\max}}{2}\right)^2$$

*holds, where the constants $\mu_0$ and $\lambda_0$ are the same as in (1.1.12).*

**Theorem 4.4.8** (Lyapunov type inequality)**.** *Let $x(t)$ be a nontrivial solution of (4.4.9) satisfying the Dirichlet boundary conditions (1.1.2). If $x(t)\neq 0$ in $(a,b)$, then the inequality*

$$\left(\int_a^b\left[p^+(\tau)+q^+(\tau)\right]d\tau\right)\left(\int_a^b\left[\mu_0p^+(\tau)+\lambda_0q^+(\tau)+|f(\tau)|\right]d\tau\right)>\left(\frac{G_{\max}}{2}\right)^2 \tag{4.4.17}$$

*holds, where the constants $\mu_0$ and $\lambda_0$ are the same as in (1.1.12).*

**Corollary 4.4.9** (Disconjugacy)**.** *If*

$$\left(\int_a^b\left[p^+(\tau)+q^+(\tau)\right]d\tau\right)\left(\int_a^b\left[\mu_0p^+(\tau)+\lambda_0q^+(\tau)+|f(\tau)|\right]d\tau\right)\leq\left(\frac{G_{\max}}{2}\right)^2 \tag{4.4.18}$$

*then (4.4.9) is disconjugate in $[a,b]$, where the constants $\mu_0$ and $\lambda_0$ are the same as in (1.1.12).*

Next we present two new results where (4.4.9) has two special type of nonlinearities, namely forced sub-linear or forced super-linear, i.e., $p(t)=0$ or $q(t)=0$, respectively.

**Theorem 4.4.10.** *Let $x(t)$ be a nontrivial solution of (4.4.9) satisfying the Dirichlet boundary conditions (1.1.2). If $x(t)\neq 0$ in $(a,b)$, then the following hold:*

*(i) Hartman type inequality*

$$\left(\int_a^b \frac{q^+(\tau)\tau^{\rho-1}d\tau}{(\tau^\rho-a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\right)\left(\int_a^b \frac{(\lambda_0 q^+ + |f|)(\tau)\tau^{\rho-1}d\tau}{(\tau^\rho-a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\right) > \left(\frac{\Gamma(\alpha)}{2}\right)^2\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2\alpha-2}$$

*(ii) Lyapunov type inequality*

$$\left(\int_a^b q^+(\tau)\,d\tau\right)\left(\int_a^b \left[\lambda_0 q^+(\tau)+|f(\tau)|\right]d\tau\right) > \left(\frac{G_{max}}{2}\right)^2$$

*where $\lambda \in (0,1)$ and the constant $\lambda_0$ is the same as in (1.1.12).*

**Theorem 4.4.11.** *Let $x(t)$ be a nontrivial solution of (4.4.9) satisfying the Dirichlet boundary conditions (1.1.2). If $x(t) \neq 0$ in $(a,b)$, then the following hold:*

*(i) Hartman type inequality*

$$\left(\int_a^b \frac{p^+(\tau)\tau^{\rho-1}d\tau}{(\tau^\rho-a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\right)\left(\int_a^b \frac{(\mu_0 p^+ + |f|)(\tau)\tau^{\rho-1}d\tau}{(\tau^\rho-a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\right) > \left(\frac{\Gamma(\alpha)}{2}\right)^2\left(\frac{b^\rho-a^\rho}{\rho}\right)^{2\alpha-2}$$

*(ii) Lyapunov type inequality*

$$\left(\int_a^b p^+(\tau)\,d\tau\right)\left(\int_a^b \left[\mu_0 p^+(\tau)+|f(\tau)|\right]d\tau\right) > \left(\frac{G_{max}}{2}\right)^2$$

*where $\mu \in (1,2)$ and the constant $\mu_0$ is the same as in (1.1.12).*

When $\lambda \to 1^-$ (or $\mu \to 1^+$), (4.1.5) (or (4.1.6) reduces to forced Riemann–Liouville linear fractional differential equation of order $\alpha \in (0,2)$

$${}^{c}\mathscr{D}_{a+}^{\alpha,\rho}x(t)+\nu(t)x(t)=f(t), \tag{4.4.19}$$

where $\nu(t) = p(t)$ or $\nu(t) = q(t)$ Since

$$\lim_{\mu \longrightarrow 1^+} \mu_0 = \lim_{\lambda \longrightarrow 1^-} \lambda_0 = \frac{1}{4},$$

we have the following result from Theorems 4.4.10 and 4.4.11.

**Corollary 4.4.12.** *Let $x(t)$ be a nontrivial solution of (4.4.19) satisfying the Dirichlet boundary conditions (1.1.2). If $x(t) \neq 0$ in $(a,b)$, then the following hold.*

*(i) Hartman type inequality*

$$\left(\int_a^b \frac{\nu^+(\tau)\tau^{\rho-1}d\tau}{(\tau^\rho - a^\rho)^{1-\alpha}(b^\rho - \tau^\rho)^{1-\alpha}}\right)\left(\int_a^b \frac{(\mu_0\nu^+ + 4|f|)(\tau)\tau^{\rho-1}d\tau}{(\tau^\rho - a^\rho)^{1-\alpha}(b^\rho - \tau^\rho)^{1-\alpha}}\right) > \Gamma(\alpha)^2\left(\frac{b^\rho - a^\rho}{\rho}\right)^{2\alpha-2}. \tag{4.4.20}$$

*(ii) Lyapunov type inequality*

$$\left(\int_a^b \nu^+(\tau)d\tau\right)\left(\int_a^b \left[\mu_0\nu^+(\tau) + 4|f(\tau)|\right]d\tau\right) > \Gamma(\alpha)^2\left(\frac{4\rho}{b^\rho - a^\rho}\right)^{2\alpha-2}, \tag{4.4.21}$$

*where $\mu \in (1,2)$ and the constant $\mu_0$ is the same as in (1.1.12).*

**Remark 4.4.13.** In the case when $x(t) > 0$ or $x(t) < 0$ for $t \in (a,b)$ , Hartman and Lyapunov type inequalities for forced sub-linear equations (4.1.5), forced super-linear equation (4.1.6) and forced linear equation (4.4.19) can be formulated by replacing the term $|f(t)|$ by $f^+(t)$ or $f^-(t)$ in Theorems 4.4.10 and 4.4.11 and Corollary 4.4.12.

**Remark 4.4.14.** When $f(t) = 0$, Ineq. (4.4.21) in Corollary 4.4.12 reduces to

$$\int_a^b \nu^+(\tau)d\tau > \Gamma(\alpha)\left(\frac{4\rho}{b^\rho - a^\rho}\right)^{\alpha-1},$$

which is sharper than (1.1.3) given by Ferreira [22]. Moreover, when $f(t) = 0$ and $\alpha = 2$, Ineq.'s (4.4.20) and (4.4.21) coincide with the classical Lyapunov and Hartman inequalities, i.e. Ineq.'s (1.1.5) and (1.1.6).

We note that the results obtained in this subsection generalize the results by R. P. Agarwal, A. Özbekler, see [30].

**Example 4.4.15.** For the fractional equation

$$\mathscr{D}_0^{3/2,\rho}x(t)+\sigma_1 x(t)|x(t)|^{1/2}+\sigma_2 x(t)|x(t)|^{-1/2}-\sigma_3=0,\ t\geq 0, \tag{4.4.22}$$

where $\sigma_j, j=1,2,3$, are real constants with $\sigma_1,\sigma_2>0$, if the solution $x(t)$ has consecutive zeros at $0$ and $b>0$, then in view of (4.4.1) the following inequality must be satisfied

$$(\sigma_1+\sigma_2)[\mu_0\sigma_2+\lambda_0\sigma_2+|\sigma_3|]\left(\int_a^b(\tau^\rho-a^\rho)^{\alpha-1}(b^\rho-\tau^\rho)^{\alpha-1}\tau^{\rho-1}d\tau\right)^2>\left(\frac{\Gamma(\alpha)}{2}\right)^2\left(-\frac{\rho}{a^\rho-b^\rho}\right)^{2-2\alpha},$$

holds, where

$$\mu_0=(2-\mu)2^{2/(\mu-2)}\mu^{\mu/(\mu-2)}>0,\lambda_0=(2-\lambda)2^{2/(\lambda-2)}\lambda^{\lambda/(\lambda-2)}>0, \tag{4.4.23}$$

and

$$(\sigma_1+\sigma_2)[\mu_0\sigma_2+\lambda_0\sigma_2+|\sigma_3|]\frac{\rho^{2-8\alpha}}{(\Gamma(2\alpha+2))^2}(\Gamma(\alpha+1))^4(b^\rho-a^\rho)^{4\alpha+2}>\left(\frac{\Gamma(\alpha)}{2}\right)^2\left(-\frac{\rho}{a^\rho-b^\rho}\right)^{2-2\alpha}$$

$$(b^\rho-a^\rho)^{2\alpha+4}>\left(\frac{\left(\frac{\Gamma(\alpha)}{2}\right)^2\rho^{2-2\alpha}}{(\sigma_1+\sigma_2)[\mu_0\sigma_2+\lambda_0\sigma_2+|\sigma_3|]\frac{\rho^{2-8\alpha}}{(\Gamma(2\alpha+2))^2}(\Gamma(\alpha+1))^4}\right)$$

when $\alpha=3/2,\ a=0$

$$b^{7\rho}>\left(\frac{\frac{1024}{9\pi}\rho^9}{(\sigma_1+\sigma_2)[\mu_0\sigma_2+\lambda_0\sigma_2+|\sigma_3|]}\right)$$

$$b>\left(\frac{\left(\frac{1024}{9}\right)^{\frac{1}{7\rho}}(\rho^9)^{\frac{1}{7\rho}}}{(\pi(\sigma_1+\sigma_2)[\mu_0\sigma_2+\lambda_0\sigma_2+|\sigma_3|])^{\frac{1}{7\rho}}}\right)$$

when $\rho=1$

$$b>\left(\frac{\frac{2}{3}\sqrt[7]{3}\sqrt[7]{648}}{(\pi(\sigma_1+\sigma_2)[\mu_0\sigma_2+\lambda_0\sigma_2+|\sigma_3|])^{\frac{1}{7}}}\right).$$

## 4.5 Conclusion

In this chapter, we have investigated generalized Lyapunov-type and Hartman-type inequalities for boundary value problems involving mixed nonlinearities with a fractional order $\alpha \in (1,2)$. Furthermore, we have examined several related disconjugacy results derived from these inequalities in the context of forced fractional differential equations. The theoretical developments presented herein provide useful analytical tools that may be effectively applied to other fractional calculus models and boundary value problems.

# CHAPTER 5

# Nonlinear generalized Caputo fractional differential equations incorporating forcing terms with more general boundary conditions

## Summary

## 5.1 Introduction

This chapter aims to establish key generalizations of Hartman-type and Lyapunov-type inequalities for forced generalized Caputo fractional differential equations with mixed nonlinearities, focusing in particular on the generalized Caputo fractional boundary value problem (1.2.3–1.2.2). To the best of our knowledge, no existing work has considered equation (1.2.3) under Dirichlet boundary conditions (1.2.2) within the framework of generalized Caputo fractional derivatives. For complementary contributions in related settings, we refer the interested reader to the references cited throughout this work.

The chapter is organized as follows. In Section 2, we derive the corresponding Green's function and establish upper bounds for it, thereby laying a rigorous foundation for the subsequent analysis. We also prove generalized Hartman-type and Lyapunov-type inequalities, together with associated disconjugacy results, for the linear generalized Caputo fractional differential problem (1.2.1–1.2.2).

Section 3 extends these results to the nonlinear setting by developing generalized Hartman-type and Lyapunov-type inequalities, as well as disconjugacy criteria, for the nonlinear generalized Caputo fractional differential problem (1.2.3–1.2.2). In Section 4, we present a comprehensive formulation and proof of the generalized Hartman-type and Lyapunov-type inequalities for (1.2.3–1.2.2), further highlighting their theoretical relevance.

Finally, we demonstrate the applicability of the developed results by applying our main theorems to the nonlinear problem (1.2.3-1.2.2), illustrating the practical impact of our contributions beyond purely theoretical considerations.

## 5.2 Generalized Hartman and Lyapunov type inequalities for (1.2.1-1.2.2)

We begin by writing problem (1.2.1)-(1.2.2) in its equivalent integral form.

**Theorem 5.2.1.** *Let* $x \in \mathscr{C}([a,b])$ *is a solution of (1.2.1) and (1.2.2) if and only if*

$$x(t) = \int_a^b G(t,\tau)h(\tau)d\tau, \tag{5.2.1}$$

*where* $G(t,\tau)$ *is the Green's function given by*

$$G(t,\tau)=\frac{\rho^{1-\alpha}}{\tau^{1-\rho}\Gamma(\alpha)}\begin{cases}G_1(t,\tau), & \text{if } a\le t\le\tau\le b,\\ G_1(t,\tau)-(t^\rho-\tau^\rho) & \text{if } a\le\tau\le t\le b,\end{cases}\tag{5.2.2}$$

*with*

$$G_1(t,\tau)=\frac{\rho\beta_0 a^{\rho-1}+(t^\rho-a^\rho)}{\rho\beta_0 a^{\rho-1}+\rho\beta_1 b^{\rho-1}+(b^\rho-a^\rho)}\left[(b^\rho-\tau^\rho)^{\alpha-1}+\beta_1(\alpha-1)(b^\rho-\tau^\rho)^{\alpha-2}\right].\tag{5.2.3}$$

**Proof.** We reduce $(1.2.1)$ to the equivalent integral equation given by

$$\begin{aligned}x(t)=&-\left(\mathscr{I}^{\alpha,\rho}_{a+}\right)h(t)+c_0+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right)\\=&\frac{-\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t(t^\rho-\tau^\rho)^{\alpha-1}h(\tau)\frac{d\tau}{\tau^{1-\rho}}+c_0+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right)\end{aligned}\tag{5.2.4}$$

and

$$\begin{aligned}x'(t)=&\frac{-\rho^{1-\alpha}}{\Gamma(\alpha)}(\alpha-1)\int_a^t(t^\rho-\tau^\rho)^{\alpha-2}h(\tau)\frac{d\tau}{\tau^{1-\rho}}+c_1t^{\rho-1}.\\x'(a)=&c_1a^{\rho-1}\end{aligned}\tag{5.2.5}$$

Substituting (5.2.4) and (5.2.5) into the boundary condition $x(a)-\beta_0x'(a)=0$, hence

$$\begin{cases}x(a)=c_0\\ c_0-\beta_0c_1a^{\rho-1}=0\end{cases}\quad\text{implies}\ \ c_0=\beta_0c_1a^{\rho-1}.\tag{5.2.6}$$

Combining (5.2.6) with the boundary condition $x(b)+\beta_1x'(b)=0$, this yields

$$\begin{aligned}x(b)=&\frac{-\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b(b^\rho-\tau^\rho)^{\alpha-1}h(\tau)\frac{d\tau}{\tau^{1-\rho}}+c_0+c_1\left(\frac{b^\rho-a^\rho}{\rho}\right),\\x'(b)=&\frac{-\rho^{1-\alpha}}{\Gamma(\alpha)}(\alpha-1)\int_a^b(b^\rho-\tau^\rho)^{\alpha-2}h(\tau)\frac{d\tau}{\tau^{1-\rho}}+c_1b^{\rho-1};\end{aligned}$$

We then obtain

$$c_1 = \frac{\rho^{2-\alpha}}{D\Gamma(\alpha)} \int_a^b R(\tau)h(\tau)\frac{d\tau}{\tau^{1-\rho}}, \tag{5.2.7}$$

where

$$\begin{aligned} D =& \rho\beta_0 a^{\rho-1} + \rho\beta_1 b^{\rho-1} + (b^\rho - a^\rho), \\ R(\tau) =& (b^\rho - \tau^\rho)^{\alpha-1} + \beta_1(\alpha-1)(b^\rho-\tau^\rho)^{\alpha-2}. \end{aligned} \tag{5.2.8}$$

Therefore, the solution of (5.2.6), (5.2.7) is

$$x(t) = -\left(\mathscr{I}_{a+}^{\alpha,\rho}\right)h(t) + c_1\beta_0 a^{\rho-1} + c_1\frac{(t^\rho - a^\rho)}{\rho}.$$

Consequently,

$$x(t) = \tfrac{-\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - \tau^\rho)^{\alpha-1} h(\tau)\tfrac{d\tau}{\tau^{1-\rho}} + \tfrac{\rho\beta_0 a^{\rho-1}(t^\rho - a^\rho)}{D} - \tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^b R(\tau)h(\tau)\tfrac{d\tau}{\tau^{1-\rho}}. \tag{5.2.9}$$

The proof is finished. ■

For $\rho = 1$, we find the results of Meng and Stynes [60].

We will assume that

$$\begin{cases} \rho\beta_0 a^{\rho-1} \geq \left(\dfrac{\rho b^{\rho-1}}{\alpha-1} - 1\right)(b^\rho - a^\rho) \\ \rho b^{\rho-1} > 1 \text{ and } \beta_1 \geq 0, \end{cases} \tag{5.2.10}$$

**Lemma 5.2.2.** *The function* $G$ *defined in Theorem 5.2.1 satisfies the following property*

$$\max\{|G(t,\tau)| : a \leq t \leq b\} \leq G(\tau,\tau) \textit{ for all } \tau \in [a,b],$$

*where* $G(\tau,\tau)$ *given by*

$$G(\tau,\tau) = \frac{\rho^{2-\alpha}\beta_0\alpha^{\rho-1}}{D\Gamma(\alpha)}\tau^{\rho-1}(b^\rho-\tau^\rho)^{\alpha-2}\left((b^\rho-\tau^\rho)+\beta_1(\alpha-1)\right) \textit{ for } a \leq \tau \leq b. \tag{5.2.11}$$

**Proof.**

i. – For $a \leq t \leq \tau \leq b$, $G(t,\tau) \geq 0$.

- – For $a \leq t < \tau \leq b$

  Set
$$G(t) = \frac{\rho^{1-\alpha}}{\tau^{1-\alpha}} g_1(t,\tau) \tag{5.2.12}$$
  where
$$g_1(t,\tau) = \frac{\rho\beta_0 a^{\rho-1}+(t^\rho-a^\rho)}{D}\left[\frac{(b^\rho-\tau^\rho)^{\alpha-1}}{\Gamma(\alpha)}+\beta_1\frac{(b^\rho-\tau^\rho)^{\alpha-2}}{\Gamma(\alpha-1)}\right]-\frac{(t^\rho-\tau^\rho)^{\alpha-1}}{\Gamma(\alpha)}. \tag{5.2.13}$$

As $\dfrac{\rho^{\alpha-1}}{\tau^{1-\rho}} > 0$ then $g_1(t,\tau) \geq 0 \rightarrow G(t,\tau) \geq 0$.

We will prove that
$$g_1(t,\tau) \geq g_1(b,\tau) \geq g_1(b,a) \geq 0.$$

First, fix $\tau \in [a,b]$, then, for $\tau < t < b$
$$\begin{aligned}\frac{\partial g_1}{\partial t}(t,\tau) &= \frac{\rho t^{\rho-1}}{\Gamma(\alpha)}\left[\frac{(b^\rho-\tau^\rho)^{\alpha-1}}{\Gamma(\alpha)}+\beta_\lambda\frac{(b^\rho-\tau^\rho)^{\alpha-2}}{\Gamma(\alpha-1)}\right]-\frac{\rho(b^\rho-\tau^\rho)^{\rho-1}}{\Gamma(\alpha)}(t^\rho-\tau^\rho)^{\alpha-2}\\ &\leq \frac{\rho t^{\rho-1}}{D}\left[\frac{(b^\rho-\tau^\rho)^{\alpha-1}}{\Gamma(\alpha)}+\beta_1\frac{(b^\rho-\tau^\rho)^{\alpha-2}}{\Gamma(\alpha-1)}\right]-\frac{\rho t^{\rho-1}}{\Gamma(\alpha-1)}(b^\rho-\tau^\rho)^{\alpha-1}.\end{aligned}$$

We obtain
$$\begin{aligned}\frac{\partial g_1}{\partial t}(t,\tau) &\leq \frac{\rho t^{\rho-1}}{D\Gamma(\alpha-1)}(b^\rho-\tau^\rho)^{\alpha-2}\left[\frac{(b^\rho-\tau^\rho)}{\alpha-1}+\beta_1-D\right]\\ &\leq \frac{\rho t^{\rho-1}}{D\Gamma(\alpha-1)}(b^\rho-\tau^\rho)^{\alpha-2}\left[\frac{(b^\rho-\tau^\rho)}{\alpha-1}+\beta_1-\rho\beta_0 a^{\rho-1}-\rho\beta_1 b^{\rho-1}-(b^\rho-a^\rho)\right]\\ &\leq \frac{\rho t^{\rho-1}}{D\Gamma(\alpha-1)}(b^\rho-\tau^\rho)^{\alpha-2}\left[\frac{(b^\rho-\tau^\rho)}{\alpha-1}+\beta_1-\rho\beta_0 a^{\rho-1}+\beta_1(1-\rho b^{\rho-1})\right].\end{aligned}$$

In the other hand

$$\begin{aligned}\rho\beta_0 a^{\rho-1} &\geq \left(\frac{\rho b^{\rho-1}}{\alpha-1}-1\right)(b^\rho-a^\rho)\\ &\geq \left(\frac{1}{\alpha-1}-1\right)(b^\rho-a^\rho)\\ &\geq \frac{b^\rho-a^\rho}{\alpha-1}-(b^\rho-a^\rho).\end{aligned}$$

We deduce that $g_1$ is a decreasing function so $g_1(t,\tau)\geq g_1(b,\tau)$.

$$g_1(b,\tau)=\frac{\rho\beta_0\alpha^{\rho-1}+(b^\rho-a^\rho)}{D}\left[\frac{(b^\rho-\tau^\rho)^{\alpha-1}}{\Gamma(\alpha)}+\beta_1\frac{(b^\rho-\tau^\rho)^{\alpha-2}}{\Gamma(\alpha-1)}\right]-\frac{(b^\rho-\tau^\rho)^{\alpha-1}}{\Gamma(\alpha)}$$

For $\tau\in[a,b]$

$$\begin{aligned}\frac{dg_1}{d\tau}(b,\tau)=\frac{N}{D}&\left[-\frac{\rho\tau^{\rho-1}(\alpha-1)(b^\rho-\tau^\rho)^{\alpha-2}}{\Gamma(\alpha)}-\frac{\beta_1(\alpha-2)\rho\tau^{\rho-1}(b^\rho-\tau^\rho)^{\alpha-3}}{\Gamma(\alpha-1)}\right]\\ &+\frac{\rho\tau^{\rho-1}(\alpha-1)(b^\rho-\tau^\rho)^{\alpha-2}}{\Gamma(\alpha)}\end{aligned}$$

with $N=\rho\beta_0 a^{\rho-1}((b^\rho-a^\rho))$.

$$\begin{aligned}\frac{dg_1}{d\tau}(b,\tau)&=\frac{\rho\tau^{\rho-1}}{D}\left[-N\frac{(b^\rho-\tau^\rho)^{\alpha-2}}{\Gamma(\alpha-1)}-N\frac{\beta_1(\alpha-2)(b^\rho-\tau^\rho)^{\alpha-3}}{\Gamma(\alpha-1)}+\frac{D(b^\rho-\tau^\rho)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\\ &=\frac{\rho\tau^{\rho-1}}{D}\left[\frac{D-N}{\Gamma(\alpha-1)}(b^\rho-\tau^\rho)^{\alpha-2}-\frac{N\beta_1(\alpha-2)}{\Gamma(\alpha-1)}(b^\rho-\tau^\rho)^{\alpha-3}\right]\\ &=\frac{\rho\tau^{\rho-1}}{D\Gamma(\alpha-1)}\left[\rho\beta_1 b^{\rho-1}(b^\rho-\tau^\rho)^{\alpha-2}-N\beta_1(\alpha-2)(b^\rho-\tau^\rho)^{\alpha-3}\right]\\ &=\frac{\beta_1\rho\tau^{\rho-1}}{D\Gamma(\alpha-1)}(b^\rho-\tau^\rho)^{\alpha-3}\left[\rho b^{\rho-1}(b^\rho-\tau^\rho)+N(2-\alpha)\right]\end{aligned}$$

This implies that

$$\frac{dg_1}{d\tau}(b,\tau)\geq 0,\forall\tau\in[a,b].$$

It follows that, $g_1(b,\tau)$ is an increasing function. Then $g_1(b,\tau) \geq g_1(b,a), \forall \tau \in [a,b]$,

$$\begin{aligned} g_1(b,a) &= \frac{N}{\Gamma(\alpha)D}\left[(b^\rho - a^\rho)^{\alpha-1} + \beta_1(\alpha-1)(b^\rho - a^\rho)^{\alpha-2}\right] - \frac{(b^\rho - a^\rho)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{N}{\Gamma(\alpha)D}\left[N(b^\rho - a^\rho)^{\alpha-1} + N\beta_1(\alpha-1)(b^\rho - a^\rho)^{\alpha-2} - D(b^\rho - a^\rho)^{\alpha-1}\right] \\ &= \frac{N}{\Gamma(\alpha)D}\left[(N-D)(b^\rho - a^\rho)^{\alpha-1} + N\beta_1(\alpha-1)(b^\rho - a^\rho)^{\alpha-2}\right] \\ &= \frac{(b^\rho - a^\rho)^{\alpha-2}}{\Gamma(\alpha)D}\left[-\rho\beta_1 b^{\rho-1}(b^\rho - a^\rho) + N\beta_1(\alpha-1)\right] \\ &= \frac{\beta_1(b^\rho - a^\rho)^{\alpha-2}}{\Gamma(\alpha)D}\left[-\rho\beta_1 b^{\rho-1}(b^\rho - a^\rho) + (\alpha-1)\left(\rho\beta_0 a^{\rho-1} + b^\rho - a^\rho\right)\right] \end{aligned}$$

$$\begin{aligned} g_1(b,a) &= \frac{\beta_1(b^\rho - a^\rho)^{\alpha-2}}{\Gamma(\alpha)D}\left[\rho\beta_1 b^{\rho-1}(b^\rho - a^\rho) - \frac{\rho b^{\rho-1}(b^\rho - a^\rho)}{\alpha-1}\right](\alpha-1) \\ &= \frac{\beta_1(b^\rho - a^\rho)^{\alpha-2}}{\Gamma(\alpha)D}(\alpha-1)\left[\rho\beta_0 a^{\rho-1} - \left(\frac{\rho b^{\rho-1}}{\alpha-1} - 1\right)\right] \end{aligned}$$

By assumption (5.2.10), we deduce that $g_1(b,a) \geq 0$. This completes the proof.

For $\tau \in [a,b]$ when $a \leq t \leq \tau$

$$\frac{\partial G}{\partial t}(t,\tau) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)\tau^{1-\rho}D}\rho t^{\rho-1}\left[(b^\rho - \tau^\rho)^{\rho-1} + \beta_1(\alpha-1)(b^\rho - \tau^\rho)^{\alpha-2}\right]$$

Then

$$\begin{aligned} \frac{\partial G}{\partial t}(t,\tau) \geq 0 &\Rightarrow G \quad \text{is an increasing function} \\ &\Rightarrow \max_{t\in[a,b]} G(t,\tau) = G(\tau,\tau) \end{aligned}$$

When $\tau < t \leq b$

$$\frac{\partial G}{\partial t}(t,\tau) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)\tau^{1-\rho}}\left[\frac{\partial G_1}{\partial t}(t,\tau) - \rho t^{\rho-1}(\alpha-1)(t^\rho - \tau^\rho)^{\alpha-2}\right]$$

.

$$\begin{aligned}\frac{\partial G}{\partial t}(t,\tau) &= \tfrac{\rho^{1-\alpha}}{\Gamma(\alpha)\tau^{1-\rho}}\left[\tfrac{\rho t^{\rho-1}}{D}\left((b^\rho-\tau^\rho)^{\alpha-1}+\beta_1(\alpha-1)(b^\rho-\tau^\rho)^{\alpha-2}\right)-\rho t^{\rho-1}(\alpha-1)(t^\rho-\tau^\rho)^{\alpha-2}\right]\\
&= \frac{\rho^{2-\alpha}t^{\rho-1}}{\Gamma(\alpha)\tau^{1-\rho}D}\left[(b^\rho-\tau^\rho)^{\alpha-1}+\beta_1(\alpha-1)(b^\rho-\tau^\rho)^{\alpha-2}-D(\alpha-1)(t^\rho-\tau^\rho)^{\alpha-2}\right]\\
&= \frac{\rho^{2-\alpha}t^{\rho-1}(\alpha-1)}{\Gamma(\alpha)\tau^{1-\rho}D}\left[\frac{(b^\rho-\tau^\rho)^{\alpha-1}}{\alpha-\Delta}+\beta_1(b^\rho-\tau^\rho)^{\alpha-2}-D(t^\rho-\tau^\rho)^{\alpha-2}\right].\end{aligned}$$

We denote $c=\dfrac{\rho^{2-\alpha}t^{\rho-1}(\alpha-1)}{\Gamma(\alpha)\tau^{1-\rho}D}$, we have

$$\begin{aligned}\frac{\partial G}{\partial t}(t,\tau) &\le c\,(t^\rho-\tau^\rho)^{\alpha-2}\left[\frac{(b^\rho-\tau^\rho)}{\alpha-\Delta}+\beta_1-D\right]\\
&\le c\,(t^\rho-\tau^\rho)^{\alpha-2}\left[+\beta_1-\rho\beta_1 b^{\rho-1}+\frac{(b^\rho-a^\rho)}{\alpha-1}-\rho\beta_0 a^{\rho-1}-(b^\rho-a^\rho)\right]\\
&\le c\,(b^\rho-\tau^\rho)^{\alpha-2}\left[\beta_1(1-\rho b^{\rho-1})+(\frac{1}{\alpha-1}-1)(b^\rho-a^\rho)-\rho\beta_0 a^{\rho-1}\right]\\
&\le c\,(b^\rho-\tau^\rho)^{\alpha-1}\left[\beta_1(1-\rho b^{\rho-1})+(\frac{1}{\alpha-1}-1)(b^\rho-a^\rho)+(1-\frac{\rho b^{\rho-2}}{\alpha-1})(b^\rho-a^\rho)\right]\\
&\le c\,(b^\rho-\tau^\rho)^{\alpha-1}\left[\beta_1(1-\rho b^{\rho-1})+\frac{1-\rho b^{\rho-1}}{\alpha-1}(b^\rho-a^\rho)\right]\end{aligned}$$

$$\frac{\partial G}{\partial t}(t,\tau)\le c\left[(1-\rho b^{\rho-1})+\left(\beta_1+\frac{(b^\rho-a^\rho)}{\alpha-1}\right)\right].$$

By assumption (5.2.5), we get that $\dfrac{dG}{dt}(t,\tau)\le 0$, G is a decreasing function we deduce that $\max_{t\in[a,b]} G(t,\tau)=G(\tau,\tau)$

$$\begin{aligned}G(\tau,\tau) =& \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\tau^{\rho-1}\left[\frac{\rho\beta_0 a^{\rho-1}+(\tau^\rho-a^\rho)}{\rho\beta_0 a^{\rho-1}+\rho\beta_1 b^{\rho-1}+(b^\rho-a^\rho)}\left((b^\rho-\tau^\rho)^{\alpha-1}+\beta_1(\alpha-1)(b^\rho-\tau^\rho)^{\alpha-2}\right)\right]\\
\le& \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\tau^{\rho-1}(b^\rho-\tau^\rho)^{\alpha-2}\frac{\rho\beta_0 a^{\rho-1}+(\tau^\rho-a^\rho)}{\rho\beta_0 a^{\rho-1}+\rho\beta_1 b^{\rho-1}+(b^\rho-a^\rho)}\left((b^\rho-\tau^\rho)+\beta_1(\alpha-1)\right)\\
\le& \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\tau^{\rho-1}(b^\rho-\tau^\rho)\frac{1}{1+\frac{\rho\beta_1 b^{\rho-1}}{\rho\beta_0 a^{\rho-1}+(b^\rho-a^\rho)}}\left((b^\rho-\tau^\rho)+\beta_1\right)\end{aligned}$$

Hence

$$G(\tau,\tau) \leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\tau^{\rho-1}(b^{\rho}-\tau^{\rho})\left((b^{\rho}-\tau^{\rho})+\beta_1\right).$$

■

Next, we establish a new some versions generalized Lyapunov and Hartman type integral inequality, which generalize previous result.

### 5.2.1 Lyapunov type inequality for the problem (1.2.1-1.2.2)

**Theorem 5.2.3.** *If a nontrivial continuous solution of the problem (1.2.1-1.2.2) exists, then the Lyapunov-type inequality is*

$$\int_a^b \tau^{\rho-1}(b^{\rho}-\tau^{\rho})^{\alpha-2}p^{+}(\tau)d\tau \geq \frac{\Gamma(\alpha)}{\rho^{1-\alpha}[(b^{\rho}-a^{\rho})+\beta_1]}, \tag{5.2.14}$$

*where* $p^{+}(\tau)=\max(p(\tau),0)$ *is the nonnegative part of* $p(\tau)$ *and in particular, for* $\alpha=2$ *and* $\rho=1$ *in (1.2.1) gives the standard Lyapunov inequality for ( 1.1.1-1.2.2).*

**Proof.** By Lemma 3.2.4, a solution $x \in C([a,b])$ to (1.2.1) and (1.2.2) has the expression (5.2.2). From this, for any $a \leq t \leq b$, we obtain

$$|x(t)| \leq \int_a^b |G(t,\tau)||p(\tau)x(\tau)|d\tau. \tag{5.2.15}$$

We get that

$$\begin{aligned} |x(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^b G_1(\tau,\tau)|p(\tau)||x(\tau)|d\tau \qquad (5.2.16)\\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}((b^{\rho}-a^{\rho})+\beta_1)\int_a^b (b^{\rho}-\tau^{\rho})^{\alpha-2}|p(\tau)||x(\tau)|\frac{d\tau}{\tau^{1-\rho}} \end{aligned}$$

we obtain

$$1 \leq \frac{\rho^{1-\alpha}((b^{\rho}-a^{\rho})+\beta_1)}{\Gamma(\alpha)}\int_a^b \tau^{\rho-1}(b^{\rho}-\tau^{\rho})^{\alpha-2}|p(\tau)|d\tau, \tag{5.2.17}$$

or equivalently,

$$\int_a^b \tau^{\rho-1}(b^{\rho}-\tau^{\rho})^{\alpha-2}|p(\tau)|d\tau \geq \frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^{\rho}-a^{\rho})+\beta_1}.$$

Then we get

$$\int_a^b \tau^{\rho-1}(b^\rho-\tau^\rho)^{\alpha-2}p^+(\tau)|d\tau \geq \frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho-a^\rho)+\beta_1}. \tag{5.2.18}$$

The proof is complete. ■

**Corollary 5.2.4** (Disconjugacy)**.** *if*

$$\int_a^b s^{\rho-1}(b^\rho-s^\rho)^{\alpha-2}|p(s)|ds < \frac{\Gamma(\alpha)}{\rho^{1-\alpha}(b^\rho-a^\rho)+\beta_1}. \tag{5.2.19}$$

*Then (1.2.1) is disconjugate in* $[a,b]$.

## 5.2.2 The special case $\beta_1 = 0$

An inspection of the formula for $g_1$ shows that

$$\lim_{\tau\to b} G(\tau,\tau) = \lim_{\tau\to b} g_1(\tau,\tau) = +\infty \Longleftrightarrow \beta_1 > 0$$

Consequently, in this subsection, we assume that $\beta_1 = 0$ with $(\gamma_0 = \rho\beta_0 a^{\rho-1})$.

Let consider

$$\begin{aligned} g(\tau,\tau) &= \frac{\rho\beta_0 a^{\rho-1}+(\tau^\rho-a^\rho)}{\rho\beta_0 a^{\rho-1}+(b^\rho-a^\rho)}(b^\rho-a^\rho)^{\alpha-1}. \\ &= \frac{\gamma_0+(\tau^\rho-a^\rho)}{\rho\beta_0 a^{\rho-1}+(b^\rho-a^\rho)}(b^\rho-a^\rho)^{\alpha-1} \end{aligned}$$

We have

$$\max_{\tau\in\{a,b\}} g(\tau,\tau) = \begin{cases} \left(\rho\beta_0 a^{\rho-1}-a^\rho+b^\rho\right)^{\alpha-1}, & \text{if} \quad \tau < \rho\beta a^{\rho-1} \leq \frac{(b^\rho-a^\rho)}{\alpha-1}, \\ \frac{\rho\beta_0 a^{\rho-1}}{\rho\beta_0 a^{\rho-1}+(b^\rho-a^\rho)}(b^\rho-a^\rho)^{\alpha-1}, & \text{if} \quad \rho\beta a^{\rho-1} > \frac{(b^\rho-a^\rho)}{\alpha-1}, \end{cases}$$

where $\tau = \left(\frac{\rho b^{\rho-1}}{\alpha-1} - 1\right)(b^\rho - a^\rho)$.

$$\begin{aligned}\frac{dg}{d\tau}(\tau,\tau) &= \tfrac{\rho\tau^{\rho-1}}{\gamma_0+b^\rho-a^\rho}(b^\rho-\tau^\rho)^{\alpha-1} - \rho\tau^{\rho-1}(\alpha-1)\tfrac{\gamma_0+\tau^\rho-a^\rho}{\gamma_0+b^\rho-a^\rho}(b^\rho-a^\rho)^{\alpha-2}\\ &= \frac{\rho\tau^{\rho-1}}{\gamma_0+b^\rho-a^\rho}(b^\rho-\tau^\rho)^{\alpha-2}\left[(b^\rho-\tau^\rho)-(\alpha-1)(\gamma_0+b^\rho-a^\rho)\right].\end{aligned}$$

Hence

$$\begin{aligned}\frac{dg}{d\tau}(\tau,\tau)=0 &\Longleftrightarrow \alpha\tau^\rho = b^\rho-(\alpha-1)(\alpha_0-a^\rho)\\ &\Longleftrightarrow \alpha\tau^\rho = b^\rho-(\alpha-1)(\gamma_0-a^\rho+b^\rho)\\ &\Longleftrightarrow \tau^\rho = b^\rho-\frac{(\alpha-1)}{\alpha}(\gamma_0-a^\rho+b^\rho)=\tau_0\end{aligned}$$

Then

$$\begin{aligned}&\frac{dg}{d\tau}(\tau,\tau)\geq 0 \Leftrightarrow \tau^\rho \leq \tau_0\\ &\tau\in[a,b]\Rightarrow \tau^\rho\in[a^\rho,b^\rho]\\ &\tau_0 = b^\rho-\frac{\alpha-1}{\alpha}(\gamma_0+b^{\rho-a^\rho})\Rightarrow \tau_0\leq b^\rho\end{aligned}$$

So, we have two cases: $\tau_0\in[a,b]$ or $\tau_0\in[-\infty,a]$

Case 1. $\tau_0\in(a^\rho,b^\rho)$ Then $\gamma_0\leq \dfrac{1}{\alpha-1}(b^\rho-a^\rho)$.

In this case, $\max_{\tau\in\{a,b\}} g(\tau,\tau) = g(\tau_0^{1/\rho},\tau_0^{1/\rho})$

$$g(\tau_0^{1/\rho},\tau_0^{1/\rho}) = \frac{\gamma_0+\tau_0-a^\rho}{\gamma_0+b^\rho-a^\rho}(b^\rho-\tau^0)^{\alpha-1}.$$

By replacing $\tau_0$, we get:

$$\begin{aligned}\max_{\tau\in\{a,b\}} g(\tau,\tau) &= (\gamma_0-a^\rho+b^\rho)^{\alpha-1}\left(\frac{\alpha-1}{\alpha}\right)^\alpha \alpha\\ &= (\gamma_0-a^\rho+b^\rho)^{\alpha-1}\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\end{aligned}$$

Case 2. $\tau_0 \in [a,b] \Rightarrow \gamma_0 > \frac{1}{\alpha-1}(b^\rho - a^\rho)$

In this case,

$$\begin{aligned}\max_{\tau\in\{a,b\}} g(\tau,\tau) &= g(a,a)\\ &= \frac{\gamma_0}{\gamma_0 + (b^\rho - a^\rho)(b^\rho - a^\rho)^{\alpha-1}}.\end{aligned}$$

**Theorem 5.2.5.** *Suppose that the fractional boundary problem (1.2.1-1.2.2) has a nontrivial solution* $x$. *Then:*

$$\int_a^b p^+(\tau)\,d\tau \geq \begin{cases} \dfrac{\rho^{\alpha-1}c^{\rho-1}\Gamma(\alpha)\alpha^\alpha}{}, & \text{if } \rho\beta_0 a^{\alpha-1} \leq \frac{b^\rho - a^\rho}{\alpha-1},\\ \frac{\rho^{\alpha-1}c^{\rho-1}(\rho\beta_0 a^{\rho-1} + b^\rho - a^\rho)}{\rho\beta_0 a^{\alpha-1}(b^\rho - a^\rho)^{\alpha-1}}, & \text{if } \rho\beta_0 a^{\rho-1} > \frac{b^\rho - a^\rho}{\alpha-1}.\end{cases}$$

*where* $(c = a \text{ if } 0 < \rho < 1; c = b \text{ if } \rho < 1)$

**Proof.**

$$\begin{aligned}|x(t)| &\leq \int_a^b G(\tau,\tau)p^+(\tau)|x(\tau)|d\tau\\ \max_{\tau\in\{a,b\}}|x(t)| &\leq \int_a^b G(\tau,\tau)p^+(\tau)|x(\tau)|d\tau\\ &\Rightarrow |x(t)| \leq \int_a^b \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\tau^{\rho-1}g(\tau,\tau)p^+(\tau)|x(\tau)|d\tau\\ \|x\|_\infty &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}c^{\rho-1}\int_a^b g(\tau,\tau)p^+(\tau)\,\|x\|_\infty\,d\tau\end{aligned}$$

$$c = \begin{cases} a, & \text{if } 0 < \rho < 1\\ b, & \text{if } \rho > 1\end{cases}$$

$$\begin{aligned}1 &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}c^{\rho-1}\int_a^b g(\tau,\tau)p^+(\tau)d\tau\\ \int_a^b p^+(\tau)d\tau &\geq \frac{\Gamma(\alpha)}{\rho^{1-\alpha}c^{\rho-1}\max_{\tau\in\{a,b\}} g(\tau,\tau)}\end{aligned}$$

■

**Remark 5.2.6.** We note that the results obtained in this subsection generalize the results by R. P. Agarwal, A. Özbekler, see [30].

## 5.3 Hartman type inequality for the problem (1.2.3-1.2.2)

**Theorem 5.3.1.** *Let* $h : [a, b] \longrightarrow \mathbb{R}^+$ *be a real nontrivial Lebesgue integrable function. If the fractional boundary value problem (1.2.3) and (1.2.2) has a nontrivial solution* $x$*. Assume that there exist nonnegative numbers* $A, B,$ *and* $\mu$ *such that (4.1.10) holds, if* $x(t) > 0$ *in* $(a, b)$ *then the inequality*

$$\frac{\left(\int_a^b \tau^{\rho-1}\left(b^a-\tau^a\right)^{\alpha-2}\left(p^+(\tau)+q^+(\tau)\right)d\tau\right)}{\left(\int_a^b \tau^{\rho-1}\left(b^\rho-\tau^\rho\right)\left(\mu_0 p^+(\tau)+\gamma_0 q^+(\tau)+f^-(\tau)\right)d\tau\right)^{-1}} > \frac{\Gamma^2(\alpha)}{4\rho^{2-2\alpha}\left(b^\rho-a^\rho+\beta_1\right)^2} \tag{5.3.1}$$

*holds, where the constant* $\mu_0$ *is defined in (1.1.12).*

**Proof.** Let $x(t)$ be a positive solution of (1.2.3) with ( 1.2.2) where where $a < b$ are consecutive zeros. Then by using Theorem 5.2.1, $x(t)$ can be expressed as

$$x(t) = \int_a^b G(t,\tau)\left[p(\tau)h(x(\tau)) - f(\tau)\right]d\tau. \tag{5.3.2}$$

By (5.3.2), we get that

$$|x(t)| \leq \int_a^b G(\tau,\tau)\left|p(\tau)h(x(\tau)) + f^-(\tau)\right|d\tau, \tag{5.3.3}$$

where

$$f^-(\tau) = \max\left(-f(\tau), 0\right). \tag{5.3.4}$$

It follows from $x'(a) + B_0 x'(a) = x(b) + B_1 x'(b) = 0$ and $x$ is not identically zero on $[a, b]$ one can choose $c \in (a, b)$ such that

$$\mathscr{M} = |x(c)| = \max_{t\in(a,b)}\left(|x(t)|\right). \tag{5.3.5}$$

By assumption: From (5.3.3) and (5.3.4), we observe that

$$\begin{aligned} |x(c)| &\leq \frac{1}{\Gamma(\alpha)}\left(\rho\left(b^{\rho}-a^{\rho}\right)\right)^{1-\alpha}\int_{a}^{b}\frac{p^{+}(\tau)\left[A\left|x^{\mu}(c)\right|+B\right]+f^{-}(\tau)}{\left(\tau^{\rho}-a^{\rho}\right)^{1-\alpha}\left(b^{\rho}-\tau^{\rho}\right)^{1-\alpha}}\frac{d\tau}{\tau^{1-\rho}} \\ &\leq P_{A}\left|x^{\mu}(c)\right|+P_{B}+F_{0}, \end{aligned} \tag{5.3.6}$$

where

$$P_{A}=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(b^{\rho}-a^{\rho}+\beta_{1}\right)\int_{a}^{b}\tau^{\rho-1}\left(b^{\rho}-\tau^{\rho}\right)^{\alpha-2}p^{+}(\tau)d\tau, \tag{5.3.7}$$

$$P_{B}=\frac{B}{A}P_{A}=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(b^{\rho}-a^{\rho}+\beta_{1}\right)\int_{a}^{b}\tau^{\rho-1}\left(b^{\rho}-\tau^{\rho}\right)^{\alpha-2}q^{+}(\tau)d\tau \tag{5.3.8}$$

and

$$F_{0}=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(b^{\rho}-a^{\rho}+\beta_{1}\right)\int_{a}^{b}\tau^{\rho-1}\left(b^{\rho}-\tau^{\rho}\right)^{\alpha-2}f^{-}(\tau)d\tau. \tag{5.3.9}$$

On the other hand, (4.1.10) in Lemma 4.1.3 with $\mathbf{A}=\mathbf{B}=1$, implies that

$$\left|x^{\mu}(c)\right|\leq x^{2}(c)+\mu_{0}, \tag{5.3.10}$$

we have

$$\begin{aligned} |x(c)| &\leq \frac{1}{\Gamma(\alpha)}\left(\rho\left(b^{\rho}-a^{\rho}\right)\right)^{1-\alpha}\int_{a}^{b}\frac{\left|p^{+}(\tau)\right|\left|Ax^{2}(c)+A\mu_{0}+B\right|+f^{-}(\tau)}{\left(\tau^{\rho}-a^{\rho}\right)^{1-\alpha}\left(b^{\rho}-\tau^{\rho}\right)^{1-\alpha}}\frac{d\tau}{\tau^{1-\rho}} \\ &\leq P_{A}\left[x^{2}(c)+\mu_{0}\right]+P_{B}+F_{0},. \end{aligned} \tag{5.3.11}$$

Using these inequalities (5.3.11), we find the following quadratic inequality

$$P_{A}x^{2}(c)-|x(c)|+\left(P_{A}\mu_{0}+P_{B}+F_{0}\right)>0. \tag{5.3.12}$$

The expression

$$\Delta=1-4P_{A}\left(P_{A}\mu_{0}+P_{B}+F_{0}\right),$$

is called the discriminant. If $\Delta<0$ then the associated quadratic equation of the inequality (5.3.12) has no real roots.

Therefore, the inequality (5.3.12) is possible only when

$$P_A\left(P_A\mu_0+P_B+F_0\right)>\frac{1}{4},$$

This previous inequality gives

$$\frac{\left(\int_a^b\left(\frac{\tau^\rho-a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}p^+(\tau)\frac{d\tau}{\tau^{1-\rho}}\right)}{\left(\int_a^b\left(\frac{\tau^\rho-a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}(p^+(\tau)(A\mu_0+B)+f^-(\tau))\frac{d\tau}{\tau^{1-\rho}}\right)^{-1}}>\left(\frac{\Gamma(\alpha)}{2A}\right)^2\left(\frac{\rho}{b^\rho-a^\rho}\right)^{2-2\alpha}$$

this is equivalent to (5.3.1). ■

We establish a some versions generalized Hartman and Lyapunov type integral inequality.

**Theorem 5.3.2** (Hartman type inequality)**.** *Assume that* $x(t)$ *be a nontrivial solution of (1.2.3) gratifying the Dirichlet boundary conditions (1.2.2). Whenever* $x(t)>0$ *in* $(a,b)$*, Let* $|x|$ *be maximized at a point* $c\in(a,b)$, *below mentioned inequality*

$$\frac{\left(\int_a^b\tau^{\rho-1}(b^a-\tau^a)^{\alpha-2}\left(p^+(\tau)+q^+(\tau)\right)d\tau\right)}{\left(\int_a^b\tau^{\rho-1}(b^\rho-\tau^\rho)(\mu_0p^+(\tau)+\gamma_0q^+(\tau)+f^-(\tau))d\tau\right)^{-1}}>\frac{\Gamma^2(\alpha)}{4\rho^{2-2\alpha}(b^\rho-a^\rho+\beta_1)^2}\tag{5.3.13}$$

*holds, where the constants* $\mu_0$ *and* $\lambda_0$ *are defined in (1.1.12).*

**Proof.** Besides that, we observe that (5.2.2) can be used to represent the solution of the generalised fractional differential equation (1.2.3) of order $\alpha\in(0,2]$, fulfilling the Dirichlet boundary conditions (1.2.2); where $G(t,\tau)$ is defined by (5.2.1 ). Now let $x(t)$ be a positive solution of (1.2.3) and (1.2.2 ), $x(t)$ can be expressed as

$$x(t)=\int_a^b G(t,\tau)\left[p(\tau)x^\mu(\tau)+q(\tau)x^\lambda(g(\tau))-f(\tau)\right]d\tau\tag{5.3.14}$$

Now, since $g(\tau)\le\tau$ we have $x(g(\tau))\le x(\tau)$ and by Theorem 5.2.1 , Lemma 3.2.4, (5.3.5) and (5.3.14), we have

$$|x(c)| = \left|\int_a^b G(c,\tau)\left[p(\tau)x^\mu(c)+q(\tau)x^\lambda(g(c))-f(\tau)\right]d\tau\right|\tag{5.3.15}$$

$$\le \int_a^b G(\tau,\tau)\left[p^+(\tau)x^\mu(c)+q^+(\tau)x^\lambda(g(c))+f^-(\tau)\right]d\tau\tag{5.3.16}$$

$$|x(c)| \leq \frac{1}{\Gamma(\alpha)}\int_a^b\left[\left(\frac{\tau^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\left(\frac{b^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\right]\left[p^+(\tau)x^\mu(c)+q^+(\tau)x^\lambda(c)+f^-(\tau)\right]\frac{d\tau}{\tau^{1-\rho}}$$

$$\leq \frac{1}{\Gamma(\alpha)}\left(\frac{\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\int_a^b\left[\left(\frac{\tau^\rho-a^\rho}{\rho}\right)^{\alpha-1}\left(\frac{b^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\right]\times$$

$$\left[p^+(\tau)x^\mu(c)+q^+(\tau)x^\lambda(c)+f^-(\tau)\right]\frac{d\tau}{\tau^{1-\rho}}$$

$$\leq P_0x^\mu(c)+Q_0x^\lambda(c)+F_0,$$

where

$$P_0=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}(b^\rho-a^\rho+\beta_1)\int_a^b\tau^{\rho-1}(b^\rho-\tau^\rho)^{\alpha-2}p^+(\tau)d\tau, \tag{5.3.17}$$

$$Q_0=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}(b^\rho-a^\rho+\beta_1)\int_a^b\tau^{\rho-1}(b^\rho-\tau^\rho)^{\alpha-2}q^+(\tau)d\tau \tag{5.3.18}$$

and

$$F_0=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}(b^\rho-a^\rho+\beta_1)\int_a^b\tau^{\rho-1}(b^\rho-\tau^\rho)^{\alpha-2}f^-(\tau)d\tau. \tag{5.3.19}$$

On the other hand, with (4.1.10), when $\mathbf{A}=\mathbf{B}=1$, thus

$$x^\mu(c)\leq x^2(c)+\mu_0 \text{ and } x^\lambda(g(c))\leq x^2(c)+\lambda_0$$

Using these inequalities and (5.3.15) we find the following quadratic inequality

$$(P_0+Q_0)x^2(c)-|x(c)|+(\mu_0P_0+\lambda_0Q_0+F_0)>0.$$

But this is possible only when

$$(P_0+Q_0)(\mu_0P_0+\lambda_0Q_0+F_0)>\frac{1}{4}. \tag{5.3.20}$$

That implies

$$\left(\frac{1}{\rho}\right)^{4(\alpha-1)}\frac{\left(\int_a^b\frac{p^+(\tau)+q^+(\tau)}{(\tau^\rho-a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\frac{d\tau}{\tau^{1-\rho}}\right)}{\left(\int_a^b\frac{(\mu_0p^+(\tau)+\lambda_0q^+(\tau)+f^-(\tau))}{(\tau^\rho-a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\frac{d\tau}{\tau^{1-\rho}}\right)^{-1}} > \left(\frac{\Gamma(\alpha)}{2}\right)^2\left(\frac{\rho}{b^\rho-a^\rho}\right)^{2-2\alpha},$$

or the following

$$\frac{\left(\int_a^b \frac{p^+(\tau)+q^+(\tau)}{(\tau^\rho-a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\frac{d\tau}{\tau^{1-\rho}}\right)}{\left(\int_a^b \frac{(\mu_0 p^+(\tau)+\lambda_0 q^+(\tau)+f^-(\tau))}{(\tau^\rho-a^\rho)^{1-\alpha}(b^\rho-\tau^\rho)^{1-\alpha}}\frac{d\tau}{\tau^{1-\rho}}\right)^{-1}} > \left(\frac{\Gamma(\alpha)}{2}\right)^2 (\rho(b^\rho-a^\rho))^{2\alpha-2},$$

this is equivalent to (5.3.13) . This concludes the proof of Theorem 5.3.2. ■

**Theorem 5.3.3** (Hartman type inequality)**.** *Assume that* $x(t)$ *be a nontrivial solution of (1.2.3) gratifying the Dirichlet boundary conditions (1.2.2). Whenever* $x(t)<0$ *in* $(a,b)$*, below mentioned*

$$\frac{\left(\int_a^b \tau^{\rho-1}(b^a-\tau^a)^{\alpha-2}\left(p^+(\tau)+q^+(\tau)\right)d\tau\right)}{\left(\int_a^b \tau^{\rho-1}(b^\rho-\tau^\rho)(\mu_0 p^+(\tau)+\gamma_0 q^+(\tau)+f^+(\tau))d\tau\right)^{-1}} > \frac{\Gamma^2(\alpha)}{4\rho^{2-2\alpha}(b^\rho-a^\rho+\beta_1)^2} \tag{5.3.21}$$

*holds, where the constants* $\mu_0$ *and* $\lambda_0$ *are the same as in ( 1.1.12).*

**Remark 5.3.4.** When $\alpha=2$ and $\rho=1$, then Theorem 5.3.3 coincides with [**?**, Theorem 2.4].

**Corollary 5.3.5** (Disconjugacy)**.** *If*

$$\frac{\left(\int_a^b \tau^{\rho-1}(b^a-\tau^a)^{\alpha-2}\left(p^+(\tau)+q^+(\tau)\right)d\tau\right)}{\left(\int_a^b \tau^{\rho-1}(b^\rho-\tau^\rho)(\mu_0 p^+(\tau)+\gamma_0 q^+(\tau)+|f(\tau)|)d\tau\right)^{-1}} \leq \frac{\Gamma^2(\alpha)}{4\rho^{2-2\alpha}(b^\rho-a^\rho+\beta_1)^2} \tag{5.3.22}$$

*then (1.2.1) is disconjugate in* $[a,b]$*, where the constants* $\mu_0$ *and* $\lambda_0$ *are the same as in (1.1.12).*

## 5.4 Conclusion

In this study, we focused on generalized Lyapunov-type and Hartman-type inequalities for boundary value problems with mixed nonlinearities, where $\alpha\in(1,2)$. We also addressed the concept of disconjugacy through these inequalities within the framework of forced differential equations. The innovative formulas we propose contribute to advancing the theoretical understanding of various models in fractional calculus and open new avenues for research and applications.

# *CHAPTER 6*

## *Conclusion and future work*

### Summary

This thesis is devoted to obtain generalized Lyapunov and Hartman type inequalities for generalized Caputo boundary value problems characterized by mixed nonlinearities with delayed and advanced arguments $g$ , specifically with $\alpha$ in the range of $(1,2)$. Moreover, we address the concept of disconjugacy through these inequalities in the framework of forced differential equations, employing the positive parts $p^+$ and $q^+$ of the functions $p$ and $q$, respectively, and utilizing the maximum of the associated Green's function $G_{max}$, we've achieved greater precision in our results compared to what's currently available in the expansive literature. The innovative formulas we present are poised to significantly advance the theoretical understanding of various models in fractional calculus, opening new avenues for research and application.

Finally, we conclude with a few remarks connecting our findings to previous results in the literature:

(i) This article addresses a significant gap in the literature concerning Lyapunov-type inequalities for forced, generalized Caputo fractional differential equations with mixed nonlinearities and deviating arguments. To the best of our knowledge, such a setting has not been previously studied with the level of generality and rigor presented here.

(ii) Compared to previous works such as [28, 27], which focused on linear, unforced fractional differential equations under Dirichlet boundary conditions, the theorems presented in this article extend the theory to forced and nonlinear problems involving generalized Caputo derivatives.

(iii) Furthermore, our results yield new Lyapunov-type inequalities involving the Caputo Katugampola fractional derivative, encompassing several previously studied models as special cases:

- If $\alpha \in (1,2)$ and $\rho \to 0^+$, our results extend those in [37, 38, 39].
- If $\alpha \in (1,2)$ and $\rho \to 1$, the boundary value problems (**??**) and (**??**) recover the results presented in [36, 37].
- If $\alpha = 2$, $\rho \to 1$ and $g(t) = t$, the boundary value problems (1.2.3) reduces to that of [21, 30].

(iv) The Caputo Katugampola fractional derivative employed in this work generalizes both the Caputo with integral of Riemann Liouville and Caputo with Hadamard operators. As a result, it provides a unified framework that may help avoid the proliferation of separate studies dedicated to individual fractional operators.

(v) This generalization is nontrivial due to two main factors:

– The nonlocal nature of the generalized Caputo derivative,

– The complexity introduced by the nonlinear and forcing terms.

(vi) The inequalities obtained in this work serve as effective analytical tools for studying the qualitative properties of solutions, particularly in establishing conditions that guarantee the nonexistence of nontrivial positive solutions.

(vii) This work opens a fruitful direction for future research, notably through the study variants of fractional differential equations and the use of other generalizations of fractional integral operators, particularly those defined with more general kernels.

(viii) It would be relevant to examine other types of inequalities related to our results, such as , Wirtinger-type or Opial-type inequalities in the fractional framework, which could contribute to deepening the qualitative theory of such problems.

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